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ABSTRACT

Recently, physical billiards were introduced where a moving particle is a hard sphere rather than a point as in standard mathematical billiards. It has been shown that in the same billiard tables, the physical billiards may have totally different dynamics than mathematical billiards. This difference appears if the boundary of a billiard table has visible singularities (internal corners if the billiard table is two-dimensional); i.e., the particle may collide with these singular points. Here, we consider the collision of a hard ball with a visible singular point and demonstrate that the motion of the smooth ball after collision with a visible singular point is indeed the one that was used in the studies of physical billiards. Therefore, such collision is equivalent to the elastic reflection of hard ball's center off a sphere with the center at the singular point and the same radius as the radius of the moving particle. However, a ball could be rough, not smooth. In difference with a smooth ball, a rough ball also acquires rotation after reflection off a point of the boundary, which leads to more complicated dynamics.

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A standard (mathematical) billiard is a dynamical system generated by a motion by inertia of a point particle within a closed region (billiard table) and its elastic collisions off the boundary. If one considers instead a motion of a hard ball (physical particle) in the same billiard table, then dynamics may drastically change. Specifically, any type of transition from regular to chaotic dynamics may occur and vice versa. Moreover, these transitions may occur as soft (i.e., for any positive radius of the moving ball) as well as hard ones (i.e., for sufficiently large balls). All these transitions occur only in the presence of singularities in the boundary of a billiard table. Here, we describe which types of singularities may result in a change of dynamics in transition from a point particle to a hard ball. We also discuss possible types of collisions of a hard ball with a point, which are consistent with the laws of mechanics. It is shown that the collision law for a smooth hard ball, used so far in the studies of physical billiards, is indeed consistent (satisfies) to mechanical conservation laws respected by collisions. It is also shown that a rough ball collision law, when a ball acquires rotation after a collision, is also consistent with the laws of mechanics.

I. INTRODUCTION

Mathematical billiards serve as relevant models of various phenomena in mechanics, geometric optics and acoustics, statistical physics, and quantum physics. Such billiards also constitute one of the most popular and arguably the most visual class of dynamical systems in the mathematical studies. In mathematical billiards, a point particle moves by inertia in a domain with a boundary. When the point particle reaches the boundary, it gets elastically reflected.

Recently, physical billiards were introduced where the moving particle is a hard ball.¹ It was shown in this paper that in the transition from a mathematical to a physical billiard in the same billiard table, any type of transition from chaotic to regular dynamics and vice versa may occur. Moreover, such transition from the point to a finite size particle can completely change the dynamics of some classical and well-studied models such as, e.g., the Ehrenfests' Wind-Tree model.² In quantum systems, a "particle" naturally has a finite size due to the uncertainty principle, which leads to some new findings in the quantum chaos theory.^{3,4}

It is worthwhile to mention that the most celebrated billiard system is a gas of hard balls (Boltzmann gas) where moving particles

are indeed hard balls. However, this system can be reduced to a mathematical billiard within a billiard table with a specific boundary. In fact, studies of dynamics of the Boltzmann gas inspired Ya. G. Sinai to introduce famous Sinai billiards, which made the foundation for the theory of chaotic billiards. However, the dynamics of Sinai billiards and other most famous mathematical billiards does not change after the transition to physical billiards in the same billiards tables. In fact, changes in dynamics occur only when the boundary of a billiard table has a visible singularity, i.e., a point in the intersection of two or more smooth components of the boundary such that a small enough physical particle can hit that point of the boundary. If a billiard table is two-dimensional, then such singularities are internal corners where two smooth components of the boundary intersect and make an angle greater than π inside the billiard table. In all the papers cited above, it was assumed that reflection of the ball off such visible singularity occurs in a natural manner corresponding to the simplest elastic collision. In the present note, we justify this assumption for a smooth hard ball. It is worthwhile to mention that there are other types of reflection of a ball off a visible singular point that correspond to a rough ball, which may acquire rotation after such collision⁵ even under the assumption that it is a no-slip collision.^{6,7} We demonstrate that both types of reflection, for a smooth ball and a hard ball, satisfy the (conservation) laws of mechanics. Therefore, physical billiards generated by the motion of a smooth or a rough ball can be considered natural realistic dynamical systems.

