Commun. Math. Phys. 390, 113–168 (2022)
Digital Object Identifier (DOI) https://doi.org/10.1007/s00220-022-04309-w

Communications in
**Mathematical
Physics**

Poisson Approximations and Convergence Rates for Hyperbolic Dynamical Systems

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Received: 22 March 2021 / Accepted: 2 January 2022
Published online: 19 January 2022

Abstract: We prove the asymptotic functional Poisson laws in the total variation norm and obtain estimates of the corresponding convergence rates for a large class of hyperbolic dynamical systems. These results generalize the ones obtained before in this area. Applications to intermittent solenoids, Axiom A attractors, Hénon attractors and to billiards, are also considered.

Contents

1. Introduction

The studies of Poisson approximations of the process of recurrences to small subsets in the phase spaces of chaotic dynamical systems, started in [40], are developed now into a large active area of the dynamical systems theory. Another view at this type of problems is a subject of the theory of open dynamical systems [39], where some positive measure subset A of the phase space is named a hole, and hitting and escape the hole processes are studied. The third view at this type of problems concerns statistics of extreme events ("record values") in the theory of random processes [23,25,29,38]. In this paper we present new advances in this area.

In a general set up, one picks a small measure subset A in the phase space $\mathcal{M}$ of hyperbolic (chaotic) ergodic dynamical system and attempts to prove that in the limit, when the measure of A approaches zero, the corresponding process of recurrences to A converges to the Poisson process.

This area received an essential boost in L-S.Young papers [45,46], where a new general framework for analysis of statistical properties of hyperbolic dynamical systems was introduced. This approach employs representation of the phase space of a dynamical system as a tower (later called Young tower, Gibbs-Markov-Young tower, etc), which allow to study dynamics by analysing recurrences to the base of this tower. Several developments of this method were proposed later, essentially all focused on the dynamical systems with weak hyperbolicity (slow decay of correlations). For such systems the method of inducing was employed, when the base of the tower is chosen as such subset of the phase space where the induced dynamics, generated by the recurrences to the base, is strongly hyperbolic [17,18,34].

Our approach to the Poisson approximations is slightly different. It employs pulling back a hole A to a nice (strongly hyperbolic) reference set in the phase space, e.g., the base of the Young tower. This pull back method gives a new approach to two main challenges for Poisson approximations: short returns and coronas (see Definition 7) which were introduced and studied in [16,37]. It allows to improve various results previously obtained in this area.

The main results (Theorems 3 and 4) of the paper are dealing with convergence of a random process, generated by the measure preserving dynamics, to the functional Poisson law in the total variation (TV) norm. We also obtain estimates of the corresponding convergence rates in the following form: for almost every $z \in \mathcal{M}$,

$$d_{TV}\left(N^{r,z,T}, P\right) \precsim_{T,z} r^a, \tag{1.1}$$

where P is a Poisson point process and $N^{r,z,T}$ is a dynamical point process which counts a number of entrances by an orbit to a metric ball $B_r(z)$ with radius r and the center z in the phase space of a dynamical system during the time interval $[0, T]$. The notation $\precsim_{T,z}$ means that a constant in (1.1) depends only on z and T (see Definition 1 for more details).

These results on convergence to the Poisson distribution are stronger than the ones obtained previously [16,22,26,29,30,37]. Namely

1. In [22,26,29] the following forms of convergence were obtained: for almost every $z \in \mathcal{M}$

   $$\lim_{r\to 0} \mathbb{P}\left\{N^{r,z,T}([0,T]) = k\right\} = \mathbb{P}\{P([0,T]) = k\}$$

   and/or when $r \to 0$,

   $$\left(N^{r,z,T}(I_1), \ldots, N^{r,z,T}(I_m)\right) \to_d (P(I_1), \ldots, P(I_m)),$$

   where $m \geq 1, k \geq 0$ and intervals $I_1, \ldots I_m \subseteq [0, T]$. Clearly, (1.1) implies these two forms.
2. In [22,26,29,37] only convergence to the Poisson law was considered, while the estimations of the convergence rates were not studied because the approaches used there did not allow for such estimates.

3. In [16,30] convergence rates were obtained in a weaker form. Namely, for any $r \in (0, 1)$ there exist positive constants a, b and a set $M_r \subseteq \mathcal{M}$ with $\mathbb{P}(M_r) \leq r^b$ such that for any $z \notin M_r$

$$\sum_k \left| \mathbb{P}\left\{ N^{r,z,T}([0,T]) = k \right\} - \mathbb{P}\{P([0,T]) = k\} \right| \precsim r^a. \tag{1.2}$$

Besides just mentioned generalizations and strengthens of previous results, we also obtain results under weaker conditions than the ones that were used previously. Namely,

1. In [16,30] a relatively high regularity (at least bounded derivatives) of dynamics was required, while we just need it to be a local C^1-diffeomorphism. Particularly, derivatives can be unbounded. Note also that the results of [16] require a bounded derivative and therefore are not applicable to dispersing billiards.
2. Unlike [16], we do not assume that unstable manifolds are one-dimensional.
3. In [16,29,30,37] sufficiently fast decay rates of return times on hyperbolic towers were required. Our proofs of the existence of the Poisson limit laws use only polynomial contraction (expansion) rate α on unstable and stable manifolds. Particularly, a simple (easy to verify) criterion for existence of the Poisson limit law is obtained:

$$\alpha > C_{\dim \gamma^u, \dim_H \mu},$$

where a constant $C_{\dim \gamma^u, \dim_H \mu}$ only depends on the dimension $\dim \gamma^u$ of unstable manifolds γ^u and on the Hausdorff dimension $\dim_H \mu$ of the SRB measure μ on a Gibbs–Markov–Young tower (see the details in Theorems 3 and 4).
This criterion allows to skip verification of the so called corona conditions (see Definition 7 or [37]), which is usually rather cumbersome even for uniformly hyperbolic dynamical systems. Such verification becomes even more involved for non-uniformly hyperbolic systems.

Now we briefly describe main theorems and applications considered in the paper. Theorem 3 deals with the systems which can be modelled by Young towers with the first return times. We apply it to smooth dynamical systems studied in [37], i.e., to Axiom A attractors and intermittent solenoids. For systems which can not be modelled by first return Young towers, our Theorem 4 gives different criteria. We apply it to non-uniformly hyperbolic dynamical systems studied in [16,37], i.e. to Billiards and Hénon attractors. Our results improve various previously known ones for these classes of dynamical systems.

Finally, it is worthwhile to mention that convergence to compound Poisson distributions was studied in [23,28] for periodic points $z \in \mathcal{M}$ of hyperbolic dynamical systems. We do not consider such limit laws in the present paper.

The structure of the paper is the following. In Sect. 2 we introduce notations, give necessary definitions and formulate main results. Section 3 presents a proof of the functional Poisson law (with the error term) for systems admitting Young towers of general type. Section 4 contains a proof of Theorem 3. A proof of Theorem 4 is in section 5. Applications to Axiom A attractors, intermittent solenoids, billiards and Henon attractors are considered in Sect. 6.

2. Definitions and Main Results

We start by introducing some notations and conventions

1. C_z denotes a constant depending on z.
2. The notation "$a_n \precsim_z b_n$" ("$a_n = O_z(b_n)$") means that there is a constant $C_z \geq 1$ such that (s.t.) $a_n \leq C_z b_n$ for all $n \geq 1$, whereas the notation "$a_n \precsim b_n$" (or "$a_n = O(b_n)$") means that there is a constant $C \geq 1$ such that $a_n \leq C b_n$ for all $n \geq 1$. Next, "$a_n \approx_z b_n$" and $a_n = C_z^{\pm 1} b_n$ mean that there is a constant $C_z \geq 1$ such that $C_z^{-1} b_n \leq a_n \leq C_z b_n$ for all $n \geq 1$. Further, the notation "$a_n \approx b_n$" means that there is a constant $C \geq 1$ such that $C^{-1} b_n \leq a_n \leq C b_n$ for all $n \geq 1$. Finally, "$a_n = o(b_n)$" means that $\lim_{n\to\infty} |a_n/b_n| = 0$.
3. The notation $\mathbb{P}$ refers to a probability distribution on the probability space, where a random variable lives, and $\mathbb{E}$ denotes the expectation of a random variable.
4. By $\mathbb{1}_A$ we denote the characteristic function of a measurable set A.
5. $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$

Definition 1 *(Dynamical point processes).* Let $(\mathcal{M}, d)$ be a Riemannian manifold (with or without boundaries, connected or non-connected, compact or non-compact), d is the Riemannian metric on $\mathcal{M}$ and $B_r(z)$ is a geodesic ball in $\mathcal{M}$ with a radius r and a center $z \in \mathcal{M}$. We assume that dynamics $f : (\mathcal{M}, \mu) \to (\mathcal{M}, \mu)$ is ergodic with respect to (w.r.t.) some invariant probability measure μ.

Let $T > 0$. Consider a dynamical point process on $[0, T]$, so that for any $t \in [0, T]$

$$N_t^{r,T,z} := \sum_{i=0}^{t/\mu(B_r(z))} \mathbb{1}_{B_r(z)} \circ f^i.$$

Thus the dynamical point process $N^{r,T,z}$ is a random counting measure on $[0, T]$.

Definition 2 *(Poisson point processes).* For any $T > 0$, we say that P is a Poisson point process on $[0, T]$ if

1. P is a random counting measure on $[0, T]$.
2. $P(A)$ is a Poisson-distributed random variable for any Borel set $A \subseteq [0, T]$.
3. If $A_1, A_2, \ldots, A_n \subseteq [0, T]$ are pairwise disjoint, then $P(A_1), \ldots, P(A_n)$ are independent.
4. $\mathbb{E}P(A) = \text{Leb}(A)$ for any Borel set $A \subseteq [0, T]$.

Definition 3 *(Total variation norms of point processes).* For any $T > 0$ consider the σ-algebra $\mathcal{C}$ on the space of point processes on $[0, T]$, defined as

$$\sigma\left\{\pi_A^{-1} B : A \subseteq [0, T], B \subseteq \mathbb{N}\right\}, \tag{2.1}$$

where A, B are Borel sets and π_A is an evaluation map defined on the space of counting measures, so that for any counting measure N

$$\pi_A N := N(A).$$

Now we can define the total variation norm for the Poisson approximation of a dynamical point process as

$$d_{TV}\left(N^{r,T,z}, P\right) := \sup_{C \in \mathcal{C}} \left|\mu(N^{r,T,z} \in C) - \mathbb{P}(P \in C)\right|$$

Remark 1. The total variation norm in [16,37] is actually

$$\sup_{C \subseteq \mathbb{N}} \left| \mu(N^{r,T,z}[0,T] \in C) - \mathbb{P}(P[0,T] \in C) \right|.$$

Obviously, our total variation norm is stronger and gives more information, for example, for any sub-interval $[T_1, T_2] \subseteq [0, T]$,

$$\sup_{C \subseteq \mathbb{N}} \left| \mu(N^{r,T,z}[T_1,T_2] \in C) - \mathbb{P}(P[T_1,T_2] \in C) \right| \leq d_{TV}\left(N^{r,T,z}, P\right).$$

Definition 4 *(Convergence rates of Poisson approximations).* Suppose that for any $T > 0$ there exists a constant $a > 0$ s.t. for almost every $z \in \mathcal{M}$

$$d_{TV}\left(N^{r,T,z}, P\right) \precsim_{T,z} r^a \to 0.$$

Then a is called a convergence rate of a Poisson approximation.

Remark 2. Our convergence rates imply that for any sub-interval $[T_1, T_2] \subseteq [0, T]$,

$$\sup_{C \subseteq \mathbb{N}} \left| \mu(N^{r,T,z}[T_1,T_2] \in C) - \mathbb{P}(P[T_1,T_2] \in C) \right| \precsim_{T,z} r^a \to 0.$$

We now turn to the definition of the Gibbs-Markov-Young structures [2,45,46]:

Definition 5 *(Gibbs-Markov-Young structures).* Introduce at first several notions concerning hyperbolic dynamics f on Riemannian manifolds $(\mathcal{M}, d)$.

1. An embedded disk γ^u is called an unstable manifold if for every $x, y \in \gamma^u$

$$\lim_{n\to\infty} d\left(f^{-n}(x), f^{-n}(y)\right) = 0$$

2. An embedded disk γ^s is called a stable manifold if for every $x, y \in \gamma^s$

$$\lim_{n\to\infty} d\left(f^{n}(x), f^{n}(y)\right) = 0$$

3. $\Gamma^u := \{\gamma^u\}$ is called a continuous family of C^1-unstable manifolds if there is a compact set K^s, a unit disk D^u in some $\mathbb{R}^n$ and a map $\phi^u : K^s \times D^u \to \mathcal{M}$ such that
   (a) $\gamma^u = \phi^u(\{x\} \times D^u)$ is an unstable manifold,
   (b) ϕ^u maps $K^s \times D^u$ homeomorphically onto its image,
   (c) $x \to \phi^u|_{\{x\}\times D^u}$ defines a continuous map from K^s to $\mathrm{Emb}^1(D^u, \mathcal{M})$, where $\mathrm{Emb}^1(D^u, \mathcal{M})$ is the space of C^1-embeddings of D^u into $\mathcal{M}$.

A continuous family of C^1-stable manifolds $\Gamma^s := \{\gamma^s\}$ is defined similarly.

We say that a compact set $\Lambda \subseteq \mathcal{M}$ has a hyperbolic product structure if there exist continuous families of stable manifolds $\Gamma^s := \{\gamma^s\}$ and of unstable manifolds $\Gamma^u := \{\gamma^u\}$ such that

1. $\Lambda = \left(\bigcup \gamma^s\right) \bigcap \left(\bigcup \gamma^u\right)$,
2. $\dim \gamma^s + \dim \gamma^u = \dim \mathcal{M}$,
3. each γ^s intersects each γ^u at exactly one point,

4. stable and unstable manifolds are transversal, and the angles between them are uniformly bounded away from 0.

A subset $\Lambda_1 \subseteq \Lambda$ is called a s-subset if Λ_1 has a hyperbolic product structure and, moreover, the corresponding families of stable and unstable manifolds Γ_1^s and Γ_1^u can be chosen so that $\Gamma_1^s \subseteq \Gamma^s$ and $\Gamma_1^u = \Gamma^u$.

Analogously, a subset $\Lambda_2 \subseteq \Lambda$ is called an u-subset if Λ_2 has a hyperbolic product structure and the families Γ_2^s and Γ_2^u can be chosen so that $\Gamma_2^u \subseteq \Gamma^u$ and $\Gamma_2^s = \Gamma^s$.

For $x \in \Lambda$, denote by $\gamma^u(x)$ (resp. $\gamma^s(x)$) the element of Γ^u (resp. Γ^s) which contains x. Also, for each $n \geq 1$, denote by $(f^n)^u$ the restriction of the map f^n to γ^u-disks, and by $\det D\,(f^n)^u$ denote the Jacobian of $(f^n)^u$.

We say that the set Λ with hyperbolic product structure has also a **Gibbs-Markov-Young structure** if the following properties are satisfied

1. Lebesgue detectability: there exists $\gamma \in \Gamma^u$ such that $\mathrm{Leb}_\gamma\left(\Lambda \bigcap \gamma\right) > 0$.
2. Markovian property: there exist pairwise disjoint s-subsets $\Lambda_1, \Lambda_2, \ldots \subseteq \Lambda$ such that
   (a) $\mathrm{Leb}_\gamma\left(\Lambda \setminus \left(\bigcup_{i\geq 1} \Lambda_i\right)\right) = 0$ on each $\gamma \in \Gamma^u$,
   (b) for each $i \geq 1$ there exists $R_i \in \mathbb{N}$ such that $f^{R_i}(\Lambda_i)$ is an u-subset, and for all $x \in \Lambda_i$

$$f^{R_i}\left(\gamma^s(x)\right) \subseteq \gamma^s\left(f^{R_i}(x)\right)$$

and

$$f^{R_i}\left(\gamma^u(x)\right) \supseteq \gamma^u(f^{R_i}(x)).$$

Define now a return time function $R : \Lambda \to \mathbb{N}$ and a return function $f^R : \Lambda \to \Lambda$, so that for each $i \geq 1$

$$R\big|_{\Lambda_i} = R_i \text{ and } f^R\big|_{\Lambda_i} = f^{R_i}\big|_{\Lambda_i}$$

The separation time $s(x, y)$ for $x, y \in \Lambda$ is defined as

$$s(x, y) := \min\{n \geq 0 : \left(f^R\right)^n(x) \text{ and } \left(f^R\right)^n(y) \text{ belong to the different sets } \Lambda_i\}.$$

We also assume that there are constants $C > 1$, $\alpha > 0$ and $0 < \beta < 1$, which depend only on f and Λ, such that the following conditions hold

3. Polynomial contraction on stable leaves: for any $\gamma^s \in \Gamma^s$, $x, y \in \gamma^s$, $n \geq 1$,

$$d\Big(f^n(x), f^n(y)\Big) \leq Cn^{-\alpha}.$$

4. Backward polynomial contraction on unstable leaves: for any $\gamma^u \in \Gamma^u$, $x, y \in \gamma^u$, $n \geq 1$,

$$d\Big(f^{-n}(x), f^{-n}(y)\Big) \leq Cn^{-\alpha}.$$

5. Bounded distortion: for any $\gamma \in \Gamma^u$ and $x, y \in \gamma \bigcap \Lambda_i$ for some Λ_i,

$$\log \frac{\det D\left(f^R\right)^u(x)}{\det D\left(f^R\right)^u(y)} \leq C\beta^{s\left(f^R(x), f^R(y)\right)}.$$

6. Regularity of the stable foliations: for each $\gamma, \gamma' \in \Gamma^u$ denote

$$\Theta_{\gamma,\gamma'} : \gamma' \bigcap \Lambda \to \gamma \bigcap \Lambda : x \to \gamma^s(x) \bigcap \gamma.$$

Then the following properties hold

(a) $\Theta_{\gamma,\gamma'}$ is absolutely continuous and for any $x \in \gamma \bigcap \Lambda$

$$\frac{d\left(\Theta_{\gamma,\gamma'}\right)_* \mathrm{Leb}_{\gamma'}}{d\,\mathrm{Leb}_\gamma}(x) = \prod_{n\geq 0} \frac{\det Df^u\left(f^n(x)\right)}{\det Df^u\left(f^n\left(\Theta_{\gamma,\gamma'}^{-1}x\right)\right)},$$

$$\frac{d\left(\Theta_{\gamma,\gamma'}\right)_* \mathrm{Leb}_{\gamma'}}{d\,\mathrm{Leb}_\gamma}(x) = C^{\pm 1},$$

(b) for any $x, y \in \gamma \bigcap \Lambda$

$$\log \frac{\frac{d\left(\Theta_{\gamma,\gamma'}\right)_* \mathrm{Leb}_{\gamma'}}{d\,\mathrm{Leb}_\gamma}(x)}{\frac{d\left(\Theta_{\gamma,\gamma'}\right)_* \mathrm{Leb}_{\gamma'}}{d\,\mathrm{Leb}_\gamma}(y)} \leq C\beta^{s(x,y)}.$$

7. Aperiodicity: $\gcd(R_i, i \geq 1) = 1$.
8. A decay rate of the return times R: there exist $\xi > 1$ and $\gamma \in \Gamma^u$ such that

$$\mathrm{Leb}_\gamma\,(R > n) \leq Cn^{-\xi}.$$

SRB measures: Let the dynamics $f : (\mathcal{M}, \mu) \to (\mathcal{M}, \mu)$ has Gibbs-Markov-Young structure. It was proved in [2,45,46] that there exists an ergodic probability measure μ such that for any unstable manifold γ^u (including Γ^u) $\mu_{\gamma^u} \ll \mathrm{Leb}_{\gamma^u}$, where μ_{γ^u} is the conditional measure of μ on an unstable manifold γ^u. Such μ is called Sinai-Ruelle-Bowen measure (SRB measure).

Assumption 1 (Geometric regularities). Assume that $f : \mathcal{M} \to \mathcal{M}$ has the Gibbs-Markov-Young structure, as described in Definition 5, and

1. f is bijective and a local C^1-diffeomorphism on $\bigcup_{i\geq 1}\bigcup_{0\leq j<R_i} f^j(\Lambda_i)$.
2. the following limit exists

$$\dim_H \mu := \lim_{r\to 0} \frac{\log \mu(B_r(z))}{\log r}$$

for almost every $z \in \mathcal{M}$. Then $dim_H \mu$ is called a Hausdorff dimension of the measure μ.

3. $\alpha \dim_H \mu > 1$, where α is the contraction rate of the (un)stable manifolds in Definition 5.

Assumption 2 (The first returns & interior assumptions on Λ). Assume that $f : \mathcal{M} \to \mathcal{M}$ has the Gibbs-Markov-Young structure, and there are constants $C > 1$ and $\beta \in (0, 1)$ (the same as that in Definition 5) such that

1. $R : \Lambda \to \mathbb{N}$ is the first return time and $f^R : \Lambda \to \Lambda$ is the first return map for Λ. This implies that f^R is actually bijective (see Lemma 4 below).

2. for any $\gamma \in \Gamma^s$, $\gamma_1 \in \Gamma^u$, $x, y \in \gamma \bigcap \Lambda$, $x_1, y_1 \in \gamma_1 \bigcap \Lambda$,

$$d\Big(\left(f^R\right)^n(x), \left(f^R\right)^n(y)\Big) \leq C\beta^n,$$

and

$$d\Big(\left(f^R\right)^{-n}(x_1), \left(f^R\right)^{-n}(y_1)\Big) \leq C\beta^n d(x_1, y_1).$$

3. $\mu\{\text{int}(\Lambda)\} > 0$ and $\mu(\partial\Lambda) = 0$, where

$$\text{int}\,\Lambda := \left\{x \in \Lambda : \text{ there exists } r_x > 0 \text{ s.t. } \mu\left(B_{r_x}(x) \setminus \Lambda\right) = 0\right\}, \quad \partial\Lambda := \Lambda \setminus \text{int}\,\Lambda.$$

In other words, $x \in \text{int}\,\Lambda$ if and only if $x \in \Lambda$ and there is a small ball $B_{r_x}(x)$ s.t. $B_{r_x}(x) \subseteq \Lambda$ μ-almost surely.

Now we can formulate the first main result of the paper.

Theorem 3 (Convergence rates for functional Poisson laws I). *Assume that the dynamics $f : (\mathcal{M}, \mu) \to (\mathcal{M}, \mu)$ has a first return Gibbs-Markov-Young structure (see Definition 5) and satisfies Assumptions 1 and 2. Then for any $T > 0$ the following results hold*

1. $\dim_H \mu \geq \dim \gamma^u$ *and*
2. *If either $\alpha > \frac{2}{\dim \gamma^u} - \frac{1}{\dim_H \mu}$ or $\mu \ll \text{Leb}_{\mathcal{M}}$ with $\frac{d\mu}{d\,\text{Leb}_{\mathcal{M}}} \in L^\infty_{loc}(\mathcal{M})$, then for almost every $z \in \mathcal{M}$*

$$d_{TV}\left(N^{r,z,T}, P\right) \precsim_{T,\xi,z} r^a,$$

where the constant $a > 0$ depends on $\xi > 1$, $\dim_H \mu$, $\dim \gamma^u$ and α, but it does not depend on $z \in \mathcal{M}$. The expression for a can be found in Lemma 19.

Definition 6 *(Induced measurable partitions).* We say a probability measure μ for the dynamics $f : \mathcal{M} \to \mathcal{M}$ has an induced measurable partition if there are constants $\beta \in (0, 1)$, $C > 1$ (the same as that in Definition 5) and $b > 0$ such that

1. There exists a subset $U \subseteq \mathcal{M}$ with $\mu\{\text{int}(U)\} > 0$, $\mu(\partial U) = 0$.
2. The subset U has a measurable partition $\Theta := \{\gamma^u(x)\}_{x\in U}$ (which could be different from Γ^u), such that the elements of Θ are disjoint connected unstable manifolds, so that μ-almost surely $U = \bigsqcup_{x\in U} \gamma^u(x)$ and for any function g

$$\mu_U(g) = \int_U \mu_{\gamma^u(x)}(g)d\mu_U(x),$$

where $\mu_U := \frac{\mu|_U}{\mu(U)}$ and $\mu_{\gamma^u(x)}$ is the conditional probability induced by μ on $\gamma^u(x) \in \Theta$.
3. Each $\gamma^u \in \Theta$ is (at least C^1) smooth.
4. All $\gamma^u \in \Theta$ have uniformly bounded sectional curvatures and the same dimensions.
5. For any $\epsilon \in (0, 1)$

$$\mu_U\{x \in U : |\gamma^u(x)| < \epsilon\} \leq C\epsilon^b,$$

where $|\gamma^u(x)|$ is the radius of the largest inscribed geodesic ball in $\gamma^u(x) \in \Theta$, and a geodesic ball is defined with respect to the distance $d_{\gamma^u(x)}$ on $\gamma^u(x)$, induced by the Riemannian metric. This property implies that almost every $\gamma^u(x) \in \Theta$ is non-degenerated, i.e., $|\gamma^u(x)| > 0$ for almost every $x \in U$.

6. For almost every point $x \in U$ we have $\mu_{\gamma^u(x)} \ll \mathrm{Leb}_{\gamma^u(x)}$, $\mu_{\gamma^u(x)}(\gamma^u(x)) > 0$, and for any $y, z \in \gamma^u(x)$

$$\frac{d\mu_{\gamma^u(x)}}{d\,\mathrm{Leb}_{\gamma^u(x)}}(y) = C^{\pm 1}\frac{d\mu_{\gamma^u(x)}}{d\,\mathrm{Leb}_{\gamma^u(x)}}(z).$$

7. Denote by $\overline{R}$ the first return time to U for f. Then the first return map $f^{\overline{R}} : U \to U$ has an exponential u-contraction, i.e., for any $\gamma^u \in \Theta$, $x, y \in \gamma^u$, $n \geq 1$

$$d\Big(\left(f^{\overline{R}}\right)^{-n}(x), \left(f^{\overline{R}}\right)^{-n}(y)\Big) \leq C\beta^n d(x, y),$$

and an exponential decay of correlation, i.e., for any $h \in \mathrm{Lip}(U)$

$$\left|\int h \circ \left(f^{\overline{R}}\right)^n h d\mu_U - \left(\int h d\mu_U\right)^2\right| \leq C\beta^n ||h||^2_{\mathrm{Lip}}.$$

Now we are able to formulate the second main result of the paper.

Theorem 4 (Convergence rates for the functional Poisson laws II). *Assume that the dynamics $f : (\mathcal{M}, \mu) \to (\mathcal{M}, \mu)$ has the Gibbs-Markov-Young structure (see Definition 5), satisfies Assumption 1 and μ has an induced measurable partition (see Definition 6). Then for any $T > 0$, the following results hold.*

1. $\dim_H \mu \geq \frac{b}{b+\dim\gamma^u}\dim\gamma^u$ *and*
2. *If either* $\alpha > \frac{2}{\dim\gamma^u}\frac{b+\dim\gamma^u}{b} - \frac{1}{\dim_H\mu}$ *or* $\mu \ll \mathrm{Leb}_{\mathcal{M}}$ *and* $\frac{d\mu}{d\,\mathrm{Leb}_{\mathcal{M}}} \in L^\infty_{loc}(\mathcal{M})$, *then for almost every (a.e.)* $z \in \mathcal{M}$,

$$d_{TV}\left(N^{r,z,T}, P\right) \precsim_{T,\xi,z,b} r^a,$$

where a constant $a > 0$ depends on $\xi > 1$, $\dim_H \mu$, $\dim\gamma^u$, b and α, but it does not depend on $z \in \mathcal{M}$. The expression for a can be found in Lemma 31.

Remark 3. 1. For many hyperbolic systems contraction (resp. expansion) along stable (resp. unstable) manifolds is exponential. Therefore, the rate α can be chosen as an arbitrary large number. Therefore, in this case, the condition for α in Theorems 3 and 4 holds automatically.

2. For a simple uniformly hyperbolic system, e.g. for an algebraic toral automorphism (Arnold's cat), $\dim_H \mu = 2$, $\dim\gamma^u = 1$ and ξ, α can be arbitarily large. Therefore it follows from Lemma 19 that the convergence rate a can be chosen as any number in the interval $(0, 24^{-4})$.
3. Our Assumption 2 that R is the first return time and f^R is the first return map of Λ is natural for hyperbolic systems that have a Markov partition. Otherwise, we assume that the system has a subset U with an induced measurable partition (see Definition 6).

   It will be shown in what follows that Theorems 3 and 4 work efficiently for various systems in applications (see section 6). Clearly, a key issue here is a choice of the reference sets Λ and U.

   Our approach is close to a standard one in Ergodic theory, which restricts dynamics as an induced map to some "good" subset. Then a result is proved for the corresponding induced map, and then it is"lifted" to entire phase space. Our approach employs

instead pulling of a hole back to good sets Λ and U, and then uses the induced map with good properties to deal with two main challenges for Poisson approximations: short returns and coronas (see Definition 7).
Therefore our approach (see the details in Sects. 4 and 5) does not work for non-invertible systems (e.g. expanding and intermittent type maps). The reason is that such non-invertible systems usually have multiple inverse branches, and a hole can not be entirely pulled back to Λ and U. We believe that our approach could be modified to handle as well non-invertible systems. However, it is a subject for future studies.

