OPTIMAL CONVERGENCE RATES FOR ELLIPTIC HOMOGENIZATION PROBLEMS IN NONDIVERGENCE-FORM: ANALYSIS AND NUMERICAL ILLUSTRATIONS*

TIMO SPREKELER† AND HUNG V. TRAN‡

Abstract. We study optimal convergence rates in the periodic homogenization of linear elliptic equations of the form $-A(x/\varepsilon) : D^2 u^\varepsilon = f$ subject to a homogeneous Dirichlet boundary condition. We show that the optimal rate for the convergence of u^ε to the solution of the corresponding homogenized problem in the $W^{1,p}$-norm is $\mathcal{O}(\varepsilon)$. We further obtain gradient and Hessian bounds with correction terms taken into account in the L^p-norm and recover the known optimal convergence rate in the L^∞-norm under weak assumptions on the data. We then provide an explicit c-bad diffusion matrix and use it to perform various numerical experiments, which demonstrate the optimality of the obtained rates. Finally, we discuss extensions of the results to the case of nonsmooth domains and their utility in regards to numerical homogenization.

Key words. Homogenization, nondivergence-form elliptic PDE, optimal convergence rates

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1. Introduction. In this work, we study optimal rates in the periodic homogenization of elliptic equations in nondivergence-form. We consider the linear prototype equation subject to a homogeneous Dirichlet boundary condition, posed on a bounded smooth domain $\Omega \subset \mathbb{R}^n$, i.e., problems of the form

$$
(1.1) \qquad
\begin{cases}
-A\left(\dfrac{\cdot}{\varepsilon}\right) : D^2 u^\varepsilon = f & \text{in } \Omega, \\
\qquad\qquad\quad u^\varepsilon = 0 & \text{on } \partial\Omega,
\end{cases}
$$

with a parameter $\varepsilon > 0$ (considered to be small), a right-hand side

$$
f \in W^{2,q}(\Omega) \quad \text{for some } q > n,
$$

and a symmetric, $\mathbb{Z}^n$-periodic, uniformly elliptic coefficient function

$$
A \in C^{0,\alpha}(\mathbb{T}^n; \mathcal{S}^n_+) \quad \text{for some } \alpha \in (0,1].
$$

Here, $\mathbb{T}^n := \mathbb{R}^n / \mathbb{Z}^n$ denotes the flat n-dimensional torus and $\mathcal{S}^n_+ \subset \mathbb{R}^{n \times n}$ the set of symmetric positive definite $n \times n$ matrices. Throughout this work, we denote the unit cell in $\mathbb{R}^n$ by

$$
Y := [0,1]^n \subset \mathbb{R}^n.
$$

In the theory of periodic homogenization, it is well-known (see e.g., Bensoussan, Lions, Papanicolaou [7], Jikov, Kozlov, Oleinik [27]) that as the parameter ε tends to

†University of Oxford, Mathematical Institute, Woodstock Road, Oxford OX2 6GG, UK (sprekeler@maths.ox.ac.uk).
‡Department of Mathematics, University of Wisconsin Madison, Van Vleck hall, 480 Lincoln drive, Madison, WI 53706, USA (hung@math.wisc.edu).

zero, the corresponding sequence of solutions $(u^\varepsilon)_{\varepsilon>0}$ to (1.1) converges uniformly on $\bar{\Omega}$ to the solution u of the homogenized problem

$$(1.2) \qquad \begin{cases} -\bar{A} : D^2 u = f & \text{in } \Omega, \\ \qquad\quad\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here, the effective coefficient $\bar{A} \in \mathcal{S}^n_+$ is a constant positive definite matrix, and can be obtained through integration against an invariant measure, that is

$$\bar{A} := \int_Y Ar$$

with the invariant measure $r : \mathbb{R}^n \to \mathbb{R}$ being the solution to the periodic problem

$$-D^2 : (Ar) = 0 \quad \text{in } Y, \qquad r \text{ is } Y\text{-periodic}, \qquad r > 0, \qquad \int_Y r = 1,$$

see e.g., Avellaneda, Lin [5], Engquist, Souganidis [14]. The effective coefficient $\bar{A}$ can be equivalently characterized via corrector functions: For $i, j \in \{1, \ldots, n\}$, the (i, j)-th entry $\bar{a}_{ij}$ of $\bar{A}$ is the unique value such that the periodic cell problem

$$(1.3) \qquad -A : D^2 v^{ij} = a_{ij} - \bar{a}_{ij} \quad \text{in } Y, \qquad v^{ij} \text{ is } Y\text{-periodic}, \qquad \int_Y v^{ij} = 0$$

admits a unique solution $v^{ij} : \mathbb{R}^n \to \mathbb{R}$, called a corrector function. We write $V = (v^{ij})_{1 \le i,j \le n} : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ for the matrix of corrector functions.

Let us briefly illustrate a formal two-scale asymptotic expansion argument, relying on an expansion

$$(1.4) \qquad\qquad u^\varepsilon(x) = \sum_{k=0}^\infty \varepsilon^k u_k \left(x, \frac{x}{\varepsilon} \right)$$

with functions $u_k = u_k(x, y)$ being Y-periodic in the y-variable for $k \ge 0$. Substituting (1.4) into the equation $\left[-A(y) : D^2 u^\varepsilon(x) \right]\big|_{y=\frac{x}{\varepsilon}} = f(x)$ and comparing coefficients of powers of ε yields the cascade of equations

$$\begin{aligned} \text{order } \varepsilon^{-2} &\quad : \quad L_{yy} u_0 = 0 &, \\ \text{order } \varepsilon^{-1} &\quad : \quad L_{yy} u_1 = -L_{xy} u_0 &, \\ \text{order } \varepsilon^0 &\quad : \quad L_{yy} u_2 = f - L_{xy} u_1 - L_{xx} u_0 &, \\ \text{order } \varepsilon^{k-2}, \ k \ge 3: &\quad L_{yy} u_k = -L_{xy} u_{k-1} - L_{xx} u_{k-2}, \end{aligned}$$

where $L_{yy} := -A(y) : D^2_y$, $L_{xy} := -2A(y) : \nabla_x \nabla_y^{\mathrm{T}}$, and $L_{xx} := -A(y) : D^2_x$. It is quickly seen that $u_0(x, y) \equiv u_0(x)$ and $u_1(x, y) \equiv u_1(x)$. The solvability condition for the equation at order ε^0 yields $-\bar{A} : D^2 u_0 = f$, i.e., the homogenized equation, and we obtain that $u_2(x, y) = V(y) : D^2 u_0(x) + \text{constant}$. At this point one might think that u_1 can be set to zero in the expansion (1.4). However, the solvability condition for the equation at order ε^1 yields $-\bar{A} : D^2 u_1 = 2 \sum_{j,k,l=1}^n c_j^{kl} \partial_{jkl}^3 u_0$ with c_j^{kl} defined in (1.5), which will generally prevent the first-order term u_1 from vanishing. Let us emphasize that this formal asymptotic expansion argument is purely heuristic and neglects the boundary condition, but it is helpful in providing useful intuition.

In this paper, we are interested in optimal rates for the convergence of the solution u^ε of (1.1) to the solution u of the homogenized problem (1.2) in appropriate function spaces. Optimal rates in $L^\infty(\Omega)$ have recently been obtained in Guo, Tran, Yu [23]. With $c_j^{kl} \in \mathbb{R}$, $j, k, l \in \{1, \ldots, n\}$, defined by

$$(1.5) \qquad c_j^{kl} = c_j^{kl}(A) := \int_Y A e_j \cdot \nabla v^{kl} \, r,$$

the function h defined by

$$(1.6) \qquad h := \sum_{j,k,l=1}^{n} c_j^{kl} \partial_{jkl}^3 u,$$

and the solution z to the problem

$$(1.7) \qquad \begin{cases} -\bar{A} : D^2 z = -h & \text{in } \Omega, \\ \qquad\quad z = 0 & \text{on } \partial\Omega, \end{cases}$$

the main result in [23] states the following:

THEOREM 1.1 (Theorem 1.2 in [23]). *Assume that $A \in C^2(\mathbb{T}^n; \mathcal{S}_+^n)$ and $f \in C^3(\bar{\Omega})$. Let u^ε, u and z be the solutions to (1.1), (1.2) and (1.7) respectively. Then we have*

$$\|u^\varepsilon - u + 2\varepsilon z\|_{L^\infty(\Omega)} = \mathcal{O}(\varepsilon^2).$$

In particular, with h given by (1.6), the following assertions hold:
  (i) If $h \equiv 0$, then $\|u^\varepsilon - u\|_{L^\infty(\Omega)} = \mathcal{O}(\varepsilon^2)$ and this rate of convergence is optimal.
  (ii) If $h \not\equiv 0$, then $\|u^\varepsilon - u\|_{L^\infty(\Omega)} = \mathcal{O}(\varepsilon)$ and this rate of convergence is optimal.

Remark 1.2. There is a typo in [23], which uses the opposite sign for the $\mathcal{O}(\varepsilon)$-term.

