Advances in Mathematics 389 (2021) 107912

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

A local Hopf lemma and unique continuation for elliptic equations ☆

S. Berhanu

Department of Mathematics, Temple University, Philadelphia, PA 19122, USA

ARTICLE INFO

Article history:
Received 15 September 2020
Received in revised form 29 June 2021
Accepted 2 July 2021
Available online xxxx
Communicated by YanYan Li

MSC:
primary 35J05, 35B60
secondary 31B25, 30B40

Keywords:
Unique continuation
Hopf lemma
Poisson kernel

ABSTRACT

We prove results on unique continuation at the boundary for the solutions of real analytic elliptic partial differential equations of the form

$$\Delta u + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = 0.$$

The work is motivated by and generalizes the main results of X. Huang et al. in [15] and [16], and M.S. Baouendi and L.P. Rothschild in [5].

1. Introduction

This paper concerns unique continuation at the boundary and a local Hopf lemma for solutions of

$$Pu = \Delta u + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = 0$$

☆ Work supported in part by NSF DMS 1855737.

E-mail address: berhanu@temple.edu.

https://doi.org/10.1016/j.aim.2021.107912

where the coefficients of P are real analytic.

The work generalizes the results proved for harmonic functions in [5], [15], [16] and those in [12] for solutions of the Helmholtz's equation $\Delta u + cu = 0,\ c \in \mathbb{R}$. See also [6], [9] and [10].

These uniqueness phenomena extend the classical Hopf lemma about the nonvanishing of the normal derivative at a boundary point where a nonconstant solution attains an extremum: the assumption is local in nature and only imposes conditions at the boundary.

For holomorphic functions of one variable with nonnegative real part on a piece of the boundary, unique continuation and local forms of Hopf's lemma were proved in [7], [15], [16], and [18]. The results were used to prove unique continuation for CR mappings for certain classes of CR manifolds. They were also used to prove a more general Schwarz reflection principle for holomorphic functions mapping the real line into a totally real manifold or a real analytic set. Earlier results along this line appeared in the works [4], [8], [2] and [3]. H. Alexander's paper [3] contains a general local Hopf lemma for holomorphic functions of one variable with applications to unique continuation for CR mappings. See also [11] for an extension of the latter results.

Further extensions of the results of Baouendi and Rothschild were proved by V. Shklover ([22]) and H.S. Shapiro ([21]). In particular, Shklover showed that Theorem 3 in [5] (Theorem 2.1 in this paper) fails in general if the normal direction is replaced with a transverse direction. Shapiro used convolution transforms as discussed in [14] to obtain new proofs and generalizations of the theorems of Baouendi and Rothschild. In the article [23], N. Suzuki established a local Hopf lemma in the spirit of [5] for the one-dimensional heat equation.

The article is organized as follows: Section 2 contains the statements of the results in this work. Section 3 is devoted to the construction of a kernel that effectively (for our purposes here) serves as a Poisson kernel for P on the flat piece of the half ball B_r^+. The proofs of the theorem and its corollaries are presented in section 4.

2. Statement of the results

We will say that a continuous function u defined on a half ball

$$B_r^+ = \{x = (x', x_n) \in \mathbb{R}^n : |x| < r, x_n > 0\}$$

is flat at 0 if for every positive integer N, there is a constant $C_N > 0$ such that

$$|u(x)| \leq C_N |x|^N.$$

Suppose now $D \subseteq \mathbb{R}^n$ is a smoothly bounded domain, $x_0 \in \partial D$. We will say a function u vanishes to infinite order in a direction v at x_0, where v is a unit vector pointing inside D and transversal to ∂D if for every N, there is a constant $C_N > 0$ such that:

$$|u(x_0 + tv)| \le C_N t^N.$$

We also say u vanishes to infinite order on a non-singular smooth curve $S: x = x(t)$, $0 \le t \le 1$, in D passing through x_0 and transversal to ∂D if for every N there is $C_N > 0$ such that:

$$|u(x(t))| \le C_N t^N.$$

It is easy to see that this latter definition is independent of the parametrization.

We recall the main result of [5]:

Theorem 2.1. *Let u be harmonic on the half ball B_r^+, continuous on the closure. Suppose*

(1) *$u(x',0) \ge 0$ for $|x'| \le r$, $x' \in \mathbb{R}^{n-1}$;*
(2) *the function $x_n \mapsto u(0', x_n)$ is flat at $x_n = 0$;*

Then $u(x',0) \equiv 0$ for x' near the origin in $\mathbb{R}^{n-1}$.

Somewhat similar but weaker results under the stronger hypothesis that u is harmonic in the upper half plane and decays exponentially along the y-axis was obtained in [19]. The theorem of Baouendi and Rothschild has the following immediate consequence on boundary unique continuation for harmonic functions:

Corollary 2.2. *Let u be harmonic in B_r^+, continuous on the closure of B_r^+. Assume that*

(1) *$u(x',0) \ge 0$ for $|x'| \le r$;*
(2) *The function u is flat at 0.*

Then $u \equiv 0$.

Our generalization is as follows:

Theorem 2.3. *Let u be a solution of*

$$Pu = \Delta u + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u(x) = 0$$

in the half ball B_r^+, C^2 on $\overline{B_r^+}$. Assume that the coefficients of P are real analytic on $\overline{B_r^+}$. Suppose

(1) *$u(x',0) \ge 0$ for $|x'| \le r$;*
(2) *the function $x_n \mapsto u(0', x_n)$ is flat at $x_n = 0$;*

(3) *for every positive integer* N*, the function*

$$|x'|^{-N}u(x',0)$$

is integrable on $|x'| \leq r$*.*

Then $u(x',0) \equiv 0$ *for* x' *small.*

We remark that by the results in [20], u then extends as a solution in a neighborhood of the origin in $\mathbb{R}^n$.

Theorem 2.3 has the following consequence on boundary unique continuation:

Corollary 2.4. *Let* u *be a solution of* $Pu = 0$ *in* B_r^+*,* C^2 *on the closure of* B_r^+*. Assume that*

(1) $u(x',0) \geq 0$ *for* $|x'| \leq r$*;*
(2) *The function* u *is flat at* 0*.*

Then $u \equiv 0$*.*

In [22] the author considered the following refinement of Theorem 2.1 of Baouendi and Rothschild:

Suppose $D \subseteq \mathbb{R}^n$ is a smoothly bounded domain, $x_0 \in \partial D$, $V \subset \partial D$ real analytic, $x_0 \in V$. If u is harmonic in D and continuous on $\overline{D}$, vanishes to infinite order in a direction v (or a curve) at x_0, and $u(x) \geq 0$ on V, then $u(x) \equiv 0$ in some neighborhood of x_0 in V.

It was shown in [22] that in the plane, this property holds (for harmonic functions) if and only if V is locally symmetric about the normal to ∂D at x_0. The author also proved further results for the situation when the normal is replaced by a transversal curve.

These results generalize to the operators P under study:

Corollary 2.5. *Let* $n = 2$ *and* u *be a solution of* $Pu = 0$ *in* B_r^+*,* C^2 *on the closure of* B_r^+*. Assume that*

(1) $u(x) \geq 0$ *for* $x \in V \subset \mathbb{R}$*,* $0 \in V$*;*
(2) u *vanishes to infinite order on an analytic curve* S *through the origin orthogonal to the* x*-axis and* S *is symmetric with respect to the* x*-axis.*

Then u *vanishes on some subinterval of* V *about the origin.*

In the case of a general domain D we have:

Corollary 2.6. *Let $n = 2$ and u be a solution of $Pu = 0$ in a domain D, C^2 on the closure of D. Suppose $V \subseteq \partial D$ is real analytic, and is tangent to the real axis at the origin. Assume that*

(1) $u(x) \geq 0$ *for* $x \in V$*;*
(2) *the function* $y \mapsto u(0, y)$ *vanishes to infinite order at* 0*;*
(3) V *is locally symmetric about the imaginary axis. Then* u *vanishes on some subinterval of* V *about the origin.*

Corollary 2.7. *Let $n = 2$ and u be a solution of $Pu = 0$ in a domain D, C^2 on the closure of D. Suppose $V \subseteq \partial D$ is real analytic and $x_0 \in V$. Assume that*

(1) $u(x) \geq 0$ *for* $x \in V$*;*
(2) *The function* u *is flat at* x_0*.*

Then u vanishes on D.