4. Under similar conditions to Definition 6, it was proved in [43] that $\mu_U\left(\overline{R} > n\right)$ characterizes the optimal bound for the decay rates of correlations for sufficiently good observables supported on U (see Theorem 1.3 in [43]); the paper [10] uses operator renewal theory as a method to prove also sharp results on polynomial decay of correlations (see Theorem 3.1 in [10]). For many purposes the aperiodicity in Definition 5 is irrelevant provided the dynamic $f : (\mathcal{M}, \mu) \to (\mathcal{M}, \mu)$ is mixing (see Remark 2.2 in [10]). Indeed all dynamical systems, which we consider in applications (Sect. 6), do have a countable Markov partition. And any hyperbolic ergodic dynamical systems with singularities (e.g. dispersing billiards) in Sect. 6 only have countably infinite Markov partition (see [14]). Also an ergodic completely hyperbolic (all Lyapunov exponents do not vanish) dynamical system is mixing. Therefore Young towers are mixing. So, to simplify the argument of our proof, we only assume aperiodicity in the Gibbs-Markov-Young structures.
5. When dealing with applications, (see Sect. 6), it is always assumed that μ is a hyperbolic measure (i.e., the Lyapunov exponents do not vanish almost everywhere, see [4]). Also, in applications most often there is an explicit natural invariant measure (sometimes called a physical measure). Therefore, Assumption 1, which requires that $\dim_H \mu := \lim_{r\to 0} \frac{\log \mu(B_r(z))}{\log r}$, is relevant to such approach. (However, another dimension conditions, like e.g. in [29], could be used as well).
6. If an SRB measure μ is explicitly known, then the Poisson approximations are usually well understood [8,9,21,26,29]. However, if it is not the case, then often essential difficulties arise, e.g. for intermittent solenoid attractors, Axiom A attractors, etc (see [37]). Our Theorem 3 provides an useful, easy to verify, criterion. Indeed, if $\alpha > 2/\dim \gamma^u$, then there is no need to know $\dim_H \mu$. Moreover, estimations of the corresponding convergence rates can be obtained as well.
7. According to Theorems 3 and 4, it is only required that $\xi > 1$. In fact, it is a minimal requirement for the existence of the SRB measures (see [2]).
8. Observe that for our approach only the contraction rate $O\left(n^{-\alpha}\right)$ along (un)stable manifolds matters, which is different from the ones employed in [16,29,37].
9. If f has a sufficiently good regularity, then $\dim_H \mu \geq \dim \gamma^u$ [4,32,33]. Our only assumption is that f is a local C^1-diffeomorphism. Observe that we do not even assume that $\mathcal{M}$ is a compact manifold (see Definition 5 and Assumption 1). Therefore Theorem 4 does not provide a good lower bound for $\dim_H \mu$. It is worthwhile to mention also that for all applications considered below (see Sect. 6) the relation $\dim_H \mu \geq \dim \gamma^u$ always holds.

Corollary 1 (The first hitting and survival probabilities). *Under the same conditions as in Theorem 3 or 4 consider first hitting moment of time* $\tau_{B_r(z)}(x) := \inf\left\{n \geq 1 : f^n(x) \in B_r(z)\right\}$. *Then for almost every* $z \in \mathcal{M}$, *any* $T > 0$ *and any* $t \leq T$ *the following relation holds for the first hitting probability*

$$\mu\Big(\tau_{B_r(z)} > t\big/\mu(B_r(z))\Big) - e^{-t} = O_{T,\xi,z}(r^a). \tag{2.2}$$

Particularly, survival probability at time T can be approximated as

$$\mu\left(\tau_{B_r(z)} > T\right) = e^{-T\mu(B_r(z))} + \min\left\{O_{T,\xi,z}(r^a), 1\right\}.$$

Moreover, the following limiting relations hold

$$\lim_{T\to\infty}\lim_{r\to 0}\frac{\log\mu\left(\tau_{B_r(z)} > T\right)}{-T\mu(B_r(z))} = 1, \tag{2.3}$$

and for any $T > 0$

$$\lim_{r\to 0}\frac{\log\mu\Big(\tau_{B_r(z)} > T\big/\mu(B_r(z))\Big)}{-T} = 1. \tag{2.4}$$

Proof. Clearly $\mu\left\{\tau_{B_r(z)} > t/\mu(B_r(z))\right\} = \mu\left\{N^{r,T,z}\left[0, t/\mu(B_r(z))\right] = 0\right\}$. Apply now a relevant one of Theorems 3 and 4. Then $O_{T,\xi,z,b}(r^a)$ is the error term with the convergence rate a. For the survival probability at time T take $t = T\mu(B_r(z))$. The relation (2.2) implies (2.4). According to Assumption 1, f is a local diffeomorphism almost everywhere. Besides, the set of all periodic points has measure zero. Hence $\mu\left(\tau_{B_r(z)} > T\right) = 1 - \mu\left\{\bigcup_{i\le T} f^{-i}B_r(z)\right\} = 1-(T+1)\mu(B_r(z))$, if r is small enough. Therefore (2.3) holds. □

Remark 4. The papers [1,24,31,44] obtained convergence rates for hitting times statistics, extreme value distributions and escape rates. Particularly, the paper [44] also provides error terms for Poisson approximations for some stochastic processes.

3. Functional Poisson Limit Laws

This section deals with the functional Poisson limit laws and convergence rates of $d_{TV}\left(N^{r,z,T}, P\right)$ for the dynamics f described in Definition 5 and satisfying Assumption 1 only. For any $n \ge 0$, $I \subseteq [0, n]$, let

$$X_i := \mathbb{1}_{B_r(z)} \circ f^i, \quad X_I := \sum_{i\in I}\mathbb{1}_{B_r(z)} \circ f^i.$$

Denote by $\{\hat{X}_i\}_{i\ge 0}$ i.i.d. random variables defined on a probability space $(\hat{\Omega}, \hat{\mathbb{P}})$, such that for each $i \ge 0$,

$$X_i =_d \hat{X}_i,$$

that is, they have the same distribution. Let

$$\hat{X}_I := \sum_{i\in I}\hat{X}_i.$$

Observe that generally $\hat{X}_I$ and X_I are not identically distributed. For any $m \ge 1$ we define

$$d_{TV}\Big(\Big(X_{I_1},\ldots,X_{I_m}\Big),\Big(\hat{X}_{I_1},\ldots,\hat{X}_{I_m}\Big)\Big)$$
$$:=\sup_{h\in[0,1]}\Big|\mathbb{E}h\Big(X_{I_1},\ldots,X_{I_m}\Big)-h\left(\hat{X}_{I_1},\ldots,\hat{X}_{I_m}\right)\Big|,$$

where h is a measurable function on $\mathbb{R}^m$ with values in [0, 1] and $\mathbb{E}$ is expectation of $\mu\otimes\hat{\mathbb{P}}$.

Throughout this section the notation $h(\underbrace{\bullet,\bullet,\ldots,\bullet}_{m})$ means that function h is defined on $\mathbb{R}^m$ for some $m\geq 1$. $h\in[0,1]$ means that a function h takes values in [0, 1].

Lemma 1. *For any disjoint sets $I_1, I_2,\ldots,I_m\subseteq[0,n]$ and any integer $p\in(0,n)$,*

$$d_{TV}\Big(\Big(X_{I_1},\ldots,X_{I_m}\Big),\Big(\hat{X}_{I_1},\ldots,\hat{X}_{I_m}\Big)\Big)\leq R_1+R_2+R_3,$$

where

$$R_1:=2\sum_{0\leq l\leq n-p}\sup_{h\in[0,1]}\Big|\mathbb{E}\Big[\mathbb{1}_{X_0=1}h(X_p,\ldots,X_{n-l})\Big]-\mathbb{E}\mathbb{1}_{X_0=1}\mathbb{E}\Big[h(X_p,\ldots,X_{n-l})\Big]\Big|$$
$$R_2:=4(n-p)\mathbb{E}\Big(\mathbb{1}_{X_0=1}\mathbb{1}_{\sum_{1\leq j\leq p-1}X_j\geq 1}\Big)$$
$$R_3:=4p(n-p)\mu(B_r(z))^2+4p\mu(B_r(z)),$$

and h is a measurable function with values in [0, 1]. *Observe that we obtain a slightly better error bound here, compared to Theorem 2.1 in* [16].

Proof. By definition of total variation norm

$$d_{TV}\Big(\Big(X_{I_1},\ldots,X_{I_m}\Big),\Big(\hat{X}_{I_1},\ldots,\hat{X}_{I_m}\Big)\Big)$$
$$=\sup_{h\in[0,1]}\Big|\mathbb{E}h\Big(X_{I_1},\ldots,X_{I_m}\Big)-h\left(\hat{X}_{I_1},\ldots,\hat{X}_{I_m}\right)\Big|$$
$$\leq\sup_{h\in[0,1]}\Big|\mathbb{E}h\Big(X_0,\ldots,X_n\Big)-h\left(\hat{X}_0,\ldots,\hat{X}_n\right)\Big|$$
$$=d_{TV}\Big(\Big(X_0,\ldots,X_n\Big),\Big(\hat{X}_0,\ldots,\hat{X}_n\Big)\Big).$$

Hence, it suffices to estimate

$$d_{TV}\Big(\Big(X_0,\ldots,X_n\Big),\Big(\hat{X}_0,\ldots,\hat{X}_n\Big)\Big)$$
$$=\sup_{h\in[0,1]}\Big|\mathbb{E}h\Big(X_0,\ldots,X_n\Big)-h\left(\hat{X}_0,\ldots,\hat{X}_n\right)\Big|$$
$$=\sup_{h\in[0,1]}\Big|\sum_{0\leq l\leq n}\mathbb{E}h\left(\hat{X}_1,\ldots,\hat{X}_{l-1},X_l,\ldots,X_n\right)$$
$$-\mathbb{E}h\left(\hat{X}_1,\ldots,\hat{X}_{l-1},\hat{X}_l,\ldots,X_n\right)\Big|$$
$$\leq\sup_{h\in[0,1]}\Big|\sum_{0\leq l\leq n}\mathbb{E}h_l\Big(X_l,,X_{l+1},\ldots,X_n\Big)-\mathbb{E}h_l\left(\hat{X}_l,X_{l+1},\ldots,X_n\right)\Big|,$$

here $h_l(\bullet) := h(\hat{X}_1, \ldots, \hat{X}_{l-1}, \bullet)$. Since $\hat{X}_1, \ldots, \hat{X}_{l-1}$ are independent of other random variables, without loss of generality, h_l can be regarded as a function which does not depend on $\hat{X}_1, \ldots, \hat{X}_{l-1}$. Note that $X_l =_d \hat{X}_l$ are $\{0, 1\}$-valued random variables. Thus

$$\begin{aligned}
&\left|\mathbb{E}h_l\Big(X_l, X_{l+1}, \ldots, X_n\Big) - \mathbb{E}h_l\left(\hat{X}_l, X_{l+1}, \ldots, X_n\right)\right| \\
&= \Big|\mathbb{E}\Big[\mathbb{1}_{X_l=0}h_l(0, X_{l+1}, \ldots, X_n)\Big] + \mathbb{E}\Big[\mathbb{1}_{X_l=1}h_l(1, X_{l+1}, \ldots, X_n)\Big] \\
&\quad - \mathbb{E}\mathbb{1}_{\hat{X}_l=0}\mathbb{E}h_l(0, X_{l+1}, \ldots, X_n) - \mathbb{E}\mathbb{1}_{\hat{X}_l=1}\mathbb{E}h_l(1, X_{l+1}, \ldots, X_n)\Big| \\
&= \Big|\mathbb{E}\Big[\mathbb{1}_{X_l=1}h_l(0, X_{l+1}, \ldots, X_n)\Big] + \mathbb{E}\Big[\mathbb{1}_{X_l=1}h_l(1, X_{l+1}, \ldots, X_n)\Big] \\
&\quad - \mathbb{E}\mathbb{1}_{\hat{X}_l=1}\mathbb{E}h_l(0, X_{l+1}, \ldots, X_n) - \mathbb{E}\mathbb{1}_{\hat{X}_l=1}\mathbb{E}h_l(1, X_{l+1}, \ldots, X_n)\Big| \\
&\le 2 \sup_{h\in[0,1]}\Big|\mathbb{E}\Big[\mathbb{1}_{X_l=1}h(X_{l+1}, \ldots, X_n)\Big] - \mathbb{E}\mathbb{1}_{X_l=1}\mathbb{E}h(X_{l+1}, \ldots, X_n)\Big|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&d_{TV}\Big(\Big(X_0, \ldots, X_n\Big), \Big(\hat{X}_0, \ldots, \hat{X}_n\Big)\Big) \\
&\quad \le 2\sum_{0\le l\le n}\sup_{h\in[0,1]}\Big|\mathbb{E}\Big[\mathbb{1}_{X_l=1}h(X_{l+1}, \ldots, X_n)\Big] - \mathbb{E}\mathbb{1}_{X_l=1}\mathbb{E}h(X_{l+1}, \ldots, X_n)\Big|. \qquad (3.1)
\end{aligned}$$

We will first estimate the terms with $l \le n - p$ in (3.1).

$$\begin{aligned}
&\Big|\mathbb{E}\Big[\mathbb{1}_{X_l=1}h(X_{l+1}, \ldots, X_n)\Big] - \mathbb{E}\mathbb{1}_{X_l=1}\mathbb{E}h(X_{l+1}, \ldots, X_n)\Big| \\
&= \Big|\mathbb{E}\Big[\mathbb{1}_{X_l=1}h(X_{l+1}, \ldots, X_n)\Big] - \mathbb{E}\Big[\mathbb{1}_{X_l=1}h(0, \ldots, 0, X_{l+p}, \ldots, X_n)\Big] \\
&\quad + \mathbb{E}\Big[\mathbb{1}_{X_l=1}h(0, \ldots, 0, X_{l+p}, \ldots, X_n)\Big] - \mathbb{E}\mathbb{1}_{X_l=1}\mathbb{E}h(X_{l+1}, \ldots, X_n) \\
&\quad + \mathbb{E}\mathbb{1}_{X_l=1}\mathbb{E}h(0, \ldots, 0, X_{l+p}, \ldots, X_n) - \mathbb{E}\mathbb{1}_{X_l=1}\mathbb{E}h(0, \ldots, 0, X_{l+p}, \ldots, X_n)\Big| \\
&= \Big|\mathbb{E}\Big\{\mathbb{1}_{X_l=1}\Big[h(X_{l+1}, \ldots, X_n) - h(0, \ldots, 0, X_{l+p}, \ldots, X_n)\Big]\Big\} \\
&\quad + \mathbb{E}\mathbb{1}_{X_l=1}\mathbb{E}\Big[h(0, \ldots, 0, X_{l+p}, \ldots, X_n) - h(X_{l+1}, \ldots, X_n)\Big] \\
&\quad + \mathbb{E}\Big[\mathbb{1}_{X_l=1}h(0, \ldots, 0, X_{l+p}, \ldots, X_n)\Big] - \mathbb{E}\mathbb{1}_{X_l=1}\mathbb{E}h(0, \ldots, 0, X_{l+p}, \ldots, X_n)\Big|
\end{aligned}$$

Observe that

$$|h(X_{l+1}, \ldots, X_n) - h(0, \ldots, 0, X_{l+p}, \ldots, X_n)| \le 2\mathbb{1}_{\sum_{l+1\le j\le l+p-1} X_j\ge 1}.$$

Now, because of stationarity of $(X_i)_{i\ge 0}$, we can continue estimates as

$$\begin{aligned}
&\le \Big|\mathbb{E}\Big[\mathbb{1}_{X_l=1}h(0, \ldots, 0, X_{l+p}, \ldots, X_n)\Big] - \mathbb{E}\mathbb{1}_{X_l=1}\mathbb{E}h\left(0, \ldots, 0, X_{l+p}, \ldots, X_n\right)\Big| \\
&\quad + 2\mathbb{E}\Big(\mathbb{1}_{X_l=1}\mathbb{1}_{\sum_{l+1\le j\le l+p-1} X_j\ge 1}\Big) + 2\mathbb{E}\mathbb{1}_{X_l=1}\mathbb{E}\mathbb{1}_{\sum_{l+1\le j\le l+p-1} X_j\ge 1} \\
&\le \Big|\mathbb{E}\Big[\mathbb{1}_{X_l=1}h(0, \ldots, 0, X_{l+p}, \ldots, X_n)\Big] - \mathbb{E}\mathbb{1}_{X_l=1}\mathbb{E}h(0, \ldots, 0, X_{l+p}, \ldots, X_n)\Big|
\end{aligned}$$

$$+ 2\mathbb{E}\Big(\mathbb{1}_{X_0=1}\mathbb{1}_{\sum_{1\le j\le p-1}X_j\ge 1}\Big) + 2\mathbb{E}\mathbb{1}_{X_0=1}\mathbb{E}\mathbb{1}_{\sum_{1\le j\le p-1}X_j\ge 1}.$$

Note that $\mathbb{1}_{\sum_{1\le j\le p-1}X_j\ge 1} = \mathbb{1}_{\cup_{1\le j\le p-1}f^{-j}B_r(z)}$. Hence, we can continue the sequence of inequalities above as

$$\le \Big|\mathbb{E}\Big[\mathbb{1}_{X_l=1}h(0,\dots,0,X_{l+p},\dots,X_n)\Big] - \mathbb{E}\mathbb{1}_{X_l=1}\mathbb{E}h(0,\dots,0,X_{l+p},\dots,X_n)\Big|$$
$$+ 2\mathbb{E}\left(\mathbb{1}_{X_0=1}\mathbb{1}_{\sum_{1\le j\le p-1}X_j\ge 1}\right) + 2(p-1)\mu(B_r(z))^2.$$

Therefore for terms with $l \le n-p$ in (3.1) we have

$$\Big|\mathbb{E}\Big[\mathbb{1}_{X_l=1}h(X_{l+1},\dots,X_n)\Big] - \mathbb{E}\mathbb{1}_{X_l=1}\mathbb{E}h(X_{l+1},\dots,X_n)\Big|$$
$$\le \Big|\mathbb{E}\Big[\mathbb{1}_{X_l=1}h(0,\dots,0,X_{l+p},\dots,X_n)\Big] - \mathbb{E}\mathbb{1}_{X_l=1}\mathbb{E}h(0,\dots,0,X_{l+p},\dots,X_n)\Big|$$
$$+ 2\mathbb{E}\left(\mathbb{1}_{X_0=1}\mathbb{1}_{\sum_{1\le j\le p-1}X_j\ge 1}\right) + 2p\mu(B_r(z))^2.$$

Consider now the terms with $l > n-p$ in (3.1). Since $||h||_\infty \le 1$, then

$$\Big|\mathbb{E}\Big[\mathbb{1}_{X_l=1}h(X_{l+1},\dots,X_n)\Big] - \mathbb{E}\mathbb{1}_{X_l=1}\mathbb{E}h(X_{l+1},\dots,X_n)\Big| \le 2\mu(B_r(z)).$$

Therefore

$$(3.1) = 2\sum_{0\le l\le n}\sup_{h\in[0,1]}\Big|\mathbb{E}\Big[\mathbb{1}_{X_l=1}h(X_{l+1},\dots,X_n)\Big] - \mathbb{E}\mathbb{1}_{X_l=1}\mathbb{E}h(X_{l+1},\dots,X_n)\Big|$$
$$\le 2\sum_{0\le l\le n-p}\sup_{h\in[0,1]}\Big|\mathbb{E}\Big[\mathbb{1}_{X_l=1}h(X_{l+p},\dots,X_n)\Big] - \mathbb{E}\mathbb{1}_{X_l=1}\mathbb{E}h(X_{l+p},\dots,X_n)\Big|$$
$$+ 4(n-p)\mathbb{E}\Big(\mathbb{1}_{X_0=1}\mathbb{1}_{\sum_{1\le j\le p-1}X_j\ge 1}\Big) + 4p(n-p)\mu(B_r(z))^2 + 4p\mu(B_r(z)).$$

By making use of stationarity of $(X_i)_{i\ge 0}$, the last expression above can be estimated as

$$\le 2\sum_{0\le l\le n-p}\sup_{h\in[0,1]}\Big|\mathbb{E}\Big[\mathbb{1}_{X_0=1}h(X_p,\dots,X_{n-l})\Big] - \mathbb{E}\mathbb{1}_{X_0=1}\mathbb{E}h(X_p,\dots,X_{n-l})\Big|$$
$$+ 4(n-p)\mathbb{E}\left(\mathbb{1}_{X_0=1}\mathbb{1}_{\sum_{1\le j\le p-1}X_j\ge 1}\right) + 4p(n-p)\mu(B_r(z))^2 + 4p\mu(B_r(z)).$$

□

For further estimates we will need the following lemma.

Lemma 2 (Hyperbolic towers, see [37,45]). *Define a tower* Δ *and a map* $F : \Delta \to \Delta$ *as*

$$\Delta := \{(x,l) \in \Lambda \times \mathbb{N} : 0 \le l < R(x)\},$$
$$F(x,l) := \begin{cases}(x, l+1), & l < R(x)-1\\ \big(f^R(x), 0\big), & l = R(x)-1\end{cases}.$$

Equivalence relation $\sim$ on Λ is then

$$x \sim y \text{ if and only if } x, y \in \gamma^s \text{ for some } \gamma^s \in \Gamma^s.$$

Now we can define a quotient tower $\widetilde{\Delta} := \Delta / \sim$, a quotient Gibbs-Markov-Young product structure $\widetilde{\Lambda} := \Lambda / \sim$, quotient maps $\widetilde{F} : \widetilde{\Delta} \to \widetilde{\Delta}$, $\widetilde{f^R} : \widetilde{\Lambda} \to \widetilde{\Lambda}$, and canonical projections $\widetilde{\pi}_\Delta : \Delta \to \widetilde{\Delta}$ and $\widetilde{\pi}_\Lambda : \Lambda \to \widetilde{\Lambda}$.

At first, we introduce a family of partitions $(\mathcal{Q}_k)_{k \geq 0}$ of Δ as

$$\mathcal{Q}_0 := \{\Lambda_i \times \{l\}, i \geq 1, l < R_i\}, \quad \mathcal{Q}_k := \bigvee_{0 \leq i \leq k} F^{-i} \mathcal{Q}_0.$$

Next, a projection $\pi : \Delta \to \mathcal{M}$ is defined as

$$\pi(x, l) := f^l(x).$$

Then there exists a constant $C > 1$ (the same as that in Definition 5) such that for any $Q \in \mathcal{Q}_{2k}$

$$\operatorname{diam}\left(\pi \circ F^k(Q)\right) \leq C k^{-\alpha}. \tag{3.2}$$

There exist also probability measures μ_Δ, μ_Λ on Δ and Λ, respectively, such that

$$\pi_* \mu_\Delta = \mu, \quad F_* \mu_\Delta = \mu_\Delta, \quad f_* \mu = \mu, \quad \left(f^R\right)_* \mu_\Lambda = \mu_\Lambda. \tag{3.3}$$

Further, there exist probability measures $\mu_{\widetilde{\Delta}}$, $\mu_{\widetilde{\Lambda}}$ on $\widetilde{\Delta}$ and $\widetilde{\Lambda}$ respectively, such that

$$(\widetilde{\pi}_\Delta)_* \mu_\Delta = \mu_{\widetilde{\Delta}}, \quad (\widetilde{\pi}_\Lambda)_* \mu_\Lambda = \mu_{\widetilde{\Lambda}}, \quad \widetilde{F}_* \mu_{\widetilde{\Delta}} = \mu_{\widetilde{\Delta}}, \quad \left(\widetilde{f^R}\right)_* \mu_{\widetilde{\Lambda}} = \mu_{\widetilde{\Lambda}}. \tag{3.4}$$

Thus μ is supported on $\bigcup_{i \geq 1} \bigcup_{j < R_i} f^j(\Lambda_i)$, i.e.,

$$\mu\Big\{\bigcup_{i \geq 1} \bigcup_{j < R_i} f^j(\Lambda_i)\Big\} = 1.$$

Moreover

$$(\mu_\Lambda)_{\gamma^u} \ll \mathrm{Leb}_{\gamma^u}, \quad \frac{d(\mu_\Lambda)_{\gamma^u}}{d\,\mathrm{Leb}_{\gamma^u}} = C^{\pm 1}, \tag{3.5}$$

where $(\mu_\Lambda)_{\gamma^u}$ is the conditional measure of μ_Λ on $\gamma^u \in \Gamma^u$. Since R is the first return time, (see Assumption 2), then

$$\mu_\Lambda = \frac{\mu|_\Lambda}{\mu(\Lambda)}.$$

Finally, for any $k \geq 1$ and any $(Q_i)_{i \geq 1} \subseteq \mathcal{Q}_k$, any $h : \Delta \to \mathbb{R}$ satisfying $||h||_\infty \leq 1$ and $h(x, l) = h(y, l)$ for any $x, y \in \gamma^s \in \Gamma^s$, and any allowable $l \in \mathbb{N}$ (i.e., h is $\sigma(\bigcup_{k \geq 0} \mathcal{Q}_k)$-measurable), we have the following estimate for decay of correlations

$$\Big| \int \mathbb{1}_{\bigcup_{i \geq 1} Q_i} h \circ F^{2k} d\mu_\Delta - \mu_\Delta\Big(\bigcup_{i \geq 1} Q_i\Big) \int h d\mu_\Delta \Big| \leq C k^{1-\xi} \mu_\Delta\Big(\bigcup_{i \geq 1} Q_i\Big). \tag{3.6}$$

Lemma 3. *For any $l \geq 0$, $p^{-\alpha} \ll r$ and any measurable function h with values in* $[0, 1]$

$$\left|\mathbb{E}\Big[\mathbb{1}_{X_0=1}h(X_p,\dots,X_{p+l})\Big] - \mathbb{E}\mathbb{1}_{X_0=1}\mathbb{E}h(X_p,\dots,X_{p+l})\right| \\ \leq 4Cp^{1-\xi}\mu\Big(B_{r+C4^\alpha p^{-\alpha}}(z)\Big) + \Big[2+4l\mu\Big(B_{r+C4^\alpha p^{-\alpha}}(z)\Big)\Big] \\ \times \mu\Big(B_{r+C4^\alpha p^{-\alpha}}(z)\setminus B_{r-C4^\alpha p^{-\alpha}}(z)\Big),$$

where a constant C is the same as that in Definition 5.

Proof. Similarly to the approach of [37], we will make use of Markov partition of hyperbolic towers. Let $m := \lfloor p/4\rfloor$. By (3.3) and the invariance of F (i.e., $F_*\mu_\Delta = \mu_\Delta$) we have

$$\mathbb{E}\Big[\mathbb{1}_{X_0=1}h(X_p,\dots,X_{p+l})\Big] \\ = \int \mathbb{1}_{B_r(z)}h\Big(\mathbb{1}_{B_r(z)}\circ f^p,\dots,\mathbb{1}_{B_r(z)}\circ f^{p+l}\Big)d\mu \\ = \int \mathbb{1}_{B_r(z)}\circ\pi\circ F^m h\Big(\mathbb{1}_{B_r(z)}\circ\pi\circ F^{p+m-p},\dots,\mathbb{1}_{B_r(z)}\circ\pi\circ F^{p+l+m-p}\Big)\circ F^p d\mu_\Delta.$$

Denote $A_1 := F^{-m}\pi^{-1}B_r(z)$, $A_0 := \bigsqcup_{Q\in\mathcal{Q}_{2m}:Q\bigcap A_1\neq\emptyset} Q$ and $A_2 := \bigsqcup_{Q\in\mathcal{Q}_{2m}:Q\bigcap(A_0\setminus A_1)\neq\emptyset} Q$. Then $A_1\bigcup A_2 = A_0$. The sets A_0, A_2 are $\sigma(\bigcup_{k\geq 0}\mathcal{Q}_k)$-measurable. Therefore we can continue the equality above as

$$= \int \mathbb{1}_{A_1}h(\mathbb{1}_{A_1},\dots,\mathbb{1}_{A_1}\circ F^l)\circ F^p d\mu_\Delta \\ = \int \mathbb{1}_{A_1}h(\mathbb{1}_{A_1\bigcup A_2},\dots,\mathbb{1}_{A_1\bigcup A_2}\circ F^l)\circ F^p d\mu_\Delta \\ + \int \mathbb{1}_{A_1}h(\mathbb{1}_{A_1},\dots,\mathbb{1}_{A_1}\circ F^l)\circ F^p d\mu_\Delta \\ - \int \mathbb{1}_{A_1}h(\mathbb{1}_{A_1\bigcup A_2},\dots,\mathbb{1}_{A_1\bigcup A_2}\circ F^l)\circ F^p d\mu_\Delta.$$

Claim: $\left|h\left(\mathbb{1}_{A_1},\dots,\mathbb{1}_{A_1}\circ F^l\right) - h\left(\mathbb{1}_{A_1\bigcup A_2},\dots,\mathbb{1}_{A_1\bigcup A_2}\circ F^l\right)\right| \leq 2\mathbb{1}_{\cup_{j\leq l}F^{-j}A_2}$.

Indeed, if $F^j(x,l)\notin A_2$ for all $j\leq l$, then $\mathbb{1}_{A_1}\circ F^{j-p}(x,l) = \mathbb{1}_{A_1\cup A_2}\circ F^{j-p}(x,l)$. On the other hand, $||h_j||_\infty \leq 1$. Hence, the claim holds.

