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Global Hilbert Expansion for the Relativistic Vlasov–Maxwell–Boltzmann System

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Abstract: Consider the relativistic Vlasov–Maxwell–Boltzmann system describing the dynamics of an electron gas in the presence of a fixed ion background. Thanks to recent works Germain and Masmoudi (Ann Sci Éc Norm Supér 47(3):469–503, 2014), Guo et al. (J Math Phys 55(12):123102, 2014) and Deng et al. (Arch Ration Mech Anal 225(2):771–871, 2017), we establish the global-in-time validity of its Hilbert expansion and derive the limiting relativistic Euler–Maxwell system as the mean free path goes to zero. Our method is based on the $L^2 - L^\infty$ framework and the Glassey–Strauss Representation of the electromagnetic field, with auxiliary H^1 estimates and $W^{1,\infty}$ estimates to control the characteristic curves and corresponding L^∞ norm.

Contents

1. Introduction

1.1. Relativistic Vlasov–Maxwell–Boltzmann system. The relativistic Vlasov–Maxwell–Boltzmann system is a fundamental and complete model describing the dynamics of a dilute collisional plasma appearing in nuclear fusion and the interior of stars, etc. Correspondingly, the relativistic Euler–Maxwell system, the foundation of the two-fluid theory in plasma physics, describes the dynamics of two compressible ion and electron fluids interacting with their own self-consistent electromagnetic field. It is also the origin of many celebrated dispersive PDE such as NLS, KP, KdV, Zaharov, etc, as various scaling limits and approximations of such a fundamental model. Since the ion mass is far larger than the electron mass in a plasma, the dynamics of ions is negligible for simplification sometimes. In this special case, the plasma can be approximately described by the one-species relativistic Vlasov–Maxwell–Boltzmann system in the mesoscopic level and treated as a single fluid in the macroscopic level. It has been an important open question if the general Euler–Maxwell system can be derived rigorously from its kinetic counter-part, the Vlasov–Maxwell–Boltzmann system, as the mean field path goes to zero.

In this paper, we are able to answer this question in the affirmative in the special case when the ions form a constant background, and the relativistic Euler–Maxwell system describes the dynamics of the electron gas. The relativistic Vlasov-Boltzmann system can be written as:

$$\partial_t F^\varepsilon + c\hat{p}\cdot\nabla_x F^\varepsilon - e_-\Big(E^\epsilon + \hat{p}\times B^\varepsilon\Big)\cdot\nabla_p F^\varepsilon = \frac{1}{\varepsilon}Q(F^\varepsilon, F^\varepsilon), \tag{1.1}$$

which is coupled with the Maxwell system

$$\begin{aligned}
&\partial_t E^\varepsilon - c\nabla_x\times B^\varepsilon = 4\pi e_-\int_{\mathbb{R}^3}\hat{p}F^\varepsilon dp,\\
&\partial_t B^\varepsilon + c\nabla_x\times E^\varepsilon = 0,\\
&\nabla_x\cdot E^\varepsilon = 4\pi e_-\Big(\bar{n} - \int_{\mathbb{R}^3}F^\varepsilon dp\Big),\\
&\nabla_x\cdot B^\varepsilon = 0.
\end{aligned} \tag{1.2}$$

Here ε is the Knudsen number (the mean free path), $F^\varepsilon = F^\varepsilon(t,x,p)$ is the number density function for electrons at time $t \geq 0$, position $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and momentum $p = (p_1, p_2, p_3) \in \mathbb{R}^3$. $p^0 = \sqrt{m^2c^2 + |p|^2}$ is the energy of an electron and $\hat{p} = \frac{p}{p^0}$. The constants $-e_-$ and m are the magnitude of the electrons' charge and rest mass, respectively. c is the speed of light, and $E(t,x)$, $B(t,x)$ are the electromagnetic fields.

Corresponding to (1.1)–(1.2), at the hydrodynamic level, the electron gas obeys the relativistic Euler–Maxwell system, which is an important 'two-fluid' model for a plasma:

$$\begin{cases} \frac{1}{c}\partial_t(nu^0) + \nabla_x \cdot (nu) = 0, \\ \frac{1}{c}\partial_t[(e+P)u^0u] + \nabla_x \cdot [(e+P)u \otimes u] + c^2\nabla_x P + ce_-n[u^0E + u \times B] = 0, \\ \partial_t E - c\nabla_x \times B = 4\pi e_- \frac{nu}{c}, \\ \partial_t B + c\nabla_x \times E = 0, \\ \nabla_x \cdot E = 4\pi e_-(\bar{n} - \frac{1}{c}nu^0), \\ \nabla_x \cdot B = 0, \end{cases} \tag{1.3}$$

where n is the particle number density, $u = (u_1, u_2, u_3)$, $u^0 = \sqrt{|u|^2 + c^2}$, P is the pressure, e is the total energy which is the sum of internal energy and the energy in the rest frame.

The purpose of this article is to rigorously prove that solutions of the relativistic Vlasov–Maxwell–Boltzmann system (1.1)–(1.2) converge to solutions of the relativistic Euler–Maxwell system (1.3) globally in time, as the Knudsen number ε tends to zero:

$$\sup_{0\le t\le \varepsilon^{-\frac{1}{2}}} \left(\left\|(F^\varepsilon - \mathbf{M})(t)\right\|_{H^1} + \|(E^\varepsilon - E_0)(t)\|_{H^1} + \|(B^\varepsilon - B_0)(t)\|_{H^1}\right) = O(\varepsilon). \tag{1.4}$$

Namely, the solution $(F^\varepsilon, E^\varepsilon, B^\varepsilon)$ to the relativistic Vlasov–Maxwell–Boltzmann system converges to $(\mathbf{M}, E_0, B_0)$ in the H^1 norm. The macroscopic variables n_0, u, T_0 of the Maxwellian $\mathbf{M}$ (1.12), and E_0, B_0 satisfy the relativistic Euler–Maxwell system (1.3).

Our contribution is a step forward to derive two-fluid models for describing a plasma from kinetic theory. On the other hand, due to complexity of different scalings, such a derivation for a general two-fluid model with both ions and electrons remains a major open problem.

Now we briefly explain the strategy of our proof. Detailed explanations will be followed after the statement of Theorem 1.1. In fact, the main part of our proof is constructing solutions to equations of remainder terms F_R^ε, E_R^ε and B_R^ε in the Hilbert expansion (1.8) for the relativistic Vlasov–Maxwell–Boltzmann system in a $H^1 - W^{1,\infty}$ framework. First, it is simple to get the L^2 estimate of the remainder terms. Under a priori assumptions (1.27) for the $W^{1,\infty}$ estimates of the remainder terms, we can proceed characteristics estimates and further obtain the L^∞ estimate of F_R^ε to close the $L^2 - L^\infty$ energy estimates. To justify the assumptions (1.27), the essential and delicate part is the $W^{1,\infty}$ estimates of the electromagnetic fields F_R^ε, E_R^ε via the Glassey–Strauss Representation. We succeed to bound the $W^{1,\infty}$ estimates of the electromagnetic fields through the combination of H^1 and $W^{1,\infty}$ estimates of F_R^ε properly. Then we modify the corresponding proof in [22] and bound the $W^{1,\infty}$ norm of F_R^ε with its H^1 norm. Finally, we derive the H^1 norm estimates of the remainder terms to close the energy estimates and verify the assumptions (1.27) for $t \in [0, \varepsilon^{-1/2}]$.

The relativistic Boltzmann collision operator $Q(\cdot,\cdot)$ in (1.1) takes the form of

$$Q(F,G) = \frac{c}{p^0}\int_{\mathbb{R}^3}\frac{dq}{q^0}\int_{\mathbb{R}^3}\frac{dq'}{q'^0}\int_{\mathbb{R}^3}\frac{dp'}{p'^0} W[F(p')G(q') - F(p)G(q)]. \tag{1.5}$$

Here the "transition rate" $W = W(p, q|p', q')$ is defined as

$$W = s\sigma(g, \theta)\delta(p^0 + q^0 - p'^0 - q'^0)\delta^{(3)}(p + q - p' - q'). \tag{1.6}$$

In a pioneering work of Glassey and Strauss [16], the collision operator Q in (1.5) was represented as follows:

$$Q(F, G) = \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} d\omega \frac{s\sigma(g, \theta)}{p^0 q^0} \mathbb{B}(p, q, \omega)[F(p')G(q') - F(p)G(q)],$$

where the kernel $\mathbb{B}(p, q, \omega)$ is

$$\mathbb{B}(p, q, \omega) = \frac{(p^0 + q^0)^2 |\omega \cdot (p^0 q - q^0 p)|}{[(p^0 + q^0)^2 - (\omega \cdot (p + q))^2]^2}.$$

Denote the four-momentums $p^\mu = (p^0, p)$ and $q^\mu = (q^0, q)$. We use the Einstein convention that repeated up-down indices are summed, and we raise and lower indices using the Minkowski metric $g_{\mu\nu} \stackrel{\text{def}}{=} diag(-1, 1, 1, 1)$. The Lorentz inner product is then given by

$$p^\mu q_\mu \stackrel{\text{def}}{=} -p^0 q^0 + \sum_{i=1}^{3} p_i q_i.$$

The quantity s in (1.6) is the square of the energy in the "center of momentum", $p+q = 0$, and is given as

$$s = s(p, q) = -(p^\mu + q^\mu)(p_\mu + q_\mu) = 2(p^0 q^0 - p \cdot q + m^2 c^2) \geq 4m^2 c^2.$$

And the relativistic momentum g in (1.6) is denoted as

$$g = g(p, q) = \sqrt{-(p^\mu - q^\mu)(p_\mu - q_\mu)} = \sqrt{2(p^0 q^0 - p \cdot q - m^2 c^2)} \geq 0.$$

The condition for elastic collision is then given by

$$p^0 + q^0 = p'^0 + q'^0, \qquad p + q = p' + q', \tag{1.7}$$

where p' and q' are the post collisional momentums given as

$$p' = p + a(p, q, \omega)\omega, \qquad q' = q - a(p, q, \omega)\omega,$$
$$a(p, q, \omega) = \frac{2(p^0 + q^0)[\omega \cdot (p^0 q - q^0 p)]}{(p^0 + q^0)^2 - (\omega \cdot (p + q))^2}.$$

The Jacobian for the transformation $(p, q) \to (p', q')$ in these variables [15] is

$$\frac{\partial(p', q')}{\partial(p, q)} = -\frac{p'^0 q'^0}{p^0 q^0}.$$

The relativistic differential cross section $\sigma(g, \theta)$ measures the interactions between particles. See [10,11] for a physical discussion of general assumptions. We use the following hypothesis.
Hypothesis on the collision kernel. We consider the "hard ball" condition

$$\sigma(g, \theta) = \text{constant}.$$

This condition is used throughout the rest of the article. In fact, without loss of generality, we will use the normalized condition $\sigma(g, \theta) = 1$ for simplicity. The Newtonian limit, as $c \to \infty$, in this situation is the Newtonian hard-sphere Boltzmann collision operator [36].

1.2. Hilbert expansion. We consider the Hilbert expansion for small Knudsen number ε,

$$
\begin{aligned}
F^{\varepsilon} &= \sum_{n=0}^{2k-1} \varepsilon^n F_n + \varepsilon^k F_R^{\varepsilon}, \\
E^{\varepsilon} &= \sum_{n=0}^{2k-1} \varepsilon^n E_n + \varepsilon^k E_R^{\varepsilon}, \\
B^{\varepsilon} &= \sum_{n=0}^{2k-1} \varepsilon^n B_n + \varepsilon^k B_R^{\varepsilon}, \qquad (k \geq 3).
\end{aligned} \tag{1.8}
$$

To determine the coefficients $F_0(t,x,p),\dots,F_{2k-1}(t,x,p)$; $E_0(t,x),\dots,E_{2k-1}(t,x)$; $B_0(t,x),\dots,B_{2k-1}(t,x)$, we plug the formal expansion (1.8) into the rescaled equations (1.1)–(1.2) to have

$$
\begin{aligned}
&\partial_t\Big(\sum_{n=0}^{2k-1} \varepsilon^n F_n + \varepsilon^k F_R^{\varepsilon}\Big) + c\hat{p}\cdot\nabla_x\Big(\sum_{n=0}^{2k-1} \varepsilon^n F_n + \varepsilon^k F_R^{\varepsilon}\Big) \\
&\quad - e_-\Big[\Big(\sum_{n=0}^{2k-1} \varepsilon^n E_n + \varepsilon^k E_R^{\varepsilon}\Big) + \hat{p}\times\Big(\sum_{n=0}^{2k-1} \varepsilon^n B_n + \varepsilon^k B_R^{\varepsilon}\Big)\Big]\cdot\nabla_p\Big(\sum_{n=0}^{2k-1} \varepsilon^n F_n + \varepsilon^k F_R^{\varepsilon}\Big) \\
&\ = \frac{1}{\varepsilon}Q\Big(\sum_{n=0}^{2k-1} \varepsilon^n F_n + \varepsilon^k F_R^{\varepsilon}, \sum_{n=0}^{2k-1} \varepsilon^n F_n + \varepsilon^k F_R^{\varepsilon}\Big)), \\
&\quad \partial_t\Big(\sum_{n=0}^{2k-1} \varepsilon^n E_n + \varepsilon^k E_R^{\varepsilon}\Big) - c\nabla_x\times\Big(\sum_{n=0}^{2k-1} \varepsilon^n B_n + \varepsilon^k B_R^{\varepsilon}\Big) \\
&\ = 4\pi e_-\int_{\mathbb{R}^3}\hat{p}\Big(\sum_{n=0}^{2k-1} \varepsilon^n F_n + \varepsilon^k F_R^{\varepsilon}\Big)dp, \\
&\quad \partial_t\Big(\sum_{n=0}^{2k-1} \varepsilon^n B_n + \varepsilon^k B_R^{\varepsilon}\Big) + c\nabla_x\times\Big(\sum_{n=0}^{2k-1} \varepsilon^n E_n + \varepsilon^k E_R^{\varepsilon}\Big) = 0, \\
&\quad \nabla_x\cdot\Big(\sum_{n=0}^{2k-1} \varepsilon^n E_n + \varepsilon^k E_R^{\varepsilon}\Big) = 4\pi e_-\Big(\bar{n} - \int_{\mathbb{R}^3}\Big(\sum_{n=0}^{2k-1} \varepsilon^n F_n + \varepsilon^k F_R^{\varepsilon}\Big)dp\Big), \\
&\quad \nabla_x\cdot\Big(\sum_{n=0}^{2k-1} \varepsilon^n B_n + \varepsilon^k B_R^{\varepsilon}\Big) = 0.
\end{aligned} \tag{1.9}
$$

Now we equate the coefficients on both sides of equation (1.9) in front of different powers of the parameter ε to obtain:

$$
\begin{aligned}
&\frac{1}{\varepsilon} : Q(F_0, F_0) = 0, \\
&\varepsilon^0 : \partial_t F_0 + c\hat{p}\cdot\nabla_x F_0 - e_-\Big(E_0 + \hat{p}\times B_0\Big)\cdot\nabla_p F_0 = Q(F_1, F_0) + Q(F_0, F_1),
\end{aligned}
$$

$$
\begin{aligned}
&\partial_t E_0 - c\nabla_x \times B_0 = 4\pi e_- \int_{\mathbb{R}^3} \hat{p} F_0 dp,\\
&\partial_t B_0 + c\nabla_x \times E_0 = 0,\\
&\nabla_x \cdot E_0 = 4\pi e_- \Big(\bar{n} - \int_{\mathbb{R}^3} F_0 dp\Big),\\
&\nabla_x \cdot B_0 = 0,\\
&\cdots\cdots\\
\varepsilon^n :&\partial_t F_n + c\hat{p}\cdot\nabla_x F_n - e_-\Big(E_n + \hat{p}\times B_n\Big)\cdot\nabla_p F_0 - e_-\Big(E_0 + \hat{p}\times B_0\Big)\cdot\nabla_p F_n\\
&= \sum_{\substack{i+j=n+1\\ i,j\geq 0}} Q(F_i, F_j) + e_- \sum_{\substack{i+j=n\\ i,j\geq 1}} \Big(E_i + \hat{p}\times B_i\Big)\cdot\nabla_p F_j,\\
&\partial_t E_n - c\nabla_x \times B_n = 4\pi e_- \int_{\mathbb{R}^3} \hat{p} F_n dp,\\
&\partial_t B_n + c\nabla_x \times E_n = 0,\\
&\nabla_x \cdot E_n = -4\pi e_- \int_{\mathbb{R}^3} F_n dp,\\
&\nabla_x \cdot B_n = 0,\\
&\cdots\cdots\\
\varepsilon^{2k-1} :&\partial_t F_{2k-1} + c\hat{p}\cdot\nabla_x F_{2k-1} - e_-\Big(E_{2k-1} + \hat{p}\times B_{2k-1}\Big)\cdot\nabla_p F_0\\
&- e_-\Big(E_0 + \hat{p}\times B_0\Big)\cdot\nabla_p F_{2k-1}\\
&= \sum_{\substack{i+j=2k\\ i,j\geq 1}} Q(F_i, F_j) + e_- \sum_{\substack{i+j=2k-1\\ i,j\geq 1}} \Big(E_i + \hat{p}\times B_i\Big)\cdot\nabla_p F_j,\\
&\partial_t E_{2k-1} - c\nabla_x \times B_{2k-1} = 4\pi e_- \int_{\mathbb{R}^3} \hat{p} F_{2k-1} dp,\\
&\partial_t B_{2k-1} + c\nabla_x \times E_{2k-1} = 0,\\
&\nabla_x \cdot E_{2k-1} = -4\pi e_- \int_{\mathbb{R}^3} F_{2k-1} dp,\\
&\nabla_x \cdot B_{2k-1} = 0.
\end{aligned}
\tag{1.10}
$$

The remainder terms F_R^ε, E_R^ε and B_R^ε satisfy the following equations:

$$
\begin{aligned}
&\partial_t F_R^\varepsilon + c\hat{p}\cdot\nabla_x F_R^\varepsilon - e_-\Big(E_R^\varepsilon + \hat{p}\times B_R^\varepsilon\Big)\cdot\nabla_p F_0\\
&\quad - e_-\Big(E_0 + \hat{p}\times B_0\Big)\cdot\nabla_p F_R^\varepsilon - \frac{1}{\varepsilon}[Q(F_R^\varepsilon, F_0) + Q(F_0, F_R^\varepsilon)]\\
&= \varepsilon^{k-1} Q(F_R^\varepsilon, F_R^\varepsilon) + \sum_{i=1}^{2k-1} \varepsilon^{i-1}[Q(F_i, F_R^\varepsilon) + Q(F_R^\varepsilon, F_i)] + \varepsilon^k e_-\Big(E_R^\varepsilon + \hat{p}\times B_R^\varepsilon\Big)\cdot\nabla_p F_R^\varepsilon\\
&\quad + \sum_{i=1}^{2k-1} \varepsilon^i e_-\Big[\Big(E_i + \hat{p}\times B_i\Big)\cdot\nabla_p F_R^\varepsilon + \Big(E_R^\varepsilon + \hat{p}\times B_R^\varepsilon\Big)\cdot\nabla_p F_i\Big] + \varepsilon^k A,
\end{aligned}
$$

$$
\begin{aligned}
&\partial_t E_R^\varepsilon - c\nabla_x \times B_R^\varepsilon = 4\pi e_- \int_{\mathbb{R}^3} \hat{p} F_R^\varepsilon dp,\\
&\partial_t B_R^\varepsilon + c\nabla_x \times E_R^\varepsilon = 0,\\
&\nabla_x \cdot E_R^\varepsilon = -4\pi e_- \int_{\mathbb{R}^3} F_R^\varepsilon dp,\\
&\nabla_x \cdot B_R^\varepsilon = 0,
\end{aligned}
\tag{1.11}
$$

where

$$
A = \sum_{\substack{i+j\geq 2k+1\\ 2\leq i,j\leq 2k-1}} \varepsilon^{i+j-2k-1} Q(F_i, F_j) + \sum_{\substack{i+j\geq 2k\\ 1\leq i,j\leq 2k-1}} \varepsilon^{i+j-2k} e_-\Big(E_i + \hat{p}\times B_i\Big)\cdot\nabla_p F_j.
$$

From the first equation in (1.10), we can obtain that F_0 should be a local Maxwellian $\mathbf{M} = F_0$:

$$
F_0(t,x,p) = \frac{n_0\gamma}{4\pi m^3 c^3 K_2(\gamma)} \exp\left\{\frac{u^\mu p_\mu}{k_B T_0}\right\},
\tag{1.12}
$$

where $K_j(\gamma)(j = 0, 1, 2, \ldots)$ are the modified second order Bessel functions:

$$
K_j(\gamma) = \frac{(2^j)j!}{(2j)!}\frac{1}{\gamma^j}\int_{\lambda=\gamma}^{\lambda=\infty} e^{-\lambda}(\lambda^2-\gamma^2)^{j-1/2}d\lambda, \quad (j\geq 0),
$$

and γ is a dimensionless variable defined as

$$
\gamma = \frac{mc^2}{k_B T_0},
\tag{1.13}
$$

k_B is Boltzmann's constant. Here $n_0(t,x)$, $u^\mu(t,x)$ and $T_0(t,x)$ are the number density, four-velocity and temperature. As proved in Proposition 3.3 of [33], it holds that

$$
\begin{aligned}
I^\alpha[\mathbf{M}] &= c\int_{\mathbb{R}^3} \frac{p^\alpha}{p^0}\mathbf{M}\, dp = n_0 u^\alpha,\\
T^{\alpha\beta}[\mathbf{M}] &= c\int_{\mathbb{R}^3} \frac{p^\alpha p^\beta}{p^0}\mathbf{M}\, dp = \frac{e_0+P_0}{c^2}u^\alpha u^\beta + P_0 g^{\alpha\beta},
\end{aligned}
\tag{1.14}
$$

where $e_0(t,x)$ is the total energy and $P_0(t,x)$ is the pressure satisfying

$$
\begin{aligned}
P_0 &= \frac{mnc^2}{\gamma} = \frac{k_B}{m}\rho T,\\
e_0 &= \frac{n_0 mc^2}{K_2(\gamma)}\left[K_3(\gamma) - \frac{1}{\gamma}K_2(\gamma)\right].
\end{aligned}
$$

Noting the relationship $p^0 = \sqrt{m^2c^2 + |p|^2}$ and

$$
\begin{aligned}
&-\int_{\mathbb{R}^3} p^0\Big(E_0 + \hat{p}\times B_0\Big)\cdot\nabla_p \mathbf{M} dp\\
&\quad = \int_{\mathbb{R}^3} E_0\cdot\hat{p}\mathbf{M} dp = \frac{n_0 u\cdot E_0}{c},
\end{aligned}
$$

we project the second equation in (1.10) onto the five collision invariants, 1, p, p^0, for the relativistic Boltzmann operator Q and use (1.14) to have

$$\begin{cases} \frac{1}{c}\partial_t(n_0u^0)+\nabla_x\cdot(n_0u)=0, \\ \frac{1}{c}\partial_t[(e_0+P_0)u^0u]+\nabla_x\cdot[(e_0+P_0)u\otimes u] \\ \qquad\qquad +c^2\nabla_x P_0+ce_-n_0[u^0E_0+u\times B_0]=0, \\ \frac{1}{c}\partial_t[(e_0+P_0)(u^0)^2-c^2|u|^2P_0]+\nabla_x\cdot[(e_0+P_0)u^0u]+ce_-n_0u\cdot E_0=0. \end{cases} \tag{1.15}$$

Then, under the conditions

$$e_0+P_0=n_0h(n_0), \qquad \text{and} \qquad P_0'(n_0)=n_0h'(n_0)>0, \tag{1.16}$$

which were indicated in [20], we can further use the equations of ε^0 power in (1.10) to obtain equations of n_0, u and E_0, B_0:

$$\begin{cases} \frac{1}{c}\partial_t(n_0u^0)+\nabla_x\cdot(n_0u)=0, \\ \frac{1}{c}\partial_t[(e_0+P_0)u^0u]+\nabla_x\cdot[(e_0+P_0)u\otimes u] \\ \qquad\qquad +c^2\nabla_x P_0+ce_-n_0[u^0E_0+u\times B_0]=0, \\ \partial_t E_0-c\nabla_x\times B_0=4\pi e_-\frac{n_0u}{c}, \\ \partial_t B_0+c\nabla_x\times E_0=0, \\ \nabla_x\cdot E_0=4\pi e_-(\bar{n}-\frac{1}{c}n_0u^0), \\ \nabla_x\cdot B_0=0. \end{cases} \tag{1.17}$$

In fact, we use $(1.15)_1$ to have

$$\begin{aligned} &\frac{n_0u^0}{c}\partial_t\Big(\frac{e_0+P_0}{n_0}u\Big)+n_0u\cdot\nabla_x\Big(\frac{e_0+P_0}{n_0}u\Big) \\ &\qquad\qquad +c^2\nabla_x P_0+ce_-n_0[u^0E_0+u\times B_0]=0, \\ &\frac{n_0u^0}{c}\partial_t\Big(\frac{e_0+P_0}{n_0}u^0\Big)-c\partial_t P_0+n_0\cdot\nabla_x\cdot\Big(\frac{e_0+P_0}{n_0}u\Big)+ce_-n_0u\cdot E_0=0. \end{aligned} \tag{1.18}$$

We further employ $u^0(1.18)_2-u\cdot(1.18)_1$ to obtain

$$u^0\Big[n_0\partial_t\Big(\frac{e_0+P_0}{n_0}\Big)-\partial_t P_0\Big]+u\cdot\Big[n_0\nabla_x\Big(\frac{e_0+P_0}{n_0}\Big)-\nabla_x P_0\Big]=0. \tag{1.19}$$

(1.19) automatically holds under the conditions (1.16). Namely, the third equation in (1.15), the energy equation, can be expressed by the first and second equations in (1.15), and (1.17) holds true.

Here and below, we assume that $[n_0(t,x), u(t,x), E_0(t,x), B_0(t,x)]$ is a global smooth solution to the relativistic Euler–Maxwell 1-fluid system (1.17) constructed in [20] with $n_0(t,x)-\bar{n}$, $u(t,x)$, $E_0(t,x)$ and $B_0(t,x)$ sufficiently small and satisfying

$$\begin{aligned} &\sup_{t\in[0,\infty]}\|[n_0(t)-\bar{n}, u(t), E_0(t), B_0(t)]\|_{H^{N_0}}+\sup_{t\in[0,\infty]}\Big[(1+t)^{\beta_0} \\ &\times\big(\sup_{|\rho|\le 3}\|D_x^\rho(n_0(t)-\bar{n})\|_\infty+\sup_{|\rho|\le 4}\|[D_x^\rho u(t), D_x^\rho E_0(t), D_x^\rho B_0(t)]\|_\infty\big)\Big]\lesssim\bar{\varepsilon}_0, \end{aligned} \tag{1.20}$$

where $N_0 = 10^4$, $\beta_0 = 101/100$ and $\bar{\varepsilon}_0$ is a small positive constant.

For $\mathbf{M}$ given in (1.12), where $(n_0, u, T_0)(t, x)$ is part of a solution to the relativistic Euler–Maxwell 1-fluid system (1.17) constructed in [20], we define the linearized collision operator Lf and nonlinear collision operator $\Gamma(f_1, f_2)$:

$$
\begin{aligned}
Lf &= -\frac{1}{\sqrt{\mathbf{M}}}[Q(\sqrt{\mathbf{M}}f, \mathbf{M}) + Q(\mathbf{M}, \sqrt{\mathbf{M}}f)] = \nu(\mathbf{M})f - K_{\mathbf{M}}(f),\\
\Gamma(f_1, f_2) &= \frac{1}{\sqrt{\mathbf{M}}}Q(\sqrt{\mathbf{M}}f_1, \sqrt{\mathbf{M}}f_2),
\end{aligned}
$$

where the collision frequency $\nu = \nu(t, x, p)$ is defined as

$$
\nu = \Gamma^{loss}(1, \sqrt{\mathbf{M}}) = \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} d\omega \frac{s}{p^0 q^0} \mathbb{B}(p, q, \omega)\sqrt{\mathbf{M}}(q),
$$

and $K_{\mathbf{M}}(f)$ takes the following form:

$$
\begin{aligned}
K_{\mathbf{M}}(f) &= \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} d\omega \frac{s}{p^0 q^0} \mathbb{B}(p, q, \omega)\sqrt{\mathbf{M}}(q)[\sqrt{\mathbf{M}}(q')f(p') + \sqrt{\mathbf{M}}(p')f(q')]\\
&\quad - \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} d\omega \frac{s}{p^0 q^0} \mathbb{B}(p, q, \omega)\sqrt{\mathbf{M}}(q)\sqrt{\mathbf{M}}(p)f(q)\\
&= K_2(f) - K_1(f).
\end{aligned}
\tag{1.21}
$$

Then from Lemma 3.1 in [37] and Lemma 6 in [25], we can similarly obtain

$$
\nu(t, x, p) \approx 1, \qquad \text{and} \qquad |D_p^\rho \nu(t, x, p)| \lesssim (p^0)^{-1}, \quad |\rho| > 0.
$$

Note that the null space of the linearized operator L is given by

$$
\mathcal{N} = \text{span}\left\{\sqrt{\mathbf{M}}, p_i\sqrt{\mathbf{M}}(1 \le i \le 3), p^0\sqrt{\mathbf{M}}\right\}.
$$

Let $\mathbf{P}$ be the orthogonal projection from L^2_p onto $\mathcal{N}$. Given $f(t, x, p)$, one can express $\mathbf{P}f$ as a linear combination of the basis in $\mathcal{N}$:

$$
\mathbf{P}f = \{a_f(t, x) + b_f(t, x) \cdot p + c_f(t, x)p^0\}\sqrt{\mathbf{M}}.
$$

Then we have [16]

$$
\langle Lf, f\rangle \ge \delta_0 \|\{\mathbf{I} - \mathbf{P}\}f\|^2,
$$

for some constant $\delta_0 > 0$. Define f^ε as

$$
F_R^\varepsilon = \sqrt{\mathbf{M}} f^\varepsilon. \tag{1.22}
$$

We further introduce a global Maxwellian

$$
J_M = \frac{n_M \gamma_M}{4\pi m^3 c^3 K_2(\gamma_M)} \exp\{-\frac{p^0}{k_B T_M}\},
$$

and define

$$
F_R^\varepsilon = (1 + |p|)^{-\beta}\sqrt{J_M} h^\varepsilon = \frac{\sqrt{J_M}}{w(|p|)} h^\varepsilon, \tag{1.23}
$$

with $w(|p|) \equiv (1+|p|)^{\beta}$ for some $\beta \geq 8$. Here $n_M, T_M = mc^2/(k_B\gamma_M)$ satisfy the conditions in (1.16) and

$$T_M < \sup_{t\in[0,\varepsilon^{-\frac{1}{2}}],x\in\mathbb{R}^3} T_0(t,x) < 2T_M. \tag{1.24}$$

Remark 1.1. Since the presence of physical constants do not cause essential mathematical difficulties, we will normalize all constants in the relativistic Vlasov–Maxwell–Boltzmann system (1.1), (1.2) and in all related quantities to be one.

1.3. Notations. Throughout the paper, C denotes a generic positive constant which may change line by line. The notation $A \lesssim B$ implies that there exists a positive constant C such that $A \leq CB$ holds uniformly over the range of parameters. The notation $A \approx B$ means $\frac{1}{\bar{C}}A \leq B \leq \bar{C}A$ for some constant $\bar{C} > 1$. We use standard notations to denote the Sobolev spaces $W^{k,2}(\mathbb{R}^3_x)$ (or $W^{k,2}(\mathbb{R}^3_x \times \mathbb{R}^3_p)$) and $W^{k,\infty}(\mathbb{R}^3_x)$ (or $W^{k,\infty}(\mathbb{R}^3_x \times \mathbb{R}^3_p)$) with corresponding norms $\|\cdot\|_{H^k}$ and $\|\cdot\|_{W^{k,\infty}}$, respectively. We also use the standard notations $\|\cdot\|$ and $\|\cdot\|_\infty$ to denote the L^2 norm and L^∞ norm in both $(x,p) \in \mathbb{R}^3 \times \mathbb{R}^3$ and $x \in \mathbb{R}^3$, respectively. The standard $L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ inner product is denoted as $\langle \cdot, \cdot \rangle$ for simplicity. D_x and D_p are used as any space and momentum derivative, respectively.