II. DIFFERENT TYPES OF BOUNDARY SINGULARITIES IN BILLIARD TABLES

Let Q be a domain in d -dimensional Euclidean space $\mathbf{R}^d$ such that its boundary ∂Q is the union of a finite number of C^1 -smooth $(d-1)$ -dimensional manifolds. A point q of the boundary ∂Q is called singular if the boundary is not differentiable at that point. This means that a singular point belongs to the intersection of some (at least two) differentiable (aka regular) components of the boundary. Note that we also call a singular point in dimension two (i.e., $\dim Q = 2$) a corner. All non-singular points of the boundary ∂Q are called regular points.

Consider a free motion of a hard ball (a disk in $\dim 2$) of radius $r > 0$ in the domain Q with elastic reflections off the boundary ∂Q . The resulting dynamical system is called a physical billiard¹ and the domain Q a billiard table. To describe the dynamics of such a ball, it is enough to follow the motion of its center. It is easy to see that the center of the ball moves in the smaller billiard table, which one gets by moving any point q of the boundary by r to the interior of the billiard table along the internal unit normal vector $n(q)$.¹

We will call a singular point q of the boundary ∂Q an invisible singular point if for any $r > 0$, the hard ball of radius r cannot hit that point. Otherwise, a singular point is called a visible singular point. Therefore, q is a visible singular point of a billiard table if a ball with a sufficiently small radius can hit q . A formal mathematical definition of a visible singular point (in any dimension) is the following one. A singular point A is a visible singular point if for any neighborhood N of A , the convex hull of $Q \cap N$ contains a neighborhood of A .

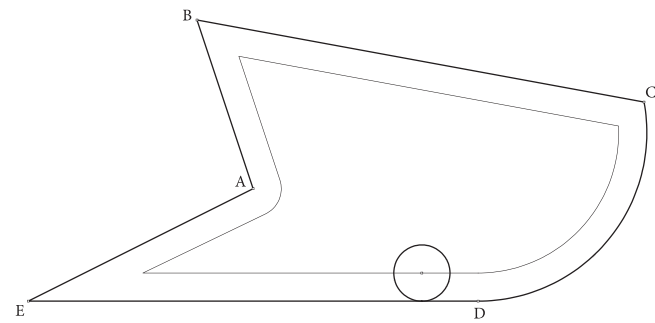


FIG. 1. Corners (singular points) B, C, and E are invisible to any disk. The point D is not singular since the boundary is differentiable at D. The corner A is an internal corner (a visible singular point).

We also call a visible singular point in dimension two an internal corner. For example, Fig. 1 shows visible and invisible singular points in dimension two. Note that being a visible singular point (an internal corner) does not mean that a hard ball of any radius $r > 0$ can reach (hit) that point. Specifically, if the radius of the particle is larger than some constant (which depends on the shape of a billiard table), then some visible singular points become invisible (see Fig. 2). Observe that at the moment of collision with a visible singular point, the center of the hard ball can be at different positions, and these possible positions depend on the shape of the boundary ∂Q (see Fig. 3). This should be contrasted with the collision of the ball off the boundary at a regular point, when the center of the ball always has one position, namely, at the distance r on the internal normal line to the boundary of a billiard table. In Fig. 3, two situations are depicted, which may happen in three dimensional billiard tables.