Therefore,

$$\left|\mathbb{E}\Big[\mathbb{1}_{X_0=1}h(X_p,\dots,X_{p+l})\Big] - \mathbb{E}\mathbb{1}_{X_0=1}\mathbb{E}h(X_p,\dots,X_{p+l})\right| \\ = \Big|\int \mathbb{1}_{A_1}h\left(\mathbb{1}_{A_1\bigcup A_2},\dots,\mathbb{1}_{A_1\bigcup A_2}\circ F^l\right)\circ F^p d\mu_\Delta - \mu_\Delta(A_1) \\ \int h\left(\mathbb{1}_{A_1\bigcup A_2},\dots,\mathbb{1}_{A_1\bigcup A_2}\circ F^l\right)d\mu_\Delta \\ + \int \mathbb{1}_{A_1}h\left(\mathbb{1}_{A_1},\dots,\mathbb{1}_{A_1}\circ F^l\right)\circ F^p d\mu_\Delta \\ - \int \mathbb{1}_{A_1}h\left(\mathbb{1}_{A_1\bigcup A_2},\dots,\mathbb{1}_{A_1\bigcup A_2}\circ F^l\right)\circ F^p d\mu_\Delta$$

$$- \int \mathbb{1}_{A_1} d\mu_\Delta \int h\left(\mathbb{1}_{A_1}, \ldots, \mathbb{1}_{A_1} \circ F^l\right) \circ F^p d\mu_\Delta + \mu_\Delta(A_1)$$
$$\int h\left(\mathbb{1}_{A_1 \bigcup A_2}, \ldots, \mathbb{1}_{A_1 \bigcup A_2} \circ F^l\right) d\mu_\Delta \Big|$$
$$\leq \left| \int \mathbb{1}_{A_1} h\left(\mathbb{1}_{A_1 \bigcup A_2}, \ldots, \mathbb{1}_{A_1 \bigcup A_2} \circ F^l\right) \circ F^p d\mu_\Delta - \mu_\Delta(A_1)\right.$$
$$\left.\int h\left(\mathbb{1}_{A_1 \bigcup A_2}, \ldots, \mathbb{1}_{A_1 \bigcup A_2} \circ F^l\right) d\mu_\Delta \right|$$
$$+ 2 \int \mathbb{1}_{A_1} \mathbb{1}_{\bigcup_{j \leq l} F^{-j} A_2} \circ F^p d\mu_\Delta + 2\mu_\Delta(A_1) \int \mathbb{1}_{\bigcup_{j \leq l} F^{-j} A_2} d\mu_\Delta$$
$$\leq \left| \int \mathbb{1}_{A_0} h\left(\mathbb{1}_{A_1 \bigcup A_2}, \ldots, \mathbb{1}_{A_1 \bigcup A_2} \circ F^l\right) \circ F^p d\mu_\Delta - \mu_\Delta(A_0)\right.$$
$$\left.\int h\left(\mathbb{1}_{A_1 \bigcup A_2}, \ldots, \mathbb{1}_{A_1 \bigcup A_2} \circ F^l\right) d\mu_\Delta \right|$$
$$+ \left| \int \mathbb{1}_{A_0 \setminus A_1} h\left(\mathbb{1}_{A_1 \bigcup A_2}, \ldots, \mathbb{1}_{A_1 \bigcup A_2} \circ F^l\right) \circ F^p d\mu_\Delta\right.$$
$$- \int h\left(\mathbb{1}_{A_1 \bigcup A_2}, \ldots, \mathbb{1}_{A_1 \bigcup A_2} \circ F^l\right) d\mu_\Delta$$
$$\left.\times\, \mu_\Delta(A_0 \setminus A_1)\right| + 2 \int \mathbb{1}_{A_1} \mathbb{1}_{\bigcup_{j \leq l} F^{-j} A_2} \circ F^p d\mu_\Delta + 2\mu_\Delta(A_1)$$
$$\int \mathbb{1}_{\bigcup_{j \leq l} F^{-j} A_2} d\mu_\Delta$$

Note that $A_0 \setminus A_1 \subseteq A_2$, $A_1 \subseteq A_0$, which means that we can continue the estimate above as

$$\leq \left| \int \mathbb{1}_{A_0} h\left(\mathbb{1}_{A_1 \bigcup A_2}, \ldots, \mathbb{1}_{A_1 \bigcup A_2} \circ F^l\right) \circ F^p d\mu_\Delta - \mu_\Delta(A_0)\right.$$
$$\left.\int h\left(\mathbb{1}_{A_1 \bigcup A_2}, \ldots, \mathbb{1}_{A_1 \bigcup A_2} \circ F^l\right) d\mu_\Delta \right|$$
$$+ \int \mathbb{1}_{A_2} h\left(\mathbb{1}_{A_1 \bigcup A_2}, \ldots, \mathbb{1}_{A_1 \bigcup A_2} \circ F^l\right) \circ F^p d\mu_\Delta + \mu_\Delta(A_2)$$
$$\int h\left(\mathbb{1}_{A_1 \bigcup A_2}, \ldots, \mathbb{1}_{A_1 \bigcup A_2} \circ F^l\right) d\mu_\Delta$$
$$+ 2 \int \mathbb{1}_{A_0} \mathbb{1}_{\bigcup_{j \leq l} F^{-j} A_2} \circ F^p d\mu_\Delta + 2\mu_\Delta(A_1)$$
$$\int \mathbb{1}_{\bigcup_{j \leq l} F^{-j} A_2} d\mu_\Delta$$
$$\leq \left| \int \mathbb{1}_{A_0} h\left(\mathbb{1}_{A_1 \bigcup A_2}, \ldots, \mathbb{1}_{A_1 \bigcup A_2} \circ F^l\right) \circ F^p d\mu_\Delta - \mu_\Delta(A_0)\right.$$
$$\left.\int h\left(\mathbb{1}_{A_1 \bigcup A_2}, \ldots, \mathbb{1}_{A_1 \bigcup A_2} \circ F^l\right) d\mu_\Delta \right|$$
$$+ \left| \int \mathbb{1}_{A_2} h\left(\mathbb{1}_{A_1 \bigcup A_2}, \ldots, \mathbb{1}_{A_1 \bigcup A_2} \circ F^l\right) \circ F^p d\mu_\Delta\right.$$

$$
\begin{aligned}
&- \int h\left(\mathbb{1}_{A_1\bigcup A_2}, \ldots, \mathbb{1}_{A_1\bigcup A_2}\circ F^l\right) d\mu_\Delta \\
&\times \mu_\Delta(A_2)\Big| + 2\mu_\Delta(A_2)\int h\left(\mathbb{1}_{A_1\bigcup A_2}, \ldots, \mathbb{1}_{A_1\bigcup A_2}\circ F^l\right) d\mu_\Delta \\
&+ 2\left|\int \mathbb{1}_{A_0}\mathbb{1}_{\bigcup_{j\le l}F^{-j}A_2}\circ F^p d\mu_\Delta - \int \mathbb{1}_{A_0} d\mu_\Delta \int \mathbb{1}_{\bigcup_{j\le l}F^{-j}A_2}\circ F^p d\mu_\Delta\right| \\
&+ 2\int \mathbb{1}_{A_0} d\mu_\Delta \int \mathbb{1}_{\bigcup_{j\le l}F^{-j}A_2}\circ F^p d\mu_\Delta + 2\int \mathbb{1}_{A_1} d\mu_\Delta \int \mathbb{1}_{\cup_{j\le l}F^{-j}A_2} d\mu_\Delta.
\end{aligned}
\tag{3.7}
$$

Claim: $h\left(\mathbb{1}_{A_1\bigcup A_2}, \ldots, \mathbb{1}_{A_1\bigcup A_2}\circ F^l\right)$ is $\sigma(\bigcup_{k\ge 0}\mathcal{Q}_k)$-measurable. Observe, that for any $(x,l), (y,l) \in \Delta$, $x, y \in \gamma^s \in \Gamma^s$ we have $F^{j-p}(x,l) = (x', l')$, $F^{j-p}(y,l) = (y', l')$ for some $l' \in \mathbb{N}$ and some $x', y' \in (\gamma^s)' \in \Gamma^s$. Since $\mathbb{1}_{A_1\bigcup A_2}$ is $\sigma(\bigcup_{k\ge 0}\mathcal{Q}_k)$-measurable, $\mathbb{1}_{A_1\cup A_2}\circ F^{j-p}(x,l) = \mathbb{1}_{A_1\cup A_2}\circ F^{j-p}(y,l)$. Therefore $h\Big(\mathbb{1}_{A_1\bigcup A_2}, \ldots, \mathbb{1}_{A_1\bigcup A_2}\circ F^l\Big)$ is $\sigma(\cup_{k\ge 0}\mathcal{Q}_k)$-measurable.

Claim: $\mathbb{1}_{\bigcup_{j\le l}F^{-j}A_2}$ is also $\sigma(\bigcup_{k\ge 0}\mathcal{Q}_k)$-measurable.

Indeed, each set $F^{-j}A_2$ is $\sigma(\bigcup_{k\ge 0}\mathcal{Q}_k)$-measurable. So their union is also $\sigma(\bigcup_{k\ge 0}\mathcal{Q}_k)$-measurable.

Claim: $\mu_\Delta(A_2) \le \mu\left(B_{r+C4^\alpha p^{-\alpha}}(z) \setminus B_{r-C4^\alpha p^{-\alpha}}(z)\right)$.

Observe that

$$\mu_\Delta(A_2) \le \mu_\Delta\left(F^{-m}\pi^{-1}\pi F^m A_2\right) = \mu\left(\pi F^m A_2\right).$$

By definition of $A_2 := \bigsqcup_{Q\in\mathcal{Q}_{2m}: Q\bigcap(A_0\setminus A_1)\ne\emptyset} Q$, for each Q, contained in A_2, there exist $x_1, x_2 \in Q$, such that $\pi(F^m x_1) \in B_r(z)$, $\pi(F^m x_2) \notin B_r(z)$. Now, by making use of (3.2) and $m = \lfloor p/4\rfloor$, we obtain $\pi(F^m A_2) \subseteq B_{r+C4^\alpha p^{-\alpha}}(z) \setminus B_{r-C4^\alpha p^{-\alpha}}(z)$. Hence the claim holds.

Having these claims and (3.6), we can continue estimate of (3.7) as

$$
\begin{aligned}
&\le Cp^{1-\xi}\mu_\Delta(A_0) + Cp^{1-\xi}\mu_\Delta(A_2) + 2\mu_\Delta(A_2) + 2Cp^{1-\xi}\mu_\Delta(A_0) \\
&\quad + 2\mu_\Delta(A_0)\mu_\Delta\Big(\bigcup_{j\le l}F^{-j}A_2\Big) + 2\mu_\Delta(A_1)\mu_\Delta\Big(\bigcup_{j\le l}F^{-j}A_2\Big) \\
&\le Cp^{1-\xi}\mu_\Delta(A_1) + Cp^{1-\xi}\mu_\Delta(A_2) + Cp^{1-\xi}\mu_\Delta(A_2) + 2\mu_\Delta(A_2) \\
&\quad + 2Cp^{1-\xi}\mu_\Delta(A_1) + 2Cp^{1-\xi}\mu_\Delta(A_2) \\
&\quad + 2\Big[\mu_\Delta(A_1) + \mu_\Delta(A_2)\Big]\mu_\Delta\Big(\bigcup_{j\le l}F^{-j}A_2\Big) + 2\mu_\Delta(A_1)\mu_\Delta\Big(\bigcup_{j\le l}F^{-j}A_2\Big).
\end{aligned}
\tag{3.8}
$$

Let $q^{-\alpha} := C4^\alpha p^{-\alpha}$, recall that $\mu(B_r(z)) = \mu_\Delta(A_1)$ and $\mu_\Delta(A_2) \le \mu\Big(B_{r+q^{-\alpha}}(z)\setminus B_{r-q^{-\alpha}}(z)\Big)$. Therefore estimate of (3.8) can be continued as

$$
\begin{aligned}
&\le Cp^{1-\xi}\mu(B_r(z)) + 2Cp^{1-\xi}\mu\Big(B_{r+q^{-\alpha}}(z)\setminus B_{r-q^{-\alpha}}(z)\Big) \\
&\quad + 2\mu\Big(B_{r+q^{-\alpha}}(z)\setminus B_{r-q^{-\alpha}}(z)\Big) + 2Cp^{1-\xi}\mu(B_r(z))
\end{aligned}
$$

$$
\begin{aligned}
&+2Cp^{1-\xi}\mu\Big(B_{r+q^{-\alpha}}(z)\setminus B_{r-q^{-\alpha}}(z)\Big)+2l\mu(B_r(z))\mu\Big(B_{r+q^{-\alpha}}(z)\setminus B_{r-q^{-\alpha}}(z)\Big)\\
&+2l\Big[\mu(B_r(z))+\mu\Big(B_{r+q^{-\alpha}}(z)\setminus B_{r-q^{-\alpha}}(z)\Big)\Big]\mu\Big(B_{r+q^{-\alpha}}(z)\setminus B_{r-q^{-\alpha}}(z)\Big)\\
\le\ &4Cp^{1-\xi}\mu\Big(B_{r+q^{-\alpha}}(z)\Big)+2\mu\Big(B_{r+q^{-\alpha}}(z)\setminus B_{r-q^{-\alpha}}(z)\Big)\\
&+4l\mu\Big(B_{r+q^{-\alpha}}(z)\Big)\mu\Big(B_{r+q^{-\alpha}}(z)\setminus B_{r-q^{-\alpha}}(z)\Big).
\end{aligned}
$$

To finish proof we replace $q^{-\alpha}$ with $C4^{\alpha}p^{-\alpha}$. □

Proposition 1 (Functional Poisson limit laws). *Let* $p := \left\lfloor \left\lfloor \frac{T}{\mu(B_r(z))} \right\rfloor^{\frac{\dim_H \mu - \epsilon}{\dim_H \mu}} \right\rfloor$, *where* ϵ *is so small that*

$$
\alpha(\dim_H \mu - \epsilon) > \frac{\dim_H \mu}{\dim_H \mu - \epsilon} > 1 \text{ (see Assumption 1).}
$$

Then for almost any $z \in \mathcal{M}$ *there is* $r_z > 0$, *such that for any* $r < r_z$

$$
d_{TV}\left(N^{r,T,z}, P\right) \precsim_{T,\xi,\epsilon} R_1(r) + R_2(r,z) + R_3(r,z) + R_4(r,z)
$$

where

$$
\begin{aligned}
R_1(r) &:= r^{\dim_H \mu - \epsilon} + r^{\frac{(\dim_H \mu - \epsilon)^2}{\dim_H \mu}(\xi - 1)} + r^{\frac{\epsilon(\dim_H \mu - \epsilon)}{\dim_H \mu}}\\
R_2(r,z) &:= \frac{1}{\mu(B_r(z))}\mu\left(B_{r+C'r^{\frac{(\dim_H \mu-\epsilon)^2\alpha}{\dim_H \mu}}}(z)\setminus B_{r-C'r^{\frac{(\dim_H \mu-\epsilon)^2\alpha}{\dim_H \mu}}}(z)\right)\\
R_3(r,z) &:= \frac{1}{\mu(B_r(z))^2}\mu\left(B_{r+C'r^{\frac{(\dim_H \mu-\epsilon)^2\alpha}{\dim_H \mu}}}(z)\setminus B_{r-C'r^{\frac{(\dim_H \mu-\epsilon)^2\alpha}{\dim_H \mu}}}(z)\right)^2\\
R_4(r,z) &:= \frac{1}{\mu(B_r(z))}\int_{B_r(z)} \mathbb{1}_{\bigcup_{1\le j\le p} f^{-j}B_r(z)}\, d\mu,
\end{aligned}
$$

and C' *depends on* T, α, ϵ *and on constant* C *in Definition 5.*

Obviously, $\lim_{r\to 0} R_1(r) = 0$ *and* $R_3(r,z) \le R_2(r,z)$. *Therefore, to claim* $d_{TV}\left(N^{r,T,z}, P\right) \to 0$ *with certain convergence rate, it suffices to prove* $\lim_{r\to 0} R_2(r,z) = \lim_{r\to 0} R_4(r,z) = 0$ *for a.e.* $z \in \mathcal{M}$ *with certain convergence rates in Sects. 4 and 5.*

Proof. Let $n := \left\lfloor \frac{T}{\mu(B_r(z))} \right\rfloor$, $p := \left\lfloor n^{\frac{\dim_H \mu - \epsilon}{\dim_H \mu}} \right\rfloor$, $q^{-\alpha} := C4^{\alpha}p^{-\alpha}$, $\eta := \frac{(\dim_H \mu - \epsilon)^2\alpha}{\dim_H \mu}$ where ϵ is so small that $\alpha(\dim_H \mu - \epsilon) > \frac{\dim_H \mu}{\dim_H \mu - \epsilon} > 1$ (in view of Assumption 1).

Therefore for a.e. $z \in \mathcal{M}$ there exists $r_z > 0$ such that for any $r < r_z$, in view of Assumption 1,

$$
\begin{aligned}
&Tr^{\epsilon - \dim_H \mu} \precsim n \precsim Tr^{-\dim_H \mu - \epsilon},\\
&p^{-\alpha} \precsim n^{-\frac{\alpha(\dim_H \mu - \epsilon)}{\dim_H \mu}} \precsim_{T,\alpha} r^{\frac{\dim_H \mu - \epsilon}{\dim_H \mu}\alpha(\dim_H \mu - \epsilon)} = r^{\eta} \ll r.
\end{aligned}
$$

Hence by Lemmas 1 and 3, for any disjoint sets $I_1, \ldots, I_m \subseteq [0, n]$ we have

$$
d_{TV}\Big(\Big(X_{I_1}, \ldots, X_{I_m}\Big), \Big(\hat{X}_{I_1}, \ldots, \hat{X}_{I_m}\Big)\Big)
$$

$$
\begin{aligned}
&\leq 2 \sum_{0\leq l\leq n-p} \sup_{h\in[0,1]} \left|\mathbb{E}\Big[\mathbb{1}_{X_0=1} h(X_p,\ldots,X_{n-l})\Big] - \mathbb{E}\mathbb{1}_{X_0=1}\mathbb{E}h(X_p,\ldots,X_{n-l})\right| \\
&\quad + 4(n-p)\mathbb{E}\left(\mathbb{1}_{X_0=1}\mathbb{1}_{\sum_{1\leq j\leq p-1} X_j\geq 1}\right) + 4p(n-p)\mu(B_r(z))^2 + 4p\mu(B_r(z)) \\
&\leq \sum_{0\leq l\leq n-p} 8Cp^{1-\xi}\mu\Big(B_{r+q^{-\alpha}}(z)\Big) + \Big[4 + 8(n-l-p)\mu\Big(B_{r+q^{-\alpha}}(z)\Big)\Big] \\
&\quad \times \mu\Big(B_{r+q^{-\alpha}}(z)\setminus B_{r-q^{-\alpha}}(z)\Big) + 4(n-p)\mathbb{E}\left(\mathbb{1}_{X_0=1}\mathbb{1}_{\sum_{1\leq j\leq p-1} X_j\geq 1}\right) \\
&\quad + 4p(n-p)\mu(B_r(z))^2 + 4p\mu(B_r(z)) \\
&\leq 8nCp^{1-\xi}\mu\Big(B_{r+q^{-\alpha}}(z)\Big) + \Big[4 + 8n\mu\Big(B_{r+q^{-\alpha}}(z)\Big)\Big] n\mu\Big(B_{r+q^{-\alpha}}(z)\setminus B_{r-q^{-\alpha}}(z)\Big) \\
&\quad + 4n\mathbb{E}\left(\mathbb{1}_{X_0=1}\mathbb{1}_{\sum_{1\leq j\leq p-1} X_j\geq 1}\right) + 4pn\mu(B_r(z))^2 + 4p\mu(B_r(z)) \\
&\leq 8Cnp^{1-\xi}\mu(B_r(z)) + 8Cnp^{1-\xi}\mu\Big(B_{r+q^{-\alpha}}(z)\setminus B_{r-q^{-\alpha}}(z)\Big) \\
&\quad + \Big[4 + 8n\mu(B_r(z)) + 8n\mu\Big(B_{r+q^{-\alpha}}(z)\setminus B_{r-q^{-\alpha}}(z)\Big)\Big] n\mu\Big(B_{r+q^{-\alpha}}(z)\setminus B_{r-q^{-\alpha}}(z)\Big) \\
&\quad + 4n\mathbb{E}\left(\mathbb{1}_{X_0=1}\mathbb{1}_{\sum_{1\leq j\leq p} X_j\geq 1}\right) + 4pn\mu(B_r(z))^2 + 4p\mu(B_r(z)).
\end{aligned}
$$

Note that

$$
p \approx_{T,\epsilon} \mu(B_r(z))^{-\frac{\dim_H \mu-\epsilon}{\dim_H \mu}}, \quad n\mu(B_r(z)) \leq T.
$$

Thus we can continue the inequality above as

$$
\begin{aligned}
&\precsim_{T,\xi,\epsilon} \mu(B_r(z))^{\frac{\dim_H \mu-\epsilon}{\dim_H \mu}(\xi-1)} + \frac{\mu\Big(B_{r+q^{-\alpha}}(z)\setminus B_{r-q^{-\alpha}}(z)\Big)}{\mu(B_r(z))} \\
&\quad + \left[\frac{\mu\Big(B_{r+q^{-\alpha}}(z)\setminus B_{r-q^{-\alpha}}(z)\Big)}{\mu(B_r(z))}\right]^2 \\
&\quad + \frac{1}{\mu(B_r(z))}\int_{B_r(z)} \mathbb{1}_{\bigcup_{1\leq j\leq p} f^{-j}B_r(z)} d\mu + \mu(B_r(z))^{\frac{\epsilon}{\dim_H \mu}}.
\end{aligned}
$$

By applying Theorems 2 and 3 of [3] to $(\hat{X}_i)_{i\geq 0}$ one gets that for any disjoint intervals $J_1,\ldots,J_m \subseteq [0,T]$,

$$
\sup_{h\in[0,1]}\left|\mathbb{E}h\Big(P(J_1),\ldots,P(J_m)\Big) - h\left(\hat{X}_{J_1'},\ldots,\hat{X}_{J_m'}\right)\right| \leq 4n\mu(B_r(z))^2 \precsim_T \mu(B_r(z)),
$$

where $J_i' := J_i/\mu(B_r(z)) := \{x : x\mu(B_r(z)) \in J_i\} \subseteq [0,n]$ for all $i = 1,\ldots,m$.

Approximate now $(X_{J_1'},\ldots,X_{J_m'})$ by the Poisson point process P. Then

$$
\begin{aligned}
&\sup_{h\in[0,1]}\left|\mathbb{E}h\left(X_{J_1'},\ldots,X_{J_m'}\right) - h\Big(P(J_1),\ldots,P(J_m)\Big)\right| \\
&\quad \leq \sup_{h\in[0,1]}\left|\mathbb{E}h\Big(P(J_1),\ldots,P(J_m)\Big) - h\left(\hat{X}_{J_1'},\ldots,\hat{X}_{J_m'}\right)\right|
\end{aligned}
$$

$$+ d_{TV}\Big((X_{J_1'},\dots,X_{J_m'}),(\hat{X}_{J_1'},\dots,\hat{X}_{J_m'})\Big)$$

$$\precsim_{T,\xi,\epsilon} \mu(B_r(z)) + \mu(B_r(z))^{\frac{\dim_H\mu-\epsilon}{\dim_H\mu}(\xi-1)} + \frac{\mu\Big(B_{r+q^{-\alpha}}(z)\setminus B_{r-q^{-\alpha}}(z)\Big)}{\mu(B_r(z))}$$

$$+ \left[\frac{\mu\Big(B_{r+q^{-\alpha}}(z)\setminus B_{r-q^{-\alpha}}(z)\Big)}{\mu(B_r(z))}\right]^2$$

$$+ \frac{1}{\mu(B_r(z))}\int_{B_r(z)} \mathbb{1}_{\bigcup_{1\le j\le p} f^{-j}B_r(z)}d\mu + \mu(B_r(z))^{\frac{\epsilon}{\dim_H\mu}}.$$

Since σ-algebra $\mathcal{C} = \sigma\{\pi_A^{-1}B :$ any Borel sets $A \subseteq [0,T]$, $B \subseteq \mathbb{N}\}$ is generated by $\mathcal{C}' := \{\pi_{J_1}^{-1}A_1 \cap\dots\cap \pi_{J_m}^{-1}A_m :$ any $A_1,\dots,A_m \subseteq \mathbb{N}$, disjoint intervals $J_1,\dots,J_m \subseteq [0,T]\}$, we obtain the following functional Poisson approximation: for any $r < r_z$

$$d_{TV}\left(N^{r,T,z},P\right)$$

$$\le \sup_{\text{disjoint } J_i\subseteq[0,T],h\in[0,1]} \left|\mathbb{E}h\Big(P(J_1),\dots,P(J_m)\Big) - h\Big(N^{r,T,z}(J_1),\dots,N^{r,T,z}(J_m)\Big)\right|$$

$$= \sup_{\text{disjoint } J_i\subseteq[0,T],h\in[0,1]} \left|\mathbb{E}h\Big(P(J_1),\dots,P(J_m)\Big) - h\left(X_{J_1'},\dots,X_{J_m'}\right)\right|$$

$$\precsim_{T,\xi,\epsilon} \mu(B_r(z)) + \mu(B_r(z))^{\frac{\dim_H\mu-\epsilon}{\dim_H\mu}(\xi-1)} + \mu(B_r(z))^{\frac{\epsilon}{\dim_H\mu}}$$

$$+ \frac{\mu\Big(B_{r+q^{-\alpha}}(z)\setminus B_{r-q^{-\alpha}}(z)\Big)}{\mu(B_r(z))} + \left[\frac{\mu\left(B_{r+q^{-\alpha}}(z)\setminus B_{r-q^{-\alpha}}(z)\right)}{\mu(B_r(z))}\right]^2$$

$$+ \frac{1}{\mu(B_r(z))}\int_{B_r(z)} \mathbb{1}_{\bigcup_{1\le j\le p} f^{-j}B_r(z)}d\mu$$

$$\precsim_{T,\xi,\epsilon} r^{\dim_H\mu-\epsilon} + r^{\frac{(\dim_H\mu-\epsilon)^2}{\dim_H\mu}(\xi-1)} + r^{\frac{\epsilon(\dim_H\mu-\epsilon)}{\dim_H\mu}} + \frac{\mu\Big(B_{r+C'r^\eta}(z)\setminus B_{r-C'r^\eta}(z)\Big)}{\mu(B_r(z))}$$

$$+ \left[\frac{\mu\left(B_{r+C'r^\eta}(z)\setminus B_{r-C'r^\eta}(z)\right)}{\mu(B_r(z))}\right]^2 + \frac{1}{\mu(B_r(z))}\int_{B_r(z)} \mathbb{1}_{\bigcup_{1\le j\le p} f^{-j}B_r(z)}d\mu,$$

where the last "$\precsim$" follows from $q^{-\alpha} \le C'r^\eta$ for some C' depending on T,α,ϵ and constant C in Definition 5. To finish proof replace η with $\frac{(\dim_H\mu-\epsilon)^2\alpha}{\dim_H\mu}$. □

Definition 7. *(Short returns and coronas)* Let p be the one in Proposition 1. Define

1. **Short returns**:

$$\int_{B_r(z)} \mathbb{1}_{\bigcup_{1\le j\le p} f^{-j}B_r(z)}d\mu.$$

2. **Coronas**:

$$\mu\left(B_{r+C'r^{\frac{(\dim_H\mu-\epsilon)^2\alpha}{\dim_H\mu}}}(z)\Big\backslash B_{r-C'r^{\frac{(\dim_H\mu-\epsilon)^2\alpha}{\dim_H\mu}}}(z)\right).$$

It will be shown below that these quantities tend to 0 for almost all $z \in \mathcal{M}$ with certain convergence rates.

4. Proof of Theorem 3

4.1. Properties of the first return to Λ. Before studying convergence rates for short returns and coronas we will prove several lemmas for the first return time R and for the first return map $f^R : \Lambda \to \Lambda$ under Assumptions 1 and 2.

Lemma 4. *The map* $f^R : \Lambda \to \Lambda$ *is bijective.*

Proof. We will show first that f^R is **one-to-one**. Suppose that $f^R(x) = f^R(y)$ for $x, y \in \Lambda$. If $x, y \in \Lambda_i$ for some i, then $f^{R_i}(x) = f^{R_i}(y)$. On the other hand, it follows from Assumption 1 that f is bijective on $\bigcup_{i\geq 1}\bigcup_{j<R_i} f^j(\Lambda_i)$. Thus we have inductively the following reduction

$$f\left(f^{R_i-1}x\right) = f\left(f^{R_i-1}y\right) \Rightarrow f\left(f^{R_i-2}x\right) = f\left(f^{R_i-2}y\right)$$
$$\Rightarrow \cdots \Rightarrow f(x) = f(y) \Rightarrow x = y.$$

Let $x \in \Lambda_i$, $y \in \Lambda_j$ for some $i \neq j$ and $f^R(x) = f^R(y)$. Without any loss of generality, we may assume that $R_i < R_j$. Then $f^{R_i}(x) = f^{R_j}(y)$. Again, by Assumptions 1

$$f\left(f^{R_i-1}x\right) = f\left(f^{R_j-1}y\right) \Rightarrow f\left(f^{R_i-2}x\right) = f\left(f^{R_j-2}y\right)$$
$$\Rightarrow \cdots \Rightarrow x = f^{R_j-R_i}y \in \Lambda.$$

But the first return time of y to Λ is R_j, i.e., $f^{R_j-R_i}y \notin \Lambda$. So we came to a contradiction, and therefore this case cannot occur.

We show now that f^R is **onto**. Let $y \in \Lambda$ and $y \in \Lambda_i$ for some i. Then $y \in \bigcup_{i\geq 1}\bigcup_{j<R_i} f^j(\Lambda_i)$. By Assumption 1 f is bijective on $\bigcup_{i\geq 1}\bigcup_{j<R_i} f^j(\Lambda_i)$. Therefore there exists $x' \in \bigcup_{i\geq 1}\bigcup_{j<R_i} f^j(\Lambda_i)$, i.e., there is $x \in \Lambda_k$ such that $f^j(x) = x'$, where $j < R_k$ and $f(x') = f^{j+1}(x) = y$. Since R_k is the first return time for x, then $j+1 = R_k$ and $f^R(x) = y$. □

Lemma 5. *The following properties hold*

$$\pi : \Delta \to \bigcup_{i\geq 1}\bigcup_{j<R_i} f^j(\Lambda_i) \text{ is bijective,}$$
$$\pi : \Delta_0 \to \Lambda \text{ is identity,}$$
$$\pi : \Delta_{\geq 1} \to \bigcup_{i\geq 1}\bigcup_{1\leq j<R_i} f^j(\Lambda_i) \text{ is bijective,}$$
$$\text{where } \Delta_{\geq 1} := \{(x,l) \in \Lambda \times \mathbb{N} : 1 \leq l < R(x)\},$$
$$\Delta_0 := \{(x,0) : x \in \Lambda\} \subseteq \Delta.$$

Proof. Clearly, it is enough to prove just the first statement. By definition of Δ the first map π is onto. Let us show now that it is actually one-to-one. For all $(x,l), (x',l') \in \Delta$ with $\pi(x,l) = \pi(x',l')$ it holds that $f^l(x) = f^{l'}(x')$. Without loss of generality, let $l \leq l'$. By Assumption 1 f is bijective on $\bigcup_{i\geq 1}\bigcup_{j<R_i} f^j(\Lambda_i)$. Then

$$f\left(f^{l-1}x\right) = f\left(f^{l'-1}y\right) \Rightarrow f\left(f^{l-2}x\right) = f\left(f^{l'-2}y\right) \Rightarrow \cdots \Rightarrow x = f^{l'-l}y.$$

Since $x, y \in \Lambda$ and $l' - l$ is less than first return time of y, one gets that $l' = l$ and $x = y$. □

For every $z \in \bigcup_{i\geq 1}\bigcup_{0\leq j<R_i} f^j(\Lambda_i)$ we define

$$j_z := \min\{n \in \mathbb{N} : f^{-n}(z) \in \text{int}\,(\Lambda)\}. \tag{4.1}$$

Recall that $\mu\{\text{int}\,(\Lambda)\} > 0$, so $j_z < \infty$ for almost every $z \in \bigcup_{i\geq 1}\bigcup_{0\leq j<R_i} f^j(\Lambda_i)$ by Birkhoff's ergodic theorem.

Lemma 6 (Pulling metric balls back to Λ). *There exists a small enough* $r > 0$ *such that*

$$\mu\Big(f^{-j_z}B_r(z)\bigcap \Lambda^c\Big) = 0, \quad \mu_\Delta\Big(\Big\{\pi^{-1}f^{-j_z}B_r(z)\Big\}\bigcap \Delta_{\geq 1}\Big) = 0,$$

and

$$\mu_\Delta\Big(\Big\{\pi^{-1}f^{-j_z}B_r(z)\Big\}\bigcap \Delta_0\Big) = 1,$$

where j_z *is defined in* (4.1).

