# ON HOLOMORPHIC EXTENDABILITY AND THE STRONG MAXIMUM PRINCIPLE FOR CR FUNCTIONS

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Abstract. We explore some links between the holomorphic extendability of CR functions on a hypersurface and the validity of the strong maximum principle for continuous CR functions.

## 1. Introduction

Let $\mathcal{M}$ be a $C^\infty$ hypersurface in $\mathbb{C}^n$. Consider the following two types of maximum principles for locally defined continuous CR functions on $\mathcal{M}$.

**Definition 1.1.** We say $\mathcal{M}$ satisfies the strong maximum principle if given any connected open set $U$ in $\mathcal{M}$ and any continuous CR function $h$ on $U$, $|h|$ can not have a weak local maximum at any point of $U$ unless $h$ is constant on $U$. That is, $p \in U$ and $|h(z)| \leq |h(p)|$ for all $z \in U$ implies that $h$ is constant on $U$.

**Definition 1.2.** We say $\mathcal{M}$ satisfies the weak maximum principle if given any connected open set $U$ in $\mathcal{M}$ and any continuous CR function $h$ on $U$, $|h|$ can not peak at any point of $U$. That is, there is no $p \in U$ satisfying $|h(z)| < |h(p)|$ for all $z \in U,\ z \neq p$.

The strong maximum principle clearly implies the weak maximum principle. The weak maximum principle is well understood even for CR submanifolds of arbitrary codimension. From the works [10] and [11], the weak maximum principle is valid on an embedded CR manifold if and only if there is no direction in which the Levi form is strictly positive definite. A Levi flat hypersurface satisfies the weak maximum principle but not the strong maximum principle. The non Levi flat hypersurfaces given by

$$\mathcal{M} = \{(z_1, \ldots, z_{n-1}, x_n + \sqrt{-1}(|z_1|^2 + \ldots + |z_q|^2)) : z_j \in \mathbb{C},\\ x_n \in \mathbb{R},\ 1 \leq q < n-1\},\ n \geq 3$$

also satisfy the weak maximum principle but not the strong maximum principle. Indeed, the CR function

$$h = \exp\left(\sqrt{-1}(x_n + \sqrt{-1}(|z_1|^2 + \ldots + |z_q|^2))\right)$$

has the property that $|h|$ attains a weak local maximum at any point of $\mathcal{M}$ where $z_1 = \ldots = z_q = 0$, and so $\mathcal{M}$ violates the strong maximum principle. The Levi form of $\mathcal{M}$ has no strictly definite points and so by the result in [10], $\mathcal{M}$ satisfies the weak maximum principle. Indeed, if a CR functions attains a strict

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local maximum at a point $p$, then by a result in [10], the point $p$ is the limit of a sequence $\{p_j\}$ where the Levi form is strictly definite. The weak maximum principle for CR functions has been studied in several papers including in [5] and [8].

To our knowledge, necessary and sufficient conditions for the validity of the strong maximum principle are not known. As we will see in Section 2, if $\mathcal{M}$ satisfies the holomorphic extendability property, then it satisfies the strong maximum principle. In the converse direction, we will show that the strong maximum principle implies holomorphic extendability on a dense open set. Moreover, we will show that if $\mathcal{M}$ satisfies the strong maximum principle, then the restriction of any nonconstant holomorphic function to $\mathcal{M}$ is an open map. In Section 3 we will present an example of a real analytic hypersurface $\mathcal{M}$ with the property that the restriction to any neighborhood of a point in $\mathcal{M}$ of any locally defined non constant holomorphic function satisfies the strong maximum principle but the principle does not hold for some continuous CR function. In Section 4 we show that for real analytic tube structures, analytic hypoellipticity ( which is holomorphic extendability of all solutions in the CR case) is equivalent to the validity of the strong maximum principle. Section 5 we present an improvement of a result of [9] on the validity of the strong maximum principle for abstract CR manifolds of hypersurface type.

## 2. Statements and proofs of the main results

**Definition 2.1.** We say $\mathcal{M}$ satisfies the extendability property if any locally defined continuous CR function extends as a holomorphic function.

We begin with the following observation.

**Proposition 2.2.** *Suppose $\mathcal{M} \subset \mathbb{C}^n$ is a smooth hypersurface that satisfies the extendability property. The if $h$ is a nonconstant continuous CR function on a connected open subset $U \subset \mathcal{M}$, the map $h : U \to \mathbb{C}$ is an open map, that is, if $W \subset U$ is open, so is $h(W)$. In particular, $\mathcal{M}$ satisfies the strong maximum principle.*

*Proof.* Let $h$ and $U$ be as in the Proposition and $p \in U$. Since extendability holds, there exist a neighborhood $U_1$ of $p$ in $U$ and a neighborhood $U_2$ of $p$ in $\mathbb{C}^n$ such that every CR function $f$ on $U_1$ extends to a a holomorphic function $\tilde{f}$ on $U_2$. This follows fom Montel's theorem and the Baire Category Theorem (see [13]). It follows that for any continuous CR function $f$ on $U_1$, $\tilde{f}(U_2) \subset f(U_1)$. Otherwise, if $\tilde{f}(w) \notin f(U_1)$ for some $w \in U_2$, the CR function

$$\frac{1}{f(z) - \tilde{f}(w)} \text{ on } U_1 \text{ would not extend to } U_2.$$

Let

$$S = \{w \in U : \text{ there is a neighborhood of } w \text{ where } h \equiv h(p)\}.$$

$S$ is clearly open. Let $\{w_j\}$ be a sequence in $S$ such that $w_j \to w \in U$. Since $\mathcal{M}$ satisfies the extendability property at each point of $U$ and $U$ is connected, by Trepreau's theorem [12], the Sussmann orbit $\Omega_p(U)$ of $p$ in $U$ coincides with $U$.

By a theorem of Treves (see [13]) on propagation of zeros, it follows that $w \in S$. Hence either $S = U$ or $S = \emptyset$. Since $h$ is nonconstant, we conclude that $S = \emptyset$. This implies that the holomorphic extension $\tilde{h}$ of $h$ is nonconstant on $U_2$ and since $\tilde{h}(U_2) \subset h(U_1)$, the set $h(U_1)$ contains a neighborhood of $h(p)$ in $\mathbb{C}$. It is now clear that $\mathcal{M}$ satisfies the strong maximum principle. □

**Theorem 2.3.** *Let $\mathcal{M}$ be a $C^\infty$ hypersurface in $\mathbb{C}^n$ that satisfies the strong maximum principle. If $f$ is a nonconstant holomorphic function on a neighborhood of $p \in \mathcal{M}$, then the restriction of $f$ to $\mathcal{M}$ is an open map into $\mathbb{C}$.*

*Proof.* We may assume $p = 0 \in \mathcal{M}$ and let

$$r(z', x_n, y_n) = \varphi(z', x_n) - y_n,\ z' \in \mathbb{C}^{n-1}$$

be a defining function of $\mathcal{M}$ with $\varphi(0) = 0$ and $d\varphi(0) = 0$. By Corollary 3.2 in [2], CR functions near 0 extend to a side of $\mathcal{M}$ (locally), say to the side $r < 0$. By the theorem of Treves in [13], there is a neighborhood $\Omega$ of 0 in $\mathcal{M}$ and $\delta > 0$ such that every continuous CR function on $\Omega$ extends as a holomorphic function to

$$\Omega_\delta = B_\delta(0) \cap \{z : r(z) < 0\}, \quad B_\delta(0) = \{z \in \mathbb{C}^n :\ |z| < \delta\}.$$