Let us recall the $\mathcal{O}$ notation, which we are going to use throughout this paper: For a function $e : (0, \infty) \ni \varepsilon \mapsto e(\varepsilon) \in [0, \infty)$ and an exponent $\gamma \geq 0$, we write

$$e(\varepsilon) = \mathcal{O}(\varepsilon^\gamma) \quad \Longleftrightarrow \quad \exists\, C, \varepsilon_0 > 0 : e(\varepsilon) \leq C\varepsilon^\gamma \quad \forall \varepsilon \in (0, \varepsilon_0).$$

As a consequence of Theorem 1.1, we can classify coefficients $A \in C^2(\mathbb{T}^n; \mathcal{S}_+^n)$ into those that give optimal rate of convergence $\mathcal{O}(\varepsilon^2)$, called the c-good coefficients, and those that give optimal rate of convergence $\mathcal{O}(\varepsilon)$, called the c-bad coefficients.

COROLLARY 1.3 (*c-good and c-bad matrices*). *Let $A \in C^2(\mathbb{T}^n; \mathcal{S}_+^n)$. Then, with $\{c_j^{kl}\}_{1 \leq j,k,l \leq n}$ given by (1.5), the following assertions hold:*
  (i) If $c_j^{kl}(A) = 0$ for all $j, k, l \in \{1, \ldots, n\}$, then the situation (i) of Theorem 1.1 occurs for any choice of f. We then say A is c-good.
  (ii) If $c_j^{kl}(A) \neq 0$ for some $j, k, l \in \{1, \ldots, n\}$, then there exists f such that the situation (ii) of Theorem 1.1 occurs. We then say A is c-bad.

It has further been shown that the set of c-bad matrices is open and dense in $C^2(\mathbb{T}^n; \mathcal{S}_+^n)$ for dimensions $n \geq 2$ (see Theorem 1.4 in [23]). Therefore, we have generically that the optimal rate is $\mathcal{O}(\varepsilon)$ in $L^\infty(\Omega)$. Related results on convergence rates and error estimates in the periodic homogenization of elliptic equations in divergence-form have been derived by various authors; see e.g., [21, 28, 29, 31, 33, 37] and the references therein.

The objective of this work are the optimal rates in higher-order norms. This question has not been studied yet and it seems that the only available result in higher-order norms is the following $W^{2,p}$ corrector estimate from Capdeboscq, Sprekeler, Süli [9]:

THEOREM 1.4 (Theorem 2.8 in [9]).   *Assume that $A \in C^{0,\alpha}(\mathbb{T}^n; \mathcal{S}_+^n)$ for some $\alpha \in (0,1]$ and $f \in W^{2,p}(\Omega)$ for some $p \in (1,\infty)$. Further, assume that the solution u to (1.2) satisfies $u \in W^{4,p}(\Omega) \cap W^{2,\infty}(\Omega)$. Then, with u^ε given by (1.1) and $V = (v^{ij})_{1 \le i,j \le n}$ given by (1.3), we have*

$$\left\| u^\varepsilon - u - \varepsilon^2 V\left(\frac{\cdot}{\varepsilon}\right) : D^2 u \right\|_{W^{2,p}(\Omega)} = \mathcal{O}(\varepsilon^{\frac{1}{p}}).$$

Note that the standing assumption in [9] is $A \in W^{1,q}(Y) \cap C^{0,\alpha}(\mathbb{T}^n; \mathcal{S}_+^n)$ for some $q > n$, which is useful for the numerical homogenization but not essential for the result of Theorem 1.4. We observe that we cannot expect strong convergence of u^ε to the homogenized solution u in $W^{2,p}(\Omega)$ and that it is necessary to add corrector terms. The optimality of the rate of convergence $\mathcal{O}(\varepsilon^{\frac{1}{p}})$ in Theorem 1.4 has not been discussed yet, and we contribute to this open question by providing numerical experiments suggesting that the bound is indeed optimal. The main contribution of this work is to derive optimal $W^{1,p}(\Omega)$ estimates for $p \in (1,\infty)$ and to provide numerical illustrations.

For the numerical homogenization of linear equations in nondivergence-form, we refer the reader to Capdeboscq, Sprekeler, Süli [9], Froese, Oberman [17], and the references therein. Let us note that the divergence-form case was the focus of active research over the past decades; see e.g., the works [1, 11, 12, 13, 26] by various authors on heterogeneous multiscale methods and multiscale finite element methods.

For some results on fully nonlinear equations of nondivergence-structure, we refer to Camilli, Marchi [8], Kim, Lee [30] for convergence rates and to Gallistl, Sprekeler, Süli [18], Finlay, Oberman [16] for numerical homogenization of Hamilton–Jacobi–Bellman equations.

1.1. Main results. The main result is the following theorem on optimal rates for the convergence of u^ε to the homogenized solution u in $W^{1,p}(\Omega)$:

THEOREM 1.5 ($W^{1,p}$ *estimate and optimal rate*).   *Assume that $A \in C^{0,\alpha}(\mathbb{T}^n; \mathcal{S}_+^n)$ for some $\alpha \in (0,1]$ and $f \in W^{2,q}(\Omega)$ for some $q > n$. Let u^ε, u and z be the solutions to (1.1), (1.2) and (1.7) respectively. Further, let $V = (v^{ij})_{1 \le i,j \le n}$ be the matrix of corrector functions given by (1.3). Then, for all $p \in (1,\infty)$, the following assertions hold true:*

(i) As $\varepsilon \searrow 0$, the sequence $(u^\varepsilon)_{\varepsilon > 0}$ converges strongly to u in the $W^{1,p}(\Omega)$-norm with convergence rate $\mathcal{O}(\varepsilon)$, i.e.,

$$(1.8) \qquad\qquad \|u^\varepsilon - u\|_{W^{1,p}(\Omega)} = \mathcal{O}(\varepsilon).$$

(ii) Moreover, if $f \in W^{3,q}(\Omega)$ for some $q > n$, we have that

$$\left\| u^\varepsilon - u + 2\varepsilon z - \varepsilon^2 V\left(\frac{\cdot}{\varepsilon}\right) : D^2 u \right\|_{W^{1,p}(\Omega)} = \mathcal{O}(\varepsilon^{1+\frac{1}{p}}).$$

In particular, the rate $\mathcal{O}(\varepsilon)$ in (1.8) is optimal in general.

Remark 1.6 (L^∞ estimate, gradient estimate and Hessian estimate).   In the situation of Theorem 1.5 (ii), the following assertions hold.

(i) L^∞ bound: An inspection of the proof, see (2.14) and (2.6), yields that

$$(1.9) \qquad \|u^\varepsilon - u + 2\varepsilon z\|_{L^\infty(\Omega)} = \mathcal{O}(\varepsilon^2).$$

(ii) Gradient bound: For all $p \in (1, \infty)$, we have

$$(1.10) \qquad \left\| \nabla u^\varepsilon - \nabla u + 2\varepsilon \nabla z - \varepsilon \sum_{i,j=1}^n \nabla v^{ij} \left(\frac{\cdot}{\varepsilon}\right) \partial_{ij}^2 u \right\|_{L^p(\Omega)} = \mathcal{O}(\varepsilon^{1+\frac{1}{p}}).$$

(iii) Hessian bound: In view of Theorem 1.4, for all $p \in (1, \infty)$, there holds

$$(1.11) \qquad \left\| D^2 u^\varepsilon - D^2 u - \sum_{i,j=1}^n D^2 v^{ij} \left(\frac{\cdot}{\varepsilon}\right) \partial_{ij}^2 u \right\|_{L^p(\Omega)} = \mathcal{O}(\varepsilon^{\frac{1}{p}}).$$

In particular, we recover the result on optimal L^∞ rates from Theorem 1.1 under these weaker assumptions on the coefficient A and the right-hand side f.

An essential role in the proof plays the boundary corrector θ^ε, which is defined to be the solution to the following problem with oscillations in the boundary data:

$$(1.12) \qquad \begin{cases} -A\left(\frac{\cdot}{\varepsilon}\right) : D^2 \theta^\varepsilon = 0 & \text{in } \Omega, \\ \theta^\varepsilon = -V\left(\frac{\cdot}{\varepsilon}\right) : D^2 u & \text{on } \partial\Omega. \end{cases}$$

We then have the following result on the asymptotic behavior of the boundary corrector under the reduced regularity $f \in W^{2,q}(\Omega)$ for some $q > n$:

LEMMA 1.7 (Boundary corrector $W^{1,p}$ bound). *Assume that* $A \in C^{0,\alpha}(\mathbb{T}^n; \mathcal{S}_+^n)$ *for some* $\alpha \in (0, 1]$ *and* $f \in W^{2,q}(\Omega)$ *for some* $q > n$. *Further, let* θ^ε *be the solution to the problem* (1.12). *Then, for all* $p \in (1, \infty)$, *we have that*

$$(1.13) \qquad \varepsilon \|\theta^\varepsilon\|_{W^{1,p}(\Omega)} = \mathcal{O}(\varepsilon^{\frac{1}{p}}).$$

Remark 1.8 (Boundary corrector $W^{2,p}$ bound [9]). In the situation of Theorem 1.4, there holds

$$(1.14) \qquad \varepsilon^2 \|\theta^\varepsilon\|_{W^{2,p}(\Omega)} = \mathcal{O}(\varepsilon^{\frac{1}{p}}).$$

Let us remark that the estimate (1.13) for $p = 2$ has been shown in [3, 34] in the context of divergence-form homogenization by energy estimates. It is worth noting here that we only obtain $W^{1,p}$ and $W^{2,p}$ bounds for the boundary corrector θ^ε, and we do not study qualitative and quantitative homogenization of (1.12) (for the latter see e.g., [2, 4, 15, 19]).