1.4. Main results. We now state our main results.

Theorem 1.1. *Let $F_0 = \mathbf{M}$ as in (1.12) and let $[n_0(0,x), u(0,x), E_0(0,x), B_0(0,x)]$ satisfies the same assumptions in Theorem 2.2 of [20] and $[n_0(t,x), u(t,x), E_0(t,x), B_0(t,x)]$ be a corresponding global solution. Then for the remainder terms F_R^ε, E_R^ε and B_R^ε in (1.8), there exists an $\varepsilon_0 > 0$ such that for $0 \leq \varepsilon \leq \varepsilon_0$,*

$$\begin{aligned}
&\sup_{0\leq t\leq\varepsilon^{-\frac{1}{2}}} \left(\varepsilon^{\frac{3}{2}} \left\| \frac{(1+|p|)^{\beta} F_R^\varepsilon}{\sqrt{J_M}}(t) \right\|_\infty + \varepsilon^2 \left\| \frac{(1+|p|)^{\beta} F_R^\varepsilon}{\sqrt{J_M}}(t) \right\|_{W^{1,\infty}} \right) \\
&\quad + \sup_{0\leq t\leq\varepsilon^{-\frac{1}{2}}} \left(\varepsilon^{\frac{5}{2}} \|[E_R^\varepsilon(t), B_R^\varepsilon(t)]\|_{W^{1,\infty}} \right) + \sup_{0\leq t\leq\varepsilon^{-\frac{1}{2}}} \left[\left\| \frac{F_R^\varepsilon}{\sqrt{\mathbf{M}}}(t) \right\| \right. \\
&\quad \left. + \|[E_R^\varepsilon(t), B_R^\varepsilon(t)]\| + \sqrt{\varepsilon} \left(\left\| \frac{F_R^\varepsilon}{\sqrt{\mathbf{M}}}(t) \right\|_{H^1} + \|[E_R^\varepsilon(t), B_R^\varepsilon(t)]\|_{H^1} \right) \right] \\
&\lesssim \varepsilon^{\frac{3}{2}} \left\| \frac{(1+|p|)^{\beta} F_R^\varepsilon}{\sqrt{J_M}}(0) \right\|_\infty + \varepsilon^2 \left\| \frac{(1+|p|)^{\beta} F_R^\varepsilon}{\sqrt{J_M}}(0) \right\|_{W^{1,\infty}} \\
&\quad + \varepsilon^{\frac{5}{2}} \|[E_R^\varepsilon(0), B_R^\varepsilon(0)]\|_{W^{1,\infty}} + \left\| \frac{F_R^\varepsilon}{\sqrt{\mathbf{M}}}(0) \right\| + \|[E_R^\varepsilon(0), B_R^\varepsilon(0)]\| \\
&\quad + \sqrt{\varepsilon} \left(\left\| \frac{F_R^\varepsilon}{\sqrt{\mathbf{M}}}(0) \right\|_{H^1} + \|[E_R^\varepsilon(0), B_R^\varepsilon(0)]\|_{H^1} \right) + 1.
\end{aligned} \tag{1.25}$$

Remark 1.2. We only require $k \geq 3$ for the expansion in (1.8). This requirement is the same as the case of the Hilbert expansion for the Boltzmann equation [23,24,33] and more relaxed than that for the Vlasov–Poisson–Boltzmann system [22]. Moreover, our uniform estimates lead to the relativistic Euler–Maxwell limit (1.4).

Our result guarantees that the Hilbert expansion for the relativistic Vlasov–Maxwell–Boltzmann system is valid for any time if ε is chosen sufficiently small. Similar result was also obtained in [22] for the Hilbert expansion of the Vlasov–Poisson–Boltzmann system. Due to shock formations in the pure compressible Euler flow, as illustrated in [5,32], corresponding Hilbert expansion, acoustic limit for the Boltzmann equation or relativistic Boltzmann equation [4,23,24,33] is only valid local in time. Different from the pure compressible Euler fluids where shock waves may develop even for smooth irrotational initial data with small amplitude, the electromagnetic interaction in the two-fluid models [17,20,21] could create stronger dispersive effects, enhance linear decay rates, and prevent formation of shock waves with small amplitude.

Our method is based on an $L^2 - L^\infty$ framework originated in [19]. In Sect. 2, we first establish the basic L^2 estimate for remainders $(f^\varepsilon, E_R^\varepsilon, B_R^\varepsilon)$. In order to close the energy estimate (2.3) in Proposition 2.1, we need L^∞ estimate of h^ε along the curved characteristic given by

$$\begin{aligned}
\frac{dX(\tau;t,x,p)}{d\tau} &= \hat{P}(\tau;t,x,p),\\
\frac{dP(\tau;t,x,p)}{d\tau} &= -E^\varepsilon(\tau, X(\tau;t,x,p)) - \hat{P}(\tau;t,x,p)\times B^\varepsilon(\tau, X(\tau;t,x,p)),
\end{aligned} \tag{1.26}$$

with $X(t;t,x,p) = x$, $P(t;t,x,p) = p$.

In Sect. 3, we study the characteristics and L^∞ estimate of h^ε under the crucial bootstrap assumptions:

$$\begin{aligned}
\sup_{t\in[0,T]} \varepsilon^2 \|h^\varepsilon(t)\|_\infty &\le \varepsilon^{\frac{3}{8}},\\
\sup_{t\in[0,T]} \varepsilon^{\frac{5}{2}} \Big(\|\nabla_x h^\varepsilon(t)\|_\infty + \|\nabla_p h^\varepsilon(t)\|_\infty \Big) &\le \varepsilon^{\frac{1}{4}},\\
\sup_{t\in[0,T]} \varepsilon^{\frac{11}{4}} \|[E_R^\varepsilon(t), B_R^\varepsilon(t)]\|_{W^{1,\infty}} &\le 1,
\end{aligned} \tag{1.27}$$

for given $T \in [0, \varepsilon^{-\frac{1}{2}}]$. We can repeat the L^∞ estimate of h^ε in Sect. 5.1 of [22] and obtain Proposition 3.1. Combing Propositions 2.1 and 3.1, we close the $L^2 - L^\infty$ energy estimates. The rest of the paper is devoted to the proof of (1.27).

In Sect. 4, we estimate the $W^{1,\infty}$ norm of E_R^ε, B_R^ε in term of the $W^{1,\infty}$ norm of f^ε, via Glassey–Strauss Representation (see Theorem 3 and Theorem 4 in [14]). In the previous work [22], the electric field E_R^ε was calculated via elliptic estimate of the Poisson system:

$$E_R^\varepsilon = -\nabla_x \phi_R^\varepsilon, \qquad \triangle \phi_R^\varepsilon = \rho_R^\varepsilon, \quad \rho_R^\varepsilon \in L^\infty.$$

It is important to note that the relativistic nature of bounded velocity $\hat{p}$ and the strong weight $(1+|p|)^k$ play key roles. We remark that the $W^{1,\infty}$ estimate (4.1) in Proposition 4.1 fails for the classical Vlasov–Maxwell–Boltzmann system with unbounded velocity. Our method relies crucially on the Glassey–Strauss representation, which fails for the non-relativistic case. Once the relativistic two-fluid model is derived, it is then possible to study the classical (non-relativistic) limit as the speed of light goes to ∞. It is possible to extend our result to the 2D case, with a similar Glassey–Strauss formulation in $2D$ [13].

In Sect. 5, we establish the $W^{1,\infty}$ estimate for h^ε. In [22], such an estimate relies on a second integration by parts

$$\begin{aligned} D_x h^\varepsilon(t,x,v) &\sim \ldots + \frac{1}{\varepsilon^2}\int_0^t\int_0^s \exp\Big\{-\frac{1}{\varepsilon}\int_s^t \nu(\tau)d\tau - \frac{1}{\varepsilon}\int_{s_1}^s \nu(\tau)d\tau\Big\} \\ &\quad\times \int_{\mathbb{R}^3\times\mathbb{R}^3} l_{M,w}(V(s),v')l_{M,w}(V(s_1),v'')D_x h^\varepsilon(s_1,X(s_1),v'')|\frac{dv'}{dy}|dv''yds_1ds + \ldots \\ &\sim \ldots - \frac{1}{\varepsilon^2}\int_0^t\int_0^{s-\kappa\varepsilon} \exp\Big\{-\frac{1}{\varepsilon}\int_s^t \nu(\tau)d\tau - \frac{1}{\varepsilon}\int_{s_1}^s \nu(\tau)d\tau\Big\} \\ &\quad\times \int_{\hat{B}} l_N(V(s),v')l_N(V(s_1),v'')h^\varepsilon(s_1,X(s_1),v'')D_x(|\frac{dv'}{dy}|)dydv''ds_1ds + \ldots, \end{aligned} \tag{1.28}$$

where $l_{M,w}$ is the kernel corresponding to $K_{M,w}$ in [22], l_N is a smooth approximation of $l_{M,w}$ with compact support, $y = X(s_1)$, and $\hat{B} = \{|y - X(s)| \le C(s-s_1)N, |v''| \le 3N\}$. Thanks to the elliptic regularity of the Poisson system, the electric field $E^\varepsilon = -\nabla_x\phi^\varepsilon$ is bounded almost in $W^{2,\infty}$ with the auxiliary $W^{1,\infty}$ bound for the distribution function, and $\|D_x(|\frac{dv'}{dy}|)\|_{L^2_{\hat{B}}}$ can also be bounded by $C_N/(\varepsilon^4)$. Unfortunately, in the presence of the magnetic field, the Maxwell system is hyperbolic, and such a gain of regularity and bound of $\|D_x(|\frac{dv'}{dy}|)\|_{L^2_{\hat{B}}}$ are impossible. Instead, we estimate the first expression in (1.28) by the H^1 norm of f^ε and $W^{1,\infty}$ norm of $(E^\varepsilon, B^\varepsilon)$.

We turn to an auxiliary H^1 estimate in Sect. 6. Even though such an H^1 estimate in Proposition 6.3 leads to a loss of $\varepsilon^{-\frac{1}{2}}$, it is still sufficient for the $W^{1,\infty}$ estimate of h^ε in the crucial bootstrap assumptions (1.27). Noting the fact

$$\|\nabla_p f^\varepsilon\| \lesssim \|\nabla_p \mathbf{P} f^\varepsilon\| + \|\nabla_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|,$$

we estimate the norm $\|\nabla_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|$ instead of a direct estimation of $\|\nabla_p f^\varepsilon\|$ to avoid estimates, such as the estimate related to the transport term

$$\langle (D_p\hat{p})\cdot\nabla_x f^\varepsilon, \nabla_p f^\varepsilon\rangle \lesssim \|D_p f^\varepsilon\|\|D_x f^\varepsilon\|, \tag{1.29}$$

which may lead to exponential growth of the H^1 norm of remainders. In fact, we proceed energy estimate after employing momentum differentiation D_p to the equation of $\{\mathbf{I}-\mathbf{P}\}f^\varepsilon$ via micro projection onto the equation of f^ε. Thanks to the dissipation of $\frac{1}{\varepsilon}\nabla_p(Lf^\varepsilon)$, instead of (1.29), we can obtain

$$\langle (D_p\hat{p})\cdot\nabla_x f^\varepsilon, \nabla_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\rangle \lesssim \frac{\kappa}{\varepsilon}\|D_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|^2 + \frac{\varepsilon}{\kappa}\|D_x f^\varepsilon\|^2$$

for some small positive constant $\kappa > 0$. Then $(\kappa/\varepsilon)\|D_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|^2$ can be absorbed by the dissipation term and the factor ε of $(\varepsilon/\kappa)\|D_x f^\varepsilon\|^2$ can kill the increase of time for $t \in [0, \varepsilon^{-\frac{1}{2}}]$.

In Sect. 7, we finally verify (1.27) and close the energy estimates via a continuity argument for $0 \le t \le \varepsilon^{-\frac{1}{2}}$. The proof of our main result Theorem 1.1 is given.

Appendix 3 is devoted to construction of coefficients $F_n(t,x,p)$, $E_n(t,x)$, $B_n(t,x)$, $(1 \le n \le 2k-1)$ in the Hilbert expansion (1.8) and estimation of their regularities (see Theorem 8.1). Due to the complex nature of the relativistic Boltzmann equation, the construction and estimates are delicate.

We remark that the time decay rate $(1+t)^{-\beta_0}(\beta_0 > 1)$ of the solutions [20] $n_0(t,x)$, $u(t,x)$, $E_0(t,x)$ and $B_0(t,x)$ in (1.20) is important. As in (2.7), (6.4) and (6.5), thanks to the time decay of $\|\nabla_x n_0\|_\infty$, $\|\nabla_x u\|_\infty$, $\|\nabla_x E_0\|_\infty$ and $\|\nabla_x B_0\|_\infty$, we obtain estimates such as $(1+t)^{-\beta_0}\|f^\varepsilon\|^2$, which is bound by Grönwall's inequality.

We briefly review works related to the relativistic Vlasov–Maxwell–Boltzmann system. For the two species relativistic Vlasov–Maxwell–Boltzmann system with $\varepsilon = 1$, global smooth solutions near a Maxwellian was constructed [25] in a periodic box with new momentum regularity estimates. This result was extended to the whole space in [30] and [29] for short range interaction. There are more studies about the non-relativistic Vlasov–Maxwell–Boltzmann system. In [18], global classical solutions near a Maxwellian were constructed in a periodic box. Since then, many works have been done in global existence and asymptotic stability of solutions, and hydrodynamic limits, such as [1,2,7–9,26–28,35].

2. L^2 Estimate for f^ε

In this section, we derive the L^2 energy estimates for the remainder $f^\varepsilon = \frac{F_R^\varepsilon}{\sqrt{\mathbf{M}}}$.

To perform the L^2 energy estimate, we first use (1.22) to rewrite (1.11) as

$$\begin{aligned}
&\partial_t f^\varepsilon + \hat{p}\cdot\nabla_x f^\varepsilon + \frac{u^0}{2T_0}\hat{p}\sqrt{\mathbf{M}}\cdot E_R^\varepsilon - \frac{u\sqrt{\mathbf{M}}}{2T_0}\cdot\left(\hat{p}\times B_R^\varepsilon\right)\\
&\quad-\left(E_0+\hat{p}\times B_0\right)\cdot\nabla_p f^\varepsilon + \frac{Lf^\varepsilon}{\varepsilon}\\
&= -\frac{f^\varepsilon}{\sqrt{\mathbf{M}}}\left[\partial_t + \hat{p}\cdot\nabla_x - \left(E_0+\hat{p}\times B_0\right)\cdot\nabla_p\right]\sqrt{\mathbf{M}} + \varepsilon^{k-1}\Gamma(f^\varepsilon, f^\varepsilon)\\
&\quad+\sum_{i=1}^{2k-1}\varepsilon^{i-1}[\Gamma(F_i,f^\varepsilon)+\Gamma(f^\varepsilon,F_i)] + \varepsilon^k\left(E_R^\varepsilon+\hat{p}\times B_R^\varepsilon\right)\cdot\nabla_p f^\varepsilon\\
&\quad-\varepsilon^k\frac{1}{2T_0}\left(u^0\hat{p}-u\right)\cdot\left(E_R^\varepsilon+\hat{p}\times B_R^\varepsilon\right)f^\varepsilon\\
&\quad+\sum_{i=1}^{2k-1}\varepsilon^i\left[\left(E_i+\hat{p}\times B_i\right)\cdot\nabla_p f^\varepsilon + \left(E_R^\varepsilon+\hat{p}\times B_R^\varepsilon\right)\cdot\nabla_p F_i\right]\\
&\quad-\sum_{i=1}^{2k-1}\varepsilon^i\left[\left(E_i+\hat{p}\times B_i\right)\cdot\frac{1}{2T_0}\left(u^0\hat{p}-u\right)f^\varepsilon\right] + \varepsilon^k\bar{A},
\end{aligned}\tag{2.1}$$

and

$$\begin{aligned}
\partial_t E_R^\varepsilon - \nabla_x\times B_R^\varepsilon &= \int_{\mathbb{R}^3}\hat{p}\sqrt{\mathbf{M}}f^\varepsilon dp,\\
\partial_t B_R^\varepsilon + \nabla_x\times E_R^\varepsilon &= 0,\\
\nabla_x\cdot E_R^\varepsilon &= -\int_{\mathbb{R}^3}F_R^\varepsilon dp, \qquad \nabla_x\cdot B_R^\varepsilon = 0,
\end{aligned}\tag{2.2}$$

where $\bar{A} = \frac{A}{\sqrt{\mathbf{M}}}$.

Proposition 2.1. *For the remainders $(f^\varepsilon, E_R^\varepsilon, B_R^\varepsilon)$, it holds that*

$$
\begin{aligned}
&\frac{d}{dt}\Big(\Big\|\sqrt{\frac{2T_0}{u^0}}f^\varepsilon(t)\Big\|^2+\|[E_R^\varepsilon(t),B_R^\varepsilon(t)]\|^2\Big)\\
&\quad+\Big(\frac{\delta_0 T_M}{2\varepsilon}-C\varepsilon^{k-2}\|h^\varepsilon\|_\infty\Big)\|\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|^2\\
&\quad\lesssim\Big[(1+t)^{-\beta_0}+\varepsilon^k\|h^\varepsilon\|_\infty+\varepsilon\mathcal{I}_1\Big]\Big(\|f^\varepsilon\|^2+\|(E_R^\varepsilon,B_R^\varepsilon)\|^2\Big)\\
&\quad+\Big[\varepsilon^{\frac{11}{4}}\|h^\varepsilon\|_\infty+\varepsilon^k\mathcal{I}_2(t)\Big]\|f^\varepsilon\|,
\end{aligned}\tag{2.3}
$$

where $\mathcal{I}_1$ and $\mathcal{I}_2$ have the following form:

$$
\begin{aligned}
\mathcal{I}_1(t)&=\sum_{i=1}^{2k-1}\varepsilon^{i-1}(1+t)^{i-1}\Big[1+\sum_{i=1}^{2k-1}\varepsilon^{i-1}(1+t)^{i-1}\Big],\\
\mathcal{I}_2(t)&=\sum_{1+2k\le i+j\le 4k-2}\varepsilon^{i+j-2k-1}(1+t)^{i+j-2}.
\end{aligned}
$$

Proof. We take the L^2 inner product with $\frac{2T_0}{u^0}f^\varepsilon$ on both sides of (2.1) to have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big\|\sqrt{\frac{2T_0}{u^0}}f^\varepsilon(t)\Big\|^2+\Big\langle\hat{p}\cdot E_R^\varepsilon\sqrt{\mathbf{M}},f^\varepsilon\Big\rangle+\frac{2\delta_0}{\varepsilon}\Big\|\sqrt{\frac{T_0}{u^0}}\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\Big\|^2\\
&\le-\Big\langle\frac{f^\varepsilon}{\sqrt{\mathbf{M}}}\Big[\partial_t+\hat{p}\cdot\nabla_x-\Big(E_0+\hat{p}\times B_0\Big)\cdot\nabla_p\Big]\sqrt{\mathbf{M}},\frac{2T_0}{u^0}f^\varepsilon\Big\rangle\\
&\quad+\frac{1}{2}\Big\langle\Big[\Big(\partial_t+\hat{p}\cdot\nabla_x\Big)\Big(\frac{2T_0}{u^0}\Big)\Big]f^\varepsilon,f^\varepsilon\Big\rangle\\
&\quad+\Big\langle u\cdot\Big(\hat{p}\times B_R^\varepsilon\Big)\sqrt{\mathbf{M}},f^\varepsilon\Big\rangle+\varepsilon^{k-1}\Big\langle\Gamma(f^\varepsilon,f^\varepsilon),\frac{2T_0}{u^0}f^\varepsilon\Big\rangle\\
&\quad+\sum_{i=1}^{2k-1}\varepsilon^{i-1}\Big\langle[\Gamma(F_i,f^\varepsilon)+\Gamma(f^\varepsilon,F_i)],\frac{2T_0}{u^0}f^\varepsilon\Big\rangle\\
&\quad-\varepsilon^k\Big\langle\Big(u^0\hat{p}-u\Big)\cdot\Big(E_R^\varepsilon+\hat{p}\times B_R^\varepsilon\Big)f^\varepsilon,\frac{f^\epsilon}{u^0}\Big\rangle\\
&\quad+\sum_{i=1}^{2k-1}\varepsilon^i\Big\langle\Big[\Big(E_i+\hat{p}\times B_i\Big)\cdot\nabla_p f^\varepsilon+\Big(E_R^\varepsilon+\hat{p}\times B_R^\varepsilon\Big)\cdot\nabla_p F_i\Big],\frac{2T_0}{u^0}f^\varepsilon\Big\rangle\\
&\quad-\sum_{i=1}^{2k-1}\varepsilon^i\Big\langle\Big[\Big(E_i+\hat{p}\times B_i\Big)\cdot\frac{1}{T_0}\Big(u^0\hat{p}-u\Big)f^\varepsilon\Big],\frac{2T_0}{u^0}f^\varepsilon\Big\rangle+\varepsilon^k\Big\langle\bar{A},\frac{2T_0}{u^0}f^\varepsilon\Big\rangle.
\end{aligned}\tag{2.4}
$$

On the other hand, from (2.2), we have

$$
\frac{1}{2}\frac{d}{dt}\Big(\|E_R^\varepsilon(t)\|^2+\|B_R^\varepsilon(t)\|^2\Big)=\Big\langle\hat{p}\cdot E_R^\varepsilon\sqrt{\mathbf{M}},f^\varepsilon\Big\rangle.\tag{2.5}
$$

Noting (1.24), we combine (2.4) and (2.5) to have

$$
\frac{1}{2}\frac{d}{dt}\Big(\Big\|\sqrt{\frac{2T_0}{u^0}}f^\varepsilon(t)\Big\|^2+\|[E_R^\varepsilon(t),B_R^\varepsilon(t)]\|^2\Big)+\frac{\delta_0 T_M}{\varepsilon}\|\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|^2
$$

$$
\begin{aligned}
&\le -\Big\langle \frac{f^\varepsilon}{\sqrt{\mathbf{M}}}\Big[\partial_t + \hat{p}\cdot\nabla_x - \Big(E_0 + \hat{p}\times B_0\Big)\cdot\nabla_p\Big]\sqrt{\mathbf{M}}, \frac{2T_0}{u^0} f^\varepsilon\Big\rangle \\
&\quad + \frac{1}{2}\Big\langle\Big[\Big(\partial_t + \hat{p}\cdot\nabla_x\Big)\Big(\frac{2T_0}{u^0}\Big)\Big] f^\varepsilon, f^\varepsilon\Big\rangle \\
&\quad + \Big\langle u\cdot\Big(\hat{p}\times B_R^\varepsilon\Big)\sqrt{\mathbf{M}}, f^\varepsilon\Big\rangle + \varepsilon^{k-1}\Big\langle\Gamma(f^\varepsilon, f^\varepsilon), \frac{2T_0}{u^0} f^\varepsilon\Big\rangle \\
&\quad + \sum_{i=1}^{2k-1}\varepsilon^{i-1}\Big\langle[\Gamma(F_i, f^\varepsilon) + \Gamma(f^\varepsilon, F_i)], \frac{2T_0}{u^0} f^\varepsilon\Big\rangle \\
&\quad - \varepsilon^k\Big\langle\Big(u^0\hat{p} - u\Big)\cdot\Big(E_R^\varepsilon + \hat{p}\times B_R^\varepsilon\Big) f^\varepsilon, \frac{f^\epsilon}{u^0}\Big\rangle \\
&\quad + \sum_{i=1}^{2k-1}\varepsilon^i\Big\langle\Big[\Big(E_i + \hat{p}\times B_i\Big)\cdot\nabla_p f^\varepsilon + \Big(E_R^\varepsilon + \hat{p}\times B_R^\varepsilon\Big)\cdot\nabla_p F_i\Big], \frac{2T_0}{u^0} f^\varepsilon\Big\rangle \\
&\quad - \sum_{i=1}^{2k-1}\varepsilon^i\Big\langle\Big[\Big(E_i + \hat{p}\times B_i\Big)\cdot\frac{1}{T_0}\Big(u^0\hat{p} - u\Big) f^\varepsilon\Big], \frac{2T_0}{u^0} f^\varepsilon\Big\rangle + \varepsilon^k\Big\langle\bar{A}, \frac{2T_0}{u^0} f^\varepsilon\Big\rangle. \qquad (2.6)
\end{aligned}
$$

Now we estimate the terms in the right hand of (2.6). For brevity, we only treat the first, fourth and fifth terms since the remained terms can be estimated in the same way as Proposition 3.1 in [22]. Note that $\frac{1}{\sqrt{\mathbf{M}}}[\partial_t + \hat{p}\cdot\nabla_x - (E_0 + \hat{p}\times B_0)\cdot\nabla_p]\sqrt{\mathbf{M}}$ is a linear polynomial of p, $(1+|p|)|f^\varepsilon| \le (1+|p|)^{-7}|h^\varepsilon|$ from (1.24), and

$$
\Big(\int_{|p|\ge\sqrt{\frac{\kappa}{\varepsilon}}}(1+|p|)^{-7\times 2}dp\Big)^{1/2} \lesssim \Big(\frac{\varepsilon}{\kappa}\Big)^{\frac{11}{4}},
$$

where κ is a sufficiently small positive constant. Then the first term can be estimated as follows:

$$
\begin{aligned}
&-\Big\langle\frac{f^\varepsilon}{\sqrt{\mathbf{M}}}\Big[\partial_t + \hat{p}\cdot\nabla_x - \Big(E_0 + \hat{p}\times B_0\Big)\cdot\nabla_p\Big]\sqrt{\mathbf{M}}, \frac{2T_0}{u^0} f^\varepsilon\Big\rangle \\
&\lesssim (\|\nabla_x n_0\|_\infty + \|\nabla_x u\|_\infty + \|\nabla_x E_0\|_\infty + \|\nabla_x B_0\|_\infty)\|f^\varepsilon\|^2 \\
&\quad + (\|\nabla_x n_0\| + \|\nabla_x u\| + \|\nabla_x E_0\| + \|\nabla_x B_0\|) \\
&\quad \times \|(1+|p|)\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|_{L_x^\infty L_p^2}\|\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\| \\
&\lesssim (1+t)^{-\beta_0}\|f^\varepsilon\|^2 + \|\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\| \\
&\quad \times \|(1+|p|)\{\mathbf{I}-\mathbf{P}\}f^\varepsilon(\mathbf{1}_{|p|\ge\sqrt{\frac{\kappa}{\varepsilon}}} + \mathbf{1}_{|p|\le\sqrt{\frac{\kappa}{\varepsilon}}})\|_{L_x^\infty L_p^2}\|\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\| \\
&\lesssim (1+t)^{-\beta_0}\|f^\varepsilon\|^2 + \frac{\kappa}{\varepsilon}\|\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|^2 + \varepsilon^{\frac{11}{4}}\|h^\varepsilon\|_\infty\|f^\varepsilon\|. \qquad (2.7)
\end{aligned}
$$

By Theorem 2 in [25], we can estimate the fourth term as

$$
\begin{aligned}
&\varepsilon^{k-1}\Big\langle\Gamma(f^\varepsilon, f^\varepsilon), \frac{2T_0}{u^0} f^\varepsilon\Big\rangle \\
&= \varepsilon^{k-1}\Big\langle\Gamma(f^\varepsilon, f^\varepsilon), \frac{2T_0}{u^0}\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\Big\rangle \\
&\lesssim \varepsilon^{k-1}\|h^\varepsilon\|_\infty\|f^\varepsilon\|\|\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|
\end{aligned}
$$

$$\lesssim \varepsilon^{k-2}\|h^\varepsilon\|_\infty\|\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|^2+\varepsilon^k\|h^\varepsilon\|_\infty\|f^\varepsilon\|^2.$$

For the fifth term, we use Theorem 2 in [25] and (8.17) in Theorem 8.1 to obtain

$$\begin{aligned}
&\sum_{i=1}^{2k-1}\varepsilon^{i-1}\Big\langle[\Gamma(F_i,f^\varepsilon)+\Gamma(f^\varepsilon,F_i)],\frac{2T_0}{u^0}f^\varepsilon\Big\rangle\\
&\quad=\sum_{i=1}^{2k-1}\varepsilon^{i-1}\Big\langle[\Gamma(F_i,f^\varepsilon)+\Gamma(f^\varepsilon,F_i)],\frac{2T_0}{u^0}\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\Big\rangle\\
&\quad\lesssim\sum_{i=1}^{2k-1}\varepsilon^{i-1}\|F_i\|_{L^\infty_xL^2_p}\|f^\varepsilon\|\|\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|\\
&\quad\lesssim\frac{\kappa}{\varepsilon}\|\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|^2+\varepsilon\Big(\sum_{i=1}^{2k-1}\varepsilon^{i-1}(1+t)^{i-1}\Big)^2\|f^\varepsilon\|^2.
\end{aligned}$$

□

3. Characteristics and L^∞ estimates

In this section, we first establish the characteristic estimates under the a priori assumptions (1.27). Then the L^∞ estimate of h^ε will be presented.

3.1. Characteristics estimates. In this subsection, we will provide several characteristics estimates for the relativistic Vlasov–Maxwell–Boltzmann system. To this aim, we need a uniform $W^{1,\infty}$ estimate for the electro-magnetic field $[E^\varepsilon(t,x),B^\varepsilon(t,x)]$.

We first note that, under the a priori assumptions (1.27),

$$\begin{aligned}
&\sup_{t\in[0,T]}\|[E^\varepsilon(t),B^\varepsilon(t)]\|_{W^{1,\infty}}\\
&\quad\le\sum_{i=0}^{2k-1}\varepsilon^i\|[E_i(t),B_i(t)]\|_{W^{1,\infty}}+\varepsilon^k\|[E_R^\varepsilon(t),B_R^\varepsilon(t)]\|_{W^{1,\infty}}\lesssim 1,
\end{aligned}\tag{3.1}$$

for $T\in[0,\frac{1}{\sqrt{\varepsilon}}]$ and $k\ge 3$.

With the uniform estimate (3.1), we can study the characteristics defined in (1.26).

Lemma 3.1. *Let $T\in[0,\varepsilon^{-\frac12}]$ and assumptions in (1.27) hold. Then there exists a small constant $\bar{T}\in[0,T]$ such that for $\tau\in[0,\bar{T}]$,*

$$|D_pX(\tau)|\lesssim\frac{|t-\tau|}{p^0},\tag{3.2}$$

$$\frac{d^2D_pX(\tau;t,x,p)}{d\tau^2}\lesssim\frac{1}{(p^0)^2},\tag{3.3}$$

$$\frac{1}{2(p^0)^5}|t-\tau|^3\le\left|\det\left(\frac{\partial X(\tau)}{\partial p}\right)\right|\le\frac{2}{(p^0)^5}|t-\tau|^3.\tag{3.4}$$

Proof. We first prove (3.2). For $i, j = 1, 2, 3$, we differentiate (1.26) w.r.t. p_i to have

$$\frac{d\partial_{p_i} X_j(\tau; t, x, p)}{d\tau} = \Big(\frac{(P^0)^2\partial_{p_i} P_j - P_j(P\cdot\partial_{p_i} P)}{(P^0)^3}\Big)(\tau; t, x, p), \tag{3.5}$$

and

$$\begin{aligned}
&\frac{d\partial_{p_i} P_j(\tau; t, x, p)}{d\tau}\\
&\quad = -[\nabla_x E_j^\varepsilon\cdot\partial_{p_i} X](\tau, X(\tau; t, x, p))\\
&\qquad - \Big[\hat{P}(\tau; t, x, p)\times[\nabla_x B^\varepsilon\cdot\partial_{p_i} X](\tau, X(\tau; t, x, p))\Big]_j\\
&\qquad - \Big[\Big(\frac{(P^0)^2\partial_{p_i} P - P(P\cdot\partial_{p_i} P)}{(P^0)^3}\Big)(\tau; t, x, p)\times B^\varepsilon(\tau, X(\tau; t, x, p))\Big]_j.
\end{aligned} \tag{3.6}$$

Note that $\partial_{p_i} P_j(t; t, x, p) = \delta_{ij}$. We integrate (3.6) over $[t, \tau]$ and use (3.1) to get

$$\|\partial_{p_i} P_j(\tau)\|_\infty \le \delta_{ij} + C|\tau - t|\Big(\sup_{\tau\in[0,\bar{T}]}\|\partial_{p_i} X(\tau)\|_\infty + \frac{\sup_{\tau\in[0,\bar{T}]}\|\partial_{p_i} P(\tau)\|_\infty}{\inf_{\tau\in[0,\bar{T}]}\|P^0(\tau)\|_\infty}\Big).$$

Then, for sufficiently small $\bar{T}$, it holds that

$$\|\partial_{p_i} P_j(\tau)\|_\infty \le \frac{5}{4}\delta_{ij} + C|\tau - t|\sup_{\tau\in[0,\bar{T}]}\|\partial_{p_i} X(\tau)\|_\infty. \tag{3.7}$$

Noting that $\partial_{p_i} X_j(t; t, x, p) = 0$, and for some $\bar{\tau}$ between τ and t,

$$P^0(\tau) = p^0 + (\tau - t)\frac{dP}{d\tau}(\bar{\tau}; t, x, p) \le p^0 + C|\tau - t|, \tag{3.8}$$

we integrate (3.5) over $[t, \tau]$ and use (3.8) to obtain

$$\|\partial_{p_i} X_j(\tau)\|_\infty \le \frac{2|t-\tau|}{\inf_{\tau\in[0,\bar{T}]}\|P^0(\tau)\|_\infty} \lesssim \frac{|t-\tau|}{p^0}, \tag{3.9}$$

for $\bar{T}$ small enough. Inserting (3.9) into (3.7) yields

$$\|\partial_{p_i} P_j(\tau)\|_\infty \le 2, \tag{3.10}$$

for sufficiently small $\bar{T}$. Moreover, from (1.26), one has

$$\begin{aligned}
\frac{d^2\partial_{p_i} X_j(\tau)}{d\tau^2} &= \partial_{p_i}\Big(\frac{1}{P^0}\frac{dP_j}{d\tau} - \frac{P_j}{(P^0)^3}\frac{dP}{d\tau}\cdot P\Big)(\tau)\\
&= \partial_{p_i}\Big[\frac{-E_j^\varepsilon - [\hat{P}\times B^\varepsilon]_j}{P^0} + \frac{P_j P\cdot E^\varepsilon}{(P^0)^3}\Big](\tau)\\
&= \Big\{\frac{-\nabla_x E_j^\varepsilon\cdot\partial_{p_i} X - [\hat{P}\times(\nabla_x B^\varepsilon\cdot\partial_{p_j} X)]_j - [\partial_{p_i}\hat{P}\times B^\varepsilon]_j}{P^0}\\
&\quad + \frac{\partial_{p_i} P_j P\cdot E^\varepsilon + P_j\partial_{p_i} P\cdot E^\varepsilon + P_j P\cdot(\nabla_x E^\varepsilon\cdot\partial_{p_i} X)}{(P^0)^3}
\end{aligned}$$

$$+\frac{E_j^\varepsilon + [\hat{P}\times B^\varepsilon]_j}{(P^0)^3} P\cdot\partial_{p_i} P - \frac{3P_j P\cdot E^\varepsilon(P\cdot\partial_{p_i}P)}{(P^0)^5}\Big\}(\tau). \tag{3.11}$$

Then we use (3.8), (3.9) and (3.10) to bound $\frac{d^2\partial_{p_i}X_j(\tau)}{d\tau^2}$ by $\frac{C}{(p^0)^2}(1+|t-\tau|)$. (3.3) follows for small $\bar{T}$.