We may assume $f$ is holomorphic on a neighborhood of $\overline{\Omega}$ in $\mathbb{C}^n$ and that $f(0) = 0$. By Corollary 3.3 in [2], we may assume that $df(0) = 0$. Moreover, if the function of one variable $z_n \longmapsto f(0', z_n)$ is identically zero, then by Theorem 3.1 in [2], the zero set of $f$ has to cross both sides of $\mathcal{M}$ which in turn would imply that the restriction of $f$ to $\Omega$ is open at 0.
Hence we may assume that $z_n \longmapsto f(0', z_n)$ is nonconstant. By the Weierstrass Preparation Theorem, we can therefore factorize $f$ near 0 as

$$f(z', z_n) = b(z', z_n)\Bigg(z_n^k + a_{k-1}(z')z_n^{k-1} + \ldots + a_0(z')\Bigg)$$

where $b(0) \neq 0$, $a_j(0') = 0\ \forall j, k \geq 2, a_j(z'), b(z', z_n)$ holomorphic.
We may assume that $b(0) = 1$.
Since $\varphi(0) = 0$ and $d\varphi(0) = 0$, there is $M > 0$ such that

$$|\varphi(0', x_n)| \leq M x_n^2.$$

Therefore, for some $\epsilon > 0$,

$$D_\epsilon = \{(0', z_n) : |z_n| < \epsilon,\ y_n \geq M x_n^2\} \subset \{z :\ r(z) \leq 0\}.$$

Suppose first $k \geq 3$:
If $\epsilon < \delta$, then $D_\epsilon \subset \overline{\Omega_\delta}$ and so since $f(\overline{\Omega_\delta}) \subset f(\Omega)$ (for a smaller $\delta$), we only have to show that $f(D_\epsilon)$ contains a neighborhood of the origin in $\mathbb{C}$. The set

$$f(D_\epsilon) = \{b(0', z_n)z_n^k : |z_n| \leq \epsilon,\ y_n \geq M x_n^2\}.$$

For any $0 < c < M\epsilon$, if we set

$$A_c = \{(0', z_n) : |z_n| \leq \frac{c}{M},\ y_n \geq c|x_n|\},$$

then $A_c \subset D_\epsilon$. Moreover, if $0 < c \leq \frac{1}{\sqrt{3}}$, since $k \geq 3$, the set $z_n^k(A_c)$ is a neighborhood of 0. Since $b(0) = 1$, there is a holomorphic function $h$ such that $b(0', z_n) = h(z_n)^k$ with $h(0) = 1$, and so

$$f(0', z_n) = g(z_n)^k \text{ where } g(z_n) = z_n h(z_n).$$

Fix $0 < c < \frac{1}{\sqrt{3}}$ and let $\epsilon_1 > 0$ such that $c + \epsilon_1 < \frac{1}{\sqrt{3}}$.
Using $g(0) = 0$ and $g'(0) = 1$, we can find $\beta > 0$ small that satisfies

$$\{z_n : |z_n| \leq \beta,\ y_n \geq (c + \epsilon_1)|x_n|\} \subset g\Big(\{z_n : |z_n| \leq 2\beta,\ y_n \geq c|x_n|\}\Big).$$

It follows that $f(D_\epsilon)$ and hence $f(\Omega)$ contains a neighborhood of 0.
Suppose now $k = 2$.
Recall that $f(z', z_n) = b(z', z_n)\Big(z_n^2 + a_1(z')z_n + a_0(z')\Big)$, $a_1(0') = a_0(0') = 0, b(0) = 1$, and $|\varphi(0', x_n)| \leq M x_n^2$.
For some $c > 0$ to be determined, we will use the change of holomorphic coordinates

$$(w', w_n) \longmapsto (w', w_n + \sqrt{-1}c^2 w_n^2).$$

In the $w$ coordinates, the defining function of $\mathcal{M}$ becomes

$$\tilde{r}(w', w_n) = r(w', w_n + \sqrt{-1}c^2 w_n^2).$$

We will show that if $w_n = u_n + \sqrt{-1}v_n$ is sufficiently small,

$$\tilde{r}(0', w_n) \leq 0 \text{ whenever } v_n \geq -c u_n^2.$$

To see this, observe that

$$\begin{aligned}
\tilde{r}(0', w_n) &= \varphi\big(0', \Re(w_n + \sqrt{-1}c^2 w_n^2)\big) - \Im(w_n + \sqrt{-1}c^2 w_n^2) \\
&= \varphi(0', u_n - 2c^2 u_n v_n) - v_n - c^2 u_n^2 + c^2 v_n^2 \\
&\leq M u_n^2 (1 - 2c^2 v_n)^2 - v_n - c^2 u_n^2 + c^2 v_n^2.
\end{aligned}$$

Suppose now $v_n \geq -c u_n^2$. Then for $w_n$ small enough, if $v_n \geq 0$, $-v_n + c^2 v_n^2 \leq 0$ and so

$$\tilde{r}(0', w_n) \leq [M(1 - 2c^2 v_n)^2 - c^2] u_n^2 \leq 0$$

if $c$ is chosen large enough. Suppose $v_n \leq 0$. Then since $v_n \geq -c u_n^2$ , $v_n^2 \leq c^2 u_n^4$ and hence

$$\begin{aligned}
\tilde{r}(0', w_n) &\leq M u_n^2 ((1 - 2c^2 v_n)^2 - v_n - c^2 u_n^2 + c^4 u_n^4 \\
&\leq M u_n^2 ((1 - 2c^2 v_n)^2 + c u_n^2 - c^2 u_n^2 + c^4 u_n^4 \\
&\leq 0
\end{aligned}$$

for $c$ big enough and $w_n$ sufficiently small.
Thus for $w_n$ small and $c$ large enough,

$$\tilde{r}(0', w_n) \leq 0 \text{ whenever } v_n \geq -c u_n^2.$$

In the $w$ coordinates, $f$ becomes

$$\tilde{f}(w', w_n) = f(w', w_n + \sqrt{-1}c^2 w_n^2),$$

in particular,

$$\tilde{f}(0', w_n) = \tilde{b}(0', w_n)(-c^4 w_n^4 + 2\sqrt{-1}c^2 w_n^3 + w_n^2),$$

where

$$\tilde{b}(0', w_n) = b(0', w_n + \sqrt{-1}c^2 w_n^2).$$

We will show that there is a holomorphic function $a(z)$ defined near $z = 0$ in $\mathbb{C}$ such that near 0,

$$\tilde{f}(0', za(z)) = \tilde{f}(0', z),\ a(0) = -1.$$

Define the function

$$G(z, a) = \tilde{b}(0', az)(-c^4 a^4 z^2 + 2\sqrt{-1}c^2 a^3 z + a^2) - \tilde{b}(0', z)(-c^4 z^2 + 2\sqrt{-1}c^2 z + 1)$$

which is holomorphic for $(z, a)$ near $(0, -1)$ in $\mathbb{C}^2$, $G(0, -1) = 0$ and since

$$\frac{\partial G}{\partial a}(z, a) = \frac{\partial \tilde{b}}{\partial w_n}(0', az)z\Big(-c^4 a^4 z^2 + 2\sqrt{-1}c^2 a^3 z + a^2\Big) + \tilde{b}(0', az)\Big(-4c^4 a^3 z^2 + 6\sqrt{-1}c^2 a^2 z + 2a\Big),$$

