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Event-triggered control using a positive systems approach ☆

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ABSTRACT

We provide a new event-triggered control strategy that relies on the notion of positive systems. Our results cover output feedback, and robustness with respect to uncertain dynamics. Our proofs are based on interval observers. Our examples show potential advantages of our approach over earlier event-triggered methods.

1. Introduction

Event-triggered control has played an increasingly important role in control theoretic research; see, for instance, the works [6,8,9,15,17,18]. One advantage of using event-triggered control is that it can reduce the computational burden associated with implementing controls, by only changing the value of the control when there is a significant event. These events can be characterized as times when the state enters some prescribed region of the state space. This contrasts with standard zero-order hold strategies, where the times at which the control values are recomputed are usually independent of the state. Much event-triggered control literature can be reformulated as an interconnected control system problem to which small gain methods can be applied; see, e.g., [7]. Small gain methods have the desirable advantage of ensuring robustness to uncertainty, but can sometimes be conservative, insofar that they can lead to unnecessarily frequent control recomputation times.

Although emerging computing methods can facilitate recomputing control values, the increasing use of shared wireless (or shared wired) networked systems calls for designing controls that take computation, communication, and energy constraints into account [6]. This led to systematic designs for event-triggered controls, e.g., in [1,2,17,20]. At the same time, much research has been done on positive systems, i.e., systems where the nonnegative orthant is positively invariant, leading to new control analysis and designs that overcome some challenges of using traditional Lyapunov methods. Some works on positive systems use interval observers (as defined, e.g., in [3,12,16]), which provide intervals containing values of unknown states when the inequalities involving vector solutions are viewed componentwise; see [12,13]. Positive systems and interval observers led to advances in aerospace engineering, mathematical biology, and other areas.

This motivates our work, where we use a new positivity based event-triggered control technique to design control strategies that yield less conservative triggering conditions than those in literature such as [7], and where we also provide a robustness result that removes the requirement that there be a constant $C > 0$ such that there is a bound $||\delta(t)|| \leq C||x(t)||$ relating the state $x(t)$ and the uncertainty $\delta(t)$ at each time $t \geq 0$ that was present in important works such as [18]. We establish an output feedback and a robustness result, covering cases where there may be time-varying uncertainties in the dynamics. Our stability proofs use interval observers as comparison systems, and are reminiscent of the results of [13] insofar that they are based on linear Lyapunov functions for positive systems. However, [13] did not cover event-triggering, and to the best of our knowledge, our work is the first systematic use of interval observers and positive systems to design event-triggered controls.

We provide our notation and preliminaries and introduce our main class of systems in Section 2. Then, in Section 3, we provide our main stability theorem, our extension to output feedback control using an observer, and our robustness theorem for uncertain models. In Section 4, we illustrate potential advantages of our

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method using an example where our method ensures less frequent control recomputations than the main small gain event-triggered result in [7] and also less than a result from [17]. We close in Section 5 by summarizing our findings and suggesting future research directions.

2. Preliminaries

We use the following notation, where the dimensions of our Euclidean spaces are arbitrary unless otherwise noted. The arguments of functions are omitted when no confusion can arise from the context. Set $\mathbb{Z}_0 = \{0, 1, 2, \ldots\}$ and $\mathbb{N} = \mathbb{Z}_0 \backslash \{0\}$. For a matrix $G = [g_{ij}] \in \mathbb{R}^{r\times s}$, we set $|G| = [|g_{ij}|]$, i.e., the entries of $|G|$ are the absolute values of the corresponding entries g_{ij} of G. Similarly, we set $G^+ = [\max\{g_{ij}, 0\}]$ and $G^- = G^+ - G$, and $\sup_{\ell\in J} |G(\ell)| = [m_{ij}]$ where $m_{ij} = \sup_{\ell\in J} |g_{ij}(\ell)|$ when G is time-varying and bounded matrix valued function and J is an interval in the domain of G. A square matrix is called Metzler provided all of its off-diagonal entries are nonnegative. For two matrices $D = [d_{ij}]$ and $E = [e_{ij}]$ of the same size, we write $D < E$ (resp., $D \le E$) provided $d_{ij} < e_{ij}$ (resp., $d_{ij} \le e_{ij}$) for all i and j. We also write $D \not\le E$ provided there is a pair (i, j) such that $d_{ij} > e_{ij}$. We adopt similar notation for vectors.

A matrix S is called positive provided $0 < S$, where 0 is the zero matrix. Let $||\cdot||$ denote the standard Euclidean 2-norm of vectors and matrices, and $||\cdot||_\infty$ (resp., $||\cdot||_J$) denote the corresponding sup norm of matrix valued functions over their entire domain (resp., over an interval J in their domain). We let I denote the identity matrix.