3. Construction of a Poisson kernel for P

For elliptic differential operators of any order with constant coefficients, Poisson kernels for the upper half space $\mathbb{R}^n$ were constructed by Agmon-Douglis-Nirenberg in the work [1]. For elliptic operators with real analytic coefficients, the existence of a local Poisson kernel was proved in [17]. However, this latter kernel is not explicit, and it doesn't serve our purpose since we will need precise estimates on arbitrarily high order derivatives of the kernel. Let

$$P = P(D_x) = \Delta + \sum_{j=1}^{n} b_j(x) \frac{\partial}{\partial x_j} + c(x)$$

be as in Theorem 2.3. In this section we will construct what will essentially serve as a Poisson kernel $K = K(x, y')$ for P using the ideas and methods of Hadamard ([13]). In [13] it is shown that P has a fundamental solution $E(x, y)$ $(x, y \in \mathbb{R}^n)$ of a form that depends on the parity of n.

Case 1: Assume n is odd. In that case, E has the form

$$E(x, y) = \frac{1}{d(x, y)^{n-2}} \sum_{k=0}^{\infty} E_k(x, y) d(x, y)^{2k}, \quad d(x, y) = |x - y|$$

for x, y in a ball B centered at the origin in $\mathbb{R}^n$.

We will use the same idea to construct first a solution

$$F(x, y) = \frac{1}{\bar{d}(x, y)^{n-2}} \sum_{k=0}^{\infty} F_k(x, y) \bar{d}(x, y)^{2k}$$

of $P(D_x)F(x,y)=0$ for $x\neq\bar{y}$, where $\bar{d}(x,y)=|x-\bar{y}|$ and for any $y=(y_1,\ldots,y_n)$, $\bar{y}=(y_1,\ldots,y_{n-1},-y_n)$. In addition, the F_k will be required to satisfy the initial condition

$$F_k(x',0,y)=E_k(x',0,y).$$

Consider

$$PF=\sum_{k=0}^{\infty}P\Big(F_k\bar{d}^{2k-n+2}\Big).$$

We have

$$\begin{aligned}
&P\Big(F_k\ \bar{d}^{2k-n+2}\Big)(x,y)\\
&=(PF_k)\bar{d}^{2k-n+2}+2\langle\nabla F_k,\nabla(\bar{d}^{2k-n+2})\rangle\\
&+F_k\Delta(\bar{d}^{2k-n+2})+F_k\Big(\sum_{j=1}^{n}b_j\frac{\partial}{\partial x_j}\bar{d}^{2k-n+2}\Big)\\
&=(PF_k)\bar{d}^{2k-n+2}+2(2k-n+2)\Big[\sum_{j=1}^{n-1}(x_j-y_j)\frac{\partial F_k}{\partial x_j}+(x_n+y_n)\frac{\partial F_k}{\partial x_n}\Big]\bar{d}^{\,2k-n}\\
&+(2k-n+2)\Big(2kF_k\bar{d}^{2k-n}+\sum_{j=1}^{n-1}b_j(x)(x_j-y_j)+b_n(x)(x_n+y_n)\Big)F_k\ \bar{d}^{2k-n}\\
&=(PF_k)\ \bar{d}^{2k-n+2}+2(2k-n+2)\\
&\Big[\sum_{j=1}^{n-1}(x_j-y_j)\frac{\partial F_k}{\partial x_j}+(x_n+y_n)\frac{\partial F_k}{\partial x_n}+(\theta(x,y)+k-m-1)F_k\Big]\bar{d}^{2k-n}
\end{aligned}$$

where $m=\frac{n-2}{2}$ and

$$\theta(x,y)=\frac{1}{4}\Big(2n+2\sum_{i=1}^{n-1}b_i(x)(x_i-y_i)+2b_n(x)(x_n+y_n)\Big).$$

Let

$$x(t)=tx+(1-t)\bar{y},\ 0<t\leq 1.$$

Setting the coefficients of the powers of $\bar{d}(x,y)$ equal to 0, we are led to the initial value problems

$$t\frac{d}{dt}F_0(x(t),y)+(\theta(x(t),y)-m-1)F_0(x(t),y)=0, \tag{3.1}$$

$$F_0(x',0,y)=E_0(x',0,y) \tag{3.2}$$

and for any $k \geq 1$,

$$t\frac{d}{dt}F_k(x(t),y)+(\theta(x(t),y)+k-m-1)F_k(x(t),y) = -\frac{1}{4(k-m)}P(F_{k-1})(x(t),y), \quad (3.3)$$

$$F_k(x',0,y) = E_k(x',0,y). \quad (3.4)$$

For $0 < t \leq 1$, define F_0 by

$$F_0(x(t),y) = E_0(x',0,y)\exp\{2\int_0^1 b(\tilde{x}(r))\cdot(x-\bar{y})dr\}\cdot\exp\{-2\int_0^t b(x(r))\cdot(x-\bar{y})dr\}$$

where $\tilde{x}(r) = r(x',0)+(1-r)\bar{y}$.

Clearly,

$$F_0(x',0,y) = E_0(x',0,y)$$

and equation (3.1) is also satisfied. Thus

$$F_0(x,y) = E_0(x',0,y)\exp\{2\int_0^1 b(\tilde{x}(r))\cdot(x-\bar{y})dr\}\cdot\exp\{-2\int_0^1 b(x(r))\cdot(x-\bar{y})dr\}.$$

For $k \geq 1$, define F_k by

$$\begin{aligned}\frac{t^k F_k(x(t),y)}{F_0(x(t),y)} &= \frac{-1}{4(k-m)}\int_0^t \frac{r^{k-1}PF_{k-1}(x(r),y)}{F_0(x(r),y)}dr \\ &\quad + \frac{1}{4(k-m)}\int_0^1 \frac{r^{k-1}PF_{k-1}(\tilde{x}(r),y)}{F_0(\tilde{x}(r),y)}dr + \frac{E_k(x',0,y)}{E_0(x',0,y)}.\end{aligned}$$

It is easy to see that $F_k(x',0,y) = E_k(x',0,y)$. To show that equation (3.3) is satisfied, note that since

$$\frac{d}{dt}\left\{\frac{t^k F_k(x(t),y)}{F_0(x(t),y)}\right\} = -\frac{t^{k-1}PF_{k-1}(x(t),y)}{4(k-m)F_0(x(t),y)},$$

$$t^k\frac{d}{dt}\left(\frac{F_k}{F_0}\right) + kt^{k-1}\frac{F_k}{F_0} = -\frac{t^{k-1}PF_{k-1}(x(t),y)}{4(k-m)F_0(x(t),y)}$$

and hence

$$t\frac{dF_k}{dt} - \frac{t}{F_0}\frac{dF_k}{dt}F_k + kF_k = -\frac{PF_{k-1}}{4(k-m)}.$$

The latter together with equation (3.1) leads to (3.3).

