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LARGE TIME BEHAVIOR AND DIFFUSION LIMIT FOR A SYSTEM OF BALANCE LAWS FROM CHEMOTAXIS IN MULTI-DIMENSIONS*

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Abstract. We consider the Cauchy problem for a system of balance laws derived from a chemotaxis model with singular sensitivity in multiple space dimensions. Utilizing energy methods, we first prove the global well-posedness of classical solutions to the Cauchy problem when only the energy of the first order spatial derivatives of the initial data is sufficiently small, and the solutions are shown to converge to the prescribed constant equilibrium states as time goes to infinity. Then we prove that the solutions of the fully dissipative model converge to those of the corresponding partially dissipative model when the chemical diffusion coefficient tends to zero.

Keywords. System of balance laws; global well-posedness; long-time behavior; diffusion limit.

AMS subject classifications. 35K55; 35K57; 35K45; 35K50; 35Q92; 92C15; 92C17.

1. Introduction

In this paper, we consider the system of balance laws:

$$\begin{cases} \partial_t p - \nabla\cdot(p\mathbf{v}) = \Delta p, \\ \partial_t \mathbf{v} - \nabla\left(p - \varepsilon|\mathbf{v}|^2\right) = \varepsilon\Delta\mathbf{v}, \end{cases} \qquad \mathbf{x}\in\mathbb{R}^n,\ t>0, \tag{1.1}$$

where $p(\mathbf{x},t)\in\mathbb{R}$ and $\mathbf{v}(\mathbf{x},t)\in\mathbb{R}^n$ are unknown functions, $n=2,3$, and $\varepsilon\geq 0$ is a constant. The purpose of this paper is to study the qualitative behavior, such as global well-posedness, long-time behavior, and zero diffusion limit (as $\varepsilon\to 0$), of classical solutions to the Cauchy problem of (1.1) in multiple space dimensions.

1.1. Background. System (1.1) can be derived from the following chemotaxis model with logarithmic sensitivity:

$$\begin{cases} \partial_t u = D\Delta u - \chi\nabla\cdot(u\nabla\log(c)), \\ \partial_t c = \varepsilon\Delta c - \mu uc - \sigma c, \end{cases} \qquad \mathbf{x}\in\mathbb{R}^n,\ t>0. \tag{1.2}$$

The chemotaxis model (1.2) was proposed in [23,37] to describe the movement of chemotactic populations, such as myxobacteria, that deposit little- or non-diffusive chemical signals that modify the local environment for succeeding passages. System (1.2) with $\sigma=0$ also appeared as a sub-model in [24] to understand the underlying mechanism of tumor angiogenesis, and in particular the role of protease inhibitors in stopping angiogenesis.

System (1.2) belongs to a family of nonlinear reaction-diffusion models, which are nowadays called the Keller-Segel type chemotaxis models. The canonical form of the

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Keller-Segel model reads:

$$\begin{cases} \partial_t u = D\Delta u - \chi \nabla \cdot (u \nabla \Phi(c)), \\ \partial_t c = \varepsilon \Delta c + f(u,c). \end{cases} \tag{1.3}$$

Inspired by the existence of traveling bands in the chemotactic movement of *E. Coli* produced by Adler [1], in the pioneering work [22] Keller and Segel successfully reproduced such an experimental result through developing their original model by taking $\Phi(c) = \log(c)$, $f(u,c) = -\mu u c^m \, (0 \le m < 1)$ with $\chi, \mu > 0$. The model's capability of describing fundamental phenomena in chemotactic movement, such as aggregation and uniform distribution (leveling out), inspired much of the later works investigating chemotaxis.

Biologically, the model (1.3) describes the movement of biological organisms in response to the chemical signals that they release in the local environment for succeeding passages, while both entities are naturally diffusing and reacting (producing, consuming, degrading, *et al.*). Because of the biological background and analytical difficulties stemming from nonlinear advection (chemotaxis), the mathematical study of (1.3) also attracted considerable attention from the community of nonlinear partial differential equations in recent decades. We refer the reader to the review papers [4, 14, 15, 44] and the references therein for more information.

System (1.2) is a special case of (1.3) when $\Phi(c) = \log(c)$ and $f(u,c) = -\mu u c - \sigma c$. The unknown functions and system parameters appearing in (1.2) are interpreted as follows: $u(x,t)$ denotes the density of cellular population at position x and time t, $c(x,t)$ the concentration of chemical signal at position x and time t, $D > 0$ the diffusion coefficient of cellular density, $\chi \neq 0$ the coefficient of chemotactic sensitivity, $\varepsilon \ge 0$ the diffusion coefficient of chemical signal, $\mu \neq 0$ the density-dependent production/degradation rate of chemical signal, and $\sigma > 0$ denotes the natural degradation rate of chemical signal.

One of the most important parameters in (1.2) is χ. The sign of χ dictates whether the chemotaxis is attractive ($\chi > 0$) or repulsive ($\chi < 0$), and $|\chi|$ measures the strength of chemotactic response. The introduction of the nonlinear advection term in (1.3) is the major contribution of the Keller-Segel type model, which captures the intrinsic features elucidating the underlying mechanisms of chemotactic movements.

Another important feature of (1.2) is the logarithmic (singular) sensitivity function in the first equation. The logarithmic sensitivity entails that the chemotactic response of cellular population to chemical signal follows the Weber-Fechner's law which is a fundamental hypothesis in psychophysics. The law states that subjective sensation is proportional to the logarithm of the stimulus intensity. It has played important roles in the modeling of biological processes (cf. [2, 3, 8, 20]). The significance of the logarithmic sensitivity was exemplified in the original Keller-Segel model through demonstrating the existence of traveling wave solutions which corroborates the experimental result of [1].

On the other hand, despite its importance in biological modeling, the possible singularity emanating from the logarithmic sensitivity function brings significant challenges to the analytical and numerical analyses of (1.2). Following the initiation of (1.2), it was observed that the possible singularity might be removed by taking the Cole-Hopf transformation [23]: $\mathbf{V} = \nabla_{\mathbf{x}} (\log(e^{\sigma t} c(\mathbf{x},t)))$. This results in a system of balance laws (also denoting $P \equiv u$):

$$\begin{cases} \partial_t P + \nabla \cdot (\chi P \mathbf{V}) = D \Delta P, \\ \partial_t \mathbf{V} + \nabla \left(\mu P - \varepsilon |\mathbf{V}|^2 \right) = \varepsilon \nabla (\nabla \cdot \mathbf{V}). \end{cases} \tag{1.4}$$

Since $\Delta\mathbf{V}=\nabla(\nabla\cdot\mathbf{V})-\nabla\times(\nabla\times\mathbf{V})$ and $\mathbf{V}$ is a gradient field, for classical solutions, the system (1.4) is equivalent to the following system of equations:

$$\begin{cases}\partial_t P+\nabla\cdot(\chi P\mathbf{V})=D\Delta P,\\ \partial_t\mathbf{V}+\nabla\left(\mu P-\varepsilon|\mathbf{V}|^2\right)=\varepsilon\Delta\mathbf{V}.\end{cases}\tag{1.5}$$

The sign of $\chi\mu$ plays an indispensable role in the qualitative study of the model. To see this, by applying the following re-scalings:

$$t\to\frac{|\chi\mu|}{D}t,\qquad \mathbf{x}\to\frac{\sqrt{|\chi\mu|}}{D}\mathbf{x},\qquad \mathbf{V}\to-\mathrm{sign}(\chi)\sqrt{\frac{|\chi|}{|\mu|}}\mathbf{V},$$

to the transformed system (1.5), one obtains a clean version of the model:

$$\begin{cases}\partial_t P-\nabla\cdot(P\mathbf{V})=\Delta P,\\ \partial_t\mathbf{V}-\nabla\left(\mathrm{sign}(\chi\mu)P-\dfrac{\varepsilon}{\chi}|\mathbf{V}|^2\right)=\dfrac{\varepsilon}{D}\Delta\mathbf{V}.\end{cases}\tag{1.6}$$

In the one-dimensional case, we can show that the characteristics associated with the flux on the left-hand sides of the equations in (1.6) are

$$\lambda_\pm=\frac{\left(\frac{2\varepsilon}{\chi}-1\right)V\pm\sqrt{\left(\frac{2\varepsilon}{\chi}-1\right)^2V^2+4\mathrm{sign}(\chi\mu)P}}{2}.$$

Hence, the principle part of (1.6) is hyperbolic when $\chi\mu>0$ in biologically relevant regimes where $P(\mathbf{x},t)>0$; while the system may change type when $\chi\mu<0$ (cf. [23]). We refer the readers to [26] for a recent study of the mixed type case, where the oscillatory traveling waves are investigated.

In this paper, we consider the case when $\chi\mu>0$, since otherwise the possible change of type may bring intractable difficulties to the underlying analysis. Formally, when $\chi>0$, the associated chemotactic flux in the first equation of (1.2) states that the chemical will attract cells to regions of high concentration of the chemical and hence drive a possible aggregation. This is in contrast to the homogenizing process driven diffusion. On the other hand, the (exponentially) rapid degradation (due to $\mu>0$) in the second equation of (1.2) illustrates that the force driving the cellular population to aggregate is diminishing as time goes on. Hence, one may expect that the system will enter into an equilibrium state in the long time run due to the balance between cellular aggregation and chemical degradation. Similarly, when $\chi<0$ and $\mu<0$, because of the interaction between chemotactic repulsion and chemical production, the system is also expected to reach into a steady state as time goes on.

Collectively, we do not anticipate the development of finite time singularities in (1.2) when $\chi\mu>0$. In this case, the synergy of diffusion, chemotactic attraction/repulsion, and chemical degradation/production makes the dynamics of the model an intriguing problem to pursue.

In this paper, we aim to understand the dynamics of the model (1.2) through studying the qualitative behavior of solutions to the transformed system (1.6) for fixed values of χ,μ and D when $\chi\mu>0$. Hence for brevity, we assume $\chi=\mu=D=1$ throughout the paper. This leads to the following system of equations:

$$\begin{cases}\partial_t P-\nabla\cdot(P\mathbf{V})=\Delta P,\\ \partial_t\mathbf{V}-\nabla\left(P-\varepsilon|\mathbf{V}|^2\right)=\varepsilon\Delta\mathbf{V}.\end{cases}\tag{1.7}$$

We remark that the model (1.7) is formally identical to the system (1.1). However, the system of balance laws (1.1) is more general than the system (1.7), since the solution component $\mathbf{V}$ in the latter one is a gradient field (hence curl free). In this paper, we consider the general model (1.1) and specify the conditions under which the model automatically generates curl-free solutions $\mathbf{V}$, from which one can recover the solutions to the original chemotaxis model (1.2).

1.2. Literature review and motivations. To put things into perspective, now we would like to point out the existing results that are related to this work. When the spatial dimension is one, the following results for (1.1) are available in the literature: global well-posedness and large-time behavior with small or large data in [10, 13, 27–29, 36, 39, 42, 47, 49], local stability of traveling waves in [7, 19, 31–35, 38], boundary layer formation and characterization in [16, 17, 28, 39], shock wave formation in [45], explicit and numerical solutions in [23], and so on. In particular, the results in [27–29, 36, 39, 42, 47] indicate that when $\chi\mu > 0$, no matter how strong the chemotactic sensitivity is and how large the energy of initial data is, the cellular population always distributes uniformly over space as time evolves.

One of the main ingredients of the proofs constructed in [27–29, 36, 39, 42, 47] is the implementation of the free energy (weak Lyapunov functional) associated with (1.1):

$$\frac{d}{dt}\left(\int E(p,\bar{p})dx + \|v\|_{L^2}^2\right) + \int \frac{(p_x)^2}{p}dx + \varepsilon\|v_x\|_{L^2}^2 = 0, \tag{1.8}$$

where $\bar{p} > 0$ is a constant equilibrium state and the "entropy expansion" is defined by

$$E(p,\bar{p}) = [p\ln(p) - p] - [\bar{p}\ln(\bar{p}) - \bar{p}] - \ln(\bar{p})(p - \bar{p}). \tag{1.9}$$

The entropy type estimate (1.8) lays down a foundation for the subsequent energy estimates that lead to the global stability of constant equilibrium states associated with the one-dimensional version of (1.1).

On the other hand, when the space dimension is greater than one, (1.8) takes a different form:

$$\frac{d}{dt}\left(\int E(p,\bar{p})d\mathbf{x} + \|\mathbf{v}\|_{L^2}^2\right) + \int \frac{|\nabla p|^2}{p}d\mathbf{x} + \varepsilon\|\nabla\mathbf{v}\|_{L^2}^2 = \varepsilon\int |\mathbf{v}|^2\nabla\cdot\mathbf{v}\ d\mathbf{x}. \tag{1.10}$$

Note that the integral on the right-hand side does not vanish, and is not sign-preserving either. Moreover, we can show that (1.1) is invariant under the scaling

$$(p,\mathbf{v}) \to (p^\xi, \mathbf{v}^\xi) := \left(\xi^2 p\left(\xi\mathbf{x}, \xi^2 t\right), \xi\mathbf{v}\left(\xi\mathbf{x}, \xi^2 t\right)\right).$$

Under the scaling, when the initial data are perturbed around the zero ground state, it holds that

$$\|p_0^\xi\|_{L^2}^2 = \xi^{4-n}\|p_0\|_{L^2}^2 \qquad \text{and} \qquad \|\mathbf{v}_0^\xi\|_{L^2}^2 = \xi^{2-n}\|\mathbf{v}_0\|_{L^2}^2.$$

This reveals that norm-inflation (especially for the $\mathbf{v}$-component) is not possible when $n \geq 2$.

The aforementioned (unfavorable) features of the multi-dimensional version of (1.1) brings substantial difficulties to the rigorous analysis of the fundamental properties of the model, such as global well-posedness of large data classical solutions. Unlike the one-dimensional case, most of the results obtained in the multi-dimensional case assume certain smallness on the initial data. Here we mention some of the results that are mostly relevant to the present work. All of the results are concerned with global existence and large-time behavior of solutions to the initial value problem of (1.1) under smallness assumptions on initial data. We list the assumptions as follows:

- $(p_0-\bar{p},\mathbf{v}_0)\in H^s(\mathbb{R}^n)$ when $\|p_0-\bar{p}\|_{H^s}+\|\mathbf{v}_0\|_{H^s}$ is small, where $s>\frac{n}{2}+1$, see [25],
- global existence when $(p_0-\bar{p})\in L^2(\mathbb{R}^3)$, $\mathbf{v}_0\in H^1(\mathbb{R}^3)$, and $\|p_0-\bar{p}\|_{L^2}+\|\mathbf{v}_0\|_{H^1}$ is small; large-time behavior when $(p_0-\bar{p})\in H^2(\mathbb{R}^3)$, $\mathbf{v}_0\in H^1(\mathbb{R}^3)$, and $\|p_0-\bar{p}\|_{H^2}+\|\mathbf{v}_0\|_{H^1}$ is small, see [9],
- global existence when $(p_0-\bar{p},\mathbf{v}_0)\in H^k(\mathbb{R}^n)$ and only $\|(p_0-\bar{p},\mathbf{v}_0)\|_{H^1}$ is small, where $n=2,3$ and $k\geq 2$; large-time behavior when additionally $(p_0-\bar{p},\mathbf{v}_0)\in \dot{H}^{-s}(\mathbb{R}^n)$, where $s\in(0,\frac{n}{2})$, see [46],
- $(p_0-\bar{p},\mathbf{v}_0)\in H^3(\mathbb{R}^3)$ when only $\|(p_0-\bar{p},\mathbf{v}_0)\|_{L^2}$ is small, see [40],
- $(p_0-\bar{p},\mathbf{v}_0)\in H^2(\mathbb{R}^2)$ when only $\|(p_0-\bar{p},\mathbf{v}_0)\|_{L^2}$ is small, see [43],
- $(p_0-\bar{p},\mathbf{v}_0)\in H^2(\mathbb{R}^n)$ when only $\int_{\mathbb{R}^n}E(p_0,\bar{p})d\mathbf{x}+\|\mathbf{v}_0\|_{L^2}^2$ is small (cf. (1.9)), where $n=2,3$, see [43].

There are also results for the initial-boundary value problems of the model in [5, 6, 18, 30, 41, 48]. Since these results are less relevant, we do not explain them in detail here.

We note that all the results listed above have assumed at least one of the norms $\|p_0-\bar{p}\|_{L^2(\mathbb{R}^n)}$ and $\|\mathbf{v}_0\|_{L^2(\mathbb{R}^n)}$ is small. None of them gives a positive answer to the question of global well-posedness and large-time behavior of classical solutions to the Cauchy problem of (1.1) when both the norms $\|p_0-\bar{p}\|_{L^2(\mathbb{R}^n)}$ and $\|\mathbf{v}_0\|_{L^2(\mathbb{R}^n)}$ are potentially *large*. The analytic techniques utilized in the above previous works are not sufficient for handling such a situation. The novelty of the current work is that we give a positive answer to the above question, namely, we show the global well-posedness of classical solutions to the Cauchy problem when only the energy of the first order spatial derivatives of the initial data is sufficiently small, see Subsection 1.4 for the novelty in our proof.

The zero-diffusion limit problem in the current work is motivated by that in certain chemotactic processes, the chemical signals deposited by the organisms that modify the local environment for succeeding passages are little- or non-diffusive (cf. [24, 37]). Hence, it is desirable to know whether the chemically diffusive (realistic) model can be approximated by the non-diffusive (ideal) one when ε is small. Equivalently, the question is "Does the solution of the slightly diffusive model converge to the solution of the non-diffusive model, as $\varepsilon\to 0$?" Such a topic has been investigated in [16, 28, 35, 36, 39, 47] for the one-dimensional case, and in [18, 40, 41, 43, 46] for the multi-dimensional case. Nevertheless, the question is open when $n\geq 2$ and $\|(p_0-\bar{p},\mathbf{v}_0)\|_{L^2(\mathbb{R}^n)}$ is large.

1.3. Statement of results. Motivated by the above facts, we study the global well-posedness, long-time behavior, and zero chemical diffusion limit of multi-dimensional classical solutions to the initial value problem of (1.1) when $\|(p_0-\bar{p},\mathbf{v}_0)\|_{L^2(\mathbb{R}^n)}$ is potentially large. The point of study is the following Cauchy problem:

$$\begin{cases}\partial_t p-\nabla\cdot(p\mathbf{v})-\bar{p}\nabla\cdot\mathbf{v}=\Delta p,\\ \partial_t\mathbf{v}-\nabla\left(p-\varepsilon|\mathbf{v}|^2\right)=\varepsilon\Delta\mathbf{v},\\ (p_0,\mathbf{v}_0)\in H^3(\mathbb{R}^n),\qquad p_0+\bar{p}>0,\qquad \nabla\times\mathbf{v}_0=\mathbf{0},\end{cases}\tag{1.11}$$

where $\bar{p}>0$ is a constant, and $(p,\mathbf{v})$ denotes the perturbation of the original solution around the constant state $(\bar{p},\mathbf{0})$. The main results are stated in the following theorems.

Notation 1.1. *Throughout the rest part of this paper, we use* $\|\cdot\|$ *to denote* $\|\cdot\|_{L^2}$.

Theorem 1.1. *Let* $n=3$, *and consider the Cauchy problem* (1.11). *Define*

$$\kappa \equiv 2(1+1/\bar{p})\left(\|\nabla p_0\|^2+\bar{p}\|\nabla\cdot\mathbf{v}_0\|^2\right), \quad N_1\equiv(1+1/\bar{p})\left(\|p_0\|^2+\bar{p}\|\mathbf{v}_0\|^2\right)+1. \tag{1.12}$$

If

$$N_1\kappa \le \min\left\{(16c_1c_2)^{-4},\ (8c_1c_2)^{-1},\ (54c_3^4)^{-1},\ 1\right\}, \tag{1.13}$$

where c_1 *and* c_2 *are generic constants appearing in the Gagliardo-Nirenberg interpolation inequalities (cf.* (2.1)–(2.2)*), then there exists a unique solution to the Cauchy problem* (1.11) *for any* $\varepsilon\ge 0$, *such that it holds that*

$$\|(p,\mathbf{v})(t)\|_{H^3}^2+\int_0^t\left(\|(\nabla p,\sqrt{\varepsilon}\nabla\cdot\mathbf{v})(\tau)\|_{H^3}^2+\|\nabla\cdot\mathbf{v}(\tau)\|_{H^2}^2\right)d\tau\le C, \quad \forall t>0, \tag{1.14}$$

where the positive constant C *is independent of* t *and remains bounded as* $\varepsilon\to 0$. *In addition, the following decay estimate holds:*

$$\lim_{t\to\infty}\left(\|(p,\mathbf{v})(t)\|_{W^{1,\infty}}^2+\|(\nabla p,\nabla\cdot\mathbf{v})(t)\|_{H^2}^2\right)=0, \tag{1.15}$$

for any $\varepsilon\ge 0$. *Furthermore, let* $(p^\varepsilon,\mathbf{v}^\varepsilon)$ *and* $(p^0,\mathbf{v}^0)$ *be the solutions to the Cauchy problem with* $\varepsilon>0$ *and* $\varepsilon=0$, *respectively, for the same initial data. Then for any* $t>0$, *it holds that*

$$\begin{aligned}&\|(p^\varepsilon-p^0,\mathbf{v}^\varepsilon-\mathbf{v}^0)(t)\|_{H^1}^2\le D_1e^t\varepsilon^2,\\ &\|(\Delta p^\varepsilon-\Delta p^0,\Delta\mathbf{v}^\varepsilon-\Delta\mathbf{v}^0)(t)\|^2\le D_2e^t(1+\varepsilon)\varepsilon,\end{aligned} \tag{1.16}$$

where the positive constants D_1 *and* D_2 *are independent of* t *and remain bounded as* $\varepsilon\to 0$.

Theorem 1.2. *Let* $n=2$, *and consider the Cauchy problem* (1.11). *Let*

$$\begin{aligned}H_1\equiv&\frac{1}{4}\|p_0\|^2+\left\|\frac{p_0}{2}-\frac{2\varepsilon+1}{6}p_0^2\right\|^2+\left(\frac{2(2\varepsilon+1)^2}{9}+\frac{8\varepsilon^2+8\varepsilon^3}{12\bar{p}}\right)\|p_0\|_{L^4}^4+\frac{\bar{p}}{4}\|\mathbf{v}_0\|^2\\ &+\frac{\bar{p}}{4}\left\|\left(1-\frac{2\varepsilon}{\bar{p}}p_0\right)\mathbf{v}_0\right\|^2+\frac{\varepsilon^2}{\bar{p}}\|p_0\mathbf{v}_0\|^2+\frac{1}{12}\left(\frac{4\varepsilon+4\varepsilon^2}{\bar{p}}+9\varepsilon+32\varepsilon^3\right)\|\mathbf{v}_0\|_{L^4}^4,\end{aligned} \tag{1.17}$$

$$H_2\equiv\frac{\bar{p}^2}{2}\|\nabla p_0\|^2+\bar{p}^3\|\nabla\cdot\mathbf{v}_0\|^2+\frac{1}{2}\|\bar{p}\nabla p_0-p_0\nabla p_0\|^2+\frac{1}{2}\|p_0\nabla p_0\|^2,$$

and define

$$\begin{aligned}&M_1\equiv 4(1+1/\bar{p})H_1+1,\\ &\delta\equiv\frac{2(2\bar{p}+1)}{\bar{p}^3}\exp\left\{\frac{4}{\bar{p}}\left(2916d_1^8\bar{p}^2M_1+3d_1^4\bar{p}^2+27d_1^8M_1\bar{p}^2+2916d_1^8d_5^4(\bar{p})^{-1}\right)H_1\right\}H_2,\end{aligned} \tag{1.18}$$

where d_1 *and* d_5 *are generic constants appearing in the Gagliardo-Nirenberg interpolation inequalities (cf.* (3.1), (3.5)*). If*

$$\max\{M_1\delta,\ M_1\delta^2,\ \delta\}\le\varrho, \tag{1.19}$$

for some positive constant ϱ *which is sufficiently small such that* (3.43) *and* (3.67) *are fulfilled, then there exists a unique solution to the Cauchy problem* (1.11) *for any* $\varepsilon \geq 0$, *such that the solution obeys similar estimates as* (1.14), (1.15) *and* (1.16).

REMARK 1.1. We see from (1.12) and (1.17)–(1.18) that the smallness assumptions, (1.13) and (1.19), can be realized by taking the L^2-norm of the first order spatial derivatives of the initial perturbations to be sufficiently small, while the L^2-norm of the initial perturbations can be potentially large. We provide explicit examples in the Appendix, which fulfill such requirements.

REMARK 1.2. In Theorems 1.1 and 1.2, we assumed that $\mathbf{v}_0$ is curl free. This is a natural assumption since (1.1) reduces to (1.7) when $\mathbf{v}$ is curl free, and the latter one originates from (1.2) through the transformation $\mathbf{V} = \nabla_{\mathbf{x}}(\log(e^{\sigma t}c(\mathbf{x},t)))$. Since $(\nabla \times \mathbf{v})_t = \varepsilon\Delta(\nabla \times \mathbf{v})$, the second equation in (1.1) automatically generates curl-free solutions when $\nabla \times \mathbf{v}_0 = 0$. Hence, one may recover the solution to (1.2) from the solution to (1.1) under the initial curl-free condition.

1.4. Difficulties and idea of proof. We prove Theorems 1.1 and 1.2 by developing L^p-based energy methods. Since we only assume the smallness of a fraction of the total Sobolev norm of the initial perturbation, the major technical difficulty consists in closing the energy estimate for each individual frequency of the solution. Unlike the case when the total Sobolev norm of the perturbation is small, one can not combine energy estimates of low and high frequencies for the problem considered in this paper. Because the energy of the zeroth frequency part of the perturbation is allowed to be potentially large, the estimation of the zeroth frequency part is challenging due to the lack of the Poincaré inequality in the whole space case. Moreover, because the Gagliardo-Nirenberg interpolation inequalities generate less powers of high frequencies of a function in $\mathbb{R}^2$ than in $\mathbb{R}^3$, the proof of the two-dimensional case is considerably more involved than the three-dimensional case.

In the three-dimensional case, we get over the barrier by taking full advantage of the dissipation mechanisms and the smallness assumption on L^2-norm of the first order spatial derivatives of the initial perturbation. The Gagliardo-Nirenberg interpolation inequality plays an important role in our analysis. For the two-dimensional problem, we overcome the additional difficulty (deficiency in interpolation inequalities) by terminating low frequencies through creating higher order nonlinearities. Furthermore, since we also aim to establish the zero chemical diffusion limit, it is vital to obtain the uniform ε-*independent* energy estimates of the chemically diffusive solution. We reach the goal by deriving a linear and inhomogeneous damping equation for the spatial divergence of $\mathbf{v}$ and taking advantage of the dissipative structures of the system.

The rest of the paper is organized as follows. In Section 2 we give a complete proof of Theorem 1.1. Since the proof of Theorem 1.2 is involved, we present the main steps of the proof in Section 3, while leaving some tedious calculations to the Appendix.

2. Proof of Theorem 1.1

In this section we prove the Theorem 1.1. First of all, the local well-posedness of (1.11) can be established by applying Kawashima's theory on a general system of balance laws [21], see also [11, 25]. Moreover, it follows from the maximum principle (cf. [12]) that the local solution satisfies $p + \bar{p} > 0$ within its lifespan. We collect the results in the following:

LEMMA 2.1 (Local Well-posedness). *Consider the Cauchy problem* (1.11). *For any* $\varepsilon \geq 0$, *there exists a unique local-in-time solution such that* $p + \bar{p} > 0$ *and for some* $T_0 \in$

$(0,\infty)$, *it holds that* $(p-\bar{p},\mathbf{v})\in L^\infty([0,T_0];H^3(\mathbb{R}^3))\cap L^2([0,T_0];H^4(\mathbb{R}^3))$ *when* $\varepsilon>0$, *and* $(p-\bar{p})\in L^\infty([0,T_0];H^3(\mathbb{R}^3))\cap L^2([0,T_0];H^4(\mathbb{R}^3))$ *and* $\mathbf{v}\in L^\infty([0,T_0];H^3(\mathbb{R}^3))$ *when* $\varepsilon=0$.