Finally, we prove (3.4). Expand $\partial_{p_i}X_j(\tau)$ at t to have

$$\begin{aligned}
\partial_{p_i}X_j(\tau) &= \partial_{p_i}X_j(t) + \frac{d\partial_{p_i}X_j(\tau;t,x,p)}{d\tau}\Big|_{\tau=t}(\tau-t) + \frac{(\tau-t)^2}{2}\frac{d^2\partial_{p_i}X_j(\bar{\tau})}{d\tau^2}\\
&= (\tau-t)\frac{(p^0)^2\delta_{ij}-p_ip_j}{(p^0)^3} + \frac{(\tau-t)^2}{2}\Big\{\frac{-\nabla_x E_i^\varepsilon\cdot\partial_{p_j}X}{P^0}\\
&\quad - \frac{[\hat{P}\times(\nabla_x B^\varepsilon\cdot\partial_{p_j}X)]_j + [\partial_{p_j}\hat{P}\times B^\varepsilon]_j}{P^0}\\
&\quad + \frac{\partial_{p_i}P_jP\cdot E^\varepsilon + P_j\partial_{p_i}P\cdot E^\varepsilon + P_jP\cdot(\nabla_x E^\varepsilon\cdot\partial_{p_i}X)}{(P^0)^3}\\
&\quad + \frac{E_j^\varepsilon+[\hat{P}\times B^\varepsilon]_j}{(P^0)^3}P\cdot\partial_{p_i}P - \frac{3P_jP\cdot E^\varepsilon(P\cdot\partial_{p_i}P)}{(P^0)^5}\Big\}(\bar{\tau}).
\end{aligned} \tag{3.12}$$

Then, we have

$$\begin{aligned}
\left|\det\left(\frac{\partial X(\tau)}{\partial p}\right)\right| &= |\tau-t|^3\left|\det\left(\frac{\mathbf{I}}{(p^0)^3} - \frac{p\otimes p}{(p^0)^5} + \frac{\tau-t}{2}\frac{d^2\partial_{p_i}X_j(\bar{\tau})}{d\tau^2}\right)\right|\\
&= |\tau-t|^3\Bigg|\frac{1}{(p^0)^5} + \frac{\tau-t}{2}\sum_{i,j=1}^3\frac{\delta_{ij}+p_ip_j}{(p^0)^4}\frac{d^2\partial_{p_i}X_j(\bar{\tau})}{d\tau^2}\\
&\quad + \frac{O(1)|\tau-t|^2}{p^0}\left\|\frac{d^2\partial_pX(\bar{\tau})}{d\tau^2}\right\|_\infty^2 + O(1)|\tau-t|^3\left\|\frac{d^2\partial_pX(\bar{\tau})}{d\tau^2}\right\|_\infty^3\Bigg|.
\end{aligned}$$

Here $\mathbf{I}$ is the 3×3 identity matrix. By (3.8) and (3.11), we can estimate $\left|\det\left(\frac{\partial X(\tau)}{\partial p}\right)\right|$ as

$$|\tau-t|^3\left|[1+O(1)|\tau-t|^2]^2\frac{1}{(p^0)^5} + \frac{\tau-t}{2}\sum_{i,j=1}^3\frac{\delta_{ij}+p_ip_j}{(p^0)^4}\frac{d^2\partial_{p_i}X_j(\bar{\tau})}{d\tau^2}\right|. \tag{3.13}$$

Inserting the expression of $\partial_{p_i}X_j(\tau)$ in (3.12) into (3.13), after a long computation, we can further estimate $\left|\det\left(\frac{\partial X(\tau)}{\partial p}\right)\right|$ as

$$\begin{aligned}
&\frac{|\tau-t|^3}{(p^0)^5}\Big|[1+O(1)|\tau-t|^2]^2 + \frac{(\tau-t)}{2p^0}\Big[\hat{P}\cdot E^\varepsilon\\
&\quad - \nabla_x\cdot E^\varepsilon + \sum_{i,j=1}^3\hat{P}_i\hat{P}_j\partial_{x_i}E_j^\varepsilon + \hat{P}\cdot(\nabla_x\times B^\varepsilon)\Big](\bar{\tau})\Big|.
\end{aligned} \tag{3.14}$$

Then we combine (3.14) and (3.8), and choose $\bar{T}$ sufficiently small to obtain (3.4). □

3.2. L^∞ estimate of h^ε. With the characteristic estimates in Lemma 3.1, we can repeat the L^∞ estimate of h^ε in subsection 5.1 [22] to obtain the following proposition.

Proposition 3.1. *For given $T \in [0, \varepsilon^{-\frac{1}{2}}]$, assume the crucial bootstrap assumptions (1.27). Then, for sufficiently small ε, we have*

$$\sup_{0\le t\le T}\left(\varepsilon^{\frac{3}{2}}\|h^\varepsilon(t)\|_\infty\right) \lesssim \varepsilon^{\frac{3}{2}}\|h^\varepsilon(0)\|_\infty + \sup_{0\le t\le T}\|f^\varepsilon(t)\| + \varepsilon^{\frac{2k+3}{2}}.$$

4. $W^{1,\infty}$ estimates for $(E_R^\varepsilon, B_R^\varepsilon)$

In this section, we use the well-known Glassey–Strauss Representation in [14] to perform $W^{1,\infty}$ estimates for $(E_R^\varepsilon, B_R^\varepsilon)$.

Proposition 4.1. *For $t \in [0, T]$, remainders $(E_R^\varepsilon, B_R^\varepsilon)$ satisfy*

$$\begin{aligned}
&\|[E_R^\varepsilon(t), B_R^\varepsilon(t)]\|_{W^{1,\infty}}\\
&\quad\lesssim \frac{1}{\varepsilon}(1+t)^{\frac{1}{4}}\sup_{0\le t\le T}\left(\|h^\varepsilon(t)\|_\infty\|f^\varepsilon(t)\|\right)^{\frac{1}{2}}\\
&\qquad+(1+t)\sup_{t\in[0,T]}(\|h^\varepsilon(t)\|_{W^{1,\infty}}+1)\\
&\qquad+(1+t)^2\sup_{t\in[0,T]}\left[\left(\varepsilon^{k-1}\|h^\varepsilon(t)\|_\infty+\mathcal{I}_1(t)+1\right)(\|h^\varepsilon(t)\|_\infty+1)\right]\\
&\qquad+\frac{\sqrt{1+t}}{\varepsilon}\sup_{t\in[0,T]}\|f^\varepsilon(t)\|_{H^1}+(1+t)^{\frac{3}{2}}\sup_{t\in[0,T]}\left(\varepsilon^k\|h^\varepsilon(t)\|_\infty+\varepsilon\mathcal{I}_1(t)+1\right)\\
&\qquad\times\sup_{t\in[0,T]}\Big[\varepsilon^{k-1}(\|f^\varepsilon(t)\|_{H^1}^2+\|[E_R^\varepsilon(t), B_R^\varepsilon(t)]\|_{H^1}^2)\\
&\qquad+(\frac{1}{\varepsilon}+\mathcal{I}_1(t))\left(\|f^\varepsilon(t)\|+\|[E_R^\varepsilon(t), B_R^\varepsilon(t)]\|\right)+\varepsilon^{k-1}\mathcal{I}_2(t)\Big]\\
&\qquad+\left(1+\sup_{t\in[0,T]}[\varepsilon^k\|h^\varepsilon(t)\|_\infty+\varepsilon\mathcal{I}_1(t)]\right)\sqrt{1+t}\sup_{t\in[0,T]}\Big[\left(\frac{1}{\varepsilon^2}+\frac{\mathcal{I}_1(t)}{\varepsilon}\right)(\|f^\varepsilon(t)\|\\
&\qquad+\|[E_R^\varepsilon(t), B_R^\varepsilon(t)]\|)+\varepsilon^{k-2}\left(\|f^\varepsilon(t)\|_{H^1}^2+\|[E_R^\varepsilon(t), B_R^\varepsilon(t)]\|_{H^1}^2\right)\Big]\\
&\qquad+(1+t)\varepsilon^k\sup_{t\in[0,T]}\Big[\|f^\varepsilon(t)\|_{H^1}+\varepsilon^{k-1}\left(\|f^\varepsilon(t)\|_{H^1}^2+\|[E_R^\varepsilon(t), B_R^\varepsilon(t)]\|_{H^1}^2\right)\\
&\qquad+[\frac{1}{\varepsilon}+\mathcal{I}_1(t)](\|f^\varepsilon\|+\|[E_R^\varepsilon(t), B_R^\varepsilon(t)]\|)+\varepsilon^k\mathcal{I}_2(t)\Big]^2.
\end{aligned}\tag{4.1}$$

Proof. Proposition 4.1 is a combination of Propositions 4.2 and 4.3. $\square$

4.1. L^∞ estimate for $(E_R^\varepsilon, B_R^\varepsilon)$. Define an operator S by

$$S = \partial_t + \hat{p}\cdot\nabla_x.$$

The remainder of the electro-magnetic field $(E_R^\varepsilon, B_R^\varepsilon)$ can be explicitly expressed in terms of F_R^ε and SF_R^ε as follows:

Lemma 4.1 [14]. *For $i = 1, 2, 3$, one has*

$$
\begin{aligned}
4\pi E_R^{\varepsilon,i}(t,x) &= E_R^{\varepsilon,i}(0,x) + E_{R,T}^{\varepsilon,i}(t,x) + E_{R,S}^{\varepsilon,i}(t,x),\\
4\pi B_R^{\varepsilon,i}(t,x) &= B_R^{\varepsilon,i}(0,x) + B_{R,T}^{\varepsilon,i}(t,x) + B_{R,S}^{\varepsilon,i}(t,x),
\end{aligned} \tag{4.2}
$$

where $E_R^{\varepsilon,i}(0,x)$ and $B_R^{\varepsilon,i}(0,x)$ are initial data of $E_R^{\varepsilon,i}$ and $B_R^{\varepsilon,i}$, and

$$
\begin{aligned}
E_{R,T}^{\varepsilon,i}(t,x) &= -\iint\limits_{|x-y|\le t} \frac{dpdy}{|y-x|^2}\frac{(\omega_i+\hat{p}_i)(1-|\hat{p}|^2)}{(1+\hat{p}\cdot\omega)^2}F_R^\varepsilon(t-|y-x|,y,p),\\
E_{R,S}^{\varepsilon,i}(t,x) &= -\iint\limits_{|x-y|\le t} \frac{dpdy}{|y-x|}\frac{(\omega_i+\hat{p}_i)}{1+\hat{p}\cdot\omega}(SF_R^\varepsilon)(t-|y-x|,y,p),\\
B_{R,T}^{\varepsilon,i}(t,x) &= \iint\limits_{|x-y|\le t} \frac{dpdy}{|y-x|^2}\frac{(\omega\times\hat{p})_i(1-|\hat{p}|^2)}{(1+\hat{p}\cdot\omega)^2}F_R^\varepsilon(t-|y-x|,y,p),\\
B_{R,S}^{\varepsilon,i}(t,x) &= \iint\limits_{|x-y|\le t} \frac{dpdy}{|y-x|}\frac{(\omega\times\hat{p})_i}{1+\hat{p}\cdot\omega}(SF_R^\varepsilon)(t-|y-x|,y,p),
\end{aligned}
$$

with $\omega = \frac{x-y}{|x-y|}$.

From the kinetic equation of F_R^ε in (1.11), we can obtain the upper bound of the electro-magnetic field.

Proposition 4.2. *For $t \in [0,T]$, the remainders E_R^ε and B_R^ε can be estimated as*

$$
\begin{aligned}
&\|[E_R^\varepsilon(t), B_R^\varepsilon(t)]\|_\infty\\
&\quad\lesssim (1+t)\sup_{t\in[0,T]}\Big[\|[E_R^\varepsilon(0),B_R^\varepsilon(0)]\|_\infty + \|h^\varepsilon(t)\|_\infty\Big]\\
&\qquad+\sqrt{1+t}\sup_{t\in[0,T]}\Big[\varepsilon^{k-1}\Big(\|f^\varepsilon(t)\|_{H^1}^2 + \|[E_R^\varepsilon(t),B_R^\varepsilon(t)]\|_{H^1}^2\Big)\\
&\qquad+[\frac{1}{\varepsilon}+\mathcal{I}_1(t)](\|f^\varepsilon\| + \|[E_R^\varepsilon(t),B_R^\varepsilon(t)]\|) + \varepsilon^k\mathcal{I}_2(t)\Big].
\end{aligned} \tag{4.3}
$$

Proof. Since the estimate of B_R^ε can be done in a same way, we only deal with E_R^ε. We first estimate $E_{R,T}^{\varepsilon,i}$ in (4.2). Note that

$$
\begin{aligned}
&1-|\hat{p}|^2 = (1+|\hat{p}|)(1-|\hat{p}|) \le 2(1-|\hat{p}|),\\
&(1+\hat{p}\cdot\omega)^{-m} \le (1-|\hat{p}|)^{-m} \le (1+|p|)^m.
\end{aligned}
$$

for any $m \ge 0$. Then we have

$$
|E_{R,T}^{\varepsilon,i}| \lesssim (1+t)\sup_{t\in[0,T]}\|h^\varepsilon(t)\|_\infty\int_{\mathbb{R}^3}(1+|p|)\frac{\sqrt{J_M}}{w(|p|)}dp \lesssim (1+t)\sup_{t\in[0,T]}\|h^\varepsilon(t)\|_\infty.
$$

For $E_{R,S}^{\varepsilon,i}$, we obtain from the definition of operator S and (1.11) that

$$
\begin{aligned}
&E^{\varepsilon,i}_{R,S}(t,x)\\
&=-\iint\limits_{|x-y|\le t}\frac{dpdy}{|y-x|}\frac{(\omega_i+\hat p_i)}{1+\hat p\cdot\omega}\Big[(E^\varepsilon_R+\hat p\times B^\varepsilon_R)\cdot\nabla_p F_0\Big](t-|y-x|,y,p)\\
&\quad-\iint\limits_{|x-y|\le t}\frac{dpdy}{|y-x|}\frac{(\omega_i+\hat p_i)}{1+\hat p\cdot\omega}\Big[(E_0+\hat p\times B_0)\cdot\nabla_p F^\varepsilon_R\Big](t-|y-x|,y,p)\\
&\quad-\frac{1}{\varepsilon}\iint\limits_{|x-y|\le t}\frac{dpdy}{|y-x|}\frac{(\omega_i+\hat p_i)}{1+\hat p\cdot\omega}[Q(F^\varepsilon_R,F_0)+Q(F_0,F^\varepsilon_R)](t-|y-x|,y,p)\\
&\quad-\varepsilon^{k-1}\iint\limits_{|x-y|\le t}\frac{dpdy}{|y-x|}\frac{(\omega_i+\hat p_i)}{1+\hat p\cdot\omega}Q(F^\varepsilon_R,F^\varepsilon_R)(t-|y-x|,y,p)\\
&\quad-\iint\limits_{|x-y|\le t}\frac{dpdy}{|y-x|}\sum_{i=1}^{2k-1}\varepsilon^{i-1}\frac{(\omega_i+\hat p_i)}{1+\hat p\cdot\omega}[Q(F_i,F^\varepsilon_R)+Q(F^\varepsilon_R,F_i)](t-|y-x|,y,p)\\
&\quad-\varepsilon^{k}\iint\limits_{|x-y|\le t}\frac{dpdy}{|y-x|}\frac{(\omega_i+\hat p_i)}{1+\hat p\cdot\omega}\Big(E^\varepsilon_R+\hat p\times B^\varepsilon_R\Big)\cdot\nabla_p F^\varepsilon_R(t-|y-x|,y,p)\\
&\quad-\iint\limits_{|x-y|\le t}\frac{dpdy}{|y-x|}\sum_{i=1}^{2k-1}\varepsilon^{i}\frac{(\omega_i+\hat p_i)}{1+\hat p\cdot\omega}\Big[\Big(E_i+\hat p\times B_i\Big)\cdot\nabla_p F^\varepsilon_R\Big](t-|y-x|,y,p)\\
&\quad-\iint\limits_{|x-y|\le t}\frac{dpdy}{|y-x|}\sum_{i=1}^{2k-1}\varepsilon^{i}\frac{(\omega_i+\hat p_i)}{1+\hat p\cdot\omega}\Big[\Big(E^\varepsilon_R+\hat p\times B^\varepsilon_R\Big)\cdot\nabla_p F_i\Big](t-|y-x|,y,p)\\
&\quad-\varepsilon^{k}\iint\limits_{|x-y|\le t}\frac{dpdy}{|y-x|}\frac{(\omega_i+\hat p_i)}{1+\hat p\cdot\omega}A(t-|y-x|,y,p).
\end{aligned}\tag{4.4}
$$

For the first term in the right hand side of (4.4), by integration by parts w.r.t. p, it can be controlled by

$$\sqrt{t}\sup_{t\in[0,T]}\Big(\|[E^\varepsilon_R(t),B^\varepsilon_R(t)]\|\|(1+|p|)F_0\|_{L^1_p}\Big)\lesssim\sqrt{1+t}\sup_{t\in[0,T]}\|[E^\varepsilon_R(t),B^\varepsilon_R(t)]\|.$$

Integrating by parts w.r.t. p, we bound the second term in the right hand side of (4.4) by

$$\sqrt{t}\|(1+|p|)\sqrt{\mathbf{M}}\|_{L^2_p}\sup_{t\in[0,T]}\Big(\|[E_0(t),B_0(t)]\|_\infty\|f^\varepsilon(t)\|\Big)\lesssim\sqrt{1+t}\sup_{t\in[0,T]}\|f^\varepsilon(t)\|.$$

Noting from (1.7) that,

$$(1+|p|)(\mathbf{M}(p'))^{\frac14}(\mathbf{M}(q'))^{\frac14}=(1+|p|)(\mathbf{M}(p))^{\frac14}(\mathbf{M}(q))^{\frac14}\lesssim 1,$$

we use (8.17) in Theorem 8.1 to control the third term in the right hand side of (4.4) by

$$\frac{1}{\varepsilon}\iint\limits_{|x-y|\le t}\frac{dpdy}{|y-x|}(1+|p|)(\mathbf{M}(p))^{\frac14}(\mathbf{M}(q))^{\frac14}\Big[Q(f^\varepsilon\mathbf{M}^{\frac14},\frac{F_0}{\mathbf{M}^{\frac14}})$$

$$
\begin{aligned}
&\quad + Q(\frac{F_0}{\mathbf{M}^{\frac14}}, f^\varepsilon \mathbf{M}^{\frac14})\Big](t-|y-x|, y, p) \\
&\lesssim \frac{1}{\varepsilon}\int_{|x-y|\le t} \frac{dy}{|y-x|}\|f^\varepsilon(t-|y-x|, y)\|_{L^2_p} \\
&\lesssim \frac{\sqrt{1+t}}{\varepsilon}\sup_{t\in[0,T]}\|f^\varepsilon(t)\|.
\end{aligned}
$$

Similarly, the fourth term and fifth term in the right hand side of (4.4) can be estimated as

$$
\begin{aligned}
&\varepsilon^{k-1}\iint_{|x-y|\le t}\frac{dpdy}{|y-x|}(1+|p|)(\mathbf{M}(p))^{\frac14}(\mathbf{M}(q))^{\frac14}Q(f^\varepsilon\mathbf{M}^{\frac14}, f^\varepsilon\mathbf{M}^{\frac14})(t-|y-x|, y, p) \\
&\lesssim \varepsilon^{k-1}\int_{|x-y|\le t}\frac{dy}{|y-x|}\|f^\varepsilon(t-|y-x|, y)\|^2_{L^2_p} \\
&\lesssim \varepsilon^{k-1}\sqrt{t}\sup_{t\in[0,T]}\Big(\|f^\varepsilon(t)\|_{L^3_x}\|f^\varepsilon(t)\|_{L^6_x}\Big) \\
&\lesssim \varepsilon^{k-1}\sqrt{1+t}\sup_{t\in[0,T]}\Big(\|f^\varepsilon(t)\|^2+\|\nabla_x f^\varepsilon(t)\|^2\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{i=1}^{2k-1}\varepsilon^{i-1}\iint_{|x-y|\le t}\frac{dpdy}{|y-x|}(1+|p|)(\mathbf{M}(p))^{\frac14}(\mathbf{M}(q))^{\frac14}\Big[Q(f^\varepsilon\mathbf{M}^{\frac14}, \frac{F_i}{\mathbf{M}^{\frac14}}) \\
&\quad + Q(\frac{F_i}{\mathbf{M}^{\frac14}}, f^\varepsilon\mathbf{M}^{\frac14})\Big](t-|y-x|, y, p) \\
&\lesssim \sum_{i=1}^{2k-1}\varepsilon^{i-1}(1+t)^{i-1}\int_{|x-y|\le t}\frac{dy}{|y-x|}\|f^\varepsilon(t-|y-x|, y)\|_{L^2_p} \\
&\lesssim \sqrt{1+t}\sup_{t\in[0,T]}\Big(\sum_{i=1}^{2k-1}\varepsilon^{i-1}(1+t)^{i-1}\|f^\varepsilon(t)\|\Big),
\end{aligned}
$$

respectively. Similar to above estimates, the sixth, seventh, eighth and ninth terms can be controlled by

$$
\begin{aligned}
&C\varepsilon^k\|(1+|p|)\sqrt{\mathbf{M}}\|_{L^2_p}\sqrt{t}\sup_{t\in[0,T]}\Big(\|[E^\varepsilon_R(t), B^\varepsilon_R(t)]\|_{L^3_x}\|f^\varepsilon(t)\|_{L^6_x}\Big) \\
&\quad + C\sum_{i=1}^{2k-1}\varepsilon^i\|(1+|p|)\sqrt{\mathbf{M}}\|_{L^2_p}\sqrt{t}\sup_{t\in[0,T]}\Big(\|[E_i(t), B_i(t)]\|_\infty\|f^\varepsilon(t)\|\Big) \\
&\quad + C\sum_{i=1}^{2k-1}\varepsilon^i\sqrt{t}\sup_{t\in[0,T]}\Big(\|(1+|p|)F_i\|_{L^1_p}\|[E^\varepsilon_R(t), B^\varepsilon_R(t)]\|\Big) + C\varepsilon^k\sup_{t\in[0,T]}\mathcal{I}_2(t)
\end{aligned}
$$

$$\lesssim \sqrt{1+t} \sup_{t\in[0,T]} \Big[\varepsilon^k \left(\|[E_R^\varepsilon(t), B_R^\varepsilon(t)]\|_{H^1}^2 + \|\nabla_x f^\varepsilon(t)\|^2 \right)$$
$$+ \varepsilon \sum_{i=1}^{2k-1} \varepsilon^{i-1}(1+t)^{i-1} \left(\|f^\varepsilon(t)\| + \|[E_R^\varepsilon(t), B_R^\varepsilon(t)]\| \right) + \varepsilon^k \mathcal{I}_2(t) \Big].$$

We collect the estimates above to get (4.3). □

4.2. $W^{1,\infty}$ estimates for $(E_R^\varepsilon, B_R^\varepsilon)$. The gradient of the remainder of the electro-magnetic field $(E_R^\varepsilon, B_R^\varepsilon)$ can be explicitly expressed in the following:

Lemma 4.2 [14]. *For $i, j = 1, 2, 3$, one has*

$$\partial_{x_j} E_R^{\varepsilon,i}(t,x) = \partial_{x_j} E_R^{\varepsilon,i}(0,x) + \iint_{|x-y|\le t} \frac{dpdy}{|y-x|^3} a(\omega, \hat{p}) F_R^\varepsilon(t-|y-x|, y, p)$$
$$+ O(1) + \iint_{|x-y|\le t} \frac{dpdy}{|y-x|^2} b(\omega, \hat{p})(SF_R^\varepsilon)(t-|y-x|, y, p) \tag{4.5}$$
$$+ \iint_{|x-y|\le t} \frac{dpdy}{|y-x|} c(\omega, \hat{p})(S^2 F_R^\varepsilon)(t-|y-x|, y, p),$$

where $\partial_{x_j} E_R^{\varepsilon,i}(0,x)$ is a derivative of the initial datum. Moreover, the kernels a, b, c above are smooth except at points where $1 + \hat{p}\cdot\omega = 0$, and satisfy

$$(i) \qquad |a(\omega, \hat{p})| \lesssim (1+\hat{p}\cdot\omega)^{-4}, \qquad \int_{|\omega|=1} d\omega a(\omega, \hat{p}) = 0,$$
$$(ii) \qquad |b(\omega, \hat{p})| \lesssim (1+\hat{p}\cdot\omega)^{-3},$$
$$(iii) \qquad |c(\omega, \hat{p})| \lesssim (1+\hat{p}\cdot\omega)^{-2}. \tag{4.6}$$

The gradient of the magnetic field $\partial_{x_j} B_R^{\varepsilon,i}$ admits a similar representation as (4.2), but with slightly different kernels a_B, b_B and c_B for which the properties (4.5) and (4.6) remain valid.

Using again the definition of the operator S and the equation satisfied by F_R^ε in (1.11), we have the following gradient bound estimate.

Proposition 4.3. *For $t \in [0, T]$, the gradient of remainders E_R^ε and B_R^ε can be estimated as*

$$\|[\nabla_x E_R^\varepsilon(t), \nabla_x B_R^\varepsilon(t)]\|_\infty$$
$$\lesssim \frac{1}{\varepsilon}(1+t)^{\frac{1}{4}} \sup_{0\le t\le T} \left(\|h^\varepsilon(t)\|_\infty \|f^\varepsilon(t)\| \right)^{\frac{1}{2}}$$
$$+ (1+t) \sup_{t\in[0,T]} (\|h^\varepsilon(t)\|_{W^{1,\infty}} + 1)$$
$$+ (1+t)^2 \sup_{t\in[0,T]} \Big[\left(\varepsilon^{k-1} \|h^\varepsilon(t)\|_\infty + \mathcal{I}_1(t) + 1 \right) (\|h^\varepsilon(t)\|_\infty + 1) \Big]$$

$$
\begin{aligned}
&+\frac{\sqrt{1+t}}{\varepsilon}\sup_{t\in[0,T]}\|f^\varepsilon(t)\|_{H^1}+(1+t)^{\frac{3}{2}}\sup_{t\in[0,T]}\Big(\varepsilon^k\|h^\varepsilon(t)\|_\infty+\varepsilon\mathcal{I}_1(t)+1\Big)\\
&\times\sup_{t\in[0,T]}\Big[\varepsilon^{k-1}(\|f^\varepsilon(t)\|_{H^1}^2+\|[E_R^\varepsilon(t),B_R^\varepsilon(t)]\|_{H^1}^2)\\
&+(\frac{1}{\varepsilon}+\mathcal{I}_1(t))\left(\|f^\varepsilon(t)\|+\|[E_R^\varepsilon(t),B_R^\varepsilon(t)]\|\right)+\varepsilon^{k-1}\mathcal{I}_2(t)\Big]\\
&+\Big(1+\sup_{t\in[0,T]}[\varepsilon^k\|h^\varepsilon(t)\|_\infty+\varepsilon\mathcal{I}_1(t)]\Big)\sqrt{1+t}\sup_{t\in[0,T]}\Big[\Big(\frac{1}{\varepsilon^2}+\frac{\mathcal{I}_1(t)}{\varepsilon}\Big)(\|f^\varepsilon(t)\|\\
&+\|[E_R^\varepsilon(t),B_R^\varepsilon(t)]\|)+\varepsilon^{k-2}\Big(\|f^\varepsilon(t)\|_{H^1}^2+\|[E_R^\varepsilon(t),B_R^\varepsilon(t)]\|_{H^1}^2\Big)\Big]\\
&+(1+t)\varepsilon^k\sup_{t\in[0,T]}\Big[\|f^\varepsilon(t)\|_{H^1}+\varepsilon^{k-1}\Big(\|f^\varepsilon(t)\|_{H^1}^2+\|[E_R^\varepsilon(t),B_R^\varepsilon(t)]\|_{H^1}^2\Big)\\
&+[\frac{1}{\varepsilon}+\mathcal{I}_1(t)](\|f^\varepsilon\|+\|[E_R^\varepsilon(t),B_R^\varepsilon(t)]\|)+\varepsilon^k\mathcal{I}_2(t)\Big]^2.
\end{aligned}\tag{4.7}
$$

Proof. We estimate the three integrals in the right hand side of (4.7) one by one. Noting that

$$
\begin{aligned}
&\iint\limits_{|x-y|\le t}\frac{dpdy}{|y-x|^3}a(\omega,\hat{p})F_R^\varepsilon(t-|y-x|,y,p)\\
&\quad=\int_0^t\int_{\mathbb{S}^2}\frac{dpd\omega d\tau}{t-\tau}a(\omega,\hat{p})F_R^\varepsilon(\tau,x+(t-\tau)\omega,p),
\end{aligned}
$$

we use (i) in (4.6) and (1.23) to have

$$
\begin{aligned}
&\iint\limits_{|x-y|\le t}\frac{dpdy}{|y-x|^3}a(\omega,\hat{p})F_R^\varepsilon(t-|y-x|,y,p)\\
&=\int_0^t\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}\frac{dpd\omega d\tau}{t-\tau}a(\omega,\hat{p})\left[F_R^\varepsilon(\tau,x+(t-\tau)\omega,p)-F_R^\varepsilon(\tau,x,p)\right]\\
&\lesssim t\sup_{t\in[0,T]}\|\nabla_x h^\varepsilon(t)\|_\infty\int_{\mathbb{R}^3}dp(1+|p|)^4(1+|p|)^{-\beta}\sqrt{J_M}\\
&\lesssim(1+t)\sup_{t\in[0,T]}\|\nabla_x h^\varepsilon(t)\|_\infty.
\end{aligned}
$$