$\frac{\partial G}{\partial a}(0, -1) = -2$. Therefore, by the implicit function theorem, there is a holomorphic function $a(z)$ defined near $z = 0$ that satisfies

$$G(z, a(z)) = 0, \text{ and } a(0) = -1.$$

It follows that

$$\tilde{f}(0', za(z)) - \tilde{f}(0', z) = z^2 G(z, a(z)) = 0.$$

From the equation $G(z, a(z)) = 0$, we get

$$a'(0) = -\frac{\partial \tilde{b}}{\partial w_n}(0) = \frac{\partial b}{\partial z_n}(0).$$

For $\epsilon > 0$, let

$$W_\epsilon = \{(0', w_n) : v_n \geq -c\, x_n^2,\ |w_n| < \epsilon\}.$$

We choose $\epsilon$ small enough so that

$$\tilde{r} \leq 0 \text{ on } W_\epsilon.$$

We know that $\tilde{f}(W_\epsilon) \subset \tilde{f}(\Omega)$. Therefore, $\tilde{f}$ will be open at 0 if for $|z| < \epsilon$, the point $(0', za(z)) \in W_\epsilon$ whenever $(0', z) \notin W_\epsilon$.
Suppose then $|z| < \epsilon$ and $(0', z) \notin W_\epsilon$. Then $y < -c\, x^2$. Notice that

$$a(z)z = -z - \frac{\partial b}{\partial z_n}(0)z^2 + O(z^3),$$

and so setting $\frac{\partial b}{\partial z_n}(0) = s + \sqrt{-1}\, t$,

$$\begin{aligned}\Im(za(z)) &= -y - 2sxy - t(x^2 - y^2) + O(|x|^3 + |v|^3)\\ &\geq -\frac{y}{2} + \frac{c}{2}x^2 - 2sxy - t(x^2 - y^2) + O(|x|^3 + |v|^3)\end{aligned}$$

and hence for $c$ large enough and $\epsilon$ small enough, $\Im(za(z)) > 0$ showing that $(0', za(z)) \in W_\epsilon$. This proves the theorem. $\square$

We recall from [2] that if $\mathcal{M}$ satisfies the strong maximum principle, then any locally defined CR function extends holomorphically to a side of $\mathcal{M}$. We now use Theorem 2.3 to show that in addition, holomorphic extendability holds on a dense open subset of $\mathcal{M}$.

**Theorem 2.4.** *Let $\mathcal{M}$ be a smooth hypersurface in $\mathbb{C}^n$ with the strong maximum principle property. Then there is a dense open subset $\Sigma \subset \mathcal{M}$ such that any CR function defined on an open subset $U \subset \mathcal{M}$ extends as a holomorphic function to a neighborhood of $U \cap \Sigma$ in $\mathbb{C}^n$.*

In the proof of Theorem 2.4 we will use the following result of Catlin ([4]).

**Lemma 2.5.** *(Lemma 3.3.2 in* [4]*) Let $z_0$ be a boundary point of an $n-$dimensional complex manifold with smooth pseudoconvex boundary. Suppose that there is an $l-$dimensional complex manifold $\gamma \subset b\,\Omega$, with $z_0 \in \gamma$, and that the rank of the Levi form at $z_0$ is $n-l-1$. Then there exists a coordinate neighborhood $V$ of $z_0$, with holomorphic coordinates $z_1, \ldots, z_n$ satisfying the following properties:*
*(1)* $\gamma = \{z \in V : \ z_{l+k} = 0, \ k = 1, 2, \ldots, n-l\}$
*(2) Writing $z' = (z_1, \ldots, z_l)$ and $z'' = (z_{l+1}, \ldots, z_n)$ the Taylor expansion of the boundary defining function $r(z)$ in the variables $z''$ has the form*

$$r(z) = r(z', z'') = \frac{\partial r}{\partial z_n}(z', 0) z_n + \frac{\partial r}{\partial \bar{z}_n}(z', 0)\bar{z}_n + O(|z''|^2).$$

*(3) For $z \in V \cap \overline{\Omega}$,*

$$2\Re z_n - |\Im z_n| + \frac{1}{2}\sum_{k=l+1}^{n-1} |z_k|^2 \le r(z) \le \frac{1}{2}\Re z_n + |\Im z_n| + 2\sum_{k=l+1}^{n-1} |z_k|^2.$$

*Proof of Theorem 2.4.* Let $\mathcal{V}$ denote the CR bundle of $\mathcal{M}$. For each $p \in \mathcal{M}$, consider the Levi form

$$\mathcal{L}_p : (\mathcal{V}_p \oplus \overline{\mathcal{V}_p}) \times (\mathcal{V}_p \oplus \overline{\mathcal{V}_p}) \to \frac{\mathbb{C}T_p\mathcal{M}}{\mathcal{V}_p \oplus \overline{\mathcal{V}_p}}$$

which is defined by

$$\mathcal{L}_p(X_p, Y_p) = \frac{1}{2\sqrt{-1}} \pi_p([X, Y]_p)$$

where $\pi_p$ is the projection map

$$\pi_p : \mathbb{C}T_p\mathcal{M} \to \frac{\mathbb{C}T_p\mathcal{M}}{\mathcal{V}_p \oplus \overline{\mathcal{V}_p}},$$

$X$ and $Y$ are smooth sections of $\mathcal{V}_p \oplus \overline{\mathcal{V}_p}$ that extend $X_p$ and $Y_p$ respectively. For $p \in \mathcal{M}$, let $\mathcal{N}_p$ denote the null space of $\mathcal{L}_p$ given by

$$\mathcal{N}_p = \{X_p \in \mathcal{V}_p \oplus \overline{\mathcal{V}_p} : \mathcal{L}_p(X_p, Y_p) = 0 \text{ for all } Y_p \in \mathcal{V}_p \oplus \overline{\mathcal{V}_p}\}.$$

$\mathcal{N}_p$ is a complex vector space and if $k = k(p)$ is the number of nonzero eigenvalues of $\mathcal{L}_p$, then

$$k + \dim_{\mathbb{C}} \mathcal{N}_p = n - 1.$$

Fix $p \in \mathcal{M}$ and let $p \in U \subset \mathcal{M}$ be a neighborhood. Since $\mathcal{M}$ satisfies the strong maximum principle, $(U, \mathcal{V})$ cannot be Levi flat and so there are points $q \in U$ where $k(q) > 0$. Let

$$\alpha = \max\{k(q) : q \in U\}.$$

Then there is an open subset $W \subset U$ satisfying

$$k(q) = \alpha \ \ \forall q \in W.$$

If there is a point $q \in W$ where the Levi form has a positive and a negative eigenvalue, then by Lewy's extension theorem, any CR function defined on a neighborhood of $q$ extends to a holomorphic function in a full neighborhood of $q$ in $\mathbb{C}^n$. We may therefore assume that all the eigenvalues in $W$ have the same sign. The set

$$\mathcal{N} = \bigcup_{q \in W} \mathcal{N}_q$$

forms a smooth subbundle of $\mathcal{V} \oplus \overline{\mathcal{V}}$ over $W$. In this case, by a theorem of Freeman ([6]), $W$ is foliated by complex submanifolds such that for each $q \in W$, $\mathcal{N}_q$ is the complex tangent space of the leaf passing through $q$.
We use the arguments given in [3] for the Levi flat case. We first show that $\mathcal{N}$ is involutive. Let $X$ and $Y$ be smooth sections of $\mathcal{N}$ over $W$, and let $Z$ be a smooth section of $\mathcal{V} \oplus \overline{V}$. Using the Jacobi identity