We now establish *a priori* estimates for the local solution, in order to obtain a global solution. First we recall the following Gagliardo-Nirenberg interpolation inequalities:

$$\|f\|_{L^3}\le c_1\|\nabla f\|^{\frac12}\|f\|^{\frac12},\qquad \forall\ f\in H^1(\mathbb{R}^3), \tag{2.1}$$

$$\|f\|_{L^6}\le c_2\|\nabla f\|,\qquad \forall\ f\in H^1(\mathbb{R}^3), \tag{2.2}$$

$$\|f\|_{L^\infty}\le c_3\|\Delta f\|^{\frac12}\|\nabla f\|^{\frac12},\qquad \forall\ f\in H^2(\mathbb{R}^3). \tag{2.3}$$

Secondly, in addition to κ and N_1 defined in (1.12), we let

$$\begin{aligned}
N_2&\equiv(1+1/\bar{p})\exp\left\{\frac{3c_4}{2\bar{p}}\left(\|p_0\|_{H^1}^2+\bar{p}\|\mathbf{v}_0\|_{H^1}^2\right)\right\}\left(\|\Delta p_0\|^2+\bar{p}\|\Delta\mathbf{v}_0\|^2\right)+1,\\
\hat{N}&\equiv\frac{3c_4}{2(\bar{p}+1)}\left(\|p_0\|_{H^1}^2+\bar{p}\|\mathbf{v}_0\|_{H^1}^2\right)(N_2-1)+\left(\|\Delta p_0\|^2+\bar{p}\|\Delta\mathbf{v}_0\|^2\right),\\
N_3&\equiv(1+1/\bar{p})\exp\left\{\frac{3c_5}{2\bar{p}}\left(\|p_0\|_{H^1}^2+\bar{p}\|\mathbf{v}_0\|_{H^1}^2\right)\right\}\Big(\|\nabla\Delta p_0\|^2+\bar{p}\|\Delta(\nabla\cdot\mathbf{v}_0)\|^2\\
&\qquad+c_5(1+N_2)\hat{N}\Big)+1,
\end{aligned}\tag{2.4}$$

where c_4 and c_5 are defined below by (2.23) and (2.29), respectively. Then we observe that

$$\begin{aligned}
&\|p_0\|^2+\|\mathbf{v}_0\|^2<N_1-1, &&\|\nabla p_0\|^2+\|\nabla\cdot\mathbf{v}_0\|^2<\frac{\kappa}{2},\\
&\|\Delta p_0\|^2+\|\Delta\mathbf{v}_0\|^2<N_2-1, &&\|\nabla\Delta p_0\|^2+\|\Delta\nabla\cdot\mathbf{v}_0\|^2<N_3-1.
\end{aligned}\tag{2.5}$$

Then it follows from Lemma 2.1 that there exists $T_1\in(0,T_0]$, such that

$$\begin{aligned}
&\sup_{0\le t\le T_1}\left(\|p(t)\|^2+\|\mathbf{v}(t)\|^2\right)\le N_1, &&\sup_{0\le t\le T_1}\left(\|\nabla p(t)\|^2+\|\nabla\cdot\mathbf{v}(t)\|^2\right)\le\kappa,\\
&\sup_{0\le t\le T_1}\left(\|\Delta p(t)\|^2+\|\Delta\mathbf{v}(t)\|^2\right)\le N_2, &&\sup_{0\le t\le T_1}\left(\|\nabla\Delta p(t)\|^2+\|\Delta\nabla\cdot\mathbf{v}(t)\|^2\right)\le N_3.
\end{aligned}\tag{2.6}$$

Next, we derive *a priori* estimates for the local solution within the time interval $[0,T_1]$.

2.1. L^2-estimate.

Proof. By testing the equations in (1.11) with the targeting functions, and using (2.1)–(2.2), we can show that

$$\begin{aligned}
&\frac12\frac{d}{dt}\left(\|p\|^2+\bar{p}\|\mathbf{v}\|^2\right)+\|\nabla p\|^2+\varepsilon\bar{p}\|\nabla\cdot\mathbf{v}\|^2\\
=&-\int_{\mathbb{R}^3}p(\mathbf{v}\cdot\nabla p)d\mathbf{x}+\varepsilon\bar{p}\int_{\mathbb{R}^3}|\mathbf{v}|^2\nabla\cdot\mathbf{v}\ d\mathbf{x}\\
\le&\ \|p\|_{L^6}\|\mathbf{v}\|_{L^3}\|\nabla p\|+\varepsilon\bar{p}\|\mathbf{v}\|_{L^3}\|\mathbf{v}\|_{L^6}\|\nabla\cdot\mathbf{v}\|\\
\le&\ c_1c_2\left(\|\nabla p\|\|\nabla\cdot\mathbf{v}\|^{\frac12}\|\mathbf{v}\|^{\frac12}\|\nabla p\|+\varepsilon\bar{p}\|\nabla\cdot\mathbf{v}\|^{\frac12}\|\mathbf{v}\|^{\frac12}\|\nabla\cdot\mathbf{v}\|^2\right)\\
\le&\ c_1c_2\left(N_1\kappa\right)^{\frac14}\left(\|\nabla p\|^2+\varepsilon\bar{p}\|\nabla\cdot\mathbf{v}\|^2\right),
\end{aligned}\tag{2.7}$$

where we applied (2.6) in the last inequality. Hence, when

$$N_1\kappa\le(2c_1c_2)^{-4}, \tag{2.8}$$

it holds that

$$\frac{d}{dt}\left(\|p\|^2+\bar{p}\|\mathbf{v}\|^2\right)+\|\nabla p\|^2+\varepsilon\bar{p}\|\nabla\cdot\mathbf{v}\|^2\leq 0, \tag{2.9}$$

which yields

$$\|p(t)\|^2+\bar{p}\|\mathbf{v}(t)\|^2+\int_0^t\left(\|\nabla p(\tau)\|^2+\varepsilon\bar{p}\|\nabla\cdot\mathbf{v}(\tau)\|^2\right)d\tau\leq\|p_0\|^2+\bar{p}\|\mathbf{v}_0\|^2. \tag{2.10}$$

Therefore, in view of (1.12), we see that

$$\|p(t)\|^2+\|\mathbf{v}(t)\|^2\leq N_1-1. \tag{2.11}$$

This completes the proof for the L^2-estimate. □

Next, we make estimates on the first order spatial derivatives of the solution.

2.2. ***H^1*-estimate.**

Proof. Testing the equations in (1.11) by the $-\Delta$ of the targeting functions, and using (2.3), we can show that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla p\|^2+\bar{p}\|\nabla\cdot\mathbf{v}\|^2\right)+\|\Delta p\|^2+\varepsilon\bar{p}\|\Delta\mathbf{v}\|^2\\
&=-\int_{\mathbb{R}^3}\nabla\cdot(p\mathbf{v})\Delta p\;d\mathbf{x}+\varepsilon\bar{p}\int_{\mathbb{R}^3}\nabla(|\mathbf{v}|^2)\Delta\mathbf{v}\;d\mathbf{x}\\
&\leq\|p\|_{L^\infty}\|\nabla\cdot\mathbf{v}\|\|\Delta p\|+\|\nabla p\|_{L^6}\|\mathbf{v}\|_{L^3}\|\Delta p\|+2\varepsilon\bar{p}\|\mathbf{v}\|_{L^3}\|\nabla\mathbf{v}\|_{L^6}\|\Delta\mathbf{v}\|\\
&\leq c_3\|\nabla p\|^{\frac{1}{2}}\|\Delta p\|^{\frac{1}{2}}\|\nabla\cdot\mathbf{v}\|\|\Delta p\|+c_1c_2\|\mathbf{v}\|^{\frac{1}{2}}\|\nabla\cdot\mathbf{v}\|^{\frac{1}{2}}\|\Delta p\|^2+c_1c_2\varepsilon\bar{p}\|\nabla\cdot\mathbf{v}\|^{\frac{1}{2}}\|\mathbf{v}\|^{\frac{1}{2}}\|\Delta\mathbf{v}\|^2\\
&\leq\left(\frac{1}{4}+c_1c_2\|\mathbf{v}\|^{\frac{1}{2}}\|\nabla\cdot\mathbf{v}\|^{\frac{1}{2}}\right)\|\Delta p\|^2+\frac{27}{4}c_3^4\|\nabla p\|^2\|\nabla\cdot\mathbf{v}\|^4+c_1c_2\varepsilon\bar{p}\|\nabla\cdot\mathbf{v}\|^{\frac{1}{2}}\|\mathbf{v}\|^{\frac{1}{2}}\|\Delta\mathbf{v}\|^2\\
&\leq\left(\frac{1}{4}+c_1c_2(N_1\kappa)^{\frac{1}{4}}\right)\|\Delta p\|^2+\frac{27}{4}c_3^4\kappa^2\|\nabla p\|^2+c_1c_2\varepsilon\bar{p}(N_1\kappa)^{\frac{1}{4}}\|\Delta\mathbf{v}\|^2,
\end{aligned} \tag{2.12}$$

where we applied Young's inequality. Hence, when

$$N_1\kappa\leq(4c_1c_2)^{-4}, \tag{2.13}$$

it holds that

$$\frac{d}{dt}\left(\|\nabla p\|^2+\bar{p}\|\nabla\cdot\mathbf{v}\|^2\right)+\|\Delta p\|^2+\varepsilon\bar{p}\|\Delta\mathbf{v}\|^2\leq\frac{27}{2}c_3^4\kappa^2\|\nabla p\|^2. \tag{2.14}$$

Integrating (2.14) with respect to time and using (2.10), we obtain

$$\begin{aligned}
&\|\nabla p(t)\|^2+\bar{p}\|\nabla\cdot\mathbf{v}(t)\|^2+\int_0^t\left(\|\Delta p(\tau)\|^2+\varepsilon\bar{p}\|\Delta\mathbf{v}(\tau)\|^2\right)d\tau\\
&\leq\|\nabla p_0\|^2+\bar{p}\|\nabla\cdot\mathbf{v}_0\|^2+\frac{27}{2}c_3^4\kappa^2\left(\|p_0\|^2+\bar{p}\|\mathbf{v}_0\|^2\right),
\end{aligned} \tag{2.15}$$

which implies

$$\|\nabla p(t)\|^2+\|\nabla\cdot\mathbf{v}(t)\|^2\leq(1+1/\bar{p})\left[\|\nabla p_0\|^2+\bar{p}\|\nabla\cdot\mathbf{v}_0\|^2+\frac{27}{2}c_3^4\kappa^2\left(\|p_0\|^2+\bar{p}\|\mathbf{v}_0\|^2\right)\right]. \tag{2.16}$$

In view of (1.12), we see that

$$\frac{27}{2}c_3^4\kappa^2\left(\|p_0\|^2+\bar{p}\|\mathbf{v}_0\|^2\right)=54c_3^4(1+1/\bar{p})^2\left[\|\nabla p_0\|^2+\bar{p}\|\nabla\cdot\mathbf{v}_0\|^2\right]^2\left(\|p_0\|^2+\bar{p}\|\mathbf{v}_0\|^2\right)$$
$$\leq 27c_3^4\kappa N_1\left[\|\nabla p_0\|^2+\bar{p}\|\nabla\cdot\mathbf{v}_0\|^2\right]\leq\frac{1}{2}\left[\|\nabla p_0\|^2+\bar{p}\|\nabla\cdot\mathbf{v}_0\|^2\right],$$

provided that

$$N_1\kappa\leq(54\ c_3^4)^{-1}. \tag{2.17}$$

Hence, we update (2.16) as

$$\|\nabla p(t)\|^2+\|\nabla\cdot\mathbf{v}(t)\|^2\leq\frac{3}{2}(1+1/\bar{p})\left[\|\nabla p_0\|^2+\bar{p}\|\nabla\cdot\mathbf{v}_0\|^2\right]=\frac{3}{4}\kappa. \tag{2.18}$$

In addition, we deduce from (2.15) that

$$\int_0^t\left(\|\Delta p(\tau)\|^2+\varepsilon\bar{p}\|\Delta\mathbf{v}(\tau)\|^2\right)d\tau\leq\frac{3}{2}\left(\|\nabla p_0\|^2+\bar{p}\|\nabla\cdot\mathbf{v}_0\|^2\right), \tag{2.19}$$

which will be utilized in the subsequent section. This completes the proof for the H^1-estimate. ∎

2.3. ***H*²-estimate.**

Proof. By computing the second order L^2 inner products, we can show that

$$\frac{1}{2}\frac{d}{dt}\left(\|\Delta p\|^2+\bar{p}\|\Delta\mathbf{v}\|^2\right)+\|\nabla\Delta p\|^2+\varepsilon\bar{p}\|\Delta(\nabla\cdot\mathbf{v})\|^2$$
$$=-\int_{\mathbb{R}^3}\nabla(\nabla\cdot(p\mathbf{v}))\cdot\nabla(\Delta p)\ d\mathbf{x}+\varepsilon\bar{p}\int_{\mathbb{R}^3}\Delta(|\mathbf{v}|^2)\Delta(\nabla\cdot\mathbf{v})\ d\mathbf{x}. \tag{2.20}$$

For the first term on the RHS of (2.20), by using Hölder, Gagliardo-Nirenberg and Young inequalities, we can show that

$$\left|-\int_{\mathbb{R}^3}\nabla(\nabla\cdot(p\mathbf{v}))\cdot\nabla(\Delta p)\ d\mathbf{x}\right|$$
$$\leq\ \left(\|p\|_{L^\infty}\|\Delta\mathbf{v}\|+\|\nabla p\|_{L^3}\|\nabla\mathbf{v}\|_{L^6}+\|\Delta p\|_{L^6}\|\mathbf{v}\|_{L^3}\right)\|\nabla\Delta p\|$$
$$\leq\ \left(c_3\|\nabla p\|^{\frac12}\|\Delta p\|^{\frac12}\|\Delta\mathbf{v}\|+c_1c_2\|\nabla p\|^{\frac12}\|\Delta p\|^{\frac12}\|\Delta\mathbf{v}\|+c_1c_2\|\nabla\Delta p\|\|\mathbf{v}\|^{\frac12}\|\nabla\cdot\mathbf{v}\|^{\frac12}\right)\|\nabla\Delta p\|$$
$$\leq\ \left(\frac14+c_1c_2\left(N_1\kappa\right)^{\frac14}\right)\|\nabla\Delta p\|^2+\left(c_3+c_1c_2\right)^2\|\nabla p\|\|\Delta p\|\|\Delta\mathbf{v}\|^2$$
$$\leq\ \left(\frac14+c_1c_2\left(N_1\kappa\right)^{\frac14}\right)\|\nabla\Delta p\|^2+\frac{\left(c_3+c_1c_2\right)^2}{2}\left(\|\nabla p\|^2+\|\Delta p\|^2\right)\|\Delta\mathbf{v}\|^2.$$

In a similar fashion, we can show that

$$\left|\varepsilon\bar{p}\int_{\mathbb{R}^3}\Delta(|\mathbf{v}|^2)\Delta(\nabla\cdot\mathbf{v})\ d\mathbf{x}\right|$$
$$\leq\ 2\varepsilon\bar{p}\left(\|\nabla\mathbf{v}\|_{L^3}\|\nabla\mathbf{v}\|_{L^6}+\|\mathbf{v}\|_{L^3}\|\nabla^2\mathbf{v}\|_{L^6}\right)\|\Delta(\nabla\cdot\mathbf{v})\|$$
$$\leq\ 2c_1c_2\varepsilon\bar{p}\left(\|\nabla\cdot\mathbf{v}\|^{\frac12}\|\Delta\mathbf{v}\|^{\frac32}+\|\mathbf{v}\|^{\frac12}\|\nabla\cdot\mathbf{v}\|^{\frac12}\|\Delta(\nabla\cdot\mathbf{v})\|\right)\|\Delta(\nabla\cdot\mathbf{v})\|$$
$$\leq\ \varepsilon\bar{p}\left(\frac14+2c_1c_2\left(N_1\kappa\right)^{\frac14}\right)\|\Delta(\nabla\cdot\mathbf{v})\|^2+4c_1^2c_2^2\varepsilon\bar{p}\|\nabla\cdot\mathbf{v}\|\|\Delta\mathbf{v}\|\|\Delta\mathbf{v}\|^2$$
$$\leq\ \varepsilon\bar{p}\left(\frac14+2c_1c_2\left(N_1\kappa\right)^{\frac14}\right)\|\Delta(\nabla\cdot\mathbf{v})\|^2+2c_1^2c_2^2\left(\varepsilon\bar{p}\|\nabla\cdot\mathbf{v}\|^2+\varepsilon\bar{p}\|\Delta\mathbf{v}\|^2\right)\|\Delta\mathbf{v}\|^2.$$

Hence, when

$$N_1\kappa \le (8c_1c_2)^{-4}, \tag{2.21}$$

it holds that

$$\begin{aligned}&\frac{d}{dt}\left(\|\Delta p\|^2+\bar{p}\|\Delta \mathbf{v}\|^2\right)+\|\nabla\Delta p\|^2+\varepsilon\bar{p}\|\Delta(\nabla\cdot\mathbf{v})\|^2\\ \le\ &\frac{c_4}{\bar{p}}\left(\|\nabla p\|^2+\|\Delta p\|^2+\varepsilon\bar{p}\|\nabla\cdot\mathbf{v}\|^2+\varepsilon\bar{p}\|\Delta\mathbf{v}\|^2\right)\left(\|\Delta p\|^2+\bar{p}\|\Delta\mathbf{v}\|^2\right),\end{aligned} \tag{2.22}$$

where

$$c_4=\max\left\{(c_3+c_1c_2)^2, 4c_1^2c_2^2\right\}. \tag{2.23}$$

Applying the Grönwall inequality to (2.22) and using (2.10) and (2.19), we have

$$\begin{aligned}&\|\Delta p(t)\|^2+\bar{p}\|\Delta\mathbf{v}(t)\|^2\\ \le\ &\exp\left\{\frac{c_4}{\bar{p}}\int_0^t\left(\|\nabla p\|^2+\|\Delta p\|^2+\varepsilon\bar{p}\|\nabla\cdot\mathbf{v}\|^2+\varepsilon\bar{p}\|\Delta\mathbf{v}\|^2\right)d\tau\right\}\left(\|\Delta p_0\|^2+\bar{p}\|\Delta\mathbf{v}_0\|^2\right)\\ \le\ &\exp\left\{\frac{3c_4}{2\bar{p}}\left(\|p_0\|_{H^1}^2+\bar{p}\|\mathbf{v}_0\|_{H^1}^2\right)\right\}\left(\|\Delta p_0\|^2+\bar{p}\|\Delta\mathbf{v}_0\|^2\right)=\frac{\bar{p}}{\bar{p}+1}(N_2-1),\end{aligned} \tag{2.24}$$

where we used the definition in (2.4). From (2.24) we deduce that

$$\|\Delta p(t)\|^2+\|\Delta\mathbf{v}(t)\|^2\le N_2-1. \tag{2.25}$$

In addition, by plugging (2.24) into (2.22), then integrating with respect to t, we can show that

$$\begin{aligned}&\int_0^t\left(\|\nabla\Delta p(\tau)\|^2+\varepsilon\bar{p}\|\Delta(\nabla\cdot\mathbf{v})(\tau)\|^2\right)d\tau\\ \le\ &\frac{3c_4}{2(\bar{p}+1)}\left(\|p_0\|_{H^1}^2+\bar{p}\|\mathbf{v}_0\|_{H^1}^2\right)(N_2-1)+\left(\|\Delta p_0\|^2+\bar{p}\|\Delta\mathbf{v}_0\|^2\right)=\hat{N}.\end{aligned} \tag{2.26}$$

We note that the constant $\hat{N}$ depends only on $\bar{p}$, the initial data and Gagliardo-Nirenberg constants. This completes the proof for the H^2-estimate. □

2.4. H^3-estimate.

Proof. For the third order estimate, we can show that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\nabla\Delta p\|^2+\bar{p}\|\Delta(\nabla\cdot\mathbf{v})\|^2\right)+\|\Delta^2 p\|^2+\varepsilon\bar{p}\|\Delta^2\mathbf{v}\|^2\\ \le\ &(\|p\|_{L^\infty}\|\Delta(\nabla\cdot\mathbf{v})\|+3\|\nabla p\|_{L^3}\|\Delta\mathbf{v}\|_{L^6}+3\|\Delta p\|_{L^6}\|\nabla\cdot\mathbf{v}\|_{L^3}+\|\nabla\Delta p\|\|\mathbf{v}\|_{L^\infty})\|\Delta^2 p\|\\ &+\varepsilon\bar{p}(2\|\Delta(\nabla\cdot\mathbf{v})\|\|\mathbf{v}\|_{L^\infty}+6\|\nabla\cdot\mathbf{v}\|_{L^3}\|\Delta\mathbf{v}\|_{L^6})\|\Delta^2\mathbf{v}\|,\end{aligned} \tag{2.27}$$

where the terms on the right-hand side are estimated by applying Gagliardo-Nirenberg inequalities and Sobolev embeddings as follows:

- $$\begin{aligned}\|p\|_{L^\infty}\|\Delta(\nabla\cdot\mathbf{v})\|\cdot\|\Delta^2 p\|&\le c_3\|\nabla p\|^{\frac{1}{2}}\|\Delta p\|^{\frac{1}{2}}\|\Delta(\nabla\cdot\mathbf{v})\|\cdot\|\Delta^2 p\|\\ &\le 2c_3^2\|\nabla p\|\cdot\|\Delta p\|\cdot\|\Delta(\nabla\cdot\mathbf{v})\|^2+\frac{1}{8}\|\Delta^2 p\|^2\end{aligned}$$

$$\le c_3^2(\|\nabla p\|^2+\|\Delta p\|^2)\|\Delta(\nabla\cdot\mathbf{v})\|^2+\frac{1}{8}\|\Delta^2 p\|^2,$$

- $$3\|\nabla p\|_{L^3}\|\Delta\mathbf{v}\|_{L^6}\|\Delta^2 p\|\le 3c_1c_2\|\nabla p\|^{\frac12}\|\Delta p\|^{\frac12}\|\Delta(\nabla\cdot\mathbf{v})\|\cdot\|\Delta^2 p\|$$
$$\le 18c_1^2c_2^2\|\nabla p\|\cdot\|\Delta p\|\cdot\|\Delta(\nabla\cdot\mathbf{v})\|^2+\frac{1}{8}\|\Delta^2 p\|^2$$
$$\le 9c_1^2c_2^2(\|\nabla p\|^2+\|\Delta p\|^2)\|\Delta(\nabla\cdot\mathbf{v})\|^2+\frac{1}{8}\|\Delta^2 p\|^2,$$

- $$3\|\Delta p\|_{L^6}\|\nabla\cdot\mathbf{v}\|_{L^3}\|\Delta^2 p\|\le 3c_1c_2\|\nabla\Delta p\|\cdot\|\nabla\cdot\mathbf{v}\|^{\frac12}\|\Delta\mathbf{v}\|^{\frac12}\|\Delta^2 p\|$$
$$\le 18c_1^2c_2^2\|\nabla\cdot\mathbf{v}\|\cdot\|\Delta\mathbf{v}\|\cdot\|\nabla\Delta p\|^2+\frac{1}{8}\|\Delta^2 p\|^2,$$
$$\le 9c_1^2c_2^2(\|\nabla\cdot\mathbf{v}\|^2+\|\Delta\mathbf{v}\|^2)\|\nabla\Delta p\|^2+\frac{1}{8}\|\Delta^2 p\|^2,$$

- $$\|\nabla\Delta p\|\cdot\|\mathbf{v}\|_{L^\infty}\|\Delta^2 p\|\le c_3\|\nabla\Delta p\|\cdot\|\nabla\cdot\mathbf{v}\|^{\frac12}\|\Delta\mathbf{v}\|^{\frac12}\|\Delta^2 p\|$$
$$\le 2c_3^2\|\nabla\cdot\mathbf{v}\|\cdot\|\Delta\mathbf{v}\|\cdot\|\nabla\Delta p\|^2+\frac{1}{8}\|\Delta^2 p\|^2,$$
$$\le c_3^2(\|\nabla\cdot\mathbf{v}\|^2+\|\Delta\mathbf{v}\|^2)\|\nabla\Delta p\|^2+\frac{1}{8}\|\Delta^2 p\|^2,$$

- $$2\varepsilon\bar p\|\Delta(\nabla\cdot\mathbf{v})\|\cdot\|\mathbf{v}\|_{L^\infty}\|\Delta^2\mathbf{v}\|\le 2c_3\varepsilon\bar p\|\nabla\cdot\mathbf{v}\|^{\frac12}\|\Delta\mathbf{v}\|^{\frac12}\|\Delta(\nabla\cdot\mathbf{v})\|\cdot\|\Delta^2\mathbf{v}\|$$
$$\le 4c_3^2\varepsilon\bar p\|\nabla\cdot\mathbf{v}\|\cdot\|\Delta\mathbf{v}\|\cdot\|\Delta(\nabla\cdot\mathbf{v})\|^2+\frac{1}{4}\varepsilon\bar p\|\Delta^2\mathbf{v}\|^2$$
$$\le 2c_3^2\varepsilon\bar p(\|\nabla\cdot\mathbf{v}\|^2+\|\Delta\mathbf{v}\|^2)\|\Delta(\nabla\cdot\mathbf{v})\|^2+\frac{1}{4}\varepsilon\bar p\|\Delta^2\mathbf{v}\|^2,$$

- $$6\varepsilon\bar p\|\nabla\cdot\mathbf{v}\|_{L^3}\|\Delta\mathbf{v}\|_{L^6}\|\Delta^2\mathbf{v}\|\le 6c_1c_2\varepsilon\bar p\|\nabla\cdot\mathbf{v}\|^{\frac12}\|\Delta\mathbf{v}\|^{\frac12}\|\Delta(\nabla\cdot\mathbf{v})\|\cdot\|\Delta^2\mathbf{v}\|$$
$$\le 36c_1^2c_2^2\varepsilon\bar p\|\nabla\cdot\mathbf{v}\|\cdot\|\Delta\mathbf{v}\|\cdot\|\Delta(\nabla\cdot\mathbf{v})\|^2+\frac{1}{4}\varepsilon\bar p\|\Delta^2\mathbf{v}\|^2$$
$$\le 18c_1^2c_2^2\varepsilon\bar p(\|\nabla\cdot\mathbf{v}\|^2+\|\Delta\mathbf{v}\|^2)\|\Delta(\nabla\cdot\mathbf{v})\|^2+\frac{1}{4}\varepsilon\bar p\|\Delta^2\mathbf{v}\|^2.$$

By substituting the above estimates into (2.27), we get

$$\begin{aligned}
&\frac{d}{dt}\big(\|\nabla\Delta p\|^2+\bar p\|\Delta(\nabla\cdot\mathbf{v})\|^2\big)+\|\Delta^2 p\|^2+\varepsilon\bar p\|\Delta^2\mathbf{v}\|^2\\
&\le c_5(\|\nabla p\|^2+\|\Delta p\|^2)\|\Delta(\nabla\cdot\mathbf{v})\|^2+2c_5(\|\nabla\cdot\mathbf{v}\|^2+\|\Delta\mathbf{v}\|^2)\times\big(\|\nabla\Delta p\|^2+\varepsilon\bar p\|\Delta(\nabla\cdot\mathbf{v})\|^2\big)\\
&\le \frac{c_5}{\bar p}(\|\nabla p\|^2+\|\Delta p\|^2)\big(\|\nabla\Delta p\|^2+\bar p\|\Delta(\nabla\cdot\mathbf{v})\|^2\big)+c_5(\kappa+N_2)\\
&\qquad\times\big(\|\nabla\Delta p\|^2+\varepsilon\bar p\|\Delta(\nabla\cdot\mathbf{v})\|^2\big),
\end{aligned}\tag{2.28}$$

where

$$c_5=9c_1^2c_2^2+c_3^2.\tag{2.29}$$

By applying the Grönwall inequality to (2.28) and using (2.10), (2.19) and (2.26), we have

$$\|\nabla\Delta p(t)\|^2+\bar p\|\Delta(\nabla\cdot\mathbf{v})(t)\|^2$$

$$\le \exp\left\{\frac{c_5}{\bar p}\int_0^t (\|\nabla p(\tau)\|^2+\|\Delta p(\tau)\|^2)d\tau\right\}\Big(\|\nabla\Delta p_0\|^2+\bar p\|\Delta(\nabla\cdot\mathbf{v}_0)\|^2$$

$$+c_5(\kappa+N_2)\int_0^t \left(\|\nabla\Delta p(\tau)\|^2+\varepsilon\bar p\|\Delta(\nabla\cdot\mathbf{v})(\tau)\|^2\right)d\tau\Big)$$

$$\le \exp\left\{\frac{3c_5}{2\bar p}\left(\|p_0\|_{H^1}^2+\bar p\|\mathbf{v}_0\|_{H^1}^2\right)\right\}\Big(\|\nabla\Delta p_0\|^2+\bar p\|\Delta(\nabla\cdot\mathbf{v}_0)\|^2+c_5(\kappa+N_2)\hat N\Big). \quad (2.30)$$