For the second integral in the right hand side of (4.7), one has

$$
\begin{aligned}
&\iint\limits_{|x-y|\le t}\frac{dpdy}{|y-x|^2}b(\omega,\hat{p})(SF_R^\varepsilon)(t-|y-x|,y,p)\\
&=\iint\limits_{|x-y|\le t}\frac{dpdy}{|y-x|^2}b(\omega,\hat{p})\Big[\Big(E_R^\varepsilon+\hat{p}\times B_R^\varepsilon\Big)\cdot\nabla_pF_0+\Big(E_0+\hat{p}\times B_0\Big)\cdot\nabla_pF_R^\varepsilon\\
&+\frac{1}{\varepsilon}[Q(F_R^\varepsilon,F_0)+Q(F_0,F_R^\varepsilon)]+\varepsilon^{k-1}Q(F_R^\varepsilon,F_R^\varepsilon)\\
&+\sum_{i=1}^{2k-1}\varepsilon^{i-1}[Q(F_i,F_R^\varepsilon)+Q(F_R^\varepsilon,F_i)]+\varepsilon^k\Big(E_R^\varepsilon+\hat{p}\times B_R^\varepsilon\Big)\cdot\nabla_pF_R^\varepsilon
\end{aligned}
$$

$$
+\sum_{i=1}^{2k-1}\varepsilon^i\Big((E_i+\hat p\times B_i)\cdot\nabla_p F_R^\varepsilon+(E_R^\varepsilon+\hat p\times B_R^\varepsilon)\cdot\nabla_p F_i\Big)
$$
$$
+\varepsilon^k A\Big](t-|y-x|,y,p). \tag{4.8}
$$

We only estimate the term $\frac{1}{\varepsilon}[Q(F_R^\varepsilon,F_0)+Q(F_0,F_R^\varepsilon)]$ in (4.8) since other terms can be estimated directly. For this term, by Hölder's inequality, we have

$$
\begin{aligned}
&\iint\limits_{|x-y|\le t}\frac{dpdy}{|y-x|^2}b(\omega,\hat p)\frac{1}{\varepsilon}[Q(F_R^\varepsilon,F_0)+Q(F_0,F_R^\varepsilon)](t-|y-x|,y,p)\\
&\quad\lesssim\frac{1}{\varepsilon}\int\limits_{|x-y|\le t}\frac{dy}{|y-x|^2}\|f^\varepsilon(t-|y-x|,y)\|_{L_p^2}^{\frac12}\sup_{0\le t\le T}\|h^\varepsilon(t)\|_\infty^{\frac12}\\
&\quad\lesssim\frac{1}{\varepsilon}t^{\frac13\times\frac34}\|f^\varepsilon(t)\|^{\frac12}\sup_{0\le t\le T}\|h^\varepsilon(t)\|_\infty^{\frac12}\\
&\quad\lesssim\frac{1}{\varepsilon}(1+t)^{\frac14}\sup_{0\le t\le T}(\|h^\varepsilon(t)\|_\infty^{\frac12}\|f^\varepsilon(t)\|^{\frac12}).
\end{aligned}
$$

Then we use (4.3) to further estimate (4.8) as

$$
\begin{aligned}
&\iint\limits_{|x-y|\le t}\frac{dpdy}{|y-x|^2}b(\omega,\hat p)(SF_R^\varepsilon)(t-|y-x|,y,p)\\
&\lesssim\frac{1}{\varepsilon}(1+t)^{\frac14}\sup_{0\le t\le T}(\|h^\varepsilon(t)\|_\infty^{\frac12}\|f^\varepsilon(t)\|^{\frac12})\\
&\quad+(1+t)\sup_{t\in[0,T]}\Big[\varepsilon^{k-1}\|h^\varepsilon(t)\|_\infty^2+(1+\mathcal{I}_1(t))\|h^\varepsilon(t)\|_\infty+(\varepsilon^k\|h^\varepsilon(t)\|_\infty+1)\\
&\quad\times\|[E_R^\varepsilon(t),B_R^\varepsilon(t)]\|_\infty+\varepsilon\mathcal{I}_1(t)(\|h^\varepsilon(t)\|_\infty+\|[E_R^\varepsilon(t),B_R^\varepsilon(t)]\|_\infty)+\varepsilon^k\mathcal{I}_2(t)\Big]\\
&\lesssim\frac{1}{\varepsilon}(1+t)^{\frac14}\sup_{0\le t\le T}\Big(\|h^\varepsilon(t)\|_\infty^{\frac12}\|f^\varepsilon(t)\|^{\frac12}\Big)\\
&\quad+(1+t)^2\sup_{t\in[0,T]}\Big[\Big(\varepsilon^{k-1}\|h^\varepsilon(t)\|_\infty+\mathcal{I}_1(t)+1\Big)\Big(1+\|h^\varepsilon(t)\|_\infty\Big)\Big]\\
&\quad+(1+t)^{\frac32}\sup_{t\in[0,T]}\Big(\varepsilon^k\|h^\varepsilon(t)\|_\infty+\varepsilon\mathcal{I}_1(t)+1\Big)\sup_{t\in[0,T]}\Big[\varepsilon^k\mathcal{I}_2(t)+\varepsilon^{k-1}(\|f^\varepsilon(t)\|_{H^1}^2\\
&\quad+\|[E_R^\varepsilon(t),B_R^\varepsilon(t)]\|_{H^1}^2)+(\frac{1}{\varepsilon}+\mathcal{I}_1(t))\Big(\|f^\varepsilon(t)\|+\|[E_R^\varepsilon(t),B_R^\varepsilon(t)]\|\Big)\Big].
\end{aligned}
$$

Now we estimate the third integral in the right hand side of (4.7). It holds that

$$
\begin{aligned}
&\iint\limits_{|x-y|\le t}\frac{dpdy}{|y-x|}c(\omega,\hat p)(S^2F_R^\varepsilon)(t-|y-x|,y,p)\\
&\quad=\iint\limits_{|x-y|\le t}\frac{dpdy}{|y-x|}c(\omega,\hat p)(\partial_t+\hat p\cdot\nabla_x)\Big[(E_R^\varepsilon+\hat p\times B_R^\varepsilon)\cdot\nabla_p F_0\Big](t-|y-x|,y,p)
\end{aligned}
$$

$$
\begin{aligned}
&+\iint\limits_{|x-y|\le t}\frac{dpdy}{|y-x|}c(\omega,\hat p)(\partial_t+\hat p\cdot\nabla_x)\Big[(E_0+\hat p\times B_0)\cdot\nabla_p F_R^\varepsilon\Big](t-|y-x|,y,p)\\
&+\frac{1}{\varepsilon}\iint\limits_{|x-y|\le t}\frac{dpdy}{|y-x|}c(\omega,\hat p)(\partial_t+\hat p\cdot\nabla_x)[Q(F_R^\varepsilon,F_0)+Q(F_0,F_R^\varepsilon)](t-|y-x|,y,p)\\
&+\varepsilon^{k-1}\iint\limits_{|x-y|\le t}\frac{dpdy}{|y-x|}c(\omega,\hat p)(\partial_t+\hat p\cdot\nabla_x)Q(F_R^\varepsilon,F_R^\varepsilon)(t-|y-x|,y,p)\\
&+\iint\limits_{|x-y|\le t}\frac{dpdy}{|y-x|}\sum_{i=1}^{2k-1}\varepsilon^{i-1}c(\omega,\hat p)(\partial_t+\hat p\cdot\nabla_x)[Q(F_i,F_R^\varepsilon)+Q(F_R^\varepsilon,F_i)](t-|y-x|,y,p)\\
&+\varepsilon^{k}\iint\limits_{|x-y|\le t}\frac{dpdy}{|y-x|}c(\omega,\hat p)(\partial_t+\hat p\cdot\nabla_x)\Big[\Big(E_R^\varepsilon+\hat p\times B_R^\varepsilon\Big)\cdot\nabla_p F_R^\varepsilon\Big](t-|y-x|,y,p)\\
&+\iint\limits_{|x-y|\le t}\frac{dpdy}{|y-x|}\sum_{i=1}^{2k-1}\varepsilon^{i}c(\omega,\hat p)(\partial_t+\hat p\cdot\nabla_x)\Big[\Big(E_i+\hat p\times B_i\Big)\cdot\nabla_p F_R^\varepsilon\Big](t-|y-x|,y,p)\\
&+\iint\limits_{|x-y|\le t}\frac{dpdy}{|y-x|}\sum_{i=1}^{2k-1}\varepsilon^{i}c(\omega,\hat p)(\partial_t+\hat p\cdot\nabla_x)\Big[\Big(E_R^\varepsilon+\hat p\times B_R^\varepsilon\Big)\cdot\nabla_p F_i\Big](t-|y-x|,y,p)\\
&+\varepsilon^{k}\iint\limits_{|x-y|\le t}\frac{dpdy}{|y-x|}c(\omega,\hat p)(\partial_t+\hat p\cdot\nabla_x)A(t-|y-x|,y,p).
\end{aligned}
\tag{4.9}
$$

For the first term in the right hand side of (4.9), we use the Maxwell system in (1.11) and (1.17) to have

$$
\begin{aligned}
&\iint\limits_{|x-y|\le t}\frac{dpdy}{|y-x|}c(\omega,\hat p)(\partial_t+\hat p\cdot\nabla_x)\Big[(E_R^\varepsilon+\hat p\times B_R^\varepsilon)\cdot\nabla_p F_0\Big](t-|y-x|,y,p)\\
&\quad=\iint\limits_{|x-y|\le t}\frac{dpdy}{|y-x|}c(\omega,\hat p)\Big[(\partial_t+\hat p\cdot\nabla_x)(E_R^\varepsilon+\hat p\times B_R^\varepsilon)\cdot\nabla_p F_0\Big](t-|y-x|,y,p)\\
&\qquad+\iint\limits_{|x-y|\le t}\frac{dpdy}{|y-x|}c(\omega,\hat p)\Big[(E_R^\varepsilon+\hat p\times B_R^\varepsilon)\cdot(\partial_t+\hat p\cdot\nabla_x)\nabla_p F_0\Big](t-|y-x|,y,p)\\
&\quad\le\int\limits_{|x-y|\le t}\frac{dy}{|y-x|}\Big(|\nabla_x E_R^\varepsilon|+|\nabla_x B_R^\varepsilon|+\int_{\mathbb{R}^3}dp\sqrt{\mathbf{M}}|f^\varepsilon|\Big)(t-|y-x|,y)\\
&\qquad\times\sup_{t,x}\int_{\mathbb{R}^3}(1+|p|)^2|\nabla_p F_0|dp\\
&\qquad+\sqrt{t}\sup_{t\in[0,t]}\|[E_R^\varepsilon(t),B_R^\varepsilon(t)]\|\sup_{t,x}\int_{\mathbb{R}^3}(1+|p|)^2|(\partial_t+\hat p\cdot\nabla_x)\nabla_p F_0|dp\\
&\quad\lesssim\sqrt{1+t}\sup_{t\in[0,t]}\Big(\|[E_R^\varepsilon(t),B_R^\varepsilon(t)]\|_{H^1}+\|f^\varepsilon(t)\|\Big).
\end{aligned}
$$

Similar to the estimation of $E_{R,S}^{\varepsilon,i}$, the second term in the right hand side of (4.9) can be estimated as

$$
\begin{aligned}
&\iint\limits_{|x-y|\le t} \frac{dpdy}{|y-x|} c(\omega,\hat{p})(\partial_t+\hat{p}\cdot\nabla_x)\Big[(E_0+\hat{p}\times B_0)\cdot\nabla_p F_R^\varepsilon\Big](t-|y-x|,y,p)\\
&\quad=\iint\limits_{|x-y|\le t} \frac{dpdy}{|y-x|} c(\omega,\hat{p})\Big[(\partial_t+\hat{p}\cdot\nabla_x)(E_0+\hat{p}\times B_0)\cdot\nabla_p F_R^\varepsilon\Big](t-|y-x|,y,p)\\
&\qquad+\iint\limits_{|x-y|\le t} \frac{dpdy}{|y-x|} c(\omega,\hat{p})\Big[(E_0+\hat{p}\times B_0)\cdot(\partial_t+\hat{p}\cdot\nabla_x)\nabla_p F_R^\varepsilon\Big](t-|y-x|,y,p)\\
&\quad\le-\iint\limits_{|x-y|\le t} \frac{dpdy}{|y-x|} F_R^\varepsilon\nabla_p\cdot\Big[c(\omega,\hat{p})(\partial_t+\hat{p}\cdot\nabla_x)(E_0+\hat{p}\times B_0)\Big](t-|y-x|,y,p)\\
&\qquad-\iint\limits_{|x-y|\le t} \frac{dpdy}{|y-x|}(\partial_t+\hat{p}\cdot\nabla_x) F_R^\varepsilon\nabla_p\cdot\Big[c(\omega,\hat{p})(E_0+\hat{p}\times B_0)\Big](t-|y-x|,y,p)\\
&\qquad+\iint\limits_{|x-y|\le t} \frac{dpdy}{|y-x|}(1+|p|)^2[|\nabla_x F_R^\varepsilon|(|E_0|+|B_0|)](t-|y-x|,y,p)\\
&\quad\lesssim\sqrt{1+t}\sup_{t\in[0,T]}\Big[\|f^\varepsilon(t)\|_{H^1}+\varepsilon^{k-1}\Big(\|f^\varepsilon(t)\|_{H^1}^2+\|[E_R^\varepsilon(t),B_R^\varepsilon(t)]\|_{H^1}^2\Big)\\
&\qquad+[\frac{1}{\varepsilon}+\mathcal{I}_1(t)](\|f^\varepsilon\|+\|[E_R^\varepsilon(t),B_R^\varepsilon(t)]\|)+\varepsilon^k\mathcal{I}_2(t)\Big].
\end{aligned}
$$

For the third term in the right hand side of (4.9), we only need to estimate the integral of $Q(F_R^\varepsilon,F_0)$ since the other integral can be estimated in the same way. Rewrite this integral in the following form:

$$
\begin{aligned}
&\frac{1}{\varepsilon}\iint\limits_{|x-y|\le t} \frac{dpdy}{|y-x|} c(\omega,\hat{p})(\partial_t+\hat{p}\cdot\nabla_x)[Q(F_R^\varepsilon,F_0)](t-|y-x|,y,p)\\
&\quad=\frac{1}{\varepsilon}\iint\limits_{|x-y|\le t} \frac{dpdy}{|y-x|} c(\omega,\hat{p})\Big[Q(F_R^\varepsilon,\partial_t F_0)+\hat{p}\cdot Q(F_R^\varepsilon,\nabla_x F_0)\Big](t-|y-x|,y,p)\\
&\qquad+\frac{1}{\varepsilon}\iint\limits_{|x-y|\le t} \frac{dpdy}{|y-x|} c(\omega,\hat{p})\Big[Q(\partial_t F_R^\varepsilon,F_0)+\hat{p}\cdot Q(\nabla_x F_R^\varepsilon,F_0)\Big](t-|y-x|,y,p)\\
&\quad=\mathcal{H}_1+\mathcal{H}_2.
\end{aligned}
\tag{4.10}
$$

Here we denote $\mathcal{H}_1$ and $\mathcal{H}_2$ as the two integrals in the right hand side of (4.10). As the estimation for the third term in the right hand side of (4.4), $\mathcal{H}_1$ can be controlled by $\frac{\sqrt{1+t}}{\varepsilon}\sup_{t\in[0,T]}\|f^\varepsilon(t)\|$. For $\mathcal{H}_2$, we have

$$
\mathcal{H}_2=\frac{1}{\varepsilon}\iint\limits_{|x-y|\le t} \frac{dpdy}{|y-x|} c(\omega,\hat{p})\Big[\hat{p}\cdot Q(\nabla_x F_R^\varepsilon,F_0)-Q(\hat{p}\cdot\nabla_x F_R^\varepsilon,F_0)\Big](t-|y-x|,y,p)
$$

$$
\begin{aligned}
&+\frac{1}{\varepsilon}\iint\limits_{|x-y|\le t}\frac{dpdy}{|y-x|}c(\omega,\hat{p})Q[(E_R^\varepsilon+\hat{p}\times B_R^\varepsilon)\cdot\nabla_p F_0, F_0](t-|y-x|,y,p)\\
&+\frac{1}{\varepsilon}\iint\limits_{|x-y|\le t}\frac{dpdy}{|y-x|}c(\omega,\hat{p})Q[(E_0+\hat{p}\times B_0)\cdot\nabla_p F_R^\varepsilon, F_0](t-|y-x|,y,p)\\
&+\frac{1}{\varepsilon^2}\iint\limits_{|x-y|\le t}\frac{dpdy}{|y-x|}c(\omega,\hat{p})Q[Q(F_R^\varepsilon,F_0)+Q(F_0,F_R^\varepsilon),F_0](t-|y-x|,y,p)\\
&+\varepsilon^{k-2}\iint\limits_{|x-y|\le t}\frac{dpdy}{|y-x|}c(\omega,\hat{p})Q[Q(F_R^\varepsilon,F_R^\varepsilon),F_0](t-|y-x|,y,p)\\
&+\iint\limits_{|x-y|\le t}\frac{dpdy}{|y-x|}\sum_{i=1}^{2k-1}\varepsilon^{i-2}c(\omega,\hat{p})Q[Q(F_i,F_R^\varepsilon)+Q(F_R^\varepsilon,F_i),F_0](t-|y-x|,y,p)\\
&+\varepsilon^{k-1}\iint\limits_{|x-y|\le t}\frac{dpdy}{|y-x|}c(\omega,\hat{p})Q[(E_R^\varepsilon+\hat{p}\times B_R^\varepsilon)\cdot\nabla_p F_R^\varepsilon, F_0](t-|y-x|,y,p)\\
&+\iint\limits_{|x-y|\le t}\frac{dpdy}{|y-x|}\sum_{i=1}^{2k-1}\varepsilon^{i-1}c(\omega,\hat{p})Q[(E_i+\hat{p}\times B_i)\cdot\nabla_p F_R^\varepsilon, F_0](t-|y-x|,y,p)\\
&+\iint\limits_{|x-y|\le t}\frac{dpdy}{|y-x|}\sum_{i=1}^{2k-1}\varepsilon^{i-1}c(\omega,\hat{p})Q[(E_R^\varepsilon+\hat{p}\times B_R^\varepsilon)\cdot\nabla_p F_i, F_0](t-|y-x|,y,p)\\
&+\varepsilon^{k-1}\iint\limits_{|x-y|\le t}\frac{dpdy}{|y-x|}c(\hat{p})Q(A,F_0)(t-|y-x|,y,p).
\end{aligned}
\tag{4.11}
$$

Similar to the estimation of the third term in the right hand side of (4.4), the first, second and third terms in the right hand side of (4.11) can be controlled by $\frac{\sqrt{1+t}}{\varepsilon}\sup_{t\in[0,T]}(\|f^\varepsilon(t)\|_{H^1}+\|[E_R^\varepsilon(t),B_R^\varepsilon(t)]\|)$, and the upper bound of the fourth term is

$$
\frac{\sqrt{1+t}}{\varepsilon^2}\sup_{t\in[0,T]}\|Q(f^\varepsilon\mathbf{M}^{\frac14},\frac{F_0}{\mathbf{M}^{\frac14}})(t)\|\lesssim\frac{\sqrt{1+t}}{\varepsilon^2}\sup_{t\in[0,T]}\|f^\varepsilon(t)\|.
$$

For the fifth term in the right hand side of (4.11), its upper bound is

$$
\varepsilon^{k-2}\sqrt{1+t}\sup_{t\in[0,T]}\|Q(f^\varepsilon\mathbf{M}^{\frac14},f^\varepsilon\mathbf{M}^{\frac14})(t)\|\lesssim\varepsilon^{k-2}\sqrt{1+t}\sup_{t\in[0,T]}\|f^\varepsilon(t)\|_{H^1}^2.
$$

Similarly, the sixth and tenth terms in the right hand side of (4.11) can be controlled by

$$
\sqrt{1+t}\sum_{i=1}^{2k-1}\varepsilon^{i-2}\sup_{t\in[0,T]}(1+t)^{i-1}\|f^\varepsilon(t)\|+\sqrt{1+t}\varepsilon^{k-1}\sup_{t\in[0,T]}\mathcal{I}_2(t).
$$

For the seventh, eighth and ninth terms in the right hand side of (4.11), after integration by parts for the momentum variable, we obtain their upper bounds as follows:

$$
\sqrt{1+t}\varepsilon^{k-1}\sup_{t\in[0,T]}\left(\|f^\varepsilon(t)\|_{H^1}^2+\|[E_R^\varepsilon(t),B_R^\varepsilon(t)]\|_{H^1}^2\right)
$$

$$+\sqrt{1+t}\sum_{i=1}^{2k-1}\varepsilon^{i-1}\sup_{t\in[0,T]}(1+t)^{i-1}\|f^{\varepsilon}(t)\|$$
$$+\sqrt{1+t}\sum_{i=1}^{2k-1}\varepsilon^{i-1}\sup_{t\in[0,T]}(1+t)^{i-1}\|[E_R^{\varepsilon}(t),B_R^{\varepsilon}(t)]\|$$
$$\lesssim\sqrt{1+t}\sup_{t\in[0,T]}\Big[\varepsilon^{k-1}(\|f^{\varepsilon}(t)\|_{H^1}^2+\|[E_R^{\varepsilon}(t),B_R^{\varepsilon}(t)]\|_{H^1}^2)$$
$$+\mathcal{I}_1(t)(\|f^{\varepsilon}(t)\|+\|[E_R^{\varepsilon}(t),B_R^{\varepsilon}(t)]\|)\Big].$$

Collecting the above estimates in (4.11), we further combine (4.11) and the estimate for $\mathcal{H}_1$ to obtain a upper bound of the third term in the right hand side of (4.9):

$$\sqrt{1+t}\Big[\frac{1}{\varepsilon^2}\sup_{t\in[0,T]}\|f^{\varepsilon}(t)\|+\sup_{t\in[0,T]}\Big[\frac{\mathcal{I}_1(t)}{\varepsilon}(\|f^{\varepsilon}(t)\|+\|[E_R^{\varepsilon}(t),B_R^{\varepsilon}(t)]\|)\Big]$$
$$+\varepsilon^{k-2}(\|f^{\varepsilon}(t)\|_{H^1}+\|[E_R^{\varepsilon}(t),B_R^{\varepsilon}(t)]\|_{H^1})+\varepsilon^{k-1}\mathcal{I}_2(t)\Big].$$

By slight modification, the fourth term in the right hand side of (4.9) can also be controlled by

$$\sqrt{1+t}\Big[\frac{1}{\varepsilon^2}\sup_{t\in[0,T]}\|f^{\varepsilon}(t)\|+\sup_{t\in[0,T]}\Big[\frac{\mathcal{I}_1(t)}{\varepsilon}(\|f^{\varepsilon}(t)\|+\|[E_R^{\varepsilon}(t),B_R^{\varepsilon}(t)]\|)\Big]$$
$$+\varepsilon^{k-2}(\|f^{\varepsilon}(t)\|_{H^1}+\|[E_R^{\varepsilon}(t),B_R^{\varepsilon}(t)]\|_{H^1})+\varepsilon^{k-1}\mathcal{I}_2(t)\Big]\sup_{t\in[0,T]}(\varepsilon^k\|h^{\varepsilon}(t)\|_{\infty}).$$

And the fifth term's upper bound is

$$\sqrt{1+t}\Big[\frac{1}{\varepsilon^2}\sup_{t\in[0,T]}\|f^{\varepsilon}(t)\|+\sup_{t\in[0,T]}\Big[\frac{\mathcal{I}_1(t)}{\varepsilon}(\|f^{\varepsilon}(t)\|+\|[E_R^{\varepsilon}(t),B_R^{\varepsilon}(t)]\|)\Big]$$
$$+\varepsilon^{k-2}(\|f^{\varepsilon}(t)\|_{H^1}+\|[E_R^{\varepsilon}(t),B_R^{\varepsilon}(t)]\|_{H^1})+\varepsilon^{k-1}\mathcal{I}_2(t)\Big]\sup_{t\in[0,T]}[\varepsilon\mathcal{I}_1(t)].$$

Slightly modifying the estimation of the first term and second term in the right hand side of (4.9), we bound the sixth term by

$$\sqrt{1+t}\sup_{t\in[0,T]}\left(\|[E_R^{\varepsilon}(t),B_R^{\varepsilon}(t)]\|_{H^1}+\|f^{\varepsilon}(t)\|\right)\sup_{t\in[0,T]}(\varepsilon^k\|h^{\varepsilon}(t)\|_{\infty})$$
$$+\sqrt{1+t}\sup_{t\in[0,T]}\Big[\|f^{\varepsilon}(t)\|_{H^1}+\varepsilon^{k-1}\left(\|f^{\varepsilon}(t)\|_{H^1}^2+\|[E_R^{\varepsilon}(t),B_R^{\varepsilon}(t)]\|_{H^1}^2\right)$$
$$+[\frac{1}{\varepsilon}+\mathcal{I}_1(t)](\|f^{\varepsilon}(t)\|+\|[E_R^{\varepsilon}(t),B_R^{\varepsilon}(t)]\|)+\varepsilon^k\mathcal{I}_2(t)\Big]\sup_{t\in[0,T]}(\varepsilon^k\|[E_R^{\varepsilon}(t),B_R^{\varepsilon}(t)]\|_{\infty})$$
$$\lesssim(1+t)^{\frac{3}{2}}\varepsilon^k\sup_{t\in[0,T]}\Big[\left(1+\|h^{\varepsilon}(t)\|_{\infty}\right)[\varepsilon^{k-1}\left(\|[E_R^{\varepsilon}(t),B_R^{\varepsilon}(t)]\|_{H^1}^2+\|f^{\varepsilon}(t)\|_{H^1}^2\right)$$
$$+\|f^{\varepsilon}(t)\|_{H^1}+\|[E_R^{\varepsilon}(t),B_R^{\varepsilon}(t)]\|_{H^1}+\Big(\frac{1}{\varepsilon}+\mathcal{I}_1(t)\Big)(\|f^{\varepsilon}(t)\|+\|[E_R^{\varepsilon}(t),B_R^{\varepsilon}(t)]\|)$$
$$+\varepsilon^k\mathcal{I}_2(t)]\Big]+(1+t)\varepsilon^k\sup_{t\in[0,T]}\Big[\varepsilon^{k-1}\left(\|[E_R^{\varepsilon}(t),B_R^{\varepsilon}(t)]\|_{H^1}^2+\|f^{\varepsilon}(t)\|_{H^1}^2\right)$$

$$+\|f^\varepsilon(t)\|_{H^1}+\Big(\frac{1}{\varepsilon}+\mathcal{I}_1(t)\Big)(\|f^\varepsilon(t)\|+\|[E_R^\varepsilon(t),B_R^\varepsilon(t)]\|)+\varepsilon^k\mathcal{I}_2(t)\Big]^2.$$

Correspondingly, the upper bound of the seventh, eighth, and ninth terms in the right hand side of (4.9) is

$$\begin{aligned}
&\sqrt{1+t}\sup_{t\in[0,T]}[\varepsilon\mathcal{I}_1(t)]\sup_{t\in[0,T]}\big(\|[E_R^\varepsilon(t),B_R^\varepsilon(t)]\|_{H^1}+\|f^\varepsilon(t)\|\big)\\
&\quad+\sqrt{1+t}\sup_{t\in[0,t]}[\varepsilon\mathcal{I}_1(t)]\sup_{t\in[0,T]}\Big[\|f^\varepsilon(t)\|_{H^1}+\varepsilon^{k-1}\big(\|f^\varepsilon(t)\|_{H^1}^2+\|[E_R^\varepsilon(t),B_R^\varepsilon(t)]\|_{H^1}^2\big)\\
&\quad+[\frac{1}{\varepsilon}+\mathcal{I}_1(t)](\|f^\varepsilon\|+\|[E_R^\varepsilon(t),B_R^\varepsilon(t)]\|)+\varepsilon^k\mathcal{I}_2(t)\Big]+\sqrt{1+t}\varepsilon^k\sup_{t\in[0,T]}\mathcal{I}_2(t)\\
&\lesssim\sqrt{1+t}\sup_{t\in[0,T]}\Big[\varepsilon\mathcal{I}_1(t)[\varepsilon^{k-1}\big(\|[E_R^\varepsilon(t),B_R^\varepsilon(t)]\|_{H^1}^2+\|f^\varepsilon(t)\|_{H^1}^2\big)+\|f^\varepsilon(t)\|_{H^1}\\
&\quad+\|[E_R^\varepsilon(t),B_R^\varepsilon(t)]\|_{H^1}+(\frac{1}{\varepsilon}+\mathcal{I}_1(t))(\|f^\varepsilon(t)\|+\|[E_R^\varepsilon(t),B_R^\varepsilon(t)]\|)+\varepsilon^k\mathcal{I}_2(t)]+\varepsilon^k\mathcal{I}_2(t)\Big].
\end{aligned}$$

□

5. $W^{1,\infty}$ Estimates for h^ε

In this section, we present the $W^{1,\infty}$ estimates for h^ε.

Proposition 5.1. *For given $T\in[0,\varepsilon^{-\frac{1}{2}}]$, assume the crucial bootstrap assumptions* (1.27)*. Then, for sufficiently small ε, we have*

$$\sup_{0\le t\le T}\Big(\varepsilon^{\frac{3}{2}}\|h^\varepsilon(t)\|_{W^{1,\infty}}\Big)\lesssim\varepsilon^{\frac{3}{2}}\|h^\varepsilon(0)\|_{W^{1,\infty}}+\sup_{0\le t\le T}\|f^\varepsilon(t)\|_{H^1}+\varepsilon^{\frac{2k+3}{2}}.$$

Proof. With the basic estimates along the curved characteristics in Lemma 3.1 and the nice estimates of $\bar{k}_1$ and $\bar{k}_2$ in Corollary 8.1, the proof can be done in a similar way as the L^∞ bound estimate in Sect. 5.1 of [22]. However, different from the $W^{1,\infty}$ estimates in [22], we don't apply integration by parts to avoid estimating the norms $\|\nabla_{x,p}f^\varepsilon(t)\|$ during treating the most difficult term $K_{M,w}(h^\varepsilon)$, and we directly use the norms $\|\nabla_{x,p}f^\varepsilon(t)\|$ instead. Take the L^∞ estimate of D_xh^ε for example. In Sect. 5.2 of [22], D_xh^ε was written as

$$\begin{aligned}
&D_xh^\varepsilon(t,x,v)\\
&=\ldots-\frac{1}{\varepsilon}\int_0^t\int_0^s\exp\Big\{-\frac{1}{\varepsilon}\int_s^t\nu(\tau)d\tau\Big\}(K_{M,w}D_xh^\varepsilon)(s,X(s),V(s))ds+\ldots\\
&=\ldots+\frac{1}{\varepsilon^2}\int_0^t\int_0^{s-\kappa\varepsilon}\exp\Big\{-\frac{1}{\varepsilon}\int_s^t\nu(\tau)d\tau-\frac{1}{\varepsilon}\int_{s_1}^s\nu(\tau)d\tau\Big\}\\
&\times\int_B l_N(V(s),v')l_N(V(s_1),v'')D_xh^\varepsilon(s_1,X(s_1),v'')dv'dv''ds_1ds+\ldots\\
&=\ldots+\frac{1}{\varepsilon^2}\int_0^t\int_0^{s-\kappa\varepsilon}\exp\Big\{-\frac{1}{\varepsilon}\int_s^t\nu(\tau)d\tau-\frac{1}{\varepsilon}\int_{s_1}^s\nu(\tau)d\tau\Big\}\\
&\times\int_{\hat{B}}l_N(V(s),v')l_N(V(s_1),v'')D_xh^\varepsilon(s_1,X(s_1),v'')\Big|\frac{dv'}{dy}\Big|dydv''ds_1ds+\ldots,
\end{aligned}$$

$$(5.1)$$

where $B = \{|v'| \le 2N, |v''| \le 3N\}$. Noting that the volume of B satisfies $|B| \lesssim N^6$ and $|\frac{dv'}{dy}| \lesssim \varepsilon^{-3}$, the upper bound of the last expression in (5.1) is

$$\frac{C_N}{\varepsilon^{3/2}} \|D_x f^\varepsilon\|.$$

It should also be pointed out that in our case, we don't need extra weighted L^∞ estimate for h^ε. In fact, for estimates related to $\bar{K}(h^\varepsilon)$, no extra weight arises due to Corollary 8.1. For a typical term $\frac{w(|p|)}{\sqrt{J_M}} D_x Q(\frac{\sqrt{J_M}}{w} h^\varepsilon, F_i)$, extra weight for h^ε is not necessary because J_M is a global Maxwellian and $\frac{D_x F_i}{\sqrt{J_M}}$ can also be controlled by a Maxwellian. While for $\frac{w(|p|)}{\sqrt{J_M}} D_p Q(\frac{\sqrt{J_M}}{w} h^\varepsilon, F_i)$, it can be estimated similarly as Lemma 2 and Lemma 5 in [25] without extra weight for h^ε arises either. □

6. Auxiliary H^1 Estimates for f^ε

In this part, we continue to perform the L^2 energy estimates for the first order derivatives of the remainders $(f^\varepsilon, E_R^\varepsilon, B_R^\varepsilon)$. We first deal with the space derivative estimate.