Finally, we demonstrate through numerical experiments that the obtained rates in the previously stated results cannot be improved in general.

Remark 1.9 (Optimality of rates). We numerically demonstrate that the rates $\mathcal{O}(\varepsilon^2)$, $\mathcal{O}(\varepsilon^{1+\frac{1}{p}})$, $\mathcal{O}(\varepsilon^{\frac{1}{p}})$ in the right-hand sides of (1.9), (1.10), (1.11) are optimal in general. Consequently, also the rate $\mathcal{O}(\varepsilon^{\frac{1}{p}})$ in the right-hand sides of (1.13), (1.14) is optimal in general. It is an open problem to verify these rigorously, but we provide some theoretical evidence in Section 4.2.

For the numerical illustrations we use an explicit c-bad matrix (recall Corollary 1.3 for the definition of c-bad) and consider a homogenization problem of the form (1.1) with $z \not\equiv 0$. This is the first direct proof of the existence of a c-bad matrix.

THEOREM 1.10 (Explicit c-bad matrix).   *The matrix-valued function* $A : \mathbb{R}^2 \to \mathbb{R}^{2 \times 2}$ *given by*

$$A(y_1, y_2) := \frac{1}{r(y_1, y_2)} \begin{pmatrix} 1 - \frac{1}{2}\sin(2\pi y_1)\sin(2\pi y_2) & 0 \\ 0 & 1 + \frac{1}{2}\sin(2\pi y_1)\sin(2\pi y_2) \end{pmatrix}$$

with $r : \mathbb{R}^2 \to \mathbb{R}$ *defined by*

$$r(y_1, y_2) := 1 + \frac{1}{4}(\cos(2\pi y_1) - 2\sin(2\pi y_1))\sin(2\pi y_2)$$

is c-*bad. More precisely, there holds* $c_1^{11} = c_1^{22} = -\frac{1}{128\pi}$ *and* $c_j^{kl} = 0$ *otherwise.*

We briefly explain the organization of the paper:

1.2. Structure of the paper. In Section 2, we prove the main result, i.e., Theorem 1.5. We start by recalling some uniform estimates from the theory of homogenization in Section 2.1. Thereafter, we prove Lemma 1.7 and Theorem 1.5 (i) in Sections 2.2 and 2.3 respectively, and finally Theorem 1.5 (ii) in Section 2.4.

In Section 3, we provide numerical illustrations of the convergence rates from Remark 1.6. We start by proving Theorem 1.10 in Section 3.1, providing an explicit c-bad matrix which we use for the numerical experiments. We illustrate the L^∞ bound from Remark 1.6 (i) in Section 3.2, the gradient bound from Remark 1.6 (ii) in Section 3.3 and the Hessian bound from Remark 1.6 (iii) in Section 3.4. Numerical illustrations comparing c-bad and c-good problems are provided in Section 3.5.

Finally in Section 4, we discuss some extensions to nonsmooth domains and give some concluding remarks.

2. Proofs of the main results.

2.1. Uniform estimates. Uniform estimates are essential in the theory of homogenization and form the basis for the proofs of the main results. The crucial uniform estimate for the proofs is the uniform $C^{1,\alpha}$ estimate from [5] for nondivergence-form homogenization problems.

LEMMA 2.1 (Theorem 1 in [5]).   *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded* $C^{1,\gamma}$ *domain. Assume that* $A \in C^{0,\alpha}(\mathbb{T}^n; \mathcal{S}_+^n)$ *for some* $\alpha \in (0,1]$ *and* $f \in L^q(\Omega)$ *for some* $q > n$. *For* $\varepsilon > 0$, *let* u^ε *be the solution to the problem* (1.1). *Then there exists* $\nu \in (0,1]$ *such that there holds*

$$\|u^\varepsilon\|_{C^{1,\nu}(\Omega)} \leq C\|f\|_{L^q(\Omega)}$$

with a constant $C > 0$ *independent of* ε.

For the proof of Lemma 1.7 it turns out to be useful to transform the problem (1.12) into divergence-form and use the uniform $W^{1,p}$ estimate from [6] for divergence-form homogenization problems.

LEMMA 2.2 (Theorem C in [6]).   *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded* $C^{2,\gamma}$ *domain. Assume that* $A^{\mathrm{div}} \in C^{0,\alpha}(\mathbb{T}^n; \mathbb{R}^{n \times n})$ *for some* $\alpha \in (0,1]$ *is a uniformly elliptic coefficient,* $F \in L^p(\Omega)$ *and* $g \in W^{1,p}(\Omega)$ *for some* $p \in (1, \infty)$. *For* $\varepsilon \in (0,1]$, *let* $\rho^\varepsilon \in W^{1,p}(\Omega)$ *be*

the solution to the problem

$$\begin{cases} -\nabla \cdot \left(A^{\mathrm{div}} \left(\frac{\cdot}{\varepsilon} \right) \nabla \rho^\varepsilon \right) = -\nabla \cdot F & in \ \Omega, \\ \rho^\varepsilon = g & on \ \partial\Omega. \end{cases}$$

Then we have the estimate

$$\|\rho^\varepsilon\|_{W^{1,p}(\Omega)} \le C \left(\|F\|_{L^p(\Omega)} + \|g\|_{W^{1,p}(\Omega)} \right)$$

with a constant $C > 0$ independent of ε.

With the uniform estimates at hand, we can prove the main results. We start with the proof of Lemma 1.7.

2.2. Proof of Lemma 1.7. The main ingredient for the proof of Theorem 1.5 is the asymptotic behavior of the boundary corrector, i.e., the solution θ^ε to the problem (1.12). We start by proving Lemma 1.7 and it turns out to be useful to transform the problem (1.12) into the divergence-form problem

$$\begin{cases} -\nabla \cdot \left(A^{\mathrm{div}} \left(\frac{\cdot}{\varepsilon} \right) \nabla \theta^\varepsilon \right) = 0 & in \ \Omega, \\ \theta^\varepsilon = -V \left(\frac{\cdot}{\varepsilon} \right) : D^2 u & on \ \partial\Omega, \end{cases}$$

with a coefficient $A^{\mathrm{div}} \in C^{0,\alpha}(\mathbb{T}^n; \mathbb{R}^{n \times n})$ for some $\alpha \in (0,1]$ that is uniformly elliptic. Indeed, this is a well-known reduction procedure and can be achieved by multiplication of the equation (1.12) with the invariant measure and addition of a suitable skew-symmetric matrix; see [5].

Proof of Lemma 1.7. Firstly note that, as $f \in W^{2,q}(\Omega)$ for some $q > n$, we have $u \in W^{4,q}(\Omega)$ for some $q > n$ and hence also $u \in W^{3,\infty}(\Omega)$. We further note that, as $A \in C^{0,\alpha}(\mathbb{R}^n)$ for some $\alpha \in (0,1]$, we have $V \in C^{2,\alpha}(\mathbb{R}^n)$ by elliptic regularity theory [20]. We need to show that

$$(2.1) \qquad \varepsilon \|\theta^\varepsilon\|_{W^{1,p}(\Omega)} = \mathcal{O}(\varepsilon^{\frac{1}{p}})$$

for any $p \in (1, \infty)$. To this end, we let $\eta \in C_c^\infty(\mathbb{R}^n)$ be a cut-off function with the properties $0 \le \eta \le 1$,

$$\eta \equiv 1 \quad in \quad \left\{ x \in \Omega : \mathrm{dist}(x, \partial\Omega) < \frac{\varepsilon}{2} \right\},$$

$$\eta \equiv 0 \quad in \quad \left\{ x \in \Omega : \mathrm{dist}(x, \partial\Omega) \ge \varepsilon \right\},$$

and $|\nabla\eta| = \mathcal{O}(\varepsilon^{-1})$. Note that this implies that

$$(2.2) \qquad \|\eta\|_{L^p(\Omega)} + \varepsilon \|\nabla\eta\|_{L^p(\Omega)} = \mathcal{O}(\varepsilon^{\frac{1}{p}})$$

for any $p \in (1, \infty)$. We then define the function

$$\tilde{\theta}^\varepsilon := \theta^\varepsilon + \eta \, V \left(\frac{\cdot}{\varepsilon} \right) : D^2 u$$

and note that it is the solution to the problem

$$\begin{cases} -\nabla \cdot \left(A^{\mathrm{div}} \left(\frac{\cdot}{\varepsilon} \right) \nabla \tilde{\theta}^\varepsilon \right) = -\nabla \cdot F_1^\varepsilon & in \ \Omega, \\ \tilde{\theta}^\varepsilon = 0 & on \ \partial\Omega, \end{cases}$$

with F_1^ε given by

$$F_1^\varepsilon := A^{\mathrm{div}}\left(\frac{\cdot}{\varepsilon}\right)\nabla\left[\eta\,V\left(\frac{\cdot}{\varepsilon}\right):D^2u\right].$$

Using the uniform $W^{1,p}$ estimate from Lemma 2.2, we find that for $\varepsilon\in(0,1]$ and any $p\in(1,\infty)$, we have

$$\|\tilde\theta^\varepsilon\|_{W^{1,p}(\Omega)}\le C\|F_1^\varepsilon\|_{L^p(\Omega)}\le C\left\|\eta\,V\left(\frac{\cdot}{\varepsilon}\right):D^2u\right\|_{W^{1,p}(\Omega)}.$$