Since $\kappa<1$ (cf. (1.12)–(1.13)), in view of (2.30) and (2.4), we see that

$$\|\nabla\Delta p(t)\|^2+\|\Delta(\nabla\cdot\mathbf{v})(t)\|^2$$

$$\le (1+1/\bar p)\exp\left\{\frac{3c_5}{2\bar p}\left(\|p_0\|_{H^1}^2+\bar p\|\mathbf{v}_0\|_{H^1}^2\right)\right\}\Big(\|\nabla\Delta p_0\|^2+\bar p\|\Delta(\nabla\cdot\mathbf{v}_0)\|^2+c_5(\kappa+N_2)\hat N\Big)$$

$$\le N_3-1. \quad (2.31)$$

This completes the proof for the H^3-estimate. ∎

2.5. Further Estimate for v.

Proof. From previous sections, we see that the temporal integral of $\|\nabla\cdot\mathbf{v}\|_{H^2}^2$ is inversely proportional to ε (cf. (2.10), (2.19), (2.26)). Now we improve the estimate of such a quantity to be proportional to ε, which is used later to prove the zero chemical diffusion limit. For this purpose, by combining the equations in (1.11), we get

$$\partial_t(\nabla\cdot\mathbf{v})+\bar p\nabla\cdot\mathbf{v}=\varepsilon\Delta(\nabla\cdot\mathbf{v})+\partial_t p-\varepsilon\Delta(|\mathbf{v}|^2)-\nabla\cdot(p\mathbf{v}). \quad (2.32)$$

Taking the L^2 inner product of (2.32) with $\nabla\cdot\mathbf{v}$, we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla\cdot\mathbf{v}\|^2+\bar p\|\nabla\cdot\mathbf{v}\|^2+\varepsilon\|\Delta\mathbf{v}\|^2$$

$$=\int_{\mathbb{R}^3}(\partial_t p)(\nabla\cdot\mathbf{v})d\mathbf{x}-\varepsilon\int_{\mathbb{R}^3}\Delta(|\mathbf{v}|^2)(\nabla\cdot\mathbf{v})d\mathbf{x}-\int_{\mathbb{R}^3}(\nabla\cdot(p\mathbf{v}))(\nabla\cdot\mathbf{v})d\mathbf{x}. \quad (2.33)$$

We note that

$$\int_{\mathbb{R}^3}(\partial_t p)(\nabla\cdot\mathbf{v})d\mathbf{x}=\frac{d}{dt}\int_{\mathbb{R}^3}p(\nabla\cdot\mathbf{v})d\mathbf{x}-\int_{\mathbb{R}^3}p(\partial_t\nabla\cdot\mathbf{v})d\mathbf{x}$$

$$=\frac{d}{dt}\int_{\mathbb{R}^3}p(\nabla\cdot\mathbf{v})d\mathbf{x}+\|\nabla p\|^2-\int_{\mathbb{R}^3}p\left(\varepsilon\Delta(\nabla\cdot\mathbf{v})-\varepsilon\Delta(|\mathbf{v}|^2)\right)d\mathbf{x},$$

where we used the second equation of (1.11). Then we update (2.33) as

$$\frac{d}{dt}\left(\frac{1}{2}\|\nabla\cdot\mathbf{v}\|^2-\int_{\mathbb{R}^3}p(\nabla\cdot\mathbf{v})d\mathbf{x}\right)+\bar p\|\nabla\cdot\mathbf{v}\|^2+\varepsilon\|\Delta\mathbf{v}\|^2$$

$$=\|\nabla p\|^2+\varepsilon\int_{\mathbb{R}^3}\nabla(|\mathbf{v}|^2)\cdot(\Delta\mathbf{v})d\mathbf{x}-\int_{\mathbb{R}^3}(\nabla\cdot(p\mathbf{v}))(\nabla\cdot\mathbf{v})d\mathbf{x}$$

$$+\int_{\mathbb{R}^3}\nabla p\cdot\left(\varepsilon\nabla(\nabla\cdot\mathbf{v})-\varepsilon\nabla(|\mathbf{v}|^2)\right)d\mathbf{x}. \quad (2.34)$$

For the second term on the RHS of (2.34), according to (2.1) and (2.2), we have

$$\left|\varepsilon\int_{\mathbb{R}^3}\nabla(|\mathbf{v}|^2)\cdot(\Delta\mathbf{v})d\mathbf{x}\right|\le 2\varepsilon\|\mathbf{v}\|_{L^3}\|\nabla\cdot\mathbf{v}\|_{L^6}\|\Delta\mathbf{v}\|$$

$$\le 2c_1c_2\varepsilon\|\mathbf{v}\|^{\frac12}\|\nabla\cdot\mathbf{v}\|^{\frac12}\|\Delta\mathbf{v}\|^2\le 2c_1c_2\varepsilon(N_1\kappa)^{\frac14}\|\Delta\mathbf{v}\|^2.$$

For the third term on the RHS of (2.34), by using similar arguments as in (2.12), we can show that

$$\begin{aligned}\left|-\int_{\mathbb{R}^3}(\nabla\cdot(p\mathbf{v}))(\nabla\cdot\mathbf{v})d\mathbf{x}\right| &\le (\|\nabla p\|_{L^3}\|\mathbf{v}\|_{L^6}+\|p\|_{L^\infty}\|\|\nabla\cdot\mathbf{v}\|\|)\|\nabla\cdot\mathbf{v}\| \\ &\le (c_1c_2+c_3)\|\nabla p\|^{\frac{1}{2}}\|\Delta p\|^{\frac{1}{2}}\|\nabla\cdot\mathbf{v}\|^2 \\ &\le \frac{(c_1c_2+c_3)^2}{2\bar{p}}\|\nabla p\|\|\Delta p\|\|\nabla\cdot\mathbf{v}\|^2+\frac{\bar{p}}{2}\|\nabla\cdot\mathbf{v}\|^2 \\ &\le \frac{(c_1c_2+c_3)^2}{\bar{p}}\left(\|\nabla p\|^2+\|\Delta p\|^2\right)\kappa+\frac{\bar{p}}{2}\|\nabla\cdot\mathbf{v}\|^2 \\ &\le \frac{(c_1c_2+c_3)^2}{\bar{p}}\left(\|\nabla p\|^2+\|\Delta p\|^2\right)+\frac{\bar{p}}{2}\|\nabla\cdot\mathbf{v}\|^2,\end{aligned}$$

provided that $\kappa\le 1$. For the fourth term on the RHS of (2.34), we can show that

$$\begin{aligned}\left|\int_{\mathbb{R}^3}\nabla p\cdot\left(\varepsilon\nabla(\nabla\cdot\mathbf{v})-\varepsilon\nabla(|\mathbf{v}|^2)\right)d\mathbf{x}\right| &\le \varepsilon\|\nabla p\|\|\nabla(\nabla\cdot\mathbf{v})\|+2\varepsilon\|\nabla p\|\|\mathbf{v}\|_{L^3}\|\nabla\mathbf{v}\|_{L^6} \\ &\le 2\varepsilon\|\nabla p\|^2+\frac{\varepsilon}{4}\|\Delta\mathbf{v}\|^2+\varepsilon\|\mathbf{v}\|_{L^3}^2\|\nabla\mathbf{v}\|_{L^6}^2 \\ &\le 2\varepsilon\|\nabla p\|^2+\frac{\varepsilon}{4}\|\Delta\mathbf{v}\|^2+c_1c_2\varepsilon\|\mathbf{v}\|\|\nabla\cdot\mathbf{v}\|\|\Delta\mathbf{v}\|^2 \\ &\le 2\varepsilon\|\nabla p\|^2+\frac{\varepsilon}{4}\|\Delta\mathbf{v}\|^2+c_1c_2\varepsilon(N_1\kappa)\|\Delta\mathbf{v}\|^2.\end{aligned}$$

Hence, when

$$N_1\kappa\le\min\left\{(16c_1c_2)^{-4},(8c_1c_2)^{-1}\right\}, \tag{2.35}$$

we update (2.34) as

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\|\nabla\cdot\mathbf{v}\|^2-\int_{\mathbb{R}^3}p(\nabla\cdot\mathbf{v})d\mathbf{x}\right)+\frac{\bar{p}}{2}\|\nabla\cdot\mathbf{v}\|^2+\frac{\varepsilon}{2}\|\Delta\mathbf{v}\|^2 \\ &\le (1+2\varepsilon)\|\nabla p\|^2+\frac{(c_1c_2+c_3)^2}{\bar{p}}\left(\|\nabla p\|^2+\|\Delta p\|^2\right).\end{aligned} \tag{2.36}$$

By multiplying (2.9) by 2, then adding the result to (2.36), we have

$$\begin{aligned}&\frac{d}{dt}E(t)+\frac{\bar{p}}{2}\|\nabla\cdot\mathbf{v}\|^2+\frac{\varepsilon}{2}\|\Delta\mathbf{v}\|^2+\|\nabla p\|^2+2\varepsilon\bar{p}\|\nabla\cdot\mathbf{v}\|^2 \\ &\le \left(2\varepsilon+\frac{(c_1c_2+c_3)^2}{\bar{p}}\right)\|\nabla p\|^2+\frac{(c_1c_2+c_3)^2}{\bar{p}}\|\Delta p\|^2,\end{aligned} \tag{2.37}$$

where

$$\begin{aligned}E(t) &= \frac{1}{2}\|\nabla\cdot\mathbf{v}\|^2-\int_{\mathbb{R}^3}p(\nabla\cdot\mathbf{v})d\mathbf{x}+2\|p\|^2+2\bar{p}\|\mathbf{v}\|^2 \\ &= \frac{1}{4}\|\nabla\cdot\mathbf{v}\|^2+\left\|\frac{1}{2}\nabla\cdot\mathbf{v}-p\right\|^2+\|p\|^2+2\bar{p}\|\mathbf{v}\|^2.\end{aligned}$$

Integrating (2.37) with respect to time and using (2.10) and (2.19) then yield, in particular,

$$\frac{\bar{p}}{2}\int_0^t\|\nabla\cdot\mathbf{v}(\tau)\|^2d\tau\le E(0)+\left(2\varepsilon+\frac{(c_1c_2+c_3)^2}{\bar{p}}\right)\left(\|p_0\|^2+\bar{p}\|\mathbf{v}_0\|^2\right)$$

$$+\frac{3(c_1c_2+c_3)^2}{2\bar{p}}\left(\|\nabla p_0\|^2+\bar{p}\|\nabla\cdot\mathbf{v}_0\|^2\right), \tag{2.38}$$

where the constant on the RHS is independent of t and remains bounded as $\varepsilon\to 0$. In a completely similar fashion, by working on the non-homogeneous damped Equation (2.32) and using the H^1-, H^2- and H^3-estimates established in the previous sections, we can show that

$$\int_0^t \left(\|\nabla(\nabla\cdot\mathbf{v})(\tau)\|^2+\|\Delta(\nabla\cdot\mathbf{v})(\tau)\|^2\right)d\tau\le C(c_1,c_2,c_3,\bar{p},p_0,\mathbf{v}_0,\varepsilon).$$

We omit the technical details to simplify the presentation. ∎

We remark that the constants appearing in the energy estimates in Sections 2.1–2.5 remain bounded as $\varepsilon\to 0$. This allows us to establish the global well-posedness of (1.11) when $\varepsilon=0$. Indeed, by repeating the arguments in Sections 2.1–2.5, one can establish similar energy estimates for the solution to (1.11) when $\varepsilon=0$, and the solution satisfies (1.14) with $\varepsilon=0$. More importantly, the energy estimates derived in Sections 2.1–2.5 allow us to take the zero chemical diffusion limit of the solution. Furthermore, in view of (2.8), (2.13), (2.17), (2.21), (2.35) and the derivation of (2.31), we see that all of the energy estimates derived in Sections 2.1–2.4 are valid when

$$N_1\kappa\le\min\left\{(16c_1c_2)^{-4},\ (8c_1c_2)^{-1},\ (54c_3^4)^{-1},\ 1\right\}. \tag{2.39}$$

Note that (2.39) is the same smallness condition as stated in Theorem 1.1.

2.6. Global well-posedness. We now prove the global well-posedness of (1.11). First, from (2.11), (2.18), (2.25) and (2.31), we see that for $\forall\ t\in[0,T_1]$,

$$\begin{aligned}&\|p(t)\|^2+\|\mathbf{v}(t)\|^2\le N_1-1, && \|\nabla p(t)\|^2+\|\nabla\cdot\mathbf{v}(t)\|^2\le\frac{3}{4}\kappa,\\ &\|\Delta p(t)\|^2+\|\Delta\mathbf{v}(t)\|^2\le N_2-1, && \|\nabla\Delta p(t)\|^2+\|\Delta(\nabla\cdot\mathbf{v})(t)\|^2\le N_3-1,\end{aligned} \tag{2.40}$$

which, in particular, imply that

$$\begin{aligned}&\|p(T_1)\|^2+\|\mathbf{v}(T_1)\|^2\le N_1-1, && \|\nabla p(T_1)\|^2+\|\nabla\cdot\mathbf{v}(T_1)\|^2\le\frac{3}{4}\kappa,\\ &\|\Delta p(T_1)\|^2+\|\Delta\mathbf{v}(T_1)\|^2\le N_2-1, && \|\nabla\Delta p(T_1)\|^2+\|\Delta(\nabla\cdot\mathbf{v})(T_1)\|^2\le N_3-1.\end{aligned} \tag{2.41}$$

From the local existence in Lemma 2.1, for some $\hat{T}\in(0,\infty)$ there exists a unique classical solution to (1.11) on the time interval $[T_1,T_1+\hat{T}]$ satisfying for $\forall t\in[T_1,T_1+\hat{T}]$,

$$\begin{aligned}&\|p(t)\|^2+\|\mathbf{v}(t)\|^2\le N_1, && \|\nabla p(t)\|^2+\|\nabla\cdot\mathbf{v}(t)\|^2\le\kappa,\\ &\|\Delta p(t)\|^2+\|\Delta\mathbf{v}(t)\|^2\le N_2, && \|\nabla\Delta p(t)\|^2+\|\Delta(\nabla\cdot\mathbf{v})(t)\|^2\le N_3,\end{aligned} \tag{2.42}$$

In view of (2.6) and (2.42), we see that for $\forall t\in[0,T_1+\hat{T}]$,

$$\begin{aligned}&\|p(t)\|^2+\|\mathbf{v}(t)\|^2\le N_1, && \|\nabla p(t)\|^2+\|\nabla\cdot\mathbf{v}(t)\|^2\le\kappa,\\ &\|\Delta p(t)\|^2+\|\Delta\mathbf{v}(t)\|^2\le N_2, && \|\nabla\Delta p(t)\|^2+\|\Delta(\nabla\cdot\mathbf{v})(t)\|^2\le N_3.\end{aligned} \tag{2.43}$$

From the standard procedure as in [41], we conclude that the solution exists globally in time. This completes the proof of the global well-posedness result recorded in Theorem 1.1.

2.7. Long-time behavior. In this section, we derive the long-time behavior of the solution. We combine the previous energy estimates with the fact that any function $f(t)$, belonging to $W^{1,1}(0,\infty)$, converges to zero as $t\to\infty$, to establish the decay estimate. For brevity, we only present the proof for the first order spatial derivatives of the solution. We note that according to (2.10) and (2.38), it holds that

$$\|\nabla p(t)\|^2+\|\nabla\cdot\mathbf{v}(t)\|^2\in L^1(0,\infty), \tag{2.44}$$

which is valid for any $\varepsilon\geq 0$. Next, by using similar arguments as in deriving (2.12), we can show that

$$\begin{aligned}\frac{1}{2}\left|\frac{d}{dt}\left(\|\nabla p\|^2+\bar{p}\|\nabla\cdot\mathbf{v}\|^2\right)\right|\leq&\left(\frac{1}{4}+c_1c_2(N_1\kappa)^{\frac{1}{4}}\right)\|\Delta p\|^2+\frac{27}{4}c_3^4\kappa^2\|\nabla p\|^2\\&+c_1c_2\varepsilon\bar{p}(N_1\kappa)^{\frac{1}{4}}\|\Delta\mathbf{v}\|^2+\|\Delta p\|^2+\varepsilon\bar{p}\|\Delta\mathbf{v}\|^2,\end{aligned}$$

which, together with (2.8) and (2.13), implies

$$\left|\frac{d}{dt}\left(\|\nabla p\|^2+\bar{p}\|\nabla\cdot\mathbf{v}\|^2\right)\right|\leq 3\|\Delta p\|^2+\frac{27}{2}c_3^4\kappa^2\|\nabla p\|^2+3\varepsilon\bar{p}\|\Delta\mathbf{v}\|^2. \tag{2.45}$$

By integrating (2.45) with respect to t and applying (2.10) and (2.19), we have

$$\frac{d}{dt}\left(\|(\nabla p)(t)\|^2+\bar{p}\|(\nabla\cdot\mathbf{v})(t)\|^2\right)\in L^1(0,\infty).$$

By combining (2.44) and (2.45), we conclude that $\|(\nabla p)(t)\|^2+\bar{p}\|(\nabla\cdot\mathbf{v})(t)\|^2\in W^{1,1}(0,\infty)$, which implies $\lim_{t\to\infty}\left(\|(\nabla p)(t)\|^2+\bar{p}\|(\nabla\cdot\mathbf{v})(t)\|^2\right)=0$. Furthermore, by working with (2.20) and (2.27) we can obtain the similar result for the second and third order derivatives of the solution.

2.8. Diffusion limit. Now we study the zero chemical diffusion limit and quantify the convergence rate in terms of ε. Let $(p^\varepsilon,\mathbf{v}^\varepsilon)$ and $(p^0,\mathbf{v}^0)$ be the solutions to (1.11) with $\varepsilon>0$ and $\varepsilon=0$, respectively, for the same initial data, and set $\tilde{p}=p^\varepsilon-p^0$ and $\tilde{\mathbf{v}}=\mathbf{v}^\varepsilon-\mathbf{v}^0$. Then $(\tilde{p},\tilde{\mathbf{v}})$ satisfies

$$\begin{cases}\partial_t\tilde{p}-\nabla\cdot\tilde{\mathbf{v}}=\Delta\tilde{p}+\nabla\cdot(\tilde{p}\mathbf{v}^\varepsilon+p^0\tilde{\mathbf{v}}),\\ \partial_t\tilde{\mathbf{v}}-\nabla\tilde{p}=\varepsilon\Delta\mathbf{v}^\varepsilon-\varepsilon\nabla\left(|\mathbf{v}^\varepsilon|^2\right);\\ (\tilde{p}_0,\tilde{\mathbf{v}}_0)=(0,\mathbf{0}),\end{cases} \tag{2.46}$$

where for simplicity, we took $\bar{p}=1$. We begin with the zeroth frequency estimate.

Step 1. By taking the L^2 inner products, we find

$$\frac{1}{2}\frac{d}{dt}\left(\|\tilde{p}\|^2+\|\tilde{\mathbf{v}}\|^2\right)+\|\nabla\tilde{p}\|^2=-\int_{\mathbb{R}^3}(\tilde{p}\mathbf{v}^\varepsilon+p^0\tilde{\mathbf{v}})\cdot\nabla\tilde{p}\,d\mathbf{x}+\int_{\mathbb{R}^3}\left[\varepsilon\Delta\mathbf{v}^\varepsilon-\varepsilon\nabla\left(|\mathbf{v}^\varepsilon|^2\right)\right]\cdot\tilde{\mathbf{v}}\,d\mathbf{x}. \tag{2.47}$$

For the first term on the RHS of (2.47), by applying (2.3), we have

$$\begin{aligned}&\left|-\int_{\mathbb{R}^3}(\tilde{p}\mathbf{v}^\varepsilon+p^0\tilde{\mathbf{v}})\cdot\nabla\tilde{p}\,d\mathbf{x}\right|\\ \leq&\ \frac{1}{2}\|\nabla\tilde{p}\|^2+\|\mathbf{v}^\varepsilon\|_{L^\infty}^2\|\tilde{p}\|^2+\|p^0\|_{L^\infty}^2\|\tilde{\mathbf{v}}\|^2\\ \leq&\ \frac{1}{2}\|\nabla\tilde{p}\|^2+c_3^2\left(\|\nabla\cdot\mathbf{v}^\varepsilon\|\cdot\|\Delta\mathbf{v}^\varepsilon\|\cdot\|\tilde{p}\|^2+\|\nabla p^0\|\cdot\|\Delta p^0\|\cdot\|\tilde{\mathbf{v}}\|^2\right)\end{aligned}$$

$$\leq \frac{1}{2}\|\nabla\tilde{p}\|^2+\frac{c_3^2}{2}\left[(\|\nabla\cdot\mathbf{v}^\varepsilon\|^2+\|\Delta\mathbf{v}^\varepsilon\|^2)\|\tilde{p}\|^2+(\|\nabla p^0\|^2+\|\Delta p^0\|^2)\|\tilde{\mathbf{v}}\|^2\right]. \tag{2.48}$$

For the second term on the RHS of (2.47), again by applying (2.3), we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3}\left[\varepsilon\Delta\mathbf{v}^\varepsilon-\varepsilon\nabla\left(|\mathbf{v}^\varepsilon|^2\right)\right]\cdot\tilde{\mathbf{v}}\,d\mathbf{x}\right|\\
\leq\ & \frac{1}{2}\|\tilde{\mathbf{v}}\|^2+\varepsilon^2\|\Delta\mathbf{v}^\varepsilon\|^2+4\varepsilon^2\|\mathbf{v}^\varepsilon\|_{L^\infty}^2\|\nabla\cdot\mathbf{v}^\varepsilon\|^2\\
\leq\ & \frac{1}{2}\|\tilde{\mathbf{v}}\|^2+\varepsilon^2\|\Delta\mathbf{v}^\varepsilon\|^2+4c_3^2\varepsilon^2\|\nabla\cdot\mathbf{v}^\varepsilon\|\cdot\|\Delta\mathbf{v}^\varepsilon\|\cdot\|\nabla\cdot\mathbf{v}^\varepsilon\|^2\\
\leq\ & \frac{1}{2}\|\tilde{\mathbf{v}}\|^2+\varepsilon^2\|\Delta\mathbf{v}^\varepsilon\|^2+2c_3^2\sqrt{3\kappa(N_2-1)}\,\varepsilon^2\|\nabla\cdot\mathbf{v}^\varepsilon\|^2.
\end{aligned} \tag{2.49}$$

where we applied (2.18) and (2.25). By substituting (2.48) and (2.49) into (2.47) then multiplying through by 2, we have

$$\begin{aligned}
&\frac{d}{dt}\left(\|\tilde{p}\|^2+\|\tilde{\mathbf{v}}\|^2\right)+\|\nabla\tilde{p}\|^2\\
\leq\ & \left[c_3^2\left(\|\nabla\cdot\mathbf{v}^\varepsilon\|^2+\|\Delta\mathbf{v}^\varepsilon\|^2+\|\nabla p^0\|^2+\|\Delta p^0\|^2\right)+1\right]\left(\|\tilde{p}\|^2+\|\tilde{\mathbf{v}}\|^2\right)\\
&+\ 2\varepsilon^2\|\Delta\mathbf{v}^\varepsilon\|^2+4c_3^2\sqrt{3\kappa(N_2-1)}\,\varepsilon^2\|\nabla\cdot\mathbf{v}^\varepsilon\|^2.
\end{aligned} \tag{2.50}$$

By applying the Grönwall inequality to (2.50), we have

$$\begin{aligned}
&\|\tilde{p}(t)\|^2+\|\tilde{\mathbf{v}}(t)\|^2\\
\leq\ & \exp\left\{c_3^2\int_0^t\left(\|\nabla\cdot\mathbf{v}^\varepsilon(\tau)\|^2+\|\Delta\mathbf{v}^\varepsilon(\tau)\|^2+\|\nabla p^0(\tau)\|^2+\|\Delta p^0(\tau)\|^2\right)d\tau+t\right\}\\
&\times\left(2\int_0^t\|\Delta\mathbf{v}^\varepsilon(\tau)\|^2d\tau+4c_3^2\sqrt{3\kappa(N_2-1)}\int_0^t\|\nabla\cdot\mathbf{v}^\varepsilon(\tau)\|^2d\tau\right)\varepsilon^2.
\end{aligned}$$

We note that according to the energy estimates recorded in Theorem 1.1, see (1.14), the integrals on the RHS of the above inequality are bounded by some constants that are independent of t and remain bounded as $\varepsilon\to 0$. We rewrite the above estimate by using short notations as

$$\|\tilde{p}(t)\|^2+\|\tilde{\mathbf{v}}(t)\|^2\leq e^{C_1+t}C_2\varepsilon^2. \tag{2.51}$$

Next, we consider the convergence of the first order derivatives of the perturbation.