Thus for $k \geq 1$,

$$\frac{F_k(x,y)}{F_0(x,y)} = \frac{-1}{4(k-m)} \int_0^1 r^{k-1} P F_{k-1}(x(r),y)dr + \frac{1}{4(k-m)} \int_0^1 r^{k-1} P F_{k-1}(\tilde{x}(r),y)dr + \frac{E_k(x',0,y)}{E_0(x',0,y)}.$$

We will next show that there is $\delta > 0$ such that the series

$$\sum_{k=0}^{\infty} F_k(x,y)\bar{d}(x,y)^{2k} \tag{3.5}$$

converges uniformly on $\{(x,y) : x \in B_\delta(0), |x-\bar{y}| < \delta\}$.

As in [13], this is accomplished by the method of majorants analogous to the Cauchy-Kowalewski technique. For the purpose of the convergence proof, we may assume that $F_0 \equiv$ constant, since the substitution $u_1 = \frac{u}{F_0}$ reduces $Pu = 0$ to a new partial differential equation

$$P_1(u_1) = P(F_0 u_1) = 0 \tag{3.6}$$

of the same type as P for which such an assumption is valid.

With $F_0 \equiv$ const, for $k \geq 1$,

$$F_k(x,y) = \frac{-1}{4(k-m)} \int_0^1 r^{k-1} P F_{k-1}(x(r),y)dr + \frac{1}{4(k-m)} \int_0^1 r^{k-1} P F_{k-1}(\tilde{x}(r),y)dr + E_k(x',0,y). \tag{3.7}$$

Assume $y = 0$ first. Let

$$\frac{K\epsilon}{\epsilon - |x_1| - \ldots - |x_n|} = \sum_{j=0}^{\infty} \frac{K}{\epsilon^j} \left(\sum_{i=1}^{k} |x_j| \right)^j$$

be a majorant for the Taylor expansions in powers of x of all of the coefficients of P. In [13] it is shown that if

$$M\{E_k\} = \frac{M_k}{(1 - \frac{|x_1|+\ldots+|x_n|}{\epsilon})^{2k}}$$

denotes a majorant of E_k, then

$$M\{E_{k+1}\} = \frac{M_{k+1}}{(1 - \frac{|x_1|+\ldots+|x_n|}{\epsilon})^{2k+2}}$$

with

$$M_{k+1} = \frac{k(2k+1)}{2(k+1)(k-m+1)}\left(1 + \frac{n}{\epsilon} + \frac{2n}{\epsilon^2}\right) K M_k$$

(note that [13] has $\frac{n^2}{\epsilon^2}$, but in our particular case, it can be replaced with $\frac{2n}{\epsilon^2}$). Suppose now

$$M\{F_k\} = \frac{\tilde{M}_k}{(1 - \frac{|x_1|+\ldots+|x_n|}{\epsilon})^{2k}}$$

for some constant $\tilde{M}_k$ with $\tilde{M}_k \geq M_k$ is a majorant of F_k.

Then

$$M\{PF_k\} = \frac{2k(2k+1)(1 + \frac{n}{\epsilon} + \frac{2n}{\epsilon^2}) K \tilde{M}_k}{(1 - \frac{\sum_{j=1}^n |x_j|}{\epsilon})^{2k+3}}. \tag{3.8}$$

We will show that for some constants C_k,

$$\tilde{M}_{k+1} = 2C_k \tilde{M}_k + M_{k+1}.$$

Recall that $y = 0$. Write

$$F_{k+1}(x, 0) = I_1 + I_2 + I_3,$$

where

$$I_1 = \frac{-1}{4(k+1-m)} \int_0^1 r^k PF_k(x(r), 0) dr,$$

$$I_2 = \frac{1}{4(k+1-m)} \int_0^1 r^k PF_k(\tilde{x}(r), 0) dr,$$

and

$$I_3 = E_{k+1}(x', 0, 0).$$

Using (3.8), with $A_k = 2k(2k+1)(1 + \frac{n}{\epsilon} + \frac{2n}{\epsilon^2}) K \tilde{M}_k$, we get:

$$
\begin{aligned}
M\{I_1\} &= \frac{A_k}{4(k+1-m)} \int_0^1 [1 + (\frac{|rx_1| + \ldots + |rx_n|}{\epsilon})] \frac{r^k}{(1 - \frac{\sum_i r|x_i|}{\epsilon})^{2k+3}} dr \\
&= \frac{A_k}{4(k+1-m)(k+1)} \int_0^1 \frac{d}{dr} \left[\frac{r^{k+1}}{(1 - \frac{\sum_i r|x_i|}{\epsilon})^{2k+2}} \right] dr \\
&= \frac{A_k}{4(k+1-m)(k+1)} \left(\frac{1}{(1 - \frac{\sum_{i=1}^n |x_i|}{\epsilon})^{2k+2}} \right).
\end{aligned}
$$

Thus

$$
M\{I_1\} = \frac{k(2k+1)(1 + \frac{n}{\epsilon} + \frac{2n}{\epsilon^2})K\tilde{M}_k}{2(k+1)(k-m+1)} \cdot \frac{1}{(1 - \frac{\sum_{i=1}^n |x_i|}{\epsilon})^{2k+2}}.
$$

Likewise,

$$
M\{I_2\} = \frac{k(2k+1)(1 + \frac{n}{\epsilon} + \frac{2n}{\epsilon^2})K\tilde{M}_k}{2(k+1)(k-m+1)} \cdot \frac{1}{(1 - \frac{\sum_{i=1}^n |x_i|}{\epsilon})^{2k+2}}
$$

and from [13], we recall that

$$
M\{I_3\} = \frac{M_{k+1}}{(1 - \frac{\sum_{i=1}^n |x_i|}{\epsilon})^{2k+2}}.
$$

It follows that when $y = 0$,

$$
M\{F_{k+1}\} = \frac{\tilde{M}_{k+1}}{(1 - \frac{\sum_{i=1}^n |x_i|}{\epsilon})^{2k+2}}
$$

with

$$
\tilde{M}_{k+1} = \frac{k(2k+1)(1 + \frac{n}{\epsilon} + \frac{2n}{\epsilon^2})K\tilde{M}_k}{(k+1)(k-m+1)} + M_{k+1}.
$$

We have

$$
\begin{aligned}
\frac{\tilde{M}_{k+1}}{\tilde{M}_k} &= \frac{k(2k+1)(1 + \frac{n}{\epsilon} + \frac{2n}{\epsilon^2})K}{(k+1)(k-m+1)} + \frac{M_{k+1}}{\tilde{M}_k} \\
&\leq \frac{k(2k+1)(1 + \frac{n}{\epsilon} + \frac{2n}{\epsilon^2})K}{(k+1)(k-m+1)} + \frac{M_{k+1}}{M_k}.
\end{aligned}
$$

Since

$$
\lim_{k \to \infty} \frac{M_{k+1}}{M_k} = (1 + \frac{n}{\epsilon} + \frac{2n}{\epsilon^2})K,
$$

given $\delta > 0$, we can get N such that for $k \geq N$,

$$\frac{\tilde{M}_{k+1}}{\tilde{M}_k} < \alpha + \delta,$$

where $\alpha = 3(1 + \frac{n}{\epsilon} + \frac{2n}{\epsilon^2})K$.

Therefore, for some $c > 0$,

$$\tilde{M}_k \leq c(\alpha + \delta)^k \ \ \forall k \geq 1.$$

If $y \neq 0$, we replace x with $x - \bar{y}$ and still arrive at the same estimate for the $\tilde{M}_k$.