We consider the system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

where x is valued in $\mathbb{R}^n$, u is valued in $\mathbb{R}^p$, and the matrices A and B are constant. We assume:

Assumption 1. There is a matrix $K \in \mathbb{R}^{p\times n}$ such that the matrix $A_{\rm cl} = A + BK$ is Hurwitz and Metzler.□

Remark 1. Assumption 1 is not restrictive because many systems satisfy it after a change of coordinates. When (A, B) is controllable, there is a change of coordinates that provides new matrices A and B that satisfy this assumption. In fact, if (A, B) is controllable, then there is a $K \in \mathbb{R}^{p\times n}$ such that all eigenvalues of $A + BK$ are negative real numbers. Then there exists a matrix $P \in \mathbb{R}^{n\times n}$ such that $P(A + BK)P^{-1}$ is Metzler because the Jordan canonical form of $A + BK$ is Metzler. Thus Assumption 1 is satisfied by the pair (PAP^{-1}, PB). □

Recall that the Zeno phenomenon is that a system with sampling has infinitely many sample times on some interval of finite length. A key ingredient in our analysis in later sections will be finding a lower bound $\nu > 0$ on the inter-sample times $t_{i+1} - t_i$ between the event triggering times in all three of our theorems which will imply that the Zeno phenomenon does not occur, which will ensure implementability of our control. This constant ν will be provided by the following lemma:

Lemma 1. *Let K satisfy Assumption 1. Then there is a positive matrix $\Gamma \in \mathbb{R}^{n\times n}$ such that the matrix*

$$M = A_{\rm cl} + |BK|\Gamma \tag{2}$$

is Metzler and Hurwitz. Also, using the function $\Omega : [0, +\infty) \to \mathbb{R}^{n\times n}$ that is defined by

$$\Omega(s) = e^{sA} + \int_0^s e^{\ell A}{\rm d}\ell BK, \tag{3}$$

there exists a constant $\nu > 0$ such that for all $s \in [0, \nu]$, the matrix $\Omega(s)$ is invertible and such that the inequality

$$\left|I - \Omega(s)^{-1}\right| \le \Gamma \tag{4}$$

is satisfied.□

Proof. First observe that Assumption 1 implies that there exists a positive matrix $\Gamma \in \mathbb{R}^{n\times n}$ such that M as defined in (2) is Metzler and Hurwitz. In fact, since $0 \le |BK|$, it follows that $0 \le |BK|\Gamma$ if $0 < \Gamma$, which implies that M is Metzler. Moreover it is Hurwitz when the entries of Γ are sufficiently small (by the continuity of eigenvalues as functions of the characteristic polynomial's coefficients, and so also of the entries of the matrix [5]). We conclude by observing that Ω is continuous, $\Omega(0) = I$, and $0 < \Gamma$. □

Remark 2. Since e^{sA} is invertible, the matrix $\Omega(s)$ in (3) with $s \ge 0$ is invertible if and only if $I + \int_0^s e^{(\ell-s)A}{\rm d}\ell BK$ is invertible. Thus, it is invertible if

$$\left|\left|\int_0^s e^{(\ell-s)A}{\rm d}\ell BK\right|\right| < 1, \tag{5}$$

which follows by checking that the nullspace of $I + \int_0^s e^{(\ell-s)A}{\rm d}\ell BK$ is trivial. The inequality (5) is satisfied if $BK = 0$. If $BK \ne 0$, then for any $\epsilon_0 \in (0, 1)$, (5) holds if $s < s_*$, where s_* is the sup of all r values such that the left side of (5) is bounded above by ϵ_0 for all $s \in [0, r]$. A more explicit bound on the allowable s values in the $BK \ne 0$ case can be obtained as follows. If $BK \ne 0$, then (5) is satisfied if

$$\int_0^s e^{(s-\ell)||A||}{\rm d}\ell < \frac{1}{||BK||}. \tag{6}$$

If $A = 0$, then this gives the condition $s < 1/||BK||$. On the other hand, if $A \ne 0$, then we instead have the condition $(e^{s||A||} - 1)/||A|| < 1/||BK||$ which is equivalent to $s < (1/||A||)\ln(1 + ||A||/||BK||)$. □

Remark 3. Notice for later use that since M defined in (2) is Metzler and Hurwitz, there are a vector $V \in \mathbb{R}^n$ satisfying $0 < V$, and a constant $p \in (0, +\infty)$, such that

$$V^\top M \le -pV^\top \tag{7}$$

holds. This follows from Haddad et al. [4, Lemma 2.3, p. 41]. □

3. Main results

3.1. State feedback event-triggered control

We next prove an event-triggered control theorem, whose event-triggered control and triggering times t_i will be defined by