Step 2. By taking the L^2 inner products of the first two equations in (2.46) with the $-\Delta$ of the targeting functions, we deduce

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla\tilde{p}\|^2+\|\nabla\cdot\tilde{\mathbf{v}}\|^2\right)+\|\Delta\tilde{p}\|^2\\
=&-\int_{\mathbb{R}^3}\left[\nabla\cdot(\tilde{p}\mathbf{v}^\varepsilon+p^0\tilde{\mathbf{v}})\right]\Delta\tilde{p}\,d\mathbf{x}+\varepsilon\int_{\mathbb{R}^3}(\Delta\nabla\cdot\mathbf{v}^\varepsilon)(\nabla\cdot\tilde{\mathbf{v}})d\mathbf{x}-\varepsilon\int_{\mathbb{R}^3}\Delta\left(|\mathbf{v}^\varepsilon|^2\right)(\nabla\cdot\tilde{\mathbf{v}})d\mathbf{x}.
\end{aligned} \tag{2.52}$$

For the first term on the RHS of (2.52), we have

$$\left|-\int_{\mathbb{R}^3}\left[\nabla\cdot(\tilde{p}\mathbf{v}^\varepsilon+p^0\tilde{\mathbf{v}})\right]\Delta\tilde{p}\,d\mathbf{x}\right|\leq\frac{1}{2}\|\Delta\tilde{p}\|^2+\frac{1}{2}\|\nabla\cdot(\tilde{p}\mathbf{v}^\varepsilon+p^0\tilde{\mathbf{v}})\|^2, \tag{2.53}$$

where the second term on the RHS can be estimated as

$$\begin{aligned}
&\frac{1}{2}\|\nabla\cdot(\tilde{p}\mathbf{v}^\varepsilon+p^0\tilde{\mathbf{v}})\|^2\\
&\le 2\left(\|\nabla\tilde{p}\cdot\mathbf{v}^\varepsilon\|^2+\|\tilde{p}(\nabla\cdot\mathbf{v}^\varepsilon)\|^2+\|\nabla p^0\cdot\tilde{\mathbf{v}}\|^2+\|p^0(\nabla\cdot\tilde{\mathbf{v}})\|^2\right)\\
&\le 2\left(\|\nabla\tilde{p}\|^2\|\mathbf{v}^\varepsilon\|_{L^\infty}^2+\|\tilde{p}\|_{L^6}^2\|\nabla\cdot\mathbf{v}^\varepsilon\|_{L^3}^2+\|\nabla p^0\|_{L^3}^2\|\tilde{\mathbf{v}}\|_{L^6}^2+\|p^0\|_{L^\infty}^2\|\nabla\cdot\tilde{\mathbf{v}}\|^2\right)\\
&\le 2\big(c_3^2\|\nabla\tilde{p}\|^2\|\nabla\cdot\mathbf{v}^\varepsilon\|\cdot\|\Delta\mathbf{v}^\varepsilon\|+c_1^2c_2^2\|\nabla\tilde{p}\|^2\|\nabla\cdot\mathbf{v}^\varepsilon\|\cdot\|\Delta\mathbf{v}^\varepsilon\|+\\
&\qquad c_1^2c_2^2\|\nabla p^0\|\cdot\|\Delta p^0\|\cdot\|\nabla\cdot\tilde{\mathbf{v}}\|^2+c_3^2\|\nabla p^0\|\cdot\|\Delta p^0\|\cdot\|\nabla\cdot\tilde{\mathbf{v}}\|^2\big)\\
&\le (c_3^2+c_1^2c_2^2)\left(\|\nabla\cdot\mathbf{v}^\varepsilon\|^2+\|\Delta\mathbf{v}^\varepsilon\|^2+\|\nabla p^0\|^2+\|\Delta p^0\|^2\right)\left(\|\nabla\tilde{p}\|^2+\|\nabla\cdot\tilde{\mathbf{v}}\|^2\right).
\end{aligned}$$

So we can update (2.53) as

$$\begin{aligned}
&\left|-\int_{\mathbb{R}^3}\left[\nabla\cdot(\tilde{p}\mathbf{v}^\varepsilon+p^0\tilde{\mathbf{v}})\right]\Delta\tilde{p}\,d\mathbf{x}\right|\\
\le\ &\frac{1}{2}\|\Delta\tilde{p}\|^2+(c_3^2+c_1^2c_2^2)\left(\|\nabla\cdot\mathbf{v}^\varepsilon\|^2+\|\Delta\mathbf{v}^\varepsilon\|^2+\|\nabla p^0\|^2+\|\Delta p^0\|^2\right)\left(\|\nabla\tilde{p}\|^2+\|\nabla\cdot\tilde{\mathbf{v}}\|^2\right).
\end{aligned} \tag{2.54}$$

For the second and third terms on the RHS of (2.52), in a similar fashion, we can show that

$$\begin{aligned}
&\left|\varepsilon\int_{\mathbb{R}^3}(\Delta\nabla\cdot\mathbf{v}^\varepsilon)(\nabla\cdot\tilde{\mathbf{v}})d\mathbf{x}-\varepsilon\int_{\mathbb{R}^3}\Delta\left(|\mathbf{v}^\varepsilon|^2\right)(\nabla\cdot\tilde{\mathbf{v}})d\mathbf{x}\right|\\
\le\ &\frac{1}{2}\|\nabla\cdot\tilde{\mathbf{v}}\|^2+\varepsilon^2\|\Delta\nabla\cdot\mathbf{v}^\varepsilon\|^2+\varepsilon^2\left\|\Delta\left(|\mathbf{v}^\varepsilon|^2\right)\right\|^2\\
\le\ &\frac{1}{2}\|\nabla\cdot\tilde{\mathbf{v}}\|^2+\varepsilon^2\|\Delta\nabla\cdot\mathbf{v}^\varepsilon\|^2+8\varepsilon^2\left(\|\Delta\mathbf{v}^\varepsilon\|^2\|\mathbf{v}^\varepsilon\|_{L^\infty}^2+\|\nabla\cdot\mathbf{v}^\varepsilon\|_{L^3}^2\|\nabla\mathbf{v}^\varepsilon\|_{L^6}^2\right)\\
\le\ &\frac{1}{2}\|\nabla\cdot\tilde{\mathbf{v}}\|^2+\varepsilon^2\|\Delta\nabla\cdot\mathbf{v}^\varepsilon\|^2+8\left(c_3^2+c_1^2c_2^2\right)\varepsilon^2\|\nabla\cdot\mathbf{v}^\varepsilon\|\cdot\|\Delta\mathbf{v}^\varepsilon\|\cdot\|\Delta\mathbf{v}^\varepsilon\|^2\\
\le\ &\frac{1}{2}\|\nabla\cdot\tilde{\mathbf{v}}\|^2+\varepsilon^2\|\Delta\nabla\cdot\mathbf{v}^\varepsilon\|^2+4\left(c_3^2+c_1^2c_2^2\right)\sqrt{3\kappa(N_2-1)}\,\|\Delta\mathbf{v}^\varepsilon\|^2\varepsilon^2.
\end{aligned} \tag{2.55}$$

By feeding (2.54) and (2.55) into (2.52), we find

$$\begin{aligned}
&\frac{d}{dt}\left(\|\nabla\tilde{p}\|^2+\|\nabla\cdot\tilde{\mathbf{v}}\|^2\right)+\|\Delta\tilde{p}\|^2\\
\le\ &\left[2(c_3^2+c_1^2c_2^2)\left(\|\nabla\cdot\mathbf{v}^\varepsilon\|^2+\|\Delta\mathbf{v}^\varepsilon\|^2+\|\nabla p^0\|^2+\|\Delta p^0\|^2\right)+1\right]\left(\|\nabla\tilde{p}\|^2+\|\nabla\cdot\tilde{\mathbf{v}}\|^2\right)\\
&+2\varepsilon^2\|\Delta\nabla\cdot\mathbf{v}^\varepsilon\|^2+8\left(c_3^2+c_1^2c_2^2\right)\sqrt{3\kappa(N_2-1)}\,\|\Delta\mathbf{v}^\varepsilon\|^2\varepsilon^2.
\end{aligned} \tag{2.56}$$

By applying the Grönwall inequality to (2.56), we deduce

$$\begin{aligned}
&\|\nabla\tilde{p}(t)\|^2+\|\nabla\cdot\tilde{\mathbf{v}}(t)\|^2\\
\le\ &\exp\left\{2(c_3^2+c_1^2c_2^2)\int_0^t\left(\|\nabla\cdot\mathbf{v}^\varepsilon(\tau)\|^2+\|\Delta\mathbf{v}^\varepsilon(\tau)\|^2+\|\nabla p^0(\tau)\|^2+\|\Delta p^0(\tau)\|^2\right)d\tau+t\right\}\\
&\times\left(2\int_0^t\|\Delta\nabla\cdot\mathbf{v}^\varepsilon(\tau)\|^2d\tau+8\left(c_3^2+c_1^2c_2^2\right)\sqrt{3\kappa(N_2-1)}\int_0^t\|\Delta\mathbf{v}^\varepsilon(\tau)\|^2d\tau\right)\varepsilon^2.
\end{aligned} \tag{2.57}$$

We note again that the integrals on the RHS of the above inequality are bounded by some constants that are independent of t and remain bounded as $\varepsilon \to 0$. We rewrite the above estimate as

$$\|\nabla\tilde{p}(t)\|^2 + \|\nabla\cdot\tilde{\mathbf{v}}(t)\|^2 \leq e^{C_3+t} C_4 \varepsilon^2. \tag{2.58}$$

Step 3. The convergence of the second order derivatives of the solution can be established in a similar fashion, except that the term $\varepsilon^2\|\Delta^2\mathbf{v}^\varepsilon(t)\|^2$ will appear in the energy estimates. According to (1.14), the temporal integral of such term, i.e., $\int_0^t \varepsilon^2\|\Delta^2\mathbf{v}^\varepsilon(\tau)\|^2 d\tau$, has the order of $O(\varepsilon)$, instead of $O(\varepsilon^2)$, as $\varepsilon\to 0$. The final energy estimate takes the form

$$\|\Delta\tilde{p}(t)\|^2 + \|\Delta\tilde{\mathbf{v}}(t)\|^2 \leq e^{C_5+t} C_6 (1+\varepsilon)\varepsilon.$$

We omit the technical details. This completes the proof for Theorem 1.1.

3. Proof of Theorem 1.2

In this section we prove Theorem 1.2. Comparing with the 3D case, the proof of the 2D case is much lengthier, as the Gagliardo-Nirenberg inequalities generate less powers of high frequencies of a function in $\mathbb{R}^2$ than in $\mathbb{R}^3$. Such a deficiency has a substantial impact on the energy estimates for all the individual frequencies of the solution, especially when $\|p\|$ and $\|\mathbf{v}\|$ are potentially large. We overcome the difficulty by creating higher order nonlinearities.

We recall the following Gagliardo-Nirenberg inequalities in 2D:

$$\|f\|_{L^4} \leq d_1 \|\nabla f\|^{\frac{1}{2}} \|f\|^{\frac{1}{2}}, \tag{3.1}$$

$$\|f\|_{L^6} \leq d_2 \|\nabla f\|^{\frac{2}{3}} \|f\|^{\frac{1}{3}}, \tag{3.2}$$

$$\|f\|_{L^8} \leq d_3 \|\nabla f\|^{\frac{3}{4}} \|f\|^{\frac{1}{4}}, \tag{3.3}$$

$$\|f\|_{L^{12}} \leq d_4 \|\nabla f\|^{\frac{5}{6}} \|f\|^{\frac{1}{6}}, \tag{3.4}$$

$$\|f\|_{L^\infty} \leq d_5 \|f\|^{\frac{1}{2}} \|\Delta f\|^{\frac{1}{2}}. \tag{3.5}$$

In what follows, we assume that for a local existence time $T>0$ the following hold:

$$\sup_{0\leq t\leq T}\left(\|p(t)\|_{L^2}^2 + \|\mathbf{v}(t)\|_{L^2}^2\right) \leq M_1, \quad \sup_{0\leq t\leq T}\left(\|\nabla p(t)\|_{L^2}^2 + \|\nabla\cdot\mathbf{v}(t)\|_{L^2}^2\right) \leq \delta, \tag{3.6}$$

where M_1 and δ are defined in Theorem 1.2.

3.1. L^2-estimate.

By testing the equations in (1.11) with the targeting functions, we have

$$\frac{1}{2}\frac{d}{dt}\left(\|p\|^2 + \bar{p}\|\mathbf{v}\|^2\right) + \|\nabla p\|^2 + \varepsilon\bar{p}\|\nabla\cdot\mathbf{v}\|^2 = -\int_{\mathbb{R}^2} p\mathbf{v}\cdot\nabla p\, d\mathbf{x} + \varepsilon\bar{p}\int_{\mathbb{R}^2} |\mathbf{v}|^2 \nabla\cdot\mathbf{v}\, d\mathbf{x}. \tag{3.7}$$

We remark that the two terms on the RHS of (3.7) can not be estimated as in the 3D case, as the interpolation inequalities in 2D do not generate enough powers of the higher frequencies of a function. We eliminate those terms through performing higher order energy estimates. During such a process, the higher order nonlinearities can be controlled by using the smallness of the L^2 norm of ∇p and $\nabla\cdot\mathbf{v}$. We divide the subsequent proof into six steps.

Step 1. Taking the L^2 inner product of the first equation in (1.11) with $-\varepsilon|\mathbf{v}|^2$, we have

$$-\int_{\mathbb{R}^2}\varepsilon|\mathbf{v}|^2\partial_t p\,d\mathbf{x}=-\varepsilon\int_{\mathbb{R}^2}|\mathbf{v}|^2\nabla\cdot(p\mathbf{v})\,d\mathbf{x}-\varepsilon\,\bar{p}\int_{\mathbb{R}^2}|\mathbf{v}|^2\nabla\cdot\mathbf{v}\,d\mathbf{x}-\varepsilon\int_{\mathbb{R}^2}|\mathbf{v}|^2\Delta p\,d\mathbf{x}. \tag{3.8}$$

Taking the L^2 inner product of the second equation in (1.11) with $-2\varepsilon p\mathbf{v}$, we have

$$-\int_{\mathbb{R}^2}\varepsilon p\partial_t(|\mathbf{v}|^2)\,d\mathbf{x}=-2\varepsilon\int_{\mathbb{R}^2}p\mathbf{v}\cdot\nabla p\,d\mathbf{x}-2\varepsilon^2\int_{\mathbb{R}^2}p\mathbf{v}\cdot\Delta\mathbf{v}\,d\mathbf{x}+2\varepsilon^2\int_{\mathbb{R}^2}p\mathbf{v}\cdot\nabla(|\mathbf{v}|^2)\,d\mathbf{x}. \tag{3.9}$$

Adding (3.8) and (3.9), we find

$$\begin{aligned}-\frac{d}{dt}\left(\varepsilon\int_{\mathbb{R}^2}p|\mathbf{v}|^2d\mathbf{x}\right)=&-\varepsilon\int_{\mathbb{R}^2}|\mathbf{v}|^2\nabla\cdot(p\mathbf{v})\,d\mathbf{x}-\varepsilon\bar{p}\int_{\mathbb{R}^2}|\mathbf{v}|^2\nabla\cdot\mathbf{v}\,d\mathbf{x}-\varepsilon\int_{\mathbb{R}^2}|\mathbf{v}|^2\Delta p\,d\mathbf{x}\\&-2\varepsilon\int_{\mathbb{R}^2}p\mathbf{v}\cdot\nabla p\,d\mathbf{x}-2\varepsilon^2\int_{\mathbb{R}^2}p\mathbf{v}\cdot\Delta\mathbf{v}\,d\mathbf{x}+2\varepsilon^2\int_{\mathbb{R}^2}p\mathbf{v}\cdot\nabla(|\mathbf{v}|^2)\,d\mathbf{x}.\end{aligned} \tag{3.10}$$

Adding (3.10) to (3.7), we have

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\|p\|^2+\frac{\bar{p}}{2}\|\mathbf{v}\|^2-\varepsilon\int_{\mathbb{R}^2}p|\mathbf{v}|^2d\mathbf{x}\right)+\|\nabla p\|^2+\varepsilon\bar{p}\|\nabla\cdot\mathbf{v}\|^2\\&=-(2\varepsilon+1)\int_{\mathbb{R}^2}p\mathbf{v}\cdot\nabla p\,d\mathbf{x}+\varepsilon(2\varepsilon+1)\int_{\mathbb{R}^2}p\mathbf{v}\cdot\nabla(|\mathbf{v}|^2)\,d\mathbf{x}\\&\quad-\varepsilon\int_{\mathbb{R}^2}|\mathbf{v}|^2\Delta p\,d\mathbf{x}-2\varepsilon^2\int_{\mathbb{R}^2}p\mathbf{v}\cdot\Delta\mathbf{v}\,d\mathbf{x}.\end{aligned} \tag{3.11}$$

Taking the L^2 inner product of the first equation in (1.11) with $-(2\varepsilon+1)p^2$, we have

$$\begin{aligned}&-\frac{d}{dt}\left(\frac{2\varepsilon+1}{6}\int_{\mathbb{R}^2}p^3d\mathbf{x}\right)-(2\varepsilon+1)\int_{\mathbb{R}^2}p|\nabla p|^2d\mathbf{x}\\&=(2\varepsilon+1)\int_{\mathbb{R}^2}p\mathbf{v}\cdot\nabla p\,d\mathbf{x}+(2\varepsilon+1)\bar{p}\int_{\mathbb{R}^2}p^2\mathbf{v}\cdot\nabla p\,d\mathbf{x},\end{aligned} \tag{3.12}$$

where the first term on the RHS terminates the first term on the RHS of (3.11) upon addition. Adding (3.12) to (3.11), we find

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\|p\|^2+\frac{\bar{p}}{2}\|\mathbf{v}\|^2-\varepsilon\int_{\mathbb{R}^2}p|\mathbf{v}|^2d\mathbf{x}-\frac{2\varepsilon+1}{6}\int_{\mathbb{R}^2}p^3d\mathbf{x}\right)\\&\quad+\|\nabla p\|^2+\varepsilon\bar{p}\|\nabla\cdot\mathbf{v}\|^2-(2\varepsilon+1)\int_{\mathbb{R}^2}p|\nabla p|^2d\mathbf{x}\\&=(2\varepsilon+1)\bar{p}\int_{\mathbb{R}^2}p^2\mathbf{v}\cdot\nabla p\,d\mathbf{x}+\varepsilon(2\varepsilon+1)\int_{\mathbb{R}^2}p\mathbf{v}\cdot\nabla(|\mathbf{v}|^2)\,d\mathbf{x}\\&\quad-\varepsilon\int_{\mathbb{R}^2}|\mathbf{v}|^2\Delta p\,d\mathbf{x}-2\varepsilon^2\int_{\mathbb{R}^2}p\mathbf{v}\cdot\Delta\mathbf{v}\,d\mathbf{x}.\end{aligned} \tag{3.13}$$

We note that the expression inside the parenthesis on the LHS of (3.13) is not necessarily positive. Hence, we need to supply terms in order to gain the positivity of the quantity.

Step 2. First, for any positive constant k_1, taking the L^2 inner product of the first equation in (1.11) with k_1p^3, we have

$$\frac{d}{dt}\left(k_1\int_{\mathbb{R}^2}p^4d\mathbf{x}\right)+12k_1\int_{\mathbb{R}^2}p^2|\nabla p|^2d\mathbf{x}=-12k_1\bar{p}\int_{\mathbb{R}^2}p^2\mathbf{v}\cdot\nabla p\,d\mathbf{x}-12k_1\int_{\mathbb{R}^2}p^3\mathbf{v}\cdot\nabla p\,d\mathbf{x}. \tag{3.14}$$

Adding (3.14) to (3.13), we have

$$\frac{d}{dt}E_1(t)+D_1(t)=(2\varepsilon+1-12k_1)\bar{p}\int_{\mathbb{R}^2}p^2\mathbf{v}\cdot\nabla p\;d\mathbf{x}+\varepsilon(2\varepsilon+1)\int_{\mathbb{R}^2}p\mathbf{v}\cdot\nabla(|\mathbf{v}|^2)\;d\mathbf{x}$$
$$-\varepsilon\int_{\mathbb{R}^2}|\mathbf{v}|^2\Delta p\;d\mathbf{x}-2\varepsilon^2\int_{\mathbb{R}^2}p\mathbf{v}\cdot\Delta\mathbf{v}\;d\mathbf{x}-12k_1\int_{\mathbb{R}^2}p^3\mathbf{v}\cdot\nabla p\;d\mathbf{x},\quad(3.15)$$

where

$$E_1(t)\equiv\frac{1}{2}\|p\|^2+\frac{\bar{p}}{2}\|\mathbf{v}\|^2-\varepsilon\int_{\mathbb{R}^2}p|\mathbf{v}|^2d\mathbf{x}-\frac{2\varepsilon+1}{6}\int_{\mathbb{R}^2}p^3d\mathbf{x}+k_1\int_{\mathbb{R}^2}p^4d\mathbf{x},$$
$$D_1(t)\equiv\|\nabla p\|^2+\varepsilon\bar{p}\|\nabla\cdot\mathbf{v}\|^2-(2\varepsilon+1)\int_{\mathbb{R}^2}p|\nabla p|^2d\mathbf{x}+12k_1\int_{\mathbb{R}^2}p^2|\nabla p|^2d\mathbf{x}.$$

Second, by taking the L^2 inner product of the first equation in (1.11) with $2\varepsilon^2|\mathbf{v}|^2p$, we have

$$\int_{\mathbb{R}^2}\varepsilon^2|\mathbf{v}|^2\partial_t(p^2)=2\varepsilon^2\int_{\mathbb{R}^2}|\mathbf{v}|^2p\nabla\cdot(p\mathbf{v})\;d\mathbf{x}+2\varepsilon^2\bar{p}\int_{\mathbb{R}^2}|\mathbf{v}|^2p\nabla\cdot\mathbf{v}\;d\mathbf{x}+2\varepsilon^2\int_{\mathbb{R}^2}|\mathbf{v}|^2p\Delta p\;d\mathbf{x}.\quad(3.16)$$

Taking the L^2 inner product of the second equation in (1.11) with $2\varepsilon^2p^2\mathbf{v}$, we have

$$\int_{\mathbb{R}^2}\varepsilon^2p^2\partial_t(|\mathbf{v}|^2)=2\varepsilon^2\int_{\mathbb{R}^2}p^2\mathbf{v}\cdot\nabla p\;d\mathbf{x}+2\varepsilon^3\int_{\mathbb{R}^2}p^2\mathbf{v}\cdot\Delta\mathbf{v}\;d\mathbf{x}-2\varepsilon^3\int_{\mathbb{R}^2}p^2\mathbf{v}\cdot\nabla(|\mathbf{v}|^2)\;d\mathbf{x}.\quad(3.17)$$

Adding (3.16) and (3.17), we find

$$\frac{d}{dt}\left(\int_{\mathbb{R}^2}\varepsilon^2p^2|\mathbf{v}|^2d\mathbf{x}\right)$$
$$=2\varepsilon^2\int_{\mathbb{R}^2}|\mathbf{v}|^2p\nabla\cdot(p\mathbf{v})\;d\mathbf{x}+2\varepsilon^2\bar{p}\int_{\mathbb{R}^2}|\mathbf{v}|^2p\nabla\cdot\mathbf{v}\;d\mathbf{x}+2\varepsilon^2\int_{\mathbb{R}^2}|\mathbf{v}|^2p\Delta p\;d\mathbf{x}$$
$$+2\varepsilon^2\int_{\mathbb{R}^2}p^2\mathbf{v}\cdot\nabla p\;d\mathbf{x}+2\varepsilon^3\int_{\mathbb{R}^2}p^2\mathbf{v}\cdot\Delta\mathbf{v}\;d\mathbf{x}-2\varepsilon^3\int_{\mathbb{R}^2}p^2\mathbf{v}\cdot\nabla(|\mathbf{v}|^2)\;d\mathbf{x}.\quad(3.18)$$

We note that the third term on the RHS of (3.18):

$$2\varepsilon^2\int_{\mathbb{R}^2}|\mathbf{v}|^2p\Delta p\;d\mathbf{x}=-2\varepsilon^2\int_{\mathbb{R}^2}|\mathbf{v}|^2|\nabla p|^2d\mathbf{x}-2\varepsilon^2\int_{\mathbb{R}^2}p\nabla(|\mathbf{v}|^2)\cdot\nabla p\;d\mathbf{x},\quad(3.19)$$

and the fifth term:

$$2\varepsilon^3\int_{\mathbb{R}^2}p^2\mathbf{v}\cdot\Delta\mathbf{v}\;d\mathbf{x}$$
$$=-2\varepsilon^3\int_{\mathbb{R}^2}p^2\left(|\nabla v_1|^2+|\nabla v_2|^2\right)d\mathbf{x}-4\varepsilon^3\int_{\mathbb{R}^2}(pv_1\nabla p\cdot\nabla v_1+pv_2\nabla p\cdot\nabla v_2)\,d\mathbf{x}.\quad(3.20)$$

Plugging (3.19) and (3.20) into (3.18), then multiplying the result by 2, we have

$$\frac{d}{dt}\left(\int_{\mathbb{R}^2}2\varepsilon^2p^2|\mathbf{v}|^2d\mathbf{x}\right)+4\varepsilon^2\int_{\mathbb{R}^2}|\mathbf{v}|^2|\nabla p|^2d\mathbf{x}+4\varepsilon^3\int_{\mathbb{R}^2}p^2\left(|\nabla v_1|^2+|\nabla v_2|^2\right)d\mathbf{x}$$
$$=4\varepsilon^2\int_{\mathbb{R}^2}|\mathbf{v}|^2p\nabla\cdot(p\mathbf{v})\;d\mathbf{x}+4\varepsilon^2\bar{p}\int_{\mathbb{R}^2}|\mathbf{v}|^2p\nabla\cdot\mathbf{v}\;d\mathbf{x}-4\varepsilon^2\int_{\mathbb{R}^2}p\nabla(|\mathbf{v}|^2)\cdot\nabla p\;d\mathbf{x}$$

$$+4\varepsilon^2\int_{\mathbb{R}^2} p^2\mathbf{v}\cdot\nabla p\ d\mathbf{x}-4\varepsilon^3\int_{\mathbb{R}^2} p^2\mathbf{v}\cdot\nabla(|\mathbf{v}|^2)\ d\mathbf{x}-8\varepsilon^3\int_{\mathbb{R}^2}(pv_1\nabla p\cdot\nabla v_1+pv_2\nabla p\cdot\nabla v_2)\,d\mathbf{x}. \tag{3.21}$$

Next, we carry out some preliminary energy estimates for both the RHS of (3.15) and (3.21).

Step 3. For the third term on the RHS of (3.15), we have

$$\begin{aligned}\left|\varepsilon\int_{\mathbb{R}^2}|\mathbf{v}|^2\Delta p\ d\mathbf{x}\right| &= \left|\varepsilon\int_{\mathbb{R}^2}\nabla\left(|\mathbf{v}|^2\right)\cdot\nabla p\ d\mathbf{x}\right| \\ &\le \frac{1}{4}\|\nabla p\|^2+8\varepsilon^2\int_{\mathbb{R}^2}\left((v_1)^2|\nabla v_1|^2+(v_2)^2|\nabla v_2|^2\right)d\mathbf{x}. \end{aligned}\tag{3.22}$$

For the fourth term on the RHS of (3.15), we have

$$\begin{aligned}&\left|2\varepsilon^2\int_{\mathbb{R}^2} p\mathbf{v}\cdot\Delta\mathbf{v}\ d\mathbf{x}\right| \\ &= \left|2\varepsilon^2\int_{\mathbb{R}^2} p\left(|\nabla v_1|^2+|\nabla v_2|^2\right)d\mathbf{x}+4\varepsilon^2\int_{\mathbb{R}^2}(v_1\nabla p\cdot\nabla v_1+v_2\nabla p\cdot\nabla v_2)\,d\mathbf{x}\right| \\ &\le \frac{\varepsilon\bar{p}}{2}\|\nabla\cdot\mathbf{v}\|^2+\frac{2\varepsilon^3}{\bar{p}}\int_{\mathbb{R}^2} p^2\left(|\nabla v_1|^2+|\nabla v_2|^2\right)d\mathbf{x}+\frac{1}{4}\|\nabla p\|^2 \\ &\quad +32\varepsilon^4\int_{\mathbb{R}^2}\left((v_1)^2|\nabla v_1|^2+(v_2)^2|\nabla v_2|^2\right)d\mathbf{x}. \end{aligned}\tag{3.23}$$

Similarly, for the third term on the RHS of (3.21), we can show that

$$\begin{aligned}\left|4\varepsilon^2\int_{\mathbb{R}^2} p\nabla(|\mathbf{v}|^2)\cdot\nabla p\ d\mathbf{x}\right| &= \left|8\varepsilon^2\int_{\mathbb{R}^2}(pv_1\nabla p\cdot\nabla v_1+pv_2\nabla p\cdot\nabla v_2)\,d\mathbf{x}\right| \\ &\le 8\varepsilon^2\int_{\mathbb{R}^2} p^2|\nabla p|^2 d\mathbf{x}+4\varepsilon^2\int_{\mathbb{R}^2}\left((v_1)^2|\nabla v_1|^2+(v_2)^2|\nabla v_2|^2\right)d\mathbf{x}. \end{aligned}\tag{3.24}$$

In the same way, for the sixth term on the RHS of (3.21), we have

$$\begin{aligned}&\left|8\varepsilon^3\int_{\mathbb{R}^2}(pv_1\nabla p\cdot\nabla v_1+pv_2\nabla p\cdot\nabla v_2)\,d\mathbf{x}\right| \\ &\le 8\varepsilon^3\int_{\mathbb{R}^2} p^2|\nabla p|^2 d\mathbf{x}+4\varepsilon^3\int_{\mathbb{R}^2}\left((v_1)^2|\nabla v_1|^2+(v_2)^2|\nabla v_2|^2\right)d\mathbf{x}. \end{aligned}\tag{3.25}$$

By feeding (3.22) and (3.23) into (3.15), we have

$$\begin{aligned}\frac{d}{dt}E_1(t)+D_1(t) &\le (2\varepsilon+1-12k_1)\int_{\mathbb{R}^2} p^2\mathbf{v}\cdot\nabla p\ d\mathbf{x}+\varepsilon(2\varepsilon+1)\int_{\mathbb{R}^2} p\mathbf{v}\cdot\nabla(|\mathbf{v}|^2)\ d\mathbf{x} \\ &\quad -12k_1\int_{\mathbb{R}^2} p^3\mathbf{v}\cdot\nabla p\ d\mathbf{x}+\frac{\varepsilon\bar{p}}{2}\|\nabla\cdot\mathbf{v}\|^2+\frac{2\varepsilon^3}{\bar{p}}\int_{\mathbb{R}^2} p^2\left(|\nabla v_1|^2+|\nabla v_2|^2\right)d\mathbf{x} \\ &\quad +\frac{1}{2}\|\nabla p\|^2+\left(8\varepsilon^2+32\varepsilon^4\right)\int_{\mathbb{R}^2}\left((v_1)^2|\nabla v_1|^2+(v_2)^2|\nabla v_2|^2\right)d\mathbf{x}. \end{aligned}\tag{3.26}$$

