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Hilbert Expansion of the Boltzmann Equation with Specular Boundary Condition in Half-Space

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Abstract

Boundary effects play an important role in the study of hydrodynamic limits in the Boltzmann theory. Based on a systematic study of the viscous layer equations and the L^2 to L^∞ framework, we establish the validity of the Hilbert expansion for the Boltzmann equation with specular reflection boundary conditions, which leads to derivations of compressible Euler equations and acoustic equations in half-space.

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1. Introduction and Main Results

1.1. Introduction

In the founding work of Maxwell [38] and Boltzmann [6], it was shown that the Boltzmann equation is closely related to the fluid dynamical systems for both compressible and incompressible flows. Great effort has been devoted to the study of the hydrodynamic limit from the Boltzmann equation to the fluid systems. In 1912, Hilbert proposed a systematic formal asymptotic expansion for Boltzmann equation with respect to Knudsen number $\mathscr{K}_n \ll 1$. A bit later Enskog and Chapman independently proposed a somewhat different formal expansion, in 1916 and 1917, respectively. Either the Hilbert or the Chapman–Enskog expansions yield the compressible and incompressible fluid equations, for example the compressible Euler and Navier–Stokes systems, the incompressible Euler and Navier–Stokes (Fourier) systems, and the acoustic system, etc., It is a challenging problem to rigorously justify these formal approximation. In fact, the purpose of Hilbert's sixth problem [26] is to establish the laws of motion of continua from more microscopic physical models, such as Boltzmann theory, from a rigorous mathematical standpoint.

Based on the truncated Hilbert expansion, Caflisch [7] rigorously established the hydrodynamic limit from the Boltzmann equation to the compressible Euler equations when solution is smooth; see also [18,36,39,45], and [22,23] via a recent L^2-L^∞ framework. As it is well known, solutions of the compressible Euler equations in general develop singularities, such as shock wave. Generally speaking, there are three basic wave patterns for compressible Euler equations: shock wave, the rarefaction wave, and contact discontinuity. The hydrodynamic limit of Boltzmann to such wave patterns have been proved [27,29–31,48,49] in one dimensional case. For multi-dimensional case, the only result is [46] for planar rarefaction wave.

The acoustic equations are the linearization of the compressible Euler equations about a spatially homogeneous fluid state. Being essentially the wave equations, they form the simplest PDE system in fluid dynamics. Bardos et al. [4] established the convergence from the DiPerna–Lions [11] solutions of Boltzmann equation to the solution of acoustic system over a periodic spatial domain with restriction on the size of fluctuation. Recently, the restriction was relaxed in [32,34], and finally removed in [23] via a L^2-L^∞ framework.

There have been extensive research efforts to derive the incompressible Navier–Stokes system; see [2,3,5,8,11,13–16,19,25,33,35,37,40,47] and the references cited therein.

All of the above-mentioned works on the compressible Euler limit and the acoustic limit were carried out in either spatially periodic domain or the whole space. However, in many important physical applications, boundaries occur naturally, and boundary effects are crucial in the hydrodynamic limit of dilute gases governed by the Boltzmann equation. Hence it is important to study the hydrodynamic limit from the Boltzmann equation to the compressible Euler equations in the presence of physical boundaries. The purpose of this paper is to justify the compressible Euler limit and the acoustic limit of Boltzmann equation with specular reflection boundary conditions by the Hilbert expansion method. The main difficulty, due to the presence of physical boundaries, is the possible appearance of both viscous and Knudsen boundary layers, and the interaction between these layers is very complicate.

More precisely, we consider the scaled Boltzmann equation

$$F_t + v \cdot \nabla_x F = \frac{1}{\mathscr{K}_n} Q(F, F), \tag{1.1}$$

where $F(t, x, v) \geqq 0$ is the density distribution function for the gas particles with position $x \in \mathbb{R}^3_+ = \{x \in \mathbb{R}^3 : x_3 > 0\}$ and velocity $v \in \mathbb{R}^3$ at time $t > 0$, and $\mathscr{K}_n > 0$ is Knudsen number which is proportional to the mean free path. The Boltzmann collision term $Q(F_1, F_2)$ on the right is defined in terms of the following bilinear form