It follows that the series

$$\sum_{k=0}^{\infty} F_k(x,y)\bar{d}(x,y)^{2k}$$

converges on the open set

$$\left\{(x,y) : |x - \bar{y}|^2 < \frac{(1 - \sum_{i=1}^{n} \frac{|x_i - \bar{y}_i|^2}{\epsilon})^2}{3(1 + \frac{n}{\epsilon} + \frac{2n}{\epsilon^2})K}\right\}$$

where we used the notation

$$\bar{y}_i = y_i \text{ for } 1 \leq i \leq n-1, \ \bar{y}_n = -y_n.$$

Case 2: Suppose n is even. This time $m = \frac{n-2}{2}$ is an integer and so the approach used under case 1 breaks down because the formula for F_k involves division by $k - m$. We recall from [13] that the fundamental solution S for P has the form

$$S = \sum_{j=0}^{m-1} U_j(x,y)d(x,y)^{2j-n+2} + V(x,y)\log d(x,y)^2 + W(x,y)$$

with the U_j, V, and W real analytic. Motivated by this, we seek a solution $H = H(x,y)$ of the form

$$H(x,y) = \sum_{j=0}^{m-1} A_j(x,y)\bar{d}(x,y)^{2j-n+2} + B(x,y)\log \bar{d}(x,y)^2 + C(x,y)$$

where A_j, B, C are real analytic,

$$P(D_x)H(x,y) = 0 \ \text{ for } x \neq \bar{y}, \tag{3.9}$$

$$A_j(x',0,y) = U_j(x',0,y), \ 0 \leq j \leq m-1, \tag{3.10}$$

$$B(x',0,y) = V(x',0,y), \tag{3.11}$$

and

$$C(x',0,y)=W(x',0,y). \tag{3.12}$$

At a point $(x(t),y)$, $x(t)=tx+(1-t)\bar{y}$, we have

$$\begin{aligned}
&P(A_j\bar{d}^{2j-n+2})\\
&=P(A_j)\bar{d}^{2j-n+2}+2(2j-n+2)t\frac{dA_j}{dt}\bar{d}^{2j-n}\\
&+A_j(j-m)4j\bar{d}^{2j-n}+A_j(j-m)\left(\sum_{j=1}^{n}b_j\frac{\partial\bar{d}^2}{\partial x_j}\right)\bar{d}^{2j-n}\\
&=P(A_j)\bar{d}^{2j-n+2}+2(2j-n+2)[t\frac{d}{dt}A_j+(\theta(x(t),y)+j-m-1)A_j]\bar{d}^{2j-n}
\end{aligned}$$

where

$$\theta(x,y)=\frac{1}{4}\left(2n+2\sum_{i=1}^{n}b_i(x)(x_i-\bar{y}_i)\right),$$

$\bar{y}_i=y_i$ for $1\leq i\leq n-1$, $\bar{y}_n=-y_n$.

$$\begin{aligned}
P(B\log\bar{d}^2)&=(\Delta B)\log\bar{d}^2+\frac{2(n-2)B}{\bar{d}^2}+\frac{4}{\bar{d}^2}\sum_{i=1}^{n}(x_i-\bar{y}_i)\frac{\partial B}{\partial x_i}\\
&+(\sum_{j=1}^{n}b_j\frac{\partial B}{\partial x_j})\log\bar{d}^2+\frac{2B}{\bar{d}^2}\sum_{j=1}^{n}b_j(x_j-\bar{y}_j)+cB\log\bar{d}^2\\
&=(PB)\log\bar{d}^2+\frac{2(n-2)}{\bar{d}^2}B+\frac{2B}{\bar{d}^2}\sum_{j=1}^{n}b_j(x_j-\bar{y}_j)+\frac{4}{\bar{d}^2}\sum_{j=1}^{n}(x_j-\bar{y}_j)\frac{\partial B}{\partial x_j}\\
&=(PB)\log\bar{d}^2+\frac{4}{\bar{d}^2}\left\{(\theta-1)B+\sum_{i=1}^{n}(x_i-\bar{y}_i)\frac{\partial B}{\partial x_i}\right\}.
\end{aligned}$$

We choose the A_j like the F_j, that is,

$$A_0(x,y)=U_0(x',0,y)\exp\{2\int_0^1 b(\tilde{x}(r)\cdot(x-\bar{y})dr\}.\exp\{-2\int_0^1 b(x(r)\cdot(x-\bar{y})dr\}$$

and for $1\leq j\leq m-1$,

$$\frac{A_j(x,y)}{A_0(x,y)} = \frac{-1}{4(j-m)} \int_0^1 r^{j-1} P A_{j-1}(x(r),y)dr$$
$$+ \frac{1}{4(j-m)} \int_0^1 r^{j-1} P A_{j-1}(\tilde{x}(r),y)dr + \frac{U_j(x',0,y)}{U_0(x',0,y)}.$$

Thus

$$PH = 0 \text{ for } x \neq \bar{y}$$

if

$$\frac{1}{\bar{d}^2}\{P(A_{m-1}) + 4[t\frac{dB}{dt} + (\theta-1)B]\} + P(B)\log \bar{d}^2 + P(W) = 0.$$

The logarithmic term has to cancel out and so we will also need B to satisfy

$$PB = 0 \tag{3.13}$$

Moreover, we need B to satisfy the equation

$$t\frac{dB}{dt} + (\theta-1)B = \frac{-P(A_{m-1})}{4} \tag{3.14}$$

on the set where $\bar{d}(x,y) = 0$.

We seek B of the form

$$B(x,y) = \sum_{k=0}^{\infty} B_k(x,y)\bar{d}(x,y)^{2k}$$

that satisfy (3.11), (3.13) and (3.14). This will be achieved if B_0 solves

$$\begin{cases} t\frac{dB_0}{dt} + (\theta-1)B_0 = \frac{-PA_{m-1}}{4} \\ B_0(x',0,y) = V_0(x',0,y) \end{cases} \tag{3.15}$$

and for $k \geq 1$

$$\begin{cases} t\frac{dB_k}{dt} + (\theta+k-1)B_k = \frac{-1}{4k}PB_{k-1} \\ B_k(x',0,y) = V_k(x',0,y). \end{cases} \tag{3.16}$$

Define B_0 by

$$t^m \frac{B_0(x(t),y)}{A_0(x(t),y)} = -\frac{1}{4}\int_0^t \frac{PA_{m-1}(x(r),y)}{A_0(x(r),y)} r^{m-1} dr + \frac{1}{4}\int_0^1 \frac{PA_{m-1}(\tilde{x}(r),y)}{A_0(\tilde{x}(r),y)} r^{m-1} dr + \frac{V_0(x',0,y)}{A_0(x',0,y)} \tag{3.17}$$

and for $k \geq 1$, B_k is defined by

$$t^{k+m} \frac{B_k(x(t),y)}{A_0(x(t),y)} = -\frac{1}{4k}\int_0^t \frac{PB_{k-1}(x(r),y)}{A_0(x(r),y)} r^{k+m-1} dr + \frac{1}{4k}\int_0^t \frac{PB_{k-1}(\tilde{x}(r),y)}{A_0(\tilde{x}(r),y)} dr + \frac{V_k(x',0,y)}{A_0(x',0,y)} \tag{3.18}$$

To see that B_0 defined by (3.17) solves (3.15) observe that since

$$\frac{d}{dt}\left(\frac{t^m B_0}{A_0}\right) = -\frac{1}{4}\frac{PA_{m-1}}{A_0} t^{m-1},$$
$$t\frac{d}{dt}\left(\frac{B_0}{A_0}\right) + \frac{mB_0}{A_0} = \frac{-PA_{m-1}}{4A_0}$$

which implies that

$$t\,\frac{dB_0}{dt} - \frac{t\,\frac{dA_0}{dt}B_0}{A_0} + mB_0 = \frac{-PA_{m-1}}{A_0}.$$