Proof. By Assumption 2 there is a small neighborhood $U_{z'} \subseteq \mathcal{M}$ of $z' \in \text{int}\,(\Lambda)$ such that

$$\mu\Big(U_{z'}\bigcap \Lambda^c\Big) = \mu\left(U_{z'}\bigcap\Big\{\bigcup_{i\geq 1}\bigcup_{1\leq j<R_i} f^j(\Lambda_i)\Big\}\right) = 0.$$

Because f^{j_z} is a local C^1-diffeomorphism, there exists a small ball $B_r(z)$ such that $f^{-j_z}B_r(z) \subseteq U_{z'}$. So $\mu\left(f^{-j_z}B_r(z)\bigcap\Lambda^c\right) = \mu\Big(f^{-j_z}B_r(z)\bigcap\Big\{\bigcup_{i\geq 1}\bigcup_{1\leq j<R_i} f^j(\Lambda_i)\Big\}\Big) = 0$. Hence by Lemma 5,

$$\mu_\Delta\Big(\Big\{\pi^{-1}f^{-j_z}B_r(z)\Big\}\bigcap\Delta_{\geq 1}\Big) = 0 \text{ and } \mu_\Delta\Big(\Big\{\pi^{-1}f^{-j_z}B_r(z)\Big\}\bigcap\Delta_0\Big) = 1.$$

□

Definition 8 *(Topological balls).* We say that a set $U_r(z') \subseteq \mathcal{M}$ is a topological ball if there is a ball $B_r(z) \subseteq \mathcal{M}$ and a map T of $B_r(z)$, such that

$$T : B_r(z) \to U_r(z') \text{ is a } C^1\text{-diffeomorphism and } T(z) = z'.$$

We say that r, z' are the radius and the center of $U_r(z')$.

For almost every $z \in \bigcup_{i\geq 1}\bigcup_{0\leq j<R_i} f^j(\Lambda_i)$ we have $z = f^{j_z}(z')$, where $z' \in \text{int}\,(\Lambda)$ (see (4.1)). Since f^{j_z} is a local diffeomorphism (by Assumption 1), then $U_r\left(f^{-j_z}z\right) := f^{-j_z}B_r(z)$ is a topological ball for sufficiently small $r > 0$.

Lemma 7 (Comparisons of topological and metric balls). *There exist constants* $C_z \geq 1$ *and* $r_z > 0$*, such that for any* $r < r_z$

$$B_{C_z^{-1}r}\left(f^{-j_z}z\right) \subseteq U_r\left(f^{-j_z}z\right) \subseteq B_{C_z r}\left(f^{-j_z}z\right).$$

Proof. Since f^{-j_z} is a local diffeomorphism near z, then $f^{-j_z}(\partial B_r(z)) = \partial U_r\left(f^{-j_z}z\right)$. We will estimate $\sup_{x\in\partial U_r(f^{-j_z}z)} d\left(x, f^{-j_z}z\right)$ and $\inf_{x\in\partial U_r(f^{-j_z}z)} d\left(x, f^{-j_z}z\right)$. For any $x \in \partial U_r\left(f^{-j_z}z\right)$ one has $f^{j_z}(x) \in \partial B_r(z)$. Let $(\gamma_t)_{0\le t\le 1}$ be the geodesic connecting x and $f^{-j_z}z$, and a curve $\hat{\gamma} := f^{j_z}\gamma$ is connecting $f^{j_z}x$ and z. Then

$$d\left(x, f^{-j_z}z\right) = \int_0^1 \sqrt{\langle \gamma_t', \gamma_t'\rangle_{\gamma_t}}dt = \int_0^1 \sqrt{\langle Df^{-j_z}\hat{\gamma}_t', Df^{-j_z}\hat{\gamma}_t'\rangle_{\gamma_t}}dt. \tag{4.2}$$

If r is sufficiently small (i.e., $r < r_z$ for some $r_z > 0$), then Df^{-j_z} and the Riemannian metric $\langle\cdot,\cdot\rangle_{\gamma_t}$ are close to $Df^{-j_z}(z)$ and $\langle\cdot,\cdot\rangle_z$, respectively. Then there exists $C_z \ge 1$ such that

$$d\left(x, f^{-j_z}z\right) \ge C_z^{-1}\int_0^1 \sqrt{\langle \hat{\gamma}_t', \hat{\gamma}_t'\rangle_{\hat{\gamma}_t}}dt \ge C_z^{-1}d\left(f^{j_z}x, z\right) = C_z^{-1}r.$$

Similarly, let $(\gamma_t)_{0\le t\le 1}$ be a curve connecting x and $f^{-j_z}z$, such that $\hat{\gamma} := f^{j_z}\gamma$ is geodesic connecting $f^{j_z}x$ and z. Then

$$\begin{aligned} d\left(x, f^{-j_z}z\right) \le& \int_0^1 \sqrt{\langle \gamma_t', \gamma_t'\rangle_{\gamma_t}}dt = \int_0^1 \sqrt{\langle Df^{-j_z}\hat{\gamma}_t', Df^{-j_z}\hat{\gamma}_t'\rangle_{\gamma_t}}dt \\ \le& C_z\int_0^1 \sqrt{\langle \hat{\gamma}_t', \hat{\gamma}_t'\rangle_{\hat{\gamma}_t}}dt = C_z r, \end{aligned}$$

which proves lemma. □

Definition 9 *(Two-sided cylinders in* Λ*).* Since $f^R : \Lambda \to \Lambda$ is bijective, we can define two-sided cylinders as

$$\begin{aligned} \xi_{i_{-n}\ldots i_0\ldots i_n} :=& \left(f^R\right)^n \Lambda_{i_{-n}} \bigcap \left(f^R\right)^{n-1}\Lambda_{i_{-(n-1)}} \bigcap\cdots\bigcap \Lambda_{i_0}\bigcap\cdots\bigcap \left(f^R\right)^{-(n-1)}\Lambda_{i_{n-1}} \\ &\bigcap \left(f^R\right)^{-n}\Lambda_{i_n}. \end{aligned}$$

Introduce now a new partition of Λ as

$$\mathcal{M}_0 := \{\Lambda_i, i \ge 1\}, \quad \mathcal{M}_k := \bigvee_{0\le i\le k}\left(f^R\right)^{-i}\mathcal{M}_0.$$

Using Assumption 2 we obtain the following estimate. A proof here is standard. Therefore, we omit it.

Lemma 8 (Diameters of two-sided cylinders).

$$\operatorname{diam}\xi_{i_{-n}\ldots i_0\ldots i_n} \le 2C\beta^n,$$

where $C \ge 1$ *and* $\beta \in (0, 1)$ *are the same as that in Assumption 2.*

It is proved in [45,46] that $\widetilde{f^R} : (\widetilde{\Lambda}, \mu_{\widetilde{\Lambda}}) \to (\widetilde{\Lambda}, \mu_{\widetilde{\Lambda}})$ is exact and, hence, mixing. Together with (3.4) and Corollary 2.3 (b) of [35], we have the following estimate for decay of correlations. Again, a proof is standard, and we omit it.

Lemma 9 (Decay of correlations for $f^R : \Lambda \to \Lambda$). *There exist constants* $C'' > 1$, $\beta_1 \in (0, 1)$*, such that for any one-sided cylinder* $\xi_{i_0\ldots i_n} \in \mathcal{M}_n$ *and any* $A \in \sigma(\bigcup_{k\ge 0}\mathcal{M}_k)$

$$\left|\int \mathbb{1}_{\xi_{i_0\ldots i_n}} 1_A \circ \left(f^R\right)^{2n} d\mu_\Lambda - \int \mathbb{1}_{\xi_{i_0\ldots i_n}} d\mu_\Lambda \int 1_A d\mu_\Lambda\right| \le C''\beta_1^n \mu_\Lambda(\xi_{i_0\ldots i_n}).$$

4.2. Short returns. Recall that $n := \left\lfloor \frac{T}{\mu(B_r(z))} \right\rfloor$, $p := \left\lfloor n^{\frac{\dim_H \mu - \epsilon}{\dim_H \mu}} \right\rfloor$ (see Proposition 1). Consider short returns to Λ. A reason to do this is that short returns problem on $\mathcal{M}$ can be turned into short returns problem on Λ (see Lemma 15). For any $z' \in \text{int}(\Lambda)$, a fixed positive integer $M > 0$, sufficiently small constants $\epsilon' > 0, r > 0$, such that $n^{\epsilon'} \ll p$, $B_{Mr}(z') \subseteq \text{int}(\Lambda)$ almost surely

$$\int_{B_{Mr}(z')} \mathbb{1}_{\bigcup_{1\le k\le p}(f^R)^{-k} B_{Mr}(z')} d\mu_\Lambda = \int_{B_{Mr}(z')} \mathbb{1}_{\bigcup_{1\le k\le N}(f^R)^{-k} B_{Mr}(z')} d\mu_\Lambda + \int_{B_{Mr}(z')} \mathbb{1}_{\bigcup_{N\le k\le n^{\epsilon'}}(f^R)^{-k} B_{Mr}(z')} d\mu_\Lambda + \int_{B_{Mr}(z')} \mathbb{1}_{\bigcup_{n^{\epsilon'}\le k\le p}(f^R)^{-k} B_{Mr}(z')} d\mu_\Lambda, \tag{4.3}$$

where $N = \left\lfloor \frac{-\log 2C}{\log \beta} \right\rfloor + 1$, and constants C and β are defined in Assumption 2.

We begin with very short returns described by

$$\int_{B_{Mr}(z')} \mathbb{1}_{\bigcup_{1\le k\le N}(f^R)^{-k} B_{Mr}(z')} d\mu_\Lambda.$$

Lemma 10 (Very short returns). *For almost every* $z' \in \text{int}(\Lambda)$ *and sufficiently small* $r_{N,M,z'} > 0$ *we have for any* $r < r_{N,M,z'}$

$$\int_{B_{Mr}(z')} \mathbb{1}_{\bigcup_{1\le k\le N}(f^R)^{-k} B_{Mr}(z')} d\mu_\Lambda = 0.$$

(Actually, a stronger result will be proved, i.e., for any $k \in \mathbb{N}$ *a map* $(f^R)^k$ *is a local diffeomorphism at almost every point* $z' \in \text{int}(\Lambda)$*).*

Proof. From Assumption 2 $\mu(\partial\Lambda) = 0$, and then $\mu\left(\bigcup_{i\in\mathbb{Z}} f^{-i}\partial\Lambda\right) = 0$. Since a quotient map $\widetilde{f^R} : \widetilde{\Lambda} \to \widetilde{\Lambda}$ is mixing, the maps $(\widetilde{f^R})^i$ are ergodic for all $i \ge 1$. Therefore the set of periodic points A_{per} of $\widetilde{f^R}$ has $\mu_{\widetilde{\Lambda}}\left(A_{per}\right) = 0$, and $\mu_\Lambda\left(\tilde{\pi}_\Lambda^{-1}\left(A_{per}\right)\right) = 0$ due to (3.4).

Choose now $z' \in \tilde{\pi}_\Lambda^{-1}\left(A_{per}^c\right) \bigcap \text{int}(\Lambda) \bigcap \left[\bigcup_{i\in\mathbb{Z}} f^{-i}\partial\Lambda\right]^c$. Then there is $r_M > 0$, s.t. for any $r < r_M$ almost surely $B_{Mr} \subseteq \text{int}(\Lambda)$. We will make now several claims.

Claim: For any $k \in \mathbb{N}$ let R^k be the k-th return time. Then $R^k\big|_{B_{Mr}(z')} = R^k(z')$ for any $k \in [1, N]$ if $r < r'_{N,M,z'}$ for small enough $r'_{N,M,z'} > 0$.

From choice of z' we have for any $k \in [0, N-1]$

$$f^m(z') \notin \Lambda \text{ for any } m \in \left[R^k(z') + 1, R^{k+1}(z') - 1\right],$$

and for any $k \in [0, N]$

$$f^{R^k(z')}(z') \in \text{int}(\Lambda).$$

Due to Assumption 1 there is $r_{N,M,z'} > 0$, such that if $r < r'_{N,M,z'}$, $B_{Mr}(z') \subseteq \text{int}(\Lambda)$ almost surely, then for any $k \in [0, N-1]$

$$f^m(B_{Mr}(z')) \subseteq \Lambda^c \text{ for any } m \in \left[R^k(z') + 1, R^{k+1}(z') - 1\right],$$

and for any $k \in [0, N]$

$$f^{R^k(z')}(B_{Mr}(z')) \subseteq \Lambda \text{ almost surely.}$$

Since $R, R^2, \ldots, R^N$ are consecutive return times to Λ, then $R^k\big|_{B_{Mr}(z')} = R^k(z')$ for any $k \in [1, N]$. Thus, this claim holds.

Claim: f^{R^k} for all $k \in [1, N]$ is a local diffeomorphism at z'.

This claim holds because f is a local diffeomorphism on $\bigcup_{i\geq 1}\bigcup_{0\leq j<R_i} f^j(\Lambda_i)$ and $R^k\big|_{B_{Mr}(z')} = R^k(z')$ for sufficiently small $r > 0$.

These two claims, together with the fact that $f^{R^k}(z') \in \operatorname{int}(\Lambda)$ are distinct for any $k \in [0, N]$, imply that there exists small enough $r_{z',M,N} > 0$, such that for any $r < r_{z',M,N}$ the sets $f^{R^k}(B_{Mr}(z'))$ are disjoint for all $k \in [0, N]$. Hence, lemma holds.
□

Before estimating moderate short returns $\int_{B_{Mr}(z')} \mathbb{1}_{\bigcup_{N\leq k\leq n^{\epsilon'}}(f^R)^{-k}B_{Mr}(z')} d\mu_\Lambda$ we will need one more lemma.

Lemma 11 (Recurrences). *There exists $r_M > 0$, such that for any $r < r_M$, $\gamma^u \in \Gamma^u$ and $i \geq N$, the following inequality holds*

$$\operatorname{Leb}_{\gamma^u}\left\{z' \in \Lambda \bigcap \gamma^u : d\left(\left(f^R\right)^{-i} z', z'\right) \leq Mr\right\} \precsim_{\dim\gamma^u} (Mr)^{\dim\gamma^u},$$

where a constant in $\precsim_{\dim\gamma^u}$ depends on $\dim\gamma^u$, but does not depend on $i \geq N$ and $\gamma^u \in \Gamma^u$.

Proof. We start with making

Claim: there are finitely many balls $\left\{B_{r_i'}(z_i') : 1 \leq i \leq N'\right\}$, where $N' \in \mathbb{N}$ depends only on Λ, such that all unstable fibers $\gamma^u \in \Gamma^u$ are almost flat in each $B_{r_i'}(z_i')$.

In view of Definition 5 of Γ^s (respectively, of Γ^u), for any $z'' \in \Lambda$ there exists a small open ball $B_{r''}(z'')$, such that all $\gamma^s \in \Gamma^s$ (respectively, $\gamma^u \in \Gamma^u$) intersecting $B_{r''}(z'')$ are almost flat and parallel. Since Λ is compact, one can find finitely many open balls $\{B_{r_1''}(z_1''), \ldots, B_{r_{N'}''}(z_{N'}'')\}$, which cover Λ. Hence, claim holds.

Take now any of these balls, say $B_{r_1''}(z_1'')$, and any $\gamma^u \in \Gamma^u$, $z_1', z_2' \in \{z' \in \Lambda \bigcap \gamma^u \bigcap B_{r_1''}(z_1'') : d((f^R)^{-i}z', z') \leq Mr\}$. Then for any $r < \tau_M := \min\{\frac{r_1''}{8M}, \ldots, \frac{r_{N'}''}{8M}\}$,

$$d\left(\left(f^R\right)^{-i} z_1', z_1'\right) \leq Mr, \quad d\left(\left(f^R\right)^{-i} z_2', z_2'\right) \leq Mr.$$

By making use of Assumption 2 together with $i \geq N$, we get that $C\beta^i \leq C\beta^N < 1/2$ and

$$\begin{aligned} d\left(z_1', z_2'\right) &\leq d\left(z_1', \left(f^R\right)^{-i} z_1'\right) + d\left(\left(f^R\right)^{-i} z_1', \left(f^R\right)^{-i} z_2'\right) + d\left(\left(f^R\right)^{-i} z_2', z_2'\right) \\ &\leq 2Mr + C\beta^i d(z_1', z_2') \leq 2Mr + d(z_1', z_2')/2. \end{aligned}$$

Thus $d(z_1', z_2') \leq 4Mr < r_1''/2$, and

$$\operatorname{diam}\left\{z' \in \Lambda \bigcap B_{r_1''}(z_1'') \bigcap \gamma^u : d\left(\left(f^R\right)^{-i} z', z'\right) \leq Mr\right\} \leq 4Mr.$$

Since γ^u in the ball $B_{r_1''}(z_1'')$ is almost flat, then its Lebesgue measure can be estimated by diameters, i.e.,

$$\mathrm{Leb}_{\gamma^u}\left\{z' \in \Lambda \bigcap B_{r_1''}(z_1'') : d\left(\left(f^R\right)^{-i} z', z'\right) \leq Mr\right\} \precsim_{\dim \gamma^u} (Mr)^{\dim \gamma^u}.$$

This estimate also holds for balls $B_{r_2''}\left(z_2''\right), \ldots, B_{r_{N'}''}\left(z_{N'}''\right)$. By summing over all balls and noting that $\Lambda \subseteq \bigcup_{1 \leq i \leq N'} B_{r_i''}\left(z_i''\right)$, we get

$$\mathrm{Leb}_{\gamma^u}\left\{z' \in \Lambda : d\left(\left(f^R\right)^{-i} z', z'\right) \leq Mr\right\} \precsim_{\dim \gamma^u} (Mr)^{\dim \gamma^u},$$

where the constant in $\precsim_{\dim \gamma^u}$ depends on $\dim \gamma^u$, but does not depend on $i \geq N$ and $\gamma^u \in \Gamma^u$. □

Lemma 12 (Moderate short returns). *Choose* $n = \left\lfloor \frac{T}{\mu(B_r(z))} \right\rfloor$. *Then for almost every* $z \in \mathcal{M}$, $z' \in \Lambda$ *there exists* $r_{z,z',M} > 0$, *such that for any* $r < r_{z,z',M}$

$$\int_{B_{Mr}(z')} \mathbb{1}_{\bigcup_{N \leq k \leq n^{\epsilon'}} (f^R)^{-k} B_{Mr}(z')} d\mu_\Lambda \precsim_{T,\epsilon} (Mr)^{\dim_H \mu - \epsilon} M^{\frac{\min\{\dim \gamma^u, \dim_H \mu\}}{6}} r^{\frac{\min\{\dim \gamma^u, \dim_H \mu\}}{12}},$$

where $\epsilon > 0$ *is the same as that in Proposition* 1, *and* $\epsilon' < \frac{\min\{\dim_H \mu, \dim \gamma^u\}}{12 \dim_H \mu + 12\epsilon}$

Proof It follows from (3.5), the relation $\left(f^R\right)_* \mu_\Lambda = \mu_\Lambda$ and Lemma 11 that

$$\begin{aligned}
\mu_\Lambda\left\{z' \in \Lambda : d\left(\left(f^R\right)^{k} z', z'\right) \leq Mr\right\} &= \mu_\Lambda\left\{z' \in \Lambda : d\left(\left(f^R\right)^{-k} z', z'\right) \leq Mr\right\} \\
&= \int \mu_{\gamma^u}\left\{z' \in \Lambda : d\left(\left(f^R\right)^{-k} z', z'\right) \leq Mr\right\} d\mu_\Lambda \\
&\precsim \int \mathrm{Leb}_{\gamma^u}\left\{z' \in \Lambda : d\left(\left(f^R\right)^{-k} z', z'\right) \leq Mr\right\} d\mu_\Lambda \\
&\precsim_{\dim \gamma^u} (Mr)^{\dim \gamma^u}.
\end{aligned}$$

By Assumption 1, for $\delta = \min\{\dim \gamma^u, \dim_H \mu\}/6 > 0$ and almost every $z' \in \Lambda$ there exists $r_{z',\delta} > 0$, such that $r^{\dim_H \mu + \delta} \leq \mu(B_r(z')) \leq r^{\dim_H \mu - \delta}$ for any $r < r_{z',\delta}$. Let $A_m := \{z' \in \Lambda : r_{z',\delta} > 1/m\}$. Then $\bigcup_m A_m = \Lambda$, and for any $z' \in A_m$ and any $r < 1/m$

$$r^{\dim_H \mu + \delta} \leq \mu\left(B_r(z')\right) \leq r^{\dim_H \mu - \delta} \tag{4.4}$$

and

$$\int_{B_{Mr}(z')} \mathbb{1}_{\bigcup_{N \leq k \leq n^{\epsilon'}} (f^R)^{-k} B_{Mr}(z')} d\mu_\Lambda \leq \int_{B_{Mr}(z')} \mathbb{1}_{\bigcup_{N \leq k \leq n^{\epsilon'}} d\left((f^R)^k y, y\right) \leq 2Mr} d\mu_\Lambda(y).$$

Let a kernel on $\Lambda \times A_m$ be $K(y, z') := \mathbb{1}_{B_{Mr}(z')}(y)$. If $r < 1/(3mM)$, then by (4.4)

$$\int K(y, z') d\mu_\Lambda(y) = \mu_\Lambda(B_{Mr}(z')) \precsim (Mr)^{\dim_H \mu - \delta}.$$

In order to estimate $\int K(y,z')\mathbb{1}_{A_m}(z')d\mu_\Lambda(z') = \mu_\Lambda\left(A_m \bigcap B_{Mr}(y)\right)$, observe that if $z'' \in A_m \bigcap B_{Mr}(y) \neq \emptyset$, then $B_{Mr}(y) \subseteq B_{3Mr}(z'')$. Again, by (4.4)

$$\int K(y,z')\mathbb{1}_{A_m}(z')d\mu_\Lambda(z') = \mu_\Lambda\left(A_m \bigcap B_{Mr}(y)\right) \leq \mu_\Lambda\left(B_{3Mr}(z'')\right) \precsim (3Mr)^{\dim_H \mu - \delta}.$$

Having the estimates of $K(y,z')$ above and Lemma 11, we can use the Schur's test (see Theorem 5.6 in [42]), i.e., for all $r < \min\{1/(3mM), r_M\}$

$$\int \mathbb{1}_{A_m}(z')d\mu_\Lambda(z') \int_{B_{Mr}(z')} \mathbb{1}_{d\left((f^R)^k y, y\right) \leq 2Mr} d\mu_\Lambda(y) \precsim_{\dim \gamma^u} (3Mr)^{\dim_H \mu - \delta}(Mr)^{\dim \gamma^u},$$

where the constant in $\precsim_{\dim \gamma^u}$ does not depend on M, r, m, k.

Choose now $r_i = i^{-\frac{4}{\min\{\dim_H \mu, \dim \gamma^u\}}} < \min\{1/(3mM), r_M\}$. Let $q := 2^{\frac{4(\dim_H \mu + \epsilon)}{\min\{\dim_H \mu, \dim \gamma^u\}}}$, $n' := Tr^{-\dim_H \mu - \epsilon}$, $n'_i := Tr_i^{-\dim_H \mu - \epsilon}$. Since $r^{\dim_H \mu + \delta} \leq \mu(B_r(z'))$, one gets

$$\begin{aligned}
&\int_{A_m} \frac{\int_{B_{Mr}(z')} \mathbb{1}_{\bigcup_{N \leq k \leq (qn')^{\epsilon'}} d\left((f^R)^k y, y\right) \leq 2Mr} d\mu_\Lambda(y)}{(n')^{\epsilon'} \mu(B_{Mr}(z'))(Mr)^\delta} d\mu_\Lambda(z') \\
&\precsim_{\dim_H \mu, \dim \gamma^u, \epsilon, \epsilon'} \frac{(Mr)^{\dim_H \mu - \delta + \dim \gamma^u}}{(Mr)^{\dim_H \mu + 2\delta}} \\
&\leq (Mr)^{\min\{\dim_H \mu, \dim \gamma^u\}/2}.
\end{aligned}$$

Therefore, by the Borel-Cantelli Lemma, for almost every $z' \in A_m$ there exists $N_{M,m,z'} > \min\{1/(3mM), r_M\}^{-\min\{\dim_H \mu, \dim \gamma^u\}/4}$, such that for any $i > N_{M,m,z'}$

$$\int_{B_{Mr_i}(z')} \mathbb{1}_{\bigcup_{N \leq k \leq (qn'_i)^{\epsilon'}} d\left((f^R)^k y, y\right) \leq 2Mr_i} d\mu_\Lambda(y) \leq (n'_i)^{\epsilon'} \mu(B_{Mr_i}(z'))(Mr_i)^\delta. \quad (4.5)$$

Hence for almost every $z' \in \Lambda$ (in particular, for $z' \in \text{int}(\Lambda)$) there is $m_{z'} > 0$, such that $z' \in A_{m_{z'}}$. Let $r_{z',M} := \min\left\{r_M, N_{M,m_{z'},z'}^{-\frac{4}{\min\{\dim_H \mu, \dim \gamma^u\}}}\right\}$. By Assumption 1, for ϵ from Proposition 1 there exists $r_{z,z',M} \in (0, r_{z',M})$, such that for any $r < r_{z,z',M}$

$$n \leq T\mu(B_r(z))^{-1} \leq n', \quad \mu(B_{Mr}(z')) \leq (Mr)^{\dim_H \mu - \epsilon}.$$

Then for any $r \in (0, r_{z,z',M})$ there exists $i > 0$, such that $r_{i+1} \leq r \leq r_i$, and the following estimates hold

$$\begin{aligned}
\int_{B_{Mr}(z')} \mathbb{1}_{\bigcup_{N \leq k \leq n^{\epsilon'}} (f^R)^{-k} B_{Mr}(z')} d\mu_\Lambda &\leq \int_{B_{Mr_i}(z')} \mathbb{1}_{\bigcup_{N \leq k \leq (n')^{\epsilon'}} d\left((f^R)^k y, y\right) \leq 2Mr_i} d\mu_\Lambda(y) \\
&\leq \int_{B_{Mr_i}(z')} \mathbb{1}_{\bigcup_{N \leq k \leq (n'_{i+1})^{\epsilon'}} d\left((f^R)^k y, y\right) \leq 2Mr_i} d\mu_\Lambda(y) \\
&\leq \int_{B_{Mr_i}(z')} \mathbb{1}_{\bigcup_{N \leq k \leq (qn'_i)^{\epsilon'}} d\left((f^R)^k y, y\right) \leq 2Mr_i} d\mu_\Lambda(y).
\end{aligned} \quad (4.6)$$

Hence, if $\epsilon' < \frac{\min\{\dim\gamma^u, \dim_H \mu\}}{12\dim_H \mu + 12\epsilon}$, then we can use (4.5) to continue the estimate (4.6). Namely,

$$\begin{aligned}
&\int_{B_{Mr}(z')} \mathbb{1}_{\bigcup_{N\le k\le n^{\epsilon'}} (f^R)^{-k} B_{Mr}(z')} d\mu_\Lambda \\
&\le (Mr_i)^{\dim_H \mu - \epsilon} (Mr_i)^{\min\{\dim\gamma^u, \dim_H \mu\}/6} (n_i')^{\epsilon'} \\
&\le (Mrr_i/r_{i+1})^{\dim_H \mu - \epsilon} (Mrr_i/r_{i+1})^{\min\{\dim\gamma^u, \dim_H \mu\}/6} (Tr^{-\dim_H \mu - \epsilon})^{\epsilon'} \\
&\precsim_{T,\epsilon} (Mr)^{\dim_H \mu - \epsilon} M^{\frac{\min\{\dim\gamma^u, \dim_H \mu\}}{6}} r^{\frac{\min\{\dim\gamma^u, \dim_H \mu\}}{12}}
\end{aligned}$$

where the last inequality holds because $r_i/r_{i+1} \precsim 1$. □

Lemma 13 (Longest short returns). *Let* $n = \left\lfloor \frac{T}{\mu(B_r(z))} \right\rfloor$, $p = \left\lfloor \left\lfloor \frac{T}{\mu(B_r(z))} \right\rfloor^{\frac{\dim_H \mu - \epsilon}{\dim_H \mu}} \right\rfloor$ *and* $n^{\epsilon'} \ll p$. *Then, for almost all* $z \in \mathcal{M}$, $z' \in \mathrm{int}\,(\Lambda)$, *there exists* $r_{z,z',M} > 0$, *such that for any* $r < r_{z,z',M}$

$$\begin{aligned}
&\int_{B_{Mr}(z')} \mathbb{1}_{\bigcup_{n^{\epsilon'}\le k\le p} (f^R)^{-k} B_{Mr}(z')} d\mu_\Lambda \\
&\precsim_{T,\epsilon} \left\{ (Mr)^{\dim_H \mu - \epsilon^3} + \mu_\Lambda \left[B_{Mr + 2C\beta_2^{r^{-(\dim_H \mu - \epsilon)\epsilon'}}}(z') \Big\backslash B_{Mr}(z') \right] \right\} \beta_2^{r^{-(\dim_H \mu - \epsilon)\epsilon'}} \\
&\quad + r^{-(\dim_H \mu + \epsilon)\frac{\dim_H \mu - \epsilon}{\dim_H \mu}} \left\{ (Mr)^{\dim_H \mu - \epsilon^3} + \mu_\Lambda \left[B_{Mr + 2C\beta_2^{r^{-(\dim_H \mu - \epsilon)\epsilon'}}}(z') \Big\backslash B_{Mr}(z') \right] \right\}^2,
\end{aligned}$$

where $C > 1$ *is the same as that in Definition 5 and* $\beta_2 \in (0, 1)$ *does not depend on* z, z', r, M.