$$\begin{aligned} Q(F_1, F_2) \equiv & \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v-u, \theta) F_1(u') F_2(v') \,\mathrm{d}\omega \mathrm{d}u \\ & - \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v-u, \theta) F_1(u) F_2(v) \,\mathrm{d}\omega \mathrm{d}u \\ := & \; Q_+(F_1, F_2) - Q_-(F_1, F_2), \end{aligned} \tag{1.2}$$

where the relationship between the post-collision velocity (v', u') of two particles with the pre-collision velocity (v, u) is given by

$$u' = u + [(v-u)\cdot\omega]\omega, \quad v' = v - [(v-u)\cdot\omega]\omega$$

for $\omega \in \mathbb{S}^2$, which can be determined by conservation laws of momentum and energy

$$u' + v' = u + v, \quad |u'|^2 + |v'|^2 = |u|^2 + |v|^2.$$

The Boltzmann collision kernel $B = B(v-u, \theta)$ in (1.2) depends only on $|v-u|$ and θ with $\cos\theta = (v-u)\cdot\omega/|v-u|$. Throughout this paper, we consider the hard sphere model, i.e.,

$$B(v-u, \theta) = |(v-u)\cdot\omega|.$$

We denote $\vec{n} = (0, 0, -1)$ to be the outward normal of $\mathbb{R}^3_+$. We denote the phase boundary in the space $\mathbb{R}^3_+ \times \mathbb{R}^3$ as $\gamma := \partial\mathbb{R}^3_+ \times \mathbb{R}^3$ and split it into outgoing boundary γ_+, incoming boundary γ_-, and grazing boundary γ_0 as follows:

$$\begin{aligned} \gamma_+ &= \{(x, v) : x \in \partial\mathbb{R}^3_+, v\cdot\vec{n} = -v_3 > 0\}, \\ \gamma_- &= \{(x, v) : x \in \partial\mathbb{R}^3_+, v\cdot\vec{n} = -v_3 < 0\}, \\ \gamma_0 &= \{(x, v) : x \in \partial\mathbb{R}^3_+, v\cdot\vec{n} = -v_3 = 0\}. \end{aligned}$$

In the present paper we consider the Boltzmann equation with specular reflection boundary conditions, i.e.,

$$F(t, x, v)|_{\gamma_-} = F(t, x, R_x v), \tag{1.3}$$

where

$$R_x v = v - 2\{v\cdot\vec{n}\}\vec{n} = (v_1, v_2, -v_3)^\top. \tag{1.4}$$

1.2. Asymptotic Expansion

Since the thickness of viscous boundary layer is $\sqrt{\mathscr{K}_n}$, for simplicity, we use the new parameter $\varepsilon = \sqrt{\mathscr{K}_n}$, and denote the Boltzmann solution to be F^ε, then the Boltzmann equation (1.1) is rewritten as

$$\partial_t F^\varepsilon + v \cdot \nabla_x F^\varepsilon = \frac{1}{\varepsilon^2} Q(F^\varepsilon, F^\varepsilon). \tag{1.5}$$

1.2.1. Interior Expansion We define the interior expansion

$$F^\varepsilon(t,x,v) \sim \sum_{k=0}^{\infty} \varepsilon^k F_k(t,x,v). \tag{1.6}$$

Substituting (1.6) into (1.5) and comparing the order of ε, one obtains

$$\begin{aligned}
\frac{1}{\varepsilon^2}:&\qquad\qquad 0 = Q(F_0, F_0),\\
\frac{1}{\varepsilon}:&\qquad\qquad 0 = Q(F_0, F_1) + Q(F_1, F_0),\\
\varepsilon^0:&\ \{\partial_t + v\cdot\nabla_x\}F_0 = Q(F_0, F_2) + Q(F_2, F_0) + Q(F_1, F_1),\\
\varepsilon:&\ \{\partial_t + v\cdot\nabla_x\}F_1 = Q(F_0, F_3) + Q(F_3, F_0) + Q(F_1, F_2) + Q(F_2, F_1),\\
&\qquad\qquad \vdots\\
\varepsilon^k:&\ \{\partial_t + v\cdot\nabla_x\}F_k = Q(F_0, F_{k+2}) + Q(F_{k+2}, F_0) + \sum_{\substack{i+j=k+2\\ i,j\geqq 1}} Q(F_i, F_j).
\end{aligned} \tag{1.7}$$