Using

$$t\,\frac{dA_0}{dt} + (\theta - m - 1)A_0 = 0$$

in the latter equation we conclude that

$$t\frac{dB_0}{dt} + (\theta - 1)B_0 = \frac{-PA_{m-1}}{4}.$$

From the formula for $B_0(x,y) = B_0(x(1),y)$, we see that

$$B_0(x',0,y) = V_0(x',0,y).$$

When $k \geq 1$, the equation

$$\frac{d}{dt}\left(\frac{t^{k+m}B_k}{A_0}\right) = \frac{-1}{4k}\frac{PB_{k-1}}{A_0} t^{k+m-1}$$

leads to

$$t\,\frac{dB_k}{dt} - \frac{t\,\frac{dA_0}{dt}B_k}{A_0} + (k+m)B_k = \frac{-PB_{k-1}}{4k}$$

which together with

$$t\,\frac{dA_0}{dt} + (\theta - m - 1)A_0 = 0$$

imply that

$$t\,\frac{dB_k}{dt} + (\theta + k - 1)B_k = \frac{-PB_{k-1}}{4k}.$$

The formula for $B_k(x,y)$ shows that

$$B_k(x',0,y) = V_k(x',0,y).$$

The convergence of the series

$$\sum_{k=0}^{\infty} B_k(x,y)\bar{d}(x,y)^{2k}$$

is proved using the majorant method as in case 1.

The functions $A_j(x,y)$ $(0 \le j \le m-1)$ and $B(x,y)$ were constructed so that

$$Q(x,y) = P(D_x)\left(\sum_{j=0}^{m-1} A_j\bar{d}^{2j-n+2} + B(x,y)\log\bar{d}^2\right)$$

is real analytic. We choose $C(x,y)$ that satisfies

$$\begin{cases} P(D_x)C(x,y) = -Q(x,y) \\ C(x',0,y) = W(x',0,y). \end{cases}$$

It follows that $H = H(x,y)$ satisfies all the requirements.

By taking $E(x,y) - F(x,y)$ when n is odd and $S(x,y) - H(x,y)$ when n is even, we have found a function $G(x,y)$ defined for $(x,y) \in \overline{B_r^+} \times \overline{B_r^+}$ (after decreasing r), $x \neq y$, such that

$$\begin{cases} P(D_x)G(x,y) = \delta(x-y) \text{ on } B_r^+ \times B_r^+ \\ G(x',0,y) = 0 \text{ when } y_n > 0. \end{cases} \tag{3.19}$$

The transpose of P denoted by ${}^tP$ has the same form as P and so we also have G^t that solves

$$\begin{cases} {}^tP(x,D_x)G^t(x,y)=\delta(x-y) \text{ on } B_r^+\times B_r^+ \\ G^t(x',0,y)=0 \text{ when } y_n>0. \end{cases} \tag{3.20}$$

We will next show that the function $G^t(x,y)-G(y,x)$ extends as a real analytic function in a neighborhood of $(0,0)$ in $\mathbb{R}^n\times\mathbb{R}^n$.

Given a C^1 domain $D\subseteq B_r^+$, for $w_1,w_2\in C^2(\overline{D})$, by Green's identity,

$$\int_D \Big(w_1(x)Pw_2(x)-w_2(x)\,{}^tPw_1(x)\Big)dx=\int_{\partial D}\Big(w_1(x)\nabla w_2(x)-w_2(x)\nabla w_1(x)\Big)\cdot n(x)d\sigma + \int_{\partial D} w_1(x)w_2(x)b(x)\cdot n(x)d\sigma \tag{3.21}$$

where $b(x)=(b_1(x),\dots,b_n(x))$ and $n(x)$ is unit outer normal vector.

Fix $p,q\in B_r^+, p\neq q$. Let $w_1(x)=G^t(x,q)$ and $w_2(x)=G(x,p)$. For $\epsilon>0$ small, let $D_\epsilon=B_r^+\setminus(B_\epsilon(p)\cup B_\epsilon(q))$.

Since ${}^tPw_1=0$ and $Pw_2=0$ in D_ϵ, (3.21) becomes

$$\int_{\partial D_\epsilon}\Big(w_1(x)\nabla w_2(x)-w_2(x)\nabla w_1(x)\Big)\cdot n(x)d\sigma=-\int_{\partial D_\epsilon} w_1(x)w_2(x)b(x)\cdot n(x)d\sigma \tag{3.22}$$

Write $\partial B_r^+=\Sigma\cup\Sigma'$ where $\Sigma=\{x\in\partial B_r^+ : x_n=0\}$.

Since $w_1(x',0)=0=w_2(x',0)$,

$$\begin{aligned}\int_{\partial D_\epsilon}&\Big(w_1(x)\nabla w_2(x)-w_2(x)\nabla w_1(x)\Big)\cdot n(x)d\sigma\\ &=\int_{\Sigma'}\Big(w_1(x)\nabla w_2(x)-w_2(x)\nabla w_1(x)\Big)\cdot n(x)d\sigma\\ &-\int_{\partial B_\epsilon(p)}\Big(w_1(x)\nabla w_2(x)-w_2(x)\nabla w_1(x)\Big)\cdot n(x)d\sigma\\ &-\int_{\partial B_\epsilon(q)}\Big(w_1(x)\nabla w_2(x)-w_2(x)\nabla w_1(x)\Big)\cdot n(x)d\sigma\end{aligned} \tag{3.23}$$

Consider the integral

$$\int_{\partial B_\epsilon(p)} w_2(x)\nabla w_1(x)\cdot n(x)d\sigma.$$

The function $\nabla w_1(x) \cdot n(x)$ is continuous on $B_\epsilon(p)$. When n is odd, $w_2(x) = \frac{e(x)}{|x-p|^{n-2}}$ with $e(x)$ continuous on $B_\epsilon(p)$. When n is even, the principal singularity is of the same form except when $n = 2$ in which case it is a logarithmic term. In both cases,

$$\lim_{\epsilon \to 0^+} \int\limits_{\partial B_\epsilon(p)} w_2(x) \nabla w_1(x) \cdot n(x) d\sigma = 0.$$

Likewise,

$$\lim_{\epsilon \to 0^+} \int\limits_{\partial B_\epsilon(q)} w_1(x) \nabla w_2(x) \cdot n(x) d\sigma = 0.$$

Therefore,

$$\begin{aligned}
&\lim_{\epsilon \to 0^+} \int\limits_{\partial B_\epsilon(p)} \Big(w_1(x) \nabla w_2(x) - w_2(x) \nabla w_1(x) \Big) \cdot n(x) d\sigma \\
&= \lim_{\epsilon \to 0^+} \int\limits_{\partial B_\epsilon(p)} w_1(x) \nabla w_2(x) \cdot n(x) d\sigma \\
&= w_1(p) = G^t(p, q) \qquad (3.24)
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{\epsilon \to 0^+} \int\limits_{\partial B_\epsilon(q)} \Big(w_1(x) \nabla w_2(x) - w_2(x) \nabla w_1(x) \Big) \cdot n(x) d\sigma \\
&= - \lim_{\epsilon \to 0^+} \int\limits_{\partial B_\epsilon(q)} w_2(x) \nabla w_1(x) \cdot n(x) d\sigma \\
&= -w_2(q) = -G(q, p) \qquad (3.25)
\end{aligned}$$

$$\int\limits_{\partial D_\epsilon} w_1(x) w_2(x) b(x) \cdot n(x) d\sigma = \int\limits_{\Sigma'} w_1(x) w_2(x) b(x) \cdot n(x) d\sigma - \int\limits_{\partial B_\epsilon(p) \cup \partial B_\epsilon(q)} w_1(x) w_2(x) b(x) \cdot n(x) d\sigma$$

and

$$\lim_{\epsilon \to 0^+} \int\limits_{\partial B_\epsilon(p) \cup \partial B_\epsilon(q)} w_1(x) w_2(x) b(x) \cdot n(x) d\sigma = 0. \qquad (3.26)$$

From (3.23)-(3.26) we conclude that

$$\begin{aligned} G^t(p,q) - G(q,p) &= -\int_{\Sigma'} \Big(G^t(x,q)\nabla G(x,p) - G(x,p)\nabla G^t(x,q) \Big) \cdot n(x) d\sigma \\ &\quad - \int_{\Sigma'} G^t(x,q) G(x,p) b(x) \cdot n(x) d\sigma \\ &\doteq \varphi(q,p) \end{aligned} \tag{3.27}$$

which is real analytic on a neighborhood of $(0,0)$ in $\mathbb{R}^n \times \mathbb{R}^n$, say $B_r(0) \times B_r(0)$ (after decreasing r).