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$

we see that $[[X, Y], Z]$ is a section of $\mathcal{V} \oplus \overline{\mathcal{V}}$ and hence $[X, Y]$ is a section of $\mathcal{N}$. Let

$$\Re\mathcal{N} = \{A : A \text{ is the real part of a smooth section of } \mathcal{N} \text{ over } W\}.$$

Then $\Re\mathcal{N}$ is involutive and so by the Frobenius theorem, $W$ is foliated by submanifolds $S$ of $W$ such that the real tangent space $T_qS = \Re\mathcal{N}_q$ for each $q \in S$.
Observe next that $\mathcal{N}$ is $J-$ invariant. To see this, let $X$ be a smooth section of $\mathcal{N}$, $X = L_1 + \overline{L_2}$ where $L_1, L_2$ are sections of $\mathcal{V}$. For any $Y_1 \in \mathcal{V}$, $[L_1 + \overline{L_2}, Y_1]$ is a section of $\mathcal{V} \oplus \overline{\mathcal{V}}$ and so $[\overline{L_2}, Y_1]$ is a section of $\mathcal{V} \oplus \overline{\mathcal{V}}$. Likewise, for any smooth section $Y_2$ of $\mathcal{V}$, $[L_1, \overline{Y_2}]$ is a section of $\mathcal{V} \oplus \overline{\mathcal{V}}$. Therefore, for such $Y_1, Y_2$,

$$[J(X), Y_1] = [-iL_1, Y_1] + [i\overline{L_2}, Y_1]$$

is a section of $\mathcal{V} \oplus \overline{\mathcal{V}}$ and

$$[J(X), \overline{Y_2}] = [-iL_1, \overline{Y_2}] + [i\overline{L_2}, \overline{Y_2}]$$

is a section of $\mathcal{V} \oplus \overline{\mathcal{V}}$ and hence the $\mathcal{N}$ is $J-$invariant.
Let $S$ now be a leaf of the foliation induced by $\Re\mathcal{N}$ and pick a point $q \in S$ which we may assume is the origin in $\mathbb{C}^n$. Since the real tangent space $T_0S = \Re\mathcal{N}_0$ is invariant under $J$, after a complex linear change of coordinates, we may assume that $T_0S$ is spanned by

$$\left.\frac{\partial}{\partial x_j}\right|_0, \left.\frac{\partial}{\partial y_j}\right|_0, \ 1 \leq j \leq n - 1 - \alpha = s.$$

Thus near 0, $S$ is the graph of a smooth map

$$g : \mathbb{C}^s \to \mathbb{C}^{n-s},$$

that is,

$$S = \{(z_1, \ldots, z_s, g(z', \bar{z}')) : z' = (z_1, \ldots, z_s) \in \mathbb{C}^s\}.$$

We will show that $g$ is holomorphic by showing that the push forward map

$$g_* : \mathbb{R}^{2s} \to \mathbb{R}^{2n-2s}$$

commutes with $J$. Let $G : \mathbb{C}^s \to \mathbb{C}^{n-s} \times \mathbb{C}^s$ be given by

$$G(z') = (z', g(z', \bar{z}')).$$

Let $v$ be in the tangent space $T_0(\mathbb{R}^{2s})$. Then for $a \in \mathbb{R}^{2s} \simeq \mathbb{C}^s$ near 0,

$$\begin{aligned} J\big(G_*(a), (v)\big) &= J\big(v, g_*(a)(v)\big) \\ &= \big(J(v), J(g_*(a)(v))\big). \end{aligned}$$

Clearly, $G_*(a)(v) \in T_{G(a)}S$ and by the $J-$invariance of $\mathcal{N}$ and hence of $\Re\mathcal{N}$, $J\big(G_*(a)(v)\big) \in T_{G(a)}S$. Since any tangent vector in $T_{G(a)}S$ has the form $(v, g_*(a)(v))$ for some $v \in T_a(\mathbb{R}^{2s})$, it follows that

$$g_*(a)(J(v)) = J(g_*(a)(v))$$

and hence $S$ is a complex manifold.

The proof of Lemma 2.5 requires that $\mathcal{M}$ be pseudoconvex only near the central point 0 and hence there are holomorphic coordinates $z_1, \ldots, z_n$ such that for $z \in \mathcal{M}$ near 0,

$$\Re z_n \leq \frac{1}{2}|\Im z_n| - \frac{1}{4}\sum_{k=s+1}^{n-1} |z_k|^2.$$

That is, for $z \in \mathcal{M}$ near 0,

$$\Re z_n \leq \frac{1}{2}|\Im z_n|$$

which means that the restriction of the holomorphic function $f(z) = z_n$ to $\mathcal{M}$ is not open at 0, thus contradicting Theorem 2.3. Thus $U$ contains a point where the Levi form has a positive and a negative eigenvalue. □

## 3. An example

We next show an example of a hypersurface $\mathcal{M}$ in $\mathbb{C}^6$ satisfying the strong maximum principle for the restrictions of nonconstant holomorphic functions but not for continuous CR functions.

We first recall the following definition from [5]:

**Definition 3.1.** Let $\mathcal{M} \subset \mathbb{C}^n$ be a smooth hypersurface. A point $p \in \mathcal{M}$ is called an extreme point of $\mathcal{M}$ if there exists a local holomorphic coordinate system $z = (z_1, \ldots, z_n)$ in a neighborhood $U$ of $p$ such that $z(p) = 0$ and $\mathcal{M} \cap U \subset \{z : \Im z_n \geq 0\}$.

Clearly, the absence of extreme points on $\mathcal{M}$ is a necessary condition for the validity of the strong maximum principle. In [5] the authors conjectured a converse statement: if $\mathcal{M}$ has no extreme points, then the strong maximum principle is valid for differentiable CR functions. Later in [8] the author proved the following:

**Theorem 3.2.** *If $\mathcal{M}$ is a CR submanifold of $\mathbb{C}^n$ without extremem points, then for any CR function $f$ on $\mathcal{M}$, $|f|$ cannot attain a strict local maximum at any point of $\mathcal{M}$.*

Although the proof in [8] assumes that $f \in C^2$, one can use the Baouendi-Treves approximation theorem to see that the theorem is valid for $f$ continuous. We also remark that this theorem follows from the results in [10] and [11]. The theorem may be viewed as a partial answer to the conjecture stated above. Indeed, it tells us that if $\mathcal{M}$ has no extreme points, then it satisfies the weak maximum principle. In [2] we gave an example of a hypersurface $\mathcal{M}$ with no extreme points but where the strong maximum principle does not hold for continuous CR functions. In this section we present a simpler example with the same properties.

Observe that a point $p \in \mathcal{M}$ is an extreme point if and only if there is a holomorphic function $f$ defined near $p$ such that $df(p) \neq 0$ and $|f|$ on $\mathcal{M}$ has a weak local maximum at $p$, that is, $|f(q)| \leq |f(p)|$ for $q$ near $p$ on $\mathcal{M}$. Therefore, the conjecture of [5] may be rephrased as follows: if the strong maximum principle is valid for the restrictions to $\mathcal{M}$ of holomorphic functions with nonzero differential, then it is also valid for differentiable CR functions. This seems plausible in light of the Baouendi-Treves approximation theorem according to which any continuous CR function is locally the uniform limit of the restrictions of holomorphic functions. Our example $\mathcal{M}$ below has the following properties:

1. $\mathcal{M}$ has no extreme points. In fact, more is true - if $f$ is holomorphic near any point of $\mathcal{M}$ (no condition on the differential of $f$), the strong maximum principle is valid for the restriction of $|f|$ to $\mathcal{M}$: that is, if $p \in \mathcal{M}$, $U$ a neighborhood of $p$ in $\mathcal{M}$, $f$ a holomorphic near $U$ and $|f(q)| \leq |f(p)|$ for all $q \in U$, then $f$ is constant.
2. There is a continuous non constant continuous CR function $h$ on $\mathcal{M}$ for which the strong maximum principle is not valid, that is, $|h|$ attains a weak local maximum.