It follows from $(1.7)_1$ and the celebrated H-theorem that F_0 should be a local Maxwellian

$$\mu(t,x,v) := F_0(t,x,v) \equiv \frac{\rho(t,x)}{[2\pi T(t,x)]^{3/2}} \exp\left\{-\frac{|v-\mathfrak{u}(t,x)|^2}{2T(t,x)}\right\},$$

where $\rho(t,x)$, $\mathfrak{u}(t,x) = (\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3)(t,x)$, and $T(t,x)$ are defined by

$$\int_{\mathbb{R}^3} F_0 \mathrm{d}v = \rho, \quad \int_{\mathbb{R}^3} v F_0 \mathrm{d}v = \rho\mathfrak{u}, \quad \int_{\mathbb{R}^3} |v|^2 F_0 \mathrm{d}v = \rho|\mathfrak{u}|^2 + 3\rho T,$$

which represent the macroscopic density, velocity and temperature, respectively. Projecting the equation $(1.7)_3$ onto 1, v, $\frac{|v|^2}{2}$, which are five collision invariants for the Boltzmann collision operator $Q(\cdot,\cdot)$, one obtains that $(\rho, \mathfrak{u}, T)$ satisfies the compressible Euler system

$$\begin{cases}
\partial_t \rho + \mathrm{div}(\rho\mathfrak{u}) = 0,\\
\partial_t(\rho\mathfrak{u}) + \mathrm{div}(\rho\mathfrak{u}\otimes\mathfrak{u}) + \nabla p = 0,\\
\partial_t\left[\rho\left(\frac{3T}{2} + \frac{|\mathfrak{u}|^2}{2}\right)\right] + \mathrm{div}\left[\rho\mathfrak{u}\left(\frac{3T}{2} + \frac{|\mathfrak{u}|^2}{2}\right)\right] + \mathrm{div}(p\mathfrak{u}) = 0,
\end{cases} \tag{1.8}$$

where $x \in \mathbb{R}^3_+$, $t > 0$ and $p = \rho T$ is the pressure function. For the compressible Euler equations (1.8), we impose the slip boundary condition

$$\mathfrak{u} \cdot \vec{n}|_{x_3=0} = \mathfrak{u}_3|_{x_3=0} = 0. \tag{1.9}$$

and the initial data

$$(\rho, \mathfrak{u}, T)(0, x) = (1 + \delta\varphi_0, \delta\Phi_0, 1 + \delta\vartheta_0)(x), \tag{1.10}$$

with $\|(\varphi_0, \Phi_0, \vartheta_0)\|_{H^{s_0}} \leqq 1$, where $\delta > 0$ is a parameter and $s_0 \geqq 3$ is some given positive number. Choose $\delta_1 > 0$ so that for any $\delta \in (0, \delta_1]$, the positivity of $1 + \delta\varphi_0$ and $1 + \delta\vartheta_0$ is guaranteed. Then for each $\delta \in (0, \delta_1]$, there is a family of classical solutions $(\rho^\delta, \mathfrak{u}^\delta, T^\delta) \in C([0, \tau^\delta]; H^{s_0}(\mathbb{R}^3_+)) \cap C^1([0, \tau^\delta]; H^{s_0-1}(\mathbb{R}^3_+))$ of the compressible Euler equations (1.8)–(1.10) such that $\rho^\delta > 0$ and $T^\delta > 0$; see Lemma 2.1 for details.

Generally, the solution of interior expansion F_i, $i = 1, 2, \ldots$ does not satisfy the specular reflection boundary conditions. To overcome the difficulty coming from the boundary condition, the boundary layer expansions is needed (see [43,44]).