(a) $\dot{x}(t) = Ax(t) + BKx(t_i)$ if $t \in [t_i, t_{i+1})$,
(b) $|x(t) - x(t_i)| \le \Gamma|x(t)|$ if $t \in [t_i, t_{i+1})$, and
(c) For each $\epsilon > 0$, and for each $i \in \mathbb{Z}_0$ such that $t_{i+1} < +\infty$, there is a $t_\ddagger \in (t_{i+1}, t_{i+1} + \epsilon)$ such that $|x(t_\ddagger) - x(t_i)| \not\le \Gamma|x(t_\ddagger)|$

for all i, where $t_0 = 0$ and K is from Assumption 1, and where condition (c) means that for each $\epsilon > 0$, there exist $t_\ddagger \in (t_{i+1}, t_{i+1} + \epsilon)$ and $j \in \{1, \ldots, n\}$ such that

$$|x_j(t_\ddagger) - x_j(t_i)| > \sum_{g=1}^{n} \Gamma_{jg}|x_g(t_\ddagger)| \tag{8}$$

is satisfied, where $\Gamma = [\Gamma_{ij}]$. The dynamics in (a)-(c) call for resetting the control values at certain times t_i, as follows. At time $t_0 = 0$, we choose the control value $u(0) = Kx(0)$. Then, we maintain the control value at $u(t) = Kx(0)$ as long as $|x(t) - x(0)| \le \Gamma|x(t)|$ continues to hold, which produces an interval $[0, t_*]$ with $t_* \ge \nu$ during which the control value stays at $u(t) = Kx(0)$, by Lemma 1 and (9) below. If, at some later time, $|x(t) - x(0)| \le \Gamma|x(t)|$ is violated (in the sense of (8)), then we change the control value to $u(t_1) = Kx(t_1)$ where t_1 is the infimum of all times $t > 0$ when such a violation occurs. We repeat this process with t_0 replaced by t_1, and argue inductively, to define the control for all $t \ge 0$.

Hence, we have a sampled control, with event-triggered sample times defined by (b). This produces an infinite sequence of triggering times t_i, or only finitely many t_i's. Moreover, after resetting the control values at the times t_i, the inequality in (b) will hold for all $t \geq 0$. Similar reasoning applies for our other theorems below. A key novel feature of our theorems is our use of the matrix of absolute values $|\cdot|$ instead of the usual Euclidean norm (e.g., in (b) above), which can reduce the number of sample times t_i on given intervals; see our illustrations below. Our event-triggered control (a)-(c) can be viewed as a novel combination of emulation (because it uses the nominal control Kx in (a)) and co-design (because it designs the matrix Γ in the triggering rule in (b)). Our first theorem is then:

Theorem 1. *Consider the system (1) under Assumption 1 and let $\Gamma > 0$ satisfy the requirements of Lemma 1 and K satisfy the requirements from Assumption 1. Consider the nonnegative t_i's defined by $t_0 = 0$ and (a)-(c) above. Then the closed loop system given by (a)-(c) admits the origin as a globally exponentially stable equilibrium point on $\mathbb{R}^n$.* □

Proof. *First Part.* Letting ν be the constant from Lemma 1, we prove that the inter-event times $t_{i+1} - t_i$ are bounded below by ν. More precisely, we prove that either there are only a finite number of instants $t_0, \ldots, t_j$ and $\min_{l \in \{0,\ldots,j-1\}}(t_{l+1} - t_l) \geq \nu$ when $j > 0$; or else there are infinitely many instants t_i and $\inf_{l \in \mathbb{Z}_0}(t_{l+1} - t_l) \geq \nu$.

Consider any $i \in \mathbb{Z}_{\geq 0}$ such that the Zeno phenomenon does not occur on $[0, t_i]$. If $x(t_i) = 0$, then the theorem on existence and uniqueness of solutions gives $x(t) = 0$ for all $t \geq t_i$. Next, consider the case where $x(t_i) \neq 0$. This implies that $t_{i+1} = +\infty$ or t_{i+1} is finite and $t_{i+1} > t_i$. Consider the case where t_{i+1} is finite. By integrating the equation in (a) on the interval $[t_i, t)$ with $t \in [t_i, t_i + \nu]$ from the initial state $x(t_i)$, we get

$$x(t) = \Omega(t - t_i)x(t_i) \tag{9}$$

where Ω is defined in (3). From (4), it follows that, for any $t \in [t_i, t_i + \nu]$, the vector inequality $\left|I - \Omega(t - t_i)^{-1}\right||x(t)| \leq \Gamma|x(t)|$ is satisfied. It follows that $\left|x(t) - \Omega(t - t_i)^{-1}x(t)\right| \leq \Gamma|x(t)|$. This inequality in combination with (9) gives $|x(t) - x(t_i)| \leq \Gamma|x(t)|$ for all $t \in [t_i, t_i + \nu]$. We conclude that $t_{i+1} \geq t_i + \nu$. Thus, the Zeno phenomenon does not occur and the inter-sample times are bounded below by ν.