Proof. Observe first that $n^{\epsilon'} \ll p$ implies $\epsilon' < \frac{\dim_H \mu - \epsilon}{\dim_H \mu}$. Cover $B_{Mr}(z')$ by two-sided cylinders $\xi_{i_{-m}\ldots i_0 \ldots i_m}$, where $m := n^{\epsilon'}/4$ and $\xi_{i_{-m}\ldots i_0\ldots i_m} \bigcap B_{Mr}(z') \neq \emptyset$. By Lemma 8 we have $\operatorname{diam} \xi_{i_{-m}\ldots i_0\ldots i_m} \le 2C\beta^m$. So $\bigcup \xi_{i_{-m}\ldots i_0\ldots i_m} \setminus B_{Mr}(z') \subseteq B_{Mr+2C\beta^m}(z') \setminus B_{Mr}(z')$. Denote $i_0 := p - n^{\epsilon'}$. Then

$$\begin{aligned}
&\int_{B_{Mr}(z')} \mathbb{1}_{\bigcup_{n^{\epsilon'}\le k\le p} (f^R)^{-k} B_{Mr}(z')} d\mu_\Lambda \\
&= \int \mathbb{1}_{B_{Mr}(z')} \mathbb{1}_{\ge 1} \circ \left[\mathbb{1}_{B_{Mr}(z')} + \cdots + \mathbb{1}_{B_{Mr}(z')} \circ \left(f^R\right)^{i_0} \right] \circ \left(f^R\right)^{n^{\epsilon'}} d\mu_\Lambda \\
&= \int \mathbb{1}_{(f^R)^{-m} B_{Mr}(z')} \mathbb{1}_{\ge 1} \circ \left[\mathbb{1}_{(f^R)^{-m} B_{Mr}(z')} + \cdots + \mathbb{1}_{(f^R)^{-m} B_{Mr}(z')} \circ \left(f^R\right)^{i_0} \right] \circ \left(f^R\right)^{n^{\epsilon'}} d\mu_\Lambda.
\end{aligned} \tag{4.7}$$

Let $A_1 := \left(f^R\right)^{-m} B_{Mr}(z')$, $A_2 := \left(f^R\right)^{-m} \bigcup_{\xi_{i_{-m}\ldots i_0\ldots i_m} \bigcap \partial B_{Mr}(z') \neq \emptyset} \xi_{i_{-m}\ldots i_0\ldots i_m}$, and $A_0 := \left(f^R\right)^{-m} \bigcup \xi_{i_{-m}\ldots i_0\ldots i_m}$. Observe that

$$A_0, A_2 \in \mathcal{M}_{2m}, \quad A_0 = A_1 \bigcup A_2, \quad \left(f^R\right)^m A_2 \subseteq B_{Mr+2C\beta^m}(z') \setminus B_{Mr-2C\beta^m}(z'),$$

and, moreover, the function $\mathbb{1}_{\geq 1} \circ [\mathbb{1}_{A_0} + \cdots + \mathbb{1}_{A_0} \circ (f^R)^{i_0}]$ is constant along each $\gamma^s \in \Gamma^s$. Then we can continue the equality (4.7) as

$$
\begin{aligned}
&= \Big[\int \mathbb{1}_{A_1} \mathbb{1}_{\geq 1} \circ \left[\mathbb{1}_{A_1} + \cdots + \mathbb{1}_{A_1} \circ \left(f^R\right)^{i_0} \right] \circ \left(f^R\right)^{n^{\epsilon'}} d\mu_\Lambda \\
&\quad - \int \mathbb{1}_{A_1} \mathbb{1}_{\geq 1} \circ \left[\mathbb{1}_{A_0} + \cdots + \mathbb{1}_{A_0} \circ \left(f^R\right)^{i_0} \right] \circ \left(f^R\right)^{n^{\epsilon'}} d\mu_\Lambda \Big] \\
&\quad - \int \mathbb{1}_{A_0 \setminus A_1} \mathbb{1}_{\geq 1} \circ \left[\mathbb{1}_{A_0} + \cdots + \mathbb{1}_{A_0} \circ \left(f^R\right)^{i_0} \right] \circ \left(f^R\right)^{n^{\epsilon'}} d\mu_\Lambda \\
&\quad + \Big[\int \mathbb{1}_{A_0} \mathbb{1}_{\geq 1} \circ \left[\mathbb{1}_{A_0} + \cdots + \mathbb{1}_{A_0} \circ \left(f^R\right)^{i_0} \right] \circ \left(f^R\right)^{n^{\epsilon'}} d\mu_\Lambda \\
&\quad - \int \mathbb{1}_{A_0} d\mu_\Lambda \int \mathbb{1}_{\geq 1} \circ \left[\mathbb{1}_{A_0} + \cdots + \mathbb{1}_{A_0} \circ \left(f^R\right)^{i_0} \right] \circ \left(f^R\right)^{n^{\epsilon'}} d\mu_\Lambda \Big] \\
&\quad + \int \mathbb{1}_{A_0} d\mu_\Lambda \int \mathbb{1}_{\geq 1} \circ \left[\mathbb{1}_{A_0} + \cdots + \mathbb{1}_{A_0} \circ \left(f^R\right)^{i_0} \right] \circ \left(f^R\right)^{n^{\epsilon'}} d\mu_\Lambda. \qquad (4.8)
\end{aligned}
$$

Apply now Lemma 9, the relation $A_0 \setminus A_1 \subseteq A_2$ and the inequality

$$
\left| \mathbb{1}_{\geq 1} \circ \left[\mathbb{1}_{A_1} + \cdots + \mathbb{1}_{A_1} \circ \left(f^R\right)^{i_0} \right] - \mathbb{1}_{\geq 1} \circ \left[\mathbb{1}_{A_0} + \cdots + \mathbb{1}_{A_0} \circ \left(f^R\right)^{i_0} \right] \right| \leq \mathbb{1}_{\bigcup_{j \leq i_0} (f^R)^{-j} A_2}
$$

to the right hand side of (4.8). Then it can be estimated as

$$
\begin{aligned}
&\leq \int \mathbb{1}_{A_1} \mathbb{1}_{\geq 1} \circ \left[\mathbb{1}_{A_2} + \cdots + \mathbb{1}_{A_2} \circ \left(f^R\right)^{i_0} \right] \circ \left(f^R\right)^{n^{\epsilon'}} \\
&\quad + \int \mathbb{1}_{A_2} \mathbb{1}_{\geq 1} \circ \left[\mathbb{1}_{A_0} + \cdots + \mathbb{1}_{A_0} \circ \left(f^R\right)^{i_0} \right] \circ \left(f^R\right)^{n^{\epsilon'}} d\mu_\Lambda + C'' \mu_\Lambda(A_0) \beta_1^{n^{\epsilon'}/2} + \mu_\Lambda(A_0)^2 i_0 \\
&\leq 2 \int \mathbb{1}_{A_0} \mathbb{1}_{\geq 1} \circ \left[\mathbb{1}_{A_0} + \cdots + \mathbb{1}_{A_0} \circ \left(f^R\right)^{i_0} \right] \circ \left(f^R\right)^{n^{\epsilon'}} d\mu_\Lambda + C'' \mu_\Lambda(A_0) \beta_1^{n^{\epsilon'}/2} + \mu_\Lambda(A_0)^2 i_0 \\
&\quad - 2 \int \mathbb{1}_{A_0} d\mu_\Lambda \int \mathbb{1}_{\geq 1} \circ \left[\mathbb{1}_{A_0} + \cdots + \mathbb{1}_{A_0} \circ \left(f^R\right)^{i_0} \right] \circ \left(f^R\right)^{n^{\epsilon'}} d\mu_\Lambda \\
&\quad + 2 \int \mathbb{1}_{A_0} d\mu_\Lambda \int \mathbb{1}_{\geq 1} \circ \left[\mathbb{1}_{A_0} + \cdots + \mathbb{1}_{A_0} \circ \left(f^R\right)^{i_0} \right] \circ \left(f^R\right)^{n^{\epsilon'}} d\mu_\Lambda. \qquad (4.9)
\end{aligned}
$$

By applying Lemma 9 again, with $i_0 \leq p$, $m = n^{\epsilon'}/4$, (4.9) can be estimated as

$$
\begin{aligned}
&\leq 3C'' \mu_\Lambda(A_0) \beta_1^{n^{\epsilon'}/2} + 3p\mu_\Lambda(A_0)^2 = 3C'' \mu_\Lambda \left[\left(f^R\right)^m A_0 \right] \beta_1^{n^{\epsilon'}/2} + 3p\mu_\Lambda \left[\left(f^R\right)^m A_0 \right]^2 \\
&\precsim \mu_\Lambda(B_{Mr}(z')) \beta_1^{n^{\epsilon'}/2} + \mu_\Lambda \left[B_{Mr + 2C\beta^{\frac{n^{\epsilon'}}{4}}}(z') \Big\backslash B_{Mr}(z') \right] \beta_1^{n^{\epsilon'}/2} \\
&\quad + p \left\{ \mu_\Lambda(B_{Mr}(z')) + \mu_\Lambda \left[B_{Mr + 2C\beta^{\frac{n^{\epsilon'}}{4}}}(z') \Big\backslash B_{Mr}(z') \right] \right\}^2. \qquad (4.10)
\end{aligned}
$$

By making use of Assumption 1, we choose now the same $\epsilon > 0$ as in Proposition 1. Then for almost all $z \in \mathcal{M}$, $z' \in \text{int}(\Lambda)$ there is $r_{z,z',M} > 0$ s.t. for any $r < r_{z,z',M}$

$$B_{Mr}(z') \subseteq \Lambda \text{ almost surely}, \quad Tr^{-\dim_H \mu+\epsilon} \precsim n \precsim Tr^{-\dim_H \mu-\epsilon},$$
$$r^{\dim_H \mu+\epsilon} \le \mu(B_r(z)) \le r^{\dim_H \mu-\epsilon}, \quad (Mr)^{\dim_H \mu+\epsilon^3} \le \mu(\Lambda)\mu_\Lambda\left(B_{Mr}(z')\right) \le (Mr)^{\dim_H \mu-\epsilon^3},$$
$$\left(\frac{T}{r^{\dim_H \mu-\epsilon}}\right)^{\frac{\dim_H \mu-\epsilon}{\dim_H \mu}} \precsim p \precsim \left(\frac{T}{r^{\dim_H \mu+\epsilon}}\right)^{\frac{\dim_H \mu-\epsilon}{\dim_H \mu}}, \quad \left(\frac{T}{r^{\dim_H \mu-\epsilon}}\right)^{\epsilon'} \precsim n^{\epsilon'} \precsim \left(\frac{T}{r^{\dim_H \mu+\epsilon}}\right)^{\epsilon'}.$$

Then we can continue the estimate in (4.10) as

$$\precsim_{T,\epsilon} (Mr)^{\dim_H \mu-\epsilon^3} \beta_2{}^{r^{-(\dim_H \mu-\epsilon)\epsilon'}} + \mu_\Lambda\left[B_{Mr+2C\beta_2^{r^{-(\dim_H \mu-\epsilon)\epsilon'}}}(z') \Big\backslash B_{Mr}(z')\right]\beta_2^{r^{-(\dim_H \mu-\epsilon)\epsilon'}}$$
$$+ r^{-(\dim_H \mu+\epsilon)\frac{\dim_H \mu-\epsilon}{\dim_H \mu}}\left\{(Mr)^{\dim_H \mu-\epsilon^3} + \mu_\Lambda\left[B_{Mr+2C\beta_2^{r^{-(\dim_H \mu-\epsilon)\epsilon'}}}(z') \Big\backslash B_{Mr}(z')\right]\right\}^2,$$

where $\beta_2 \in (0, 1)$ only depends on $T, \epsilon', \beta, \beta_1$. □

By combining Lemmas 10, 12 and 13, we obtain the following summary of obtained results.

Proposition 2 (Rates of short returns). *Let* $\epsilon, \epsilon' > 0$ *satisfy relations* $\alpha(\dim_H \mu - \epsilon) > \frac{\dim_H \mu}{\dim_H \mu-\epsilon} > 1$, $p = \left\lfloor \left\lfloor \frac{T}{\mu(B_r(z))} \right\rfloor^{\frac{\dim_H \mu-\epsilon}{\dim_H \mu}} \right\rfloor$, $\epsilon' < \min\left\{\frac{\min\{\dim_H \mu, \dim \gamma^u\}}{12 \dim_H \mu+12\epsilon}, \frac{\dim_H \mu-\epsilon}{\dim_H \mu}\right\}$. *Then for almost all* $z \in \mathcal{M}$, $z' \in \text{int}(\Lambda)$ *and for each integer* $M > 0$ *there exists a small enough* $r_{z,z',M} > 0$ *such that for any* $r < r_{z,z',M}$

$$\int_{B_{Mr}(z')} \mathbb{1}_{\bigcup_{1\le k\le p}(f^R)^{-k}B_{Mr}(z')} d\mu_\Lambda$$
$$\precsim_{T,\epsilon} (Mr)^{\dim_H \mu-\epsilon} M^{\frac{\min\{\dim \gamma^u, \dim_H \mu\}}{6}} r^{\frac{\min\{\dim \gamma^u, \dim_H \mu\}}{12}} + (Mr)^{\dim_H \mu-\epsilon^3} \beta_2{}^{r^{-(\dim_H \mu-\epsilon)\epsilon'}}$$
$$+ \mu_\Lambda\left[B_{Mr+2C\beta_2^{r^{-(\dim_H \mu-\epsilon)\epsilon'}}}(z') \Big\backslash B_{Mr}(z')\right]\beta_2^{r^{-(\dim_H \mu-\epsilon)\epsilon'}}$$
$$+ r^{-(\dim_H \mu+\epsilon)\frac{\dim_H \mu-\epsilon}{\dim_H \mu}}\left\{(Mr)^{\dim_H \mu-\epsilon^3} + \mu_\Lambda\left[B_{Mr+2C\beta_2^{r^{-(\dim_H \mu-\epsilon)\epsilon'}}}(z') \Big\backslash B_{Mr}(z')\right]\right\}^2,$$

where the constant $C > 1$ *is the same as that in Definition 5, the constant* $\beta_2 \in (0, 1)$ *is independent of* z, z', r, M.

Proof. Recall that in Lemmas 10, 12 and 13 we fixed an integer $M > 0$. Then for almost all $z \in \mathcal{M}$ and $z' \in \text{int}(\Lambda)$ desired estimates hold. The set of such points in $\mathcal{M} \times \text{int}(\Lambda)$ has the full measure $\mu(\mathcal{M}) \times \mu(\text{int}(\Lambda))$ and it depends on M. However, since M is an integer, we can find a smaller set of points $(z, z') \in \mathcal{M} \times \text{int}(\Lambda)$, which does not depend on $M > 0$ and has full measure with respect to $\mu(\mathcal{M}) \times \mu(\text{int}(\Lambda))$. Therefore the required estimate holds. □

4.3. Coronas. Here we study two coronas, which appeared in previous sections. One of them is in $\mathcal{M}$ and has measure $\mu\left[B_{r+C'r^{\frac{(\dim_H\mu-\epsilon)^2\alpha}{\dim_H\mu}}}(z)\Big\backslash B_{r-C'r^{\frac{(\dim_H\mu-\epsilon)^2\alpha}{\dim_H\mu}}}(z)\right]$ (see Proposition 1). The other one is in Λ, and its measure is $\mu_\Lambda\left[B_{Mr+2C\beta_2^{r^{-(\dim_H\mu-\epsilon)\epsilon'}}}(z')\Big\backslash B_{Mr}(z')\right]$ (see Proposition 2).

Proposition 3 (Coronas in $\mathcal{M}$ and Λ). *For almost every $z \in \mathcal{M}$, $z' \in \operatorname{int}(\Lambda)$ there are $r_z, r_{z',M} > 0$ such that for any $r < \min\{r_z, r_{z',M}\}$*

$$\mu\left[B_{r+C'r^{\frac{(\dim_H\mu-\epsilon)^2\alpha}{\dim_H\mu}}}(z)\Big\backslash B_{r-C'r^{\frac{(\dim_H\mu-\epsilon)^2\alpha}{\dim_H\mu}}}(z)\right] \precsim_{z,\dim\gamma^u} r^{\left(1+\frac{(\dim_H\mu-\epsilon)^2\alpha}{\dim_H\mu}\right)\frac{\dim\gamma^u}{2}},$$
$$\mu_\Lambda\left[B_{Mr+2C\beta_2^{r^{-(\dim_H\mu-\epsilon)\epsilon'}}}(z')\Big\backslash B_{Mr}(z')\right] \precsim_{z,\dim\gamma^u} (Mr)^{\frac{\dim\gamma^u}{2}}\beta_2^{(r^{-(\dim_H\mu-\epsilon)\epsilon'})\frac{\dim\gamma^u}{2}}.$$

Proof. Let $q := \frac{(\dim_H\mu-\epsilon)^2\alpha}{\dim_H\mu}$. For corona in $\mathcal{M}$ we have from Lemma 6 for almost all $z \in \bigcup_{i\geq 1}\bigcup_{0\leq j<R_i} f^j(\Lambda_i)$, $z = f^{j_z}(z')$, $z' \in \operatorname{int}(\Lambda)$ that there exists $r'_z > 0$ such that for any $r < r'_z$

$$\mu\Big(f^{-j_z}\Big[B_{r+C'r^q}(z)\setminus B_{r-C'r^q}(z)\Big]\bigcap\Big\{\bigcup_{i\geq 1}\bigcup_{1\leq j<R_i} f^j(\Lambda_i)\Big\}\Big) = 0.$$

Hence, by invariance of μ (i.e., $f_*\mu = \mu$) we have

$$\mu\Big(\Big[B_{r+C'r^q}(z)\setminus B_{r-C'r^q}(z)\Big]\bigcap f^{j_z}\Big\{\bigcup_{i\geq 1}\bigcup_{1\leq j<R_i} f^j(\Lambda_i)\Big\}\Big) = 0.$$

Therefore

$$\Big[B_{r+C'r^q}(z)\setminus B_{r-C'r^q}(z)\Big] \subseteq f^{j_z}\Lambda \text{ almost surely.}$$

Because f^{j_z} is a local diffeomorphism, then for sufficiently small r all manifolds $f^{j_z}\gamma^u$ ($\gamma^u \in \Gamma^u$) in any non-empty set $(f^{j_z}\gamma^u)\bigcap\big[B_{r+C'r^q}(z)\setminus B_{r-C'r^q}(z)\big]$ are almost flat. From Gauss lemma for exponential map $\exp_z$ we have, that in a neughborhood of $z \in \mathcal{M}$

$$\begin{aligned}(\exp_z)^{-1}\left[B_{r+C'r^q}(z)\setminus B_{r-C'r^q}(z)\right] &\subseteq \mathrm{Cor}\\ &:= \left\{v \in \mathbb{R}^{\dim\mathcal{M}} : \langle v, v\rangle_z \in \left[r - C'r^q, r + C'r^q\right]\right\},\end{aligned}$$

where $\langle\cdot,\cdot\rangle_z$ is Riemannian metric at $z \in \mathcal{M}$. This is corona in an ellipse. If r is sufficiently small, say $r < r_z < r'_z$ for some $r_z > 0$, then all manifolds $(\exp_z)^{-1}f^{j_z}\gamma^u$ in Cor are almost flat. Hence, their diameters (in Euclidean norm) satisfy following inequality

$$\operatorname{diam}\Big\{\Big[(\exp_z)^{-1}f^{j_z}\gamma^u\Big]\bigcap\mathrm{Cor}\Big\} \precsim_z \sqrt{(r + C'r^q)^2 - (r - C'r^q)^2} \precsim r^{(q+1)/2}.$$

Therefore, Lebesgue measure of $(\exp_z)^{-1} f^{j_z}\gamma^u$ in Cor can be controlled by diameters, i.e.,

$$\mathrm{Leb}_{(\exp_z)^{-1} f^{j_z}\gamma^u}\left\{\left[(\exp_z)^{-1} f^{j_z}\gamma^u\right]\bigcap \mathrm{Cor}\right\} \precsim_{z,\dim\gamma^u} r^{(1+q)\dim\gamma^u/2}.$$

Since $\exp_z$ is a local diffeomorphism, then

$$\mathrm{Leb}_{f^{j_z}\gamma^u}\left[B_{r+C'r^q}(z)\setminus B_{r-C'r^q}(z)\right] \precsim_{z,\dim\gamma^u} r^{(1+q)\dim\gamma^u/2}.$$

Now, since f^{j_z} is also a local diffeomorphism, we have

$$\mathrm{Leb}_{\gamma^u}\left\{f^{-j_z}\left[B_{r+C'r^q}(z)\setminus B_{r-C'r^q}(z)\right]\right\} \precsim_{z,\dim\gamma^u} r^{(1+q)\dim\gamma^u/2}.$$

By making use of (3.5) and integrating over all $\gamma^u \in \Gamma^u$, we get

$$\mu\left\{f^{-j_z}\left[B_{r+C'r^q}(z)\setminus B_{r-C'r^q}(z)\right]\right\} \precsim_{z,\dim\gamma^u} r^{(1+q)\dim\gamma^u/2}.$$

Hence, thanks to invariance of measure μ ($f_*\mu = \mu$) and $q = \frac{(\dim_H\mu-\epsilon)^2\alpha}{\dim_H\mu}$,

$$\mu\left[B_{r+C'r^q}(z)\setminus B_{r-C'r^q}(z)\right] \precsim_{z,\dim\gamma^u} r^{\left(1+\frac{(\dim_H\mu-\epsilon)^2\alpha}{\dim_H\mu}\right)\frac{\dim\gamma^u}{2}}.$$

For corona in Λ, i.e. for $B_{Mr+2C\beta_2^{r^{-(\dim_H\mu-\epsilon)\epsilon'}}}(z')\setminus B_{Mr}(z')$, trick is the same and even more straightforward. Indeed, because $\gamma^u \in \Gamma^u$ intersects with this corona, there is $r_{z',M} > 0$ such that for any $r < r_{z',M}$

$$\mu_\Lambda\left[B_{Mr+2C\beta_2^{r^{-(\dim_H\mu-\epsilon)\epsilon'}}}(z')\Big\backslash B_{Mr}(z')\right] \precsim_{z',\dim\gamma^u} (Mr)^{\frac{\dim\gamma^u}{2}}\beta_2^{\frac{\dim\gamma^u}{2}r^{-(\dim_H\mu-\epsilon)\epsilon'}}.$$

□

4.4. Conclusion of proof of Theorem 3.

4.4.1. Proof that $\dim_H\mu \geq \dim\gamma^u$

Lemma 14. *It follows from Assumptions 1 and 2 that* $\dim_H\mu \geq \dim\gamma^u$.

Proof. Due to ergodicity of μ, it is enough to consider $z' \in \mathrm{int}(\Lambda)$ and a ball $B_r(z')$ in Λ. If r is sufficiently small then $\gamma^u \in \Gamma^u$ is almost flat in $B_r(z')\bigcap\gamma^u$. Therefore $\mathrm{diam}\left\{B_r(z')\bigcap\gamma^u\right\} \precsim_{z'} r$. By the same trick as in Proposition 3 we get

$$\mathrm{Leb}_{\gamma^u}(B_r(z')) \precsim_{z',\dim\gamma^u} r^{\dim\gamma^u}, \quad \mu(B_r(z')) \precsim_{z',\dim\gamma^u} r^{\dim\gamma^u}.$$

On another hand, by Assumption 1 for almost all $z' \in \mathcal{M}$ and any $m \geq 1$ there is $r_{z',m} > 0$, such that $\mu(B_r(z')) \geq r^{\dim_H\mu+1/m}$ for any $r < r_{z',m}$. This implies that $\dim\gamma^u \leq \dim_H\mu + 1/m$ for any $m \geq 1$. By taking limit $m \to \infty$ one gets $\dim_H\mu \geq \dim\gamma^u$. □

We will address now convergence rates in Theorem 3. According to Proposition 1 it suffices to find convergence rates for short returns and coronas. A proof will consist of several steps.

4.4.2. Pull back short returns $\int_{B_r(z)} \mathbb{1}_{\bigcup_{1\le j\le p} f^{-j}B_r(z)} d\mu$ For almost any $z \in \bigcup_{i\ge 1} \bigcup_{0\le j<R_i} f^j(\Lambda_i)$ we have $z = f^{j_z}(z')$ for some $z' \in \operatorname{int}\Lambda$. By Lemma 7 there is a topological ball $U_r(z') = U_r\left(f^{-j_z}(z)\right)$ and a constant $C_z > 1$, such that $B_{C_z^{-1}r}\left(f^{-j_z}z\right) \subseteq U_r\left(f^{-j_z}z\right) \subseteq B_{C_z r}\left(f^{-j_z}z\right)$.

Lemma 15 (Pull short returns back to Λ). *There exists a small enough* $r_z > 0$ *such that for any* $r < r_z$ *the following inequality holds*

$$\int_{B_r(z)} \mathbb{1}_{\bigcup_{1\le k\le p} f^{-k}B_r(z)} d\mu \le \mu(\Lambda) \int_{B_{C_z r}(f^{-j_z}z)} \mathbb{1}_{\bigcup_{1\le k\le p}(f^R)^{-k}B_{C_z r}(f^{-j_z}z)} d\mu_\Lambda.$$

Proof. It follows from invariance of μ (i.e., $f_*\mu = \mu$) that

$$\int_{B_r(z)} \mathbb{1}_{\bigcup_{1\le k\le p} f^{-k}B_r(z)} d\mu = \int_{U_r(f^{-j_z}z)} \mathbb{1}_{\bigcup_{1\le k\le p} f^{-k}U_r(f^{-j_z}z)} d\mu.$$

By Lemma 6 we have $U_r(f^{-j_z}z) \subseteq \Lambda$ almost surely for any $r < r_z$, where $r_z > 0$ is small enough. Then the equality above can be continued as

$$= \int_{U_r(f^{-j_z}z)} \mathbb{1}_{\bigcup_{1\le k\le p} f^{-k}U_r(f^{-j_z}z)} d\mu|_\Lambda = \int_{U_r(f^{-j_z}z)} \mathbb{1}_{\bigcup_{1\le k\le p} f^{-k}U_r(f^{-j_z}z)} d\mu|_\Lambda. \tag{4.11}$$

Denote $R^i := R^{i-1} + R\circ f^R$, $R^1 := R \ge 1$. Then $p \le R^p$. Also note that, since R is first return time, then $f^k \notin \Lambda$ almost surely (a.s.) if $R^i < k < R^{i+1}$. Therefore

$$\mathbb{1}_{U_r(f^{-j_z}z)} \mathbb{1}_{\bigcup_{1\le k\le R^p} f^{-k}U_r(f^{-j_z}z)} = \mathbb{1}_{U_r(f^{-j_z}z)} \mathbb{1}_{\bigcup_{1\le k\le p}(f^R)^{-k}U_r(f^{-j_z}z)} \text{ a.s.}$$

Then we can estimate (4.11) as

$$\le \int_{U_r(f^{-j_z}z)} \mathbb{1}_{\bigcup_{1\le k\le R^p} f^{-k}U_r(f^{-j_z}z)} d\mu|_\Lambda = \int_{U_r(f^{-j_z}z)} \mathbb{1}_{\bigcup_{1\le k\le p}(f^R)^{-k}U_r(f^{-j_z}z)} d\mu|_\Lambda. \tag{4.12}$$

By Lemma 7 there is a constant $C_z \ge 1$, such that $U_r\left(f^{-j_z}z\right) \subseteq B_{C_z r}\left(f^{-j_z}z\right)$, and we can continue (4.12) as

$$\le \mu(\Lambda) \int_{B_{C_z r}(f^{-j_z}z)} \mathbb{1}_{\bigcup_{1\le k\le p}(f^R)^{-k}B_{C_z r}(f^{-j_z}z)} d\mu_\Lambda.$$

□

So the short returns problem on $\mathcal{M}$ becomes short returns problem on Λ. The next task is to

4.4.3. Estimate $\frac{1}{\mu(B_r(z))}\int_{B_{C_z r}(f^{-j_z}z)} \mathbb{1}_{\bigcup_{1\le k\le p}(f^R)^{-k}B_{C_z r}(f^{-j_z}z)}d\mu_\Lambda$

Lemma 16. *If* $\epsilon < \min\left\{\dim\gamma^u/24, \dim_H\mu/24, (3\dim_H\mu)^{-1}\right\}$ *satisfies* $\alpha(\dim_H\mu - \epsilon) > \frac{\dim_H\mu}{\dim_H\mu-\epsilon} > 1$*, then for almost every* $z\in\mathcal{M}$ *there is* $r_z > 0$*, such that for any* $r < r_z 9+$

$$\frac{1}{\mu(B_r(z))}\int_{B_{C_z r}(f^{-j_z}z)} \mathbb{1}_{\bigcup_{1\le k\le p}(f^R)^{-k}B_{C_z r}(f^{-j_z}z)}d\mu_\Lambda$$
$$\precsim_{T,z,\epsilon} r^{\frac{\min\{\dim\gamma^u,\dim_H\mu\}}{12}-2\epsilon} + r^{\frac{\epsilon^2}{\dim_H\mu}-3\epsilon^3} \to 0.$$

Proof. Observe that $C_z < \infty$ for almost all $z\in\mathcal{M}$. Denote $F_1 := \{C_z < \infty\}$, $F_2\times F_3 := \{(z,z')\in\mathcal{M}\times\mathrm{int}(\Lambda) :$ Propositions 2, 3 hold$\}$. Define a new measure one set in $\mathcal{M}$ as $F := F_1\bigcap F_2\bigcap\left\{\bigcup_{j\ge 0} f^j[\mathrm{int}(\Lambda)\bigcap F_3^c]\right\}^c$. Then for any $z\in F$ we have $C_z<\infty$ and $\left(z, f^{-j_z}z\right)\in F_2\times F_3$. Let $M := \lfloor C_z\rfloor + 1$, $z' = f^{-j_z}z$. Then by Proposition 2 there is $r_{z,z',M} = r_{z,f^{-j_z}z,\lfloor C_z\rfloor+1} > 0$ such that for any $r < r_{z,f^{-j_z}z,\lfloor C_z\rfloor+1}$

$$\int_{B_{C_z r}(f^{-j_z}z)} \mathbb{1}_{\bigcup_{1\le k\le p}(f^R)^{-k}B_{C_z r}(f^{-j_z}z)}d\mu_\Lambda$$
$$\le \int_{B_{Mr}(z')} \mathbb{1}_{\bigcup_{1\le k\le p}(f^R)^{-k}B_{Mr}(z')}d\mu_\Lambda$$
$$\precsim_{T,\epsilon} (Mr)^{\dim_H\mu-\epsilon} M^{\frac{\min\{\dim\gamma^u,\dim_H\mu\}}{6}} r^{\frac{\min\{\dim\gamma^u,\dim_H\mu\}}{12}}$$
$$+ (Mr)^{\dim_H\mu-\epsilon^3}\beta_2{}^{r^{-(\dim_H\mu-\epsilon)\epsilon'}} + \mu_\Lambda\left[B_{Mr+2C\beta_2^{r^{-(\dim_H\mu-\epsilon)\epsilon'}}}(z')\Big\backslash B_{Mr}(z')\right]\beta_2^{r^{-(\dim_H\mu-\epsilon)\epsilon'}}$$
$$+ r^{-(\dim_H\mu+\epsilon)\frac{\dim_H\mu-\epsilon}{\dim_H\mu}}\left\{(Mr)^{\dim_H\mu-\epsilon^3} + \mu_\Lambda\left[B_{Mr+2C\beta_2^{r^{-(\dim_H\mu-\epsilon)\epsilon'}}}(z')\Big\backslash B_{Mr}(z')\right]\right\}^2.$$