Let $\mathcal{M}$ be the real analytic hypersurface in $\mathbb{C}^6$ defined by

$$\mathcal{M} = \{z : y_6 = x_6(|z_1|^2 - |z_2|^2) + |z_5|^2(1 + |z_3|^2 - |z_4|^2)\}.$$

A basis of the tangential Cauchy-Riemann vector fields on $\mathcal{M}$ is given by

$$L_1 = \frac{\partial}{\partial \bar{z}_1} - \left( \frac{2\sqrt{-1}x_6 z_1}{1 + \sqrt{-1}(|z_1|^2 - |z_2|^2)} \right) \frac{\partial}{\partial \bar{z}_6},$$

$$L_2 = \frac{\partial}{\partial \bar{z}_2} + \left( \frac{2\sqrt{-1}x_6 z_2}{1 + \sqrt{-1}(|z_1|^2 - |z_2|^2)} \right) \frac{\partial}{\partial \bar{z}_6},$$

$$L_3 = \frac{\partial}{\partial \bar{z}_3} - \left( \frac{2\sqrt{-1}|z_5|^2 z_3}{1+\sqrt{-1}(|z_1|^2 - |z_2|^2)} \right) \frac{\partial}{\partial \bar{z}_6},$$

$$L_4 = \frac{\partial}{\partial \bar{z}_4} + \left( \frac{2\sqrt{-1}|z_5|^2 z_4}{1+\sqrt{-1}(|z_1|^2 - |z_2|^2)} \right) \frac{\partial}{\partial \bar{z}_6},$$

and

$$L_5 = \frac{\partial}{\partial \bar{z}_5} - \left( \frac{2\sqrt{-1} z_5 (1+|z_3|^2 - |z_4|^2)}{1+\sqrt{-1}(|z_1|^2 - |z_2|^2)} \right) \frac{\partial}{\partial \bar{z}_6}.$$

We next consider the Levi form:

$$[L_1, \overline{L_1}] = \left( \frac{2\sqrt{-1}x_6}{1-\sqrt{-1}(|z_1|^2 - |z_2|^2)} + x_6 O(|z|^2) \right) \frac{\partial}{\partial z_6} + \left( \frac{2\sqrt{-1}x_6}{1+\sqrt{-1}(|z_1|^2 - |z_2|^2)} + x_6 O(|z|^2) \right) \frac{\partial}{\partial \bar{z}_6}$$

$$= 2\sqrt{-1}x_6 \frac{\partial}{\partial z_6} + 2\sqrt{-1}x_6 \frac{\partial}{\partial \bar{z}_6} + x_6\ O(|z|^2)\big(\frac{\partial}{\partial z_6} + \frac{\partial}{\partial \bar{z}_6}\big),$$

$$[L_2, \overline{L_2}] = \left( \frac{-2\sqrt{-1}x_6}{1-\sqrt{-1}(|z_1|^2 - |z_2|^2)} + x_6 O(|z|^2) \right) \frac{\partial}{\partial z_6} + \left( \frac{-2\sqrt{-1}x_6}{1+\sqrt{-1}(|z_1|^2 - |z_2|^2)} + x_6 O(|z|^2) \right) \frac{\partial}{\partial \bar{z}_6},$$

$$[L_3, \overline{L_3}] = \left( \frac{2\sqrt{-1}|z_5|^2}{1-\sqrt{-1}(|z_1|^2 - |z_2|^2)} \right) \frac{\partial}{\partial z_6} + \left( \frac{2\sqrt{-1}|z_5|^2}{1+\sqrt{-1}(|z_1|^2 - |z_2|^2)} \right) \frac{\partial}{\partial \bar{z}_6},$$

$$[L_4, \overline{L_4}] = \left( \frac{-2\sqrt{-1}|z_5|^2}{1-\sqrt{-1}(|z_1|^2 - |z_2|^2)} \right) \frac{\partial}{\partial z_6} + \left( \frac{-2\sqrt{-1}|z_5|^2}{1+\sqrt{-1}(|z_1|^2 - |z_2|^2)} \right) \frac{\partial}{\partial \bar{z}_6},$$

$$[L_5, \overline{L_5}] = (2\sqrt{-1} + O(|z|^2)\Big) \frac{\partial}{\partial z_6} + \Big(2\sqrt{-1} + O(|z|^2)\Big) \frac{\partial}{\partial \bar{z}_6}.$$

Let

$$r(z, \overline{z}) = y_6 - x_6(|z_1|^2 - |z_2|^2) - |z_5|^2(1 + |z_3|^2 - |z_4|^2$$

and define

$$\theta = \bar{\partial} r = \sum_{j=1}^{5} \frac{\partial \rho}{\partial \bar{z}_j} d\bar{z}_j + \frac{\sqrt{-1}}{2}\big(1 + \sqrt{-1}(|z_1|^2 - |z_2|^2)\big) d\bar{z}_6.$$

We have:

$$\langle \theta, [L_1, \overline{L}_1] \rangle = -x_6 + x_6 O(|z|^2),$$

$$\langle \theta, [L_2, \overline{L}_2] \rangle = x_6 + x_6 O(|z|^2),$$

$$\langle \theta, [L_3, \overline{L}_3] \rangle = |z_5|^2 + O(|z_5|^2|z|^4),$$

$$\langle \theta, [L_4, \overline{L}_4] \rangle = -|z_5|^2 + O(|z_5|^2|z|^4),$$

$$\langle \theta, [L_5, \overline{L}_5] \rangle = -1 + O(|z|^2).$$

Observe that on $\mathcal{M}$, when $x_6 = 0$, $|z_5|^2$ is comparable to $y_6$.
Therefore, after shrinking $\mathcal{M}$ near 0, we conclude that at every point in the set

$$S = \{z \in \mathcal{M} : z_6 \neq 0\},$$

the Levi form has at least one positive and one negative eigenvalue. This implies that no nonconstant locally defined continuous CR function can attain a weak local maximum at a point in $S$. Next note that if $z \in \mathcal{M}$ and $z_6 = 0$, then $z_5 = 0$.