Since the particle is a hard ball, it will keep its shape at the moment of collision. Hence, the center of the hard ball is at the distance r from a collision point (regardless of whether this point is a regular or singular point of the boundary). Therefore, the boundary of the reduced billiard table of the mathematical billiard, which has the same dynamics as the considered physical billiard,¹ acquires a piece of a sphere (or an arc of a circle if the dimension of the billiard table is two) of radius r with the center at the visible singular point. Hence, the reduced billiard table of the equivalent mathematical billiard has a dispersing component in the boundary, which generates a chaotic (hyperbolic) dynamics in case if a moving particle is a hard ball.

In Fig. 4, it is easy to see that the boundary of the reduced billiard table of the mathematical billiard acquires a dispersing

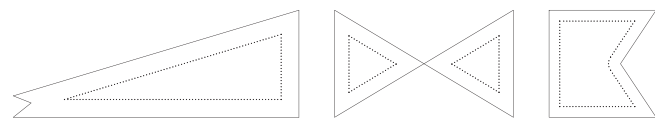


FIG. 2. An internal corner becomes invisible when the radius of the disk is larger than some constant.

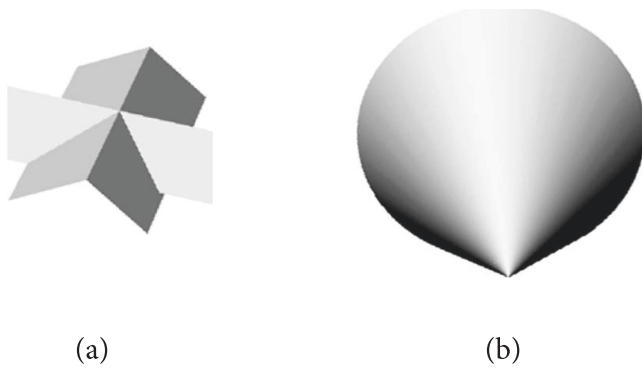


FIG. 3. (a) There are two lines of visible singular points. When a hard ball hits a point on those lines, its center is on an arc of a circle centered at that point and orthogonal to the corresponding line. However, at the moment of collision with the intersection point of those two lines, the center of the hard ball can be only in one position. (b) Here is one isolated visible singular point. At the moment of collision with such singularity, the center of a hard ball is on a piece of a two-sphere centered at that singular point.

component because of the case of dimension two depicted. Here, the center of a disk can be located at any point of an arc of the circle with the center at the singular point and with the radius being equal to the radius of the disk.

The fact that a reduced billiard table of the equivalent mathematical billiard acquires a dispersing (or semi-dispersing) component holds true for any types of collisions of the physical ($r > 0$) particle with the boundary at a visible singular point. However, such collisions can generally be elastic or inelastic (i.e., the angle of incident is equal to the angle of reflection or not) and with or without slip (i.e., the particle slips on the boundary at reflection moment or only touches the boundary at a single point).^{5,7} Dynamics of a rough ball even in the case of no-slip collisions is much more complicated than the dynamics of a smooth ball.

It is also worthwhile to mention that if a billiard table has an internal visible corner, in the transition from a mathematical to a physical billiard, the boundary of a reduced billiard table becomes smoother than the boundary of the initial billiard

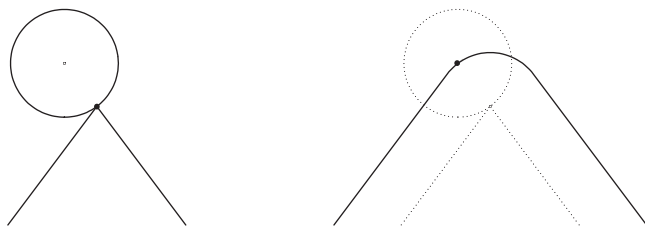


FIG. 4. A collision between a disk and a visible singular point (here, an internal corner) is shown in the left picture. On the right, one can see its equivalent for a (virtual) collision between disk's center and an arc of a circle centered at the internal corner with the same radius as the disk's radius.

table. Indeed, instead of a corner, there appears an arc of a circle that has common tangents with neighboring components of the boundary. However, this smoothening does not influence the dynamics. Dynamics gets changed because of the appearance of a (new) dispersing component of the boundary, which makes it more chaotic.