For later use, we define the linearized collision operator $\mathbf{L}$ by

$$\mathbf{L}g = -\frac{1}{\sqrt{\mu}}\Big\{Q(\mu, \sqrt{\mu}g) + Q(\sqrt{\mu}g, \mu)\Big\}.$$

The null space $\mathcal{N}$ of $\mathbf{L}$ is generated by

$$\begin{aligned}
\chi_0(v) &\equiv \frac{1}{\sqrt{\rho}}\sqrt{\mu},\\
\chi_i(v) &\equiv \frac{v_i - \mathfrak{u}_i}{\sqrt{\rho T}}\sqrt{\mu}, \quad i = 1, 2, 3,\\
\chi_4(v) &\equiv \frac{1}{\sqrt{6\rho}}\left\{\frac{|v - \mathfrak{u}|^2}{T} - 3\right\}\sqrt{\mu}.
\end{aligned}$$

Clearly, we have $\int_{\mathbb{R}^3} \chi_i \cdot \chi_j \mathrm{d}v = \delta_{ij}$ for $0 \leqq i, j \leqq 4$. We also define the collision frequency ν:

$$\nu(t, x, v) \equiv \nu(\mu) := \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v - u, \theta)\mu(u)\mathrm{d}\omega\mathrm{d}u. \tag{1.11}$$

Furthermore it holds that

$$\frac{1}{C}(1 + |v|) \leqq \nu(t, x, v) \leqq C(1 + |v|),$$

where $C > 0$ is some given positive constant. Let $\mathbf{P}g$ be the L^2_v projection with respect to $[\chi_0, \ldots, \chi_4]$. It is well-known that there exists a positive number $c_0 > 0$ such that for any function g

$$\langle \mathbf{L}g, g\rangle \geqq c_0\|\{\mathbf{I} - \mathbf{P}\}g\|^2_\nu,$$

where the weighted L^2-norm $\|\cdot\|_\nu$ is defined as

$$\|g\|_\nu^2 := \int_{\mathbb{R}_x^3\times\mathbb{R}_v^3} g^2(x,v)\nu(v)\,\mathrm{d}x\mathrm{d}v.$$

For each $k \geqq 1$, we define the macroscopic and microscopic part of $\frac{F_k}{\sqrt{\mu}}$ as

$$\begin{aligned}\frac{F_k}{\sqrt{\mu}} &= \mathbf{P}\left(\frac{F_k}{\sqrt{\mu}}\right) + \{\mathbf{I}-\mathbf{P}\}\left(\frac{F_k}{\sqrt{\mu}}\right)\\ &\equiv \left\{\frac{\rho_k}{\sqrt{\rho}}\chi_0 + \sum_{j=1}^3 \sqrt{\frac{\rho}{T}}u_{k,j}\cdot\chi_j + \sqrt{\frac{\rho}{6}}\frac{\theta_k}{T}\chi_4\right\} + \{\mathbf{I}-\mathbf{P}\}\left(\frac{F_k}{\sqrt{\mu}}\right)\\ &\equiv \left\{\frac{\rho_k}{\rho} + u_k\cdot\frac{v-\mathfrak{u}}{T} + \frac{\theta_k}{6T}\left(\frac{|v-\mathfrak{u}|^2}{T}-3\right)\right\}\sqrt{\mu} + \{\mathbf{I}-\mathbf{P}\}\left(\frac{F_k}{\sqrt{\mu}}\right).\end{aligned} \tag{1.12}$$

1.2.2. Viscous Boundary Layer Expansion We define the scaled normal coordinate:

$$y := \frac{x_3}{\varepsilon}. \tag{1.13}$$

For notational simplicity, we denote

$$x_{\shortparallel} = (x_1, x_2), \quad \nabla_{\shortparallel} = (\partial_{x_1}, \partial_{x_2}) \quad \text{and} \quad v_{\shortparallel} = (v_1, v_2). \tag{1.14}$$

Noting (1.9), we know that the local Maxwellian μ satisfies the specular reflection boundary conditions. However, in general, F_1 may not satisfy the specular reflection boundary conditions, therefore we need to construct viscous boundary layer to compensate the boundary conditions starting from the first order of ε.