Proposition 6.1. *For the remainders $(f^\varepsilon, E_R^\varepsilon, B_R^\varepsilon)$, it holds that*

$$\begin{aligned}
&\frac{d}{dt}\Big(\Big\|\sqrt{\frac{2T_0}{u^0}}\nabla_x f^\varepsilon(t)\Big\|^2 + \|[\nabla_x E_R^\varepsilon(t), \nabla_x B_R^\varepsilon(t)]\|^2\Big) \\
&\quad + \Big(\frac{\delta_0 T_M}{2\varepsilon} - C\varepsilon^{k-2}\|h^\varepsilon\|_\infty\Big)\|\nabla_x\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|^2 \\
&\lesssim \Big[(1+t)^{-\beta_0}(1+\mathcal{I}_1(t)) + \varepsilon^k\|h^\varepsilon\|_{W^{1,\infty}} + \varepsilon\mathcal{I}_1\Big]\Big(\|f^\varepsilon\|_{H^1}^2 + \|(E_R^\varepsilon, B_R^\varepsilon)\|_{H^1}^2\Big) \\
&\quad + \Big(\frac{1}{\varepsilon}(1+t)^{-2\beta_0} + (1+t)^{-\beta_0}[\varepsilon^{k-1}\|h^\varepsilon\|_\infty]^2\Big)\|f^\varepsilon\|^2 \\
&\quad + \varepsilon^{-\frac{7}{4}}(1+t)^{-\beta_0}\|h^\varepsilon\|_\infty\|f^\varepsilon\| + \varepsilon^k\mathcal{I}_2(t)\|\nabla_x f^\varepsilon\|.
\end{aligned} \tag{6.1}$$

Proof. We take D_x to (2.1) and (2.2), and proceed similar derivation as (2.6) to have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\Big\|\sqrt{\frac{2T_0}{u^0}}D_x f^\varepsilon(t)\Big\|^2 + \|[D_x E_R^\varepsilon(t), D_x B_R^\varepsilon(t)]\|^2\Big) + \frac{1}{\varepsilon}\langle D_x L\{\mathbf{I}-\mathbf{P}\}f^\varepsilon, D_x f^\varepsilon\rangle \\
&\quad - \Big\langle D_x\Big[\Big(E_0 + \hat{p}\times B_0\Big)\cdot\nabla_p f^\varepsilon\Big], \frac{2T_0}{u^0}D_x f^\varepsilon\Big\rangle \\
&= -\Big\langle \frac{D_x f^\varepsilon}{\sqrt{\mathbf{M}}}\Big[\partial_t + \hat{p}\cdot\nabla_x - \Big(E_0 + \hat{p}\times B_0\Big)\cdot\nabla_p\Big]\sqrt{\mathbf{M}}, \frac{2T_0}{u^0}D_x f^\varepsilon\Big\rangle \\
&\quad - \Big\langle f^\varepsilon D_x\Big\{\frac{1}{\sqrt{\mathbf{M}}}\Big[\partial_t + \frac{p}{p^0}\cdot\nabla_x - (E_0 + \hat{p}\times B_0)\cdot\nabla_p\Big]\sqrt{\mathbf{M}}\Big\}, \frac{2T_0}{u^0}D_x f^\varepsilon\Big\rangle \\
&\quad + \frac{1}{2}\Big\langle\Big[\Big(\partial_t + \hat{p}\cdot\nabla_x\Big)\Big(\frac{2T_0}{u^0}\Big)\Big]D_x f^\varepsilon, D_x f^\varepsilon\Big\rangle \\
&\quad - \Big\langle D_x\Big[\frac{(u^0\hat{p} - u)\cdot E_R^\varepsilon}{T_0}\Big], \frac{T_0}{u^0}D_x f^\varepsilon\Big\rangle + \Big\langle D_x\Big[\frac{u\sqrt{\mathbf{M}}}{T_0}\cdot\Big(\hat{p}\times B_R^\varepsilon\Big)\Big], \frac{T_0}{u^0}D_x f^\varepsilon\Big\rangle
\end{aligned}$$

$$
\begin{aligned}
&+\Big\langle D_x\Big[(E_0+\hat{p}\times B_0)\Big]\cdot\nabla_p f^\varepsilon,\frac{2T_0}{u^0}D_x f^\varepsilon\Big\rangle\\
&+\varepsilon^{k-1}\Big\langle D_x\Gamma(f^\varepsilon,f^\varepsilon),\frac{2T_0}{u^0}D_x f^\varepsilon\Big\rangle\\
&+\sum_{i=1}^{2k-1}\varepsilon^{i-1}\Big\langle D_x[\Gamma(F_i,f^\varepsilon)+\Gamma(f^\varepsilon,F_i)],\frac{2T_0}{u^0}D_x f^\varepsilon\Big\rangle\\
&-\varepsilon^k\Big\langle D_x\Big[\frac{1}{T_0}\Big(u^0\hat{p}-u\Big)\cdot\Big(E_R^\varepsilon+\hat{p}\times B_R^\varepsilon\Big)f^\varepsilon\Big],\frac{2T_0}{u^0}D_x f^\varepsilon\Big\rangle\\
&+\varepsilon^k\Big\langle D_x\Big[\Big(E_R^\varepsilon+\hat{p}\times B_R^\varepsilon\Big)\cdot\nabla_p f^\varepsilon\Big],\frac{2T_0}{u^0}D_x f^\varepsilon\Big\rangle\\
&+\sum_{i=1}^{2k-1}\varepsilon^i\Big\langle D_x\Big[\Big(E_i+\hat{p}\times B_i\Big)\cdot\nabla_p f^\varepsilon\Big],\frac{2T_0}{u^0}D_x f^\varepsilon\Big\rangle\\
&+\sum_{i=1}^{2k-1}\varepsilon^i\Big\langle D_x\Big[\Big(E_R^\varepsilon+\hat{p}\times B_R^\varepsilon\Big)\cdot\nabla_p F_i\Big],\frac{2T_0}{u^0}D_x f^\varepsilon\Big\rangle\\
&-\sum_{i=1}^{2k-1}\varepsilon^i\Big\langle D_x\Big[\Big(E_i+\hat{p}\times B_i\Big)\cdot\frac{1}{2T_0}\Big(u^0\hat{p}-u\Big)f^\varepsilon\Big],\frac{2T_0}{u^0}D_x f^\varepsilon\Big\rangle\\
&+\varepsilon^k\Big\langle D_x\bar{A},\frac{2T_0}{u^0}D_x f^\varepsilon\Big\rangle,
\end{aligned}
\tag{6.2}
$$

For brevity, we only estimate the second term in the left hand side of (6.2), the second, seventh, eighth terms in the right hand side of (6.2). The rest of terms can be estimated similarly as these terms or the corresponding terms in Proposition 2.1.

We first estimate the second term in the left hand side of (6.2). Noting

$$
\begin{aligned}
&-D_xL(\{\mathbf{I}-\mathbf{P}\}f^\varepsilon)\\
&\quad=D_x\Big(\int_{\mathbb{R}^3}dq\int_{\mathbb{S}^2}d\omega\frac{s}{p^0q^0}\mathbb{B}(p,q,\omega)\sqrt{\mathbf{M}}(q)[\{\mathbf{I}-\mathbf{P}\}f^\varepsilon(p')\sqrt{\mathbf{M}}(q')\\
&\qquad+\{\mathbf{I}-\mathbf{P}\}f^\varepsilon(q')\sqrt{\mathbf{M}}(p')-\{\mathbf{I}-\mathbf{P}\}f^\varepsilon(p)\sqrt{\mathbf{M}}(q)-\{\mathbf{I}-\mathbf{P}\}f^\varepsilon(q)\sqrt{\mathbf{M}}(p)]\Big)\\
&\quad=\int_{\mathbb{R}^3}dq\int_{\mathbb{S}^2}d\omega\frac{s}{p^0q^0}\mathbb{B}(p,q,\omega)D_x\sqrt{\mathbf{M}}(q)\Big(\{\mathbf{I}-\mathbf{P}\}f^\varepsilon(p')\sqrt{\mathbf{M}}(q')\\
&\qquad+\{\mathbf{I}-\mathbf{P}\}f^\varepsilon(q')\sqrt{\mathbf{M}}(p')-\{\mathbf{I}-\mathbf{P}\}f^\varepsilon(p)\sqrt{\mathbf{M}}(q)-\{\mathbf{I}-\mathbf{P}\}f^\varepsilon(q)\sqrt{\mathbf{M}}(p)\Big)\\
&\qquad+\int_{\mathbb{R}^3}dq\int_{\mathbb{S}^2}d\omega\frac{s}{p^0q^0}\mathbb{B}(p,q,\omega)\sqrt{\mathbf{M}}(q)\Big(\{\mathbf{I}-\mathbf{P}\}f^\varepsilon(p')[D_x\sqrt{\mathbf{M}}(q')]\\
&\qquad+\{\mathbf{I}-\mathbf{P}\}f^\varepsilon(q')[D_x\sqrt{\mathbf{M}}(p')]-\{\mathbf{I}-\mathbf{P}\}f^\varepsilon(p)[D_x\sqrt{\mathbf{M}}(q)]\\
&\qquad-\{\mathbf{I}-\mathbf{P}\}f^\varepsilon(q)[D_x\sqrt{\mathbf{M}}(p)]\Big)-L(D_x\{\mathbf{I}-\mathbf{P}\}f^\varepsilon)
\end{aligned}
$$

by the expression of the operator L, and

$$
\|\{\mathbf{I}-\mathbf{P}\}D_xf^\varepsilon-D_x\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|=\|\mathbf{P}D_xf^\varepsilon-D_x\mathbf{P}f^\varepsilon\|\lesssim(\|\nabla_xn_0\|_\infty+\|\nabla_xu\|_\infty)\|\mathbf{P}f^\varepsilon\|,
$$

we have

$$
\begin{aligned}
&\frac{1}{\varepsilon}\langle D_x L(\{\mathbf{I}-\mathbf{P}\}f^\varepsilon), \frac{2T_0}{u^0}D_x f^\varepsilon\rangle\\
&\quad\geq \frac{2\delta_0 T_M}{u^0\varepsilon}\|D_x\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|^2 - \frac{C}{\varepsilon}\left(\|\nabla_x n_0\|_\infty + \|\nabla_x u\|_\infty\right)\|\mathbf{P}f^\varepsilon\|\|D_x\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|\\
&\qquad - \frac{C}{\varepsilon}\left(\|\nabla_x n_0\|_\infty + \|\nabla_x u\|_\infty\right)\|f^\varepsilon\|(\|D_x\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\| + \|\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|)\\
&\quad\geq \frac{3\delta_0 T_M}{2\varepsilon}\|D_x\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|^2 - \frac{C}{\varepsilon}(1+t)^{-2\beta_0}\|f^\varepsilon\|^2.
\end{aligned}
$$

Here we used the integration by parts w.r.t. x when the operator D_x hits the local Maxwellian in the operator L. Note that $D_x^2\{\frac{1}{\sqrt{\mathbf{M}}}[\partial_t + \frac{p}{p^0}\cdot\nabla_x - (E_0+\hat{p}\times B_0)\cdot\nabla_p]\sqrt{\mathbf{M}}\}$ is a cubic polynomial of p, $(1+|p|)^3 f^\varepsilon \leq (1+|p|)^{-5}h^\varepsilon$, and

$$
\Big(\int_{|p|\geq\sqrt{\frac{\kappa}{\varepsilon}}}(1+|p|)^{-5\times 2}dp\Big)^{1/2} \lesssim \Big(\frac{\varepsilon}{\kappa}\Big)^{\frac{7}{4}}.
$$

Then, similar to the estimation of (2.7), the second term in the right hand side of (6.2) can be controlled by

$$
(1+t)^{-\beta_0}\|f^\varepsilon\|_{H^1}^2 + \frac{\kappa}{\varepsilon}\|\{\mathbf{I}-\mathbf{P}\}D_x f^\varepsilon\|^2 + \varepsilon^{\frac{7}{4}}(1+t)^{-\beta_0}\|h^\varepsilon\|_\infty\|f^\varepsilon\|.
$$

Now we estimate the seventh term in the right hand side of (6.2). Note that

$$
\begin{aligned}
&D_x\Gamma(f^\varepsilon, f^\varepsilon)\\
&\quad= D_x\Big(\int_{\mathbb{R}^3}dq\int_{\mathbb{S}^2}d\omega\frac{s}{p^0q^0}\mathbb{B}(p,q,\omega)\sqrt{\mathbf{M}}(q)[f^\varepsilon(p')f^\varepsilon(q') - f^\varepsilon(p)f^\varepsilon(q)]\Big)\\
&\quad= \Gamma(f^\varepsilon, D_x f^\varepsilon) + \Gamma(D_x f^\varepsilon, f^\varepsilon)\\
&\qquad+\int_{\mathbb{R}^3}dq\int_{\mathbb{S}^2}d\omega\frac{s}{p^0q^0}\mathbb{B}(p,q,\omega)D_x\sqrt{\mathbf{M}}(q)[f^\varepsilon(p')f^\varepsilon(q') - f^\varepsilon(p)f^\varepsilon(q)].
\end{aligned}
$$

Though $\mathbf{M}$ is a local Maxwellian, we can obtain from (1.24) that

$$
\sqrt{\mathbf{M}} + |D_x\sqrt{\mathbf{M}}| \lesssim J_M^{\frac{1}{4}}, \tag{6.3}
$$

for $\bar{\varepsilon}_0$ sufficiently small in (1.20). In Theorem 2 of [25], energy estimates for derivatives of the nonlinear operator Γ corresponding to a global Maxwellian were given. Then, we can use (6.3), Theorem 2 in [25] to obtain

$$
\begin{aligned}
&\varepsilon^{k-1}\Big\langle D_x\Gamma(f^\varepsilon, f^\varepsilon), \frac{2T_0}{u^0}D_x f^\varepsilon\Big\rangle\\
&\quad\leq \varepsilon^{k-1}\Big\langle[\Gamma(f^\varepsilon, D_x f^\varepsilon) + \Gamma(D_x f^\varepsilon, f^\varepsilon)], \frac{2T_0}{u^0}\{\mathbf{I}-\mathbf{P}\}D_x f^\varepsilon\Big\rangle\\
&\qquad + C\varepsilon^{k-1}(\|\nabla_x n_0\|_\infty + \|\nabla_x u\|_\infty)\|h^\varepsilon\|_\infty\|f^\varepsilon\|\|D_x f^\varepsilon\|\\
&\quad\lesssim \varepsilon^{k-2}\|h^\varepsilon\|_\infty\|\nabla_x\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|^2 + \varepsilon^k\|h^\varepsilon\|_\infty\|\nabla_x f^\varepsilon\|^2\\
&\qquad + (1+t)^{-\beta_0}\|D_x f^\varepsilon\|^2 + (1+t)^{-\beta_0}[\varepsilon^{k-1}\|h^\varepsilon\|_\infty]^2\|f^\varepsilon\|^2.
\end{aligned} \tag{6.4}
$$

Similarly, by (8.17) in Theorem 8.1, the eighth term in the right hand side of (6.2) can be estimated by

$$
\begin{aligned}
&\sum_{i=1}^{2k-1}\varepsilon^{i-1}\Big\langle[\Gamma(F_i, D_x f^\varepsilon)+\Gamma(D_x f^\varepsilon, F_i)], \frac{2T_0}{u^0}\{\mathbf{I}-\mathbf{P}\}D_x f^\varepsilon\Big\rangle\\
&\quad+\sum_{i=1}^{2k-1}\varepsilon^{i-1}\Big\langle[\Gamma(D_x F_i, f^\varepsilon)+\Gamma(f^\varepsilon, D_x F_i)], \frac{2T_0}{u^0}\{\mathbf{I}-\mathbf{P}\}D_x f^\varepsilon\Big\rangle\\
&\quad+C(\|\nabla_x n_0\|_\infty+\|\nabla_x u\|_\infty)\sum_{i=1}^{2k-1}\varepsilon^{i-1}\|F_i\|_{L^\infty_x L^2_p}\|f^\varepsilon\|\|D_x f^\varepsilon\|\\
&\lesssim\frac{\kappa}{\varepsilon}\|\{\mathbf{I}-\mathbf{P}\}D_x f^\varepsilon\|^2+\varepsilon\Big(\sum_{i=1}^{2k-1}\varepsilon^{i-1}(1+t)^{i-1}\Big)^2(\|f^\varepsilon\|^2+\|\nabla_x f^\varepsilon\|^2)\\
&\quad+(1+t)^{-\beta_0}\sum_{i=1}^{2k-1}\varepsilon^{i-1}(1+t)^{i-1}(\|f^\varepsilon\|^2+\|\nabla_x f^\varepsilon\|^2).
\end{aligned}\tag{6.5}
$$

□

Next we deal with the momentum derivative estimate for f^ε. Due to the Maxwellian structure in the Macroscopic part, it is enough to perform momentum derivative estimate for $\{\mathbf{I}-\mathbf{P}\}f^\varepsilon$. Moreover, as illustrated in the Introduction part, this technique is also necessary for us to make use of the strong dissipation term $\frac{Lf^\varepsilon}{\varepsilon}$ and close our energy estimates. Take microscopic projection onto (2.1) to have

$$
\begin{aligned}
&\partial_t\{\mathbf{I}-\mathbf{P}\}f^\varepsilon+\hat{p}\cdot\nabla_x\{\mathbf{I}-\mathbf{P}\}f^\varepsilon+\{\mathbf{I}-\mathbf{P}\}\Big[\frac{\sqrt{\mathbf{M}}}{2T_0}\Big(u^0\hat{p}-u\Big)\cdot\Big(E_R^\varepsilon+\hat{p}\times B_R^\varepsilon\Big)\Big]\\
&\quad-\Big(E_0+\hat{p}\times B_0\Big)\cdot\nabla_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon+\frac{Lf^\varepsilon}{\varepsilon}\\
&=\Big(\mathbf{P}\partial_t f^\varepsilon-\partial_t\mathbf{P}f^\varepsilon\Big)-\frac{\{\mathbf{I}-\mathbf{P}\}f^\varepsilon}{\sqrt{\mathbf{M}}}\Big[\partial_t+\hat{p}\cdot\nabla_x-\Big(E_0+\hat{p}\times B_0\Big)\cdot\nabla_p\Big]\sqrt{\mathbf{M}}\\
&\quad+\varepsilon^{k-1}\Gamma(f^\varepsilon, f^\varepsilon)+\sum_{i=1}^{2k-1}\varepsilon^{i-1}[\Gamma(F_i, f^\varepsilon)+\Gamma(f^\varepsilon, F_i)]+\varepsilon^k\Big(E_R^\varepsilon+\hat{p}\times B_R^\varepsilon\Big)\\
&\quad\cdot\nabla_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon-\varepsilon^k\frac{1}{2T_0}\Big(u^0\hat{p}-u\Big)\cdot\Big(E_R^\varepsilon+\hat{p}\times B_R^\varepsilon\Big)\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\\
&\quad+\sum_{i=1}^{2k-1}\varepsilon^i\Big[\Big(E_i+\hat{p}\times B_i\Big)\cdot\nabla_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon+\{\mathbf{I}-\mathbf{P}\}\Big((E_R^\varepsilon+\hat{p}\times B_R^\varepsilon)\cdot\nabla_p F_i\Big)\Big]\\
&\quad-\sum_{i=1}^{2k-1}\varepsilon^i\Big[\Big(E_i+\hat{p}\times B_i\Big)\cdot\frac{1}{2T_0}\Big(u^0\hat{p}-u\Big)\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\Big]\\
&\quad+\varepsilon^k\{\mathbf{I}-\mathbf{P}\}\bar{A}+[[\mathbf{P}, \tau_{E,B}]]f^\varepsilon,
\end{aligned}\tag{6.6}
$$

where $[[\mathbf{P}, \tau_{E,B}]] = \mathbf{P}\tau_{E,B} - \tau_{E,B}\mathbf{P}$ denotes the commutator of two operators $\mathbf{P}$ and $\tau_{E,B}$ given by

$$\begin{aligned}\tau_{E,B} = &\ \hat{p}\cdot\nabla_x - \Big(E_0 + \hat{p}\times B_0\Big)\cdot\nabla_p \\ &+ \frac{1}{\sqrt{\mathbf{M}}}\Big[\partial_t + \hat{p}\cdot\nabla_x - \Big(E_0 + \hat{p}\times B_0\Big)\cdot\nabla_p\Big]\sqrt{\mathbf{M}} \\ &- \varepsilon^k\Big(E_R^\varepsilon + \hat{p}\times B_R^\varepsilon\Big)\cdot\nabla_p + \frac{\varepsilon^k}{2T_0}\Big(u^0\hat{p} - u\Big)\cdot\Big(E_R^\varepsilon + \hat{p}\times B_R^\varepsilon\Big) \\ &- \sum_{i=1}^{2k-1}\varepsilon^i\Big(E_i + \hat{p}\times B_i\Big)\cdot\Big[\nabla_p - \frac{1}{2T_0}\Big(u^0\hat{p} - u\Big)\Big].\end{aligned}$$

The momentum derivative $\nabla_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon$ can be estimated as follows:

Proposition 6.2. *For the remainders $(f^\varepsilon, E_R^\varepsilon, B_R^\varepsilon)$, it holds that*

$$\begin{aligned}&\frac{d}{dt}\|\nabla_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon(t)\|^2 + \Big(\frac{\delta_0}{2\varepsilon} - C\varepsilon^{k-2}\|h^\varepsilon\|_\infty\Big)\|\nabla_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|^2 \\ &\lesssim \Big[(1+t)^{-\beta_0} + \varepsilon[1+\mathcal{I}_1(t)] + \varepsilon^k\|h^\varepsilon\|_{W^{1,\infty}}\Big]\Big(\|f^\varepsilon\|_{H^1}^2 + \|(E_R^\varepsilon, B_R^\varepsilon)\|_{H^1}^2\Big) \\ &\quad + \frac{1}{\varepsilon}\|\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|^2 + \varepsilon^{\frac{11}{4}}\|h^\varepsilon\|_\infty\|f^\varepsilon\| + \varepsilon^k\mathcal{I}_2(t)\|\nabla_p f^\varepsilon\|. \qquad (6.7)\end{aligned}$$

Proof. We apply D_p to (6.6) and take the L^2 inner product with $D_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon$ on both sides to have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|D_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon(t)\|^2 + \frac{1}{\varepsilon}\langle D_pLf^\varepsilon, D_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\rangle \\ &\quad + \langle (D_p\hat{p})\cdot\nabla_x\{\mathbf{I}-\mathbf{P}\}f^\varepsilon, D_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\rangle \\ &\quad + \Big\langle D_p\{\mathbf{I}-\mathbf{P}\}\Big[\frac{(u^0\hat{p}-u)}{2T_0}\cdot\Big(E_R^\varepsilon + \hat{p}\times B_R^\varepsilon\Big)\Big], D_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\Big\rangle \\ &\quad - \Big\langle D_p\Big[\Big(E_0 + \hat{p}\times B_0\Big)\cdot\nabla_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\Big], D_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\Big\rangle \\ &= -\Big\langle D_p\Big[\frac{\{\mathbf{I}-\mathbf{P}\}f^\varepsilon}{\sqrt{\mathbf{M}}}\Big(\partial_t + \hat{p}\cdot\nabla_x - (E_0 + \hat{p}\times B_0)\cdot\nabla_p\Big)\sqrt{\mathbf{M}}\Big], D_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\Big\rangle \\ &\quad + \Big\langle D_p\left(\mathbf{P}\partial_t f^\varepsilon - \partial_t\mathbf{P}f^\varepsilon\right), D_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\Big\rangle + \varepsilon^{k-1}\Big\langle D_p\Gamma(f^\varepsilon, f^\varepsilon), D_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\Big\rangle \\ &\quad + \sum_{i=1}^{2k-1}\varepsilon^{i-1}\Big\langle D_p[\Gamma(F_i, f^\varepsilon) + \Gamma(f^\varepsilon, F_i)], D_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\Big\rangle \\ &\quad + \varepsilon^k\Big\langle\frac{1}{2T_0}D_p\Big[(u^0\hat{p}-u)\cdot\Big(E_R^\varepsilon + \hat{p}\times B_R^\varepsilon\Big)\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\Big], D_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\Big\rangle \\ &\quad + \varepsilon^k\Big\langle D_p\Big[\Big(E_R^\varepsilon + \hat{p}\times B_R^\varepsilon\Big)\cdot\nabla_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\Big], D_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\Big\rangle \\ &\quad + \sum_{i=1}^{2k-1}\varepsilon^i\Big\langle D_p\Big[\Big(E_i + \hat{p}\times B_i\Big)\cdot\nabla_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\Big], D_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\Big\rangle\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{2k-1}\varepsilon^i\Big\langle D_p\{\mathbf{I}-\mathbf{P}\}\Big[\Big(E_R^\varepsilon+\hat{p}\times B_R^\varepsilon\Big)\cdot\nabla_p F_i\Big], D_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\Big\rangle\\
&-\sum_{i=1}^{2k-1}\varepsilon^i\Big\langle D_p\Big[\Big(E_i+\hat{p}\times B_i\Big)\cdot\frac{1}{2T_0}\Big(u^0\hat{p}-u\Big)\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\Big], D_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\Big\rangle\\
&+\varepsilon^k\Big\langle D_p\{\mathbf{I}-\mathbf{P}\}\bar{A}, D_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\Big\rangle+\Big\langle D_p\{\mathbf{I}-\mathbf{P}\}\left([[\mathbf{P},\tau_{E,B}]]f^\varepsilon\right), D_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\Big\rangle.
\end{aligned}
\tag{6.8}
$$

For brevity, we only prove the second, fourth terms in the left hand side of (6.8) and the first, second, third, fourth terms and the last term in the right hand side. We first estimate the second term in the left hand side of (6.8). Similar to (6.3), one has

$$
\sqrt{\mathbf{M}}+|D_p\sqrt{\mathbf{M}}|\lesssim J_M^{\frac{1}{4}}. \tag{6.9}
$$

Note that energy estimates for the momentum derivative of the collision frequency ν and operator K, which correspond to a global Maxwellian were proved in Propositions 7, 8 [25] via a splitting technique and different expressions of the relativistic collision operator in the Glassey–Strauss frame and the center of mass frame. We combine (6.9) and Propositions 7, 8 in [25] to estimate the third term in the left hand side of (6.8) as

$$
\frac{1}{\varepsilon}\langle D_pLf^\varepsilon, D_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\rangle\geq\frac{7\delta_0}{8\varepsilon}\|D_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|^2-\frac{C}{\varepsilon}\|\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|^2.
$$

For the fourth term in the left hand side, it can be estimated by

$$
\frac{\delta_0}{8\varepsilon}\|D_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|^2+C\varepsilon\left(\|E_R^\varepsilon\|^2+\|B_R^\varepsilon\|^2\right).
$$

For the first term in the right hand side, we integrate by parts w.r.t. p when D_p doesn't hit f^ε and bound it by

$$
\frac{\kappa}{\varepsilon}\|D_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|_{H^1}^2+\varepsilon^{\frac{11}{4}}\|h^\varepsilon\|_\infty\|f^\varepsilon\|.
$$

Noting

$$
\|\mathbf{P}\partial_t f^\varepsilon-\partial_t\mathbf{P}f^\varepsilon\|\lesssim(\|\partial_t n_0\|_\infty+\|\partial_t u\|_\infty)\|\mathbf{P}f^\varepsilon\|,
$$

we can bound the second term in the right hand side of (6.8) by

$$
\frac{\delta_0}{8\varepsilon}\|D_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|^2+C\varepsilon(1+t)^{-2\beta_0}\|f^\varepsilon\|^2.
$$

By Theorem 2 in [25] and (6.9), we can bound the third term in the right hand side by

$$
\varepsilon^{k-2}\|h^\varepsilon\|_\infty\|\nabla_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|^2+C\varepsilon^k\|h^\varepsilon\|_\infty\|f^\varepsilon\|_{H^1}^2.
$$

Similarly, we control the fourth term in the right hand side by

$$
\frac{\delta_0}{8\varepsilon}\|D_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|^2+C\varepsilon\Big[\sum_{i=1}^{2k-1}\varepsilon^{i-1}\left(\|F_i\|_{L_x^\infty L_p^2}+\|D_pF_i\|_{L_x^\infty L_p^2}\right)\Big]^2\|f^\varepsilon\|_{H^1}^2
$$

$$\leq \frac{\delta_0}{8\varepsilon}\|D_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|^2 + C\varepsilon\Big(\sum_{i=1}^{2k-1}\varepsilon^{i-1}(1+t)^{i-1}\Big)^2\|f^\varepsilon\|_{H^1}^2.$$

Similar to the above estimates, the upper bound of the last term in the right hand side is

$$\frac{\delta_0}{8\varepsilon}\|D_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|^2 + \Big[\varepsilon[1+\mathcal{I}_1(t)] + \varepsilon^k\|h^\varepsilon\|_\infty\Big]\Big(\|f^\varepsilon\|_{H^1}^2 + \|(E_R^\varepsilon, B_R^\varepsilon)\|_{H^1}^2\Big).$$

□

Finally, we multiply (2.3) in Proposition 2.1 by a sufficiently large constant C_0, combine it and (6.1) in Proposition 6.1, (6.7) in Proposition 6.2, and use (1.27) to obtain

Proposition 6.3. *Let* $0 \leq t \leq T \leq \varepsilon^{-\frac{1}{2}}$. *Under the assumptions* (1.27), *one has*

$$\begin{aligned}
&\frac{d}{dt}\Big[C_0\Big(\Big\|\sqrt{\frac{2T_0}{u^0}}f^\varepsilon(t)\Big\|^2 + \|[E_R^\varepsilon(t), B_R^\varepsilon(t)]\|^2\Big) + \Big\|\sqrt{\frac{2T_0}{u^0}}\nabla_x f^\varepsilon(t)\Big\|^2 \\
&\quad + \|[\nabla_x E_R^\varepsilon(t), \nabla_x B_R^\varepsilon(t)]\|^2 + \|\nabla_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon(t)\|^2\Big] \\
&\lesssim \Big[[(1+t)^{-\beta_0}+\varepsilon](1+\mathcal{I}_1(t)) + \varepsilon^k\|h^\varepsilon\|_{W^{1,\infty}}\Big]\Big(\|f^\varepsilon\|_{H^1}^2 + \|(E_R^\varepsilon, B_R^\varepsilon)\|_{H^1}^2\Big) \\
&\quad + \frac{1}{\varepsilon}(1+t)^{-2\beta_0}\|f^\varepsilon\|^2 + (1+t)^{-\beta_0}[\varepsilon^{k-1}\|h^\varepsilon\|_\infty]^2\|f^\varepsilon\|^2 \\
&\quad + [\varepsilon^{\frac{7}{4}}(1+t)^{-\beta_0} + \varepsilon^{\frac{11}{4}}]\|h^\varepsilon\|_\infty\|f^\varepsilon\| + \varepsilon^k\mathcal{I}_2(t)\|f^\varepsilon\|_{H^1}.
\end{aligned} \tag{6.10}$$

7. Proof of the Main Result

This section is devoted to the proof of Theorem 1.1.