Hence we only have to show that the restriction of a nonconstant holomorphic function can not attain a weak local maximum at a point in the set

$$\Sigma = \{(z_1, z_2, z_3, z_4, 0, 0)\} \subset \mathcal{M}.$$

Suppose $h$ is a nonconstant holomorphic function on a neighborhood of $p = (z_1^0, z_2^0, z_3^0, z_4^0, 0, 0) \in \Sigma$ and assume that $|h(z)| \leq |h(p)|$ for $z$ in $\mathcal{M}$ near $p$. We may assume that $\Im h(z) \leq \Im h(p)$ for $z$ near $p$ in $\mathcal{M}$. We will first show that the differential $dh(p) \neq 0$. Indeed, since $\langle \theta, [L_5, \overline{L}_5] \rangle \neq 0$, CR functions near 0 extend holomorphically to the side

$$r(z, \bar{z}) = y_6 - x_6(|z_1|^2 - |z_2|^2) - |z_5|^2(1 + |z_3|^2 - |z_4|^2) < 0.$$

Suppose $\Omega_p \subset \mathcal{M}$ is a neighborhood of $p$ where

$$\Im h(z) \leq \Im h(p) \quad \text{for every } z \in \Omega_p.$$

There is $\delta > 0$ such that any CR function on $\Omega_p$ extends as a holomorphic function to the open set

$$B_\delta = \{z \in \mathbb{C}^6 : |z - p| < \delta\} \cap \{z : r(z) < 0\}.$$

Since $h(B_\delta) \subset h(\Omega_p)$, and $h$ is nonconstant, it follows that

$$\Im h(z) < \Im h(p) \text{ for every } z \in B_\delta.$$

By Hopf's lemma, the latter inequality implies that

$$\frac{\partial}{\partial y_6} \Im h(p) \neq 0.$$

Next observe that since $\Sigma$ is a complex manifold contained in $\mathcal{M}$,

$$\Im h|_\Sigma \equiv \Im h(p)$$

and so $h$ is constant on $\Sigma$. Without loss of generality we may assume that

$$h|_\Sigma \equiv 0.$$

We next consider the Taylor expansion of the function $h(z_1, z_2, z_3, z_4, 0, z_6)$ in the variable $z_6$ near the point $p = (z_1^0, z_2^0, z_3^0, z_4^0, 0, 0)$. Write $z' = (z_1, z_2, z_3, z_4)$ for the variables in $\mathbb{C}^4$. We have:

$$\begin{aligned} h(z', 0, z_6) &= \frac{\partial h}{\partial z_6}(z', 0, 0) z_6 + O(|z_6|^2) \quad (\text{since } h(z', 0, 0) = 0) \\ &= \frac{\partial h}{\partial z_6}(p) z_6 + a(z') z_6 + O(|z_6|^2), \end{aligned}$$

where $a(z')$ is a holomorphic function near $p' = (z_1^0, z_2^0, z_3^0, z_4^0)$ and $a(p') = 0$. Hence after using the Cauchy-Riemann equations we get:

$$\Im h(z', 0, z_6) = \left( \frac{\partial \Im h}{\partial y_6}(p) + \Re a(z') \right) y_6 + \left( \frac{\partial \Im h}{\partial x_6}(p) + \Im a(z') \right) x_6 + O(x_6^2 \;+\; y_6^2).$$

Next, restrict $\Im h$ to the set $\mathcal{M} \cap \{z : z_5 = 0\}$. Note that on $\mathcal{M} \cap \{z : z_5 = 0\}$,

$$y_6 = k(z_1, \overline{z_1}, z_2, \overline{z_2}) x_6 \quad \text{where } k(z_1, \overline{z_1}, z_2, \overline{z_2}) = |z_1|^2 - |z_2|^2.$$

Therefore,

$$\Im h|_{\mathcal{M}\cap\{z_5=0\}} = \left(\frac{\partial \Im h}{\partial y_6}(p) + \Re a(z')\right)k(z_1,z_2)x_6 + \left(\frac{\partial \Im h}{\partial x_6}(p) + \Im a(z')\right)x_6 + O(x_6^2)$$
$$\leq \Im h(p) = 0.$$

It follows that

$$\left(\frac{\partial \Im h}{\partial y_6}(p) + \Re a(z')\right)k(z') = -\frac{\partial \Im h}{\partial x_6}(p) - \Im a(z').$$

Since

$$\Re a(p') = 0 \text{ and } \frac{\partial \Im h}{\partial y_6}(p) \neq 0,$$

the latter equation implies that there is a nonzero pluriharmonic function $v(z')$ defined near $p'$ such that $v(z')k(z_1,z_2)$ is pluriharmonic. We will reach a contradiction by showing that $v$ is constant.

The pluriharmonicity of $v$ and $vk$ lead to

$$\begin{aligned} 0 &= \frac{\partial^2}{\partial z_1 \partial \bar{z}_2}(vk) \\ &= \frac{\partial v}{\partial z_1}\frac{\partial k}{\partial \bar{z}_2} + \frac{\partial v}{\partial \bar{z}_2}\frac{\partial k}{\partial z_1} \\ &= -z_2\frac{\partial v}{\partial z_1} + \bar{z}_1\frac{\partial v}{\partial \bar{z}_2}. \end{aligned}$$

Hence

$$\bar{z}_1\frac{\partial v}{\partial \bar{z}_2} = z_2\frac{\partial v}{\partial z_1}.$$

But both $z_1\frac{\partial v}{\partial z_2}$ and $z_2\frac{\partial v}{\partial z_1}$ are holomorphic. Therefore, for some constants $c_1, c_2$,

$$z_2\frac{\partial v}{\partial z_1} = c_1,\ z_1\frac{\partial v}{\partial z_2} = c_2 = \bar{c}_1.$$

This in turn implies that

$$\frac{\partial^2 v}{\partial z_2 \partial z_1} = \frac{-c_1}{z_2^2}, \text{ and } \frac{\partial^2 v}{\partial z_1 \partial z_2} = \frac{-c_2}{z_1^2}$$

which is impossible unless $c_1 = c_2 = 0$.

Thus

$$\frac{\partial v}{\partial z_1} = \frac{\partial v}{\partial z_2} \equiv 0.$$

Recall that

$$v(z') = \frac{\partial \Im h}{\partial y_6}(p) + \Re a(z')$$

and

$$v(z')(|z_1|^2 - |z_2|^2) = \frac{-\partial \Im h}{\partial x_6}(p) - \Im a(z').$$

Since $v(z')$ is independent of $z_1$ and $z_2$, so are $\Re a(z')$ and hence $\Im a(z')$. This leads to the contradiction that $|z_1|^2 - |z_2|^2$ is independent of $z_1$ and $z_2$.

Thus we have shown that no nonconstant holomorphic function can attain a weak local maximum at any point of $\mathcal{M}$. That is, the strong maximum principle is valid for the restrictions of holomorphic functions.

Next consider the holomorphic function $f(z) = z_6$.

$$f(z)|_{\mathcal{M}} = x_6 + \sqrt{-1}\big(|z_5|^2 g(z) + x_6(|z_1|^2 - |z_2|^2)\big)$$

where $g(0) = 1$. Therefore, the image of $\mathcal{M}$ under $f$ does not intersect the negative $y-$axis. Hence, the continuous CR function

$$\exp\left(-\sqrt{iz_6}\right)$$

(with the principal branch) attains a local maximum at 0.

## 4. Tube Structures

In this section we will show that for real analytic tube structures, the strong maximum principle is equivalent to analytic hypoellipticity, in the CR case, equivalently, to the holomorphic extendability of every solution.

Let $m$ and $n$ be positive integers. We will denote by $x = (x_1, \ldots, x_m)$ and $t = (t_1, \ldots, t_n)$ variable points in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. Let $V$ be a domain in $\mathbb{R}^n$ and

$$\varphi(t) = (\varphi_1(t), \ldots, \varphi_m(t))$$

a real analytic mapping, $\varphi : V \to \mathbb{R}^m$.