Therefore, by the triangle inequality, we obtain the estimate

$$(2.3)\qquad\qquad\|\theta^\varepsilon\|_{W^{1,p}(\Omega)}\le C\left\|\eta\,V\left(\frac{\cdot}{\varepsilon}\right):D^2u\right\|_{W^{1,p}(\Omega)}.$$

As we have the bound

$$\left\|V\left(\frac{\cdot}{\varepsilon}\right):D^2u\right\|_{L^\infty(\Omega)}+\varepsilon\left\|\nabla\left[V\left(\frac{\cdot}{\varepsilon}\right):D^2u\right]\right\|_{L^\infty(\Omega)}=\mathcal{O}(1),$$

and the asymptotic behavior of the cut-off (2.2), we deduce from (2.3) that there holds

$$\varepsilon\|\theta^\varepsilon\|_{W^{1,p}(\Omega)}\le C\varepsilon\left(\|\nabla\eta\|_{L^p(\Omega)}+\left(\varepsilon^{-1}+1\right)\|\eta\|_{L^p(\Omega)}\right)=\mathcal{O}(\varepsilon^{\frac{1}{p}}),$$

which is precisely the claimed bound (2.1).   □

2.3. Proof of assertion (i) in Theorem 1.5. With Lemma 1.7 at hand, we can now prove the first part of Theorem 1.5, i.e., the $\mathcal{O}(\varepsilon)$ rate for the convergence of u^ε to u under the reduced regularity assumption $f\in W^{2,q}(\Omega)$ for some $q>n$.

Proof of assertion (i) in Theorem 1.5. We need to show that for any $p\in(1,\infty)$, there holds

$$(2.4)\qquad\qquad\|u^\varepsilon-u\|_{W^{1,p}(\Omega)}=\mathcal{O}(\varepsilon).$$

With the corrector matrix $V=(v^{ij})_{1\le i,j\le n}$ given by (1.3) and the boundary corrector θ^ε given by (1.12), we let

$$(2.5)\qquad\qquad\phi^\varepsilon:=\varepsilon^2\left[V\left(\frac{\cdot}{\varepsilon}\right):D^2u+\theta^\varepsilon\right].$$

Then we have that the function $u^\varepsilon-u-\phi^\varepsilon$ satisfies the problem

$$\begin{cases}-A\left(\dfrac{\cdot}{\varepsilon}\right):D^2(u^\varepsilon-u-\phi^\varepsilon)=\varepsilon F_2^\varepsilon & \text{in }\Omega,\\[2mm]\qquad\qquad\quad u^\varepsilon-u-\phi^\varepsilon=0 & \text{on }\partial\Omega,\end{cases}$$

with F_2^ε given by

$$F_2^\varepsilon:=\sum_{i,j,k,l=1}^n a_{ij}\left(\frac{\cdot}{\varepsilon}\right)\left[2\partial_i v^{kl}\left(\frac{\cdot}{\varepsilon}\right)\partial_{jkl}^3 u+\varepsilon v^{kl}\left(\frac{\cdot}{\varepsilon}\right)\partial_{ijkl}^4 u\right].$$

As $f\in W^{2,q}(\Omega)$ for some $q>n$, we have $u\in W^{4,q}(\Omega)$ and hence F_2^ε is uniformly bounded in $L^q(\Omega)$. By the uniform estimate from Lemma 2.1, we have that

$$\|u^\varepsilon-u-\phi^\varepsilon\|_{W^{1,\infty}(\Omega)}\le C\varepsilon\|F_2^\varepsilon\|_{L^q(\Omega)}=\mathcal{O}(\varepsilon).$$

Finally, by the triangle inequality and Lemma 1.7, we can conclude that

$$\|u^\varepsilon - u\|_{W^{1,p}(\Omega)} \leq \|u^\varepsilon - u - \phi^\varepsilon\|_{W^{1,p}(\Omega)} + \|\phi^\varepsilon\|_{W^{1,p}(\Omega)} = \mathcal{O}(\varepsilon),$$

which is precisely the claimed convergence rate (2.4). □

Note that Remark 1 in [23] gives that, generically, the optimal rate of convergence of u^ε to u is $\mathcal{O}(\varepsilon)$ in $L^p(\Omega)$. Therefore, the rate $\mathcal{O}(\varepsilon)$ in (1.8) (or (2.4)) is optimal in general.

2.4. Proof of assertion (ii) in Theorem 1.5. For the proof of the second part of the theorem, let us introduce the function z^ε to be the solution to the problem

$$\begin{cases} -A\left(\dfrac{\cdot}{\varepsilon}\right) : D^2 z^\varepsilon = -h & \text{in } \Omega, \\ \qquad\qquad z^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

with the function h defined by (1.6). Observe that the function z given by (1.7) is precisely the homogenized solution corresponding to $(z^\varepsilon)_{\varepsilon>0}$. We note that as $f \in W^{3,q}(\Omega)$ for some $q > n$, we have $u \in W^{5,q}(\Omega)$ and hence $h \in W^{2,q}(\Omega)$. Therefore, we can apply Theorem 1.5 (i) to the sequence $(z^\varepsilon)_{\varepsilon>0}$ to find that for any $p \in (1,\infty)$, there holds

$$(2.6) \qquad\qquad \|z^\varepsilon - z\|_{W^{1,p}(\Omega)} = \mathcal{O}(\varepsilon).$$

We further introduce the functions χ^{jkl}, $j, k, l \in \{1, \ldots, n\}$ to be the solutions to the periodic problems

$$(2.7) \quad -A : D^2 \chi^{jkl} = Ae_j \cdot \nabla v^{kl} - c_j^{kl} \quad \text{in } Y, \quad \chi^{jkl} \text{ is } Y\text{-periodic}, \quad \int_Y \chi^{jkl} = 0.$$

Note that the functions χ^{jkl} are well-defined as by definition (1.5) of c_j^{kl}, the right-hand side integrated against the invariant measure equals zero, i.e., there holds

$$\int_Y \left(Ae_j \cdot \nabla v^{kl} - c_j^{kl}\right) r = 0.$$

We also introduce a corresponding boundary corrector θ_χ^ε to be the solution to the following problem:

$$(2.8) \qquad \begin{cases} -A\left(\dfrac{\cdot}{\varepsilon}\right) : D^2 \theta_\chi^\varepsilon = 0 & \text{in } \Omega, \\ \qquad\qquad \theta_\chi^\varepsilon = -\displaystyle\sum_{j,k,l=1}^n \chi^{jkl}\left(\dfrac{\cdot}{\varepsilon}\right) \partial_{jkl}^3 u & \text{on } \partial\Omega. \end{cases}$$

As we have done for the boundary corrector θ^ε, we can transform the problem (2.8) into the divergence-form problem

$$\begin{cases} -\nabla \cdot \left(A^{\mathrm{div}}\left(\dfrac{\cdot}{\varepsilon}\right)\nabla\theta_\chi^\varepsilon\right) = 0 & \text{in } \Omega, \\ \qquad\qquad \theta_\chi^\varepsilon = -\displaystyle\sum_{j,k,l=1}^n \chi^{jkl}\left(\dfrac{\cdot}{\varepsilon}\right) \partial_{jkl}^3 u & \text{on } \partial\Omega, \end{cases}$$

with a coefficient $A^{\mathrm{div}} \in C^{0,\alpha}(\mathbb{T}^n; \mathbb{R}^{n \times n})$ for some $\alpha \in (0,1]$ that is uniformly elliptic. Let us note that since $u \in W^{5,q}(\Omega)$ for some $q > n$ and $\chi^{jkl} \in C^{2,\beta}(\mathbb{R}^n)$ for some $\beta \in (0,1]$ by elliptic regularity theory [20], we can apply Lemma 2.2 to find the bound

$$(2.9) \qquad \varepsilon \|\theta_\chi^\varepsilon\|_{W^{1,p}(\Omega)} \le C\varepsilon \sum_{j,k,l=1}^n \left\| \chi^{jkl}\left(\frac{\cdot}{\varepsilon}\right) \partial_{jkl}^3 u \right\|_{W^{1,\infty}(\Omega)} = \mathcal{O}(1)$$

for any $p \in (1, \infty)$. Finally, we introduce the function w^ε to be the solution to the problem

$$(2.10) \qquad \begin{cases} -A\left(\dfrac{\cdot}{\varepsilon}\right) : D^2 w^\varepsilon = \displaystyle\sum_{i,j,k,l=1}^n a_{ij}\left(\dfrac{\cdot}{\varepsilon}\right) \partial_i v^{kl}\left(\dfrac{\cdot}{\varepsilon}\right) \partial_{jkl}^3 u & \text{in } \Omega, \\ \qquad\qquad w^\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

Now we are in a position to prove the main result.