Proof. For $0 \leq t \leq T \leq \varepsilon^{-\frac{1}{2}}$, one has

$$\begin{aligned}
\mathcal{I}_1(t) &\lesssim \sum_{1\leq i\leq 2k-1}[(1+t)\varepsilon]^{i-1}\Big(\sum_{1\leq i\leq 2k-1}[(1+t)\varepsilon]^{i-1}+1\Big) \lesssim 1, \\
\varepsilon^k\mathcal{I}_2(t) &\lesssim \varepsilon^k\sum_{1+2k\leq i+j\leq 4k-2}\varepsilon^{1-2k}[(1+t)\varepsilon]^{i+j-2} \\
&\lesssim \sum_{1+2k\leq i+j\leq 4k-2}\varepsilon^{1-k}\varepsilon^{\frac{i+j-2}{2}} \lesssim \varepsilon^{\frac{1}{2}}.
\end{aligned} \tag{7.1}$$

We combine the a priori Assumptions (1.27), (2.3) and Proposition 3.1 to have

$$\begin{aligned}
&\frac{d}{dt}\Big(\Big\|\sqrt{\frac{2T_0}{u^0}}f^\varepsilon(t)\Big\|^2 + \|[E_R^\varepsilon(t), B_R^\varepsilon(t)]\|^2\Big) + \frac{\delta_0 T_M}{4\varepsilon}\|\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|^2 \\
&\lesssim \Big(\varepsilon^{\frac{3}{2}}\|h^\varepsilon(0)\|_\infty + \sup_{0\leq t\leq\varepsilon^{-\frac{1}{2}}}\|f^\varepsilon(t)\| + \varepsilon^{\frac{2k+3}{2}}\Big) \\
&\times\Big[\varepsilon^{k-\frac{3}{2}}\Big(\|f^\varepsilon\|^2 + \|(E_R^\varepsilon, B_R^\varepsilon)\|^2\Big) + \varepsilon^{\frac{5}{4}}\|f^\varepsilon\|\Big] \\
&+ \Big[(1+t)^{-\beta_0} + \varepsilon\mathcal{I}_1(t)\Big]\Big(\|f^\varepsilon\|^2 + \|(E_R^\varepsilon, B_R^\varepsilon)\|^2\Big) + \varepsilon^k\mathcal{I}_2(t)\|f^\varepsilon\|.
\end{aligned}$$

Then, for $T \le \varepsilon^{-\frac{1}{2}}$, Grönwall's inequality yields

$$\begin{aligned}
&\|f^\varepsilon(t)\| + \|[E_R^\varepsilon(t), B_R^\varepsilon(t)]\| + 1 \\
&\lesssim \left[\|f^\varepsilon(0)\| + \|[E_R^\varepsilon(0), B_R^\varepsilon(0)]\| + 1\right] \exp\left\{\int_0^t \left[(\varepsilon^{k-\frac{3}{2}} + \varepsilon^{\frac{5}{4}})\right.\right. \\
&\quad \times \left(\varepsilon^{\frac{3}{2}}\|h^\varepsilon(0)\|_\infty + \sup_{0\le s\le \varepsilon^{-\frac{1}{2}}} \|f^\varepsilon(s)\| + \varepsilon^{\frac{2k+3}{2}}\right) \\
&\quad \left.\left. + [(1+s)^{-\beta_0} + \varepsilon \mathcal{I}_1(s) + \varepsilon^k \mathcal{I}_2(t)\right] ds\right\} \\
&\lesssim \left[\|f^\varepsilon(0)\| + \|[E_R^\varepsilon(0), B_R^\varepsilon(0)]\| + 1\right] \\
&\quad \times \exp\left\{C(1+\sqrt{\varepsilon}) + [\varepsilon^{\frac{3}{4}} + \varepsilon^{k-2}]\left(\varepsilon^{\frac{3}{2}}\|h^\varepsilon(0)\|_\infty + \sup_{0\le s\le \varepsilon^{-\frac{1}{2}}} \|f^\varepsilon(s)\| + 1\right)\right\}.
\end{aligned}$$

Here we used $\int_0^t (1+s)^{-\beta_0} ds \lesssim 1$ and (7.1). For ε sufficiently small and $T \le \varepsilon^{-\frac{1}{2}}$, we further obtain

$$\begin{aligned}
&\|f^\varepsilon(t)\| + \|[E_R^\varepsilon(t), B_R^\varepsilon(t)]\| + 1 \\
&\lesssim \left[\|f^\varepsilon(0)\| + \|[E_R^\varepsilon(0), B_R^\varepsilon(0)]\| + 1\right] \\
&\quad \times \left[1 + (\varepsilon^{\frac{3}{4}} + \varepsilon^{k-2})\left(\varepsilon^{\frac{3}{2}}\|h^\varepsilon(0)\|_\infty + \sup_{0\le s\le T} \|f^\varepsilon(s)\| + 1\right)\right].
\end{aligned}$$

Letting ε be small enough, we obtain that for $t \le T \le \varepsilon^{-\frac{1}{2}}$, $k \ge 3$,

$$\begin{aligned}
&\sup_{0\le t\le \varepsilon^{-\frac{1}{2}}} (\|f^\varepsilon(t)\| + \|[E_R^\varepsilon(t), B_R^\varepsilon(t)]\|) + 1 \\
&\lesssim \|f^\varepsilon(0)\| + \|[E_R^\varepsilon(0), B_R^\varepsilon(0)]\| + \varepsilon^{\frac{3}{2}}\|h^\varepsilon(0)\|_\infty + 1.
\end{aligned}$$

From Proposition 3.1, we further obtain

$$\begin{aligned}
&\sup_{0\le t\le \varepsilon^{-\frac{1}{2}}} \left(\varepsilon^{\frac{3}{2}}\|h^\varepsilon(t)\|_\infty\right) + \sup_{0\le t\le \varepsilon^{-\frac{1}{2}}} (\|f^\varepsilon(t)\| + \|[E_R^\varepsilon(t), B_R^\varepsilon(t)]\|) + 1 \\
&\lesssim \|f^\varepsilon(0)\| + \|[E_R^\varepsilon(0), B_R^\varepsilon(0)]\| + \varepsilon^{\frac{3}{2}}\|h^\varepsilon(0)\|_\infty + 1.
\end{aligned} \tag{7.2}$$

Now we combine (6.10) and (7.2) to have

$$\begin{aligned}
&\frac{d}{dt}\left[C_0\left(\left\|\sqrt{\frac{2T_0}{u^0}} f^\varepsilon(t)\right\|^2 + \|[E_R^\varepsilon(t), B_R^\varepsilon(t)]\|^2\right) + \left\|\sqrt{\frac{2T_0}{u^0}} \nabla_x f^\varepsilon(t)\right\|^2\right. \\
&\qquad \left. + \|[\nabla_x E_R^\varepsilon(t), \nabla_x B_R^\varepsilon(t)]\|^2 + \|\nabla_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon(t)\|^2\right] \\
&\lesssim \left[[(1+t)^{-\beta_0} + \varepsilon](1+\mathcal{I}_1(t)) + \varepsilon^k\|h^\varepsilon\|_{W^{1,\infty}}\right] \\
&\quad \times \left(\|f^\varepsilon\|_{H^1}^2 + \|(E_R^\varepsilon, B_R^\varepsilon)\|_{H^1}^2 + \frac{1}{\varepsilon}\right) + \varepsilon^k \mathcal{I}_2(t)\|f^\varepsilon\|_{H^1},
\end{aligned}$$

Noting

$$\|f^\varepsilon\|_{H^1}^2 + \|(E_R^\varepsilon, B_R^\varepsilon)\|_{H^1}^2 \approx \left\|\sqrt{\frac{2T_0}{u^0}} f^\varepsilon\right\|^2 + \|(E_R^\varepsilon, B_R^\varepsilon)\|^2$$

$$+\left\|\sqrt{\frac{2T_0}{u^0}}\nabla_x f^\varepsilon\right\|^2+\|(\nabla_x E_R^\varepsilon,\nabla_x B_R^\varepsilon)\|^2+\|\nabla_p\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|^2,$$

we further apply Grönwall's inequality in the above inequality to obtain that for $T\le\varepsilon^{-\frac{1}{2}}$, and ε sufficiently small,

$$\begin{aligned}
&\|f^\varepsilon(t)\|_{H^1}+\|[E_R^\varepsilon(t),B_R^\varepsilon(t)]\|_{H^1}+\frac{1}{\sqrt{\varepsilon}}\\
&\lesssim\left(\|f^\varepsilon(0)\|_{H^1}+\|[E_R^\varepsilon(0),B_R^\varepsilon(0)]\|_{H^1}+\frac{1}{\sqrt{\varepsilon}}\right)\\
&\quad\times\exp\left\{C(1+t\sqrt{\varepsilon})+t\varepsilon^{k-\frac{3}{2}}\left(\varepsilon^{\frac{3}{2}}\|h^\varepsilon(0)\|_{W^{1,\infty}}+\sup_{0\le s\le 1/\sqrt{\varepsilon}}\|f^\varepsilon(s)\|_{H^1}\right)\right\}\\
&\lesssim\left(\|f^\varepsilon(0)\|_{H^1}+\|[E_R^\varepsilon(0),B_R^\varepsilon(0)]\|_{H^1}+\frac{1}{\sqrt{\varepsilon}}\right)\\
&\quad\times\left[1+\varepsilon^{k-2}\left(\varepsilon^{\frac{3}{2}}\|h^\varepsilon(0)\|_{W^{1,\infty}}+\sup_{0\le s\le 1/\sqrt{\varepsilon}}\|f^\varepsilon(s)\|_{H^1}\right)\right].
\end{aligned}\tag{7.3}$$

Taking ε sufficiently small in (7.3), we get for $k\ge 3$ that

$$\begin{aligned}
&\sup_{0\le t\le 1/\sqrt{\varepsilon}}\left(\|f^\varepsilon(t)\|_{H^1}+\|[E_R^\varepsilon(t),B_R^\varepsilon(t)]\|_{H^1}\right)+\frac{1}{\sqrt{\varepsilon}}\\
&\lesssim\|f^\varepsilon(0)\|_{H^1}+\|[E_R^\varepsilon(0),B_R^\varepsilon(0)]\|_{H^1}+\frac{1}{\sqrt{\varepsilon}}+\varepsilon^{k-\frac{5}{2}}\left(\varepsilon^{\frac{3}{2}}\|h^\varepsilon(0)\|_{W^{1,\infty}}\right).
\end{aligned}\tag{7.4}$$

From Proposition 5.1, we further obtain

$$\sup_{0\le t\le T}\left(\varepsilon^2\|h^\varepsilon(t)\|_{W^{1,\infty}}\right)\lesssim\varepsilon^2\|h^\varepsilon(0)\|_{W^{1,\infty}}+\sqrt{\varepsilon}(\|f^\varepsilon(0)\|_{H^1}+\|[E_R^\varepsilon(0),B_R^\varepsilon(0)]\|_{H^1})+1.\tag{7.5}$$

We combine (4.1), (7.2), (7.4) and (7.5) to obtain (1.25). Note that (1.25) implies the assumptions in (1.27) and these assumptions can be verified by a continuity argument for all time $t\le T\le\varepsilon^{-\frac{1}{2}}$. □

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8. Appendix

Our Appendix is about three problems: the derivation of the kernel of operator $K_{\mathbf{M}}$ in (1.21), estimates of the kernels related to K_1 and K_2, and the construction and regularity estimates of the coefficients in the Hilbert expansion (1.8).

8.1. Appendix 1: Derivation of the kernel k_2. In this part, we derive the kernel k_2 of the operator K_2. For K_i, $(i = 1, 2)$ given in (1.21), their kernels are defined as

$$K_i(f) = \int_{\mathbb{R}^3} dq k_i(p,q) f(q), \qquad i = 1, 2.$$

Following the derivation of $k_2(p,q)$ for a global Maxwellian case in the Appendix of [37], we can obtain the following expression of $k_2(p,q)$ for our case:

$$k_2(p,q) = \frac{C_1 s^{\frac{3}{2}}}{g p^0 q^0} U_1(p,q) \exp\{-U_2(p,q)\}, \tag{8.1}$$

where C_1 is some positive constant, $U_2(p,q)$ and $U_1(p,q)$ are smooth functions satisfying

$$U_2(p,q) = [1 + O(1)|u|] \frac{\sqrt{s}|p-q|}{2T_0 g},$$
$$U_1(p,q) = \frac{1}{U_2(p,q)} + \frac{(p^0+q^0)[1+O(1)|u|]}{2[U_2(p,q)]^2} + \frac{(p^0+q^0)[1+O(1)|u|]}{2[U_2(p,q)]^3}.$$

The proof of (8.1) can be proceeded similarly as in the Appendix [37] except for the estimation of additional terms w.r.t. the velocity u, which arises due to the local Maxwellian **M**.

As in [37], we first write the operator K_2 as

$$K_2(f) = \frac{1}{p^0} \int_{\mathbb{R}^3} \frac{dq}{q^0} \int_{\mathbb{R}^3} \frac{dq'}{q'^0} \int_{\mathbb{R}^3} \frac{dp'}{p'^0} W \sqrt{\mathbf{M}}(q) [\sqrt{\mathbf{M}}(q') f(p') + \sqrt{\mathbf{M}}(p') f(q')].$$

After the same exchanges of variables q, p', q' and changes of integration variables as in [37], we obtain the following form of kernel k_2

$$k_2(p,q) = \frac{C}{p^0 q^0} \int_{\mathbb{R}^3} \frac{d\bar{p}}{\bar{p}^0} \delta((q^\mu - p^\mu)\bar{p}_\mu) \bar{s} \exp\left\{\frac{u^\mu \bar{p}_\mu}{2T_0}\right\},$$

where the integration variable $\bar{p}_\mu = p'_\mu + q'_\mu$ and $\bar{s} = \bar{g}^2 + 4$ with

$$\bar{g}^2 = g^2 + \frac{1}{2}(p^\mu + q^\mu)(p_\mu + q_\mu - \bar{p}_\mu).$$

Introduce a Lorentz transformation Λ which maps into the center-of-momentum system:

$$A^\mu = \Lambda^\mu_\nu (p^\nu + q^\nu) = (\sqrt{s}, 0, 0, 0), \quad B^\mu = -\Lambda^\mu_\nu (p^\nu - q^\nu) = (0, 0, 0, g).$$

Here Λ is the following Lorentz transform derived in [34]:

$$\Lambda = (\Lambda^\mu_\nu) = \begin{pmatrix} \frac{p^0+q^0}{\sqrt{s}} & -\frac{p_1+q_1}{\sqrt{s}} & -\frac{p_2+q_2}{\sqrt{s}} & -\frac{p_3+q_3}{\sqrt{s}} \\ \Lambda^1_0 & \Lambda^1_1 & \Lambda^1_2 & \Lambda^1_3 \\ 0 & \frac{(p\times q)_1}{|p\times q|} & \frac{(p\times q)_2}{|p\times q|} & \frac{(p\times q)_3}{|p\times q|} \\ \frac{p^0-q^0}{g} & -\frac{p_1-q_1}{g} & -\frac{p_2-q_2}{g} & -\frac{p_3-q_3}{g} \end{pmatrix},$$

where $\Lambda_0^1 = \frac{2|p\times q|}{g\sqrt{s}}$ and

$$\Lambda_i^1 = \frac{2[p_i(p^0+q^0p^\mu q_\mu)+q_i(q^0+p^0p^\mu q_\mu)]}{g\sqrt{s}|p\times q|}, \quad i=1,2,3.$$

Define $\bar{u}^\mu = \Lambda_\nu^\mu u^\nu$:

$$\bar{u}^0 = \Lambda_\nu^0 u^\nu = \frac{(p^0+q^0)u^0}{\sqrt{s}} - \frac{(p+q)\cdot u}{\sqrt{s}},$$
$$\bar{u}^1 = \Lambda_\nu^1 u^\nu = \frac{2|p\times q|u^0}{g\sqrt{s}} + \frac{2[p(p^0+q^0p^\mu q_\mu)+q(q^0+p^0p^\mu q_\mu)]\cdot u}{g\sqrt{s}|p\times q|},$$
$$\bar{u}^2 = \Lambda_\nu^2 u^\nu = \frac{(p\times q)\cdot u}{|p\times q|},$$
$$\bar{u}^3 = \Lambda_\nu^3 u^\nu = \frac{(p^0-q^0)u^0}{g} - \frac{(p-q)\cdot u}{g}.$$

Then we have

$$\int_{\mathbb{R}^3}\frac{d\bar{p}}{\bar{p}^0}\delta((q^\mu-p^\mu)\bar{p}_\mu)\bar{s}\exp\left\{\frac{u^\mu\bar{p}_\mu}{2T_0}\right\} = \int_{\mathbb{R}^3}\frac{d\bar{p}}{\bar{p}^0}\delta(B^\mu\bar{p}_\mu)\bar{s}_\Lambda\exp\left\{\frac{u^\mu\bar{p}_\mu}{2T_0}\right\},$$

where $B^\mu\bar{p}_\mu = \bar{p}_3 g$,

$$\bar{s}_\Lambda = \bar{g}^2+4 = g^2+\frac{1}{2}A^\mu(A_\mu-\bar{p}_\mu) = g^2+\frac{1}{2}\sqrt{s}(\bar{p}^0-\sqrt{s}),$$

$$\begin{aligned}\bar{u}^\mu\bar{p}_\mu =& \bar{p}^0\Big(-\frac{(p^0+q^0)u^0}{\sqrt{s}}+\frac{(p+q)\cdot u}{\sqrt{s}}\Big)\\ &+\bar{p}_1\Big(\frac{2|p\times q|u^0}{g\sqrt{s}}+\frac{2[p(p^0+q^0p^\mu q_\mu)+q(q^0+p^0p^\mu q_\mu)]\cdot u}{g\sqrt{s}|p\times q|}\Big)\\ &+\bar{p}_2\frac{(p\times q)\cdot u}{|p\times q|}+\bar{p}_3\Big(\frac{(p^0-q^0)u^0}{g}-\frac{(p-q)\cdot u}{g}\Big).\end{aligned}$$

Switch to polar coordinates

$$d\bar{p} = |\bar{p}|^2d|\bar{p}|\sin\phi d\phi d\theta, \quad \bar{p} = |\bar{p}|(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi),$$

and use $\cos\phi = 0$ for $\phi = \frac{\pi}{2}$ to rewrite $k_2(p,q)$ as

$$\begin{aligned}&\frac{C}{gp^0q^0}\int_0^{2\pi}d\phi\int_0^\infty\frac{|\bar{p}|d\bar{p}}{\bar{p}^0}\bar{s}_\Lambda\exp\Big\{\frac{1}{2T_0}\Big[\bar{p}^0\Big(-\frac{(p^0+q^0)u^0}{\sqrt{s}}+\frac{(p+q)\cdot u}{\sqrt{s}}\Big)\\ &\quad+\Big(\frac{2|p\times q|u^0}{g\sqrt{s}}+\frac{2[p(p^0+q^0p^\mu q_\mu)+q(q^0+p^0p^\mu q_\mu)]\cdot u}{g\sqrt{s}|p\times q|}\Big)|\bar{p}|\cos\phi\\ &\quad+\frac{(p\times q)\cdot u}{|p\times q|}|\bar{p}|\sin\phi\Big]\Big\}\\ &=\frac{C}{gp^0q^0}\int_0^\infty\frac{|\bar{p}|d\bar{p}}{\bar{p}^0}\bar{s}_\Lambda\exp\Big\{\frac{\bar{p}^0}{2T_0}\Big(-\frac{(p^0+q^0)u^0}{\sqrt{s}}+\frac{(p+q)\cdot u}{\sqrt{s}}\Big)\Big\}\\ &\quad\times I_0\Big(\frac{\sqrt{g^2s|(p\times q)\cdot u|^2+4[|p\times q|^2u^0+(p(p^0+q^0p^\mu q_\mu)+q(q^0+p^0p^\mu q_\mu))\cdot u]^2}}{2T_0g\sqrt{s}|p\times q|}|\bar{p}|\Big),\end{aligned}$$

where I_0 is the first kind modified Bessel function of index zero:

$$I_0(y) = \frac{1}{2\pi}\int_0^{2\pi} e^{y\cos\phi}d\phi.$$

By further changing variables of integration and applying some known integrals as in the Appendix [37], we obtain (8.1) with the following exact form of $U_2(p,q)$:

$$\begin{aligned}
4T_0^2 sU_2^2 &= |(p^0+q^0)u^0 - (p+q)\cdot u|^2 - \frac{s|(p\times q)\cdot u|^2}{|p\times q|^2}\\
&\quad - \frac{4[|p\times q|^2u^0 + (p(p^0+q^0p^\mu q_\mu) + q(q^0+p^0p^\mu q_\mu))\cdot u]^2}{g^2|p\times q|^2}\\
&= \frac{g^2(p^0+q^0)^2 - 4|p\times q|^2}{g^2}(u^0)^2 + |(p+q)\cdot u|^2 - \frac{s|(p\times q)\cdot u|^2}{|p\times q|^2}\\
&\quad - \frac{2u^0[g^2(p^0+q^0)(p+q) + 4(p(p^0+q^0p^\mu q_\mu) + q(q^0+p^0p^\mu q_\mu))]\cdot u}{g^2}\\
&\quad - \frac{|(p(p^0+q^0p^\mu q_\mu) + q(q^0+p^0p^\mu q_\mu))\cdot u|^2}{g^2|p\times q|^2}.
\end{aligned} \tag{8.2}$$

Now we estimate the terms in the right hand side of the second equality in (8.2). For the first term, we have

$$\begin{aligned}
&g^2(p^0+q^0)^2 - 4|p\times q|^2\\
&\quad = 2[(2p^0q^0 + 2 + |p|^2 + |q|^2)(p^0q^0 - p\cdot q - 1) - 4|p|^2|q|^2 + 4(p\cdot q)^2\\
&\quad = 4[(|p|^2+|q|^2) - (p^0q^0+1)p\cdot q]\\
&\qquad + 2(|p|^2+|q|^2)(p^0q^0 - p\cdot q - 1) + 4(p\cdot q)^2\\
&\quad = 2(|p|^2+|q|^2)(p^0q^0 - p\cdot q + 1) - 4p\cdot q(p^0q^0 - p\cdot q + 1)\\
&\quad = s|p-q|^2.
\end{aligned} \tag{8.3}$$

It is straightforward to see that the upper bound of the second term is $s|u|^2$. Noting

$$\begin{aligned}
&g^2(p^0+q^0)(p+q) + 4(p(p^0+q^0p^\mu q_\mu) + q(q^0+p^0p^\mu q_\mu))\\
&\quad = 2p[(p^0+q^0)(p^0q^0 - p\cdot q - 1) - 2p^0|q|^2 + 2q^0p\cdot q)]\\
&\qquad + 2q[(p^0+q^0)(p^0q^0 - p\cdot q - 1) - 2q^0|p|^2 + 2p^0p\cdot q)]\\
&\quad = 2p(p^0-q^0)(p^0q^0 - p\cdot q + 1) - 2q(p^0-q^0)(p^0q^0 - p\cdot q + 1)\\
&\quad = s(p^0-q^0)(p-q),
\end{aligned}$$

the fourth term can be bounded by

$$\frac{2u^0|u|s|p-q|^2}{g^2}.$$

For the corresponding fifth term, we rewrite its numerator as

$$|p(p^0+q^0p^\mu q_\mu) + q(q^0+p^0p^\mu q_\mu)|^2$$

$$
\begin{aligned}
&= |p|^2[(p^0)^2|q|^4 - 2p^0q^0|q|^2 p\cdot q + (q^0)^2(p\cdot q)^2] \\
&\quad + |q|^2[(q^0)^2|p|^4 - 2p^0q^0|p|^2 p\cdot q + (p^0)^2(p\cdot q)^2] \\
&\quad + 2p\cdot q[p^0q^0|p|^2|q|^2 - ((p^0)^2|q|^2 + (q^0)^2|p|^2)p\cdot q + p^0q^0(p\cdot q)^2] \\
&= |p|^2|q|^2\left\{[(p^0)^2|q|^2 + (q^0)^2|p|^2 + 2p^0q^0 p\cdot q] - 4p^0q^0 p\cdot q\right\} \\
&\quad + [|p|^2(q^0)^2 + |q|^2(p^0)^2 - 2(|q|^2(p^0)^2 + |p|^2(q^0)^2)](p\cdot q)^2 + 2p^0q^0(p\cdot q)^3 \\
&= |p|^2|q|^2|p^0q - q^0p|^2 - (p\cdot q)^2[(p^0)^2|q|^2 + (q^0)^2|p|^2 - 2p^0q^0(p\cdot q)] \\
&= |p\times q|^2|p^0q - q^0p|^2.
\end{aligned}
$$

Then it can be bounded by

$$
\frac{|u|^2|p\times q|^2|p^0q - q^0p|^2}{g^2|p\times q|^2} = \frac{|u|^2|p^0q - q^0p|^2}{g^2}.
$$

On the other hand, we have

$$
\begin{aligned}
&s|p-q|^2 - |p^0q - q^0p|^2 \\
&\quad = 2(p^0q^0 - p\cdot q + 1)(|p|^2 + |q|^2 - 2p\cdot q) - |q|^2(p^0)^2 - |p|^2(q^0)^2 + 2p^0q^0 p\cdot q \\
&\quad = 4(p\cdot q)^2 - 2[(p^0)^2 + (q^0)^2 + p^0q^0]p\cdot q \\
&\qquad + 2(|p|^2 + |q|^2)(p^0q^0 + 1) - [(p^0)^2|q|^2 + (q^0)^2|p|^2] \\
&\qquad \geq 4|p|^2|q|^2 - (2|p|^2 + 2|q|^2 + 2p^0q^0 + 4)|p||q| \\
&\qquad + 2(|p|^2 + |q|^2)p^0q^0 + |p|^2 + |q|^2 - 2|p|^2|q|^2 \\
&\quad = 2(|p|^2 + |q|^2 - |p||q|)(p^0q^0 - |p||q|) + (|p|^2 + |q|^2 - 4|p||q|) \\
&\quad \geq 3(|p| - |q|)^2
\end{aligned}
$$

since $p^0q^0 - |p||q| \geq 1$. We combine the above estimates in (8.2) to get

$$
4T_0^2U_2^2(p,q) = [1 + O(1)|u|]^2\frac{s|p-q|^2}{g^2}.
$$

8.2. Appendix 2: Estimates of kernels. We first prove some important estimates for kernels k_1 and k_2. As a corollary, estimates for corresponding kernels of $\bar{K}(h^\varepsilon) = \frac{w(|p|)\sqrt{\mathbf{M}}}{\sqrt{J_M}}K(\frac{\sqrt{J_M}}{w\sqrt{\mathbf{M}}}h^\varepsilon)$ will also be established.

Lemma 8.1. *For k_2 given in* (8.1) *and $i = 1, 2, 3$, it holds that*

$$
k_2(p,q) \lesssim \frac{1}{p^0|p-q|}\exp\left\{-\frac{\sqrt{s}|p-q|}{8T_0 g}\right\}, \tag{8.4}
$$

and

$$
\begin{aligned}
\int_{\mathbb{R}^3} dq k_2(p,q) &\lesssim \frac{1}{p^0}, \\
\int_{\mathbb{R}^3} dq\partial_{x_i} k_2(p,q) &\lesssim \frac{1}{p^0},
\end{aligned}
$$

$$\int_{\mathbb{R}^3} dq \partial_{p_i} k_2(p,q) \lesssim \frac{1}{p^0}. \tag{8.5}$$

It is straightforward to verify that $k_1(p,q)$ also satisfies the same estimates.

Proof. We first prove (8.4). Note that, by Lemma 3.1 in [16], $s = g^2 + 4 \le 4p^0q^0 + 4$ and

$$\frac{|p-q|}{\sqrt{p^0q^0}} \le g \le |p-q|.$$

We use the smallness assumption of $|u|$ in (1.20) to have

$$U_2(p,q) \ge \frac{\sqrt{s}|p-q|}{4T_0 g}, \qquad U_1(p,q) \lesssim \frac{p^0+q^0}{2[U_2(p,q)]^2}.$$

Then we can further estimate the kernel k_2 as

$$\begin{aligned}
k_2(p,q) &\lesssim \frac{s^{\frac{3}{2}}}{gp^0q^0}(p^0+q^0) \times \frac{g^2}{s|p-q|^2} \exp\{-\frac{\sqrt{s}|p-q|}{4T_0 g}\} \\
&= \frac{(p^0+q^0)g^2}{p^0q^0|p-q|^3} \frac{\sqrt{s}|p-q|}{g} \exp\{-\frac{\sqrt{s}|p-q|}{4T_0 g}\} \\
&\lesssim \frac{1+|p-q|+q^0}{p^0q^0|p-q|} \exp\{-\frac{|p-q|}{6T_0}\} \\
&\lesssim \frac{1}{p^0|p-q|} \exp\{-\frac{|p-q|}{8T_0}\}.
\end{aligned}$$

Now we prove (8.5). Noting that

$$|p|^2 \le 2|p-q|^2 + 2|q|^2, \qquad |q|^2 \le 2|p-q|^2 + 2|p|^2,$$

we get from (8.4) that

$$\int_{\mathbb{R}^3} dq k_2(p,q) \lesssim \int_{\mathbb{R}^3} dq \frac{1}{p^0|p-q|} \exp\left\{-\frac{|p-q|}{8T_0}\right\} \lesssim \frac{1}{p^0}.$$

For the second inequality in (8.5), again from (8.4), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3} dq \partial_{x_i} k_2(p,q) \\
&\quad\lesssim \int_{\mathbb{R}^3} dq k_2(p,q) \frac{\sqrt{s}|p-q|}{g} \exp\{-\frac{\sqrt{s}|p-q|}{4T_0 g}\} \\
&\quad\lesssim \int_{\mathbb{R}^3} dq k_2(p,q) \exp\{-\frac{\sqrt{s}|p-q|}{8T_0 g}\} \\
&\quad\lesssim \frac{1}{p^0}.
\end{aligned}$$

Now we prove the third inequality in (8.5). Note that

$$\partial_{p_i} U_2(p,q) = [1+O(1)|u|]\partial_{p_i}\Big(\frac{\sqrt{s}|p-q|}{2T_0 g}\Big)$$

$$
\begin{aligned}
&= [1+O(1)|u|]\Big(\frac{|p-q|\partial_{p_i}s}{4T_0 g\sqrt{s}} + \frac{\sqrt{s}(p_i-q_i)}{2T_0 g|p-q|} - \frac{\sqrt{s}|p-q|\partial_{p_i}g}{2T_0 g^2}\Big)\\
&= [1+O(1)|u|]\Big[\frac{-4|p-q|}{2T_0 g^2\sqrt{s}}\frac{q^0}{g}\Big(\frac{p_i}{p^0}-\frac{q_i}{q^0}\Big) + \frac{\sqrt{s}(p_i-q_i)}{2T_0 g|p-q|}\Big]\\
&\lesssim \frac{s^{\frac{3}{2}}|p-q|^3}{g^3}\frac{q^0}{s^2|p-q|\min\{p_0,q_0\}} + \frac{\sqrt{s}|p-q|}{2T_0 g}\frac{1}{|p-q|}\\
&\lesssim \frac{s^{\frac{3}{2}}|p-q|^3}{g^3}\frac{\min\{p_0,q_0\}+|p-q|}{|p-q|\min\{p_0,q_0\}} + \frac{\sqrt{s}|p-q|}{2T_0 g}\frac{1}{|p-q|},
\end{aligned}
$$

where we have used the following inequality:

$$
\frac{p_i}{p^0} - \frac{q_i}{q^0} \le \frac{|p-q|}{\min\{p_0,q_0\}}.
$$

Then we use (8.4) to obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^3} dq\,\partial_{p_i}k_2(p,q)\\
&\quad\lesssim \int_{\mathbb{R}^3} dq\Big(\frac{s^{\frac{3}{2}}|p-q|^3}{g^3}\frac{\min\{p_0,q_0\}+|p-q|}{|p-q|\min\{p_0,q_0\}} + \frac{\sqrt{s}|p-q|}{2T_0 g}\frac{1}{|p-q|}\Big)k_2(p,q)\\
&\quad\lesssim \int_{\mathbb{R}^3} dq\,\frac{1}{p^0}\Big(\frac{s^{\frac{3}{2}}|p-q|^3}{g^3}+1\Big)\Big(1+\frac{1}{|p-q|^2}\Big)\exp\Big\{-\frac{\sqrt{s}|p-q|}{8T_0 g}\Big\}\\
&\quad\lesssim \int_{\mathbb{R}^3} dq\,\frac{1}{p^0}\Big(1+\frac{1}{|p-q|^2}\Big)\exp\Big\{-\frac{\sqrt{s}|p-q|}{16T_0 g}\Big\}\\
&\quad\lesssim \frac{1}{p^0}.
\end{aligned}
$$

□

According to the definition of h^ε in (1.23), the operator K corresponding to the equation satisfied by h^ε is $\bar{K}(h^\varepsilon) = \frac{w(|p|)\sqrt{\mathbf{M}}}{\sqrt{J_M}}K(\frac{\sqrt{J_M}}{w\sqrt{\mathbf{M}}}h^\varepsilon)$. Correspondingly, we can also define $\bar{K}_i$, $(i=1,2)$ and kernels $\bar{k}_i$ as follows:

$$
\bar{K}_i(f) = \int_{\mathbb{R}^3} dq\bar{k}_i(p,q)f(q) = \int_{\mathbb{R}^3} dq\frac{w(|p|)\sqrt{J_M(q)}\sqrt{\mathbf{M}(p)}}{w(|q|)\sqrt{J_M(p)}\sqrt{\mathbf{M}(q)}}k_i(p,q)f(q), \qquad i=1,2.
$$

Namely, $\bar{k}_i(p,q) = \frac{w(|p|)\sqrt{J_M(q)}\sqrt{\mathbf{M}(p)}}{w(|q|)\sqrt{J_M(p)}\sqrt{\mathbf{M}(q)}}k_i(p,q)$, $i=1,2$. Then $\bar{k}_i(p,q)$ also satisfy the same estimates in Lemma 8.1.