III. NO-SLIP COLLISIONS OF A HARD BALL WITH A VISIBLE SINGULAR POINT

In the case of no-slip collisions, each reflection of the moving particle (hard ball) off the boundary occurs at a single point. Hence, a collision at any point of the boundary does not depend on the shape of the boundary elsewhere. Therefore, the collision problem can be actually considered a reflection of a hard ball off a point.

At the moment of the collision, the impulse ΔP decomposes into two components, which are the normal impulse ΔP_N acting toward the center of the hard ball and the tangent impulse ΔP_T based on friction, which is tangent to the hard ball at the collision point. The tangent impulse can result in either loss of kinetic energy or exchange between linear and angular momentum while the total kinetic energy is preserved. We will consider the friction-free (elastic) collision and the case when the impulse ΔP_T results in an exchange between linear and angular momentum without loss of energy. In other words, we consider only conservative (Hamiltonian) dynamics.

Let a hard ball of radius $r > 0$ with the center at a point O hit a visible singular point A of the boundary of a billiard table Q . Without any loss of generality, we assume that the mass of the hard ball is $m = 1$. Denote the linear velocity of hard ball's center just before (after) the collision by V^b (V^a). Consider now a decomposition of V^b to two components V_N^b and V_T^b , where $V_N^b = \text{Proj}_{\vec{OA}} V^b$ and $V_T^b = V^b - V_N^b$. Note that we will use the superscript a instead of b to denote velocity components at a moment of time right after the reflection. Denote also the vector form of angular velocity just before (after) the collision about the point O by ω^b (ω^a).

The collision map S at point A will map linear components and the angular component of the velocity just before collision (V_N^b, V_T^b, ω^b) to those right after collision (V_N^a, V_T^a, ω^a). The map S has the following properties:

1. The map S is an orthogonal map because of the assumption that the system in question is Hamiltonian.
2. Because of time reversibility of dynamics, S^2 is the identity map.
3. The normal component of the linear velocity with respect to the boundary of the hard ball at the contact point A (i.e., V_N^b) always reverses under the map S .

Conditions 1 and 2 imply that the eigenvalues of the map S are 1 or -1 . In view of condition 3, one gets $S(V_N^b, V_T^b, \omega^b) = (-V_N^b, V_T^b, \omega^b)$, or equivalently, $V = (V_N^b, \vec{0}, \vec{0})$ is an eigenvector of S corresponding to the eigenvalue -1 . It also implies that $\Delta P_N = -2V_N^b$.

The Hamiltonian system under consideration satisfies three conservation laws of the kinetic energy K , the linear momentum P , and the angular momentum L about the point O . These conservation laws in dimension 3 are given by the relations

$$\begin{aligned} K^b &= \frac{1}{2} (|V_N^b|^2 + |V_T^b|^2 + I|\omega^b|^2) \\ &= \frac{1}{2} (|V_N^a|^2 + |V_T^a|^2 + I|\omega^a|^2) = K^a, \\ P^b + \Delta P &= V_N^b + V_T^b + \Delta P_N + \Delta P_T = V_N^a + V_T^a = P^a, \quad (1) \\ L^b + \Delta P_T \times \vec{AO} &= I\omega^b + \Delta P_T \times \vec{AO} = I\omega^a = L^a, \end{aligned}$$

where I is the moment of inertia of the hard ball.