Motivated by [44, Section 3.4.1], we define the viscous boundary layer expansion as

$$\bar{F}^\varepsilon(t, x_{\shortparallel}, y) \sim \sum_{k=1}^\infty \varepsilon^k \bar{F}_k(t, x_{\shortparallel}, y, v).$$

Plugging $F^\varepsilon + \bar{F}^\varepsilon$ into the Boltzmann equation (1.5) and comparing the order of ε, then using (1.7), in the neighborhood of physical boundary, we have

$$
\begin{aligned}
\frac{1}{\varepsilon}: &\quad 0 = Q(\mu_0, \bar{F}_1) + Q(\bar{F}_1, \mu_0),\\
\varepsilon^0: &\quad v_3 \frac{\partial \bar{F}_1}{\partial y} = [Q(\mu_0, \bar{F}_2) + Q(\bar{F}_2, \mu_0)] + y[Q(\partial_3\mu_0, \bar{F}_1) + Q(\bar{F}_1, \partial_3\mu_0)]\\
&\qquad + Q(F_1^0, \bar{F}_1) + Q(\bar{F}_1, F_1^0) + Q(\bar{F}_1, \bar{F}_1),\\
&\qquad \vdots\\
\varepsilon^k: &\quad \{\partial_t + v_{\shortparallel}\cdot\nabla_{\shortparallel}\}\bar{F}_k + v_3\frac{\partial \bar{F}_{k+1}}{\partial y} = Q(\mu_0, \bar{F}_{k+2}) + Q(\bar{F}_{k+2}, \mu_0)\\
&\qquad + \sum_{\substack{l+j=k+2\\ 1\leqq l\leqq \mathfrak{b},\, j\geqq 1}} \frac{y^l}{l!}\big[Q(\partial_3^l\mu_0, \bar{F}_j) + Q(\bar{F}_j, \partial_3^l\mu_0)\big]\\
&\qquad + \sum_{\substack{i+j=k+2\\ i,j\geqq 1}} \big[Q(F_i^0, \bar{F}_j) + Q(\bar{F}_j, F_i^0) + Q(\bar{F}_i, \bar{F}_j)\big]\\
&\qquad + \sum_{\substack{i+j+l=k+2\\ 1\leqq l\leqq \mathfrak{b},\, i,j\geqq 1}} \frac{y^l}{l!}\big[Q(\partial_3^l F_i^0, \bar{F}_j) + Q(\bar{F}_j, \partial_3^l F_i^0)\big], \quad \text{for } k \geqq 1,
\end{aligned}
\tag{1.15}
$$

where we have used the Taylor expansions of μ and F_i at $x_3 = 0$, i.e.,

$$
\mu(t, x_1, x_2, x_3, v) = \mu_0 + \sum_{l=1}^{\mathfrak{b}} \frac{1}{l!}\partial_3^l\mu_0 \cdot x_3^l + \frac{x_3^{\mathfrak{b}+1}}{(\mathfrak{b}+1)!}\partial_3^{\mathfrak{b}+1}\tilde{\mu}, \tag{1.16}
$$

and for $i \geqq 1$,

$$
F_i(t, x_1, x_2, x_3, v) = F_i^0 + \sum_{l=1}^{\mathfrak{b}} \frac{1}{l!}\partial_3^l F_i^0 \cdot x_3^l + \frac{x_3^{\mathfrak{b}+1}}{(\mathfrak{b}+1)!}\partial_3^{\mathfrak{b}+1}\mathfrak{F}_i. \tag{1.17}
$$

Here we have used the simplified notations

$$
\begin{aligned}
\partial_3^l\mu_0 &:= (\partial_3^l\mu)(t, x_1, x_2, 0, v), \quad \partial_3^{\mathfrak{b}+1}\tilde{\mu} := (\partial_3^{\mathfrak{b}+1}\mu)(t, x_1, x_2, \xi_0, v),\\
\partial_3^l F_i^0 &:= (\partial_3^l F_i)(t, x_1, x_2, 0, v), \quad \partial_3^{\mathfrak{b}+1}\mathfrak{F}_i := (\partial_3^{\mathfrak{b}+1}F_i)(t, x_1, x_2, \xi_i, v)
\end{aligned}
\tag{1.18}
$$

for some $\xi_i \in (0, x_3)$ with $i \geqq 0$. The number $\mathfrak{b} \in \mathbb{N}_+$ will be chosen later.