Corollary 8.1. *For $i=1,2,3$, it holds that*

$$
\bar{k}_2(p,q) \lesssim \frac{1}{p^0|p-q|}\exp\Big\{-\frac{c_0\sqrt{s}|p-q|}{T_0 g}\Big\}, \tag{8.6}
$$

for some small constant c_0, and

$$
\int_{\mathbb{R}^3} dq\bar{k}_2(p,q) \lesssim \frac{1}{p^0},
$$

$$\int_{\mathbb{R}^3} dq \partial_{x_i} \bar{k}_2(p,q) \lesssim \frac{1}{p^0},$$
$$\int_{\mathbb{R}^3} dq \partial_{p_i} \bar{k}_2(p,q) \lesssim \frac{1}{p^0}. \tag{8.7}$$

$\bar{k}_1(p,q)$ also satisfies the same estimates.

Proof. In fact, we only need to note that

$$\begin{aligned}
\bar{k}_2(p,q) \leq & \frac{(1+|p|)^\beta}{(1+|q|)^\beta} \frac{s^{\frac{3}{2}}}{gp^0q^0} U_1(p,q) \\
& \times \exp\Big\{ -[1+O(1)|u|] \frac{\sqrt{s}|p-q|}{2T_0 g} + \frac{(T_0-T_M)(p^0-q^0)}{2T_M T_0} \Big\} \\
\lesssim & \frac{s^{\frac{3}{2}}}{gp^0q^0} U_1(p,q)(1+|p-q|)^\beta \exp\Big\{ [(T_0-2T_M)+O(1)|u|] \frac{\sqrt{s}|p-q|}{2T_0T_M g} \Big\} \\
\lesssim & \frac{s^{\frac{3}{2}}}{gp^0q^0} U_1(p,q) \exp\Big\{ -c_0 \frac{\sqrt{s}|p-q|}{T_0T_M g} \Big\}
\end{aligned}$$

by (1.24). Then, similar to the proof of Lemma 8.1, we can obtain (8.6) and (8.7). $\square$

8.3. Appendix 3: Construction and estimates of coefficients. We will present the existence of coefficients F_n, $(1 \leq n \leq 2k-1)$, and their momentum and time regularities estimates. For $i \in [1, 2k-1]$, we decompose $\frac{F_n}{\sqrt{\mathbf{M}}}$ as the sum of macroscopic and microscopic parts:

$$\begin{aligned}
\frac{F_n}{\sqrt{\mathbf{M}}} &= \mathbf{P}\Big(\frac{F_n}{\sqrt{\mathbf{M}}}\Big) + \{\mathbf{I}-\mathbf{P}\}\Big(\frac{F_n}{\sqrt{\mathbf{M}}}\Big) \\
&= [a_n(t,x) + b_n(t,x)\cdot p + c_n(t,x)p^0]\sqrt{\mathbf{M}} + \{\mathbf{I}-\mathbf{P}\}\Big(\frac{F_n}{\sqrt{\mathbf{M}}}\Big).
\end{aligned}$$

To obtain the linear system satisfied by the abstract functions $a_n(t,x)$, $b_n(t,x)$, $c_n(t,x)$ and $E_n(t,x)$, $B_n(t,x)$, we first derive the explicit expression of the third momentum $T^{\alpha\beta\gamma}[\mathbf{M}]$, $(\alpha,\beta,\gamma \in \{0,1,2,3\})$:

$$T^{\alpha\beta\gamma}[\mathbf{M}] = \int_{\mathbb{R}^3} \frac{p^\alpha p^\beta p^\gamma}{p^0} \mathbf{M}\, dp.$$

We will first get the expression of $T^{\alpha\beta\gamma}[\mathbf{M}]$ in the rest frame where $(u^0,u^1,u^2,u^3) = (1,0,0,0)$. For convenience, we denote $T^{\alpha\beta\gamma}[\mathbf{M}]$ as $\bar{T}^{\alpha\beta\gamma}$ in the rest frame, p as $\bar{p}$ in the rest frame, and $-v$ as the velocity of general reference frame relative to the rest frame. Then, the corresponding boost matrix $\bar{\Lambda}$ is

$$\bar{\Lambda} = (\bar{\Lambda}^\mu_\nu) = \begin{pmatrix}
\tilde{r} & \tilde{r}v_1 & \tilde{r}v_2 & \tilde{r}v_3 \\
\tilde{r}v_1 & 1+(\tilde{r}-1)\frac{v_1^2}{|v|^2} & (\tilde{r}-1)\frac{v_1v_2}{|v|^2} & (\tilde{r}-1)\frac{v_1v_3}{|v|^2} \\
\tilde{r}v_2 & (\tilde{r}-1)\frac{v_1v_2}{|v|^2} & 1+(\tilde{r}-1)\frac{v_2^2}{|v|^2} & (\tilde{r}-1)\frac{v_2v_3}{|v|^2} \\
\tilde{r}v_3 & (\tilde{r}-1)\frac{v_1v_3}{|v|^2} & (\tilde{r}-1)\frac{v_2v_3}{|v|^2} & 1+(\tilde{r}-1)\frac{v_3^2}{|v|^2}
\end{pmatrix},$$

where $\tilde{r} = u^0$, $v_i = \frac{u_i}{u^0}$. Noting

$$p^\mu = \bar{\Lambda}^\mu_\nu \bar{p}^\nu, \qquad \frac{dp}{p^0} = \frac{d\bar{p}}{\bar{p}^0},$$

we have

$$T^{\alpha\beta\gamma}[\mathbf{M}] = \bar{\Lambda}^\alpha_{\alpha'} \bar{\Lambda}^\beta_{\beta'} \bar{\Lambda}^\gamma_{\gamma'} \bar{T}^{\alpha'\beta'\gamma'}. \tag{8.8}$$

Then, we can obtain the expression of $T^{\alpha\beta\gamma}[\mathbf{M}]$ by the expression of $\bar{T}^{\alpha\beta\gamma}$ and (8.8). Now we give the expression of $\bar{T}^{\alpha\beta\gamma}$ as follows:

Lemma 8.2. *Let $i, j, k \in \{1, 2, 3\}$. For the third momentum $\bar{T}^{\alpha\beta\gamma}$ which corresponds to $T^{\alpha\beta\gamma}[\mathbf{M}]$ in the rest frame, we have*

$$\bar{T}^{000} = \frac{n_0[3K_3(\gamma) + \gamma K_2(\gamma)]}{\gamma K_2(\gamma)}, \tag{8.9}$$

$$\bar{T}^{0ii} = \bar{T}^{ii0} = \bar{T}^{i0i} = \frac{n_0 K_3(\gamma)}{\gamma K_2(\gamma)}, \tag{8.10}$$

$$\bar{T}^{\alpha\beta\gamma} = 0, \qquad \textit{if } (\alpha, \beta, \gamma) \neq (0, 0, 0), (0, i, i), (i, i, 0), (i, 0, i). \tag{8.11}$$

Proof. It is straightforward to verify (8.11) by the symmetry of the variable of integration. Now we prove (8.9). It holds that

$$\bar{T}^{000} = \int_{\mathbb{R}^3} (p^0)^2 \frac{n_0\gamma}{4\pi K_2(\gamma)} \exp\{-\gamma \bar{p}_0\}\, dp.$$

As the proof of Proposition 3.3 in [33], we let $y = \gamma \bar{p}_0$ to get $\bar{p}^0 = \frac{y}{\gamma}$, $|\bar{p}| = \frac{\sqrt{y^2-\gamma^2}}{\gamma}$, and $d|\bar{p}| = \frac{1}{\gamma}\frac{ydy}{\sqrt{y^2-\gamma^2}}$. Then we have

$$\begin{aligned}
\bar{T}^{000} &= \int_\gamma^\infty \frac{1}{\gamma^2} y^2 \frac{n_0\gamma}{K_2(\gamma)} e^{-y} \frac{1}{\gamma^2}(y^2-\gamma^2)\frac{1}{\gamma}\frac{ydy}{\sqrt{y^2-\gamma^2}} \\
&= \frac{n_0}{\gamma^4 K_2(\gamma)} \int_\gamma^\infty [(y^2-\gamma^2)^{\frac{3}{2}} + \gamma^2\sqrt{y^2-\gamma^2}] y e^{-y} dy \\
&= \frac{n_0[3K_3(\gamma) + \gamma K_2(\gamma)]}{\gamma K_2(\gamma)}.
\end{aligned}$$

Similarly, (8.10) can be proved as follows:

$$\begin{aligned}
\bar{T}^{0ii} &= c\int_{\mathbb{R}^3} \frac{|p|^2}{3} \frac{n_0\gamma}{4\pi K_2(\gamma)} \exp\{-\gamma \bar{p}_0\}\, dp \\
&= \int_\gamma^\infty \frac{1}{3\gamma^2}(y^2-\gamma^2)\frac{n_0\gamma}{K_2(\gamma)} e^{-y} \frac{1}{\gamma^2}(y^2-\gamma^2)\frac{1}{\gamma}\frac{ydy}{\sqrt{y^2-\gamma^2}} \\
&= \frac{n_0}{3\gamma^4 K_2(\gamma)} \int_\gamma^\infty (y^2-\gamma^2)^{\frac{3}{2}} y e^{-y} dy \\
&= \frac{n_0 K_3(\gamma)}{\gamma K_2(\gamma)}.
\end{aligned}$$

$\square$

Now we can use Lemma 8.2 and (8.8) to derive the explicit expression of $T^{\alpha\beta\gamma}[\mathbf{M}]$.

Lemma 8.3. *For $i, j, k \in \{1, 2, 3\}$, we have*

$$\begin{aligned}
T^{000}[\mathbf{M}] &= \frac{n_0}{\gamma K_2(\gamma)}[(3K_3(\gamma) + \gamma K_2(\gamma))(u^0)^3 + 3K_3(\gamma)u^0|u|^2],\\
T^{00i}[\mathbf{M}] &= \frac{n_0}{\gamma K_2(\gamma)}\left[(5K_3(\gamma) + \gamma K_2(\gamma))(u^0)^2u_i + K_3(\gamma)|u|^2u_i\right],\\
T^{0ij}[\mathbf{M}] &= \frac{n_0}{\gamma K_2(\gamma)}\left[(6K_3(\gamma) + \gamma K_2(\gamma))u^0u_iu_j + \delta_{ij}K_3(\gamma)u^0\right],\\
T^{ijk}[\mathbf{M}] &= \frac{n_0}{\gamma K_2(\gamma)}(6K_3(\gamma) + \gamma K_2(\gamma))u_iu_ju_k\\
&\quad + \frac{n_0K_3(\gamma)}{\gamma K_2(\gamma)}(u_i\delta_{jk} + u_j\delta_{ik} + u_k\delta_{ij}).
\end{aligned} \tag{8.12}$$

Proof. We first prove $T^{000}[\mathbf{M}]$ in (8.12). By Lemma 8.2 and (8.8), we have

$$\begin{aligned}
T^{000}[\mathbf{M}] &= \bar{\Lambda}^0_\alpha\bar{\Lambda}^0_\beta\bar{\Lambda}^0_\gamma\bar{T}^{\alpha\beta\gamma}\\
&= \sum_{\alpha=0}\tilde{r}\left(\tilde{r}^2\frac{n_0[3K_3(\gamma)+\gamma K_2(\gamma)]}{\gamma K_2(\gamma)} + \tilde{r}^2|v|^2\frac{n_0K_3(\gamma)}{\gamma K_2(\gamma)}\right)\\
&\quad + \sum_{\alpha=1}^{3}\tilde{r}v_\alpha \times 2\tilde{r}^2v_\alpha\frac{n_0K_3(\gamma)}{\gamma K_2(\gamma)}\\
&= \frac{n_0}{\gamma K_2(\gamma)}\left([3K_3(\gamma)+\gamma K_2(\gamma)](u^0)^3 + K_3(\gamma)u^0|u|^2\right) + \frac{2n_0K_3(\gamma)}{\gamma K_2(\gamma)}u^0|u|^2\\
&= \frac{n_0}{\gamma K_2(\gamma)}[(3K_3(\gamma)+\gamma K_2(\gamma))(u^0)^3 + 3K_3(\gamma)u^0|u|^2].
\end{aligned}$$

For $T^{00i}[\mathbf{M}]$, we have

$$\begin{aligned}
T^{00i}[\mathbf{M}] &= \bar{\Lambda}^0_\alpha\bar{\Lambda}^0_\beta\bar{\Lambda}^i_\gamma\bar{T}^{\alpha\beta\gamma}\\
&= \sum_{\alpha=0}\tilde{r}\left(\tilde{r}^2v_i\frac{n_0[3K_3(\gamma)+\gamma K_2(\gamma)]}{\gamma K_2(\gamma)}\right.\\
&\quad \left. + \sum_{j=1}^{3}\tilde{r}v_j\left(\frac{\tilde{r}-1}{|v|^2}v_iv_j + \delta_{ij}\right)\frac{n_0K_3(\gamma)}{\gamma K_2(\gamma)}\right)\\
&\quad + \sum_{\alpha=1}^{3}\tilde{r}v_\alpha\left[\tilde{r}^2v_\alpha v_i + \tilde{r}\left(\frac{\tilde{r}-1}{|v|^2}v_iv_\alpha + \delta_{i\alpha}\right)\right]\frac{n_0K_3(\gamma)}{\gamma K_2(\gamma)}\\
&= \frac{n_0}{\gamma K_2(\gamma)}[4K_3(\gamma)+\gamma K_2(\gamma)](u^0)^2u_i + \frac{n_0K_3(\gamma)}{\gamma K_2(\gamma)}u_i[|u|^2 + (u^0)^2]\\
&= \frac{n_0}{\gamma K_2(\gamma)}\left[(5K_3(\gamma)+\gamma K_2(\gamma))(u^0)^2u_i + K_3(\gamma)|u|^2u_i\right].
\end{aligned}$$

Similarly, we obtain

$$
\begin{aligned}
T^{0ij}[\mathbf{M}] &= \bar{\Lambda}^0_\alpha \bar{\Lambda}^i_\beta \bar{\Lambda}^j_\gamma \bar{T}^{\alpha\beta\gamma} \\
&= \sum_{\alpha=0} \tilde{r}\Big(\tilde{r}^2 v_i v_j \frac{n_0[3K_3(\gamma)+\gamma K_2(\gamma)]}{\gamma K_2(\gamma)} \\
&\quad + \sum_{k=1}^{3}\Big(\frac{\tilde{r}-1}{|v|^2} v_i v_k + \delta_{ik}\Big)\Big(\frac{\tilde{r}-1}{|v|^2} v_j v_k + \delta_{jk}\Big)\frac{n_0 K_3(\gamma)}{\gamma K_2(\gamma)}\Big) \\
&\quad + \sum_{\alpha=1}^{3} \tilde{r} v_\alpha \Big[\tilde{r}\frac{v_i}{c}\Big(\frac{\tilde{r}-1}{|v|^2} v_j v_\alpha + \delta_{j\alpha}\Big) + \tilde{r} v_j \Big(\frac{\tilde{r}-1}{|v|^2} v_i v_\alpha + \delta_{i\alpha}\Big)\Big]\frac{n_0 K_3(\gamma)}{\gamma K_2(\gamma)} \\
&= \frac{n_0}{\gamma K_2(\gamma)}\{[4K_3(\gamma)+\gamma K_2(\gamma)]u^0 u_i u_j + \delta_{ij} K_3(\gamma) u^0\} + 2\frac{n_0 K_3(\gamma)}{\gamma K_2(\gamma)} u^0 u_i u_j \\
&= \frac{n_0}{\gamma K_2(\gamma)}\Big[(6K_3(\gamma)+\gamma K_2(\gamma))u^0 u_i u_j + \delta_{ij} K_3(\gamma) u^0\Big],
\end{aligned}
$$

and

$$
\begin{aligned}
T^{ijk}[\mathbf{M}] &= \bar{\Lambda}^i_\alpha \bar{\Lambda}^j_\beta \bar{\Lambda}^k_\gamma \bar{T}^{\alpha\beta\gamma} \\
&= \sum_{\alpha=0} \tilde{r} v_i \Big(\tilde{r}^2 v_j v_k \frac{n_0[3K_3(\gamma)+\gamma K_2(\gamma)]}{\gamma K_2(\gamma)} \\
&\quad + \sum_{l=1}^{3}\Big(\frac{\tilde{r}-1}{|v|^2} v_j v_l + \delta_{jl}\Big)\Big(\frac{\tilde{r}-1}{|v|^2} v_k v_l + \delta_{kl}\Big)\frac{n_0 K_3(\gamma)}{\gamma K_2(\gamma)}\Big) \\
&\quad + \sum_{\alpha=1}^{3}\Big(\frac{\tilde{r}-1}{|v|^2} v_i v_\alpha + \delta_{i\alpha}\Big)\Big[\tilde{r} v_j \Big(\frac{\tilde{r}-1}{|v|^2} v_k v_\alpha + \delta_{k\alpha}\Big) \\
&\quad + \tilde{r} v_k \Big(\frac{\tilde{r}-1}{|v|^2} v_j v_\alpha + \delta_{j\alpha}\Big)\Big]\frac{n_0 K_3(\gamma)}{\gamma K_2(\gamma)} \\
&= \frac{n_0}{\gamma K_2(\gamma)}\{[3K_3(\gamma)+\gamma K_2(\gamma)]u_i u_j u_k + u_i(u_j u_k + \delta_{jk})K_3(\gamma)\} \\
&\quad + \frac{n_0 K_3(\gamma)}{\gamma K_2(\gamma)}[u_j(u_i u_k + \delta_{ik}) + u_k(u_i u_j + \delta_{ij})] \\
&= \frac{n_0}{\gamma K_2(\gamma)}[6K_3(\gamma)+\gamma K_2(\gamma)]u_i u_j u_k \\
&\quad + \frac{n_0 K_3(\gamma)}{\gamma K_2(\gamma)}(u_i\delta_{jk} + u_j\delta_{ik} + u_k\delta_{ij}).
\end{aligned}
$$

$\square$

With the preparation, now we construct the coefficients (F_n, E_n, B_n), $1 \le n \le 2k-1$ in a conductive way, and estimate their regularities.

Theorem 8.1. *For any $n \in [0, 2k-2]$, assume that (F_i, E_i, B_i) have been constructed for all $0 \le i \le n$. Then the microscopic part $\{\mathbf{I}-\mathbf{P}\}\Big(\frac{F_{n+1}}{\sqrt{\mathbf{M}}}\Big)$ can be written as:*

$$\{\mathbf{I}-\mathbf{P}\}\Big(\frac{F_{n+1}}{\sqrt{\mathbf{M}}}\Big) = L^{-1}\Big[-\frac{1}{\sqrt{\mathbf{M}}}\Big(\partial_t F_n + \hat{p}\cdot\nabla_x F_n - \sum_{\substack{i+j=n+1\\ i,j\ge 1}} Q(F_i, F_j) \\ - \sum_{\substack{i+j=n\\ i,j\ge 0}} \Big(E_i + \hat{p}\times B_i\Big)\cdot\nabla_p F_j\Big)\Big].$$

And $a_{n+1}(t,x)$, $b_{n+1}(t,x)$, $c_{n+1}(t,x)$, $E_{n+1}(t,x)$, $B_{n+1}(t,x)$ satisfy the following system:

$$\begin{aligned}
&\partial_t\Big(n_0u^0a_{n+1} + (e_0+P_0)u^0(u\cdot b_{n+1}) + [e_0(u^0)^2 + P_0|u|^2]c_{n+1}\Big)\\
&\quad + \nabla_x\cdot\Big(n_0ua_{n+1} + (e_0+P_0)u(u\cdot b_{n+1}) + P_0b_{n+1} + (e_0+P_0)u^0uc_{n+1}\Big)\\
&\quad + \nabla_x\cdot\int_{\mathbb{R}^3}\frac{p}{p^0}\sqrt{\mathbf{M}}\{\mathbf{I}-\mathbf{P}\}\Big(\frac{F_{n+1}}{\sqrt{\mathbf{M}}}\Big)\,dp = 0,
\end{aligned}\tag{8.13}$$

$$\begin{aligned}
&\partial_t\Big((e_0+P_0)u^0u_ja_{n+1} + \frac{n_0}{\gamma K_2(\gamma)}[(6K_3(\gamma)+\gamma K_2(\gamma))u^0u_j(u\cdot b_{n+1})\\
&\quad + K_3(\gamma)u^0b_{n+1,j}] + \frac{n_0}{\gamma K_2(\gamma)}[(5K_3(\gamma)+\gamma K_2(\gamma))(u^0)^2 + K_3(\gamma)|u|^2]u_jc_{n+1}\Big)\\
&\quad + \nabla_x\cdot\Big((e_0+P_0)u_jua_{n+1} + \frac{n_0}{\gamma K_2(\gamma)}(6K_3(\gamma)+\gamma K_2(\gamma))u_ju[(u\cdot b_{n+1}) + u^0c_{n+1}]\Big)\\
&\quad + \partial_{x_j}(P_0a_{n+1}) + \nabla_x\cdot\Big[\frac{n_0K_3(\gamma)}{\gamma K_2(\gamma)}(ub_{n+1,j} + u_jb_{n+1})\Big]\\
&\quad + \partial_{x_j}\Big(\frac{n_0K_3(\gamma)}{\gamma K_2(\gamma)}[(u\cdot b_{n+1}) + u^0c_{n+1}]\Big)\\
&\quad + E_{0,j}\Big(n_0u^0a_{n+1} + (e_0+P_0)u^0(u\cdot b_{n+1}) + [e_0(u^0)^2 + P_0|u|^2]c_{n+1}\Big)\\
&\quad + \Big[\Big(n_0ua_{n+1} + (e_0+P_0)u(u\cdot b_{n+1}) + P_0b_{n+1} + (e_0+P_0)u^0uc_{n+1}\Big)\times B_0\Big]_j\\
&\quad + n_0u^0E_{n+1,j} + \Big(n_0u\times B_{n+1}\Big)_j\\
&\quad + \sum_{\substack{k+l=n+1\\ k,l\ge 1}} E_{k,j}\Big(n_0u^0a_l + (e_0+P_0)u^0(u\cdot b_l) + P_0b_l + [e_0(u^0)^2 + P_0|u|^2]c_l\Big)\\
&\quad + \sum_{\substack{k+l=n+1\\ k,l\ge 1}}\Big[\Big(n_0ua_l + (e_0+P_0)u(u\cdot b_l) + (e_0+P_0)u^0uc_l\Big)\times B_k\Big]_j\\
&\quad + \nabla_x\cdot\int_{\mathbb{R}^3}\frac{p_jp}{p^0}\sqrt{\mathbf{M}}\{\mathbf{I}-\mathbf{P}\}\Big(\frac{F_{n+1}}{\sqrt{\mathbf{M}}}\Big)\,dp + \Big[\int_{\mathbb{R}^3}\hat{p}\times B_0\sqrt{\mathbf{M}}\{\mathbf{I}-\mathbf{P}\}\Big(\frac{F_{n+1}}{\sqrt{\mathbf{M}}}\Big)\,dp\Big]_j\\
&\quad + \sum_{\substack{k+l=n+1\\ k,l\ge 1}}\Big[\int_{\mathbb{R}^3}\hat{p}\times B_k\sqrt{\mathbf{M}}\{\mathbf{I}-\mathbf{P}\}\Big(\frac{F_l}{\sqrt{\mathbf{M}}}\Big)\,dp\Big]_j = 0,
\end{aligned}\tag{8.14}$$

for $j = 1, 2, 3$ *with* $b_{n+1} = (b_{n+1,1}, b_{n+1,2}, b_{n+1,3})$, $E_{n+1} = (E_{n+1,1}, E_{n+1,2}, E_{n+1,3})$,

$$\begin{aligned}
&\partial_t\Big([e_0(u^0)^2 + P_0|u|^2]a_{n+1} + \frac{n_0}{\gamma K_2(\gamma)}[(5K_3(\gamma) + \gamma K_2(\gamma))(u^0)^2 + K_3(\gamma)|u|^2] \\
&\quad \times (u \cdot b_{n+1}) + \frac{n_0}{\gamma K_2(\gamma)}[(3K_3(\gamma) + \gamma K_2(\gamma))(u^0)^2 + 3K_3(\gamma)|u|^2]u^0 c_{n+1}\Big) \\
&\quad + \nabla_x \cdot \Big((e_0 + P_0)u^0 u a_{n+1} + \frac{n_0}{\gamma K_2(\gamma)}(6K_3(\gamma) + \gamma K_2(\gamma))u^0 u(u \cdot b_{n+1}) \\
&\quad + \frac{n_0 K_3(\gamma)}{\gamma K_2(\gamma)} u^0 b_{n+1} + \frac{n_0}{\gamma K_2(\gamma)}[(5K_3(\gamma) + \gamma K_2(\gamma))(u^0)^2 \\
&\quad + K_3(\gamma)|u|^2]u c_{n+1}\Big) + n_0 u \cdot E_{n+1} + n_0 u \cdot E_0 a_{n+1} \\
&\quad + (e_0 + P_0)(u \cdot b_{n+1})(u \cdot E_0) + P_0 E_0 \cdot b_{n+1} \\
&\quad + (e_0 + P_0)u^0(u \cdot E_0)c_{n+1} + \int_{\mathbb{R}^3} \hat{p}\sqrt{\mathbf{M}}\{\mathbf{I} - \mathbf{P}\}\Big(\frac{F_{n+1}}{\sqrt{\mathbf{M}}}\Big)\, dp \cdot E_0 \\
&\quad + \sum_{\substack{k+l=n+1 \\ k,l \geq 1}} \Big[n_0 u \cdot E_k a_l + (e_0 + P_0)(u \cdot b_l)(u \cdot E_k) + P_0 E_k \cdot b_l \\
&\quad + (e_0 + P_0)u^0(u \cdot E_k)c_l + \int_{\mathbb{R}^3} \hat{p}\sqrt{\mathbf{M}}\{\mathbf{I} - \mathbf{P}\}\Big(\frac{F_l}{\sqrt{\mathbf{M}}}\Big)\, dp \cdot E_k\Big] = 0,
\end{aligned} \tag{8.15}$$

$$\begin{aligned}
&\partial_t E_{n+1} - \nabla_x \times B_{n+1} \\
&\quad = n_0 u a_{n+1} + (e_0 + P_0)u(u \cdot b_{n+1}) \\
&\quad + (e_0 + P_0)u^0 u c_{n+1} + \int_{\mathbb{R}^3} \hat{p}\sqrt{\mathbf{M}}\{\mathbf{I} - \mathbf{P}\}\Big(\frac{F_{n+1}}{\sqrt{\mathbf{M}}}\Big)\, dp, \\
&\quad \partial_t B_{n+1} + \nabla_x \times E_{n+1} = 0, \\
&\quad \nabla_x \cdot E_{n+1} = -\Big(n_0 u^0 a_{n+1} + (e_0 + P_0)u^0(u \cdot b_{n+1}) \\
&\quad + [e_0(u^0)^2 + P_0|u|^2]c_{n+1}\Big), \\
&\nabla_x \cdot B_{n+1} = 0.
\end{aligned} \tag{8.16}$$

Furthermore, assume $a_{n+1}(0, x)$, $b_{n+1}(0, x)$, $c_{n+1}(0, x)$, $E_{n+1}(0, x)$, $B_{n+1}(0, x) \in H^N$, $N \geq 0$ *be given initial data to the system consisted of equations* (8.13), (8.14), (8.15) *and* (8.16)*. Then the linear system is well-posed in* $C^0([0, \infty); H^N)$*. Moreover, it holds that*

$$\begin{aligned}
&|F_{n+1}| \lesssim (1+t)^n \mathbf{M}^{1-}, \qquad |\nabla_p F_{n+1}| \lesssim (1+t)^n \mathbf{M}^{1-}, \\
&|\nabla_x F_{n+1}| \lesssim (1+t)^n \mathbf{M}^{1-}, \qquad |\nabla_p^2 F_{n+1}| \lesssim (1+t)^n \mathbf{M}^{1-}, \\
&|\nabla_x \nabla_p F_{n+1}| \lesssim (1+t)^n \mathbf{M}^{1-}, \\
&|E_{n+1}| + |B_{n+1}| + |\nabla_x E_{n+1}| + |\nabla_x E_{n+1}| \lesssim (1+t)^n.
\end{aligned} \tag{8.17}$$

Proof. From the equation of F_n in (1.10), the microscopic part $\{\mathbf{I}-\mathbf{P}\}\Big(\frac{F_{n+1}}{\sqrt{\mathbf{M}}}\Big)$ can be written as

$$
\begin{aligned}
\{\mathbf{I}-\mathbf{P}\}\Big(\frac{F_{n+1}}{\sqrt{\mathbf{M}}}\Big) = L^{-1}\Big[-\frac{1}{\sqrt{\mathbf{M}}}\Big(\partial_t F_n + \hat{p}\cdot\nabla_x F_n - \sum_{\substack{i+j=n+1\\ i,j\geq 1}} Q(F_i,F_j) \\
- \sum_{\substack{i+j=n\\ i,j\geq 0}} \Big(E_i + \hat{p}\times B_i\Big)\cdot\nabla_p F_j\Big)\Big].
\end{aligned}
$$

We first prove the equation (8.13). Note that, from (1.14),

$$
\begin{aligned}
\int_{\mathbb{R}^3} F_{n+1}dp &= \int_{\mathbb{R}^3}[a_{n+1} + b_{n+1}\cdot p + c_{n+1}p_0]\mathbf{M}dp \\
&= n_0u^0a_{n+1} + (e_0+P_0)u^0(u\cdot b_{n+1}) + [e_0(u^0)^2 + P_0|u|^2]c_{n+1}, \\
\int_{\mathbb{R}^3}\hat{p}_jF_{n+1}\,dp &= n_0u_ja_{n+1} + (e_0+P_0)u_j(u\cdot b_{n+1}) + P_0b_{n+1,j} \\
&\quad + (e_0+P_0)u^0u_jc_{n+1} + \int_{\mathbb{R}^3}\hat{p}_j\sqrt{\mathbf{M}}\{\mathbf{I}-\mathbf{P}\}\Big(\frac{F_{n+1}}{\sqrt{\mathbf{M}}}\Big)\,dp, \qquad (8.18)
\end{aligned}
$$

for $j=1,2,3$, and

$$
\int_{\mathbb{R}^3}\Big(E_{n+1}+\hat{p}\times B_{n+1}\Big)\cdot\nabla_pF_0dp = \int_{\mathbb{R}^3}\Big(E_0+\hat{p}\times B_0\Big)\cdot\nabla_pF_{n+1}dp = 0.
$$

Then, we integrate the equation of F_{n+1} in (1.10) w.r.t. p to get (8.13).