Let

$$Z_i(x,t) = x_i + \sqrt{-1}\varphi_i(t),\ 1 \le i \le m$$

and consider the associated $n$ complex vector fields on $\mathbb{R}^m \times V$ given by

$$L_j = \frac{\partial}{\partial t_j} - \sqrt{-1}\sum_{k=1}^{m} \frac{\partial \varphi_k}{\partial t_j}(t)\frac{\partial}{\partial x_k},\ 1 \le j \le n.$$

Clearly,

$$L_j Z_i = 0, \quad 1 \le i \le m,\ 1 \le j \le n.$$

Let $\Omega = \mathbb{R}^m \times V$. We will consider continuous solutions $h = h(x,t)$ of the system of equations

$$L_j h = 0,\ 1 \le j \le n \tag{4.1}$$

on open subsets of $\Omega$.

We denote by $\mathcal{L}$ the system of vector fields $L_1, \ldots, L_n$. Observe that when the mapping $\varphi : V \to \mathbb{R}^m$ is an immersion, the $L_j$ define a system of CR vector fields which are not necessarily of hypersurface type.

**Definition 4.1.** We say that $\mathcal{L}$ is analytic hypoelliptic at $(x_0, t_0) \in \mathbb{R}^m \times V$ if for any distribution $u$, whenever $L_j u$ $(j = 1, \ldots, n)$ is real analytic in a neighborhood of $(x_0, t_0)$, $u$ itself is real analytic in a possibly smaller neighborhood of $(x_0, t_0)$. We say that $\mathcal{L}$ is analytic hypoelliptic on a subset of $\mathbb{R}^m \times V$ if $\mathcal{L}$ is analytic hypoelliptic at each point of the subset.

Since the coefficients of the $L_j$ are independent of $x$, it is clear that $\mathcal{L}$ is analytic hypoelliptic at $(x_0, t_0) \in \mathbb{R}^m \times V$ if and only if $\mathcal{L}$ is analytic hypoelliptic on $\mathbb{R}^m \times \{t_0\}$.

We will use the following result from [1]:

**Proposition 4.2.** *The system $\mathcal{L}$ is analytic hypoelliptic at $(x_0, t_0) \in \mathbb{R}^m \times V$ if and only if for every distribution $h$ defined in some neighborhood of $(x_0, t_0)$ which is a solution, the distribution $x \longmapsto h(x, t_0)$ is real analytic in some neighborhood of $x_0$.*

**Definition 4.3.** We say $\mathcal{L}$ satisfies the strong maximum principle if given any connected open set $D$ in $\mathbb{R}^m \times V$ and any continuous solution $h$ on $D$, $|h|$ can not have a weak local maximum at any point of $D$ unless $h$ is constant on $D$. That is, $p \in D$ and $|h(z)| \leq |h(p)|$ for all $z \in D$ implies that $h$ is constant on $U$.

We recall the characterization of analytic hypoellipticity proved by Baouendi and Treves (Theorem 2.1 in [1]):

**Theorem 4.4.** *The system $\mathcal{L}$ is analytic hypoelliptic at $(x_0, t_0) \in \mathbb{R}^m \times V$ if and only if for every $\xi \in \mathbb{R}^m \setminus \{0\}$, $t_0$ is not a local extremum of the function $t \longmapsto \varphi(t) \cdot \xi$.*

We have:

**Theorem 4.5.** *The system $\mathcal{L}$ satisfies the strong maximum principle on $\Omega = \mathbb{R}^m \times V$ if and only if $\mathcal{L}$ is analytic hypoelliptic on $\Omega$.*

*Proof.* Suppose $\mathcal{L}$ satisfies the strong maximum principle on $\Omega$. Let $(x_0, t_0) \in \Omega$ and $\xi \in \mathbb{R}^m \setminus \{0\}$. The function $u(x, t) = \exp\left(-\sqrt{-1}\xi \cdot \varphi(t)\right)$ is a solution of $\mathcal{L}$ on $\Omega$ and $|u(x, t)| = \exp(\xi \cdot \varphi(t))$. It follows that the function $t \longmapsto \varphi(t)$ cannot have a local extremum at $t_0$. Hence by Theorem 2.1 of [1], $\mathcal{L}$ is analytic hypoelliptic at $(x_0, t_0)$.

Conversely, assume that $\mathcal{L}$ is analytic hypoelliptic on $\Omega$. Let $h(x, t)$ be a solution on some open connected subset $\Omega_1 \subset \Omega$ which for some $(x_0, t_0) \in \Omega_1$ satisfies $|h(x, t)| \leq |h(x_0, t_0)|$ for all $(x, t) \in \Omega_1$. We may assume that $h(x_0, t_0) = 1$. Then for every positive integer $N = 1, 2, \ldots$, the principal branches $(h(x_0, t_0) - h(x, t))^{\frac{1}{N}}$ are solutions on $\Omega_1$ and hence are analytic. This cannot hold unless $h$ is constant on $\Omega_1$. □

## 5. The Strong Maximum Principle for abstract CR manifolds

Let $\mathcal{M}$ be a $C^\infty$ abstract CR manifold of hypersurface type with CR bundle $\mathcal{V}$. The Strong Maximum Principle for $\mathcal{M}$ is defined as in Definition 1.1. In the abstract case, it was shown in [7] that the strong maximum principle is valid when the CR manifold satisfies certain conditions. To describe their result, we will first recall some of their definitions, notations and concepts:

An abstract smooth almost CR manifold of type $(n, k)$ consists of a connected smooth paracompact manifold $\mathcal{M}$ of dimension $2n + k$, a smooth real subbundle $H\mathcal{M}$ of the real tangent bundle $T\mathcal{M}$ of rank $2n$, and a smooth complex structure

$J$ on the fibers of $H\mathcal{M}$. Let $\mathcal{V}$ be the complex subbundle of the complexification $\mathbb{C}H\mathcal{M}$ of $H\mathcal{M}$, which corresponds to the $-\sqrt{-1}$ eigenspace of $J$:

$$\mathcal{V} = \{X + \sqrt{-1}JX : X \in H\mathcal{M}\}.$$

The bundle $\mathcal{V}$ satisfies the formal integrability condition: $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$.

The characteristic bundle $H^0\mathcal{M}$ is defined to be the annihilator of $H\mathcal{M}$ in the real cotangent bundle $T^*\mathcal{M}$. The Levi form at $x \in \mathcal{M}$ is defined for $\xi \in H^0_x\mathcal{M}$ and $X \in H_x\mathcal{M}$ by

$$\mathcal{L}(\xi; X) = d\tilde{\xi}(X, JX) = \langle \xi, [J\tilde{X}, \tilde{X}] \rangle,$$

where $\tilde{\xi} \in C^\infty(\mathcal{M}, H^0\mathcal{M})$ and $\tilde{X} \in C^\infty(\mathcal{M}, H\mathcal{M})$ are smooth extensions of $\xi$ and $X$.