By feeding (3.24) and (3.25) into (3.21), we have

$$\frac{d}{dt}\left(\int_{\mathbb{R}^2} 2\varepsilon^2 p^2|\mathbf{v}|^2 d\mathbf{x}\right)+4\varepsilon^2\int_{\mathbb{R}^2}|\mathbf{v}|^2|\nabla p|^2 d\mathbf{x}+4\varepsilon^3\int_{\mathbb{R}^2} p^2\left(|\nabla v_1|^2+|\nabla v_2|^2\right)d\mathbf{x}$$

$$
\begin{aligned}
&\le 4\varepsilon^2\int_{\mathbb{R}^2}|\mathbf{v}|^2p\nabla\cdot(p\mathbf{v})\ d\mathbf{x}+4\varepsilon^2\int_{\mathbb{R}^2}|\mathbf{v}|^2p\nabla\cdot\mathbf{v}\ d\mathbf{x}+4\varepsilon^2\int_{\mathbb{R}^2}p^2\mathbf{v}\cdot\nabla p\ d\mathbf{x}\\
&\quad -4\varepsilon^3\int_{\mathbb{R}^2}p^2\mathbf{v}\cdot\nabla(|\mathbf{v}|^2)\ d\mathbf{x}+\left(8\varepsilon^2+8\varepsilon^3\right)\int_{\mathbb{R}^2}p^2|\nabla p|^2d\mathbf{x}\\
&\quad +\left(4\varepsilon^2+4\varepsilon^3\right)\int_{\mathbb{R}^2}\left((v_1)^2|\nabla v_1|^2+(v_2)^2|\nabla v_2|^2\right)d\mathbf{x}. \qquad (3.27)
\end{aligned}
$$

By adding (3.27)$\times\frac{1}{\bar p}$ to (3.26), we find

$$
\begin{aligned}
\frac{d}{dt}E_2(t)+D_2(t)&\le(2\varepsilon+1-12k_1)\int_{\mathbb{R}^2}p^2\mathbf{v}\cdot\nabla p\ d\mathbf{x}+\varepsilon(2\varepsilon+1)\int_{\mathbb{R}^2}p\mathbf{v}\cdot\nabla(|\mathbf{v}|^2)\ d\mathbf{x}\\
&\quad -12k_1\int_{\mathbb{R}^2}p^3\mathbf{v}\cdot\nabla p\ d\mathbf{x}+\frac{4\varepsilon^2}{\bar p}\int_{\mathbb{R}^2}|\mathbf{v}|^2p\nabla\cdot(p\mathbf{v})\ d\mathbf{x}+\frac{4\varepsilon^2}{\bar p}\int_{\mathbb{R}^2}|\mathbf{v}|^2p\nabla\cdot\mathbf{v}\ d\mathbf{x}\\
&\quad +\frac{4\varepsilon^2}{\bar p}\int_{\mathbb{R}^2}p^2\mathbf{v}\cdot\nabla p\ d\mathbf{x}-\frac{4\varepsilon^3}{\bar p}\int_{\mathbb{R}^2}p^2\mathbf{v}\cdot\nabla(|\mathbf{v}|^2)\ d\mathbf{x}+\frac{\left(8\varepsilon^2+8\varepsilon^3\right)}{\bar p}\int_{\mathbb{R}^2}p^2|\nabla p|^2d\mathbf{x}\\
&\quad +\left(\frac{\left(4\varepsilon^2+4\varepsilon^3\right)}{\bar p}+8\varepsilon^2+32\varepsilon^4\right)\int_{\mathbb{R}^2}\left((v_1)^2|\nabla v_1|^2+(v_2)^2|\nabla v_2|^2\right)d\mathbf{x}, \qquad (3.28)
\end{aligned}
$$

where

$$
\begin{aligned}
E_2(t)&\equiv\frac{1}{2}\|p\|^2+\frac{\bar p}{2}\|\mathbf{v}\|^2-\int_{\mathbb{R}^2}\left(\varepsilon p|\mathbf{v}|^2+\frac{2\varepsilon+1}{6}p^3\right)d\mathbf{x}+k_1\int_{\mathbb{R}^2}p^4d\mathbf{x}+\frac{2\varepsilon^2}{\bar p}\int_{\mathbb{R}^2}p^2|\mathbf{v}|^2d\mathbf{x},\\
D_2(t)&\equiv\frac{1}{2}\|\nabla p\|^2+\frac{\varepsilon\bar p}{2}\|\nabla\cdot\mathbf{v}\|^2-(2\varepsilon+1)\int_{\mathbb{R}^2}p|\nabla p|^2d\mathbf{x}+12k_1\int_{\mathbb{R}^2}p^2|\nabla p|^2d\mathbf{x}\\
&\quad +\frac{4\varepsilon^2}{\bar p}\int_{\mathbb{R}^2}|\mathbf{v}|^2|\nabla p|^2d\mathbf{x}+\frac{2\varepsilon^3}{\bar p}\int_{\mathbb{R}^2}p^2\left(|\nabla v_1|^2+|\nabla v_2|^2\right)d\mathbf{x}.
\end{aligned}
$$

We observe that the last term on the RHS of (3.28) is troublesome, as the expression of $D_2(t)$ contains only the square of $\nabla\mathbf{v}$. Next, we cook up a quantity to dominate such a bad term.

Step 4. We write the second equation in (1.11) in the component form as

$$
\begin{aligned}
&\partial_t v_1-\partial_x p=\varepsilon\Delta v_1-\varepsilon\partial_x(|\mathbf{v}|^2),\\
&\partial_t v_2-\partial_y p=\varepsilon\Delta v_2-\varepsilon\partial_y(|\mathbf{v}|^2).
\end{aligned} \qquad (3.29)
$$

For any positive constant k_2, taking the L^2 inner product of the first equation of (3.29) with $4k_2(v_1)^3$, the second equation with $4k_2(v_2)^3$, then adding the results, we can show that

$$
\begin{aligned}
&\frac{d}{dt}\left(k_2\int_{\mathbb{R}^2}(v_1)^4+(v_2)^4d\mathbf{x}\right)+12k_2\varepsilon\int_{\mathbb{R}^2}\left((v_1)^2|\nabla v_1|^2+(v_2)^2|\nabla v_2|^2\right)d\mathbf{x}\\
&=4k_2\int_{\mathbb{R}^2}\left((v_1)^3\partial_x p+(v_2)^3\partial_y p\right)d\mathbf{x}-4k_2\varepsilon\int_{\mathbb{R}^2}\left((v_1)^3\partial_x(|\mathbf{v}|^2)+(v_2)^3\partial_y(|\mathbf{v}|^2)\right)d\mathbf{x}. \qquad (3.30)
\end{aligned}
$$

By adding (3.30) to (3.28), we have

$$
\frac{d}{dt}E_3(t)+D_3(t)\le\left(2\varepsilon+1-12k_1+\frac{4\varepsilon^2}{\bar p}\right)\int_{\mathbb{R}^2}p^2\mathbf{v}\cdot\nabla p\ d\mathbf{x}+\varepsilon(2\varepsilon+1)\int_{\mathbb{R}^2}p\mathbf{v}\cdot\nabla(|\mathbf{v}|^2)\ d\mathbf{x}
$$

$$
\begin{aligned}
&-12k_1\int_{\mathbb{R}^2}p^3\mathbf{v}\cdot\nabla p\ d\mathbf{x}+\frac{4\varepsilon^2}{\bar{p}}\int_{\mathbb{R}^2}|\mathbf{v}|^2p\nabla\cdot(p\mathbf{v})\ d\mathbf{x}+\frac{4\varepsilon^2}{\bar{p}}\int_{\mathbb{R}^2}|\mathbf{v}|^2p\nabla\cdot\mathbf{v}\ d\mathbf{x}\\
&-\frac{4\varepsilon^3}{\bar{p}}\int_{\mathbb{R}^2}p^2\mathbf{v}\cdot\nabla(|\mathbf{v}|^2)\ d\mathbf{x}+4k_2\int_{\mathbb{R}^2}\big((v_1)^3\partial_xp+(v_2)^3\partial_yp\big)\,d\mathbf{x}\\
&-4k_2\varepsilon\int_{\mathbb{R}^2}\big((v_1)^3\partial_x(|\mathbf{v}|^2)+(v_2)^3\partial_y(|\mathbf{v}|^2)\big)\,d\mathbf{x}\ \equiv\ \sum_{k=1}^{8}I_k(t),
\end{aligned}
\tag{3.31}
$$

where

$$
\begin{aligned}
E_3(t)\equiv\ &\frac{1}{2}\|p\|^2+\frac{\bar{p}}{2}\|\mathbf{v}\|^2-\varepsilon\int_{\mathbb{R}^2}p|\mathbf{v}|^2d\mathbf{x}-\frac{2\varepsilon+1}{6}\int_{\mathbb{R}^2}p^3d\mathbf{x}+k_1\int_{\mathbb{R}^2}p^4d\mathbf{x}\\
&+\frac{2\varepsilon^2}{\bar{p}}\int_{\mathbb{R}^2}p^2|\mathbf{v}|^2d\mathbf{x}+k_2\int_{\mathbb{R}^2}(v_1)^4+(v_2)^4d\mathbf{x},\\
D_3(t)\equiv\ &\frac{1}{2}\|\nabla p\|^2+\frac{\varepsilon\bar{p}}{2}\|\nabla\cdot\mathbf{v}\|^2-(2\varepsilon+1)\int_{\mathbb{R}^2}p|\nabla p|^2d\mathbf{x}+\frac{4\varepsilon^2}{\bar{p}}\int_{\mathbb{R}^2}|\mathbf{v}|^2|\nabla p|^2d\mathbf{x}\\
&+\left(12k_1-\frac{(8\varepsilon^2+8\varepsilon^3)}{\bar{p}}\right)\int_{\mathbb{R}^2}p^2|\nabla p|^2d\mathbf{x}+\frac{2\varepsilon^3}{\bar{p}}\int_{\mathbb{R}^2}p^2\left(|\nabla v_1|^2+|\nabla v_2|^2\right)d\mathbf{x}\\
&+\left(12k_2\varepsilon-\left[\frac{(4\varepsilon^2+4\varepsilon^3)}{\bar{p}}+8\varepsilon^2+32\varepsilon^4\right]\right)\int_{\mathbb{R}^2}\left((v_1)^2|\nabla v_1|^2+(v_2)^2|\nabla v_2|^2\right)d\mathbf{x}.
\end{aligned}
$$

By choosing

$$
12k_1=\frac{8\varepsilon^2+8\varepsilon^3}{\bar{p}}+3(2\varepsilon+1)^2,\qquad 12k_2=\frac{4\varepsilon+4\varepsilon^2}{\bar{p}}+9\varepsilon+32\varepsilon^3,
\tag{3.32}
$$

we have

$$
\begin{aligned}
E_3(t)=\ &\frac{1}{4}\|p\|^2+\left\|\frac{p}{2}-\frac{2\varepsilon+1}{6}p^2\right\|^2+\left(\frac{2(2\varepsilon+1)^2}{9}+\frac{8\varepsilon^2+8\varepsilon^3}{12\bar{p}}\right)\|p\|_{L^4}^4\\
&+\frac{\bar{p}}{4}\|\mathbf{v}\|^2+\frac{\bar{p}}{4}\left\|\left(1-\frac{2\varepsilon}{\bar{p}}p\right)\mathbf{v}\right\|^2+\frac{\varepsilon^2}{\bar{p}}\|p\mathbf{v}\|^2+\frac{1}{12}\left(\frac{4\varepsilon+4\varepsilon^2}{\bar{p}}+9\varepsilon+32\varepsilon^3\right)\|\mathbf{v}\|_{L^4}^4,\\
D_3(t)=\ &\frac{3}{8}\|\nabla p\|^2+\frac{1}{8}\|(1-4(2\varepsilon+1)p)\nabla p\|^2+(2\varepsilon+1)^2\|p\nabla p\|^2\\
&+\frac{\varepsilon\bar{p}}{2}\|\nabla\cdot\mathbf{v}\|^2+\frac{4\varepsilon^2}{\bar{p}}\||\mathbf{v}||\nabla p|\|^2+\frac{2\varepsilon^3}{\bar{p}}\|p\nabla\mathbf{v}\|^2+\varepsilon^2\left(\|v_1\nabla v_1\|^2+\|v_2\nabla v_2\|^2\right).
\end{aligned}
\tag{3.33}
$$

Note that $k_1=\frac{1}{4}, k_2=0$ when $\varepsilon=0$, and $E_3(0)$ is the same as H_1 in the statement of Theorem 1.2. Next, we carry out energy estimates for the RHS of (3.31) by applying (3.1)–(3.5).

Step 5. By using (3.1), (3.3) and (3.6), we can show that

$$
\begin{aligned}
|I_1(t)|&=\left|\left(2\varepsilon+1-12k_1+\frac{4\varepsilon^2}{\bar{p}}\right)\int_{\mathbb{R}^2}p^2\mathbf{v}\cdot\nabla p\ d\mathbf{x}\right|\\
&\le\left(2\varepsilon+1+12k_1+\frac{4\varepsilon^2}{\bar{p}}\right)\|p\|_{L^8}^2\|\mathbf{v}\|_{L^4}\|\nabla p\|\\
&\le d_1d_3^2\left(2\varepsilon+1+12k_1+\frac{4\varepsilon^2}{\bar{p}}\right)\|\nabla p\|^{\frac{3}{2}}\|p\|^{\frac{1}{2}}\|\nabla\cdot\mathbf{v}\|^{\frac{1}{2}}\|\mathbf{v}\|^{\frac{1}{2}}\|\nabla p\|
\end{aligned}
$$

$$\leq d_1 d_3^2 \left(2\varepsilon+1+12k_1+\frac{4\varepsilon^2}{\bar{p}}\right)(M_1\delta)^{\frac{1}{2}}\|\nabla p\|^2. \tag{3.34}$$

Similarly, we can show that

$$\begin{aligned}
|I_2(t)| &= \left|\varepsilon(2\varepsilon+1)\int_{\mathbb{R}^2} p\mathbf{v}\cdot\nabla(|\mathbf{v}|^2)\ d\mathbf{x}\right| \leq 2(2\varepsilon+1)\varepsilon\|p\|_{L^4}\|\mathbf{v}\|_{L^8}^2\|\nabla\cdot\mathbf{v}\| \\
&\leq 2d_1d_3^2(2\varepsilon+1)\varepsilon\|\nabla p\|^{\frac{1}{2}}\|p\|^{\frac{1}{2}}\|\nabla\cdot\mathbf{v}\|^{\frac{3}{2}}\|\mathbf{v}\|^{\frac{1}{2}}\|\nabla\cdot\mathbf{v}\| \\
&\leq 2d_1d_3^2(2\varepsilon+1)(M_1\delta)^{\frac{1}{2}}(\varepsilon\|\nabla\cdot\mathbf{v}\|^2).
\end{aligned} \tag{3.35}$$

By using (3.4), we can show that

$$\begin{aligned}
|I_3(t)| &= \left|12k_1\int_{\mathbb{R}^2} p^3\mathbf{v}\cdot\nabla p\ d\mathbf{x}\right| \leq 12k_1\|p\|_{L^{12}}^3\|\mathbf{v}\|_{L^4}\|\nabla p\| \\
&\leq 12d_1d_4^2k_1\|\nabla p\|^{\frac{5}{2}}\|p\|^{\frac{1}{2}}\|\nabla\cdot\mathbf{v}\|^{\frac{1}{2}}\|\mathbf{v}\|^{\frac{1}{2}}\|\nabla p\| \\
&\leq 12d_1d_4^2k_1(M_1\delta^2)^{\frac{1}{2}}\|\nabla p\|^2 = d_1d_4^2\left(\frac{8\varepsilon^2+8\varepsilon^3}{\bar{p}}+3(2\varepsilon+1)^2\right)(M_1\delta^2)^{\frac{1}{2}}\|\nabla p\|^2.
\end{aligned} \tag{3.36}$$

In a similar fashion, we can show that

$$\begin{aligned}
|I_4(t)| &= \left|\frac{4\varepsilon^2}{\bar{p}}\int_{\mathbb{R}^2}|\mathbf{v}|^2 p\nabla\cdot(p\mathbf{v})\ d\mathbf{x}\right| \leq \frac{4\varepsilon^2}{\bar{p}}\left(\|\mathbf{v}\|_{L^8}^2\|p\|_{L^8}^2\|\nabla\cdot\mathbf{v}\| + \|\mathbf{v}\|_{L^{12}}^3\|p\|_{L^4}\|\nabla p\|\right) \\
&\leq \frac{4\varepsilon^2}{\bar{p}}\left(d_3^4\|\nabla\cdot\mathbf{v}\|^{\frac{3}{2}}\|\mathbf{v}\|^{\frac{1}{2}}\|\nabla p\|^{\frac{3}{2}}\|p\|^{\frac{1}{2}}\|\nabla\cdot\mathbf{v}\| + d_1d_4^3\|\nabla\cdot\mathbf{v}\|^{\frac{5}{2}}\|\mathbf{v}\|^{\frac{1}{2}}\|\nabla p\|^{\frac{1}{2}}\|p\|^{\frac{1}{2}}\|\nabla p\|\right) \\
&\leq \frac{4(d_3^4+d_1d_4^3)\varepsilon}{\bar{p}}(M_1\delta^2)^{\frac{1}{2}}(\varepsilon\|\nabla\cdot\mathbf{v}\|^2).
\end{aligned} \tag{3.37}$$

Similar to (3.34), we can show that

$$|I_5(t)| = \left|\frac{4\varepsilon^2}{\bar{p}}\int_{\mathbb{R}^2}|\mathbf{v}|^2 p\nabla\cdot\mathbf{v}\ d\mathbf{x}\right| \leq \frac{4d_1d_3^2\varepsilon}{\bar{p}}(M_1\delta)^{\frac{1}{2}}(\varepsilon\|\nabla\cdot\mathbf{v}\|^2). \tag{3.38}$$

Similar to (3.37), we can show that

$$|I_6(t)| = \left|\frac{4\varepsilon^3}{\bar{p}}\int_{\mathbb{R}^2}p^2\mathbf{v}\cdot\nabla(|\mathbf{v}|^2)\ d\mathbf{x}\right| \leq \frac{8d_3^4\varepsilon^2}{\bar{p}}(M_1\delta^2)^{\frac{1}{2}}(\varepsilon\|\nabla\cdot\mathbf{v}\|^2). \tag{3.39}$$

By using (3.2), we can show that

$$\begin{aligned}
|I_7(t)| &= \left|4k_2\int_{\mathbb{R}^2}\left((v_1)^3\partial_x p+(v_2)^3\partial_y p\right)d\mathbf{x}\right| \leq 8k_2\|\mathbf{v}\|_{L^6}^3\|\nabla p\| \\
&\leq 8d_2^3k_2\|\nabla\cdot\mathbf{v}\|^2\|\mathbf{v}\|\|\nabla p\| \leq \frac{2}{3}d_2^3\left(\frac{4+4\varepsilon}{\bar{p}}+9+32\varepsilon^2\right)(M_1\delta)^{\frac{1}{2}}(\varepsilon\|\nabla\cdot\mathbf{v}\|^2).
\end{aligned} \tag{3.40}$$

Similarly, we have

$$\begin{aligned}
|I_8(t)| &= \left|4k_2\varepsilon\int_{\mathbb{R}^2}\left((v_1)^3\partial_x(|\mathbf{v}|^2)+(v_2)^3\partial_y(|\mathbf{v}|^2)\right)d\mathbf{x}\right| \leq 32k_2\varepsilon\|\mathbf{v}\|_{L^8}^4\|\nabla\cdot\mathbf{v}\| \\
&\leq 32d_3^4k_2\varepsilon\|\nabla\cdot\mathbf{v}\|^3\|\mathbf{v}\|\|\nabla\cdot\mathbf{v}\| \leq 32d_3^4k_2(M_1\delta^2)^{\frac{1}{2}}(\varepsilon\|\nabla\cdot\mathbf{v}\|^2).
\end{aligned} \tag{3.41}$$

This completes the energy estimation for the entire RHS of (3.31).

Step 6. By substituting (3.34)–(3.41) into (3.31), we obtain

$$\frac{d}{dt}E_3(t)+D_3(t)\le(\mathcal{B}_1+\mathcal{B}_3)\|\nabla p\|^2+(\mathcal{B}_2+\mathcal{B}_4+\mathcal{B}_5+\mathcal{B}_6+\mathcal{B}_7+\mathcal{B}_8)(\varepsilon\|\nabla\cdot\mathbf{v}\|^2), \quad (3.42)$$

where

$$\mathcal{B}_1=d_1d_3^2\left(2\varepsilon+1+12k_1+\frac{4\varepsilon^2}{\bar p}\right)(M_1\delta)^{\frac12}, \qquad \mathcal{B}_2=2d_1d_3^2(2\varepsilon+1)(M_1\delta)^{\frac12},$$

$$\mathcal{B}_3=d_1d_4^2\left(\frac{8\varepsilon^2+8\varepsilon^3}{\bar p}+3(2\varepsilon+1)^2\right)(M_1\delta^2)^{\frac12}, \quad \mathcal{B}_4=\frac{4(d_3^4+d_1d_4^3)\varepsilon}{\bar p}(M_1\delta^2)^{\frac12},$$

$$\mathcal{B}_5=\frac{4d_1d_3^2\varepsilon}{\bar p}(M_1\delta)^{\frac12}, \qquad \mathcal{B}_6=\frac{8d_3^4\varepsilon^2}{\bar p}(M_1\delta^2)^{\frac12},$$

$$\mathcal{B}_7=\frac{2}{3}d_2^3\left(\frac{4+4\varepsilon}{\bar p}+9+32\varepsilon^2\right)(M_1\delta)^{\frac12}, \quad \mathcal{B}_8=32d_3^4k_2(M_1\delta^2)^{\frac12}.$$

Hence, when

$$\mathcal{B}_i\le\frac{1}{16}, \quad i=1,3, \quad \mathcal{B}_i \quad \le\frac{\bar p}{24}, \quad i=2,4,5,6,7,8, \quad (3.43)$$

we get from (3.42) that

$$\frac{d}{dt}E_3(t)+\left(D_3(t)-\frac18\|\nabla p\|^2-\frac{\varepsilon\bar p}{4}\|\nabla\cdot\mathbf{v}\|^2\right)\le 0. \quad (3.44)$$

In view of (3.33) we see that

$$D_3(t)-\frac18\|\nabla p\|^2-\frac{\varepsilon\bar p}{4}\|\nabla\cdot\mathbf{v}\|^2\ge\frac14\|\nabla p\|^2+\frac{\varepsilon\bar p}{4}\|\nabla\cdot\mathbf{v}\|^2. \quad (3.45)$$

By integrating (3.44) with respect to time and using the definition of $E_3(t)$ (cf. (3.33)), we find in particular that

$$\frac14\|p(t)\|^2+\frac{\bar p}{4}\|\mathbf{v}(t)\|^2+\int_0^t\left(\frac14\|\nabla p(\tau)\|^2+\frac{\varepsilon\bar p}{4}\|\nabla\cdot\mathbf{v}(\tau)\|^2\right)d\tau\le E_3(0).$$

This implies that

$$\|p(t)\|^2+\|\mathbf{v}(t)\|^2\le 4(1+1/\bar p)E_3(0),$$

and

$$\int_0^t\left(\|\nabla p(\tau)\|^2+\varepsilon\bar p\|\nabla\cdot\mathbf{v}(\tau)\|^2\right)d\tau\le 4E_3(0). \quad (3.46)$$

In view of (1.18) we see that

$$\|p(t)\|^2+\|\mathbf{v}(t)\|^2\le M_1-1. \quad (3.47)$$

Next, we estimate the first order derivatives of the solution.

3.2. H^1-estimate. We remark that the estimates of the first order derivatives of the solution bear the same level of difficulty as those for the zeroth frequency. Hence, we shall continue with the process of cancellation and coupling through higher order estimates.

Step 1. By testing the equations in (1.11) with the $-\Delta$ of the targeting functions, we can show that

$$\begin{aligned}&\frac{d}{dt}\left(\|\nabla p\|^2+\bar{p}\|\nabla\cdot\mathbf{v}\|^2\right)+2\|\Delta p\|^2+2\varepsilon\bar{p}\|\Delta\mathbf{v}\|^2\\ &=-\int_{\mathbb{R}^2}[2p(\nabla\cdot\mathbf{v})+2\nabla p\cdot\mathbf{v}]\Delta p\,d\mathbf{x}+2\varepsilon\bar{p}\int_{\mathbb{R}^2}\nabla(|\mathbf{v}|^2)\cdot(\Delta\mathbf{v})\,d\mathbf{x}.\end{aligned}\tag{3.48}$$

A direct calculation by using the first equation in (1.11) shows that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}p^2\Delta p\,d\mathbf{x}\\ &=\int_{\mathbb{R}^2}p\Delta p[\Delta p+\nabla\cdot(p\mathbf{v})+\bar{p}\nabla\cdot\mathbf{v}]\,d\mathbf{x}+\int_{\mathbb{R}^2}\frac{p^2}{2}\Delta[\Delta p+\nabla\cdot(p\mathbf{v})+\bar{p}\nabla\cdot\mathbf{v}]\,d\mathbf{x}.\end{aligned}\tag{3.49}$$

We note that the second term on the RHS of (3.49)

$$\int_{\mathbb{R}^2}\frac{p^2}{2}\Delta[\Delta p+\nabla\cdot(p\mathbf{v})+\bar{p}\nabla\cdot\mathbf{v}]\,d\mathbf{x}=\int_{\mathbb{R}^2}\left(p\Delta p+|\nabla p|^2\right)[\Delta p+\nabla\cdot(p\mathbf{v})+\bar{p}\nabla\cdot\mathbf{v}]\,d\mathbf{x}.$$

So we update (3.49), after integrating by parts, as

$$-\frac{d}{dt}\int_{\mathbb{R}^2}p|\nabla p|^2\,d\mathbf{x}=\int_{\mathbb{R}^2}\left(2p\Delta p+|\nabla p|^2\right)[\Delta p+\nabla\cdot(p\mathbf{v})+\bar{p}\nabla\cdot\mathbf{v}]\,d\mathbf{x}.\tag{3.50}$$

By summing up (3.48)$\times\bar{p}$ and (3.50), we have

$$\begin{aligned}&\frac{d}{dt}\left(\bar{p}\|\nabla p\|^2+\bar{p}^2\|\nabla\cdot\mathbf{v}\|^2-\int_{\mathbb{R}^2}p|\nabla p|^2\,d\mathbf{x}\right)\\ &\quad+2\bar{p}\|\Delta p\|^2-2\int_{\mathbb{R}^2}p(\Delta p)^2d\mathbf{x}+2\varepsilon\bar{p}^2\|\Delta\mathbf{v}\|^2=N_1(t),\end{aligned}\tag{3.51}$$

where

$$\begin{aligned}N_1(t)=&-2\bar{p}\int_{\mathbb{R}^2}(\nabla p\cdot\mathbf{v})\Delta p\,d\mathbf{x}+2\int_{\mathbb{R}^2}p\Delta p\nabla\cdot(p\mathbf{v})\,d\mathbf{x}+\int_{\mathbb{R}^2}|\nabla p|^2\Delta p\,d\mathbf{x}\\ &+\int_{\mathbb{R}^2}|\nabla p|^2\nabla\cdot(p\mathbf{v})\,d\mathbf{x}+\bar{p}\int_{\mathbb{R}^2}|\nabla p|^2\nabla\cdot\mathbf{v}\,d\mathbf{x}+2\varepsilon\bar{p}^2\int_{\mathbb{R}^2}\nabla(|\mathbf{v}|^2)\cdot(\Delta\mathbf{v})\,d\mathbf{x}.\end{aligned}\tag{3.52}$$

We note that the expression inside the parenthesis on the LHS of (3.51) is not necessarily positive. Hence, we need to supply a positive term in order to gain the positivity of the quantity. For this purpose, a direct calculation by using the first equation in (1.11) shows that

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^2}p^2|\nabla p|^2\,d\mathbf{x}+2\int_{\mathbb{R}^2}p^2|\Delta p|^2d\mathbf{x}\\ &=-2\int_{\mathbb{R}^2}p^2\Delta p[\nabla\cdot(p\mathbf{v})+\bar{p}\nabla\cdot\mathbf{v}]\,d\mathbf{x}-2\int_{\mathbb{R}^2}p|\nabla p|^2[\Delta p+\nabla\cdot(p\mathbf{v})+\bar{p}\nabla\cdot\mathbf{v}]\,d\mathbf{x}.\end{aligned}\tag{3.53}$$

By adding (3.53) to (3.51)×$\bar{p}$, we find

$$\frac{d}{dt}\left(\bar{p}^2\|\nabla p\|^2+\bar{p}^3\|\nabla\cdot\mathbf{v}\|^2-\bar{p}\int_{\mathbb{R}^2}p|\nabla p|^2\,d\mathbf{x}+\int_{\mathbb{R}^2}p^2|\nabla p|^2\,d\mathbf{x}\right)$$
$$+2\bar{p}^2\|\Delta p\|^2-2\bar{p}\int_{\mathbb{R}^2}p(\Delta p)^2d\mathbf{x}+2\varepsilon\bar{p}^3\|\Delta\mathbf{v}\|^2+2\int_{\mathbb{R}^2}p^2|\Delta p|^2d\mathbf{x}=N_2(t), \tag{3.54}$$

where

$$N_2(t)=\bar{p}N_1(t)-2\int_{\mathbb{R}^2}p^2\Delta p\nabla\cdot(p\mathbf{v})d\mathbf{x}-2\bar{p}\int_{\mathbb{R}^2}p^2\Delta p\nabla\cdot\mathbf{v}\,d\mathbf{x}-2\int_{\mathbb{R}^2}p|\nabla p|^2\Delta p\,d\mathbf{x}$$
$$-2\int_{\mathbb{R}^2}p|\nabla p|^2\nabla\cdot(p\mathbf{v})d\mathbf{x}-2\bar{p}\int_{\mathbb{R}^2}p|\nabla p|^2\nabla\cdot\mathbf{v}\,d\mathbf{x}\equiv\sum_{k=1}^{11}J_k(t).$$

Next, we carry out energy estimates for $N_2(t)$.