Second Part. We study the stability of the closed loop system from (a) to (c). The case where there is j such that $x(t_j) = 0$ is trivial. Thus, we consider the case where $x(t_j) \neq 0$ for all $j \in \mathbb{Z}_{\geq 0}$. For convenience, we introduce the function μ defined by $\mu(t) = x(t_i) - x(t)$ for all $t \in [t_i, t_{i+1})$ and all $i \in \mathbb{Z}_0$. Then (a)-(c) give

$$\begin{aligned} &\dot{x}(t) = A_{\rm cl}x(t) + BK\mu(t) \\ &|\mu(t)| \leq \Gamma|x(t)| \end{aligned} \tag{10}$$

for almost all $t \geq 0$, where $A_{\rm cl} = A + BK$ as before.

To study (10), we exploit the fact that $A_{\rm cl}$ is Metzler, to adopt an analysis approach that is based on interval observers. We introduce the dynamic extension

$$\begin{cases} \dot{\bar{x}}(t) = A_{\rm cl}\bar{x}(t) + (BK\mu(t))^+ \\ \dot{\underline{x}}(t) = A_{\rm cl}\underline{x}(t) - (BK\mu(t))^- \end{cases} \tag{11}$$

Consider a solution of (10) with $x(0) \in \mathbb{R}^n$ as its initial state and any initial states $\bar{x}(0) \in \mathbb{R}^n$ and $\underline{x}(0) \in \mathbb{R}^n$ for (11) such that $\underline{x}(0) < x(0) < \bar{x}(0)$ and $\underline{x}(0) < 0 < \bar{x}(0)$. Observe that $\bar{e}(t) = \bar{x}(t) - x(t)$ and $\underline{e}(t) = x(t) - \underline{x}(t)$ satisfy

$$\begin{cases} \dot{\bar{e}}(t) = A_{\rm cl}\bar{e}(t) + (BK\mu(t))^- \\ \dot{\underline{e}}(t) = A_{\rm cl}\underline{e}(t) + (BK\mu(t))^+ \end{cases} \tag{12}$$

Since $A_{\rm cl}$ is Metzler, it follows (e.g., from Mazenc et al. [11, Lemma 1]) that $\bar{e}(t) \geq 0$ and $\underline{e}(t) \geq 0$ for all $t \geq 0$. This gives

$$\underline{x}(t) \leq x(t) \leq \bar{x}(t) \ \text{ and } \ \underline{x}(t) \leq 0 \leq \bar{x}(t) \tag{13}$$

for all $t \geq 0$, where the last two inequalities in (13) follow because $\bar{x}$ and $-\underline{x}$ are solutions of $\dot{z} = A_{\rm cl}z + (BK\mu)^{\pm}$ and from again applying [11, Lemma 1] using the Metzler matrix $A_{\rm cl}$. We deduce from (13) that $\underline{x}(t) - \bar{x}(t) \leq x(t) \leq \bar{x}(t) - \underline{x}(t)$, i.e.,

$$|x(t)| \leq s(t) \tag{14}$$

for all $t \geq 0$, where $s(t) = \bar{x}(t) - \underline{x}(t)$.

We next analyze the behavior of s. We have

$$\begin{aligned} \dot{s}(t) &= A_{\rm cl}s(t) + (BK\mu(t))^+ + (BK\mu(t))^- \\ &= A_{\rm cl}s(t) + |BK\mu(t))|. \end{aligned} \tag{15}$$

Consider the linear function

$$W(s) = V^\top s, \tag{16}$$

where V satisfies the requirements of Remark 3. At each $t \geq 0$, the time derivative of W along (15) satisfies

$$\begin{aligned} \dot{W}(t) &= V^\top A_{\rm cl}s(t) + V^\top|BK\mu(t))| \\ &\leq V^\top A_{\rm cl}s(t) + V^\top|BK|\Gamma|x(t)|, \end{aligned} \tag{17}$$

where the last inequality is a consequence of the inequality in (10). Substituting (14) in (17), we obtain

$$\begin{aligned} \dot{W}(t) &\leq V^\top A_{\rm cl}s(t) + V^\top|BK|\Gamma s(t) \\ &= V^\top Ms(t), \end{aligned} \tag{18}$$

where M is the matrix defined in (2). Since s is nonnegative valued, it follows from (7) that

$$\dot{W}(t) \leq -pV^\top s(t) = -pW(s(t)). \tag{19}$$

Since $V > 0$ and $s(t) \geq 0$ for all $t \geq 0$, we deduce that $s(t)$ exponentially converges to zero as $t \to +\infty$. We can now convert the exponential stability estimate for $s(t)$ into the one for $x(t)$. To this end, we can assume that $s(0) \leq 4|x(0)|$, e.g., by requiring $\bar{x}(0) \leq 2|x(0)|$ and $\underline{x}(0) \geq -2|x(0)|$. Hence, from (14), we deduce that $x(t)$ goes exponentially to the origin as $t \to +\infty$. □