Proof of assertion (ii) in Theorem 1.5. Let ϕ^ε be given by (2.5) and w^ε be the solution to (2.10). Then we have that the function $u^\varepsilon - u - \phi^\varepsilon - 2\varepsilon w^\varepsilon$ satisfies the problem

$$\begin{cases} -A\left(\dfrac{\cdot}{\varepsilon}\right) : D^2(u^\varepsilon - u - \phi^\varepsilon - 2\varepsilon w^\varepsilon) = \varepsilon^2 F_3^\varepsilon & \text{in } \Omega, \\ \qquad\quad u^\varepsilon - u - \phi^\varepsilon - 2\varepsilon w^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

with F_3^ε given by

$$F_3^\varepsilon := \sum_{i,j,k,l=1}^n a_{ij}\left(\frac{\cdot}{\varepsilon}\right) v^{kl}\left(\frac{\cdot}{\varepsilon}\right) \partial_{ijkl}^4 u.$$

As $u \in W^{5,q}(\Omega)$ for $q > n$, we have that F_3^ε is uniformly bounded in $L^q(\Omega)$ and hence, by the uniform estimate from Lemma 2.1, we find

$$(2.11) \qquad \|u^\varepsilon - u - \phi^\varepsilon - 2\varepsilon w^\varepsilon\|_{W^{1,\infty}(\Omega)} \le C\varepsilon^2 \|F_3^\varepsilon\|_{L^q(\Omega)} = \mathcal{O}(\varepsilon^2).$$

Now, let us define the function

$$(2.12) \qquad \psi^\varepsilon := \varepsilon^2 \left[\sum_{j,k,l=1}^n \chi^{jkl}\left(\frac{\cdot}{\varepsilon}\right) \partial_{jkl}^3 u + \theta_\chi^\varepsilon \right]$$

with χ^{jkl} given by (2.7) and θ_χ^ε given by (2.8). Then we have that the function $w^\varepsilon + z^\varepsilon - \psi^\varepsilon$ satisfies the problem

$$\begin{cases} -A\left(\dfrac{\cdot}{\varepsilon}\right) : D^2(w^\varepsilon + z^\varepsilon - \psi^\varepsilon) = \varepsilon F_4^\varepsilon & \text{in } \Omega, \\ \qquad\quad w^\varepsilon + z^\varepsilon - \psi^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

with F_4^ε given by

$$F_4^\varepsilon := \sum_{d,i,j,k,l=1}^n a_{ij}\left(\frac{\cdot}{\varepsilon}\right) \left[2\, \partial_i \chi^{dkl}\left(\frac{\cdot}{\varepsilon}\right) \partial_{djkl}^4 u + \varepsilon \chi^{dkl}\left(\frac{\cdot}{\varepsilon}\right) \partial_{dijkl}^5 u \right].$$

As $u \in W^{5,q}(\Omega)$ for some $q > n$, we have that F_4^ε is uniformly bounded in $L^q(\Omega)$ and hence, by the uniform estimate from Lemma 2.1, we find

$$(2.13) \qquad \|w^\varepsilon + z^\varepsilon - \psi^\varepsilon\|_{W^{1,\infty}(\Omega)} \le C\varepsilon \|F_4^\varepsilon\|_{L^q(\Omega)} = \mathcal{O}(\varepsilon).$$

Combining the bounds (2.11) and (2.13), we obtain

$$\|u^\varepsilon - u + 2\varepsilon z^\varepsilon - \phi^\varepsilon - 2\varepsilon\psi^\varepsilon\|_{W^{1,\infty}(\Omega)} = \mathcal{O}(\varepsilon^2),$$

and therefore, using the definitions of ϕ^ε and ψ^ε from (2.5) and (2.12), we have that

$$(2.14) \qquad \left\|u^\varepsilon - u + 2\varepsilon z^\varepsilon - \varepsilon^2 V\left(\frac{\cdot}{\varepsilon}\right) : D^2 u - \varepsilon^2 \theta^\varepsilon - 2\varepsilon^3 \theta_\chi^\varepsilon\right\|_{W^{1,\infty}(\Omega)} = \mathcal{O}(\varepsilon^2).$$

Finally, using the rate of convergence of z^ε to z given by (2.6), and Lemma 1.7 and the estimate (2.9) to bound the boundary correctors, we conclude that

$$\left\|u^\varepsilon - u + 2\varepsilon z - \varepsilon^2 V\left(\frac{\cdot}{\varepsilon}\right) : D^2 u\right\|_{W^{1,p}(\Omega)} = \mathcal{O}(\varepsilon^{1+\frac{1}{p}})$$

for any $p \in (1, \infty)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3. Numerical experiments.

3.1. An explicit c-bad matrix. In this section, we prove that the matrix-valued function $A : \mathbb{R}^2 \to \mathbb{R}^{2\times2}$ given by

$$(3.1) \qquad A(y) := \frac{1}{r(y)} \begin{pmatrix} 1 - \frac{1}{2}\sin(2\pi y_1)\sin(2\pi y_2) & 0 \\ 0 & 1 + \frac{1}{2}\sin(2\pi y_1)\sin(2\pi y_2) \end{pmatrix}$$

with $r : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$(3.2) \qquad r(y) := 1 + \frac{1}{4}(\cos(2\pi y_1) - 2\sin(2\pi y_1))\sin(2\pi y_2)$$

is c-bad (recall the notion of c-bad from Corollary 1.3). We observe the following:

Remark 3.1. The function $r : \mathbb{R}^2 \to \mathbb{R}$ given by (3.2) is the invariant measure of $A : \mathbb{R}^2 \to \mathbb{R}^{2\times2}$ given by (3.1). Further note that the problem (1.1) can then be transformed into the divergence-form problem

$$(3.3) \qquad \begin{cases} -\nabla \cdot \left(A^{\mathrm{div}}\left(\frac{\cdot}{\varepsilon}\right)\nabla u^\varepsilon\right) = r\left(\frac{\cdot}{\varepsilon}\right)f & \text{in } \Omega, \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

with the matrix-valued function $A^{\mathrm{div}} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ given by

$$A^{\mathrm{div}}(y) := \begin{pmatrix} 1 - \frac{1}{2}\sin(2\pi y_1)\sin(2\pi y_2) & \frac{1}{2}\cos(2\pi y_1)\cos(2\pi y_2) \\ -\frac{1}{2}\cos(2\pi y_1)\cos(2\pi y_2) & 1 + \frac{1}{2}\sin(2\pi y_1)\sin(2\pi y_2) \end{pmatrix}.$$

We can check that A is c-bad by explicitly computing the matrix of corrector functions $V = (v^{ij})_{1\le i,j\le2}$ given by (1.3) and computing the values $\{c_j^{kl}\}_{1\le j,k,l\le2}$ given by (1.5).

Proof of Theorem 1.10. The effective coefficient $\bar{A} \in \mathcal{S}_+^2$ is given by

$$(3.4) \qquad \bar{A} = \int_Y Ar = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and it is a straightforward calculation to check that the matrix of corrector functions $V = (v^{ij})_{1 \leq 1, j \leq 2} : \mathbb{R}^2 \to \mathbb{R}^{2 \times 2}$ is given by

$$V(y) = -\frac{\sin(2\pi y_2)}{32\pi^2} \begin{pmatrix} \cos(2\pi y_1) & 0 \\ 0 & \cos(2\pi y_1) - 4\sin(2\pi y_1) \end{pmatrix}.$$

Computation of the values c_j^{kl} for $j, k, l \in \{1, 2\}$ given by (1.5) yields that

$$c_1^{11} = \int_Y ra_{11}\partial_1 v^{11} = -\frac{1}{128\pi} = \int_Y ra_{11}\partial_1 v^{22} = c_1^{22}$$

for the values of c_1^{11}, c_1^{22}, and that

$$c_2^{11} = \int_Y ra_{22}\partial_2 v^{11} = 0 = \int_Y ra_{22}\partial_2 v^{22} = c_2^{22}$$

for the values of c_2^{11}, c_2^{22}. Clearly we have that $c_j^{kl} = 0$ for any $(j, k, l) \in \{1, 2\}^3$ with $k \neq l$. □

Let us note that the effective coefficient (3.4) is the identity matrix and hence, the homogenized problem for this c-bad matrix is the Poisson problem

$$(3.5) \qquad \begin{cases} -\Delta u = f & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases}$$

Further, we have that the function z defined by (1.7) is given as the solution to the Poisson problem

$$(3.6) \qquad \begin{cases} -\Delta z = -\dfrac{\partial_1 f}{128\pi} & \text{in } \Omega, \\ \quad z = 0 & \text{on } \partial\Omega. \end{cases}$$

Finally, let us note that the factor $\frac{1}{r}$ in the definition of the c-bad matrix (3.1) is crucial for c-badness. Indeed, removing this factor we obtain a c-good matrix:

Remark 3.2. The matrix-valued function $A : \mathbb{R}^2 \to \mathbb{R}^{2 \times 2}$ given by

$$A(y) := \begin{pmatrix} 1 - \frac{1}{2}\sin(2\pi y_1)\sin(2\pi y_2) & 0 \\ 0 & 1 + \frac{1}{2}\sin(2\pi y_1)\sin(2\pi y_2) \end{pmatrix}$$

is c-good.