Step 2. By using (3.1) and Young's inequality, we can show that

$$|J_1(t)|=2\bar{p}^2\left|\int_{\mathbb{R}^2}(\nabla p\cdot\mathbf{v})\Delta p\,d\mathbf{x}\right|\le 2\bar{p}^2\|\nabla p\|_{L^4}\|\mathbf{v}\|_{L^4}\|\Delta p\|$$
$$\le 2d_1^2\bar{p}^2\|\nabla p\|^{\frac12}\|\Delta p\|^{\frac32}\|\mathbf{v}\|^{\frac12}\|\nabla\cdot\mathbf{v}\|^{\frac12}\le\frac{\bar{p}^2}{12}\|\Delta p\|^2+2916d_1^8\bar{p}^2\|\nabla p\|^2\|\mathbf{v}\|^2\|\nabla\cdot\mathbf{v}\|^2$$
$$\le\frac{\bar{p}^2}{12}\|\Delta p\|^2+2916d_1^8\bar{p}^2M_1\|\nabla p\|^2\|\nabla\cdot\mathbf{v}\|^2. \tag{3.55}$$

By using (3.1) and (3.5), we can show that

$$|J_2(t)|=2\bar{p}\left|\int_{\mathbb{R}^2}p\Delta p\nabla\cdot(p\mathbf{v})\,d\mathbf{x}\right|=2\bar{p}\left|\int_{\mathbb{R}^2}p\Delta p(p\nabla\cdot\mathbf{v}+\mathbf{v}\cdot\nabla p)\,d\mathbf{x}\right|$$
$$\le 2\bar{p}\left(\|\Delta p\|\|p\|_{L^\infty}^2\|\nabla\cdot\mathbf{v}\|+\|\Delta p\|\|p\|_{L^\infty}\|\mathbf{v}\|_{L^4}\|\nabla p\|_{L^4}\right)$$
$$\le 2\bar{p}\left(d_5^2\|\Delta p\|^2\|p\|\|\nabla\cdot\mathbf{v}\|+d_1^2d_5\|\Delta p\|^2\|p\|^{\frac12}\|\mathbf{v}\|^{\frac12}\|\nabla\cdot\mathbf{v}\|^{\frac12}\|\nabla p\|^{\frac12}\right)$$
$$\le 2(d_5^2+d_1^2d_5)\bar{p}(M_1\delta)^{\frac12}\|\Delta p\|^2. \tag{3.56}$$

By using (3.1), we can show that

$$|J_3(t)|=\bar{p}\left|\int_{\mathbb{R}^2}|\nabla p|^2\Delta p\,d\mathbf{x}\right|\le\bar{p}\|\nabla p\|_{L^4}^2\|\Delta p\|\le d_1^2\bar{p}\|\nabla p\|\|\Delta p\|^2\le d_1^2\bar{p}\delta^{\frac12}\|\Delta p\|^2. \tag{3.57}$$

By using similar arguments as in (3.56), we can show that

$$|J_4(t)|=\bar{p}\left|\int_{\mathbb{R}^2}|\nabla p|^2\nabla\cdot(p\mathbf{v})\,d\mathbf{x}\right|=2\bar{p}\left|\int_{\mathbb{R}^2}(\nabla p\cdot\mathbb{H}(p))\cdot(p\mathbf{v})\,d\mathbf{x}\right|$$
$$\le 2\bar{p}\|\Delta p\|\|p\|_{L^\infty}\|\nabla p\|_{L^4}\|\mathbf{v}\|_{L^4}\le 2d_1^2d_5\bar{p}\|\Delta p\|^2\|p\|^{\frac12}\|\nabla p\|^{\frac12}\|\mathbf{v}\|^{\frac12}\|\nabla\cdot\mathbf{v}\|^{\frac12}$$
$$\le 2d_1^2d_5\bar{p}(M_1\delta)^{\frac12}\|\Delta p\|^2, \tag{3.58}$$

where $\mathbb{H}(p)$ denotes the Hessian matrix of p. Similar to (3.57), we can show that

$$|J_5(t)|=\bar{p}^2\left|\int_{\mathbb{R}^2}|\nabla p|^2\nabla\cdot\mathbf{v}\,d\mathbf{x}\right|\le\bar{p}^2\|\nabla p\|_{L^4}^2\|\nabla\cdot\mathbf{v}\|$$

$$\leq d_1^2\bar{p}^2\|\nabla p\|\|\Delta p\|\|\nabla\cdot\mathbf{v}\|\leq\frac{\bar{p}^2}{12}\|\Delta p\|^2+3d_1^4\bar{p}^2\|\nabla p\|^2\|\nabla\cdot\mathbf{v}\|^2. \tag{3.59}$$

Again, by using (3.1), we can show that

$$\begin{aligned}|J_6(t)|&=2\varepsilon\bar{p}^3\left|\int_{\mathbb{R}^2}\nabla(|\mathbf{v}|^2)\cdot(\Delta\mathbf{v})\ d\mathbf{x}\right|\leq 4\varepsilon\bar{p}^3\|\mathbf{v}\|_{L^4}\|\nabla\mathbf{v}\|_{L^4}\|\Delta\mathbf{v}\|\\&\leq 4d_1^2\varepsilon\bar{p}^3\|\mathbf{v}\|^{\frac12}\|\nabla\cdot\mathbf{v}\|\|\Delta\mathbf{v}\|^{\frac32}\leq\varepsilon\bar{p}^3\|\Delta\mathbf{v}\|^2+27d_1^8\varepsilon\bar{p}^3\|\mathbf{v}\|^2\|\nabla\cdot\mathbf{v}\|^4\\&\leq\varepsilon\bar{p}^3\|\Delta\mathbf{v}\|^2+27d_1^8M_1\bar{p}^2(\varepsilon\bar{p}\|\nabla\cdot\mathbf{v}\|^2)\|\nabla\cdot\mathbf{v}\|^2.\end{aligned} \tag{3.60}$$

Similar to (3.56), we can show that

$$\begin{aligned}|J_7(t)|&=2\left|\int_{\mathbb{R}^2}p^2\Delta p\nabla\cdot(p\mathbf{v})d\mathbf{x}\right|\leq 2\left(\|p\Delta p\|\|p\|_{L^\infty}^2\|\nabla\cdot\mathbf{v}\|+\|p\Delta p\|\|p\|_{L^\infty}\|\nabla p\|_{L^4}\|\mathbf{v}\|_{L^4}\right)\\&\leq 2\left(d_5^2\|p\Delta p\|\|p\|\|\Delta p\|\|\nabla\cdot\mathbf{v}\|+d_1^2d_5\|p\Delta p\|\|p\|^{\frac12}\|\Delta p\|\|\nabla p\|^{\frac12}\|\mathbf{v}\|^{\frac12}\|\nabla\cdot\mathbf{v}\|^{\frac12}\right)\\&\leq 2(d_5^2+d_1^2d_5)(M_1\delta)^{\frac12}\|p\Delta p\|\|\Delta p\|\leq(d_5^2+d_1^2d_5)(M_1\delta)^{\frac12}\left(\|p\Delta p\|^2+\|\Delta p\|^2\right).\end{aligned} \tag{3.61}$$

By using (3.5), we can show that

$$\begin{aligned}|J_8(t)|&=2\bar{p}\left|\int_{\mathbb{R}^2}p^2\Delta p\nabla\cdot\mathbf{v}\ d\mathbf{x}\right|\leq 2\bar{p}\|\Delta p\|\|p\|_{L^\infty}^2\|\nabla\cdot\mathbf{v}\|\\&\leq 2d_5^2\bar{p}\|\Delta p\|\|p\|\|\Delta p\|\|\nabla\cdot\mathbf{v}\|\leq 2d_5^2\bar{p}(M_1\delta)^{\frac12}\|\Delta p\|^2.\end{aligned} \tag{3.62}$$

By using (3.1), we can show that

$$\begin{aligned}|J_9(t)|&=2\left|\int_{\mathbb{R}^2}p|\nabla p|^2\Delta p\ d\mathbf{x}\right|\leq 2\|p\Delta p\|\|\nabla p\|_{L^4}^2\\&\leq 2d_1^2\|p\Delta p\|\|\nabla p\|\|\Delta p\|\leq d_1^2\delta^{\frac12}\left(\|p\Delta p\|^2+\|\Delta p\|^2\right).\end{aligned} \tag{3.63}$$

By using (3.1) and (3.5), we can show that

$$\begin{aligned}&|J_{10}(t)|=2\left|\int_{\mathbb{R}^2}p|\nabla p|^2\nabla\cdot(p\mathbf{v})d\mathbf{x}\right|\\&\leq 2\left(\|p\|_{L^\infty}^2\|\nabla p\|_{L^4}^2\|\nabla\cdot\mathbf{v}\|+\|p\|_{L^\infty}\|\nabla p\|_{L^4}^3\|\mathbf{v}\|_{L^4}\right)\\&\leq 2\left(d_1^2d_5^2\|p\|\|\Delta p\|\|\nabla p\|\|\Delta p\|\|\nabla\cdot\mathbf{v}\|+d_1^4d_5\|p\|^{\frac12}\|\Delta p\|^{\frac12}\|\nabla p\|^{\frac32}\|\Delta p\|^{\frac32}\|\mathbf{v}\|^{\frac12}\|\nabla\cdot\mathbf{v}\|^{\frac12}\right)\\&\leq 2(d_1^2d_5^2+d_1^4d_5)(M_1\delta^2)^{\frac12}\|\Delta p\|^2.\end{aligned} \tag{3.64}$$

Again, by using (3.1) and (3.5), we can show that

$$\begin{aligned}|J_{11}(t)|&=2\bar{p}\left|\int_{\mathbb{R}^2}p|\nabla p|^2\nabla\cdot\mathbf{v}\ d\mathbf{x}\right|\leq 2\bar{p}\|p\|_{L^\infty}\|\nabla p\|_{L^4}^2\|\nabla\cdot\mathbf{v}\|\\&\leq 2d_1^2d_5\bar{p}\|p\|^{\frac12}\|\Delta p\|^{\frac12}\|\nabla p\|\|\Delta p\|\|\nabla\cdot\mathbf{v}\|\\&\leq\frac{\bar{p}^2}{12}\|\Delta p\|^2+2916d_1^8d_5^4(\bar{p})^{-1}\|p\|^2\|\nabla p\|^4\|\nabla\cdot\mathbf{v}\|^4\\&\leq\frac{\bar{p}^2}{12}\|\Delta p\|^2+2916d_1^8d_5^4(\bar{p})^{-1}(M_1\delta^2)\|\nabla p\|^2\|\nabla\cdot\mathbf{v}\|^2.\end{aligned} \tag{3.65}$$

This completes the estimate for the entire RHS of (3.54).

Step 3. By assembling, (3.55)–(3.65), we find

$$|N_2(t)| \le \varepsilon \bar{p}^3 \|\Delta \mathbf{v}\|^2 + \left[(\mathcal{C}_1 + \mathcal{C}_2 + \mathcal{D}_1)\|\nabla p\|^2 + \mathcal{C}_3(\varepsilon \bar{p}\|\nabla\cdot\mathbf{v}\|^2)\right]\|\nabla\cdot\mathbf{v}\|^2$$
$$\left(\frac{\bar{p}}{4} + \mathcal{D}_2 + \mathcal{D}_3 + \mathcal{D}_4 + \mathcal{D}_5 + \mathcal{D}_6 + \mathcal{D}_7 + \mathcal{D}_8\right)\|\Delta p\|^2 + (\mathcal{D}_5 + \mathcal{D}_7)\|p\Delta p\|^2,$$

where

$$\begin{aligned} &\mathcal{C}_1 = 2916\, d_1^8 \bar{p}^2 M_1, && \mathcal{C}_2 = 3 d_1^4 \bar{p}^2, && \mathcal{C}_3 = 27 d_1^8 M_1 \bar{p}^2, \\ &\mathcal{D}_1 = 2916\, d_1^8 d_5^4 (\bar{p})^{-1}(M_1\delta^2), && \mathcal{D}_2 = 2(d_5^2 + d_1^2 d_5)\bar{p}(M_1\delta)^{\frac{1}{2}}, && \mathcal{D}_3 = d_1^2 \bar{p}\delta^{\frac{1}{2}}, \\ &\mathcal{D}_4 = 2 d_1^2 d_5 \bar{p}(M_1\delta)^{\frac{1}{2}}, && \mathcal{D}_5 = (d_5^2 + d_1^2 d_5)(M_1\delta)^{\frac{1}{2}}, && \mathcal{D}_6 = 2 d_5^2 \bar{p}(M_1\delta)^{\frac{1}{2}}, \\ &\mathcal{D}_7 = d_1^2 \delta^{\frac{1}{2}}, && \mathcal{D}_8 = 2(d_1^2 d_5^2 + d_1^4 d_5)(M_1\delta^2)^{\frac{1}{2}}. \end{aligned} \tag{3.66}$$

Hence, when

$$M_1\delta^2 \le 1, \qquad \mathcal{D}_j \le \frac{\bar{p}}{28}, \quad j = 2,\dots,8, \qquad \text{and} \quad \mathcal{D}_j \quad \le \min\left\{\frac{\bar{p}}{28}, \frac{1}{4}\right\}, \quad j = 5,7, \tag{3.67}$$

it holds that

$$\begin{aligned} |N_2(t)| \le\; & \varepsilon \bar{p}^3 \|\Delta \mathbf{v}\|^2 + \frac{\bar{p}}{2}\|\Delta p\|^2 + \frac{1}{2}\|p\Delta p\|^2 \\ & + \left[(\mathcal{C}_1 + \mathcal{C}_2 + 2916\, d_1^8 d_5^4 (\bar{p})^{-1})\|\nabla p\|^2 + \mathcal{C}_3(\varepsilon \bar{p}\|\nabla\cdot\mathbf{v}\|^2)\right]\|\nabla\cdot\mathbf{v}\|^2. \end{aligned} \tag{3.68}$$

By substituting (3.68) into (3.54), we obtain

$$\frac{d}{dt}E_4(t) + D_4(t) \le \left[(\mathcal{C}_1 + \mathcal{C}_2 + 2916\, d_1^8 d_5^4 (\bar{p})^{-1})\|\nabla p\|^2 + \mathcal{C}_3(\varepsilon \bar{p}\|\nabla\cdot\mathbf{v}\|^2)\right]\|\nabla\cdot\mathbf{v}\|^2, \tag{3.69}$$

where

$$\begin{aligned} E_4(t) &\equiv \bar{p}^2\|\nabla p\|^2 + \bar{p}^3\|\nabla\cdot\mathbf{v}\|^2 - \bar{p}\int_{\mathbb{R}^2} p|\nabla p|^2\, d\mathbf{x} + \int_{\mathbb{R}^2} p^2|\nabla p|^2\, d\mathbf{x} \\ &= \frac{\bar{p}^2}{2}\|\nabla p\|^2 + \bar{p}^3\|\nabla\cdot\mathbf{v}\|^2 + \frac{1}{2}\|\bar{p}\nabla p - p\nabla p\|^2 + \frac{1}{2}\|p\nabla p\|^2, \\ D_4(t) &\equiv \frac{3\bar{p}^2}{2}\|\Delta p\|^2 - 2\bar{p}\int_{\mathbb{R}^2} p(\Delta p)^2 d\mathbf{x} + \varepsilon \bar{p}^3\|\Delta\mathbf{v}\|^2 + \frac{3}{2}\int_{\mathbb{R}^2} p^2|\Delta p|^2 d\mathbf{x} \\ &= \frac{\bar{p}^2}{2}\|\Delta p\|^2 + \|\bar{p}\Delta p - p\Delta p\|^2 + \varepsilon \bar{p}^3\|\Delta\mathbf{v}\|^2 + \frac{1}{2}\|p\Delta p\|^2. \end{aligned} \tag{3.70}$$

We note that $E_4(0)$ is the same as H_2 in the statement of Theorem 1.2. We also note that $\bar{p}^3\|\nabla\cdot\mathbf{v}\|^2 \le E_4(t)$. Hence, we update (3.69) as

$$\frac{d}{dt}E_4(t) + D_4(t) \le \frac{1}{\bar{p}}\left[(\mathcal{C}_1 + \mathcal{C}_2 + 2916\, d_1^8 d_5^4 (\bar{p})^{-1})\|\nabla p\|^2 + \mathcal{C}_3(\varepsilon \bar{p}\|\nabla\cdot\mathbf{v}\|^2)\right]E_4(t). \tag{3.71}$$

By applying the Grönwall inequality to (3.71) and using (3.46), we find

$$E_4(t) \le E_4(0)\exp\left\{\frac{4}{\bar{p}}(\mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3 + 2916\, d_1^8 d_5^4 (\bar{p})^{-1})\, E_3(0)\right\} \equiv \mathcal{C}_4. \tag{3.72}$$

Therefore, by the definition of $E_4(t)$, it holds that

$$\|\nabla p(t)\|^2+\|\nabla\cdot\mathbf{v}(t)\|^2\le\frac{2\bar{p}+1}{\bar{p}^3}\mathcal{C}_4. \tag{3.73}$$

In view of (1.18) and (3.73) we see that

$$\|\nabla p(t)\|^2+\|\nabla\cdot\mathbf{v}(t)\|^2\le\frac{\delta}{2}. \tag{3.74}$$

In addition, by substituting (3.72) into (3.71) then integrating with respect to t, we have

$$\int_0^t D_4(\tau)d\tau\le E_4(0)+\frac{4}{\bar{p}}(\mathcal{C}_1+\mathcal{C}_2+\mathcal{C}_3+2916\,d_1^8d_5^4(\bar{p})^{-1})\,E_3(0)\mathcal{C}_4.$$

According to (3.70), we have

$$\begin{aligned}&\int_0^t\left(\|\Delta p(\tau)(t)\|^2+\varepsilon\bar{p}\|\Delta\mathbf{v}(\tau)\|^2\right)d\tau\\ &\le\frac{3}{\bar{p}^3}\left(E_4(0)+\frac{4}{\bar{p}}(\mathcal{C}_1+\mathcal{C}_2+\mathcal{C}_3+2916\,d_1^8d_5^4(\bar{p})^{-1})\,E_3(0)\mathcal{C}_4\right)\equiv\mathcal{C}_5,\end{aligned} \tag{3.75}$$

where the constant $\mathcal{C}_5$ is independent of t and remains bounded as $\varepsilon\to0$. Next, we estimate the second order spatial derivatives of the solution.

3.3. H^2-estimate. By applying Δ to the equations in (1.11), then taking the L^2 inner products of the resulting equations with Δ of the targeting functions, we obtain

$$\begin{aligned}&\frac{d}{dt}\left(\|\Delta p\|^2+\bar{p}\|\Delta\mathbf{v}\|^2\right)+2\|\nabla\Delta p\|^2+2\varepsilon\bar{p}\|\Delta(\nabla\cdot\mathbf{v})\|^2\\ &=-2\int_{\mathbb{R}^2}\nabla(\nabla\cdot(p\mathbf{v}))\cdot(\nabla\Delta p)\ d\mathbf{x}+2\varepsilon\bar{p}\int_{\mathbb{R}^2}\Delta(|\mathbf{v}|^2)\Delta(\nabla\cdot\mathbf{v})d\mathbf{x}\\ &=-2\int_{\mathbb{R}^2}(\nabla(\nabla p\cdot\mathbf{v})+(\nabla\cdot\mathbf{v})\nabla p)\cdot(\nabla\Delta p)\ d\mathbf{x}-2\int_{\mathbb{R}^2}p\nabla(\nabla\cdot\mathbf{v})\cdot(\nabla\Delta p)\ d\mathbf{x}\\ &\quad+4\varepsilon\bar{p}\int_{\mathbb{R}^2}\left(\mathbf{v}\cdot\Delta\mathbf{v}+|\nabla v_1|^2+|\nabla v_2|^2\right)\Delta(\nabla\cdot\mathbf{v})d\mathbf{x}.\end{aligned} \tag{3.76}$$

By direct calculations, we can show that

$$\begin{aligned}&-\frac{d}{dt}\int_{\mathbb{R}^2}p(\Delta p)^2d\mathbf{x}-2\int_{\mathbb{R}^2}p|\nabla\Delta p|^2d\mathbf{x}\\ &=2\int_{\mathbb{R}^2}\Delta p\nabla p\cdot\nabla(\nabla\cdot(p\mathbf{v}))d\mathbf{x}+2\int_{\mathbb{R}^2}p\nabla\Delta p\cdot\nabla(\nabla\cdot(p\mathbf{v}))d\mathbf{x}\\ &\quad+2\bar{p}\int_{\mathbb{R}^2}\Delta p\nabla p\cdot\nabla(\nabla\cdot\mathbf{v})d\mathbf{x}+2\bar{p}\int_{\mathbb{R}^2}p\nabla\Delta p\cdot\nabla(\nabla\cdot\mathbf{v})d\mathbf{x}\\ &\quad+2\int_{\mathbb{R}^2}\Delta p\nabla p\cdot\nabla\Delta p\ d\mathbf{x}-\int_{\mathbb{R}^2}(\Delta p)^2\nabla\cdot(p\mathbf{v})d\mathbf{x}\\ &\quad-\bar{p}\int_{\mathbb{R}^2}(\Delta p)^2\nabla\cdot\mathbf{v}\ d\mathbf{x}-\int_{\mathbb{R}^2}(\Delta p)^3d\mathbf{x},\end{aligned} \tag{3.77}$$

and

$$\frac{d}{dt}\int_{\mathbb{R}^2}p^2(\Delta p)^2d\mathbf{x}+2\int_{\mathbb{R}^2}p^2|\nabla\Delta p|^2d\mathbf{x}$$

$$
\begin{aligned}
= &-4\int_{\mathbb{R}^2} p\Delta p\nabla p\cdot\nabla(\nabla\cdot(p\mathbf{v}))d\mathbf{x}-2\int_{\mathbb{R}^2} p^2\nabla\Delta p\cdot\nabla(\nabla\cdot(p\mathbf{v}))d\mathbf{x}\\
&-4\bar{p}\int_{\mathbb{R}^2} p\Delta p\nabla p\cdot\nabla(\nabla\cdot\mathbf{v})d\mathbf{x}-2\bar{p}\int_{\mathbb{R}^2} p^2\nabla\Delta p\cdot\nabla(\nabla\cdot\mathbf{v})d\mathbf{x}\\
&-4\int_{\mathbb{R}^2} p\Delta p\nabla p\cdot\nabla\Delta p\ d\mathbf{x}+2\int_{\mathbb{R}^2} p(\Delta p)^2\nabla\cdot(p\mathbf{v})d\mathbf{x}\\
&+2\bar{p}\int_{\mathbb{R}^2} p(\Delta p)^2\nabla\cdot\mathbf{v}\ d\mathbf{x}+2\int_{\mathbb{R}^2} p(\Delta p)^3 d\mathbf{x}. \qquad (3.78)
\end{aligned}
$$

The operation: (3.76)$\times\bar{p}$ + (3.77) + (3.78)$\times 1/\bar{p}$ then yields

$$
\begin{aligned}
&\frac{d}{dt}\left(\bar{p}\|\Delta p\|^2+\bar{p}^2\|\Delta\mathbf{v}\|^2-\int_{\mathbb{R}^2} p(\Delta p)^2 d\mathbf{x}+\frac{1}{\bar{p}}\|p\Delta p\|^2\right)\\
&+2\bar{p}\|\nabla\Delta p\|^2-2\int_{\mathbb{R}^2} p|\nabla\Delta p|^2 d\mathbf{x}+\frac{2}{\bar{p}}\|p\nabla\Delta p\|^2+2\varepsilon\bar{p}^2\|\Delta(\nabla\cdot\mathbf{v})\|^2=\sum_{k=1}^{19}R_k(t), \quad (3.79)
\end{aligned}
$$

where

$$
\begin{aligned}
&R_1(t)=-2\bar{p}\int_{\mathbb{R}^2}(\nabla(\nabla p\cdot\mathbf{v}))\cdot(\nabla\Delta p)\ d\mathbf{x}, && R_2(t)=-2\bar{p}\int_{\mathbb{R}^2}((\nabla\cdot\mathbf{v})\nabla p)\cdot(\nabla\Delta p)\ d\mathbf{x},\\
&R_3(t)=4\varepsilon\bar{p}^2\int_{\mathbb{R}^2}(\mathbf{v}\cdot\Delta\mathbf{v})\Delta(\nabla\cdot\mathbf{v})d\mathbf{x}, && R_4(t)=4\varepsilon\bar{p}^2\int_{\mathbb{R}^2}\left(|\nabla\mathbf{v}|^2\right)\Delta(\nabla\cdot\mathbf{v})d\mathbf{x},\\
&R_5(t)=2\int_{\mathbb{R}^2}\Delta p\nabla p\cdot\nabla(\nabla\cdot(p\mathbf{v}))d\mathbf{x}, && R_6(t)=2\int_{\mathbb{R}^2} p\nabla\Delta p\cdot\nabla(\nabla\cdot(p\mathbf{v}))d\mathbf{x},\\
&R_7(t)=2\bar{p}\int_{\mathbb{R}^2}\Delta p\nabla p\cdot\nabla(\nabla\cdot\mathbf{v})d\mathbf{x}, && R_8(t)=2\int_{\mathbb{R}^2}\Delta p\nabla p\cdot\nabla\Delta p\ d\mathbf{x},\\
&R_9(t)=-\int_{\mathbb{R}^2}(\Delta p)^2\nabla\cdot(p\mathbf{v})d\mathbf{x}, && R_{10}(t)=-\bar{p}\int_{\mathbb{R}^2}(\Delta p)^2\nabla\cdot\mathbf{v}\ d\mathbf{x},\\
&R_{11}(t)=-\int_{\mathbb{R}^2}(\Delta p)^3 d\mathbf{x}, && R_{12}(t)=-\frac{4}{\bar{p}}\int_{\mathbb{R}^2} p\Delta p\nabla p\cdot\nabla(\nabla\cdot(p\mathbf{v}))d\mathbf{x},\\
&R_{13}(t)=-\frac{2}{\bar{p}}\int_{\mathbb{R}^2} p^2\nabla\Delta p\cdot\nabla(\nabla\cdot(p\mathbf{v}))d\mathbf{x}, && R_{14}(t)=-4\int_{\mathbb{R}^2} p\Delta p\nabla p\cdot\nabla(\nabla\cdot\mathbf{v})d\mathbf{x},\\
&R_{15}(t)=-2\int_{\mathbb{R}^2} p^2\nabla\Delta p\cdot\nabla(\nabla\cdot\mathbf{v})d\mathbf{x}, && R_{16}(t)=-\frac{4}{\bar{p}}\int_{\mathbb{R}^2} p\Delta p\nabla p\cdot\nabla\Delta p\ d\mathbf{x},\\
&R_{17}(t)=\frac{2}{\bar{p}}\int_{\mathbb{R}^2} p(\Delta p)^2\nabla\cdot(p\mathbf{v})d\mathbf{x}, && R_{18}(t)=2\int_{\mathbb{R}^2} p(\Delta p)^2\nabla\cdot\mathbf{v}\ d\mathbf{x},\\
&R_{19}(t)=\frac{2}{\bar{p}}\int_{\mathbb{R}^2} p(\Delta p)^3 d\mathbf{x}.
\end{aligned}
$$