3.2. Output feedback control

The event-triggered control in Theorem 1 requires measurements of the state $x(t)$, and it is nontrivial to generalize it to cases where the state is not available for measurement. This motivates this subsection, where we consider the system (1) with an output, namely,

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) \tag{20}$$

with y valued in $\mathbb{R}^q$ and C being a constant matrix. We use this classical assumption:

Assumption 2. The pair (A, C) is observable.□

Assumption 2 provides a matrix $L \in \mathbb{R}^{n\times q}$ such that

$$G = A + LC \tag{21}$$

is Hurwitz. This allows us to prove the following, where solutions to (a')-(d') are defined in a recursive way that is analogous to the solutions in Theorem 1:

Theorem 2. *Consider the system (20) under Assumptions 1–2. Let $\Gamma > 0$ and $\nu > 0$ satisfy the requirements of Lemma 1, and K satisfy the requirements of Assumption 1. Consider the sequence of nonnegative numbers t_i and the feedback $K\hat{x}(t_i)$ defined by $t_0 = 0$,*

(a') $\dot{x}(t) = Ax(t) + BK\hat{x}(t_i)$ *if* $t \in (t_i, t_{i+1})$,
(b') $\dot{\hat{x}}(t) = A\hat{x}(t) + BK\hat{x}(t_i) + L[C\hat{x}(t) - y(t)]$ *if* $t \in (t_i, t_{i+1})$,
(c') $|\hat{x}(t) - \hat{x}(t_i)| \leq \Gamma|\hat{x}(t)| + \mathcal{B}\int_{\max\{0,t-\nu\}}^{t}|C\hat{x}(\ell) - y(\ell)|{\rm d}\ell$ *for all* $t \in [t_i, t_{i+1})$, *and*

(d') *For each* $\epsilon > 0$, *and each* i *such that* $t_{i+1} < +\infty$, *there is a* $t_\ddagger \in (t_{i+1}, t_{i+1} + \epsilon)$ *such that* $|\hat{x}(t_\ddagger) - \hat{x}(t_i)| \not\leq \Gamma|\hat{x}(t_\ddagger)| + \mathcal{B}\int_{\max\{0,t_\ddagger-\nu\}}^{t_\ddagger} |C\hat{x}(\ell) - y(\ell)|d\ell$

for all $i \in \mathbb{Z}_0$, *where* $\hat{x}(0) \neq 0$ *and*

$$\mathcal{B} = \sup_{\ell\in[0,\nu]} \left|\Omega(\ell)^{-1}\right| \sup_{\ell\in[0,\nu]} \left|e^{\ell A}L\right|. \tag{22}$$

Then the system given by (a')-(d') admits the origin as a globally exponentially stable equilibrium on $\mathbb{R}^n$. □

Proof. (Summary.) We only summarize the proof here; see [14] for a complete proof.

First part. We prove that the inter-event times $t_{i+1} - t_i$ are bounded below by ν. More precisely, we prove that either there are only a finite number of instants $t_0, \dots, t_j$ and $\min_{l\in\{0,\dots,j-1\}}(t_{l+1} - t_l) \geq \nu$ when $j > 0$; or there are an infinite number of instants t_i and $\inf_{l\in\mathbb{Z}_0}(t_{l+1} - t_l) \geq \nu$.

By integrating the $\hat{x}$-subsystem in (b') between $[t_i, t)$ with $t \in [t_i, t_i + \nu)$ with the initial state $\hat{x}(t_i)$, we can then obtain [14]

$$|\hat{x}(t) - \hat{x}(t_i)| \leq \Gamma|\hat{x}(t)| + \mathcal{B}\int_{\max\{0,t-\nu\}}^{t} \hat{C}(\ell)d\ell \tag{23}$$

for all $t \in [t_i, t_i + \nu]$, where

$$\hat{C}(\ell) = |C\hat{x}(\ell) - y(\ell)|. \tag{24}$$

It follows that $t_{i+1} - t_i \geq \nu$ for all $i \in \mathbb{Z}_0$.