Now we prove (8.14). From (1.14) and Lemma 8.3, it holds that

$$
\begin{aligned}
&\int_{\mathbb{R}^3}p_jF_{n+1}dp \\
&= \int_{\mathbb{R}^3}p_j[a_{n+1}+b_{n+1}\cdot p + c_{n+1}p_0]\mathbf{M}dp \\
&= (e_0+P_0)u^0u_ja_{n+1} + \frac{n_0}{\gamma K_2(\gamma)}[(6K_3(\gamma)+\gamma K_2(\gamma))u^0u_j(u\cdot b_{n+1}) \\
&\quad + K_3(\gamma)u^0b_{n+1,j}] + \frac{n_0}{\gamma K_2(\gamma)}[(5K_3(\gamma)+\gamma K_2(\gamma))(u^0)^2 + K_3(\gamma)|u|^2]u_jc_{n+1}, \\
&\int_{\mathbb{R}^3}\frac{p_jp}{p^0}F_{n+1}dp \\
&= \int_{\mathbb{R}^3}\frac{p_jp}{p^0}[a_{n+1}+b_{n+1}\cdot p + c_{n+1}p_0]\mathbf{M}dp + \int_{\mathbb{R}^3}\frac{p_jp}{p^0}\sqrt{\mathbf{M}}\{\mathbf{I}-\mathbf{P}\}\Big(\frac{F_{n+1}}{\sqrt{\mathbf{M}}}\Big)\,dp \\
&= (e_0+P_0)u_jua_{n+1} + \frac{n_0}{\gamma K_2(\gamma)}(6K_3(\gamma)+\gamma K_2(\gamma))u_ju[(u\cdot b_{n+1}) + u^0c_{n+1}] \\
&\quad + e_jP_0a_{n+1} + \frac{n_0K_3(\gamma)}{\gamma K_2(\gamma)}(ub_{n+1,j} + u_jb_{n+1}) \\
&\quad + e_j\frac{n_0K_3(\gamma)}{\gamma K_2(\gamma)}[(u\cdot b_{n+1}) + u^0c_{n+1}] + \int_{\mathbb{R}^3}\frac{p_jp}{p^0}\sqrt{\mathbf{M}}\{\mathbf{I}-\mathbf{P}\}\Big(\frac{F_{n+1}}{\sqrt{\mathbf{M}}}\Big)\,dp,
\end{aligned}
$$

where e_j, $(j = 1, 2, 3)$, are the unit base vectors in $\mathbb{R}^3$, and

$$
\begin{aligned}
&-\int_{\mathbb{R}^3} p_j\Big(E_{n+1} + \hat{p} \times B_{n+1}\Big) \cdot \nabla_p F_0 dp \\
&= \int_{\mathbb{R}^3} E_{n+1,j} F_0 dp + \int_{\mathbb{R}^3} \Big(\hat{p} \times B_{n+1}\Big)_j F_0 dp \\
&= n_0 u^0 E_{n+1,j} + \Big(n_0 u \times B_{n+1}\Big)_j, \\
&-\int_{\mathbb{R}^3} p_j\Big(E_0 + \hat{p} \times B_0\Big) \cdot \nabla_p F_{n+1} dp \\
&= \int_{\mathbb{R}^3} E_{0,j} F_{n+1} dp + \int_{\mathbb{R}^3} \Big(\hat{p} \times B_0\Big)_j F_{n+1} dp \\
&= E_{0,j}\Big(n_0 u^0 a_{n+1} + (e_0 + P_0)u^0(u \cdot b_{n+1}) + [e_0(u^0)^2 + P_0|u|^2]c_{n+1}\Big) \\
&\quad + \Big[\Big(n_0 u a_{n+1} + (e_0 + P_0)u(u \cdot b_{n+1}) + P_0 b_{n+1} + (e_0 + P_0)u^0 u c_{n+1}\Big) \times B_0\Big]_j \\
&\quad + \int_{\mathbb{R}^3} \Big(\hat{p} \times B_0\Big)_j \sqrt{\mathbf{M}}\{\mathbf{I} - \mathbf{P}\}\Big(\frac{F_{n+1}}{\sqrt{\mathbf{M}}}\Big) dp.
\end{aligned}
$$

Then, we multiply the equation of F_{n+1} in (1.10) by p_j and integrate the resulting equation w.r.t. p to get (8.14).

Next, we show that (8.15) holds. By (1.14) and Lemma 8.3, one has

$$
\begin{aligned}
&\int_{\mathbb{R}^3} p_0 F_{n+1} dp \\
&= \int_{\mathbb{R}^3} p_0[a_{n+1} + b_{n+1} \cdot p + c_{n+1} p_0]\mathbf{M} dp \\
&= [e_0(u^0)^2 + P_0|u|^2]a_{n+1} + \frac{n_0}{\gamma K_2(\gamma)}[(5K_3(\gamma) + \gamma K_2(\gamma))(u^0)^2 + K_3(\gamma)|u|^2] \\
&\quad \times (u \cdot b_{n+1}) + \frac{n_0}{\gamma K_2(\gamma)}[(3K_3(\gamma) + \gamma K_2(\gamma))(u^0)^2 + 3K_3(\gamma)|u|^2]u^0 c_{n+1},
\end{aligned}
$$

and

$$
\begin{aligned}
&-\int_{\mathbb{R}^3} p^0\Big(E_{n+1} + \hat{p} \times B_{n+1}\Big) \cdot \nabla_p F_0 dp \\
&= \int_{\mathbb{R}^3} E_{n+1} \cdot \hat{p} F_0 dp = n_0 u \cdot E_{n+1}, \\
&-\int_{\mathbb{R}^3} p^0\Big(E_0 + \hat{p} \times B_0\Big) \cdot \nabla_p F_{n+1} dp \\
&= \int_{\mathbb{R}^3} E_0 \cdot \hat{p} F_{n+1} dp \\
&= n_0 u \cdot E_0 a_{n+1} + (e_0 + P_0)(u \cdot b_{n+1})(u \cdot E_0) + P_0 E_0 \cdot b_{n+1} \\
&\quad + (e_0 + P_0)u^0(u \cdot E_0)c_{n+1} + \int_{\mathbb{R}^3} \hat{p}\sqrt{\mathbf{M}}\{\mathbf{I} - \mathbf{P}\}\Big(\frac{F_{n+1}}{\sqrt{\mathbf{M}}}\Big) dp \cdot E_0.
\end{aligned}
$$

We integrate the equation of F_n in (1.10) with p^0 over $\mathbb{R}^3_p$ to obtain (8.15). Finally, it is straightforward to obtain the Maxwell system (8.16) of E_{n+1}, B_{n+1} from (8.18).

Now we prove the well-posedness of the system (8.13), (8.14), (8.15) and (8.16). By conditions (1.16) and the equation $\partial_t(n_0u^0) + \nabla_x \cdot (n_0u) = 0$ in (1.17), we simplify equations (8.13), (8.14), (8.15) as follows:

$$
\begin{aligned}
&n_0u^0\partial_t\Big(a_{n+1} + h(u\cdot b_{n+1}) + hu^0c_{n+1}\Big) - \partial_t(P_0c_{n+1})\\
&\quad + n_0u\cdot\nabla_x\Big(a_{n+1} + h(u\cdot b_{n+1}) + hu^0c_{n+1}\Big) + \nabla_x\cdot(P_0b_{n+1})\\
&\quad + \nabla_x\cdot\int_{\mathbb{R}^3}\hat{p}\sqrt{\mathbf{M}}\{\mathbf{I}-\mathbf{P}\}\Big(\frac{F_{n+1}}{\sqrt{\mathbf{M}}}\Big)\,dp = 0, \qquad (8.19)
\end{aligned}
$$

$$
\begin{aligned}
&n_0u^0\partial_t\Big(hu_ja_{n+1} + \frac{1}{\gamma K_2(\gamma)}[(6K_3(\gamma)+\gamma K_2(\gamma))u_j(u\cdot b_{n+1} + u^0c_{n+1})\\
&\quad + K_3(\gamma)b_{n+1,j}]\Big) - \partial_t\Big(\frac{n_0K_3(\gamma)}{\gamma K_2(\gamma)}u_jc_{n+1}\Big)\\
&\quad + n_0u\cdot\nabla_x\Big(hu_ja_{n+1} + \frac{1}{\gamma K_2(\gamma)}[(6K_3(\gamma)+\gamma K_2(\gamma))u_j[(u\cdot b_{n+1}) + u^0c_{n+1}]\\
&\quad + K_3(\gamma)b_{n+1,j}]\Big) + \partial_{x_j}(P_0a_{n+1}) + \nabla_x\cdot\Big[\frac{n_0K_3(\gamma)}{\gamma K_2(\gamma)}u_jb_{n+1}\Big]\\
&\quad + \partial_{x_j}\Big(\frac{n_0K_3(\gamma)}{\gamma K_2(\gamma)}[(u\cdot b_{n+1}) + u^0c_{n+1}]\Big)\\
&\quad + E_{0,j}\Big(n_0u^0a_{n+1} + (e_0+P_0)u^0(u\cdot b_{n+1}) + [e_0(u^0)^2 + P_0|u|^2]c_{n+1}\Big)\\
&\quad + \Big[\Big(n_0ua_{n+1} + (e_0+P_0)u(u\cdot b_{n+1}) + P_0b_{n+1} + (e_0+P_0)u^0uc_{n+1}\Big)\times B_0\Big]_j\\
&\quad + n_0u^0E_{n+1,j} + \Big(n_0u^0\times B_{n+1}\Big)_j\\
&\quad + \sum_{\substack{k+l=n+1\\k,l\ge1}}E_{k,j}\Big(n_0u^0a_l + (e_0+P_0)u^0(u\cdot b_l) + [e_0(u^0)^2 + P_0|u|^2]c_l\Big)\\
&\quad + \sum_{\substack{k+l=n+1\\k,l\ge1}}\Big[\Big(n_0ua_l + (e_0+P_0)u(u\cdot b_l) + P_0b_l + (e_0+P_0)u^0uc_l\Big)\times B_k\Big]_j\\
&\quad + \nabla_x\cdot\int_{\mathbb{R}^3}\frac{p_jp}{p^0}\sqrt{\mathbf{M}}\{\mathbf{I}-\mathbf{P}\}\Big(\frac{F_{n+1}}{\sqrt{\mathbf{M}}}\Big)\,dp + \Big[\int_{\mathbb{R}^3}\hat{p}\times B_0\sqrt{\mathbf{M}}\{\mathbf{I}-\mathbf{P}\}\Big(\frac{F_{n+1}}{\sqrt{\mathbf{M}}}\Big)\,dp\Big]_j\\
&\quad + \sum_{\substack{k+l=n+1\\k,l\ge1}}\Big[\int_{\mathbb{R}^3}\hat{p}\times B_k\sqrt{\mathbf{M}}\{\mathbf{I}-\mathbf{P}\}\Big(\frac{F_l}{\sqrt{\mathbf{M}}}\Big)\,dp\Big]_j = 0, \qquad (8.20)
\end{aligned}
$$

and

$$
\begin{aligned}
&n_0u^0\partial_t\Big(hu^0a_{n+1} + \frac{1}{\gamma K_2(\gamma)}[(6K_3(\gamma)+\gamma K_2(\gamma))u^0](u\cdot b_{n+1})\\
&\quad + \frac{1}{\gamma K_2(\gamma)}[(3K_3(\gamma)+\gamma K_2(\gamma))(u^0)^2 + 3K_3(\gamma)|u|^2]c_{n+1}\Big)\\
&\quad - \partial_t(P_0a_{n+1}) - \partial_t\Big(\frac{n_0K_3(\gamma)}{\gamma K_2(\gamma)}(u\cdot b_{n+1})\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ n_0 u \cdot \nabla_x \Big(h u^0 a_{n+1} + \frac{1}{\gamma K_2(\gamma)} (6K_3(\gamma) + \gamma K_2(\gamma)) u^0 (u \cdot b_{n+1}) \\
&+ \frac{1}{\gamma K_2(\gamma)} [(3K_3(\gamma) + \gamma K_2(\gamma))(u^0)^2 + 3K_3(\gamma)|u|^2] c_{n+1} \Big) \\
&+ \nabla_x \cdot \Big(\frac{n_0 K_3(\gamma)}{\gamma K_2(\gamma)} u^0 b_{n+1} \Big) + \nabla_x \cdot \Big(\frac{2 n_0 K_3(\gamma)}{\gamma K_2(\gamma)} u c_{n+1} \Big) \\
&+ n_0 u \cdot E_{n+1} + n_0 u \cdot E_0 a_{n+1} + (e_0 + P_0)(u \cdot b_{n+1})(u \cdot E_0) \\
&+ P_0 E_0 \cdot b_{n+1} + (e_0 + P_0) u^0 (u \cdot E_0) c_{n+1} \\
&+ \int_{\mathbb{R}^3} \hat{p} \sqrt{\mathbf{M}} \{\mathbf{I} - \mathbf{P}\} \Big(\frac{F_{n+1}}{\sqrt{\mathbf{M}}} \Big) \, dp \cdot E_0 \\
&+ \sum_{\substack{k+l=n+1 \\ k,l \geq 1}} \Big[n_0 u \cdot E_k a_l + (e_0 + P_0)(u \cdot b_l)(u \cdot E_k) + P_0 E_k \cdot b_l \\
&+ (e_0 + P_0) u^0 (u \cdot E_k) c_l + \int_{\mathbb{R}^3} \hat{p} \sqrt{\mathbf{M}} \{\mathbf{I} - \mathbf{P}\} \Big(\frac{F_l}{\sqrt{\mathbf{M}}} \Big) \, dp \cdot E_k \Big] = 0. \qquad (8.21)
\end{aligned}
$$

Equations (8.19), (8.20), (8.21) can be further written as:

$$
\begin{aligned}
& n_0 u^0 \Big(\partial_t a_{n+1} + h(u \cdot \partial_t b_{n+1}) + h u^0 \partial_t c_{n+1} \Big) - P_0 \partial_t c_{n+1} \\
&+ n_0 u^0 \Big(\partial_t \Big(h u \Big) \cdot b_{n+1}) + \partial_t (h u^0) c_{n+1} \Big) - (\partial_t P_0) c_{n+1} \\
&+ n_0 u \cdot \Big(\nabla_x a_{n+1} + h \nabla_x b_{n+1} \cdot u + h u^0 \nabla_x c_{n+1} \Big) + P_0 \nabla_x \cdot b_{n+1} \\
&+ n_0 u \cdot \Big(\nabla_x \Big(h u \Big) \cdot b_{n+1}) + \nabla_x \Big(h u^0 \Big) c_{n+1} \Big) + b_{n+1} \cdot \nabla_x P_0 \\
&+ \nabla_x \cdot \int_{\mathbb{R}^3} \hat{p} \sqrt{\mathbf{M}} \{\mathbf{I} - \mathbf{P}\} \Big(\frac{F_{n+1}}{\sqrt{\mathbf{M}}} \Big) \, dp = 0, \qquad (8.22)
\end{aligned}
$$

$$
\begin{aligned}
& n_0 u^0 \Big(h u_j \partial_t a_{n+1} + \frac{1}{\gamma K_2(\gamma)} [(6K_3(\gamma) + \gamma K_2(\gamma)) u_j (u \cdot \partial_t b_{n+1} + u^0 \partial_t c_{n+1}) \\
&+ K_3(\gamma) \partial_t b_{n+1,j}] \Big) - \frac{n_0 K_3(\gamma)}{\gamma K_2(\gamma)} u_j \partial_t c_{n+1} - \partial_t \Big(\frac{n_0 K_3(\gamma)}{\gamma K_2(\gamma)} u_j \Big) c_{n+1} \\
&+ n_0 u^0 \Big[\partial_t \Big(h u_j \Big) a_{n+1} + \partial_t \Big(\frac{m^2 (6K_3(\gamma) + \gamma K_2(\gamma))}{\gamma K_2(\gamma)} u_j u \Big) \cdot b_{n+1} \\
&+ \partial_t \Big(\frac{1}{\gamma K_2(\gamma)} (6K_3(\gamma) + \gamma K_2(\gamma)) u_j u^0 \Big) c_{n+1} + \partial_t \Big(\frac{K_3(\gamma)}{\gamma K_2(\gamma)} \Big) b_{n+1,j} \Big] \\
&+ n_0 u \cdot \Big(h u_j \nabla_x a_{n+1} + \frac{1}{\gamma K_2(\gamma)} (6K_3(\gamma) + \gamma K_2(\gamma)) u_j [(\nabla_x b_{n+1} \cdot u) + u^0 \nabla_x c_{n+1}] \\
&+ K_3(\gamma) \nabla_x b_{n+1,j}] \Big) + P_0 \partial_{x_j} a_{n+1} + \partial_{x_j} P_0 a_{n+1} \\
&+ n_0 u \cdot \Big[\nabla_x \Big(h u_j \Big) a_{n+1} + \nabla_x \Big(\frac{1}{\gamma K_2(\gamma)} (6K_3(\gamma) + \gamma K_2(\gamma)) u_j u \Big) \cdot b_{n+1} \\
&+ \nabla_x \Big(\frac{1}{\gamma K_2(\gamma)} (6K_3(\gamma) + \gamma K_2(\gamma)) u_j u^0 \Big) c_{n+1}
\end{aligned}
$$

$$
\begin{aligned}
&+ \nabla_x\Big(\frac{K_3(\gamma)}{\gamma K_2(\gamma)}\Big)b_{n+1,j}\Big] + \frac{n_0K_3(\gamma)}{\gamma K_2(\gamma)}u_j\nabla_x\cdot b_{n+1}\\
&+ \frac{n_0K_3(\gamma)}{\gamma K_2(\gamma)}[(u\cdot\partial_{x_j}b_{n+1}) + u^0\partial_{x_j}c_{n+1}]\\
&+ \nabla_x\cdot\Big(\frac{n_0K_3(\gamma)}{\gamma K_2(\gamma)}u\Big)b_{n+1,j} + \nabla_x\Big(\frac{n_0K_3(\gamma)}{\gamma K_2(\gamma)}u_j\Big)\cdot b_{n+1}\\
&+ \partial_{x_j}\Big(\frac{n_0K_3(\gamma)}{\gamma K_2(\gamma)}u\Big)\cdot b_{n+1} + \partial_{x_j}\Big(\frac{n_0K_3(\gamma)}{\gamma K_2(\gamma)}u^0\Big)c_{n+1}\\
&+ E_{0,j}\Big(n_0u^0a_{n+1} + (e_0+P_0)u^0(u\cdot b_{n+1}) + [e_0(u^0)^2 + P_0|u|^2]c_{n+1}\Big)\\
&+ \Big[\Big(n_0ua_{n+1} + (e_0+P_0)u(u\cdot b_{n+1}) + (e_0+P_0)u^0uc_{n+1}\Big)\times B_0\Big]_j\\
&+ n_0u^0E_{n+1,j} + \Big(n_0u^0\times B_{n+1}\Big)_j\\
&+ \sum_{\substack{k+l=n+1\\k,l\geq 1}} E_{k,j}\Big(n_0u^0a_l + (e_0+P_0)u^0(u\cdot b_l) + [e_0(u^0)^2 + P_0|u|^2]c_l\Big)\\
&+ \sum_{\substack{k+l=n+1\\k,l\geq 1}} \Big[\Big(n_0ua_l + (e_0+P_0)u(u\cdot b_l) + (e_0+P_0)u^0uc_l\Big)\times B_k\Big]_j\\
&+ \nabla_x\cdot\int_{\mathbb{R}^3}\frac{p_jp}{p^0}\sqrt{\mathbf{M}}\{\mathbf{I}-\mathbf{P}\}\Big(\frac{F_{n+1}}{\sqrt{\mathbf{M}}}\Big)\,dp\\
&+ \Big[\int_{\mathbb{R}^3}\hat{p}\times B_0\sqrt{\mathbf{M}}\{\mathbf{I}-\mathbf{P}\}\Big(\frac{F_{n+1}}{\sqrt{\mathbf{M}}}\Big)\,dp\Big]_j\\
&+ \sum_{\substack{k+l=n+1\\k,l\geq 1}}\Big[\int_{\mathbb{R}^3}\hat{p}\times B_k\sqrt{\mathbf{M}}\{\mathbf{I}-\mathbf{P}\}\Big(\frac{F_l}{\sqrt{\mathbf{M}}}\Big)\,dp\Big]_j = 0, \qquad (8.23)
\end{aligned}
$$

and

$$
\begin{aligned}
&n_0u^0\Big(hu^0\partial_t a_{n+1} + \frac{1}{\gamma K_2(\gamma)}[(6K_3(\gamma)+\gamma K_2(\gamma))u^0](u\cdot\partial_t b_{n+1})\\
&\quad + \frac{1}{\gamma K_2(\gamma)}[(3K_3(\gamma)+\gamma K_2(\gamma))(u^0)^2 + 3K_3(\gamma)|u|^2]\partial_t c_{n+1}\Big)\\
&\quad + n_0u^0\Big[\partial_t(hu^0)a_{n+1} + \partial_t\Big(\frac{1}{\gamma K_2(\gamma)}[(6K_3(\gamma)+\gamma K_2(\gamma))u^0]u\Big)\cdot b_{n+1}\\
&\quad + \partial_t\Big(\frac{1}{\gamma K_2(\gamma)}[(3K_3(\gamma)+\gamma K_2(\gamma))(u^0)^2 + 3K_3(\gamma)|u|^2]\Big)c_{n+1}\Big]\\
&\quad - P_0\partial_t a_{n+1} - \frac{n_0K_3(\gamma)}{\gamma K_2(\gamma)}(u\cdot\partial_t b_{n+1})\\
&\quad - \partial_t P_0 a_{n+1} - \partial_t\Big(\frac{n_0K_3(\gamma)}{\gamma K_2(\gamma)}u\Big)\cdot b_{n+1}\\
&\quad + n_0u\cdot\Big(hu^0\nabla_x a_{n+1} + \frac{1}{\gamma K_2(\gamma)}(6K_3(\gamma)+\gamma K_2(\gamma))u^0(\nabla_x b_{n+1}\cdot u)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\gamma K_2(\gamma)}[(3K_3(\gamma)+\gamma K_2(\gamma))(u^0)^2+3K_3(\gamma)|u|^2]\nabla_x c_{n+1}\Big)\\
&+n_0 u\cdot\Big[\nabla_x\Big(hu^0\Big)a_{n+1}+\nabla_x\Big(\frac{1}{\gamma K_2(\gamma)}(6K_3(\gamma)+\gamma K_2(\gamma))u^0 u\Big)\cdot b_{n+1}\\
&+\nabla_x\Big(\frac{1}{\gamma K_2(\gamma)}[(3K_3(\gamma)+\gamma K_2(\gamma))(u^0)^2+3K_3(\gamma)|u|^2]\Big)c_{n+1}\Big]\\
&+\Big(\frac{n_0K_3(\gamma)}{\gamma K_2(\gamma)}u^0\Big)\nabla_x\cdot b_{n+1}+\Big(\frac{2n_0K_3(\gamma)}{\gamma K_2(\gamma)}u\Big)\cdot\nabla_x c_{n+1}\\
&+\nabla_x\Big(\frac{n_0K_3(\gamma)}{\gamma K_2(\gamma)}u^0\Big)\cdot b_{n+1}+\nabla_x\cdot\Big(\frac{2n_0K_3(\gamma)}{\gamma K_2(\gamma)}u\Big)c_{n+1}+n_0u\cdot E_{n+1}\\
&+n_0u\cdot E_0a_{n+1}+(e_0+P_0)(u\cdot b_{n+1})(u\cdot E_0)+P_0E_0\cdot b_{n+1}\\
&+(e_0+P_0)u^0(u\cdot E_0)c_{n+1}+\int_{\mathbb{R}^3}\hat{p}\sqrt{\mathbf{M}}\{\mathbf{I}-\mathbf{P}\}\Big(\frac{F_{n+1}}{\sqrt{\mathbf{M}}}\Big)dp\cdot E_0\\
&+\sum_{\substack{k+l=n+1\\k,l\geq 1}}\Big[n_0u\cdot E_ka_l+(e_0+P_0)(u\cdot b_l)(u\cdot E_k)+P_0E_k\cdot b_l\\
&+(e_0+P_0)u^0(u\cdot E_k)c_l+\int_{\mathbb{R}^3}\hat{p}\sqrt{\mathbf{M}}\{\mathbf{I}-\mathbf{P}\}\Big(\frac{F_l}{\sqrt{\mathbf{M}}}\Big)dp\cdot E_k\Big]=0.
\end{aligned}\tag{8.24}
$$

Now we can write the equations (8.22), (8.3), (8.24) as a linear symmetric hyperbolic system:

$$
\mathbf{A}_0\partial_t U+\sum_{i=1}^{3}\mathbf{A}_i\partial_i U+\mathbf{B}_1U+\mathbf{B}_2\bar{U}=S,\tag{8.25}
$$

where U and $\bar{U}$ are

$$
U=\begin{pmatrix}a_{n+1}\\ b_{n+1}\\ c_{n+1}\end{pmatrix},\qquad \bar{U}=\begin{pmatrix}E_{n+1}\\ B_{n+1}\end{pmatrix}.
$$

For simplicity, denote

$$
h_1=h_1(t,x)\equiv\frac{n_0}{\gamma K_2(\gamma)}(6K_3(\gamma)+\gamma K_2(\gamma)),\quad h_2=h_2(t,x)\equiv\frac{n_0K_3(\gamma)}{\gamma K_2(\gamma)}.
$$

5×5 Matrixes $\mathbf{A}_0$, $\mathbf{A}_i$, $(i=1,2,3)$ in (8.25) are

$$
\mathbf{A}_0=\begin{pmatrix}n_0u^0 & n_0u^0hu^t & [e_0(u^0)^2+P_0|u|^2]\\ n_0u^0hu & (h_1u\otimes u+h_2\mathbf{I})u^0 & (h_1(u^0)^2-h_2)u\\ [e_0(u^0)^2+P_0|u|^2] & (h_1(u^0)^2-h_2)u^t & (h_1(u^0)^2-3h_2)u^0\end{pmatrix},
$$

and

$$
\mathbf{A}_i=\begin{pmatrix}n_0u_i & n_0hu_iu^t+P_0e_i^t & n_0hu^0u_i\\ n_0hu_iu+P_0e_i & h_1u_iu\otimes u+h_2(u_i\mathbf{I}+\tilde{\mathbf{A}}_i) & (h_1u_iu+h_2e_i)u^0\\ n_0hu^0u_i & (h_1u_iu^t+h_2e_i^t)u^0 & (h_1(u^0)^2-h_2)u_i\end{pmatrix},
$$

where $(\cdot)^t$ denotes the transpose of a vector in $\mathbb{R}^3$, $\mathbf{I}$ is the 3×3 identity matrix and $\tilde{\mathbf{A}}_i$ is a 3×3 matrix with its components

$$(\tilde{\mathbf{A}}_i)_{jk} = \delta_{ij}u_k + \delta_{ik}u_j, \qquad 1 \le j, k \le 3.$$

The components of matrixes $\mathbf{B}_1$ and $\mathbf{B}_2$, and the remainder terms S can be written explicitly as functions of F_i, E_i, B_i, $(0 \le i \le n)$ and their first order derivatives. For the matrix $\mathbf{A}_0$, its determinant is

$$\begin{aligned}
&n_0^5 \begin{vmatrix} u_0 & u^0 u^t h & h(u^0)^2 - \frac{P_0}{n_0} \\ \mathbf{0} & u^0[h_3 u\otimes u + h_4\mathbf{I}] & (u^0)^2 h_3 u \\ 0 & (u^0)^2 h_3 u^t & (u^0)^3 h_3 - u^0 h_4 + \frac{1}{\gamma^2 u^0} \end{vmatrix} \\
&= n_0^5 (u^0)^6 \begin{vmatrix} h_3 u\otimes u + h_4 \mathbf{I} & h_3 u \\ h_3 u^0 u^t & u^0 h_3 - \frac{1}{u^0}h_4 + \frac{1}{\gamma^2 (u^0)^3} \end{vmatrix} \\
&= n_0^5 (u^0)^6 \begin{vmatrix} h_4\mathbf{I} & h_3 u \\ \frac{u^t}{u^0}\Big[h_4 - \frac{1}{\gamma^2(u^0)^2}\Big] & u^0 h_3 - \frac{1}{u^0}h_4 + \frac{1}{\gamma^2(u^0)^3} \end{vmatrix} \\
&= n_0^5 (u^0)^6 \begin{vmatrix} h_4\mathbf{I} & h_3 u \\ \mathbf{0}^t & \Big[\frac{-|u|^2}{u^0 h_4}\Big(h_4 - \frac{1}{\gamma^2(u^0)^2}\Big) + u^0\Big]h_3 - \frac{1}{u^0}h_4 + \frac{1}{\gamma^2(u^0)^3} \end{vmatrix} \\
&= n_0^5 (u^0)^6 h_4^3 \Big\{\Big[\frac{-|u|^2}{u^0 h_4}\Big(h_4 - \frac{1}{\gamma^2(u^0)^2}\Big) + u^0\Big]h_3 - \frac{1}{u^0}h_4 + \frac{c^4}{\gamma^2(u^0)^3}\Big\},
\end{aligned}$$

where $\mathbf{0}$ denotes the column vector $(0,0,0)^t$, h_3 and h_4 are functions with the following forms:

$$h_3 = -\Big(\frac{K_1(\gamma)}{K_2(\gamma)}\Big)^2 - \frac{2}{\gamma}\frac{K_1(\gamma)}{K_2(\gamma)} + 1 + \frac{8}{\gamma^2}, \quad h_4 = \frac{K_1(\gamma)}{\gamma K_2(\gamma)} + \frac{4}{\gamma^2}.$$

Noting

$$-\Big(\frac{K_1(\gamma)}{K_2(\gamma)}\Big)^2 - \frac{3}{\gamma}\frac{K_1(\gamma)}{K_2(\gamma)} + 1 + \frac{3}{\gamma^2} > 0$$

by Proposition 6.3 in Appendix 3 [31], we can further obtain

$$\begin{aligned}
|\mathbf{A}_0| &> n_0^5 (u^0)^6 h_4^3 \frac{1}{u^0}(h_3 - h_4) \\
&= n_0^5 (u^0)^5 h_4^3 \Big[-\Big(\frac{K_1(\gamma)}{K_2(\gamma)}\Big)^2 - \frac{3}{\gamma}\frac{K_1(\gamma)}{K_2(\gamma)} + 1 + \frac{4}{\gamma^2}\Big] \\
&> \frac{n_0^5 (u^0)^5 h_4^3}{\gamma^2}.
\end{aligned}$$

On the other hand, the system (8.16) can also be written as a linear symmetric hyperbolic system of (E_{n+1}, B_{n+1}):

$$\partial_t \bar{U} + \sum_{i=1}^3 \bar{\mathbf{A}}_i \partial_i \bar{U} + \mathbf{B}U = 0, \tag{8.26}$$

where $\mathbf{B}$ is a 6×5 matrix whose components are functions of n_0, u. Denote $\mathbf{O}$ as a 3×3 matrix with all components be 0, and define matrixes:

$$\bar{\mathbf{A}}_{11} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \bar{\mathbf{A}}_{12} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \bar{\mathbf{A}}_{21} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\bar{\mathbf{A}}_{22} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \bar{\mathbf{A}}_{31} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{\mathbf{A}}_{32} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $\bar{\mathbf{A}}_i$ in (8.26) can be expressed as

$$\bar{\mathbf{A}}_i = \begin{pmatrix} \mathbf{O} & \bar{\mathbf{A}}_{i1} \\ \bar{\mathbf{A}}_{i2} & \mathbf{O} \end{pmatrix}.$$

Combining (8.25) and (8.26), we can obtain the wellposedness of $(a_{n+1}, b_{n+1}, c_{n+1}, E_{n+1}, B_{n+1}) \in C^0([0,\infty); H^N)$ with initial data $a_{n+1}(0,x)$, $b_{n+1}(0,x)$, $c_{n+1}(0,x)$, $E_{n+1}(0,x)$, $B_{n+1}(0,x) \in H^N$, $N \geq 0$ by the standard theorem of linear symmetric system [3] (Chapter 4.2). Moreover, for $n = 0$, one has

$$\|\partial_t \mathbf{A}_0\|_\infty + \sum_{i=1}^{3} \|\nabla_x \mathbf{A}_i\|_\infty + \sum_{i=1}^{2} \|\mathbf{B}_i\|_\infty + \|S\|_\infty \lesssim (1+t)^{-\beta_0}.$$

Therefore, standard energy estimates on systems (8.25) and (8.26) yield

$$\begin{aligned} &\frac{d}{dt}\Big(\|(a_1(t), b_1(t), c_1(t))\|_{H^N}^2 + \|[E_1(t), B_1(t)]\|_{H^N}^2\Big) \\ &\quad \lesssim (1+t)^{-\beta_0}\Big[\Big(\|(a_1(t), b_1(t), c_1(t))\|_{H^N}^2 + \|[E_1(t), B_1(t)]\|_{H^N}^2\Big) \\ &\qquad + \Big(\|(a_1(t), b_1(t), c_1(t))\|_{H^N} + \|[E_1(t), B_1(t)]\|_{H^N}\Big)\Big]. \end{aligned}$$

Namely,

$$\|(a_1(t), b_1(t), c_1(t))\|_{H^N} + \|[E_1(t), B_1(t)]\|_{H^N} \lesssim 1 \tag{8.27}$$

by Grönwall's inequality. Let $Lf_1 = \nu f_1 + K(f_1) = f_2 = w_0(p)\mathbf{M}$, where $w_0(p)$ is any polynomial of p. Note that, by (1.2), (1.21) and (8.1),

$$|K(f_1)| \lesssim \int_{\mathbb{R}^3} \frac{s^{\frac{3}{2}}}{gp^0q^0} U_1(p,q) \exp\{-[1 + O(1)|u|]\frac{\sqrt{s}|p-q|}{2T_0 g}\}|f_1(q)|dq,$$

and $f_1 = \frac{f_2}{\nu} - \frac{K(f_1)}{\nu}$. Then, we can verify that $|f_1| \lesssim \mathbf{M}^{1-}$. Therefore, it holds that

$$\{\mathbf{I} - \mathbf{P}\}\Big(\frac{F_1}{\sqrt{\mathbf{M}}}\Big) \lesssim (1+|p|)\mathbf{M}^{1-}.$$

It is straightforward to obtain (8.17) by the structure of F_1 and (8.27). Assuming (8.17) holds for $1 \leq i \leq n$, the case $i = n+1$ for (8.17) holds again by the structure of the equation for F_{n+1} in (1.10), the induction assumption and similar analysis as the case $i = 1$ ($n = 0$). $\square$

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