Let $g(x,y)$ be a real analytic function on $B_r(0) \times B_r(0)$ (r may have to be decreased) that is a solution of

$$\begin{cases} P(D_x)g(x,y) = P(D_x)(\varphi_{y_n}(x,y)) \\ g(x',0,y) = 0 \end{cases}$$

Define

$$K(x,y') = \frac{\partial G^t}{\partial y_n}(y',0,x) - g(x,y',0).$$

We will show that $K(x,y')$ behaves like a local Poisson Kernel for P for the upper half space.

Indeed, first note that for $x, y \in B_r^+,\ x \neq y$,

$$\begin{aligned} P(D_x)\Big(\frac{\partial}{\partial y_n} G^t(y,x) - g(x,y)\Big) &= P(D_x)\Big(\frac{\partial}{\partial y_n} G^t(y,x) - \varphi_{y_n}(x,y)\Big) \\ &= \frac{\partial}{\partial y_n} P(D_x) G(x,y) \equiv 0. \end{aligned} \tag{3.28}$$

Second, in Section 4 we will show that if we define the function v on B_r^+ by

$$v(x) = -\int_{\mathbb{R}^{n-1}} K(x,y')\psi(y')u(y',0)\, dy',$$

where ψ is a smooth cutoff function identically equal to 1 near the origin, and supported in $\{x' : |x'| < r\}$, then v differs from u by a function that actually is real analytic in a neighborhood of the origin in $\mathbb{R}^n$. In particular, the trace $u(x',0) - v(x',0)$ is real analytic in a neighborhood of the origin in $\mathbb{R}^{n-1}$.

4. Proofs of Theorem 2.3 and its corollaries

We begin by estimating the derivatives

$$\partial_{x_n}^k \left[\frac{\partial G^t}{\partial y_n}(y',0,x) \right] \text{ at } x = 0,\ y' \neq 0.$$

Assume first that n is odd. Recall that in that case

$$G^t(x,y) = E(x,y) - F(x,y)$$

with

$$E(x,y) = \sum_{k=0}^{\infty} E_k(x,y) d(x,y)^{2k-2m}$$

and

$$F(x,y) = \sum_{k=0}^{\infty} F_k(x,y) \bar{d}(x,y)^{2k-2m}.$$

Using the identity

$$E_k(y',0,x) = F_k(y',0,x) \ \ \forall k,$$

we get

$$\begin{aligned}
\frac{\partial G^t}{\partial y_n}(y',0,x) &= \sum_{k=0}^{\infty} \left(\frac{\partial E_k}{\partial y_n}(y',0,x) - \frac{\partial F_k}{\partial y_n}(y',0,x) \right) d(y',0,x)^{2k-2m} \\
&\quad - 4 \left[\sum_{k=0}^{\infty} (k-m) E_k(y',0,x) d(y',0,x)^{2k-2m-2} \right] x_n \\
&= \frac{p(x,y')}{d(y',0,x)^{2m}} - \frac{q(x,y') x_n}{d(y',0,x)^{2m+2}}
\end{aligned}$$

with $p(x,y')$ and $q(x,y')$ real analytic.

For an odd integer M, we will estimate first

$$\partial_{x_n}^M \left(\frac{p}{d^{2m}} \right) = \sum_{N=0}^{M} \binom{M}{N} \partial_{x_n}^N d^{-2m} \partial_{x_n}^{M-N} p$$

at $x = 0, y' \neq 0$. To compute higher order derivatives, we use the formula of Faá di Bruno according to which

$$\frac{d^N}{dt^N} Q(f(t)) = \sum \frac{N!}{N_1! \dots N_N!} Q^{(N_1 + \dots + N_N)}(f(t)) \prod_{j=1}^{N} \left(\frac{f^{(j)}(t)}{j!} \right)^{N_j}$$

where the sum is over all N-tuples of nonnegative integers $(N_1, \dots, N_N)$ that satisfy the constraint

$$N_1 + 2N_2 + \ldots + NN_N = N.$$

Let

$$f(t) = |y'|^2 + t^2 \text{ and } Q(s) = s^{-m}.$$

At $t = 0$ all the terms in the sum above equal zero except when $2N_2 = N$ in which case we get

$$\frac{N!}{N_2!} Q^{(N_2)}(f(0)).$$

Hence, at $x = 0, y' \neq 0$, if $N = 2N_2$,

$$\partial_{x_n}^N d^{-2m} = (-1)^{N_2} \frac{N!}{N_2!} \frac{m(m+1)\ldots(m+N_2-1)}{|y'|^{N+2m}}.$$

Since p and q are real analytic, for some $C > 0$, and for all $k \geq 1$,

$$|\partial_{x_n}^k p| + |\partial_{x_n}^k q| \leq C^{k+1} k!.$$

It follows that at $x = 0, y' \neq 0$,

$$\begin{aligned} \left| \partial_{x_n}^{2k+1} \left(\frac{p}{d^{2m}} \right) \right| &\leq \sum_{N=0,\ N \text{ even}}^{2k} \frac{(2k+1)!}{N_2!} C^{2k-N+2} \frac{m(m+1)\ldots(m+N_2-1)}{|y'|^{N+2m}} \\ &= \frac{(2k+1)! C^{2k+2}}{|y'|^{2m}} \sum_{N=0, N \text{ even}}^{2k} \frac{m(m+1)\ldots(m+N_2-1)}{N_2!(C|y'|)^N}. \end{aligned}$$

Using

$$\frac{m(m+1)\cdots(m+N_2-1)}{N_2!} \leq \frac{(m+1)\cdots(m+k)}{k!},$$

the sum above

$$\leq \frac{C^2(2k+1)!(m+1)\cdots(m+k)}{k!|y'|^{2m+2k}} \sum_{N=0}^{2k} (C|y'|)^{2k-N}$$

and hence choosing $r < \frac{1}{C}$ so that $|y'| < \frac{1}{C}$, for some $C_1 > 0$,

$$\left| \partial_{x_n}^{2k+1} \left(\frac{p}{d^{2m}} \right) \right| \leq C_1 \frac{(2k+1)!(m+1)\cdots(m+k)}{k!|y'|^{2m+2k}}. \tag{4.1}$$