By Proposition 3, the right hand side of inequality above can be estimated as

$$\precsim_{T,C_z,z,\epsilon} r^{\dim_H\mu-\epsilon+\frac{\min\{\dim\gamma^u,\dim_H\mu\}}{12}} + r^{\dim_H\mu-\epsilon^3}\beta_2{}^{r^{-(\dim_H\mu-\epsilon)\epsilon'}} + \beta_2^{r^{-(\dim_H\mu-\epsilon)\epsilon'}\left(\frac{\dim\gamma^u}{2}+1\right)}$$
$$\times r^{\frac{\dim\gamma^u}{2}} + r^{-(\dim_H\mu+\epsilon)\frac{\dim_H\mu-\epsilon}{\dim_H\mu}}\left\{r^{\dim_H\mu-\epsilon^3} + r^{\frac{\dim\gamma^u}{2}}\beta_2^{r^{-(\dim_H\mu-\epsilon)\epsilon'}\frac{\dim\gamma^u}{2}}\right\}^2.$$

By Assumption 1 for almost all $z\in\mathcal{M}$ there is $r_z > 0$ such that for any $r < r_z$

$$r^{\dim_H\mu+\epsilon^3} \le \mu(B_r(z)) \le r^{\dim_H\mu-\epsilon^3}.$$

Then for any $r < \min\left\{r_{z,f^{-j_z}z,\lfloor C_z\rfloor+1}, r_z\right\}$,

$$\frac{1}{\mu(B_r(z))}\int_{B_{C_z r}(f^{-j_z}z)} \mathbb{1}_{\bigcup_{1\le k\le p}(f^R)^{-k}B_{C_z r}(f^{-j_z}z)}d\mu_\Lambda$$
$$\precsim_{T,z,\epsilon} r^{-\dim_H\mu-\epsilon^3}\left\{r^{\dim_H\mu-\epsilon+\frac{\min\{\dim\gamma^u,\dim_H\mu\}}{12}} + r^{\dim_H\mu-\epsilon}\beta_2{}^{r^{-(\dim_H\mu-\epsilon)\epsilon'}} + r^{\frac{\dim\gamma^u}{2}}\right.$$
$$\left.\times\beta_2^{r^{-(\dim_H\mu-\epsilon)\epsilon'}\left(\frac{\dim\gamma^u}{2}+1\right)} + r^{-(\dim_H\mu+\epsilon)\frac{\dim_H\mu-\epsilon}{\dim_H\mu}}\left(r^{\dim_H\mu-\epsilon^3} + r^{\frac{\dim\gamma^u}{2}}\beta_2^{r^{-(\dim_H\mu-\epsilon)\epsilon'}\frac{\dim\gamma^u}{2}}\right)^2\right\}$$
$$\le r^{\frac{\min\{\dim\gamma^u,\dim_H\mu\}}{12}-2\epsilon} + r^{-\epsilon-\epsilon^3}\beta_2{}^{r^{-(\dim_H\mu-\epsilon)\epsilon'}} + \beta_2^{r^{-(\dim_H\mu-\epsilon)\epsilon'}\frac{\dim\gamma^u}{2}+r^{-(\dim_H\mu-\epsilon)\epsilon'}}$$

$$\times r^{\frac{\dim\gamma^u}{2}-\epsilon^3-\dim_H\mu} + r^{-\dim_H\mu-\epsilon^3-(\dim_H\mu+\epsilon)\frac{\dim_H\mu-\epsilon}{\dim_H\mu}}$$
$$\left\{r^{\dim_H\mu-\epsilon^3} + r^{\frac{\dim\gamma^u}{2}}\beta_2^{\frac{r^{-(\dim_H\mu-\epsilon)\epsilon'}\dim\gamma^u}{2}}\right\}^2$$
$$\precsim r^{\frac{\min\{\dim\gamma^u,\dim_H\mu\}}{12}-2\epsilon} + r^{\frac{\epsilon^2}{\dim_H\mu}-3\epsilon} \to 0,$$

where the last inequality holds because $\beta_2^{r^{-c}} \ll r^{c'}$ for any $c, c' > 0$. □

By combining Lemmas 15 and 16 we get

4.4.4. Estimate for short returns rates $\frac{1}{\mu(B_r(z))}\int_{B_r(z)}\mathbb{1}_{\bigcup_{1\le j\le p}f^{-j}B_r(z)}d\mu$

Lemma 17. *If* $\epsilon < \min\left\{\dim\gamma^u/24, \dim_H\mu/24, (3\dim_H\mu)^{-1}\right\}$ *satisfies* $\alpha(\dim_H\mu-\epsilon) > \frac{\dim_H\mu}{\dim_H\mu-\epsilon} > 1$, *then for almost all* $z \in \mathcal{M}$

$$\frac{1}{\mu(B_r(z))}\int_{B_r(z)}\mathbb{1}_{\bigcup_{1\le j\le p}f^{-j}B_r(z)}d\mu \precsim_{z,T,\epsilon} r^{\frac{\min\{\dim\gamma^u,\dim_H\mu\}}{12}-2\epsilon} + r^{\frac{\epsilon^2}{\dim_H\mu}-3\epsilon^3} \to 0.$$

We finished now estimates for short returns and move to

4.4.5. Estimate for coronas rates $\frac{1}{\mu(B_r(z))}\mu\left[B_{r+C'r^{\frac{(\dim_H\mu-\epsilon)^2\alpha}{\dim_H\mu}}}(z)\Big\backslash B_{r-C'r^{\frac{(\dim_H\mu-\epsilon)^2\alpha}{\dim_H\mu}}}(z)\right]$

Lemma 18. *If* $\alpha > \frac{2}{\dim\gamma^u}-\frac{1}{\dim_H\mu}$ *and* $\epsilon < \min\left\{\dim\gamma^u/24, \dim_H\mu/24, (3\dim_H\mu)^{-1}\right\}$ *is small enough, so that* $\alpha > (\frac{2}{\dim\gamma^u}-\frac{1}{\dim_H\mu}+\frac{2\epsilon}{\dim\gamma^u\dim_H\mu})(1-\frac{\epsilon}{\dim_H\mu})^{-2}$, *then for almost all* $z \in \mathcal{M}$

$$\frac{\mu\left[B_{r+C'r^{\frac{(\dim_H\mu-\epsilon)^2\alpha}{\dim_H\mu}}}(z)\Big\backslash B_{r-C'r^{\frac{(\dim_H\mu-\epsilon)^2\alpha}{\dim_H\mu}}}(z)\right]}{\mu(B_r(z))} \precsim_z r^{\left(1+\frac{(\dim_H\mu-\epsilon)^2\alpha}{\dim_H\mu}\right)\frac{\dim\gamma^u}{2}-\dim_H\mu-\epsilon} \to 0.$$

Proof. By Assumption 1 for almost all $z \in \mathcal{M}$ there exists $r_z > 0$, such that for any $r < r_z$

$$r^{\dim_H\mu+\epsilon} \le \mu(B_r(z)) \le r^{\dim_H\mu-\epsilon}.$$

Now, in view of Proposition 3 and choice of ϵ, we have

$$\left(1+\frac{(\dim_H\mu-\epsilon)^2\alpha}{\dim_H\mu}\right)\frac{\dim\gamma^u}{2}-\dim_H\mu-\epsilon > 0,$$

and the result holds. □

By that we finished estimation of rates for coronas, and can now conclude a proof of Theorem 3.

4.4.6. Convergence rates $a > 0$ in $d_{TV}(N^{r,T,z}, P) \precsim_{T,z} r^a$

Lemma 19. *If $\alpha > \frac{2}{\dim \gamma^u} - \frac{1}{\dim_H \mu}$, choose a small $\epsilon < \min\left\{\frac{\dim \gamma^u}{24}, \frac{\dim_H \mu}{24}, \frac{1}{3\dim_H \mu}\right\}$ such that $\alpha > \max\left\{\frac{1}{\dim_H \mu}\left(\frac{\dim_H \mu - \epsilon}{\dim_H \mu}\right)^{-2}, \left(\frac{2}{\dim \gamma^u} - \frac{1}{\dim_H \mu} + \frac{2\epsilon}{\dim \gamma^u \dim_H \mu}\right)\left(\frac{\dim_H \mu - \epsilon}{\dim_H \mu}\right)^{-2}\right\}$, then we obtain the following convergence rate*

$$a = \min\left\{\frac{(\dim_H \mu - \epsilon)^2(\xi - 1)}{\dim_H \mu}, \left(1 + \frac{(\dim_H \mu - \epsilon)^2 \alpha}{\dim_H \mu}\right)\frac{\dim \gamma^u}{2} - \dim_H \mu - \epsilon, \frac{\epsilon^2}{\dim_H \mu} - 3\epsilon^3, \frac{\min\{\dim \gamma^u, \dim_H \mu\}}{12} - 2\epsilon\right\}.$$

If $\frac{d\mu}{d\,\mathrm{Leb}_{\mathcal{M}}} \in L^\infty_{loc}(\mathcal{M})$, choose a small $\epsilon < \min\left\{\dim \gamma^u/24, \dim_H \mu/24, (3\dim_H \mu)^{-1}\right\}$ such that $\alpha > \frac{\dim_H \mu}{(\dim_H \mu - \epsilon)^2}$, then we obtain the convergence rate

$$a := \min\left\{\frac{(\dim_H \mu - \epsilon)^2(\xi - 1)}{\dim_H \mu}, \frac{(\dim_H \mu - \epsilon)^2 \alpha}{\dim_H \mu} - 1, \frac{\epsilon^2}{\dim_H \mu} - 3\epsilon^3, \frac{\min\{\dim \gamma^u, \dim_H \mu\}}{12} - 2\epsilon\right\}.$$

Proof. From Proposition 1

$$d_{TV}\left(N^{r,T,z}, P\right) \precsim_{T,\xi,\epsilon} R_1(r) + R_2(r,z) + R_3(r,z) + R_4(r,z),$$

where

$$R_1(r) := r^{\dim_H \mu - \epsilon} + r^{\frac{(\dim_H \mu - \epsilon)^2}{\dim_H \mu}(\xi - 1)} + r^{\frac{\epsilon(\dim_H \mu - \epsilon)}{\dim_H \mu}}$$

$$R_2(r,z) := \frac{1}{\mu(B_r(z))}\mu\left(B_{r + C' r^{\frac{(\dim_H \mu - \epsilon)^2 \alpha}{\dim_H \mu}}}(z) \setminus B_{r - C' r^{\frac{(\dim_H \mu - \epsilon)^2 \alpha}{\dim_H \mu}}}(z)\right)$$

$$R_3(r,z) := \frac{1}{\mu(B_r(z))^2}\mu\left(B_{r + C' r^{\frac{(\dim_H \mu - \epsilon)^2 \alpha}{\dim_H \mu}}}(z) \setminus B_{r - C' r^{\frac{(\dim_H \mu - \epsilon)^2 \alpha}{\dim_H \mu}}}(z)\right)^2$$

$$R_4(r,z) := \frac{1}{\mu(B_r(z))}\int_{B_r(z)} \mathbb{1}_{\bigcup_{1 \le j \le p} f^{-j} B_r(z)} d\mu.$$

From Lemma 17 we have

$$R_4(r,z) \precsim_{z,T,\epsilon} r^{\frac{\min\{\dim \gamma^u, \dim_H \mu\}}{12} - 2\epsilon} + r^{\frac{\epsilon^2}{\dim_H \mu} - 3\epsilon^3}.$$

If $\alpha > \frac{2}{\dim \gamma^u} - \frac{1}{\dim_H \mu}$,then from Lemma 18,

$$R_3(r,z) \le R_2(r,z) \precsim_z r^{\left(1 + \frac{(\dim_H \mu - \epsilon)^2 \alpha}{\dim_H \mu}\right)\frac{\dim \gamma^u}{2} - \dim_H \mu - \epsilon}.$$

Therefore, $d_{TV}\left(N^{r,T,z}, P\right) \precsim_{T,\xi,\epsilon} r^a$,

$$
\begin{aligned}
a :=& \min\Bigg\{\frac{(\dim_H \mu - \epsilon)^2(\xi - 1)}{\dim_H \mu}, \frac{\epsilon(\dim_H \mu - \epsilon)}{\dim_H \mu},\\
&\left(1 + \frac{(\dim_H \mu - \epsilon)^2\alpha}{\dim_H \mu}\right)\frac{\dim \gamma^u}{2} - \dim_H \mu - \epsilon,\\
&\frac{\epsilon^2}{\dim_H \mu} - 3\epsilon^3, \frac{\min\{\dim \gamma^u, \dim_H \mu\}}{12} - 2\epsilon, \dim_H \mu - \epsilon\Bigg\}\\
=& \min\Bigg\{\frac{(\dim_H \mu - \epsilon)^2(\xi - 1)}{\dim_H \mu}, \left(1 + \frac{(\dim_H \mu - \epsilon)^2\alpha}{\dim_H \mu}\right)\frac{\dim \gamma^u}{2} - \dim_H \mu - \epsilon,\\
&\frac{\epsilon^2}{\dim_H \mu} - 3\epsilon^3, \frac{\min\{\dim \gamma^u, \dim_H \mu\}}{12} - 2\epsilon\Bigg\},
\end{aligned}
$$

where the last equality comes from relations $\frac{\epsilon^2}{\dim_H \mu} - 3\epsilon^3 \le \frac{\epsilon(\dim_H \mu - \epsilon)}{\dim_H \mu} \le \dim_H \mu - \epsilon$.

If $\frac{d\mu}{d\,\mathrm{Leb}_{\mathcal{M}}} \in L^\infty_{loc}(\mathcal{M})$, then

$$
\begin{aligned}
R_3(r,z) \le R_2(r,z) &= \frac{1}{\mu(B_r(z))}\mu\left(B_{r+Cr^{\frac{(\dim_H \mu-\epsilon)^2\alpha}{\dim_H \mu}}}(z) \Big\backslash B_{r-Cr^{\frac{(\dim_H \mu-\epsilon)^2\alpha}{\dim_H \mu}}}(z)\right)\\
&\precsim_z r^{\frac{(\dim_H \mu-\epsilon)^2\alpha}{\dim_H \mu} - 1}.
\end{aligned}
$$

Therefore, $d_{TV}\left(N^{r,T,z}, P\right) \precsim_{T,\xi,\epsilon} r^a$,

$$
\begin{aligned}
a :=& \min\Bigg\{\frac{(\dim_H \mu - \epsilon)^2(\xi - 1)}{\dim_H \mu}, \frac{\epsilon(\dim_H \mu - \epsilon)}{\dim_H \mu}, \frac{(\dim_H \mu - \epsilon)^2\alpha}{\dim_H \mu} - 1, \frac{\epsilon^2}{\dim_H \mu} - 3\epsilon^3,\\
&\frac{\min\{\dim \gamma^u, \dim_H \mu\}}{12} - 2\epsilon, \dim_H \mu - \epsilon\Bigg\}\\
=& \min\Bigg\{\frac{(\dim_H \mu - \epsilon)^2(\xi - 1)}{\dim_H \mu}, \frac{(\dim_H \mu - \epsilon)^2\alpha}{\dim_H \mu} - 1, \frac{\epsilon^2}{\dim_H \mu}\\
&- 3\epsilon^3, \frac{\min\{\dim \gamma^u, \dim_H \mu\}}{12} - 2\epsilon\Bigg\},
\end{aligned}
$$

where the last equality follows from $\frac{\epsilon^2}{\dim_H \mu} - 3\epsilon^3 \le \frac{\epsilon(\dim_H \mu - \epsilon)}{\dim_H \mu} \le \dim_H \mu - \epsilon$.

This completes the proof of Theorem 3. □

5. Proof of Theorem 4

The scheme of a proof of Theorem 4 is analogous to the one of Theorem 3, i.e., it relies on estimating the probability of having short returns and the measure of the coronas. However, we establish the estimates in another way because of the different assumptions in Theorem 4.

5.1. Properties of first return map $f^{\overline{R}}$.

Lemma 20 (Properties of first returns). *The map $f^{\overline{R}}$ is ergodic with respect to probability $\mu_U := \frac{\mu|_U}{\mu(U)}$, and it is bijective on $U \bigcap \bigcup_{i\geq 1}\bigcup_{0\leq j<R_i} f^j(\Lambda_i)$.*

Proof. A proof that $f^{\overline{R}}$ is one-to-one is the same as that of Lemma 4, which uses the first return $\overline{R}$ and replaces R_i, R_j by $\overline{R}(x)$, $\overline{R}(y)$. Clearly, the map $f^{\overline{R}}$ is ergodic due to exponential decay of correlations, and it is also onto due to ergodicity. □

Now, since $\mu\{\text{int}\,(U)\} > 0$, then by Birkhoff's ergodic theorem for almost every $z \in \bigcup_{i\geq 1}\bigcup_{0\leq j<R_i} f^j(\Lambda_i)$ we have $z = f^{j_z}(z')$ for some $z' \in \text{int}\,(U)$, and

$$j_z := \min\{n \in \mathbb{N} : f^{-n}(z) \in \text{int}\,(U)\} < \infty.$$

Analogously to the proof of Lemma 6, we obtain

Lemma 21 (Pull metric balls back to U). *There exists sufficiently small $r > 0$ such that $\mu\left(f^{-j_z}B_r(z)\bigcap U^c\right) = 0$.*

5.2. Short returns. Let $z' \in \text{int}\,(U)$. Take now any fixed positive integer $M > 0$, a sufficiently small constant $\epsilon' > 0$ such that $n^{\epsilon'} \ll p$, where $n = \left\lfloor \frac{T}{\mu(B_r(z))} \right\rfloor$, $p = \left\lfloor \left\lfloor \frac{T}{\mu(B_r(z))} \right\rfloor^{\frac{\dim_H \mu - \epsilon}{\dim_H \mu}} \right\rfloor$, and the same ϵ as in Proposition 1. Then $B_{Mr}(z') \subseteq \text{int}\,(U)$ almost surely for a sufficiently small $r > 0$. We now consider short returns for induced map $f^{\overline{R}} : U \to U$, namely

$$\begin{aligned}\int_{B_{Mr}(z')} \mathbb{1}_{\bigcup_{1\leq k\leq p}\left(f^{\overline{R}}\right)^{-k} B_{Mr}(z')} d\mu_U &= \int_{B_{Mr}(z')} \mathbb{1}_{\bigcup_{1\leq k\leq N}\left(f^{\overline{R}}\right)^{-k} B_{Mr}(z')} d\mu_U \\ &+ \int_{B_{Mr}(z')} \mathbb{1}_{\bigcup_{N\leq k\leq n^{\epsilon'}}\left(f^{\overline{R}}\right)^{-k} B_{Mr}(z')} d\mu_U \\ &+ \int_{B_{Mr}(z')} \mathbb{1}_{\bigcup_{n^{\epsilon'}\leq k\leq p}\left(f^{\overline{R}}\right)^{-k} B_{Mr}(z')} d\mu_U, ,\end{aligned}$$

where $N = \lfloor \frac{-\log 2C}{\log \beta} \rfloor + 1$, and C, β are the ones from Definition 6.

Lemma 22 (Very short returns). *For almost every $z' \in \text{int}\,(U)$, sufficiently small $r_{N,M,z'} > 0$ and any $r < r_{N,M,z'}$, we have*

$$\int_{B_{Mr}(z')} \mathbb{1}_{\bigcup_{1\leq k\leq N}\left(f^{\overline{R}}\right)^{-k} B_{Mr}(z')} d\mu_U = 0.$$

(Actually, a stronger result will be proved, i.e. for any $k \in \mathbb{N}$ a map $f^{\overline{R}^k}$ is a local diffeomorphism at almost every point $z' \in \text{int}\, U$).

Proof. According to Definition 6, we have $\mu(\partial U) = 0$. Also, exponential decay of correlations holds for $f^{\overline{R}}$. Let A_{per} be the set of all periodic points for $f^{\overline{R}}$. Then $\mu\left(A_{per}\right) = 0$ and $\mu(\bigcup_{n\in\mathbb{Z}} f^{-n}\partial U) = 0$. Choose $z' \notin \bigcup_{n\in\mathbb{Z}} f^{-n}\partial U \bigcup A_{per}$. The rest of the proof is exactly the same as that of Lemma 10, replacing R by $\overline{R}$ and Λ by U. □

Before estimating moderate short returns $\int_{B_{Mr}(z')} \mathbb{1}_{\bigcup_{N\le k\le n^{\epsilon'}}\left(f^{\overline{R}}\right)^{-k} B_{Mr}(z')} d\mu_U$ we will need one more lemma.

Lemma 23 (Recurrences). *There exists $r_M > 0$, such that for any $r < r_M$ and $i \ge N$*

$$\mu_U\left\{z' \in U : d\left((f^{\overline{R}})^i z', z'\right) \le Mr\right\} \precsim_{\dim\gamma^u} (Mr)^{\frac{b\dim\gamma^u}{b+\dim\gamma^u}},$$

where a constant in $\precsim_{\dim\gamma^u}$ depends on $\dim\gamma^u$*, but does not depend on $i \ge N$ and $\gamma^u \in \Theta$.*

Proof. For any $\gamma^u \in \Theta$, $z_1', z_2' \in \{z' \in U \bigcap \gamma^u : d((f^{\overline{R}})^{-i}z', z') \le Mr\}$ we have

$$d\left((f^{\overline{R}})^{-i}z_1', z_1'\right) \le Mr, \quad d\left((f^{\overline{R}})^{-i}z_2', z_2'\right) \le Mr.$$

The u-contraction (see Definition 6), together with $i \ge N$, give that $C\beta^i \le C\beta^N < 1/2$ and

$$\begin{aligned} d\left(z_1', z_2'\right) &\le d\left(z_1', (f^{\overline{R}})^{-i}z_1'\right) + d\left((f^{\overline{R}})^{-i}z_1', (f^{\overline{R}})^{-i}z_2'\right) + d\left((f^{\overline{R}})^{-i}z_2', z_2'\right) \\ &\le 2Mr + C\beta^i d\left(z_1', z_2'\right) \le 2Mr + d\left(z_1', z_2'\right)/2. \end{aligned}$$

So $d(z_1', z_2') \le 4Mr \implies \text{diam}\{z' \in U \bigcap \gamma^u : d((f^{\overline{R}})^{-i}z', z') \le Mr\} \le 4Mr$.

Since each $\gamma^u \in \Theta$ has uniformly bounded sectional curvature, then $\text{Leb}_{\gamma^u}\left(B_{4Mr}(z')\right) \precsim (4Mr)^{\dim\gamma^u}$ for any $z' \in \gamma^u$ and

$$\text{Leb}_{\gamma^u}\left\{z' \in U : d\left((f^{\overline{R}})^{-i}z', z'\right) \le Mr\right\} \precsim_{\dim\gamma^u} (Mr)^{\dim\gamma^u},$$

where a constant in $\precsim_{\dim\gamma^u}$ depends only on $\dim\gamma^u$.

From Definition 6 we get $r_M := 1/M$ such that for any $r < r_M$, $\mu_U\{x \in U : |\gamma^u(x)| < (Mr)^{\frac{\dim\gamma^u}{b+\dim\gamma^u}}\} \le C(Mr)^{\frac{b\dim\gamma^u}{b+\dim\gamma^u}}$, and for any $y \in \gamma^u \in \Theta$, $\frac{d\mu_{\gamma^u}}{d\,\text{Leb}_{\gamma^u}}(y) = C^{\pm}\frac{1}{\text{Leb}_{\gamma^u}(\gamma^u)}$. Hence

$$\begin{aligned} &\mu_U\left\{z' \in U : d\left((f^{\overline{R}})^{-i}z', z'\right) \le Mr\right\} \\ &= \int \mu_{\gamma^u(x)}\left\{z' \in U : d\left((f^{\overline{R}})^{-i}z', z'\right) \le Mr\right\} d\mu_U(x) \\ &= \int_{|\gamma^u(x)| \le (Mr)^{\frac{\dim\gamma^u}{b+\dim\gamma^u}}} \mu_{\gamma^u(x)}\left\{z' \in U : d\left((f^{\overline{R}})^{-i}z', z'\right) \le Mr\right\} d\mu_U(x) \\ &\quad + \int_{|\gamma^u(x)| \ge (Mr)^{\frac{\dim\gamma^u}{b+\dim\gamma^u}}} \mu_{\gamma^u(x)}\left\{z' \in U : d\left((f^{\overline{R}})^{-i}z', z'\right) \le Mr\right\} d\mu_U(x) \\ &\le C(Mr)^{\frac{b\dim\gamma^u}{b+\dim\gamma^u}} + \int_{|\gamma^u(x)| \ge (Mr)^{\frac{\dim\gamma^u}{b+\dim\gamma^u}}} \mu_{\gamma^u(x)}\left\{z' \in U : d\left((f^{\overline{R}})^{-i}z', z'\right) \le Mr\right\} d\mu_U(x) \\ &\precsim_{\dim\gamma^u} (Mr)^{\frac{b\dim\gamma^u}{b+\dim\gamma^u}} + \int_{|\gamma^u(x)| \ge (Mr)^{\frac{\dim\gamma^u}{b+\dim\gamma^u}}} \frac{\text{Leb}_{\gamma^u(x)}\left\{z' \in U : d\left((f^{\overline{R}})^{-i}z', z'\right) \le Mr\right\}}{\text{Leb}_{\gamma^u(x)}(\gamma^u(x))} d\mu_U(x) \\ &\precsim_{\dim\gamma^u} (Mr)^{\frac{b\dim\gamma^u}{b+\dim\gamma^u}} + (Mr)^{\dim\gamma^u - \dim\gamma^u\frac{\dim\gamma^u}{b+\dim\gamma^u}} \precsim_{\dim\gamma^u} (Mr)^{\frac{b\dim\gamma^u}{b+\dim\gamma^u}}, \end{aligned}$$

where in the last two inequalities we used that sectional curvature of $\gamma^u(x) \in \Theta$ is uniformly bounded, and $\dim \gamma^u - \dim \gamma^u \frac{\dim \gamma^u}{b+\dim \gamma^u} = \frac{b \dim \gamma^u}{b+\dim \gamma^u}$.

Finally, $(f^{\overline{R}})_* \mu_U = \mu_U$ implies that

$$\mu_U \left\{ z' \in U : d\left((f^{\overline{R}})^i z', z' \right) \leq Mr \right\} \precsim_{\dim \gamma^u} (Mr)^{\frac{b \dim \gamma^u}{b+\dim \gamma^u}}.$$

□

Lemma 24 (Moderate short returns). *For almost every $z' \in \operatorname{int}(U)$, $z \in \mathcal{M}$ there exists $r_{z,z',M} > 0$, such that for any $r < r_{z,z',M}$*

$$\int_{B_{Mr}(z')} \mathbb{1}_{\bigcup_{N \leq k \leq n^{\epsilon'}} (f^{\overline{R}})^{-k} B_{Mr}(z')} d\mu_U \precsim_{T,b} (Mr)^{\dim_H \mu - \epsilon} (M\sqrt{r})^{\min\{\frac{b \dim \gamma^u}{6(b+\dim \gamma^u)}, \frac{\dim_H \mu}{6}\}},$$

where $\epsilon > 0$ is the same as that in Proposition 1 and $\epsilon' < \frac{\min\{\dim_H \mu, \frac{b \dim \gamma^u}{b+\dim \gamma^u}\}}{12 \dim_H \mu + 12\epsilon}$.

Proof. In Lemma 23 we already proved that

$$\mu_U \left\{ z' \in U : d\left((f^{\overline{R}})^i z', z' \right) \leq Mr \right\} \precsim_{\dim \gamma^u} (Mr)^{\frac{b \dim \gamma^u}{b+\dim \gamma^u}}.$$

The rest of the proof is exactly the same as that of Lemma 12, where one should replace $\dim \gamma^u$, Λ, R by $\frac{b \dim \gamma^u}{b+\dim \gamma^u}$, U, $\overline{R}$. □

Lemma 25 (Longest short returns). *Let $n = \left\lfloor \frac{T}{\mu(B_r(z))} \right\rfloor$, $p = \left\lfloor \left\lfloor \frac{T}{\mu(B_r(z))} \right\rfloor^{\frac{\dim_H \mu - \epsilon}{\dim_H \mu}} \right\rfloor$ and $n^{\epsilon'} \ll p$. Then for almost all $z' \in \operatorname{int}(U)$ and $z \in \mathcal{M}$ there is $r_{z,z',M} > 0$, such that for any $r < r_{z,z',M}$*

$$\begin{aligned}
&\int_{B_{Mr}(z')} \mathbb{1}_{\bigcup_{n^{\epsilon'} \leq k \leq p} \left(f^{\overline{R}}\right)^{-k} B_{Mr}(z')} d\mu_U \\
&\precsim_{T,\epsilon} r^{(-\dim_H \mu - \epsilon)\frac{\dim_H \mu - \epsilon}{\dim_H \mu}} \left[(Mr)^{2\dim_H \mu - 2\epsilon^3} + \mu_U \left(B_{Mr + (\sqrt[4]{\beta})^{\left[Tr^{-(\dim_H \mu - \epsilon)} \right]^{\epsilon'}}}(z') \Big\backslash B_{Mr}(z') \right) \right],
\end{aligned}$$

where $\beta \in (0, 1)$ is the same as that in Definition 6.

Proof. The approach to proving required estimate is quite standard. It uses Lipschitz functions to approximate $B_{Mr}(z')$. However, we will write it down for completeness.