Second part. This proves exponential stability of the equilibrium point of the closed loop system. It uses the fact that with the choice $\tilde{x}(t) = \hat{x}(t) - x(t)$, the dynamics

$$\dot{\tilde{x}}(t) = G\tilde{x}(t) \tag{25}$$

are globally exponentially stable at the origin, where G is the matrix defined in (21). For details, see [14]. □

3.3. Robustness

We next generalize our results to cases with an additive disturbance $A_\delta(t)$ on A, which produces

$$\dot{x}(t) = (A + A_\delta(t))x(t) + Bu. \tag{26}$$

We impose conditions on A_δ that hold when the sup norm of A_δ is small enough, because our goal is to propose a stabilizing feedback that is robust with respect to the term $A_\delta(t)x(t)$. This motivates the next assumption, which will be satisfied under our Assumption 1 when the entries of A_δ and Γ are small enough in absolute value:

Assumption 3. The bounded function $A_\delta : [0, +\infty) \to \mathbb{R}^{n\times n}$ is locally Lipschitz. Also, for a matrix K satisfying the requirements of Assumption 1, and for a Γ satisfying the requirements of Lemma 1, there is a constant $a > 0$ such that the system

$$\dot{\xi}(t) = \left[A_{\rm cl} + \sup_{t\in[0,\infty)} |A_\delta(t)| + (1+a)|BK|\Gamma\right]\xi(t) \tag{27}$$

with the choice $A_{\rm cl} = A + BK$ admits the origin as a globally exponentially stable equilibrium point on $\mathbb{R}^n$.□

Using Lemmas A.1 and A.2 from the appendix below, we can then prove the following result, where solutions of the event-triggered system (a")-(c") are defined in an analogous recursive way, like in the preceding two theorems:

Theorem 3. *Let Assumptions 1 and 3 be satisfied for some* K *and* $\Gamma = [\Gamma_{ij}]$. *Assume that* K, Γ, *and* $\nu > 0$ *satisfy the requirements of Lemma 1. Let* A_δ *be such that*

$$||A_\delta||_\infty \nu e^{\nu(||A||+||A_\delta||_\infty)}(1 + \nu||BK||)||\Omega^{-1}||_{[0,\nu]} < c_0, \tag{28}$$

where $c_0 \in (0, 1)$ *is any constant such that*

$$\frac{c_0}{1-c_0}||\Omega^{-1}||_{[0,\nu]} \leq a \min\left\{\Gamma_{ij} : 1 \leq i \leq n, 1 \leq j \leq n\right\}. \tag{29}$$

Then the system with sampling times t_i, *defined by*

(a") $\dot{x}(t) = (A + A_\delta(t))x(t) + BKx(t_i)$ *if* $t \in (t_i, t_{i+1})$,
(b") $|x(t) - x(t_i)| \leq (1 + a)\Gamma|x(t)|$ *if* $t \in [t_i, t_{i+1})$, *and*
(c") *For each* $\epsilon > 0$, *and for each* i *such that* $t_{i+1} < +\infty$, *there is a* $t_\ddagger \in (t_{i+1}, t_{i+1} + \epsilon)$ *such that* $|x(t_\ddagger) - x(t_i)| \not\leq (1 + a)\Gamma|x(t_\ddagger)|$

for all i *and* $t_0 = 0$, *admits the origin as a globally exponentially stable equilibrium point on* $\mathbb{R}^n$. □

For a proof of the preceding theorem, see [14] (in the special case where $\hat{x} = x$, $\Gamma_a = \Gamma$, and $\Gamma_b = a\Gamma$ for the preceding choice of Γ).

4. Illustrations

To illustrate Theorem 1, consider the system

$$\begin{cases} \dot{x}_1(t) = x_1(t) + \frac{1}{2}x_2(t) + u \\ \dot{x}_2(t) = \dfrac{3}{2}x_1(t) + u \end{cases} \tag{30}$$

where x_1 and x_2 are valued in $\mathbb{R}$ and u is the input. Setting $K = [K_1\ K_2]$, Assumption 1 will be satisfied if

$$A_{\rm cl} = \begin{bmatrix} 1 + K_1 & \dfrac{1}{2} + K_2 \\ \dfrac{3}{2} + K_1 & K_2 \end{bmatrix} \tag{31}$$

is Hurwitz and Metzler. Since (31) is Hurwitz and Metzler if $K_1 + K_2 < -3/2$, $K_1 \geq -3/2$, and $K_2 \geq -1/2$, Assumption 1 is satisfied with $K_1 = -4/3$ and $K_2 = -1/3$. Then, with the notation of Section 2,

$$A_{\rm cl} = \begin{bmatrix} -\dfrac{1}{3} & \dfrac{1}{6} \\ \dfrac{1}{6} & -\dfrac{1}{3} \end{bmatrix}, \quad BK = \begin{bmatrix} -\dfrac{4}{3} & -\dfrac{1}{3} \\ -\dfrac{4}{3} & -\dfrac{1}{3} \end{bmatrix},$$