For $x = 0, y' \neq 0$,

$$
\begin{aligned}
&\partial_{x_n}^{2k+1}\left(\frac{qx_n}{d^{2m+2}}\right)\\
&\quad= (2k+1)\partial_{x_n}^{2k}\left(\frac{q}{d^{2m+2}}\right)\\
&\quad= (2k+1)\sum_{N=0}^{2k-1}\partial_{x_n}^{N}(d^{-2m-2})\partial_{x_n}^{2k-N}q + (2k+1)\partial_{x_n}^{2k}(d^{-2m-2})q\\
&\quad\geq -\frac{(2k+1)!C^{2k+1}}{|y'|^{2m+2}}\sum_{N=0,\ N\ \text{even}}^{2k-1}\frac{(m+1)\cdots(m+N_2)}{N_2!}\frac{1}{(C|y'|)^{N}}\\
&\quad+ (2k+1)\partial_{x_n}^{2k}(d^{-2m-2})q\\
&\quad\geq -\frac{(2k+1)!(m+1)\cdots(m+k)}{k!|y'|^{2m+2}}\sum_{N=0}^{2k-1}\frac{C^{2k+1}}{(C|y'|)^{N}} + (2k+1)\partial_{x_n}^{2k}(d^{-2m-2})q\\
&\quad= -\frac{C(2k+1)!(m+1)\cdots(m+k)}{k!|y'|^{2k+2m+2}}\sum_{N=0}^{2k-1}(C|y'|)^{2k+2-N}\\
&\quad+\frac{(-1)^k(2k+1)!(m+1)\cdots(m+k)q}{k!|y'|^{2k+2m+2}}\\
&\quad\geq -\frac{A(2k+1)!(m+1)\cdots(m+k)}{k!|y'|^{2m+2k}}+\frac{(-1)^k(2k+1)!(m+1)\cdots(m+k)q}{k!|y'|^{2k+2m+2}} \qquad (4.2)
\end{aligned}
$$

for some $A > 0$. In the second inequality above, we have used the fact that for any $N_2 \leq k$,

$$\frac{(m+1)\cdots(m+N_2)}{N_2!} \leq \frac{(m+1)\cdots(m+k)}{k!}.$$

From (4.1) and (4.2), for k even, since $q(0,0') = E_0(0,0) \neq 0$, we get at $x = 0, y' \neq 0$

$$\left|\partial_{x_n}^{2k+1}\frac{\partial G^t}{\partial y_n}(y',0,x)\right| \geq \frac{C(2k+1)!}{|y'|^{2k+2m+2}}. \qquad (4.3)$$

Suppose now n is even. In that case recall that $G^t = S - H$ Where

$$S(x,y) = \sum_{j=0}^{m-1} U_j(x,y)d(x,y)^{2j-n+2} + V(x,y)\log d(x,y)^2 + W(x,y)$$

and

$$H(x,y) = \sum_{j=0}^{m-1} A_j(x,y)\bar{d}(x,y)^{2j-n+2} + B(x,y)\log \bar{d}(x,y)^2 + C(x,y),$$

$$A_j(x',0,y) = U_j(x',0,y), B(x',0,y) = V(x',0,y) \text{ and } C(x',0,y) = W(x',0,y).$$

Since

$$U_j(x',0,y) = A_j(x',0,y) \ (0 \le j \le m-1),$$

the arguments for the odd case show that for some $C_3 > 0$, at $x = 0, y' \neq 0$,

$$\partial_{x_n}^{2k+1}\partial_{y_n}\left(\sum_{j=0}^{m-1} U_j d^{2j-n+2} - \sum_{j=0}^{m-1} A_j \overline{d}^{2j-n+2}\right) \ge C_3 \frac{(2k+1)!}{|y'|^{2k+2m+2}}. \tag{4.4}$$

We have:

$$\partial_{x_n}^{2k+1}\partial_{y_n}(V \ \log d^2) = \partial_{x_n}^{2k+1}(V_{y_n}\log d^2) + 2\ \partial_{x_n}^{2k+1}\left(\frac{Vx_n}{d^2}\right).$$

The term

$$\partial_{x_n}^{2k+1}(V_{y_n}\log d^2) = \sum_{N=0}^{2k+1}\binom{2k+1}{N}\partial_{x_n}^{N}\log d^2\ \partial_{x_n}^{2k+1-N}V_{y_n}.$$

Faá di Bruno's formula shows that

$$\partial_{x_n}^{N}\log d^2 = 0 \ \text{ if } N \text{ is odd}$$

and when $N = 2N_2$,

$$\partial_{x_n}^{N}\log d^2 = \frac{(-1)^{N_2+1}N!}{N_2|y'|^N}.$$

Choose $C > 0$ such that

$$\left|\partial_{x_n}^{2k+1-N}V_{y_n}\right| \le C^{2k+2-N}(2k+1-N)!.$$

It then follows that

$$\begin{aligned}
|\partial_{x_n}^{2k+1}(V_{y_n}\log d^2)| &\le \sum_{N=0}^{2k}\frac{(2k+1)!C^{2k+2-N}}{N_2|y'|^N} \\
&\le \frac{(2k+1)!}{|y'|^{2k}}C^2\sum_{N=0}^{2k}(C|y'|)^{2k-N} \\
&\le \frac{C_4(2k+1)!}{|y'|^{2k}}
\end{aligned} \tag{4.5}$$

where we have assumed that $C|y'| < 1$.

We also have

$$\left|\partial_{x_n}^{2k+1}\left(\frac{V_{x_n}}{d^2}\right)\right| = (2k+1)\left|\partial_{x_n}^{2k}\left(\frac{V}{d^2}\right)\right|$$
$$\leq C\frac{(2k+1)!}{|y'|^{2k+2}} \tag{4.6}$$

for some $C > 0$. Thus for some $C > 0$,

$$\left|\partial_{x_n}^{2k+1}\partial_{y_n}(V\log d^2)\right| \leq C\frac{(2k+1)!}{|y'|^{2k+2}}. \tag{4.7}$$

Likewise,

$$\left|\partial_{x_n}^{2k+1}\partial_{y_n}(V\log \bar{d}^2)\right| \leq C\frac{(2k+1)!}{|y'|^{2k+2}}. \tag{4.8}$$

Since $W(x,y)$ and $C(x,y)$ are real analytic, from (4.4), (4.7) and (4.8), we conclude that for some $C > 0$, at $x = 0, y' \neq 0$,

$$\partial_{x_n}^{2k+1}\partial_{y_n}G^t(y',0,x) \geq C\frac{(2k+1)!}{|y'|^{2k+2m+2}}. \tag{4.9}$$

It can easily be checked that (4.9) also holds when $n = 2$. Thus from (4.3) and (4.9), we see that for any n, at $x = 0, y' \neq 0$, when k is even,

$$\partial_{x_n}^{2k+1}\partial_{y_n}G^t(y',0,x) \geq C\frac{(2k+1)!}{|y'|^{2k+2m+2}}. \tag{4.10}$$

Recall that

$$K(x,y') = \frac{\partial G^t}{\partial y_n}(y',0,x) - g(x,y',0).$$

Let $\psi = \psi(x') \in C_0^\infty(\mathbb{R}^{n-1})$ supported in $|x'| < r$, $\psi(x') \equiv 1$ on $|x'| \leq \frac{r}{2}$ and $0 \leq \psi \leq 1$.