Using that $V_N^a = -V_N^b$ and $\Delta P_N = -2V_N^b$, one can simplify (1) as

$$\begin{aligned} |V_T^b|^2 + I|\omega^b|^2 &= |V_T^a|^2 + I|\omega^a|^2, \\ V_T^b + \Delta P_T &= V_T^a, \quad (2) \\ I\omega^b + \Delta P_T \times \vec{AO} &= I\omega^a. \end{aligned}$$

In the following, we consider two cases about the impulse ΔP_T : friction-free collision (i.e., $\Delta P_T = \vec{0}$) or collisions with friction (i.e., $\Delta P_T \neq \vec{0}$). By solving (2) for ΔP_T , we get

$$\left\langle \Delta P_T, \frac{r^2 + I}{I} \Delta P_T + 2V_T^b + 2\vec{AO} \times \omega^b \right\rangle = 0, \quad (3)$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbf{R}^3$ and r is the radius of the hard ball.

Observe that the conservation laws in dimension 2 are the same as in (1) under the assumption that the billiard table Q is a subset of the xy -plane in $\mathbf{R}^3$.

A. Friction-free collision (a smooth ball)

In this section, we study a friction-free (i.e., $\Delta P_T = \vec{0}$) Hamiltonian system. In this case, (2) implies

$$V_T^a = V_T^b, \quad \omega^a = \omega^b.$$

Here, the solution $(V_N^a, V_T^a, \omega^a) = (-V_N^b, V_T^b, \omega^b)$ of (1) corresponds to the case of a smooth hard ball⁵ when the ball does not acquire rotation upon collision. Thus, in this case, we have an elastic reflection where the angle of incidence is equal to the angle of reflection.

Also, this friction-free collision is equivalent to the elastic reflection of the hard ball's center O off a piece of a two-sphere (it can be an arc of a circle) centered at the visible singular point A with the same radius as the radius of the hard ball.^{1,2}

In case of dimension 3, the collision map S is a linear map from a six-dimensional subspace of $\mathbf{R}^9$ to itself with eigenvalues 1 and -1 . When $\Delta P_T = \vec{0}$, the eigenvectors that correspond to these eigenvalues have the forms $(\vec{0}, V_T^b, \omega^b)$ and $(c\vec{AO}, \vec{0}, \vec{0})$, respectively, where c is a constant. Also, the eigenspaces corresponding to the eigenvalues 1 and -1 have dimensions five and one, respectively.

B. Collisions with friction (a rough ball)

For the Hamiltonian system under consideration, the presence of the frictional force means that $|\Delta P_T| \neq 0$. The corresponding solution of (1) when $|\Delta P_T| \neq 0$ describes the dynamics of a rough ball,⁵ which has no-slip ultra-elastic reflections off the boundary. (After an ultra-elastic reflection, the ball will acquire rotation; therefore, the incident angle is not equal to the reflection angle.⁵) In this case, the tangential component of the linear velocity partially transfers to the angular velocity and vice versa.

A nontrivial solution for ΔP_T in (3) is given by

$$\Delta P_T = -\frac{2I}{r^2 + I} (V_T^b + \vec{AO} \times \omega^b). \quad (4)$$

Let S be the collision map in dimension 3 when the tangent impulse ΔP_T is given by (4). Then, $(\vec{0}, V_T^b, \omega^b)$ is an eigenvector of the collision map S corresponding to the eigenvalue 1 if $V_T^b + \vec{AO} \times \omega^b = \vec{0}$. The solution set of the vector equation $V_T^b + \vec{AO} \times \omega^b = \vec{0}$ is a three dimensional space. Hence, the eigenspace corresponding to eigenvalue 1 of the collision map S is a three-dimensional space.

Moreover, $(\vec{0}, V_T^b, \omega^b)$ is an eigenvector of the collision map S , which corresponds to the eigenvalue -1 if $V_T^b \times \vec{AO} - I\omega^b = \vec{0}$. In this case, the solution set of the vector equation $V_T^b \times \vec{AO} - I\omega^b = \vec{0}$ is a two-dimensional space. This implies that the eigenspace corresponding to the eigenvalue -1 of the collision map S is a three-dimensional space [we know that $(\vec{AO}, \vec{0}, \vec{0})$ is another eigenvector for eigenvalue -1].

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DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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