Let $q := T^{\epsilon'} r^{-(\dim_H \mu - \epsilon)\epsilon'}$, $L \in \operatorname{Lip}(U)$, such that $L = 1$ on $B_{Mr}(z')$, $L = 0$ on $B^c_{Mr+(\sqrt[4]{\beta})^q}(z')$, and L is linear on $B_{Mr+(\sqrt[4]{\beta})^q}(z') \setminus B_{Mr}(z')$, where ϵ is the same as that in Proposition 1. Hence $L \in \operatorname{Lip}(U)$ with Lipschitz constant $(\sqrt[4]{\beta})^{-q}$. Therefore for any $k \in [n^{\epsilon'}, p]$

$$\begin{aligned}
&\int_{B_{Mr}(z')} \mathbb{1}_{B_{Mr}(z')} \circ (f^{\overline{R}})^k d\mu_U \\
&\leq \int LL \circ (f^{\overline{R}})^k d\mu_U + 2\mu_U \left(B_{Mr+(\sqrt[4]{\beta})^q}(z') \setminus B_{Mr}(z') \right) \\
&\leq \left| \int LL \circ (f^{\overline{R}})^k d\mu_U - \int L d\mu_U \int L \circ (f^{\overline{R}})^k d\mu_U \right|
\end{aligned}$$

$$+\int Ld\mu_U\int L\circ(f^{\overline{R}})^k d\mu_U+2\mu_U\left(B_{Mr+(\sqrt[4]{\beta})^q}(z')\setminus B_{Mr}(z')\right).$$

Now, by making use of exponential decay of correlation in Definition 6, we can continue the estimate as

$$\begin{aligned}&\precsim \mathrm{Lip}(L)^2\beta^{n^{\epsilon'}}+\mu_U\left(B_{Mr+(\sqrt[4]{\beta})^q}(z')\right)^2+2\mu_U\left(B_{Mr+(\sqrt[4]{\beta})^q}(z')\setminus B_{Mr}(z')\right)\\&=(\sqrt[4]{\beta})^{-2q}\beta^{n^{\epsilon'}}+\mu_U\left(B_{Mr+(\sqrt[4]{\beta})^q}(z')\right)^2+2\mu_U\left(B_{Mr+(\sqrt[4]{\beta})^q}(z')\setminus B_{Mr}(z')\right).\end{aligned}$$

Hence

$$\begin{aligned}\int_{B_{Mr}(z')}\mathbb{1}_{\bigcup_{n^{\epsilon'}\le k\le p}\left(f^{\overline{R}}\right)^{-k}B_{Mr}(z')}d\mu_U&\precsim p(\sqrt{\beta})^{-q}\beta^{n^{\epsilon'}}+p\mu_U\left(B_{Mr+(\sqrt[4]{\beta})^q}(z')\right)^2\\&\quad+2p\mu_U\left(B_{Mr+(\sqrt[4]{\beta})^q}(z')\setminus B_{Mr}(z')\right).\end{aligned}\tag{5.1}$$

Choose now from Assumption 1 the same $\epsilon>0$ as in Proposition 1. Then, for almost all $z\in\mathcal{M}$ and $z'\in\mathrm{int}\,(U)$, there exists $r_{z,z',M}>0$ such that for any $r<r_{z,z',M}$

$$\begin{aligned}&B_{Mr+(\sqrt[4]{\beta})^q}(z')\subseteq U,\quad Tr^{-\dim_H\mu+\epsilon}\precsim n\precsim Tr^{-\dim_H\mu-\epsilon},\\&r^{\dim_H\mu+\epsilon}\le\mu(B_r(z))\le r^{\dim_H\mu-\epsilon},\\&(Mr)^{\dim_H\mu+\epsilon^3}\le\mu_U(B_{Mr}(z'))\mu(U)\le(Mr)^{\dim_H\mu-\epsilon^3},\\&(Tr^{\epsilon-\dim_H\mu})^{\frac{\dim_H\mu-\epsilon}{\dim_H\mu}}\precsim p\precsim(Tr^{-\dim_H\mu-\epsilon})^{\frac{\dim_H\mu-\epsilon}{\dim_H\mu}},\\&(Tr^{\epsilon-\dim_H\mu})^{\epsilon'}\precsim n^{\epsilon'}\precsim(Tr^{-\dim_H\mu-\epsilon})^{\epsilon'}.\end{aligned}$$

Therefore, the estimates from (5.1) can be continued as

$$\begin{aligned}&\precsim(Tr^{-\dim_H\mu-\epsilon})^{\frac{\dim_H\mu-\epsilon}{\dim_H\mu}}(\sqrt{\beta})^{-q}\beta^q+\left(Tr^{-\dim_H\mu-\epsilon}\right)^{\frac{\dim_H\mu-\epsilon}{\dim_H\mu}}\\&\times\left[\mu_U\left(B_{Mr+(\sqrt[4]{\beta})^q}(z')\right)^2+\mu_U\left(B_{Mr+(\sqrt[4]{\beta})^q}(z')\setminus B_{Mr}(z')\right)\right]\\&\precsim_{T,\epsilon}r^{(-\dim_H\mu-\epsilon)\frac{\dim_H\mu-\epsilon}{\dim_H\mu}}\left[(Mr)^{2\dim_H\mu-2\epsilon^3}+\mu_U\left(B_{Mr+(\sqrt[4]{\beta})^{T^{\epsilon'}r^{-(\dim_H\mu-\epsilon)\epsilon'}}}(z')\setminus B_{Mr}(z')\right)\right].\end{aligned}$$

The last inequality holds because $\beta^{r^{-c}}\precsim r^{c'}$ for any $c',c>0$ and $q=T^{\epsilon'}r^{-(\dim_H\mu-\epsilon)\epsilon'}$. □

Combining now Lemmas 22, 24, 25, and arguing as in Proposition 2, we can formulate the following summary of obtained results.

Proposition 4 (Short returns rates). *Let* $\epsilon,\epsilon'>0$ *satisfy the relations* $\alpha(\dim_H\mu-\epsilon)>\frac{\dim_H\mu}{\dim_H\mu-\epsilon}>1$, $p=\left\lfloor\left\lfloor\frac{T}{\mu(B_r(z))}\right\rfloor^{\frac{\dim_H\mu-\epsilon}{\dim_H\mu}}\right\rfloor$, $\epsilon'<\min\left\{\frac{\min\{\frac{b\dim\gamma^u}{b+\dim\gamma^u},\dim_H\mu\}}{12\dim_H\mu+12\epsilon},\frac{\dim_H\mu-\epsilon}{\dim_H\mu}\right\}$. *Then for almost all* $z\in\mathcal{M}$, $z'\in\mathrm{int}\,(U)$ *and for each integer* $M>0$ *there exists small enough* $r_{z,z',M}>0$*, such that for any* $r<r_{z,z',M}$

$$\int_{B_{Mr}(z')}\mathbb{1}_{\bigcup_{1\le k\le p}\left(f^{\overline{R}}\right)^{-k}B_{Mr}(z')}d\mu_U$$

$$\precsim_{\epsilon,T,b} (Mr)^{\dim_H \mu-\epsilon}(M\sqrt{r})^{\min\left\{\frac{b\dim\gamma^u}{6b+6\dim\gamma^u},\frac{\dim_H\mu}{6}\right\}} + r^{-(\dim_H\mu+\epsilon)\frac{\dim_H\mu-\epsilon}{\dim_H\mu}}$$
$$\times\left[(Mr)^{2\dim_H\mu-2\epsilon^3} + \mu_U\left(B_{Mr+(\sqrt[4]{\beta})\left[Tr^{-(\dim_H\mu-\epsilon)}\right]^{\epsilon'}}(z')\setminus B_{Mr}(z')\right)\right],$$

where $\beta\in(0,1)$ is the same as that in Definition 6.

5.3. Coronas.

Proposition 5 (Coronas in $\mathcal{M}$ and U). *For almost every $z\in\mathcal{M}$ and $z'\in\operatorname{int}(U)$ there exist $r_z, r_{z',M}>0$, such that for any $r<\min\{r_z,r_{z',M}\}$ the following estimate holds for corona in $\mathcal{M}$*

$$\mu\left\{B_{r+C'r^{\frac{(\dim_H\mu-\epsilon)^2\alpha}{\dim_H\mu}}}(z)\Big\backslash B_{r-C'r^{\frac{(\dim_H\mu-\epsilon)^2\alpha}{\dim_H\mu}}}(z)\right\}\precsim_{z,\dim\gamma^u} r^{\left(1+\frac{(\dim_H\mu-\epsilon)^2\alpha}{\dim_H\mu}\right)\frac{\dim\gamma^u b}{2(b+\dim\gamma^u)}},$$

and the estimate for the corona in U is

$$\mu_U\left[B_{Mr+(\sqrt[4]{\beta})T^{\epsilon'}r^{-(\dim_H\mu-\epsilon)\epsilon'}}(z')\Big\backslash B_{Mr}(z')\right]$$
$$\precsim_{z,\dim\gamma^u}(Mr)^{\frac{\dim\gamma^u b}{2(b+\dim\gamma^u)}}\beta^{\left(T^{\epsilon'}r^{-(\dim_H\mu-\epsilon)\epsilon'}\right)\frac{\dim\gamma^u b}{8b+8\dim\gamma^u}}.$$

Proof. Let $q:=\frac{(\dim_H\mu-\epsilon)^2\alpha}{\dim_H\mu}$. For the corona in $\mathcal{M}$ we have from Lemma 21 for almost all $z\in\bigcup_{i\geq1}\bigcup_{0\leq j<R_i}f^j(\Lambda_i)$ that $z=f^{j_z}(z')$ for some $z'\in\operatorname{int}(U)$. Moreover, there exists $r_z'>0$, such that for any $r<r_z'$

$$\mu\left(f^{-j_z}\left[B_{r+C'r^q}(z)\setminus B_{r-C'r^q}(z)\right]\bigcap U^c\right)=0.$$

Since $f_*\mu=\mu$ and f is bijective on $\bigcup_{i\geq1}\bigcup_{0\leq j<R_i}f^j(\Lambda_i)$, then

$$\mu\left(\left[B_{r+C'r^q}(z)\setminus B_{r-C'r^q}(z)\right]\bigcap f^{j_z}U^c\right)=0.$$

Therefore

$$\left[B_{r+C'r^q}(z)\setminus B_{r-C'r^q}(z)\right]\subseteq f^{j_z}U \text{ almost surely.}$$

The rest of the proof is exactly the same as that of Proposition 3, where one should replace $\gamma^u\in\Gamma^u$ by $\gamma^u\in\Theta$, and then use that sectional curvatures of all $\gamma^u\in\Theta$ are uniformly bounded. Then there exists $r_z>0$ such that for any $r<r_z$

$$\mathrm{Leb}_{\gamma^u}\{f^{-j_z}[B_{r+C'r^q}(z)\setminus B_{r-C'r^q}(z)]\}\precsim_{z,\dim\gamma^u} r^{\frac{(1+q)\dim\gamma^u}{2}}.$$

Similarly to the proof of Lemma 23, we have

$$\mu\{f^{-j_z}[B_{r+C'r^q}(z)\setminus B_{r-C'r^q}(z)]\}$$
$$=\int\mu_{\gamma^u(x)}\{f^{-j_z}[B_{r+C'r^q}(z)\setminus B_{r-C'r^q}(z)]\}d\mu(x)$$

$$
\begin{aligned}
&= \int_{|\gamma^u(x)|\geq r^{\frac{(1+q)\dim\gamma^u}{2(b+\dim\gamma^u)}}} \mu_{\gamma^u(x)}\{f^{-j_z}[B_{r+C'r^q}(z)\setminus B_{r-C'r^q}(z)]\}d\mu(x)\\
&\quad+\int_{|\gamma^u(x)|\leq r^{\frac{(1+q)\dim\gamma^u}{2(b+\dim\gamma^u)}}} \mu_{\gamma^u(x)}\{f^{-j_z}[B_{r+C'r^q}(z)\setminus B_{r-C'r^q}(z)]\}d\mu(x)\\
&\precsim_{z,\dim\gamma^u} r^{\frac{(1+q)\dim\gamma^u b}{2(b+\dim\gamma^u)}}+r^{\frac{(1+q)\dim\gamma^u}{2}}r^{-\frac{(1+q)\dim\gamma^u\dim\gamma^u}{2(b+\dim\gamma^u)}} \precsim_{z,\dim\gamma^u} r^{\frac{(1+q)\dim\gamma^u b}{2(b+\dim\gamma^u)}}.
\end{aligned}
$$

Then, using that $f_*\mu=\mu$ and $q=\frac{(\dim_H\mu-\epsilon)^2\alpha}{\dim_H\mu}$, we obtain

$$
\mu\left\{B_{r+C'r^{\frac{(\dim_H\mu-\epsilon)^2\alpha}{\dim_H\mu}}}(z)\Big\backslash B_{r-C'r^{\frac{(\dim_H\mu-\epsilon)^2\alpha}{\dim_H\mu}}}(z)\right\}\precsim_{z,\dim\gamma^u} r^{\left(1+\frac{(\dim_H\mu-\epsilon)^2\alpha}{\dim_H\mu}\right)\frac{\dim\gamma^u b}{2(b+\dim\gamma^u)}}.
$$

For the corona in U let $q:=[Tr^{-(\dim_H\mu-\epsilon)}]^{\epsilon'}$, then there is $r_{z',M}\in(0,1/M)$ such that for any $r<r_{z',M}$

$$
B_{Mr+(\sqrt[4]{\beta})^q}(z')\subseteq U,
$$

.

Now, using the same argument as in Lemma 23, we have

$$
\begin{aligned}
&\mu_U[B_{Mr+(\sqrt[4]{\beta})^q}(z')\setminus B_{Mr}(z')]\\
&=\int_{|\gamma^u(x)|\geq (Mr)^{\frac{\dim\gamma^u}{2(b+\dim\gamma^u)}}\beta^{\frac{q\dim\gamma^u}{8(b+\dim\gamma^u)}}} \mu_{\gamma^u(x)}\{B_{Mr+(\sqrt[4]{\beta})^q}(z')\setminus B_{Mr}(z')\}d\mu(x)\\
&\quad+\int_{|\gamma^u(x)|\leq (Mr)^{\frac{\dim\gamma^u}{2(b+\dim\gamma^u)}}\beta^{\frac{q\dim\gamma^u}{8(b+\dim\gamma^u)}}} \mu_{\gamma^u(x)}\{B_{Mr+(\sqrt[4]{\beta})^q}(z')\setminus B_{Mr}(z')\}d\mu(x)\\
&\precsim_{z,\dim\gamma^u} (Mr)^{\frac{\dim\gamma^u b}{2(b+\dim\gamma^u)}}\beta^{\left(T^{\epsilon'}r^{-(\dim_H\mu-\epsilon)\epsilon'}\right)\frac{\dim\gamma^u b}{8(b+\dim\gamma^u)}}.
\end{aligned}
$$

□

5.4. Conclusion of proof of Theorem 4.

5.4.1. Proof that $\dim_H\mu\geq\frac{b}{b+\dim\gamma^u}\dim\gamma^u$

Lemma 26. *It follows from Assumptions 1 and Definition 6 that* $\dim_H\mu\geq\frac{b}{b+\dim\gamma^u}$ dim γ^u.

Proof. Due to ergodicity of μ, it is enough to consider $z'\in\operatorname{int}(U)$ and a ball $B_r(z')$ in U. By the same trick as in Proposition 5 we get for any $\gamma^u\in\Theta$

$$
\mathrm{Leb}_{\gamma^u}(B_r(z'))\precsim_{z',\dim\gamma^u} r^{\dim\gamma^u}\text{ and }\mu(B_r(z'))\precsim_{z',\dim\gamma^u} r^{\frac{\dim\gamma^u b}{b+\dim\gamma^u}}.
$$

On another hand, by Assumption 1 for almost all $z'\in\mathcal{M}$ and any $m\geq1$ there is $r_{z',m}>0$, such that $\mu(B_r(z'))\geq r^{\dim_H\mu+1/m}$ for any $r<r_{z',m}$. It implies that $\dim\gamma^u\frac{b}{b+\dim\gamma^u}\leq\dim_H\mu+1/m$ for any $m\geq1$. By taking limit $m\to\infty$ one gets $\dim_H\mu\geq\dim\gamma^u\frac{b}{b+\dim\gamma^u}$. □

We will obtain now convergence rates in Theorem 4. According to Proposition 1 it suffices to find convergence rates for short returns and coronas. A proof will consist of several steps.

5.4.2. Pull back short returns $\int_{B_r(z)} \mathbb{1}_{\bigcup_{1\le j\le p} f^{-j}B_r(z)} d\mu$ For almost any $z \in \bigcup_{i\ge1} \bigcup_{0\le j<R_i} f^j(\Lambda_i)$ we have $z = f^{j_z}(z')$ for some $z' \in \mathrm{int}\,(U)$. By Lemma 7, there exist a topological ball $U_r(z') = f^{-j_z}B_r(z)$ and a constant $C_z > 1$, such that $B_{C_z^{-1}r}(f^{-j_z}z) \subseteq U_r(f^{-j_z}z) \subseteq B_{C_z r}(f^{-j_z}z)$ for sufficiently small $r > 0$.

Lemma 27 (Pull short returns back to U). *There exists small enough $r_z > 0$, such that for any $r < r_z$ one has*

$$\int_{B_r(z)} \mathbb{1}_{\bigcup_{1\le k\le p} f^{-k}B_r(z)} d\mu \le \mu(U) \int_{B_{C_z r}(f^{-j_z}z)} \mathbb{1}_{\bigcup_{1\le k\le p} (f^{\overline{R}})^{-k} B_{C_z r}(f^{-j_z}z)} d\mu_U.$$

Proof. It follows from invariance of μ (i.e. $f_*\mu = \mu$) that

$$\int_{B_r(z)} \mathbb{1}_{\bigcup_{1\le k\le p} f^{-k}B_r(z)} d\mu = \int_{U_r(f^{-j_z}z)} \mathbb{1}_{\bigcup_{1\le k\le p} f^{-k}U_r(f^{-j_z}z)} d\mu.$$

By Lemma 21 we have that $U_r(f^{-j_z}z) \subseteq U$ for any $r < r_z$ if $r_z > 0$ is small enough. The rest of the proof is the same as that of Lemma 15, where one should use first return map $f^{\overline{R}}$. □

5.4.3. Estimate $\frac{1}{\mu(B_r(z))} \int_{B_{C_z r}(f^{-j_z}z)} \mathbb{1}_{\bigcup_{1\le k\le p} (f^{\overline{R}})^{-k} B_{C_z r}(f^{-j_z}z)} d\mu_U$

Lemma 28. *For any small* $\epsilon < \min\left\{\frac{b\dim\gamma^u}{24b+24\dim\gamma^u}, \frac{\dim_H\mu}{24}, (3\dim_H\mu)^{-1}\right\}$, *satisfying* $\alpha(\dim_H\mu - \epsilon) > \frac{\dim_H\mu}{\dim_H\mu-\epsilon} > 1$, *and for almost every $z \in \mathcal{M}$ there exists $r_z > 0$, such that for any $r < r_z$*

$$\frac{\int_{B_{C_z r}(f^{-j_z}z)} \mathbb{1}_{\bigcup_{1\le k\le p} (f^{\overline{R}})^{-k} B_{C_z r}(f^{-j_z}z)} d\mu_U}{\mu(B_r(z))} \precsim_{T,z,\epsilon,b} r^{\min\left\{\frac{b\dim\gamma^u}{12b+12\dim\gamma^u}, \frac{\dim_H\mu}{12}\right\}-2\epsilon} + r^{\frac{\epsilon^2}{\dim_H\mu}-3\epsilon^3}.$$

Proof. Note that $C_z < \infty$ for almost all $z \in \mathcal{M}$. Denote $F_1 := \{C_z < \infty\}$, $F_2 \times F_3 := \{(z, z') \in \mathcal{M} \times \mathrm{int}\,(U) :$ Propositions 4, 5 hold$\}$. Define a measure one subset in $\mathcal{M}$ as $F := F_1 \bigcap F_2 \bigcap \{\bigcup_{j\ge0} f^j(\mathrm{int}\,(U) \bigcap F_3^c)\}^c$. Then for any $z \in F$ we have $C_z < \infty$ and $(z, f^{-j_z}z) \in F_2 \times F_3$. Let now $M := \lfloor C_z \rfloor + 1$, $z' = f^{-j_z}z$. By Proposition 4, there exists $r_{z,z',M} = r_{z,f^{-j_z}z,\lfloor C_z\rfloor+1} > 0$, such that for any $r < r_{z,f^{-j_z}z,\lfloor C_z\rfloor+1}$

$$\begin{aligned}
&\int_{B_{C_z r}(f^{-j_z}z)} \mathbb{1}_{\bigcup_{1\le k\le p} (f^{\overline{R}})^{-k} B_{C_z r}(f^{-j_z}z)} d\mu_U \le \int_{B_{Mr}(z')} \mathbb{1}_{\bigcup_{1\le k\le p} (f^{\overline{R}})^{-k} B_{Mr}(z')} d\mu_U \\
&\precsim_{T,b,\epsilon} (Mr)^{\dim_H\mu-\epsilon} M^{\min\{\frac{b\dim\gamma^u}{6b+6\dim\gamma^u}, \frac{\dim_H\mu}{6}\}} r^{\min\{\frac{b\dim\gamma^u}{12b+12\dim\gamma^u}, \frac{\dim_H\mu}{12}\}} \\
&\quad + r^{(-\dim_H\mu-\epsilon)\frac{\dim_H\mu-\epsilon}{\dim_H\mu}} \left[(Mr)^{2\dim_H\mu-2\epsilon^3} + \mu_U\left(B_{Mr+(\sqrt[4]{\beta})\left[T^{\epsilon'} r^{-(\dim_H\mu-\epsilon)\epsilon'}\right]}(z') \Big\backslash B_{Mr}(z') \right) \right].
\end{aligned}$$

Next, by Proposition 5, the right hand side of inequality above can be estimated as

$$\begin{aligned}
&\precsim_{T,C_z,z,\epsilon,b} r^{\dim_H\mu-\epsilon} r^{\min\{\frac{b\dim\gamma^u}{12b+12\dim\gamma^u}, \frac{\dim_H\mu}{12}\}} + r^{-(\dim_H\mu+\epsilon)\frac{\dim_H\mu-\epsilon}{\dim_H\mu}} r^{2\dim_H\mu-2\epsilon^3} \\
&\quad + r^{-(\dim_H\mu+\epsilon)\frac{\dim_H\mu-\epsilon}{\dim_H\mu}} r^{\frac{\dim\gamma^u b}{2b+2\dim\gamma^u}} \beta^{\left(T^{\epsilon'} r^{-(\dim_H\mu-\epsilon)\epsilon'}\right)\frac{\dim\gamma^u b}{8b+8\dim\gamma^u}}.
\end{aligned}$$

Now, by Assumption 1 for almost all $z \in \mathcal{M}$ there exists $r_z > 0$, such that for any $r < r_z$

$$r^{\dim_H \mu + \epsilon^3} \leq \mu(B_r(z)) \leq r^{\dim_H \mu - \epsilon^3}.$$

Then for any $r < \min\{r_{z, f^{-jz}z, \lfloor C_z \rfloor + 1}, r_z\}$

$$\begin{aligned}
&\frac{1}{\mu(B_r(z))} \int_{B_{C_z r}(f^{-jz}z)} \mathbb{1}_{\bigcup_{1\leq k\leq p}(f^{\overline{R}})^{-k} B_{C_z r}(f^{-jz}z)} d\mu_\Lambda \\
&\precsim_{T,z,\epsilon,b} r^{-\dim_H \mu - \epsilon^3} \Big\{ r^{\dim_H \mu - \epsilon} r^{\min\{\frac{b \dim \gamma^u}{12b+12\dim \gamma^u}, \frac{\dim_H \mu}{12}\}} + r^{-(\dim_H \mu + \epsilon)\frac{\dim_H \mu - \epsilon}{\dim_H \mu}} r^{2\dim_H \mu - 2\epsilon^3} \\
&\quad + r^{-(\dim_H \mu + \epsilon)\frac{\dim_H \mu - \epsilon}{\dim_H \mu}} r^{\frac{\dim \gamma^u b}{2b+2\dim \gamma^u}} \beta^{\left(T^{\epsilon'} r^{-(\dim_H \mu - \epsilon)\epsilon'}\right)\frac{\dim \gamma^u b}{8b+8\dim \gamma^u}} \Big\} \\
&\precsim_{T,z,\epsilon,b} r^{\min\{\frac{\dim \gamma^u b}{12b+12\dim \gamma^u}, \frac{\dim_H \mu}{12}\} - 2\epsilon} + r^{\frac{\epsilon^2}{\dim_H \mu} - 3\epsilon^3} \to 0,
\end{aligned}$$

where the last inequality holds because $\beta_2^{r^{-c}} \ll r^{c'}$ for any $c, c' > 0$. □

By combining Lemmas 27 and 28 we get an

5.4.4. Estimate for short returns rates $\frac{1}{\mu(B_r(z))} \int_{B_r(z)} \mathbb{1}_{\bigcup_{1\leq j\leq p} f^{-j} B_r(z)} d\mu$

Lemma 29. *For any small* $\epsilon < \min\left\{\frac{b \dim \gamma^u}{24b+24\dim \gamma^u}, \frac{\dim_H \mu}{24}, (3\dim_H \mu)^{-1}\right\}$, *satisfying* $\alpha(\dim_H \mu - \epsilon) > \frac{\dim_H \mu}{\dim_H \mu - \epsilon} > 1$, *and for almost every* $z \in \mathcal{M}$

$$\frac{1}{\mu(B_r(z))} \int_{B_r(z)} \mathbb{1}_{\bigcup_{1\leq j\leq p} f^{-j} B_r(z)} d\mu \precsim_{T,z,\epsilon,b} r^{\min\left\{\frac{\dim \gamma^u b}{12b+12\dim \gamma^u}, \frac{\dim_H \mu}{12}\right\} - 2\epsilon} + r^{\frac{\epsilon^2}{\dim_H \mu} - 3\epsilon^3} \to 0.$$

5.4.5. Estimate for coronas rates $\frac{1}{\mu(B_r(z))} \mu\Big[B_{r + C' r^{\frac{(\dim_H \mu - \epsilon)^2 \alpha}{\dim_H \mu}}}(z) \Big\backslash B_{r - C' r^{\frac{(\dim_H \mu - \epsilon)^2 \alpha}{\dim_H \mu}}}(z)\Big]$

Lemma 30. *Let* $\alpha > \frac{2}{\dim \gamma^u} \frac{b + \dim \gamma^u}{b} - \frac{1}{\dim_H \mu}$ *and* $\epsilon < \min\Big\{\frac{b \dim \gamma^u}{24b+24\dim \gamma^u}, \frac{\dim_H \mu}{24}, (3\dim_H \mu)^{-1}\Big\}$ *is small enough, so that* $\alpha > \left[\left(\frac{2}{\dim \gamma^u} + \frac{2\epsilon}{\dim \gamma^u \dim_H \mu}\right) \frac{b + \dim \gamma^u}{b} - \frac{1}{\dim_H \mu}\right] \left(1 - \frac{\epsilon}{\dim_H \mu}\right)^{-2}$. *Then for almost all* $z \in \mathcal{M}$

$$\frac{\mu\Big[B_{r + C' r^{\frac{(\dim_H \mu - \epsilon)^2 \alpha}{\dim_H \mu}}}(z) \Big\backslash B_{r - C' r^{\frac{(\dim_H \mu - \epsilon)^2 \alpha}{\dim_H \mu}}}(z)\Big]}{\mu(B_r(z))} \precsim_z r^{\left(1 + \frac{(\dim_H \mu - \epsilon)^2 \alpha}{\dim_H \mu}\right)\frac{\dim \gamma^u b}{2b+2\dim \gamma^u} - \dim_H \mu - \epsilon} \to 0,$$

Calculations here are exactly the same as that in Lemma 18, using Proposition 5. Therefore we will not repeat them.