and $M = A_{\rm cl} + |BK|\Gamma$

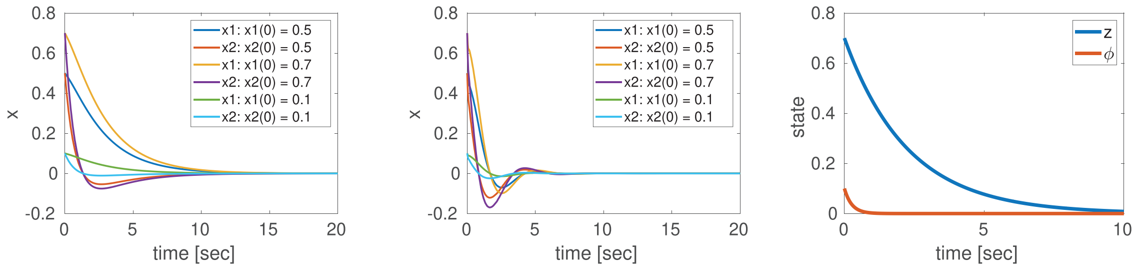
$$= \begin{bmatrix} -\dfrac{1}{3} & \dfrac{1}{6} \\ \dfrac{1}{6} & -\dfrac{1}{3} \end{bmatrix} + \begin{bmatrix} \dfrac{4}{3} & \dfrac{1}{3} \\ \dfrac{4}{3} & \dfrac{1}{3} \end{bmatrix}\Gamma. \tag{32}$$

Then simple Mathematica calculations show that the assumptions of Lemma 1 and Theorem 1 are satisfied with all entries of Γ being $\Gamma_{ij} = 0.045$ and the lower bound $\nu = 0.122$ on the sampling intervals $t_{i+1} - t_i$.

We next compare the preceding lower bound with one that can be obtained from Jiang and Liu [7, p.72, Theorem 5.2] using the small gain approach. In the linear time invariant case, the triggering times in [7, Theorem 5.2] are such that when t_{i+1} is finite, it is the smallest time $t \geq t_i$ such that $\rho(||x(t)||) = ||x(t) - x(t_i)||$ when $x(t_i) \neq 0$, for any class $\mathcal{K}$ function ρ such that $\rho(\gamma(s)) < s$ for all $s > 0$, where $\gamma \in \mathcal{K}_\infty$ is the overshoot function in a suitable input-to-state stability estimate for

$$\dot{x} = (A + BK)x + BKw, \tag{33}$$

i.e., there is a class $\mathcal{KL}$ function β such that along all solutions of (33) for all $t \geq 0$, we have $||x(t)|| \leq \max\{\beta(||x(0)||, t), \gamma(||w||_\infty)\}$; see [7] for the standard definitions of input-to-state stability and the classes $\mathcal{K}$, $\mathcal{K}_\infty$, and $\mathcal{KL}$ of comparison functions. In the linear case, we can apply variation of parameters to (33) to show that the



(a) The positive system approach (b) The small gain approach (c) AUV: positive systems approach

Fig. 1. MATLAB simulation results.

least conservative choice of γ that satisfies the preceding requirements is $\gamma(s) = 2\overline{M}s$, where

$$\overline{M} = \sup_{t\geq 0} \int_0^t ||e^{\ell A_{\rm cl}}BK||{\rm d}\ell. \tag{34}$$

Hence, the least conservative choice of ρ is $\rho(s) = s/(2\overline{M})$, so when t_{i+1} is finite, it is the supremum of all $t > t_i$ such that $\sup_{\ell\in[t_i,t]} ||I - \Omega^{-1}(\ell - t_i)|| \leq 1/(2\overline{M})$, by (9). Thus, the lower bound on the inter-sample times $t_{i+1} - t_i$ guaranteed in this case is the largest q such that $\sup_{\ell\in[0,q]} ||I - \Omega^{-1}(\ell)|| \leq 1/(2\overline{M})$, which we computed to be $q = 0.0838$, using Mathematica. Since this is significantly below the lower bound $\nu = 0.122$ that we obtained from our positive systems approach, it illustrates a potential advantage of our positive systems approach, namely, its ability to ensure less frequent event triggering times. If we had instead used [17, Corollary IV.1] to obtain a lower bound on the $t_{i+1} - t_i$'s (with $\sigma = \min\{\Gamma_{ij} : 1 \leq i \leq n, 1 \leq j \leq n\} = 0.045$ in [17, Corollary IV.1], which corresponds to our trigger condition $|e| \leq \Gamma|x|$), then we would have obtained $\nu = 0.0811$, so again our method provides a larger ν.