Denote by $H^{1,1}\mathcal{M}$ the smooth subbundle of the tensor bundle $H\mathcal{M} \otimes_{\mathcal{M}} H\mathcal{M}$ whose fiber $H^{1,1}_x\mathcal{M}$ at $x \in \mathcal{M}$ is the real vector subspace of $H_x\mathcal{M} \otimes H_x\mathcal{M}$ generated by the tensors of the form $v \otimes v + (Jv) \otimes (Jv)$ for $v \in H_x\mathcal{M}$. $H^{1,1}\mathcal{M}$ is the bundle of Hermitian symmetric tensors in $H\mathcal{M} \otimes_{\mathcal{M}} H\mathcal{M}$. For each $x \in \mathcal{M}$ and $\xi \in H^0\mathcal{M}$, the Levi form $\mathcal{L}(\xi, \cdot)$ defines a linear form $\mathcal{L}_\xi : H^{1,1}\mathcal{M} \to \mathbb{R}$ such that

$$\mathcal{L}_\xi(v \otimes v + (Jv) \otimes (Jv)) = \mathcal{L}(\xi, v) \quad \forall v \in H_x\mathcal{M}.$$

For $x \in \mathcal{M}$ denote by $\overline{\Gamma} H^{1,1}_x\mathcal{M}$ the convex hull of

$$\{v \otimes v + (Jv) \otimes (Jv);\ v \in H_x\mathcal{M}\}$$

and by $\Gamma H^{1,1}\mathcal{M}$ its interior ($H^{1,1}_x\mathcal{M} \simeq \mathbb{R}^{n^2}$). They are the closed cone of nonnegative Hermitian symmetric tensors and the open cone of positive Hermitian symmetric tensors of $H_x\mathcal{M} \otimes H_x\mathcal{M}$, respectively. The disjoint union $\Gamma H^{1,1}\mathcal{M} = \bigcup_{x \in \mathcal{M}} \gamma H^{1,1}_x\mathcal{M}$ is an open subset of $H^{1,1}\mathcal{M}$.

**Definition 5.1.** We say that the abstract almost CR manifold is essentially pseudoconcave if:

(i) $\mathcal{M}$ is minimal at each of its points;
(ii) for every $x \in \mathcal{M}$ there is an open neighborhood $U$ of $x$ in $\mathcal{M}$ and a smooth section $\Omega \in C^\infty(U, \Gamma H^{1,1}\mathcal{M})$ such that

$$\mathcal{L}_\xi(\Omega) = 0 \quad \forall x \in U, \quad \xi \in H^0_x\mathcal{M}.$$

It is shown in [7] that if $\mathcal{M}$ is essentially pseudoconcave, then for each $\xi \in H^0\mathcal{M}$, the Levi form $\mathcal{L}(\xi, \cdot)$ is either 0 or has at least one positive and one negative eigenvalue. It is also shown in [7] (see Theorem 4.1) that if $\mathcal{M}$ is essentially pseudoconcave, then every $L^2_{\text{loc}}$ CR distribution is $C^\infty$. In fact, essential pseudoconcavity implies a stronger result, namely, a subelliptic estimate for the CR complex (Theorem 4.2 in [7]).

The authors also proved that $\mathcal{M}$ satisfies the strong maximum principle if it is essentially pseudoconcave and of finite type.

**Theorem 5.2.** *(Theorem 6.4 in [7]). Assume that $\mathcal{M}$ is a connected essentially pseudoconcave abstract almost CR manifold of type $(n, k)$ of finite type. Let $u$ be a CR function on $\mathcal{M}$. If $|u|$ has a weak local maximum at some point of $\mathcal{M}$, then $u$ is constant on $\mathcal{M}$.*

In the following result, we show that for $\mathcal{M}$ of hypersurface type, the strong maximum principle holds under weaker assumptions. In particular, we don't assume essential pseudoconcavity, the finite type condition, and the existence of a complex structure $J$.

By the main result in [9], in the embedded case, the $C^\infty-$ hypoellipticity of the tangential Cauchy-Riemann vector fields is equivalent to the holomorphic extendability of all CR distributions. Therefore, the result below may be viewed as an analogue of Proposition 2.2 for abstract CR manifolds.

**Theorem 5.3.** *If every locally integrable CR distribution defined on an open subset is continuous, then $\mathcal{M}$ satisfies the strong maximum principle.*

*Proof.* Suppose every locally defined and locally integrable CR distribution is smooth. Let $U\mathcal{M}$ be a connected open subset and $h$ a continuous CR function on $U$. Assume that for some $p \in U$, $|h(x)| \le |h(p)|$ for all $x \in U$. Without loss of generality, we may assume that $|h(p)| = h(p) = 1$. For $k = 1, 2, \dots,$ let $h_k(x) = h(x)^k$. Then $\{h_k\}$ is a bounded sequence and so by Alaoglu's theorem, there is a subsequence $\{h_{k_j}\}$ that converges to $h$ weakly in $L^2_{\text{loc}}(U)$. Hence $h$ is a locally integrable CR function and so by assumption, it is smooth on $U$. By the Banach-Saks theorem, there is a further subsequence, which by abuse of notation we still denote by $\{h_{k_j}\}$ such that for some $g \in L^2_{\text{loc}}(U)$, the Cesaro means

$$f_N = \frac{1}{N}\sum_{j=1}^{N} h_{k_j} \to g \quad \text{in} \quad L^2_{\text{loc}}(U).$$

This in turn implies that there is a subsequence $\{f_{N_j}\}$ such that

$$f_{N_j}(x) \to g(x) \quad \text{pointwise on a dense subset of } U.$$

Since $g$ is continuous, it follows that $|g(x)| \le 1$ for all $x \in U$. Suppose $q \in U$ such that $|h(q)| < 1$. Let $|h(q) < \delta < 1$. For any positive integer $m$, we have

$$|f_N(q)| \le \frac{1}{N}\sum_{j=1}^{m} |h_{k_j}(q)| + \frac{1}{N}\sum_{j=m+1}^{N} |h_{k_j}(q)| \le \frac{m}{N} + \frac{\delta^m}{N}\left(\frac{1}{1-\delta}\right).$$

Letting $N$ tend to $\infty$, we see that $g(q) = 0$. Thus at any point $x \in U$, $|g(x)| = 0$ or $|g(x)| = 1$. Since $g(p) = 1$, by continuity and the connectedness of $U$ we conclude that $|g(x)| \equiv 1$ on $U$. We next observe that each point of $U$ (and hence of $\mathcal{M}$) is minimal. To see this, suppose $q$ is a point in $U$ and the Sussmann orbit $\Sigma$ of $q$ in $U$ is is of dimension $2n$. Let $\Sigma' \subset \Sigma$ be an embedded neighborhood of $q$. Then the locally integrable function which is defined to be 1 on one side of $\Sigma'$ and 0 on the other side is a CR function which contradicts the hypothesis. Thus each point is minimal. Write $g(x) = u(x) + \sqrt{-1}v(x)$ where $u$ and $v$ are the real and imaginary parts. If $L = X + \sqrt{-1}Y$ is a smooth section of $\mathcal{U}$, since $u(x)^2 + v(x)^2 \equiv 1$, we have $u(Lu) + v(Lv) \equiv 0$ which together with the equation $L(u + \sqrt{-1}v) = 0$ leads to

$$Lv \equiv 0, \quad \text{and hence} \quad Xv \equiv 0 \equiv Yv.$$

Likewise,

$$Lu \equiv 0, \quad \text{and hence} \quad Xu \equiv 0 \equiv Yu.$$

Define

$$S = \{x \in U : u \quad \text{is constant in a neighborhood of } x.$$

Clearly, the set $S$ is open. Let $\mathcal{O}_p$ be the orbit of $p$ in $U$, and let $q \in \mathcal{O}_p$. Then there is a path $\gamma$ in $U$ from $p$ to $q$ that consists of integral curves of real parts of smooth sections of $\mathcal{V}$. Without loss of generality, assume that $\gamma$ is an integral curve of $X = \Re L$ where $L$ is a smooth section of $\mathcal{V}$. Then since $Xu \equiv 0$ on $U$, the function $u$ has to be constant on $\gamma$, and hence, on all of $\mathcal{O}_p$. Therefore, by minimality, $p \in S$. This argument also shows that $S$ is closed and hence $S = U$, proving that $u$, and hence $v$ are constant on $U$.

□

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