Proof. The invariant measure is the constant function $r \equiv 1$ and hence, the effective coefficient $\bar{A} \in \mathcal{S}_+^2$ is given by

$$\bar{A} = \int_Y A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is a straightforward calculation to check that the matrix of corrector functions $V = (v^{ij})_{1 \leq 1, j \leq 2} : \mathbb{R}^2 \to \mathbb{R}^{2 \times 2}$ is given by

$$V(y) = -\frac{\sin(2\pi y_1)\sin(2\pi y_2)}{16\pi^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

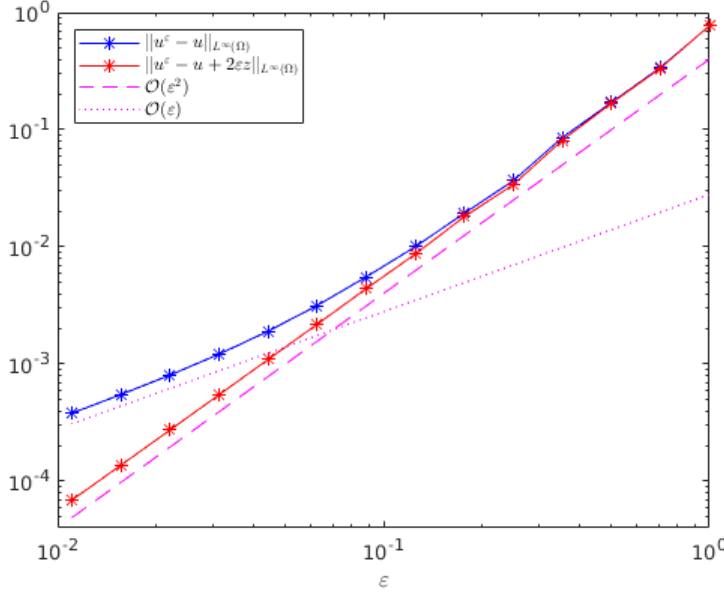Computation of the values c_j^{kl} given by (1.5) yields $c_j^{kl} = 0$ for all $j, k, l \in \{1, 2\}$. □

FIGURE 1. *blue: Plot of $\|u^\varepsilon - u\|_{L^\infty(\Omega)}$, red: Plot of $E^\varepsilon_{0,\infty}$ (see (3.7)). We observe $\|u^\varepsilon - u\|_{L^\infty(\Omega)} = \mathcal{O}(\varepsilon)$ and $E^\varepsilon_{0,\infty} = \mathcal{O}(\varepsilon^2)$ as expected from Remark 1.6.*

Note that the effective problem for this c-good matrix is again the Poisson problem (3.5), i.e., the homogenized solution coincides with the one from the c-bad problem.

3.2. Numerical illustration of the $L^\infty(\Omega)$ rates. We consider the problem (1.1) with the c-bad coefficient matrix A from Theorem 1.10, the domain $\Omega := (0,1)^2$ and the right-hand side

$$f : \bar\Omega \to \mathbb{R}, \quad f(x_1, x_2) := 8\pi^2 \sin(2\pi x_1) \sin(2\pi x_2).$$

Then, the solution to the homogenized problem (3.5) is given by

$$u : \bar\Omega \to \mathbb{R}, \quad u(x_1, x_2) = \sin(2\pi x_1) \sin(2\pi x_2),$$

and the solution z to the problem (3.6) is given by

$$z : \bar\Omega \to \mathbb{R}, \quad z(x) = \frac{1}{64} \left(\frac{\cosh(2\pi x_1 - \pi)}{\cosh(\pi)} - \cos(2\pi x_1) \right) \sin(2\pi x_2).$$

Figure 1 illustrates the estimate (1.9) from Remark 1.6, i.e., for several values of ε, we plot

$$(3.7) \qquad\qquad E^\varepsilon_{0,\infty} := \|u^\varepsilon - u + 2\varepsilon z\|_{L^\infty(\Omega)}.$$

We approximate the solution u^ε to (1.1) with $\mathbb{P}_1$ finite elements on a fine mesh, based on the natural variational formulation of the divergence-form problem (3.3). We observe the rate $E^\varepsilon_{0,\infty} = \mathcal{O}(\varepsilon^2)$ as ε tends to zero, as expected from Remark 1.6.
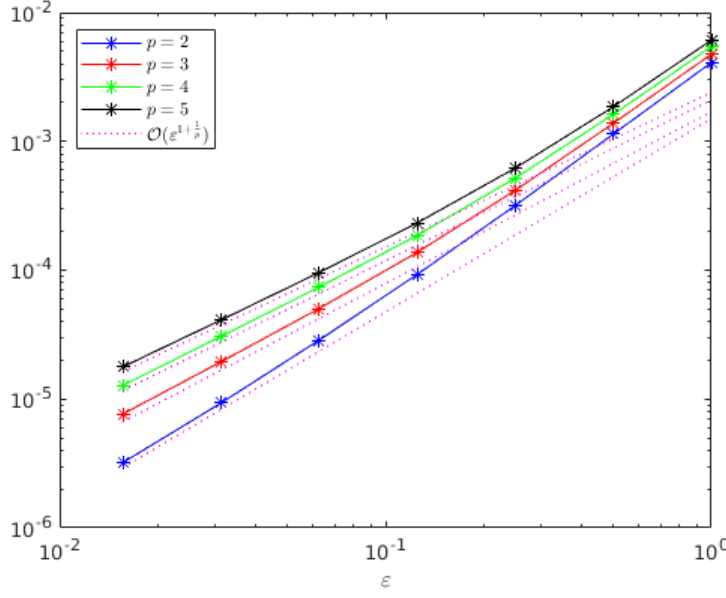
FIGURE 2. *Plot of $E_{1,p}^{\varepsilon}$ (see (3.10)) for $p = 2, 3, 4, 5$. We observe the rate $E_{1,p}^{\varepsilon} = \mathcal{O}(\varepsilon^{1+\frac{1}{p}})$ as expected from Remark 1.6.*

3.3. Numerical illustration of the $W^{1,p}(\Omega)$ rates. We consider the problem (1.1) with the c-bad coefficient matrix A from Theorem 1.10, the domain $\Omega := (0,1)^2$ and the right-hand side

$$(3.8) \qquad f : \bar{\Omega} \to \mathbb{R}, \quad f(x_1, x_2) := x_1(1 - x_1) + x_2(1 - x_2).$$

Then, the solution of the homogenized problem (3.5) is given by

$$(3.9) \qquad u : \bar{\Omega} \to \mathbb{R}, \quad u(x_1, x_2) = \frac{1}{2}x_1(1 - x_1)x_2(1 - x_2).$$

Figure 2 illustrates the estimate (1.10) from Remark 1.6, i.e., for several values of ε, we plot

$$(3.10) \qquad E_{1,p}^{\varepsilon} := \left\| \nabla u^{\varepsilon} - \nabla u + 2\varepsilon \nabla z - \varepsilon \sum_{i,j=1}^{n} \nabla v^{ij}\left(\frac{\cdot}{\varepsilon}\right) \partial_{ij}^2 u \right\|_{L^p(\Omega)}$$

for the values $p = 2, 3, 4, 5$. We approximate the solution u^{ε} to (1.1) and the solution z to (3.6) with $\mathbb{P}_2$ finite elements on a fine mesh, based on the natural variational formulation of the divergence-form problems (3.3) and (3.6). We observe the rate $E_{1,p}^{\varepsilon} = \mathcal{O}(\varepsilon^{1+\frac{1}{p}})$ as ε tends to zero, as expected from Remark 1.6.

3.4. Numerical illustration of the $W^{2,p}(\Omega)$ rates. We consider the problem (1.1) with the c-bad coefficient matrix A from Theorem 1.10, the domain $\Omega := (0,1)^2$ and f given by (3.8). As before, the homogenized solution is given by (3.9). Figure 3
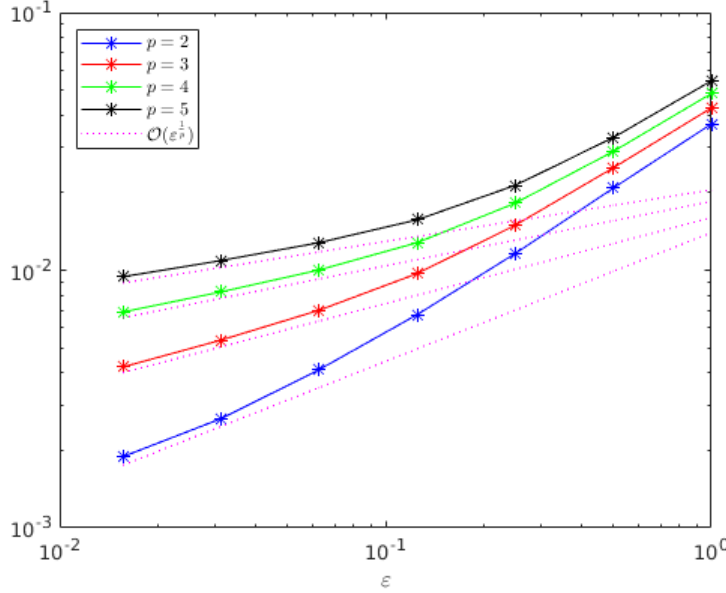
FIGURE 3. *Plot of $E_{2,p}^\varepsilon$ (see (3.11)) for $p = 2, 3, 4, 5$. We observe the rate $E_{2,p}^\varepsilon = \mathcal{O}(\varepsilon^{\frac{1}{p}})$ as expected from Remark 1.6.*

illustrates the estimate (1.11) from Remark 1.6, i.e., for several values of ε, we plot