Define

$$v(x) = -\int\limits_{\mathbb{R}^{n-1}} K(x,y')\psi(y')u(y',0)dy', \quad x \in B_r^+.$$

Let $w(x) = u(x) - v(x)$ for $x \in B_r^+$. The function w extends as a real analytic function to a neighborhood of the origin. To see this, recall from (3.21) that since u is a solution on B_r^+ and $G^t(y',0,x) \equiv 0$, for $x \in B_r^+$, $\Sigma = \{(x',0) \in \partial B_r^+\}$, we have the following representation formula:

$$
\begin{aligned}
u(x) &= \int\limits_{\partial B_r^+} u(y)\frac{\partial G^t}{\partial \eta}(y,x)d\sigma - \int\limits_{\partial B_r^+} \frac{\partial u}{\partial \eta}(y)G^t(y,x)d\sigma \\
&- \int\limits_{\partial B_r^+} u(y)G^t(y,x)b(y)\cdot n(y)d\sigma \\
&= -\int\limits_{\Sigma} u(y',0)\frac{\partial G^t}{\partial y_n}(y',0,x)dy' - \int\limits_{\partial B_r^+\setminus\Sigma} u(y)\frac{\partial G^t}{\partial \eta}(y,x)d\sigma \\
&- \int\limits_{\partial B_r^+\setminus\Sigma} \frac{\partial u}{\partial \eta}(y)G^t(y,x)d\sigma - \int\limits_{\partial B_r^+\setminus\Sigma} u(y)G^t(y,x)b(y)\cdot n(y)d\sigma \\
&= -\int\limits_{\Sigma} u(y',0)K(x,y')dy' + f(x) \\
&= v(x) + w(x)
\end{aligned}
$$

where clearly $w(x)$ is real analytic on some ball $B_\delta(0)$. Note also that by (3.28) $Pw = 0$ in B_r^+ and so because of analyticity, $Pw = 0$ in $B_\delta(0)$.

The integrability of $|x'|^{-N}u(x',0)$ for all N and the nature of the singularity of $K(x,y')$ imply that the function

$$
v(0',x_n) = -\int K(0',x_n,y')\psi(y')u(y',0)dy
$$

and hence

$$
u(0',x_n) = v(0',x_n) + w(0',x_n)
$$

are C^∞ up to $x_n = 0$. Since $u(0,x_n)$ is flat at $x_n = 0$, and $u - v = w$ is real analytic on $B_\delta(0)$, we can find a constant $D > 0$ such that $\forall k$

$$
|\partial_{x_n}^{2k+1}v(0)| = |\partial_{x_n}^{2k+1}(u-v)(0)| \le D^{2k+2}(2k+1)! \tag{4.11}
$$

On the other hand, since $u(y',0) \ge 0$, for k even, by (4.10), we have, for ϵ small,

$$
\begin{aligned}
|\partial_{x_n}^{2k+1}v(0)| &= \int\limits_{\Sigma} \partial_{x_n}^{2k+1}K(0',x_n,y')|_{x_n=0}\psi(y')u(y',0)dy' \\
&\ge C(2k+1)!\int\limits_{\Sigma} \frac{\psi(y')u(y',0)}{|y'|^{2k+2m+2}}dy' \\
&\ge C(2k+1)!\int\limits_{|y'|<\epsilon} \frac{u(y',0)}{|y'|^{2k+2m+2}}dy'
\end{aligned}
$$

$$\geq \frac{C(2k+1)!}{\epsilon^{2k+2m+2}} \int\limits_{|y'|<\epsilon} u(y',0)dy' \tag{4.12}$$

Since (4.11) and (4.12) hold for any even integer k, by choosing ϵ small enough (depending only on D), taking the $(2k+2m+2)$th root and letting $k \to \infty$, we conclude that $u(x',0) \equiv 0$ for x' in a neighborhood of the origin in $\mathbb{R}^{n-1}$.

Proof of Corollary 2.4. The hypotheses of Theorem 2.3 are satisfied and so $u(x',0) \equiv 0$ for x' small. By the boundary analyticity result of [20], u extends as a real analytic function to some neighborhood of the origin. Since it is flat at an interior point, $u \equiv 0$ in that neighborhood. But since u is real analytic on B_r^+, u vanishes everywhere.

Proofs of Corollaries 2.5 and 2.6. Observe that a conformal map preserves the form of the operator P. Therefore, the proofs of both corollaries follow from Theorem 2.3 and the following lemma known in connection with Poincare's local problem of conformal geometry (see [22] for the proof).

Lemma 4.1. *Given an arc V (or S), we can find a conformal map sending V into the x-axis (S into the y-axis), $z=0$ to $z=0$ and the y-axis (x-axis) into itself if and only if V is locally symmetric about the $y-$ axis (S is locally symmetric about the x-axis).*

Proof of Corollary 2.7. The proof follows from an application of Corollary 2.2 and the boundary regularity of the Riemann mapping.

References

[1] S. Agmon, A. Douglis, L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I, Commun. Pure Appl. Math. 12 (1959) 623–727.

[2] H. Alexander, Boundary behavior of certain holomorphic maps, Mich. Math. J. 38 (1991) 1117–1128.

[3] H. Alexander, A weak Hopf lemma for holomorphic mappings, Indag. Math. 6 (1995) 1–5.

[4] S. Alinhac, M.S. Baouendi, L.P. Rothschild, Unique continuation and regularity at the boundary for holomorphic functions, Duke Math. J. 61 (1990) 635–654.

[5] M.S. Baouendi, L.P. Rothschild, A local Hopf lemma and unique continuation for harmonic functions, Int. Math. Res. Not. 8 (1993) 245–251.

[6] M.S. Baouendi, L.P. Rothschild, Harmonic functions satisfying weighted signed conditions on the boundary, Ann. Inst. Fourier 43 (1993) 1311–1318.

[7] M.S. Baouendi, L.P. Rothschild, Unique continuation and Schwarz reflection principle for analytic sets, Commun. Partial Differ. Equ. 18 (1993) 1961–1970.

[8] S. Bell, L. Lempert, A C^∞ Schwarz reflection principle in one and several complex variables, J. Differ. Geom. 32 (1990) 889–915.

[9] S. Berhanu, Boundary unique continuation for a class of elliptic equations, Am. J. Math. 143 (2021) 783–810.

[10] S. Berhanu, Boundary unique continuation for the Laplace equation and the biharmonic operator, Commun. Anal. Geom. (2021), to appear.

[11] S. Berhanu, J. Hounie, A Hopf lemma for holomorphic functions in Hardy spaces and applications to CR mappings, J. Anal. Math. 138 (2019) 835–855.

[12] S. Berhanu, J. Hounie, A local Hopf lemma and unique continuation for the Helmholtz equation, Commun. Partial Differ. Equ. 43 (2018) 448–466.

[13] J. Hadamard, Lectures on Cauchy's Problem in Linear Differential Equations, Yale University Press, 1923.

[14] I.I. Hirschmann, D.V. Widder, The Convolution Transform, Princeton Univ. Press, 1955.

[15] X. Huang, S. Krantz, A unique continuation problem for holomorphic mappings, Commun. Partial Differ. Equ. 18 (1993) 241–263.

[16] X.J. Huang, S.G. Krantz, D. Ma, Y. Pan, A Hopf lemma for holomorphic functions and applications, Complex Var. Theory Appl. 26 (1995) 273–276.

[17] Y. Kato, The Poisson kernels and the Cauchy problem for elliptic equations with analytic coefficients, Trans. Am. Math. Soc. 144 (1969) 321–331.

[18] M. Lakner, Finite order vanishing of boundary values of holomorphic mappings, Proc. Am. Math. Soc. 112 (1991) 521–527.

[19] B.F. Logan Jr., Properties of high pass signals, Dissertation submitted to the Electrical Engineering Faculty of Columbia University, New York, 1965.

[20] C.B. Morrey, L. Nirenberg, On the analyticity of the solutions of linear elliptic systems of partial differential equations, Commun. Pure Appl. Math. 10 (1957) 271–290.

[21] H.S. Shapiro, Notes on a theorem of Baouendi and Rothschild, Expo. Math. 13 (2–3) (1995) 247–275.

[22] V. Shklover, Remarks on the local Hopf's lemma, Proc. Am. Math. Soc. 124 (9) (1996) 2711–2716.

[23] N. Suzuki, A local Hopf lemma for solutions of the one dimensional heat equation, Nagoya Math. J. 146 (1997) 1–12.