Next, we shall estimate the RHS of (3.79) term by term. However, due to a large number of terms to be estimated, the detailed analysis is long and not reader-friendly. To simplify the presentation, we only present the resulting estimate, while the detailed arguments are left to the Appendix. Indeed, after a series of energy estimates by using (3.1)–(3.5), we can show that

$$
\sum_{k=1}^{19}|R_k(t)|\le\frac{\bar{p}}{2}\|\nabla\Delta p\|^2+\frac{1}{2\bar{p}}\|p\nabla\Delta p\|^2+\varepsilon\bar{p}^2\|\Delta(\nabla\cdot\mathbf{v})\|^2+\mathcal{K}_1\left(\|\nabla p\|^2+\|\Delta p\|^2\right)
$$

$$+\mathcal{K}_2\left(\|\nabla p\|^2+\|\Delta p\|^2+\varepsilon\bar{p}\|\nabla\cdot\mathbf{v}\|^2\right)\left(\bar{p}\|\Delta p\|^2+\bar{p}^2\|\Delta\mathbf{v}\|^2\right), \tag{3.80}$$

for some constants $\mathcal{K}_1$ and $\mathcal{K}_2$ which depend only on $M_1,\delta,\bar{p}$ and the constants in (3.1)–(3.5), and remain bounded as $\varepsilon\to 0$. By substituting (3.80) into (3.79), we have

$$\begin{aligned}
&\frac{d}{dt}\left(\bar{p}\|\Delta p\|^2+\bar{p}^2\|\Delta\mathbf{v}\|^2-\int_{\mathbb{R}^2}p(\Delta p)^2d\mathbf{x}+\frac{1}{\bar{p}}\|p\Delta p\|^2\right)\\
&+2\bar{p}\|\nabla\Delta p\|^2-2\int_{\mathbb{R}^2}p|\nabla\Delta p|^2d\mathbf{x}+\frac{2}{\bar{p}}\|p\nabla\Delta p\|^2+2\varepsilon\bar{p}^2\|\Delta(\nabla\cdot\mathbf{v})\|^2\\
\leq&\frac{\bar{p}}{2}\|\nabla\Delta p\|^2+\frac{1}{2\bar{p}}\|p\nabla\Delta p\|^2+\varepsilon\bar{p}^2\|\Delta(\nabla\cdot\mathbf{v})\|^2+\mathcal{K}_1\left(\|\nabla p\|^2+\|\Delta p\|^2\right)\\
&+\mathcal{K}_2\left(\|\nabla p\|^2+\|\Delta p\|^2+\varepsilon\bar{p}\|\nabla\cdot\mathbf{v}\|^2\right)\left(\bar{p}\|\Delta p\|^2+\bar{p}^2\|\Delta\mathbf{v}\|^2\right),
\end{aligned}$$

which yields

$$\begin{aligned}
&\frac{d}{dt}E_5(t)+D_5(t)\\
&\leq\mathcal{K}_1\left(\|\nabla p\|^2+\|\Delta p\|^2\right)+\mathcal{K}_2\left(\|\nabla p\|^2+\|\Delta p\|^2+\varepsilon\bar{p}\|\nabla\cdot\mathbf{v}\|^2\right)\left(\bar{p}\|\Delta p\|^2+\bar{p}^2\|\Delta\mathbf{v}\|^2\right),
\end{aligned} \tag{3.81}$$

where

$$\begin{aligned}
E_5(t)&=\frac{\bar{p}}{2}\|\Delta p\|^2+\bar{p}^2\|\Delta\mathbf{v}\|^2+\frac{1}{2\bar{p}}\|\bar{p}\Delta p-p\Delta p\|^2+\frac{1}{2\bar{p}}\|p\Delta p\|^2,\\
D_5(t)&=\frac{\bar{p}}{2}\|\nabla\Delta p\|^2+\frac{1}{\bar{p}}\|\nabla\Delta p-p\nabla\Delta p\|^2+\frac{1}{2\bar{p}}\|p\nabla\Delta p\|^2+\varepsilon\bar{p}^2\|\Delta(\nabla\cdot\mathbf{v})\|^2.
\end{aligned}$$

Note that $\bar{p}\|\Delta p\|^2+\bar{p}^2\|\Delta\mathbf{v}\|^2\leq 2E_5(t)$. Hence, we update (3.81) as

$$\frac{d}{dt}E_5(t)+D_5(t)\leq\mathcal{K}_1\left(\|\nabla p\|^2+\|\Delta p\|^2\right)+\mathcal{K}_2\left(\|\nabla p\|^2+\|\Delta p\|^2+\varepsilon\bar{p}\|\nabla\cdot\mathbf{v}\|^2\right)E_5(t). \tag{3.82}$$

Applying the Grönwall inequality to (3.82), we find

$$\begin{aligned}
E_5(t)\leq&\exp\left\{\mathcal{K}_2\int_0^t\left(\|\nabla p\|^2+\|\Delta p\|^2+\varepsilon\bar{p}\|\nabla\cdot\mathbf{v}\|^2\right)d\tau\right\}\\
&\times\left(E_5(0)+\mathcal{K}_1\int_0^t\left(\|\nabla p\|^2+\|\Delta p\|^2\right)d\tau\right).
\end{aligned}$$

In view of the uniform-in-time integrability of $\|\nabla p\|^2$, $\|\Delta p\|^2$ and $\varepsilon\bar{p}\|\nabla\cdot\mathbf{v}\|^2$ (cf. (3.46), (3.75)), we see that $E_5(t)\leq\mathcal{K}_3$, for some constant $\mathcal{K}_3$ which is independent of t and remains bounded as $\varepsilon\to 0$. This implies that

$$\|\Delta p(t)\|^2+\|\Delta\mathbf{v}(t)\|^2\leq 2\left(\frac{1}{\bar{p}}+\frac{1}{\bar{p}^2}\right)\mathcal{K}_3. \tag{3.83}$$

By substituting the upper bound of $E_5(t)$ into (3.82), then integrating the result with respect to t, we have, in particular,

$$\int_0^t\left(\|\nabla\Delta p(\tau)\|^2+\varepsilon\bar{p}\|\Delta(\nabla\cdot\mathbf{v}(\tau))\|^2\right)d\tau\leq\mathcal{K}_4$$

for some constant which is independent of t and remains bounded as $\varepsilon \to 0$.

Next, we shall work on the third order spatial derivatives of the solution, in order to gain the desired energy estimates stated in Theorem 1.2. The proof is in exactly the same spirit as the H^1- and H^2-estimates. However, the detailed arguments involve the estimation of more than 50 nonlinear terms, whose presentation is not reader friendly. For the sake of brevity, we omit the technical details. Moreover, by applying similar arguments as in Sections 2.5–2.8 for the 3D case, we can improve the temporal integrability of $\|\nabla\cdot\mathbf{v}\|_{H^2}^2$, and establish the global well-posedness, long-time behavior and zero chemical diffusion limit results for the 2D case. The detailed proofs are omitted to simplify the presentation. This completes the proof for Theorem 1.2.

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Appendix A. Derivation of (3.80). In this appendix, we first provide the detailed derivation of (3.80). The proof involves the estimates of more than 30 nonlinear terms, which is achieved by using (3.1)–(3.5), Cauchy-Schwarz and Young's inequalities. In the subsequent energy estimates, a generic constant, denoted by η, which is to be specified at the end of the proof, will appear frequently. To simplify the presentation, we use d to denote a generic constant which has sole dependence on η and the constants in (3.1)–(3.5).

For $R_1(t)$, we have

$$|R_1(t)| = 2\bar{p}\left|\int_{\mathbb{R}^2}(\nabla(\nabla p\cdot\mathbf{v}))\cdot(\nabla\Delta p)\,d\mathbf{x}\right| = 2\bar{p}\left|\int_{\mathbb{R}^2}\left[\mathbb{H}(p)\cdot\mathbf{v} + (\nabla\mathbf{v})^{\mathrm{T}}\cdot\nabla p\right]\cdot(\nabla\Delta p)\,d\mathbf{x}\right|$$
$$\equiv \left|\tilde{R}_{11} + \tilde{R}_{12}\right|.$$

For $\tilde{R}_{11}$, we have

$$\begin{aligned}\left|\tilde{R}_{11}\right| &= 2\bar{p}\left|\int_{\mathbb{R}^2}[\mathbb{H}(p)\cdot\mathbf{v}]\cdot(\nabla\Delta p)\,d\mathbf{x}\right| \\ &\le 2\bar{p}\|\mathbb{H}(p)\|_{L^4}\|\mathbf{v}\|_{L^4}\|\nabla\Delta p\| \le d\bar{p}\|\Delta p\|^{\frac{1}{2}}\|\nabla\Delta p\|^{\frac{3}{2}}\|\mathbf{v}\|^{\frac{1}{2}}\|\nabla\cdot\mathbf{v}\|^{\frac{1}{2}} \\ &\le \eta\|\nabla\Delta p\|^2 + d\bar{p}^4\|\Delta p\|^2\|\mathbf{v}\|^2\|\nabla\cdot\mathbf{v}\|^2 \le \eta\|\nabla\Delta p\|^2 + d\bar{p}^4 M_1\delta\|\Delta p\|^2,\end{aligned}$$

for some positive generic constant η which will be specified later.

For $\tilde{R}_{12}$, we have

$$\begin{aligned}\left|\tilde{R}_{12}\right| &= 2\bar{p}\left|\int_{\mathbb{R}^2}\left[(\nabla\mathbf{v})^{\mathrm{T}}\cdot\nabla p\right]\cdot(\nabla\Delta p)\,d\mathbf{x}\right| \\ &\le 2\bar{p}\|\nabla\mathbf{v}\|_{L^4}\|\nabla p\|_{L^4}\|\nabla\Delta p\| \le d\bar{p}\|\nabla\cdot\mathbf{v}\|^{\frac{1}{2}}\|\Delta\mathbf{v}\|^{\frac{1}{2}}\|\nabla p\|^{\frac{1}{2}}\|\Delta p\|^{\frac{1}{2}}\|\nabla\Delta p\|\end{aligned}$$

$$\leq \eta\|\nabla\Delta p\|^2 + d\bar{p}^2\|\nabla\cdot\mathbf{v}\|\|\Delta\mathbf{v}\|\|\nabla p\|\|\Delta p\| \leq \eta\|\nabla\Delta p\|^2 + d\bar{p}^2\delta\left(\|\Delta\mathbf{v}\|^2 + \|\Delta p\|^2\right).$$

For $R_2(t)$, similar to the estimate of R_{12}, we have

$$|R_2(t)| = 2\bar{p}\left|\int_{\mathbb{R}^2}(\nabla\cdot\mathbf{v})\nabla p\cdot(\nabla\Delta p)\ d\mathbf{x}\right| \leq \eta\|\nabla\Delta p\|^2 + d\bar{p}^2\delta\left(\|\Delta\mathbf{v}\|^2 + \|\Delta p\|^2\right).$$

For $R_3(t)$, we have

$$\begin{aligned}|R_3(t)| &= 4\varepsilon\bar{p}^2\left|\int_{\mathbb{R}^2}(\mathbf{v}\cdot\Delta\mathbf{v})\Delta(\nabla\cdot\mathbf{v})d\mathbf{x}\right| \\ &\leq 4\varepsilon\bar{p}^2\|\mathbf{v}\|_{L^4}\|\Delta\mathbf{v}\|_{L^4}\|\Delta(\nabla\cdot\mathbf{v})\| \leq d\varepsilon\bar{p}^2\|\mathbf{v}\|^{\frac{1}{2}}\|\nabla\cdot\mathbf{v}\|^{\frac{1}{2}}\|\Delta\mathbf{v}\|^{\frac{1}{2}}\|\Delta(\nabla\cdot\mathbf{v})\|^{\frac{3}{2}} \\ &\leq \frac{\varepsilon\bar{p}^2}{2}\|\Delta(\nabla\cdot\mathbf{v})\|^2 + d\varepsilon\bar{p}^2\|\mathbf{v}\|^2\|\nabla\cdot\mathbf{v}\|^2\|\Delta\mathbf{v}\|^2 \\ &\leq \frac{\varepsilon\bar{p}^2}{2}\|\Delta(\nabla\cdot\mathbf{v})\|^2 + dM_1\bar{p}(\varepsilon\bar{p}\|\nabla\cdot\mathbf{v}\|^2)\|\Delta\mathbf{v}\|^2.\end{aligned}$$

For $R_4(t)$, we have

$$\begin{aligned}|R_4(t)| &= 4\varepsilon\bar{p}^2\left|\int_{\mathbb{R}^2}\left(|\nabla\mathbf{v}|^2\right)\Delta(\nabla\cdot\mathbf{v})d\mathbf{x}\right| \leq 4\varepsilon\bar{p}^2\|\nabla\mathbf{v}\|_{L^4}^2\|\Delta(\nabla\cdot\mathbf{v})\| \\ &\leq d\varepsilon\bar{p}^2\|\nabla\cdot\mathbf{v}\|\|\Delta\mathbf{v}\|\|\Delta(\nabla\cdot\mathbf{v})\| \leq \frac{\varepsilon\bar{p}^2}{2}\|\Delta(\nabla\cdot\mathbf{v})\|^2 + d\bar{p}(\varepsilon\bar{p}\|\nabla\cdot\mathbf{v}\|^2)\|\Delta\mathbf{v}\|^2.\end{aligned}$$

For $R_5(t)$, we have

$$\begin{aligned}|R_5(t)| &= 2\left|\int_{\mathbb{R}^2}\Delta p\nabla p\cdot\nabla(\nabla\cdot(p\mathbf{v}))d\mathbf{x}\right| \\ &= 2\left|\int_{\mathbb{R}^2}\Delta p\nabla p\cdot\left[\mathbb{H}(p)\cdot\mathbf{v} + (\nabla\mathbf{v})^{\mathrm{T}}\cdot\nabla p + (\nabla\cdot\mathbf{v})\nabla p + p\Delta\mathbf{v}\right]d\mathbf{x}\right| \\ &\equiv 2|R_{51} + R_{52} + R_{53} + R_{54}|.\end{aligned}$$

For R_{51}, we have

$$\begin{aligned}|R_{51}| &= 2\left|\int_{\mathbb{R}^2}\Delta p\nabla p\cdot[\mathbb{H}(p)\cdot\mathbf{v}]\,d\mathbf{x}\right| \leq 2\|\Delta p\|_{L^4}\|\nabla p\|_{L^4}\|\mathbb{H}(p)\|_{L^4}\|\mathbf{v}\|_{L^4} \\ &\leq d\|\nabla\Delta p\|\|\Delta p\|^{\frac{3}{2}}\|\nabla p\|^{\frac{1}{2}}\|\mathbf{v}\|^{\frac{1}{2}}\|\nabla\cdot\mathbf{v}\|^{\frac{1}{2}} \leq \eta\|\nabla\Delta p\|^2 + d\|\Delta p\|^2\|\nabla p\|\|\Delta p\|\|\mathbf{v}\|\|\nabla\cdot\mathbf{v}\| \\ &\leq \eta\|\nabla\Delta p\|^2 + d(M_1\delta)^{\frac{1}{2}}\left(\|\nabla p\|^2 + \|\Delta p\|^2\right)\|\Delta p\|^2.\end{aligned}$$

For R_{52}, we have

$$\begin{aligned}|R_{52}| &= 2\left|\int_{\mathbb{R}^2}\Delta p\nabla p\cdot\left[(\nabla\mathbf{v})^{\mathrm{T}}\cdot\nabla p\right]d\mathbf{x}\right| \leq 2\|\nabla p\|_{L^4}^2\|\Delta p\|_{L^4}\|\nabla\mathbf{v}\|_{L^4} \\ &\leq d\|\nabla p\|\|\Delta p\|^{\frac{3}{2}}\|\nabla\Delta p\|^{\frac{1}{2}}\|\nabla\cdot\mathbf{v}\|^{\frac{1}{2}}\|\Delta\mathbf{v}\|^{\frac{1}{2}} \\ &\leq \eta\|\nabla\Delta p\|^2 + d\|\nabla p\|^{\frac{4}{3}}\|\Delta p\|^2\|\nabla\cdot\mathbf{v}\|^{\frac{2}{3}}\|\Delta\mathbf{v}\|^{\frac{2}{3}} \\ &\leq \eta\|\nabla\Delta p\|^2 + d\delta^{\frac{1}{3}}\|\nabla p\|^{\frac{4}{3}}\|\Delta p\|^{\frac{4}{3}}\|\Delta p\|^{\frac{2}{3}}\|\Delta\mathbf{v}\|^{\frac{2}{3}} \\ &\leq \eta\|\nabla\Delta p\|^2 + d\delta^{\frac{1}{3}}\left(\|\nabla p\|^2\|\Delta p\|^2 + \|\Delta p\|^2\|\Delta\mathbf{v}\|^2\right).\end{aligned}$$

For R_{53}, we have

$$\begin{aligned}|R_{53}| &= 2\left|\int_{\mathbb{R}^2} \Delta p \nabla p \cdot [(\nabla\cdot\mathbf{v})\nabla p]\,d\mathbf{x}\right| \le 2\,\|\nabla p\|_{L^4}^2\,\|\Delta p\|_{L^4}\,\|\nabla\cdot\mathbf{v}\|_{L^4}\\ &\le \eta\,\|\nabla\Delta p\|^2 + d\,\delta^{\frac{1}{3}}\left(\|\nabla p\|^2\|\Delta p\|^2 + \|\Delta p\|^2\|\Delta\mathbf{v}\|^2\right).\end{aligned}$$

For R_{54}, we have

$$\begin{aligned}|R_{54}| &= 2\left|\int_{\mathbb{R}^2} \Delta p \nabla p \cdot [p\Delta\mathbf{v}]\,d\mathbf{x}\right| \le 2\,\|p\|_{L^\infty}\,\|\nabla p\|_{L^4}\,\|\Delta p\|_{L^4}\,\|\Delta\mathbf{v}\|\\ &\le d\,\|p\|^{\frac{1}{2}}\,\|\nabla p\|^{\frac{1}{2}}\,\|\Delta p\|^{\frac{3}{2}}\,\|\nabla\Delta p\|^{\frac{1}{2}}\,\|\Delta\mathbf{v}\| \le \eta\,\|\nabla\Delta p\|^2 + d\,\|p\|^{\frac{2}{3}}\,\|\nabla p\|^{\frac{2}{3}}\,\|\Delta p\|^2\,\|\Delta\mathbf{v}\|^{\frac{4}{3}}\\ &\le \eta\,\|\nabla\Delta p\|^2 + d\,(M_1\delta)^{\frac{1}{3}}\,\|\Delta p\|^{\frac{2}{3}}\,\|\Delta p\|^{\frac{4}{3}}\,\|\Delta\mathbf{v}\|^{\frac{4}{3}}\\ &\le \eta\,\|\nabla\Delta p\|^2 + d\,(M_1\delta)^{\frac{1}{3}}\left(\|\Delta p\|^2 + \|\Delta p\|^2\|\Delta\mathbf{v}\|^2\right).\end{aligned}$$

For $R_6(t)$, we have

$$\begin{aligned}|R_6(t)| &= 2\left|\int_{\mathbb{R}^2} p\nabla\Delta p \cdot \left[\mathbb{H}(p)\cdot\mathbf{v} + (\nabla\mathbf{v})^{\mathrm{T}}\cdot\nabla p + (\nabla\cdot\mathbf{v})\nabla p + p\Delta\mathbf{v}\right]d\mathbf{x}\right|\\ &\equiv 2\,|R_{61}+R_{62}+R_{63}+R_{64}|.\end{aligned}$$

For R_{61}, we have

$$\begin{aligned}|R_{61}| &= 2\left|\int_{\mathbb{R}^2} p\nabla\Delta p \cdot [\mathbb{H}(p)\cdot\mathbf{v}]\,d\mathbf{x}\right| \le 2\|p\|_{L^\infty}\,\|\nabla\Delta p\|\|\mathbb{H}(p)\|_{L^4}\,\|\mathbf{v}\|_{L^4}\\ &\le d\,\|p\|^{\frac{1}{2}}\,\|\Delta p\|\|\nabla\Delta p\|^{\frac{3}{2}}\,\|\mathbf{v}\|^{\frac{1}{2}}\,\|\nabla\cdot\mathbf{v}\|^{\frac{1}{2}} \le \eta\,\|\nabla\Delta p\|^2 + d\,\|p\|^2\|\Delta p\|^4\|\mathbf{v}\|^2\|\nabla\cdot\mathbf{v}\|^2\\ &\le \eta\,\|\nabla\Delta p\|^2 + d\,M_1^2\delta\,\|\Delta p\|^2\|\Delta p\|^2.\end{aligned}$$

For R_{62}, we have

$$\begin{aligned}|R_{62}| &= 2\left|\int_{\mathbb{R}^2} p\nabla\Delta p \cdot \left[(\nabla\mathbf{v})^{\mathrm{T}}\cdot\nabla p\right]d\mathbf{x}\right| \le 2\|p\|_{L^\infty}\,\|\nabla\Delta p\|\|\nabla p\|_{L^4}\,\|\nabla\mathbf{v}\|_{L^4}\\ &\le d\,\|p\|^{\frac{1}{2}}\,\|\nabla p\|^{\frac{1}{2}}\,\|\nabla\Delta p\|\|\Delta p\|\|\nabla\cdot\mathbf{v}\|^{\frac{1}{2}}\,\|\Delta\mathbf{v}\|^{\frac{1}{2}} \le \eta\,\|\nabla\Delta p\|^2 + c\,\|p\|\|\nabla p\|\|\Delta p\|^2\|\nabla\cdot\mathbf{v}\|\|\Delta\mathbf{v}\|\\ &\le \eta\,\|\nabla\Delta p\|^2 + d\,(M_1\delta^2)^{\frac{1}{2}}\,\|\Delta p\|^2\|\Delta\mathbf{v}\| \le \eta\,\|\nabla\Delta p\|^2 + d\,(M_1\delta^2)^{\frac{1}{2}}\left(\|\Delta p\|^2 + \|\Delta p\|^2\|\Delta\mathbf{v}\|^2\right).\end{aligned}$$

For R_{63}, we have

$$\begin{aligned}|R_{63}| &= 2\left|\int_{\mathbb{R}^2} p\nabla\Delta p \cdot [(\nabla\cdot\mathbf{v})\nabla p]\,d\mathbf{x}\right| \le 2\,\|p\|_{L^\infty}\,\|\nabla\Delta p\|\|\nabla p\|_{L^4}\,\|\nabla\cdot\mathbf{v}\|_{L^4}\\ &\le \eta\,\|\nabla\Delta p\|^2 + d\,(M_1\delta^2)^{\frac{1}{2}}\left(\|\Delta p\|^2 + \|\Delta p\|^2\|\Delta\mathbf{v}\|^2\right).\end{aligned}$$

For R_{64}, we have

$$\begin{aligned}|R_{64}| &= 2\left|\int_{\mathbb{R}^2} p\nabla\Delta p \cdot [p\Delta\mathbf{v}]\,d\mathbf{x}\right| \le 2\,\|p\|_{L^\infty}^2\,\|\nabla\Delta p\|\|\Delta\mathbf{v}\| \le d\,\|p\|\|\Delta p\|\|\nabla\Delta p\|\|\Delta\mathbf{v}\|\\ &\le \eta\,\|\nabla\Delta p\|^2 + d\,\|p\|^2\|\Delta p\|^2\|\Delta\mathbf{v}\|^2 \le \eta\,\|\nabla\Delta p\|^2 + d\,M_1\,\|\Delta p\|^2\|\Delta\mathbf{v}\|^2.\end{aligned}$$

For $R_7(t)$, we have

$$\begin{aligned}|R_7(t)| &= 2\bar{p}\left|\int_{\mathbb{R}^2}\Delta p\nabla p\cdot\nabla(\nabla\cdot\mathbf{v})d\mathbf{x}\right| \le 2\bar{p}\|\Delta\mathbf{v}\|\|\nabla p\|_{L^4}\|\Delta p\|_{L^4}\\ &\le d\bar{p}\|\Delta\mathbf{v}\|\|\nabla p\|^{\frac{1}{2}}\|\Delta p\|\|\nabla\Delta p\|^{\frac{1}{2}} \le \eta\|\nabla\Delta p\|^2 + d\bar{p}^{\frac{4}{3}}\|\Delta\mathbf{v}\|^{\frac{4}{3}}\|\nabla p\|^{\frac{2}{3}}\|\Delta p\|^{\frac{4}{3}}\\ &\le \eta\|\nabla\Delta p\|^2 + d\bar{p}^{\frac{4}{3}}\left(\|\Delta p\|^2\|\Delta\mathbf{v}\|^2 + \|\nabla p\|^2\right).\end{aligned}$$

For $R_8(t)$, we have

$$\begin{aligned}|R_8(t)| &= 2\left|\int_{\mathbb{R}^2}\Delta p\nabla p\cdot\nabla\Delta p\ d\mathbf{x}\right| \le 2\|\Delta p\|_{L^4}\|\nabla p\|_{L^4}\|\nabla\Delta p\| \le d\|\nabla\Delta p\|^{\frac{3}{2}}\|\nabla p\|^{\frac{1}{2}}\|\Delta p\|\\ &\le \eta\|\nabla\Delta p\|^2 + d\|\nabla p\|^2\|\Delta p\|^4 \le \eta\|\nabla\Delta p\|^2 + d\delta\|\Delta p\|^2\|\Delta p\|^2.\end{aligned}$$

For $R_9(t)$, we have

$$|R_9(t)| = \left|\int_{\mathbb{R}^2}(\Delta p)^2[p\nabla\cdot\mathbf{v} + \nabla p\cdot\mathbf{v}]d\mathbf{x}\right| \equiv |R_{91} + R_{92}|.$$

For R_{91}, we have

$$\begin{aligned}&|R_{91}| = \left|\int_{\mathbb{R}^2}(\Delta p)^2[p\nabla\cdot\mathbf{v}]d\mathbf{x}\right| \le \|\Delta p\|_{L^4}^2\|p\|_{L^4}\|\nabla\cdot\mathbf{v}\|_{L^4}\\ &\le d\|\Delta p\|\|\nabla\Delta p\|\|p\|^{\frac{1}{2}}\|\nabla p\|^{\frac{1}{2}}\|\nabla\cdot\mathbf{v}\|^{\frac{1}{2}}\|\Delta\mathbf{v}\|^{\frac{1}{2}} \le \eta\|\nabla\Delta p\|^2 + d\|\Delta p\|^2\|p\|\|\nabla p\|\|\nabla\cdot\mathbf{v}\|\|\Delta\mathbf{v}\|\\ &\le \eta\|\nabla\Delta p\|^2 + d(M_1\delta^2)^{\frac{1}{2}}\|\Delta p\|^2\|\Delta\mathbf{v}\| \le \eta\|\nabla\Delta p\|^2 + d(M_1\delta^2)^{\frac{1}{2}}\left(\|\Delta p\|^2 + \|\Delta p\|^2\|\Delta\mathbf{v}\|^2\right).\end{aligned}$$

For R_{92}, we have

$$\begin{aligned}|R_{92}| &= \left|\int_{\mathbb{R}^2}(\Delta p)^2[\nabla p\cdot\mathbf{v}]d\mathbf{x}\right| \le \|\Delta p\|_{L^4}^2\|\nabla p\|_{L^4}\|\mathbf{v}\|_{L^4}\\ &\le d\|\Delta p\|\|\nabla\Delta p\|\|\nabla p\|^{\frac{1}{2}}\|\Delta p\|^{\frac{1}{2}}\|\mathbf{v}\|^{\frac{1}{2}}\|\nabla\cdot\mathbf{v}\|^{\frac{1}{2}} \le \eta\|\nabla\Delta p\|^2 + d\|\Delta p\|^3\|\nabla p\|\|\mathbf{v}\|\|\nabla\cdot\mathbf{v}\|\\ &\le \eta\|\nabla\Delta p\|^2 + d(M_1\delta^2)^{\frac{1}{2}}\|\Delta p\|^3 \le \eta\|\nabla\Delta p\|^2 + d(M_1\delta^2)^{\frac{1}{2}}\left(\|\Delta p\|^2 + \|\Delta p\|^2\|\Delta p\|^2\right).\end{aligned}$$