In Fig. 1, we used MATLAB to compare the performance of the event-triggered controller from our Theorem 1 with the event-triggered control method from Jiang and Liu [7, p.72, Theorem 5.2], using the above parameter values. For our 20 s time horizon, our controller from Theorem 1 produced 73 sample times when the control value was recomputed, while the small gain approach produced 198 sample times. Therefore, although the settling times were slightly larger for our control, our simulations illustrate the substantial savings in control recomputation times that is afforded by our method. Also, the undershoot in our approach is significantly less, as compared to the small gain method, which speaks to the viability of our method for real-world applications. See also [14] for a scalar example where our approach performs better than the small gain approach.

To illustrate the effects of different choices of the matrix K in an applied example, we next illustrate Theorem 1 using a linearized representation of the BlueRov2 Heavy underwater vehicle. Consider the case of two degrees-of-freedom being actuated, namely, the z-axis and yaw ϕ of the vehicle. We obtained the model parameters using the approach in [19], leading to the diagonal matrix $A = {\rm diag}\{-0.387, -1.800\}$ and $B = [0.038 \;\; 1.500]^\top$. We chose the entries $\Gamma_{11} = 0.195$ $\Gamma_{12} = 0.795$, $\Gamma_{21} = 0.995$, and $\Gamma_{22} = 0.590$ of the matrix $\Gamma = [\Gamma_{ij}]$ for our control. With the preceding choices and $K = [-1.49 \;\; -1.37]$, the largest ν for which the requirements of Theorem 1 are satisfied is $\nu = 0.105$, which is our guaranteed lower bound on the inter-sample intervals $t_{i+1} - t_i$. Our MATLAB simulation in Fig. 1c shows the resulting performance of our event-triggered control from Theorem 1 with these parameters. If we change K to $K = [-0.65201 \;\; -0.845482]$ and keep all other parameters the same, then the largest ν for which our assumptions are satisfied is $\nu = 0.136$, hence a 29.52% increase from $\nu = 0.105$ by changing K. With this change in K, the simulation was similar to Fig. 1c.

5. Conclusion

We proposed new event-triggered control designs, where instead of small gain or other standard approaches, we used positive systems and interval observers. This allowed us to cover large classes of linear systems with outputs, or with time-varying uncertainty in the coefficients of the systems. Our main example illustrated the trade-off between performance and control updating, by producing a significantly larger lower bound on the intersampling times, significantly less frequent control updates (which is an advantage in applications), slower convergence, but less undershoot, compared with the small gain method. We aim to study such trade-offs in more cases, and ways to change coordinates (to meet our Hurwitzness and Metzler requirements) or to tune the design parameters K and Γ to further reduce the number of control update times t_i. We also aim to study applications to adaptive dynamic programming [21].

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Lemmas from Mazenc et al. [10]

We used the following lemmas (which are Lemmas 1–2 from Mazenc et al. [10], respectively) in our proof of Theorem 3:

Lemma A.1. *Let $M_0 \in \mathbb{R}^{n\times n}$ be an invertible matrix. Let $N_0 \in \mathbb{R}^{n\times n}$ be a matrix. Let $\overline{n}$ and $\overline{m}$ be two constants such that $||M_0^{-1}|| \leq \overline{m}$ and $||N_0|| \leq \overline{n}$. Assume that $\overline{mn} < 1$. Then $M_0 + N_0$ is invertible and*

$$\left|\left|M_0^{-1} - (M_0 + N_0)^{-1}\right|\right| \leq \frac{\overline{m}^2\overline{n}}{1 - \overline{m}\,\overline{n}} \tag{A.1}$$

is satisfied.□

Lemma A.2. *Let $\mathcal{A} \in \mathbb{R}^{n\times n}$. Consider the system*

$$\dot{\zeta}(t) = [\mathcal{A} + \mathcal{E}(t)]\zeta(t) \tag{A.2}$$

where ζ is valued in $\mathbb{R}^n$ and $\mathcal{E} : [0, +\infty) \to \mathbb{R}^{n\times n}$ is a bounded locally Lipschitz function. Let ϕ denote the fundamental solution of the system (A.2). Then for all $t_1 \in \mathbb{R}$ and $t_2 \in \mathbb{R}$ such that $t_1 \geq t_2 \geq 0$, the inequality

$$\left|\left|\phi(t_1, t_2) - e^{\mathcal{A}(t_1 - t_2)}\right|\right| \leq ||\mathcal{E}||_\infty (t_1 - t_2) e^{(||\mathcal{A}|| + ||\mathcal{E}||_\infty)(t_1 - t_2)} \tag{A.3}$$

is satisfied.□

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