$$(3.11) \qquad E_{2,p}^\varepsilon := \left\| D^2 u^\varepsilon - D^2 u - \sum_{i,j=1}^n D^2 v^{ij}\left(\frac{\cdot}{\varepsilon}\right) \partial_{ij}^2 u \right\|_{L^p(\Omega)}$$

for the values $p = 2, 3, 4, 5$. We approximate the solution u^ε to (1.1) with an H^2 conforming finite element method on a fine mesh, using the HCT element in FreeFem++ [24]. We multiply the equation (1.1) by the invariant measure and use the variational formulation from the framework of linear nondivergence-form equations with Cordes coefficients (see [36]): The solution u^ε to (1.1) is the unique function in $H := H^2(\Omega) \cap H_0^1(\Omega)$ such that there holds

$$\int_\Omega \frac{\operatorname{tr}\left([rA]\left(\frac{\cdot}{\varepsilon}\right)\right)}{\left|[rA]\left(\frac{\cdot}{\varepsilon}\right)\right|^2}\left(-[rA]\left(\frac{\cdot}{\varepsilon}\right) : D^2 u^\varepsilon\right) \Delta v = \int_\Omega \frac{\operatorname{tr}\left([rA]\left(\frac{\cdot}{\varepsilon}\right)\right)}{\left|[rA]\left(\frac{\cdot}{\varepsilon}\right)\right|^2} r\left(\frac{\cdot}{\varepsilon}\right) f \Delta v$$

for any $v \in H$. We observe the rate $E_{2,p}^\varepsilon = \mathcal{O}(\varepsilon^{\frac{1}{p}})$ as ε tends to zero, as expected from Remark 1.6.

3.5. Comparison of c-bad and c-good problems. We refer to the problem (1.1) with the c-bad coefficient matrix from Theorem 1.10 as the c-bad problem and to the problem (1.1) with the c-good coefficient matrix from Remark 3.2 as the c-good problem. We perform experiments for these two problems with two different choices of right-hand sides, one with known homogenized solution u and one with unknown homogenized solution u. All experiments are performed on the domain $\Omega := (0,1)^2$.

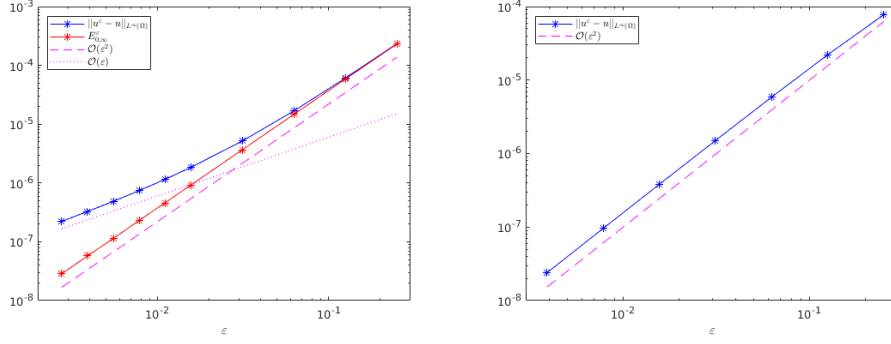We recall that the homogenized problems corresponding to the c-bad and the

FIGURE 4. *Illustration of the L^∞-rates $\|u^\varepsilon - u\|_{L^\infty(\Omega)} = \mathcal{O}(\varepsilon)$ and $E_{0,\infty}^\varepsilon = \mathcal{O}(\varepsilon^2)$ for the c-bad problem (left), and $\|u^\varepsilon - u\|_{L^\infty(\Omega)} = \mathcal{O}(\varepsilon^2)$ for the c-good problem (right) with the right-hand side (3.8).*

c-good problem coincide and that the homogenized solution u is the solution to the Poisson problem (3.5).

Let us note that the numerical experiments performed in [17] only involve c-good diffusion matrices, thus leading to L^∞ convergence rates of order $\mathcal{O}(\varepsilon^2)$.

3.5.1. c-bad and c-good problems with known (common) homogenized function u. We consider the right-hand side f given by (3.8). Then, the solution u of the homogenized problem is known and given by (3.9).

Figure 4 illustrates the L^∞ convergence rate $\mathcal{O}(\varepsilon)$ for the c-bad problem and the convergence rate $\mathcal{O}(\varepsilon^2)$ for the c-good problem. We also illustrate the corrected L^∞ bound $E_{0,\infty}^\varepsilon = \mathcal{O}(\varepsilon^2)$ for the c-bad problem. We approximate the solution u^ε to (1.1) and the solution z to (3.6) with $\mathbb{P}_2$ finite elements on a fine mesh, based on the natural variational formulation of the divergence-form problems (3.3) (note $r \equiv 1$ for the c-good problem) and (3.6).

3.5.2. c-bad and c-good problems with unknown (common) homogenized function u. We consider the right-hand side f given by

$$(3.12) \qquad f : \bar{\Omega} \to \mathbb{R}, \quad f(x) := x_1^3 (1 - x_1)^3 \sin(2\pi(x_1 - 2x_2)).$$

Let us note that we do not know the homogenized solution u exactly, we have however that $u \in H^6(\Omega) \cap H_0^1(\Omega)$ as the right-hand side $f \in H^4(\Omega)$ satisfies the compatibility conditions $f = 0$ and $\partial_1^2 f - \partial_2^2 f = 0$ at the corners of the square $(0, 1)^2 = \Omega$; see [25].

Figure 5 illustrates the L^∞ convergence rate $\mathcal{O}(\varepsilon)$ for the c-bad problem and the convergence rate $\mathcal{O}(\varepsilon^2)$ for the c-good problem. We also illustrate the corrected L^∞ bound $E_{0,\infty}^\varepsilon = \mathcal{O}(\varepsilon^2)$ for the c-bad problem. We approximate the functions u^ε, u and z with $\mathbb{P}_2$ finite elements as before.

4. Extensions and concluding remarks.

4.1. Nonsmooth domains. The smoothness assumption on the domain Ω is used to deduce regularity of u from the regularity assumption on f, and it ensures that the uniform estimates from Section 2.1 hold. We briefly discuss extensions to nonsmooth domains.
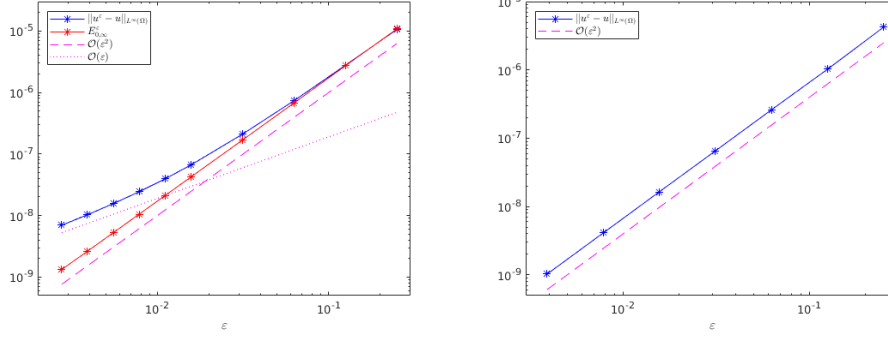
FIGURE 5. *Illustration of the L^∞-rates $\|u^\varepsilon - u\|_{L^\infty(\Omega)} = \mathcal{O}(\varepsilon)$ and $E_{0,\infty}^\varepsilon = \mathcal{O}(\varepsilon^2)$ for the c-bad problem (left), and $\|u^\varepsilon - u\|_{L^\infty(\Omega)} = \mathcal{O}(\varepsilon^2)$ for the c-good problem (right) with the right-hand side (3.12).*

4.1.1. $C^{2,\gamma}$ domains. We note that the uniform estimates from Section 2.1 still hold for bounded $C^{2,\gamma}$ domains. Therefore, the result of Theorem 1.5 remains valid with the additional assumption $u \in W^{4,q}(\Omega)$ for some $q > n$ (for assertion (ii), require $u \in W^{5,q}(\Omega)$), which has previously been deduced from the regularity of the right-hand side f.

4.1.2. Convex domains. We would like to briefly discuss the case of convex domains. Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain in dimension $n \geq 2$ and assume that the homogenized solution is of regularity $u \in W^{4,q}(\Omega)$ for some $q > n$ (for assertion (ii), require $u \in W^{5,q}(\Omega)$). Let us further assume that the coefficient is of regularity $A \in C^{0,\alpha}(\mathbb{T}^n; \mathcal{S}_+^n)$ for some $\alpha \in (0,1]$ and satisfies the Cordes condition (which dates back to [10]), i.e., that there exists a constant $\delta \in (0,1]$ such that there holds

$$(4.1) \qquad \frac{|A|^2}{(\operatorname{tr}A)^2} \leq \frac{1}{n-1+\delta} \quad \text{in } \mathbb{R}^n.$$

Let us note that the Cordes condition (4.1) is a consequence of uniform ellipticity in two dimensions, i.e., (4.1) holds for any $A \in C^{0,\alpha}(\mathbb{T}^2; \mathcal{S}_+^2)$. Let us also note that Theorem 1.4 holds in this situation for $p = 2$; see [9].