5.4.6. Convergence rates $a > 0$ in $d_{TV}(N^{r,T,z}, P) \precsim_{T,z,b} r^a$

Lemma 31. *If* $\alpha > \frac{2}{\dim \gamma^u} - \frac{1}{\dim_H \mu}$, *choose a small* $\epsilon < \min\left\{\frac{b \dim \gamma^u}{24b+24\dim\gamma^u}, \frac{\dim_H \mu}{24}, \frac{1}{3\dim_H \mu}\right\}$, *such that* $\alpha > \max\left\{\frac{\dim_H \mu}{(\dim_H \mu-\epsilon)^2}, \left[\frac{(2\dim_H \mu+2\epsilon)(b+\dim\gamma^u)}{b\dim\gamma^u \dim_H \mu} - \frac{1}{\dim_H \mu}\right]\left(1-\frac{\epsilon}{\dim_H \mu}\right)^{-2}\right\}$, *then we have convergence rate*

$$a := \min\left\{\frac{(\dim_H \mu - \epsilon)^2(\xi-1)}{\dim_H \mu}, \frac{\epsilon^2}{\dim_H \mu} - 3\epsilon^3, \left(1 + \frac{(\dim_H \mu - \epsilon)^2\alpha}{\dim_H \mu}\right)\frac{\dim \gamma^u b}{2b + 2\dim\gamma^u} - \dim_H \mu - \epsilon, \min\left\{\frac{\dim\gamma^u b}{12b+12\dim\gamma^u}, \frac{\dim_H \mu}{12}\right\} - 2\epsilon\right\},$$

If $\frac{d\mu}{d\,\mathrm{Leb}_{\mathcal{M}}} \in L^\infty_{loc}(\mathcal{M})$, *choose a small* $\epsilon < \min\left\{\frac{b\dim\gamma^u}{24b+24\dim\gamma^u}, \frac{\dim_H \mu}{24}, (3\dim_H \mu)^{-1}\right\}$, *such that* $\alpha > \frac{\dim_H \mu}{(\dim_H \mu-\epsilon)^2}$. *Then we obtain convergence rate*

$$a := \min\left\{\frac{(\dim_H \mu - \epsilon)^2(\xi-1)}{\dim_H \mu}, \frac{(\dim_H \mu - \epsilon)^2\alpha}{\dim_H \mu} - 1, \frac{\epsilon^2}{\dim_H \mu} - 3\epsilon^3, \min\left\{\frac{\dim\gamma^u b}{12b+12\dim\gamma^u}, \frac{\dim_H \mu}{12}\right\} - 2\epsilon\right\},$$

Proof. From Proposition 1,

$$d_{TV}\left(N^{r,T,z}, P\right) \precsim_{T,\xi,\epsilon} R_1(r) + R_2(r,z) + R_3(r,z) + R_4(r,z),$$

where

$$R_1(r) := r^{\dim_H \mu - \epsilon} + r^{\frac{(\dim_H \mu-\epsilon)^2}{\dim_H \mu}(\xi-1)} + r^{\frac{\epsilon(\dim_H \mu-\epsilon)}{\dim_H \mu}}$$

$$R_2(r,z) := \frac{1}{\mu(B_r(z))}\mu\left(B_{r+C'r^{\frac{(\dim_H \mu-\epsilon)^2\alpha}{\dim_H \mu}}}(z) \Big\backslash B_{r-C'r^{\frac{(\dim_H \mu-\epsilon)^2\alpha}{\dim_H \mu}}}(z)\right)$$

$$R_3(r,z) := \frac{1}{\mu(B_r(z))^2}\mu\left(B_{r+C'r^{\frac{(\dim_H \mu-\epsilon)^2\alpha}{\dim_H \mu}}}(z) \Big\backslash B_{r-C'r^{\frac{(\dim_H \mu-\epsilon)^2\alpha}{\dim_H \mu}}}(z)\right)^2$$

$$R_4(r,z) := \frac{1}{\mu(B_r(z))}\int_{B_r(z)} \mathbb{1}_{\bigcup_{1\le j\le p} f^{-j}B_r(z)}d\mu.$$

From Lemma 29 we have

$$R_4(r,z) \precsim_{T,z,\epsilon,b} r^{\min\left\{\frac{\dim\gamma^u b}{12b+12\dim\gamma^u}, \frac{\dim_H \mu}{12}\right\}-2\epsilon} + r^{\frac{\epsilon^2}{\dim_H \mu}-3\epsilon^3}.$$

If $\alpha > \frac{2}{\dim\gamma^u} - \frac{1}{\dim_H \mu}$, then from Lemma 30, we obtain

$$R_3(r,z) \le R_2(r,z) \precsim_z r^{\left(1+\frac{(\dim_H \mu-\epsilon)^2\alpha}{\dim_H \mu}\right)\frac{\dim\gamma^u b}{2b+2\dim\gamma^u} - \dim_H \mu - \epsilon}.$$

Therefore, $d_{TV}\left(N^{r,T,z}, P\right) \precsim_{T,\xi,\epsilon,b} r^a$, and

$$a := \min\left\{\frac{(\dim_H \mu-\epsilon)^2(\xi-1)}{\dim_H \mu}, \frac{\epsilon(\dim_H \mu-\epsilon)}{\dim_H \mu},\right.$$
$$\left(1+\frac{(\dim_H \mu-\epsilon)^2\alpha}{\dim_H \mu}\right)\frac{b\dim\gamma^u}{2b+2\dim\gamma^u}-\dim_H \mu$$
$$\left.-\epsilon, \frac{\epsilon^2}{\dim_H \mu}-3\epsilon^3, \min\left\{\frac{\dim\gamma^u b}{12b+12\dim\gamma^u}, \frac{\dim_H \mu}{12}\right\}-2\epsilon, \dim_H \mu-\epsilon\right\}$$
$$=\min\left\{\frac{(\dim_H \mu-\epsilon)^2(\xi-1)}{\dim_H \mu}, \frac{\epsilon^2}{\dim_H \mu}-3\epsilon^3,\right.$$
$$\left(1+\frac{(\dim_H \mu-\epsilon)^2\alpha}{\dim_H \mu}\right)\frac{\dim\gamma^u b}{2b+2\dim\gamma^u}-\dim_H \mu$$
$$\left.-\epsilon, \min\left\{\frac{\dim\gamma^u b}{12b+12\dim\gamma^u}, \frac{\dim_H \mu}{12}\right\}-2\epsilon\right\},$$

where the last equality comes from relations $\frac{\epsilon^2}{\dim_H \mu}-3\epsilon^3 \leq \frac{\epsilon(\dim_H \mu-\epsilon)}{\dim_H \mu} \leq \dim_H \mu-\epsilon$.

If $\frac{d\mu}{d\,\mathrm{Leb}_{\mathcal{M}}} \in L^\infty_{loc}(\mathcal{M})$, then

$$R_3(r,z) \leq R_2(r,z) = \frac{1}{\mu(B_r(z))}\mu\left(B_{r+Cr^{\frac{(\dim_H \mu-\epsilon)^2\alpha}{\dim_H \mu}}}(z)\Big\backslash B_{r-Cr^{\frac{(\dim_H \mu-\epsilon)^2\alpha}{\dim_H \mu}}}(z)\right)$$
$$\precsim_z r^{\frac{(\dim_H \mu-\epsilon)^2\alpha}{\dim_H \mu}-1}.$$

Therefore, $d_{TV}\left(N^{r,T,z}, P\right) \precsim_{T,\xi,\epsilon,b} r^a$, and

$$a := \min\left\{\frac{(\dim_H \mu-\epsilon)^2(\xi-1)}{\dim_H \mu}, \frac{\epsilon(\dim_H \mu-\epsilon)}{\dim_H \mu}, \frac{(\dim_H \mu-\epsilon)^2\alpha}{\dim_H \mu}-1, \frac{\epsilon^2}{\dim_H \mu}-3\epsilon^3,\right.$$
$$\left.\min\left\{\frac{\dim\gamma^u b}{12b+12\dim\gamma^u}, \frac{\dim_H \mu}{12}\right\}-2\epsilon, \dim_H \mu-\epsilon\right\}$$
$$=\min\left\{\frac{(\dim_H \mu-\epsilon)^2(\xi-1)}{\dim_H \mu}, \frac{(\dim_H \mu-\epsilon)^2\alpha}{\dim_H \mu}-1, \frac{\epsilon^2}{\dim_H \mu}-3\epsilon^3,\right.$$
$$\left.\min\left\{\frac{\dim\gamma^u b}{12b+12\dim\gamma^u}, \frac{\dim_H \mu}{12}\right\}-2\epsilon\right\},$$

where the last equality follows from $\frac{\epsilon^2}{\dim_H \mu}-3\epsilon^3 \leq \frac{\epsilon(\dim_H \mu-\epsilon)}{\dim_H \mu} \leq \dim_H \mu-\epsilon$.

This completes the proof of Theorem 4. □

6. Applications

For all classes of dynamical systems, which will be considered in this section, it is known that there exist Gibbs-Markov-Young structures (see Definition 5) and SRB measures μ. Our Assumption 1 also holds, except possibly for the condition $\alpha\dim_H \mu > 1$. Therefore, only the following conditions must be verified:

1. For Theorem 3

(a) R is the first return time for Λ and f,
(b) there exist constants $\alpha > 0$ and $C > 0$ such that

$$\sup_{x,y\in\gamma^s\in\Gamma^s, x',y'\in\gamma^u\in\Gamma^u} \left\{d(f^n x, f^n y), d(f^{-n}x', f^{-n}y')\right\} \le Cn^{-\alpha},$$

(c) $\mu\{\text{int}(\Lambda)\} > 0$ and $\mu(\partial\Lambda) = 0$,
(d) verify, whether or not $\alpha > \frac{2}{\dim \gamma^u} - \frac{1}{\dim_H \mu} > \frac{1}{\dim_H \mu}$, and whether or not $\frac{d\mu}{d\,\text{Leb}_{\mathcal{M}}} \in L^\infty_{loc}(\mathcal{M})$.

2. For Theorem 4
(a) find such reference subset $U \subseteq \mathcal{M}$ that its first return map $f^{\overline{R}}$ has exponential decay of correlations,
(b) find a measurable partition $\Theta := \{\gamma^u(x)\}_{x\in U}$ with required properties,
(c) check that estimate $\mu_U\{x \in U : |\gamma^u(x)| < \epsilon\} \le C\epsilon^b$ holds,
(d) get an estimate for the distortion, i.e., $\frac{d\mu_{\gamma^u(x)}}{d\,\text{Leb}_{\gamma^u(x)}}(y) = C^{\pm 1}\frac{d\mu_{\gamma^u(x)}}{d\,\text{Leb}_{\gamma^u(x)}}(z)$ for any $y, z \in \gamma^u(x) \in \Theta$,
(e) show that there exist constants $\alpha > 0$ and $C > 0$, such that

$$\sup_{x,y\in\gamma^s\in\Gamma^s, x',y'\in\gamma^u\in\Gamma^u} \{d(f^n x, f^n y), d(f^{-n}x', f^{-n}y')\} \le Cn^{-\alpha},$$

(f) verify, whether or not $\alpha > \frac{2}{\dim \gamma^u}\frac{b+\dim\gamma^u}{b} - \frac{1}{\dim_H \mu} > \frac{1}{\dim_H \mu}$, and whether or not $\frac{d\mu}{d\,\text{Leb}_{\mathcal{M}}} \in L^\infty_{loc}(\mathcal{M})$.

All other required conditions hold for the classes of dynamical systems we present below.

6.1. Intermittent solenoids. Following [2,37] let $\mathcal{M} = S^1 \times \mathbb{D}$, $f_\gamma(x, z) = (g_\gamma(x), \theta z + e^{2\pi i x}/2)$, where $g_\gamma : S^1 \to S^1$ is a continuous map of degree $d \ge 2$ and $\gamma \in (0, +\infty)$ such that

1. g_γ is C^2 on $S^1 \setminus \{0\}$ and $Dg_\gamma > 1$ on $S^1 \setminus \{0\}$,
2. $g_\gamma(0) = 0$, $Dg_\gamma(0+) = 1$ and $xD^2g_\gamma(x) \sim x^\gamma$ for sufficiently small positive x,
3. $Dg_\gamma(0-) > 1$,
4. $\theta > 0$ is so small that $\theta||Dg_\gamma||_\infty < 1 - \theta$.

It was proved in [2] that the SRB probability measure μ exists iff $\gamma \in (0, 1)$, the attractor is $A := \bigcap_{i\ge 0} f^i_\gamma(\mathcal{M})$, $\xi = 1/\gamma > 1$, $\alpha = 1 + 1/\gamma$, $\Lambda = (I \times \mathbb{D}) \bigcap A$, where I is one of intervals of hyperbolicity, and $f_\gamma : I \to S^1$ is a C^2-diffeomorphism. Then $\partial\Lambda = (\partial I \times \mathbb{D}) \bigcap A$ and $\mu(\partial\Lambda) \precsim \text{Leb}_{S^1}(\partial I) = 0$ due to (3.4) and (3.5). Let R be the first return time constructed in [2], and $f^R_\gamma : \Lambda \to \Lambda$ is the corresponding first return map. Clearly, $\dim \gamma^u = 1$.

With all of these, we have that $\alpha = 1 + 1/\gamma > 2 > \frac{2}{\dim \gamma^u} - \frac{1}{\dim_H \mu}$. The condition $\mu\{\text{int}(\Lambda)\} > 0$ holds also. Hence, by Theorem 3 we have the following

Corollary 2. *Functional Poisson limit laws hold for f_γ for any $\gamma \in (0, 1)$ with convergence rates specified in Lemma 19.*

Remark 5. 1. We could also use here Theorem 4. Indeed, let $U = I \times \mathbb{D}$, $\overline{R}$ is the first return time to U, Θ is the set of all unstable manifolds in U (observe that their union is, actually, Λ). The lengths of all $\gamma^u \subseteq \Lambda$ are uniformly bounded from below. Therefore, if ϵ is small enough, then $\mu_U\{x \in U : |\gamma^u(x)| < \epsilon\} = 0 \leq \epsilon^b$ (note, that here b is arbitrarily large). For each $\gamma^u \in \Theta$, $\mu_{\gamma^u} \approx \text{Leb}_{\gamma^u}$ and $\alpha = 1 + 1/\gamma$. It is well known that correlations for $f^{\overline{R}} : U \to U$ decay exponentially. Therefore Corollary 2 holds.

2. In [37] a "maximum" metric was chosen, instead of Riemannian metric. It was proved there that Poisson limit laws hold for $\gamma \in (0, \sqrt{2}/2)$. After checking details therein, we found that our approach allows to improve the results obtained there to $\gamma \in (0, 1)$, i.e., by using their metric. Note however, that we consider only Riemannian metric everywhere in the present paper. Therefore we omit here these calculations.

6.2. Axiom A attractors. It was proved in [16] that for Axiom A attractors $\Sigma \subset \mathcal{M}$ with $\dim \gamma^u = 1$ Poisson limit laws hold. Later, in [37], Poisson limit laws were established for ergodic dynamics $f : \Sigma \to \Sigma$ if $\dim_H \mu > \dim \mathcal{M} - 1$. We will show that conditions on $\dim_H \mu$ and $\dim \gamma^u$ can be dropped.

Definition 10 *(Axiom A attractors, see* [7,41,45]). Let $f : \mathcal{M} \to \mathcal{M}$ be a C^2-diffeomorphism. A compact set $\Sigma \subseteq \mathcal{M}$ is called an Axiom A attractor if

1. There is a neighborhood U of Σ, called its basin, such that $f^n(x) \to \Sigma$ for every $x \in U$.
2. The tangent bundle over Σ is split into $E^u \oplus E^s$, where E^u and E^s are df-invariant subspaces.
3. $df|_{E^u}$ is uniformly expanding and $df|_{E^s}$ is uniformly contracting.
4. $f : \Sigma \to \Sigma$ is topologically mixing.

Before turning to proofs, we need one lemma from [7]:

Lemma 32 (Markov partitions, see the chapter 3 of [7]). *The set Σ has a Markov partition $\{\Sigma_1, \Sigma_2, \ldots, \Sigma_m\}$ into elements with arbitrarily small diameters. Here the sets Σ_i are proper rectangles (i.e., $\Sigma_i = \overline{\text{int}\,(\Sigma_i)}$ and $\text{int}\,(\Sigma_i) \bigcap \text{int}\,(\Sigma_j) = \emptyset$ for $i \neq j$, where interior and closure are taken with respect to topology of Σ, rather than to topology of $\mathcal{M}$).*

We will verify now conditions imposed in our main theorems.

1. Let a horseshoe Λ coincides with Σ_1. Then return time for a hyperbolic tower Δ is actually the first return due to existence of Markov partition.
2. Contraction rates of (un)stable manifolds are exponential, i.e., faster than required $O(n^{-\alpha})$.
3. A constant α can be, in this case, arbitrary large (namely, $\alpha > \max\{2, (\dim_H \mu)^{-1}\}$). Then $\alpha > 2 > \frac{2}{\dim \gamma^u} - \frac{1}{\dim_H \mu} > \frac{1}{\dim_H \mu}$.
4. Finally, from existence of a finite Markov partition follows that $\mu(\text{int}\,(\Lambda)) > 0$. And $\mu(\partial \Lambda) = 0$ due to the structure of $\partial \Sigma_1$, according to Lemma 3.11 of [7].

Therefore Theorem 3 holds, and we obtain the following

Corollary 3. *Functional Poisson limit laws hold for Axiom A attractors with convergence rate specified in Lemma 19.*

6.3. Dispersing billiards with and without a finite horizon. Existence of the Gibbs-Markov-Young structure for dispersing billiards was established in [17]. Denote by D a billiard table, i.e., a closed region on the Euclidean plane with piecewise C^3-smooth boundary ∂D. The phase space of a billiard $\mathcal{M}$ is $\partial D \times [-\pi/2, \pi/2]$. In dispersing billiards the boundary is convex inwards. These billiards are hyperbolic dynamical systems with singularities, which appear because of orbits tangent to the boundary and orbits hitting singularities of the boundary. For technical reasons (see [15]) it is convenient to introduce some extra (artificial) singularities, and represent the phase space $\mathcal{M}$ as $\bigcup_{k\geq k_0} \partial D \times [-\pi/2+(k+1)^{-2}, -\pi/2+k^{-2}] \bigcup \bigcup_{-k\geq k_0} \partial D \times [\pi/2-k^{-2}, \pi/2-(k+1)^{-2}] \bigcup \partial D \times [-\pi/2+k_0^{-2}, \pi/2-k_0^{-2}]$, where $k_0 \gg 1$. Then $\mathcal{M}$ becomes formally closed, non-compact and disconnected. Moreover, the billiard map $f : \partial\mathcal{M} \to \partial\mathcal{M}$ becomes multi-valued because the phase space $\mathcal{M}$ gets partitioned into infinitely many pieces, and the boundary $\partial\mathcal{M}$ acquires infinitely many new components. As a result, the billiard map acquires additional singularities. However, this trick allows to get proper estimates of distortions, probability densities and Jacobian of the holonomy map due to partition of unstable manifolds into homogeneous ones (see the details in [15] or in chapter 5 of [19]). Denote by $\mathbb{S}$ the union of all singular manifolds.

We will verify now for dispersing billiards conditions of our main theorems.

1. $\overline{R} = 1, U = \mathcal{M}, \mu \ll \mathrm{Leb}_{\mathcal{M}}$. It was proved in [17,45] that correlations decay exponentially.
2. $\alpha > 0$ is arbitrarily large, $\dim_H \mu = 2$, $\dim \gamma^u = 1$.
3. Let $\mathcal{Q}_n^{\mathbb{H}}(x)$ be a connected component of $\mathcal{M} \setminus \bigcup_{m=0}^{n} f^m(\mathbb{S})$ containing a point x. The partition $\Theta := (\bigcap_{n\geq 1} \overline{\mathcal{Q}_n^{\mathbb{H}}(x)})_{x\in\mathcal{M}}$, which consists of maximal homogeneous unstable manifolds, is measurable.
4. A required distortion's estimate holds for each $\gamma^u(x) \in \Theta$ by Corollary 5.30 in [19].
5. Theorem 5.17 of [19] gives estimate $\mu_U\{x \in U : |\gamma^u(x)| < \epsilon\} \leq C\epsilon$. (Observe that here $b = 1$).

Hence Theorem 4 can be applied, and we have

Corollary 4. *The functional Poisson limit laws hold for two-dimensional dispersing billiards with or without a finite horizon, and corresponding convergence rates satisfy estimates from Lemma 31.*

6.4. Billiards with focusing components of the boundary. In this section we consider two-dimensional hyperbolic billiards, which have convex outwards of billiard table circular boundary components together with dispersing and neutral (zero curvature) components of the boundary. The main assumption is that the entire circle, which contains any focusing component, belongs to a billiard table D. This class of billiards was introduced and studied in [11,12]. Standard coordinates for the billiard map f are (r, ϕ), where r fixes a point on the boundary of a billiard table and ϕ is an angle of reflection off the boundary at this point. To simplify the exposition, we will consider now only the most studied and popular example in this class, called a stadium. (Actually, all the reasoning for a general case is the same [13]).

The boundary of a stadium consists of two semicircles of the same radius connected by two tangent to them neutral components. Existence of the Gibbs-Markov-Young structure for a stadium was proved in [20,34]. The phase space in this case is $\mathcal{M} := \partial D \times [-\pi/2, \pi/2]$, where ∂D is the boundary of a stadium.

1. Let $U \subseteq \mathcal{M}$ consists of all points, where the first or the last collision of billiard orbits with the semicircles occur. By the first (resp., last) collision we mean here the first (resp., last) collision with a circular component of the boundary, which occur after (resp., before) the last (resp., first) collision of the orbit in a series of consecutive collisions with the neutral part of the boundary or with another focusing component. Clearly this set is a disjoint union of two similar hexagons. Hence, it is enough to consider one of them, say the hexagon attached to $\{(r, \phi) \in \mathcal{M} : r = 0\}$, (see the Figure 8.10 of [19]). We have $\mu \ll \mathrm{Leb}_{\mathcal{M}}$ and $\mu(\partial U) = 0$. Let $\overline{R}$ be the first return time to U. Using Theorems 4 and 5 in [20] one can prove that the first return map $f^{\overline{R}} : U \to U$ has an exponential decay of correlations. Consider the set of singular points $\mathbb{S}$ which correspond to hitting four singular points of the boundary, where focusing and neutral components meet and generate jumps of the curvature. Let $\mathbb{S}_1 := (f^{\overline{R}})^{-1}(\mathbb{S})$.
2. Let $\mathcal{Q}_n(x)$ be the connected component of $\mathcal{M} \setminus \bigcup_{m=0}^{n}(f^{\overline{R}})^m(\mathbb{S} \bigcup \mathbb{S}_1)$ containing a point x. The partition $\Theta := (\bigcap_{n\geq 1} \overline{\mathcal{Q}_n(x)})_{x\in\mathcal{M}}$ is measurable.
3. A required estimate of distortion holds for each $\gamma^u(x) \in \Theta$ by Corollary 8.53 in [19].
4. We will prove now that $\mu_U\{x \in U : |\gamma^u(x)| < r, \gamma^u(x) \in \Theta\} \precsim \sqrt{r}$. Denote by $\mathcal{Q}'_n(x)$ the connected component of the set $\mathcal{M} \setminus \bigcup_{m=0}^{n}(f^{\overline{R}})^m(\mathbb{S})$ which contains a point x. Then some smooth unstable manifolds $\gamma^{u'}(x) \in \Theta' := (\bigcap_{n\geq 1} \overline{\mathcal{Q}'_n(x)})_{x\in\mathcal{M}}$ are cut by the set $\mathbb{S}_1$ into smaller pieces, which belong to Θ. (Observe that some of them could be disjoint with $\mathbb{S}_1$). It follows from Theorem 8.42 of [19] that $\mu_U\{x \in U : |\gamma^{u'}(x)| < r, \gamma^{u'}(x) \in \Theta'\} \leq Cr$.
   Connected components of the set $\mathbb{S}_1$ are of two types:
   (a) L_k is a straight (increasing in the (r, ϕ)- coordinates) segment in U with slope $1/k$, representing k successive reflections at one and the same semicircle (see e.g. the Figure 8.11 in [19]).
   (b) F_m is an increasing curve (in (r, ϕ)-coordinates) in U with slope ≈ 1 (i.e., it is bounded away from 0 and $+\infty$), which corresponds to m successive bounces on flat sides of the boundary (see e.g. Figure 8.12 in [19]).
   Moreover, L_k is located at the distance $\approx 1/k$ from the set $\{(r, \phi) \in \mathcal{M} : \phi = \pm\pi/2\}$, and F_m is at the distance $\approx 1/m$ from $\{(r, \phi) \in \mathcal{M} : \phi = 0\}$. Let
$$V_1 := \bigcup_{k>1/\sqrt{r}} B_r(L_k) \bigcup \bigcup_{m>1/\sqrt{r}} B_r(F_m), \quad V_2 := \bigcup_{k\leq 1/\sqrt{r}} B_r(L_k) \bigcup \bigcup_{m\leq 1/\sqrt{r}} B_r(F_m),$$
   where $B_r(L_k)$, $B_r(F_m)$ are the r-neighborhoods of L_k, F_m. Then $\mu_U(V_1 \bigcup V_2) \precsim \sqrt{r} + r/\sqrt{r} \precsim \sqrt{r}$. For almost every $x \in U \bigcap (V_1 \bigcup V_2)^c$, a curve $\gamma^u(x) \in \Theta$ is decreasing (in (r, ϕ) coordinates), and if its length $|\gamma^u(x)| < r$, then $\gamma^u(x)$ is disjoint with $\mathbb{S}_1$, and thus $\gamma^u(x) \in \Theta'$. Therefore,
$$\begin{aligned} &\mu_U \left\{x : |\gamma^u(x)| < r, \gamma^u(x) \in \Theta\right\} \\ &\precsim \mu_U \left(V_1 \bigcup V_2\right) + \mu_U \left\{x \in \left(V_1 \bigcup V_2\right)^c : |\gamma^u(x)| < r\right\} \\ &\precsim \sqrt{r} + \mu_U \left\{x \in \left(V_1 \bigcup V_2\right)^c : |\gamma^{u'}(x)| < r, \gamma^{u'}(x) \in \Theta'\right\} \precsim \sqrt{r}. \end{aligned}$$
5. It was proved in [37] that $\alpha = 1$. Also, $\dim_H \mu = 2$ and $\dim \gamma^u = 1$.

   Therefore all conditions of Theorem 4 are satisfied, and we have

Corollary 5. *Functional Poisson limit laws hold for stadium-type billiards, and corresponding convergence rates are provided by Lemma 31.*

Remark 6 (A general remark on billiards). All considerations in our paper were traditionally dealing with hitting small sets (e.g. small balls) in phase spaces of hyperbolic (chaotic) dynamical systems. However, in case of billiards, the most interesting and natural questions are about hitting (or escape through) some small sets (particularly “holes") on the boundary of billiard tables, rather than in the interior of a billiard table. These sets are small in the space (e.g. r) coordinate, but they are large (have a “full" size) along the angle (ϕ) coordinate.

It is worthwhile to mention though, that there are some real life situations, when actual escape (radiation, emission) from various physical devices (cavities, lasers, etc) occurs only in some small range of angles (see e.g. [27,36]). Our results could be directly applied to such cases. However, when a target set is a strip (or a cylinder) with a finite fixed height in the angle ϕ-coordinate, results of the present paper can also be used/adapted by cutting a cylinder into small sets. Then the obtained estimates are valid for these pieces of a cylinder. Clearly, this approach does not generally work for recurrences, but it could be applied for the first hitting probabilities because an orbit cannot escape through one hole and then again escape through another hole. (By holes we mean here disjoint pieces of a cylinder). Therefore, one can take in such cases a relevant maximum or minimum of obtained estimates for “small" sets, i.e., for pieces of a cylinder in the phase space of a billiard.

It is worthwhile to mention though, that functional Poisson limit laws for billiards with holes in the boundary (“cylindric" holes in the phase space) is a work in progress. It requires some new arguments and lengthy computations.

6.5. Hénon attractors. The Poisson limit laws for certain Hénon attractors (see [5,45]) have been proved in [16]. However, convergence rate for this class of dynamical systems was obtained in a weaker form (1.2). Here we give a simpler proof than the one in [16] and derive stronger rate of convergence (1.1) by using Theorem 4.

1. Let $\overline{R} = 1, U = \mathcal{M}$. It is proved in [6,45] that f has exponential decay of correlation, and a constant α can be, in this case, arbitrarily large.
2. Each Hénon attractor is a closure of an unstable manifold. In order to construct a measurable partition we consider a certain family of unstable leaves by making use of Young towers. Recall that a hyperbolic Young tower is generated by a horseshoe $\Lambda = \bigcup_i \Lambda_i$. Let $\mu_\Lambda|_{\Lambda_i} := \mu_{\Lambda_i}$. For any measurable $A \subseteq \mathcal{M}$,

$$\mu(A) = \mu_\Delta(\pi^{-1}A) = \sum_i \sum_{j<R_i} \mu_\Lambda(f^{-j}A \bigcap \Lambda_i) = \sum_i \sum_{j<R_i} (f^j)_*\mu_{\Lambda_i}(A).$$

   Since only measures play roles when dealing with measurable partitions, then, without loss of generality, we can identify μ as $\sum_i \sum_{j<R_i} (f^j)_*\mu_{\Lambda_i}$, and identify $\mathcal{M}$ as a disjoint union $\bigcup_i \bigcup_{j<R_i} f^j(\Lambda_i)$. Thus $\Theta := \{f^j\gamma_i^u : \gamma_i^u \in \Gamma^u \bigcap \Lambda_i, j < R_i\}$ is a measurable partition of $\mathcal{M}$.
3. From (P4) of [45] we have $\frac{\det Df^j(x)}{\det Df^j(y)} = C^{\pm 1}$ for any $x, y \in \gamma_i^u \in \Gamma^u \bigcap \Lambda_i$ and $j < R_i$. Therefore a slop of $f^j\gamma_i^u$ is almost constant. Hence all unstable leaves of Θ

are almost flat. Moreover,

$$\frac{d\mu_{f^j\gamma_i^u}}{d\,\mathrm{Leb}_{f^j\gamma_i^u}}(f^jy) = \frac{d(f^j)_*\mu_{\gamma_i^u}}{d(f^j)_*\,\mathrm{Leb}_{\gamma_i^u}}(f^jy) = C^{\pm 1}$$
$$= \frac{d(f^j)_*\mu_{\gamma_i^u}}{d(f^j)_*\,\mathrm{Leb}_{\gamma_i^u}}(f^jx) = \frac{d\mu_{f^j\gamma_i^u}}{d\,\mathrm{Leb}_{f^j\gamma^u}}(f^jx),$$

where the second and third "=" hold because the density of $\mu_{\gamma_i^u}$ is bounded from above and from below.

4. Estimate now $\mu\{x : |\gamma(x)| < \epsilon, \gamma(x) \in \Theta\}$. Let $M := \max|Df| + 1 < \infty$, $m := (\max|Df|+1)^{-1} > 0$. For any $\gamma_i^u \in \Gamma^u \bigcap \Lambda_i$. We know that $f^{R_i}\gamma_i^u = \gamma^u$ for some $\gamma^u \in \Gamma^u$. Therefore $|\gamma^u| \le M^{R_i}|\gamma_i^u|$, i.e. $|\gamma_i^u| \ge |\gamma^u|M^{-R_i} \ge C'M^{-R_i}$ for some constant $C' > 0$. If $|\gamma_i^u| \le \delta < 1$, then $R_i \ge -\log_M \delta$, i.e., $R_i \le -\log_M \delta$, and therefore $|\gamma_i^u| \ge \delta$.
The next step is to obtain estimation of size of $f^j\gamma_i^u \in \Theta$ for any $j < R_i \le -\log_M \delta$. We have

$$|f^j\gamma_i^u| \ge |\gamma_i^u|m^j \ge \delta m^j \ge \delta m^{R_i} \ge \delta m^{-\log_M \delta} = \delta^{1-\frac{\log m}{\log M}}.$$

Therefore, if $f^j\gamma_i^u \in \Theta$ has length $|f^j\gamma_i^u| < \delta^{1-\frac{\log m}{\log M}}$, then $R_i > -\log_M \delta$. The last inequality implies that

$$\mu\{x : |\gamma(x)| < \delta^{1-\frac{\log m}{\log M}}, \gamma(x) \in \Theta\} \le \mu\{x : x \in f^j\gamma_i^u \in \Theta \text{ for some } j < R_i, R_i > -\log_M \delta\}$$
$$= \sum_{n \ge -\log_M \delta} n\mu_\Lambda\{x : R(x) = n\}$$
$$\lesssim \sum_{n \ge -\log_M \delta} n\rho_0^n \lesssim \rho_0^{-\log_M \delta} = \delta^{-\frac{\log \rho_0}{\log M}},$$

where $\rho_0 \in (0, 1)$ in view of the exponential decay of return time R on Hénon attractors. Let $\epsilon = \delta^{1-\frac{\log m}{\log M}}$, then $\mu\{x : |\gamma(x)| < \epsilon, \gamma(x) \in \Theta\} \lesssim \epsilon^{\frac{-\log \rho_0}{\log M - \log m}}$. Thus all conditions of Theorem 4 are verified, and the following result holds.

Corollary 6. *Functional Poisson limit laws with convergence rates* (1.1) *hold for Hénon attractors that can be modelled by Young towers.*

Acknowledgement. We are indebted to I. Melbourne for useful comments and for pointing to the paper [10].

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Communicated by C. Liverani