For $R_{10}(t)$, we have

$$\begin{aligned}|R_{10}(t)| &= \bar{p}\left|\int_{\mathbb{R}^2}(\Delta p)^2\nabla\cdot\mathbf{v}\ d\mathbf{x}\right| \le \bar{p}\|\Delta p\|_{L^4}^2\|\nabla\cdot\mathbf{v}\| \le d\bar{p}\|\Delta p\|\|\nabla\Delta p\|\|\nabla\cdot\mathbf{v}\|\\ &\le \eta\|\nabla\Delta p\|^2 + d\bar{p}^2\|\Delta p\|^2\|\nabla\cdot\mathbf{v}\|^2 \le \eta\|\nabla\Delta p\|^2 + d\bar{p}^2\delta\|\Delta p\|^2.\end{aligned}$$

For $R_{11}(t)$, we have

$$\begin{aligned}|R_{11}(t)| &= \left|\int_{\mathbb{R}^2}(\Delta p)^3 d\mathbf{x}\right| \le \|\Delta p\|_{L^4}^2\|\Delta p\| \le d\|\Delta p\|^2\|\nabla\Delta p\|\\ &\le \eta\|\nabla\Delta p\|^2 + d\|\Delta p\|^2\|\Delta p\|^2.\end{aligned}$$

For $R_{12}(t)$, we have

$$|R_{12}(t)| = \frac{4}{\bar{p}}\left|\int_{\mathbb{R}^2} p\Delta p\nabla p\cdot\left[\mathbb{H}(p)\cdot\mathbf{v}+(\nabla\mathbf{v})^{\mathrm{T}}\cdot\nabla p+(\nabla\cdot\mathbf{v})\nabla p+p\Delta\mathbf{v}\right]d\mathbf{x}\right|$$
$$\equiv \frac{4}{\bar{p}}|R_{121}+R_{122}+R_{123}+R_{124}|.$$

For R_{121}, we have

$$|R_{121}| = \frac{4}{\bar{p}}\left|\int_{\mathbb{R}^2} p\Delta p\nabla p\cdot[\mathbb{H}(p)\cdot\mathbf{v}]\,d\mathbf{x}\right|$$
$$\le \frac{4}{\bar{p}}\|p\|_{L^\infty}\|\Delta p\|_{L^4}\|\nabla p\|_{L^4}\|\mathbb{H}(p)\|_{L^4}\|\mathbf{v}\|_{L^4} \le \frac{4}{\bar{p}}\|p\|^{\frac{1}{2}}\|\nabla\Delta p\|\|\Delta p\|^2\|\nabla p\|^{\frac{1}{2}}\|\mathbf{v}\|^{\frac{1}{2}}\|\nabla\cdot\mathbf{v}\|^{\frac{1}{2}}$$
$$\le \eta\|\nabla\Delta p\|^2+\frac{d}{\bar{p}^2}\|p\|\|\Delta p\|^4\|\nabla p\|\|\mathbf{v}\|\|\nabla\cdot\mathbf{v}\| \le \eta\|\nabla\Delta p\|^2+\frac{d}{\bar{p}^2}M_1\delta\|\Delta p\|^2\|\Delta p\|^2.$$

For R_{122}, we have

$$|R_{122}| = \frac{4}{\bar{p}}\left|\int_{\mathbb{R}^2} p\Delta p\nabla p\cdot\left[(\nabla\mathbf{v})^{\mathrm{T}}\cdot\nabla p\right]d\mathbf{x}\right| \le \frac{4}{\bar{p}}\|p\|_{L^\infty}\|\nabla p\|_{L^\infty}^2\|\Delta p\|\|\nabla\cdot\mathbf{v}\|$$
$$\le \frac{d}{\bar{p}}\|p\|^{\frac{1}{2}}\|\nabla p\|\|\Delta p\|^{\frac{3}{2}}\|\nabla\Delta p\|\|\nabla\cdot\mathbf{v}\| \le \eta\|\nabla\Delta p\|^2+\frac{d}{\bar{p}^2}\|p\|\|\nabla p\|^2\|\Delta p\|^3\|\nabla\cdot\mathbf{v}\|^2$$
$$\le \eta\|\nabla\Delta p\|^2+\frac{d}{\bar{p}^2}(M_1\delta^4)^{\frac{1}{2}}\left(\|\Delta p\|^2\|\Delta p\|^2+\|\Delta p\|^2\right).$$

For R_{123}, we have

$$|R_{123}| = \frac{4}{\bar{p}}\left|\int_{\mathbb{R}^2} p\Delta p\nabla p\cdot[(\nabla\cdot\mathbf{v})\nabla p]\,d\mathbf{x}\right| \le \frac{4}{\bar{p}}\|p\|_{L^\infty}\|\nabla p\|_{L^\infty}^2\|\Delta p\|\|\nabla\cdot\mathbf{v}\|$$
$$\le \eta\|\nabla\Delta p\|^2+\frac{d}{\bar{p}^2}(M_1\delta^4)^{\frac{1}{2}}\left(\|\Delta p\|^2\|\Delta p\|^2+\|\Delta p\|^2\right).$$

For R_{124}, we have

$$|R_{124}| = \frac{4}{\bar{p}}\left|\int_{\mathbb{R}^2} p\Delta p\nabla p\cdot[p\Delta\mathbf{v}]\,d\mathbf{x}\right| \le \frac{4}{\bar{p}}\|p\|_{L^\infty}^2\|\nabla p\|_{L^\infty}\|\Delta p\|\|\Delta\mathbf{v}\|$$
$$\le \frac{d}{\bar{p}}\|p\|\|\nabla p\|^{\frac{1}{2}}\|\Delta p\|^2\|\nabla\Delta p\|^{\frac{1}{2}}\|\Delta\mathbf{v}\| \le \eta\|\nabla\Delta p\|^2+\frac{d}{\bar{p}^{\frac{4}{3}}}\|p\|^{\frac{4}{3}}\|\nabla p\|^{\frac{2}{3}}\|\Delta p\|^{\frac{8}{3}}\|\Delta\mathbf{v}\|^{\frac{4}{3}}$$
$$\le \eta\|\nabla\Delta p\|^2+\frac{d}{\bar{p}^{\frac{4}{3}}}(M_1^2\delta)^{\frac{1}{3}}\|\Delta p\|^{\frac{4}{3}}\|\Delta p\|^{\frac{4}{3}}\|\Delta\mathbf{v}\|^{\frac{4}{3}}$$
$$\le \eta\|\nabla\Delta p\|^2+\frac{d}{\bar{p}^{\frac{4}{3}}}(M_1^2\delta)^{\frac{1}{3}}\left(\|\Delta p\|^2\|\Delta p\|^2+\|\Delta p\|^2\|\Delta\mathbf{v}\|^2\right).$$

For $R_{13}(t)$, we have

$$|R_{13}(t)| = \frac{2}{\bar{p}}\left|\int_{\mathbb{R}^2} p^2\nabla\Delta p\cdot\left[\mathbb{H}(p)\cdot\mathbf{v}+(\nabla\mathbf{v})^{\mathrm{T}}\cdot\nabla p+(\nabla\cdot\mathbf{v})\nabla p+p\Delta\mathbf{v}\right]d\mathbf{x}\right|$$
$$\equiv \frac{2}{\bar{p}}|R_{131}+R_{132}+R_{133}+R_{134}|.$$

For R_{131}, we have

$$
\begin{aligned}
|R_{131}| &= \frac{2}{\bar{p}}\left|\int_{\mathbb{R}^2} p^2\nabla\Delta p\cdot[\mathbb{H}(p)\cdot\mathbf{v}]\,d\mathbf{x}\right| \le \frac{2}{\bar{p}}\|p\|_{L^\infty}\|p\nabla\Delta p\|\|\nabla^2 p\|_{L^4}\|\mathbf{v}\|_{L^4}\\
&\le \frac{d}{\bar{p}}\|p\|^{\frac{1}{2}}\|\Delta p\|\|p\nabla\Delta p\|\nabla\Delta p\|^{\frac{1}{2}}\|\mathbf{v}\|^{\frac{1}{2}}\|\nabla\cdot\mathbf{v}\|^{\frac{1}{2}}\\
&\le 3\eta\|p\nabla\Delta p\|^{\frac{4}{3}}\|\nabla\Delta p\|^{\frac{2}{3}} + \frac{d}{\bar{p}^4}\|p\|^2\|\Delta p\|^4\|\mathbf{v}\|^2\|\nabla\cdot\mathbf{v}\|^2\\
&\le \eta\|\nabla\Delta p\|^2 + 2\eta\|p\nabla\Delta p\|^2 + \frac{d}{\bar{p}^4}M_1^2\delta\|\Delta p\|^2\|\Delta p\|^2.
\end{aligned}
$$

For R_{132}, we have

$$
\begin{aligned}
&|R_{132}| = \frac{2}{\bar{p}}\left|\int_{\mathbb{R}^2} p^2\nabla\Delta p\cdot\left[(\nabla\mathbf{v})^{\mathrm{T}}\cdot\nabla p\right]d\mathbf{x}\right|\\
&\le \frac{2}{\bar{p}}\|p\|_{L^\infty}\|p\nabla\Delta p\|\|\nabla p\|_{L^4}\|\nabla\mathbf{v}\|_{L^4} \le \frac{d}{\bar{p}}\|p\|^{\frac{1}{2}}\|\nabla p\|^{\frac{1}{2}}\|p\nabla\Delta p\|\|\Delta p\|\|\nabla\cdot\mathbf{v}\|^{\frac{1}{2}}\|\Delta\mathbf{v}\|^{\frac{1}{2}}\\
&\le \eta\|p\nabla\Delta p\|^2 + \frac{d}{\bar{p}^2}\|p\|\|\nabla p\|\|\Delta p\|^2\|\nabla\cdot\mathbf{v}\|\|\Delta\mathbf{v}\| \le \eta\|p\nabla\Delta p\|^2 + \frac{d}{\bar{p}^2}(M_1\delta^2)^{\frac{1}{2}}\|\Delta p\|^2\|\Delta\mathbf{v}\|\\
&\le \eta\|p\nabla\Delta p\|^2 + \frac{d}{\bar{p}^2}(M_1\delta^2)^{\frac{1}{2}}\left(\|\Delta p\|^2 + \|\Delta p\|^2\|\Delta\mathbf{v}\|^2\right).
\end{aligned}
$$

For R_{133}, we have

$$
\begin{aligned}
|R_{133}| &= \frac{2}{\bar{p}}\left|\int_{\mathbb{R}^2} p^2\nabla\Delta p\cdot[(\nabla\cdot\mathbf{v})\nabla p]\,d\mathbf{x}\right| \le \frac{2}{\bar{p}}\|p\|_{L^\infty}\|p\nabla\Delta p\|\|\nabla p\|_{L^4}\|\nabla\cdot\mathbf{v}\|_{L^4}\\
&\le \eta\|p\nabla\Delta p\|^2 + \frac{d}{\bar{p}^2}(M_1\delta^2)^{\frac{1}{2}}\left(\|\Delta p\|^2 + \|\Delta p\|^2\|\Delta\mathbf{v}\|^2\right).
\end{aligned}
$$

For R_{134}, we have

$$
\begin{aligned}
|R_{134}| &= \frac{2}{\bar{p}}\left|\int_{\mathbb{R}^2} p^2\nabla\Delta p\cdot[p\Delta\mathbf{v}]\,d\mathbf{x}\right| \le \frac{2}{\bar{p}}\|p\|_{L^\infty}^2\|p\nabla\Delta p\|\|\Delta\mathbf{v}\| \le \frac{d}{\bar{p}}\|p\|\|\Delta p\|\|p\nabla\Delta p\|\|\Delta\mathbf{v}\|\\
&\le \eta\|p\nabla\Delta p\|^2 + \frac{d}{\bar{p}^2}\|p\|^2\|\Delta p\|^2\|\Delta\mathbf{v}\|^2 \le \eta\|p\nabla\Delta p\|^2 + \frac{d}{\bar{p}^2}M_1\|\Delta p\|^2\|\Delta\mathbf{v}\|^2.
\end{aligned}
$$

For $R_{14}(t)$, we have

$$
\begin{aligned}
|R_{14}(t)| &= 4\left|\int_{\mathbb{R}^2} p\Delta p\nabla p\cdot\nabla(\nabla\cdot\mathbf{v})d\mathbf{x}\right| \le 4\|p\|_{L^\infty}\|\nabla p\|_{L^\infty}\|\Delta p\|\|\Delta\mathbf{v}\|\\
&\le d\|p\|^{\frac{1}{2}}\|\Delta p\|^{\frac{1}{2}}\|\nabla p\|^{\frac{1}{2}}\|\nabla\Delta p\|^{\frac{1}{2}}\|\Delta p\|\|\Delta\mathbf{v}\|\\
&\le \eta\|\nabla\Delta p\|^2 + d\|p\|^{\frac{2}{3}}\|\Delta p\|^{\frac{2}{3}}\|\nabla p\|^{\frac{2}{3}}\|\Delta p\|^{\frac{4}{3}}\|\Delta\mathbf{v}\|^{\frac{4}{3}}\\
&\le \eta\|\nabla\Delta p\|^2 + d(M_1\delta)^{\frac{1}{3}}\|\Delta p\|^{\frac{2}{3}}\|\Delta p\|^{\frac{4}{3}}\|\Delta\mathbf{v}\|^{\frac{4}{3}}\\
&\le \eta\|\nabla\Delta p\|^2 + d(M_1\delta)^{\frac{1}{3}}\left(\|\Delta p\|^2 + \|\Delta p\|^2\|\Delta\mathbf{v}\|^2\right).
\end{aligned}
$$

For $R_{15}(t)$, we have

$$|R_{15}(t)|=2\left|\int_{\mathbb{R}^2} p^2\nabla\Delta p\cdot\nabla(\nabla\cdot\mathbf{v})d\mathbf{x}\right|\le 2\|p\|_{L^\infty}^2\|\nabla\Delta p\|\|\Delta\mathbf{v}\|\le d\|p\|\|\Delta p\|\|\nabla\Delta p\|\|\Delta\mathbf{v}\|$$
$$\le\eta\|\nabla\Delta p\|^2+d\|p\|^2\|\Delta p\|^2\|\Delta\mathbf{v}\|^2\le\eta\|\nabla\Delta p\|^2+dM_1\|\Delta p\|^2\|\Delta\mathbf{v}\|^2.$$

For $R_{16}(t)$, we have

$$\begin{aligned}|R_{16}(t)|&=\frac{4}{\bar p}\left|\int_{\mathbb{R}^2} p\Delta p\nabla p\cdot\nabla\Delta p\ d\mathbf{x}\right|\le\frac{4}{\bar p}\|p\nabla\Delta p\|\|\nabla p\|_{L^4}\|\Delta p\|_{L^4}\\&\le\frac{d}{\bar p}\|p\nabla\Delta p\|\|\nabla p\|^{\frac12}\|\Delta p\|^{\frac12}\|\Delta p\|^{\frac12}\|\nabla\Delta p\|^{\frac12}\\&\le 3\eta\|p\nabla\Delta p\|^{\frac43}\|\nabla\Delta p\|^{\frac32}+\frac{d}{\bar p^4}\|\nabla p\|^2\|\Delta p\|^2\|\Delta p\|^2\\&\le\eta\|\nabla\Delta p\|^2+2\eta\|p\nabla\Delta p\|^2+\frac{d}{\bar p^4}\delta\|\Delta p\|^2\|\Delta p\|^2.\end{aligned}$$

For $R_{17}(t)$, we have

$$|R_{17}(t)|=\frac{2}{\bar p}\left|\int_{\mathbb{R}^2} p(\Delta p)^2\left[p\nabla\cdot\mathbf{v}+\nabla p\cdot\mathbf{v}\right]d\mathbf{x}\right|\equiv\frac{2}{\bar p}|R_{171}+R_{172}|.$$

For R_{171}, we have

$$\begin{aligned}|R_{171}|&=\frac{2}{\bar p}\left|\int_{\mathbb{R}^2} p(\Delta p)^2\left[p\nabla\cdot\mathbf{v}\right]d\mathbf{x}\right|\le\frac{2}{\bar p}\|p\|_{L^\infty}^2\|\Delta p\|_{L^4}^2\|\nabla\cdot\mathbf{v}\|\\&\le\frac{d}{\bar p}\|p\|\|\Delta p\|^2\|\nabla\Delta p\|\|\nabla\cdot\mathbf{v}\|\le\eta\|\nabla\Delta p\|^2+\frac{d}{\bar p^2}M_1\delta\|\Delta p\|^2\|\Delta p\|^2.\end{aligned}$$

For R_{172}, we have

$$|R_{172}|=\frac{2}{\bar p}\left|\int_{\mathbb{R}^2} p(\Delta p)^2\left[\nabla p\cdot\mathbf{v}\right]d\mathbf{x}\right|\le\frac{2}{\bar p}\|p\|_{L^\infty}\|\Delta p\|_{L^4}^2\|\nabla p\|_{L^4}\|\mathbf{v}\|_{L^4}$$
$$\le\frac{d}{\bar p}\|p\|^{\frac12}\|\Delta p\|^2\|\nabla\Delta p\|\|\nabla p\|^{\frac12}\|\mathbf{v}\|^{\frac12}\|\nabla\cdot\mathbf{v}\|^{\frac12}\le\eta\|\nabla\Delta p\|^2+\frac{d}{\bar p^2}M_1\delta\|\Delta p\|^2\|\Delta p\|^2.$$

For $R_{18}(t)$, we have

$$\begin{aligned}|R_{18}(t)|&=2\left|\int_{\mathbb{R}^2} p(\Delta p)^2\nabla\cdot\mathbf{v}\ d\mathbf{x}\right|\le 2\|p\|_{L^\infty}\|\Delta p\|_{L^4}^2\|\nabla\cdot\mathbf{v}\|\\&\le d\|p\|^{\frac12}\|\Delta p\|^{\frac32}\|\nabla\Delta p\|\|\nabla\cdot\mathbf{v}\|\le\eta\|\nabla\Delta p\|^2+d(M_1\delta^2)^{\frac12}\|\Delta p\|^3\\&\le\eta\|\nabla\Delta p\|^2+d(M_1\delta^2)^{\frac12}\left(\|\Delta p\|^2+\|\Delta p\|^2\|\Delta p\|^2\right).\end{aligned}$$

For $R_{19}(t)$, we have

$$\begin{aligned}|R_{19}(t)| &= \frac{2}{\bar{p}}\left|\int_{\mathbb{R}^2} p(\Delta p)^3 d\mathbf{x}\right| = \frac{2}{\bar{p}}\left|-\int_{\mathbb{R}^2}(\Delta p)^2|\nabla p|^2 d\mathbf{x} - 2\int_{\mathbb{R}^2} p\Delta p\nabla p\cdot\nabla\Delta p\ d\mathbf{x}\right| \\ &\le \frac{2}{\bar{p}}\|\Delta p\|_{L^4}^2\|\nabla p\|_{L^4}^2 + \eta\|\nabla\Delta p\|^2 + 2\eta\|p\nabla\Delta p\|^2 + \frac{d}{\bar{p}^4}\delta\|\Delta p\|^2\|\Delta p\|^2 \\ &\le \frac{d}{\bar{p}}\|\nabla p\|\|\Delta p\|^2\|\nabla\Delta p\| + \eta\|\nabla\Delta p\|^2 + 2\eta\|p\nabla\Delta p\|^2 + \frac{d}{\bar{p}^4}\delta\|\Delta p\|^2\|\Delta p\|^2 \\ &\le 2\eta\|\nabla\Delta p\|^2 + 2\eta\|p\nabla\Delta p\|^2 + d\left(\frac{1}{\bar{p}^2} + \frac{1}{\bar{p}^4}\right)\delta\|\Delta p\|^2\|\Delta p\|^2.\end{aligned}$$

By assembling the above estimates, we find that the RHS of (3.79) is controlled by

$$\begin{aligned}&30\eta\|\nabla\Delta p\|^2 + 7\eta\|p\nabla\Delta p\|^2 + \varepsilon\bar{p}^2\|\Delta(\nabla\cdot\mathbf{v})\|^2 + \mathcal{K}_1\left(\|\nabla p\|^2 + \|\Delta p\|^2\right) \\ &\quad + \mathcal{K}_2\left(\|\nabla p\|^2 + \|\Delta p\|^2 + \varepsilon\bar{p}\|\nabla\cdot\mathbf{v}\|^2\right)\left(\bar{p}\|\Delta p\|^2 + \bar{p}^2\|\Delta\mathbf{v}\|^2\right),\end{aligned}$$

where the constants $\mathcal{K}_1$ and $\mathcal{K}_2$ depend only on $\eta, M_1, \delta, \bar{p}$ and the constants in (3.1)–(3.5), and therefore are independent of t and remain bounded as $\varepsilon \to 0$. By choosing $\eta = \min\{\bar{p}/60, 1/(14\bar{p})\}$, we arrive at (3.80).

Appendix B. Explicit examples. Next, we provide explicit examples of initial data which fulfill the requirements of Theorems 1.1–1.2. In the **three-dimensional** case, let us consider the following functions

$$p_0(\mathbf{x}) = \begin{cases} n^{-\frac{5}{4}}\left[\sin\left(\frac{r}{n} - \frac{\pi}{2}\right) + 1\right] + B, & 2n\pi \le r \le 4n\pi, \\ B, & r \in (-\infty, 2n\pi)\cup(4n\pi, \infty); \end{cases}$$

$$\mathbf{v}_0(\mathbf{x}) = \begin{cases} n^{-\frac{5}{4}}\left[\sin\left(\frac{r}{n} - \frac{\pi}{2}\right) + 1\right]\cdot\frac{\mathbf{x}}{r}, & 2n\pi \le r \le 4n\pi, \\ \mathbf{0}, & r \in (-\infty, 2n\pi)\cup(4n\pi, \infty), \end{cases}$$

where $B > 0$ is any fixed constant, $n \in \mathbb{N}$ and $r = |\mathbf{x}|$. Then there hold that $\nabla\times\mathbf{v}_0 = \mathbf{0}$, and

$$\|p_0 - B\|^2 \cong n^{\frac{1}{2}}, \quad \|\mathbf{v}_0\|^2 \cong n^{\frac{1}{2}}, \quad \|\nabla p_0\|^2 \cong n^{-\frac{3}{2}}, \quad \|\nabla\cdot\mathbf{v}_0\|^2 \cong n^{-\frac{3}{2}}. \tag{B.1}$$

At the beginning of Section 2, we assumed that

$$\sup_{0\le t\le T}\left(\|p(t) - B\|^2 + \|\mathbf{v}(t)\|^2\right) \le N_1, \quad \sup_{0\le t\le T}\left(\|\nabla p(t)\|^2 + \|\nabla\cdot\mathbf{v}(t)\|^2\right) \le \kappa.$$

As the proof proceeded, we required that $N_1\kappa$ be smaller than some absolute constant and obtained the following

$$N_1 = (1 + 1/B)\left(\|p_1 - B\|^2 + B\|\mathbf{v}_0\|^2\right) + 1, \quad \kappa = 2(1 + 1/B)\left[\|\nabla p_0\|^2 + B\|\nabla\cdot\mathbf{v}_0\|^2\right].$$

From (B.1) we see that $N_1 \cong n^{\frac{1}{2}}$, $\kappa \cong n^{-\frac{3}{2}}$, from which we see that when $n \in \mathbb{N}$ is sufficiently large, there holds that $N_1\kappa \cong n^{-1}$. Hence, the smallness of $N_1\kappa$ can be realized by choosing $n \in \mathbb{N}$ to be sufficiently large.

In the **two-dimensional** case, let us consider the following functions

$$
\begin{aligned}
p_0(\mathbf{x}) &= \begin{cases} \frac{n}{f(n)}\left[\sin\left(\frac{r}{f(n)}-\frac{\pi}{2}\right)+1\right]+A, & 2f(n)\pi\le r\le 4f(n)\pi, \\ A, & r\in(-\infty,2f(n)\pi)\cup(4f(n)\pi,\infty); \end{cases} \\
\mathbf{v}_0(\mathbf{x}) &= \begin{cases} \frac{n}{f(n)}\left[\sin\left(\frac{r}{f(n)}-\frac{\pi}{2}\right)+1\right]\cdot\frac{\mathbf{x}}{r}, & 2f(n)\pi\le r\le 4f(n)\pi, \\ \mathbf{0}, & r\in(-\infty,2f(n)\pi)\cup(4f(n)\pi,\infty), \end{cases}
\end{aligned}
\tag{B.2}
$$

where $A>0$ is any fixed constant, $n\in\mathbb{N}$, $r=|\mathbf{x}|$ and $f(n)>0$ is to be determined. Then there holds that $\nabla\times\mathbf{v}_0=0$. As the proof in Section 3 proceeded, we obtained the following qualitative relations:

- $M_1=4(1+1/A)\ E_3(0)+1$,
- $\|\nabla p(t)\|^2+\|\nabla\cdot\mathbf{v}(t)\|^2\le\frac{2(1+1/A)}{A^2}E_4(0)\exp\left\{\frac{c}{A}(2M_1+1)E_3(0)\right\}$,

where

- $$
\begin{aligned}
E_3(0)=&\frac{1}{2}\|p_0-A\|^2+\frac{A}{2}\|\mathbf{v}\|^2-\varepsilon\int_{\mathbb{R}^2}(p_0-A)|\mathbf{v}_0|^2d\mathbf{x}-\frac{2\varepsilon+1}{6}\int_{\mathbb{R}^2}(p_0-A)^3d\mathbf{x}+\\
&k_1\int_{\mathbb{R}^2}(p_0-A)^4d\mathbf{x}+\frac{2\varepsilon^2}{A}\int_{\mathbb{R}^2}(p_0-A)^2|\mathbf{v}_0|^2d\mathbf{x}+k_2\int_{\mathbb{R}^2}(v_{01})^4+(v_{02})^4d\mathbf{x}\\
\cong&\|p_0-A\|^2+\|\mathbf{v}_0\|^2+\|p_0-A\|_{L^4}^4+\varepsilon^2\|(p_0-A)\mathbf{v}_0\|^2+\|\mathbf{v}_0\|_{L^4}^4,
\end{aligned}
$$
- $$
\begin{aligned}
E_4(0)=&\frac{A}{2}\|\nabla p_0\|^2+A^3\|\nabla\cdot\mathbf{v}_0\|^2+\frac{1}{2}\|A\nabla p_0-(p_0-A)\nabla p_0\|^2+\frac{1}{2}\|(p_0-A)\nabla p_0\|^2\\
\cong&\|\nabla p_0\|^2+\|\nabla\cdot\mathbf{v}_0\|^2+\|(p_0-A)\nabla p_0\|^2.
\end{aligned}
$$

For the functions in (B.2), direct calculations show that

$$
E_3(0)\cong n^2+\frac{n^4}{[f(n)]^2},\quad E_4(0)\cong\frac{n^2}{[f(n)]^2}+\frac{n^4}{[f(n)]^4},
\tag{B.3}
$$

which imply

$$
M_1\cong n^2+\frac{n^4}{[f(n)]^2}.
\tag{B.4}
$$

Let

$$
\delta=\frac{2(1+1/A)}{A^2}E_4(0)\exp\left\{\frac{c}{A}(2M_1+1)E_3(0)\right\}.
$$

According to (B.3), we see that

$$
\delta\cong\frac{1}{[f(n)]^2}\left(n^2+\frac{n^4}{[f(n)]^2}\right)\exp\left\{\left(n^2+\frac{n^4}{[f(n)]^2}\right)^2+\left(n^2+\frac{n^4}{[f(n)]^2}\right)\right\}.
\tag{B.5}
$$

In the proof in Section 3, we required that δ, $M_1\delta$ and $M_1\delta^2$ are smaller than some absolute constants. From (B.4) and (B.5) we see that

$$M_1\delta \cong \frac{1}{[f(n)]^2}\left(n^2+\frac{n^4}{[f(n)]^2}\right)^2 \exp\left\{\left(n^2+\frac{n^4}{[f(n)]^2}\right)^2+\left(n^2+\frac{n^4}{[f(n)]^2}\right)\right\}.$$

Hence, the smallness of δ, $M_1\delta$ and $M_1\delta^2$ can be realized by choosing $f(n)\cong e^{n^5}$ and $n\in\mathbb{N}$ to be sufficiently large. In addition, by direct calculations, we can show that $\|p_0-A\|_{L^2}^2\cong n^2$ and $\|\mathbf{v}_0\|_{L^2}^2\cong n^2$. Therefore, the L^2-norm of the zeroth frequency of the initial perturbation can be potentially large.

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