In the situation described above, there exists a unique solution $u^\varepsilon \in H^2(\Omega) \cap H_0^1(\Omega)$ to (1.1) and we have a uniform H^2 estimate [9, Theorem 2.5]. Therefore, by the Sobolev embedding, we have the uniform $W^{1,p}$ estimate

$$\|u^\varepsilon\|_{W^{1,p}(\Omega)} \leq C\|u^\varepsilon\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}$$

for any $p < 2^*$ with constants independent of ε. Here, we write $2^* := \frac{2n}{n-2}$ to denote the critical Sobolev exponent (with the convention that $2^* := \infty$ if $n = 2$). This uniform estimate replaces the need for the uniform $C^{1,\alpha}$ estimate from Lemma 2.1.

Finally, in order to estimate the boundary corrector, we transformed the problem (1.12) into divergence-form and used that for problems of the form

$$\begin{cases} -\nabla \cdot \left(A^{\mathrm{div}}\left(\frac{\cdot}{\varepsilon}\right) \nabla \rho^\varepsilon \right) = -\nabla \cdot F & \text{in } \Omega, \\ \rho^\varepsilon = g & \text{on } \partial\Omega, \end{cases}$$

we have (Lemma 2.2) the uniform $W^{1,p}$ estimate

$$(4.2) \qquad \|\rho^\varepsilon\|_{W^{1,p}(\Omega)} \le C \left(\|F\|_{L^p(\Omega)} + \|g\|_{W^{1,p}(\Omega)} \right)$$

with a constant $C > 0$ independent of ε, assuming that $A^{\mathrm{div}} \in C^{0,\alpha}(\mathbb{T}^n; \mathbb{R}^{n \times n})$ for some $\alpha \in (0,1]$ is uniformly elliptic and that Ω is sufficiently smooth.

Now as Ω is merely assumed to be convex, we still have (4.2) for $p = 2$ by standard arguments and hence, we find that the result of Theorem 1.5 remains true for $p = 2$ under the assumptions made in this section. Uniform $W^{1,p}$ estimates for divergence-form problems for a wider range of values p require a more sophisticated approach. With a symmetry assumption on A^{div}, uniform $W^{1,p}$ estimates for divergence-form problems on Lipschitz domains (recall that bounded convex domains are Lipschitz [22]) have been obtained in [35] for values of p in a certain range around $p = 2$.

4.1.3. Numerical Homogenization. Nonsmooth domains, specifically convex polygonal domains, are of particular interest in the area of numerical analysis and we would like to highlight the utility of the results of Section 4.1.2 in view of numerical homogenization, i.e., the efficient and accurate numerical approximation of the homogenized solution u and the solution u^ε of the original problem (1.1) for values $\varepsilon \ll 1$. For an overview of the numerical homogenization of linear equations in nondivergence-form, we refer the reader to [9].

In the situation of Section 4.1.2, we can now use the H^1-bound

$$\|u^\varepsilon - u\|_{H^1(\Omega)} = \mathcal{O}(\varepsilon)$$

in combination with the finite element numerical homogenization scheme from [9] to obtain an H^1-approximation to u, which in turn provides an H^1-approximation to u^ε together with a, generally optimal, quantification of the approximation error in the parameter ε. Similarly, our numerical experiments suggest that the bound

$$\left\| u^\varepsilon - u - \varepsilon^2 V \left(\frac{\cdot}{\varepsilon} \right) : D^2 u \right\|_{H^2(\Omega)} = \mathcal{O}(\sqrt{\varepsilon}),$$

and hence also the error analysis of the H^2-approximation of u^ε via the finite element method proposed in [9] is generally optimal in the parameter ε.

4.2. Optimality of Lemma 1.7. We would like to readdress the bound (1.13) on the $W^{1,p}$-norm of the boundary corrector θ^ε, that is,

$$(4.3) \qquad \varepsilon \left\| \theta^\varepsilon \right\|_{W^{1,p}(\Omega)} = \mathcal{O}(\varepsilon^{\frac{1}{p}}).$$

In this paragraph, we provide a theoretical indication that the rate $\mathcal{O}(\varepsilon^{\frac{1}{p}})$ in the right-hand side of (4.3) is optimal.

To this end, let us consider the following Poisson problem in the upper half-plane $\Omega := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$ with oscillating Dirichlet boundary data:

$$(4.4) \qquad \begin{cases} -\Delta \theta^\varepsilon = 0 & \text{in } \Omega, \\ \theta^\varepsilon(x) = g \left(\dfrac{x_1}{\varepsilon} \right) & \text{for } x = (x_1, x_2) \in \partial\Omega. \end{cases}$$

Here, the boundary data $g \in C(\mathbb{R})$ is 1-periodic. By the Poisson integral formula for the half-plane, we have that the solution to (4.4) is given by

$$\theta^\varepsilon(x_1, x_2) = \frac{x_2}{\pi} \int_{-\infty}^{\infty} \frac{g\left(\frac{t}{\varepsilon}\right)}{(x_1 - t)^2 + x_2^2} \, \mathrm{d}t = \frac{x_2}{\pi} \int_{-\infty}^{\infty} \frac{\varepsilon \, g(t)}{(x_1 - \varepsilon t)^2 + x_2^2} \, \mathrm{d}t.$$

Then, for any $x_2 = \varepsilon \tilde{x}_2 \in (\frac{\varepsilon}{2}, \varepsilon)$ we have that $\tilde{x}_2 \in (\frac{1}{2}, 1)$ and

$$\varepsilon \, \partial_1 \theta^\varepsilon(x_1, x_2) = \frac{2\tilde{x}_2}{\pi} \int_{-\infty}^{\infty} \frac{-(\frac{x_1}{\varepsilon} - t) g(t)}{\left((\frac{x_1}{\varepsilon} - t)^2 + \tilde{x}_2^2\right)^2} \, \mathrm{d}t = \frac{2\tilde{x}_2}{\pi} \int_0^\infty t \, \frac{g(\frac{x_1}{\varepsilon} + t) - g(\frac{x_1}{\varepsilon} - t)}{(t^2 + \tilde{x}_2^2)^2} \, \mathrm{d}t,$$

which is non-zero for almost all $x_1 \in \mathbb{R}$ if we choose g to be the 1-periodic extension of the map $[0, 1) \ni y \mapsto \frac{1}{2}(1 - |2y - 1|)$ for example. We deduce that

$$(4.5) \qquad\qquad \int_{(0,1)^2} |\varepsilon \, \partial_1 \theta^\varepsilon|^p \geq C\varepsilon,$$

which demonstrates that (4.3) is optimal.

Besides, if we choose $g(y) = \cos(2\pi y)$, it is easily checked that the solution to (4.4) is given by

$$\theta^\varepsilon(x_1, x_2) = \cos\left(\frac{2\pi}{\varepsilon} x_1\right) \exp\left(-\frac{2\pi}{\varepsilon} x_2\right),$$

from which we obtain (4.5) and

$$\int_{(0,1)^2} |\varepsilon^2 \partial_1^2 \theta^\varepsilon|^p \geq C\varepsilon$$

by direct computation. This indicates that both (1.13) and (1.14) are optimal.

4.3. Interpolation. Let us revisit Remark 1.6 and note that, for values $p \geq 2$, the gradient bound (1.10) follows from the L^∞ bound (1.9) and the Hessian bound (1.11) via the Gagliardo–Nirenberg interpolation inequality [32] applied to the function

$$\varphi^\varepsilon := u^\varepsilon - u + 2\varepsilon z - \varepsilon^2 \, V\left(\frac{\cdot}{\varepsilon}\right) : D^2 u.$$

Indeed, let us assume that $\|\varphi^\varepsilon\|_{L^\infty(\Omega)} = \mathcal{O}(\varepsilon^2)$ and $\|D^2\varphi^\varepsilon\|_{L^p(\Omega)} = \mathcal{O}(\varepsilon^{\frac{1}{p}})$ for any $p \in (1, \infty)$. Then the Gagliardo–Nirenberg inequality yields

$$\|\nabla \varphi^\varepsilon\|_{L^p(\Omega)} \leq C\left(\|D^2\varphi^\varepsilon\|_{L^{\frac{p}{2}}(\Omega)}^{\frac{1}{2}} \|\varphi^\varepsilon\|_{L^\infty(\Omega)}^{\frac{1}{2}} + \|\varphi^\varepsilon\|_{L^\infty(\Omega)}\right) = \mathcal{O}(\varepsilon^{1 + \frac{1}{p}})$$

for any $p \geq 2$. Note that this interpolation argument cannot be used to prove Theorem 1.5 (ii) for $p < 2$, and that it does not yield assertion (i) of Theorem 1.5.

This shows once again that the optimality of the bounds (1.9)–(1.11) is natural to expect. We conclude this paper with a review of the main results.

4.4. Conclusion. In this paper we derived optimal rates of convergence in the periodic homogenization of linear elliptic equations in nondivergence-form. As a result of a $W^{1,p}$ corrector estimate, we obtained that the optimal rate of convergence of u^ε to the homogenized solution in the $W^{1,p}$-norm is $\mathcal{O}(\varepsilon)$ and also recovered that the optimal convergence rate in the L^∞-norm is $\mathcal{O}(\varepsilon)$. Moreover, we obtained optimal estimates for the gradient and the Hessian of the solution with correction terms taken into account in L^p-norm numerically.

In the final part of the paper, we provided an example of an explicit c-bad matrix and presented several numerical experiments matching the theoretical results and illustrating the optimality of the obtained rates.

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