The main reason to use (1.16) is to make the coefficients of linearized operator of (1.15) be independent of y. The reason for (1.17) is to make the coefficients of viscous boundary layer system be independent of ε; see (1.27)–(1.28) for details. Noting the polynomial growth coefficients y^l in (1.15), it is imperative to prove that they decay with enough polynomial rate as $y \to +\infty$ provided the initial data decay sufficiently fast. For later use, we define $\bar{F}_0 = 0$.

For the macro-micro decomposition of viscous and Knudsen boundary layers, we define the corresponding linearized collision operator, macroscopic projection, and null space as

$$\mathbf{L}_0 = \mathbf{L}(t, x_{\shortparallel}, 0, v), \qquad \mathbf{P}_0 = \mathbf{P}(t, x_{\shortparallel}, 0, v), \qquad \mathcal{N}_0 = \mathcal{N}(t, x_{\shortparallel}, 0, v).$$

It is noted that $\mathbf{L}_0$, $\mathbf{P}_0$ and $\mathcal{N}_0$ are independent of normal variable. We define

$$\bar{f}_k := \frac{\bar{F}_k}{\sqrt{\mu_0}}. \tag{1.19}$$

Then it holds that

$$\begin{aligned}\bar{f}_k &= \mathbf{P}_0 \bar{f}_k + \{\mathbf{I} - \mathbf{P_0}\}\bar{f}_k \\ &= \left\{\frac{\bar{\rho}_k}{\rho^0} + \bar{u}_k \cdot \frac{v - \mathfrak{u}^0}{T^0} + \frac{\bar{\theta}_k}{6T^0}\left(\frac{|v - \mathfrak{u}^0|^2}{T^0} - 3\right)\right\}\sqrt{\mu_0} + \{\mathbf{I} - \mathbf{P_0}\}\bar{f}_k,\end{aligned}$$

where and whereafter we always use the notation $(\rho^0, \mathfrak{u}^0, T^0) := (\rho, \mathfrak{u}, T)(t, x_{\shortparallel}, 0)$.

Throughout the present paper, we always assume the far field condition

$$\bar{f}_k(t, x_{\shortparallel}, y, v) \to 0, \quad \text{as } y \to +\infty. \tag{1.20}$$

In fact, it follows from $(1.15)_1$ that $\bar{f}_1 \in \mathcal{N}_0$, i.e.,

$$\bar{f}_1 \equiv \mathbf{P}_0 \bar{f}_1 = \left\{\frac{\bar{\rho}_1}{\rho^0} + \bar{u}_1 \cdot \frac{v - \mathfrak{u}^0}{T^0} + \frac{\bar{\theta}_1}{6T^0}\left(\frac{|v - \mathfrak{u}^0|^2}{T^0} - 3\right)\right\}\sqrt{\mu_0}. \tag{1.21}$$

We denote

$$\bar{p}_k = \frac{\rho^0 \bar{\theta}_k + 3T^0 \bar{\rho}_k}{3}. \tag{1.22}$$

Multiplying $(1.15)_2$ by $\sqrt{\mu_0}$, $v_3\sqrt{\mu_0}$ and integrating over $\mathbb{R}^3$ with respect to v, one obtains

$$\partial_y \bar{u}_{1,3} = 0, \quad \partial_y \bar{p}_1 = 0, \tag{1.23}$$

where $\bar{p}_1$ is defined in (1.22) with $k = 1$. Noting from (1.20) and (1.23), we have

$$\bar{u}_{1,3}(t, x_{\shortparallel}, y) \equiv 0 \quad \text{and} \quad \bar{p}_1(t, x_{\shortparallel}, y) \equiv 0, \quad \forall\, (t, x_{\shortparallel}, y) \in [0, \tau] \times \mathbb{R}^2 \times \mathbb{R}_+. \tag{1.24}$$

Note that from (1.22), $(1.24)_2$ is similar to the Boussinesq relation in the diffusive limit of Boltzmann equation.

For later use, we define the Burnett functions $\mathcal{A}_{ij}$ and $\mathcal{B}_i$

$$\begin{aligned}\mathcal{A}_{ij} &:= \left\{\frac{(v_i - \mathfrak{u}_i)(v_j - \mathfrak{u}_j)}{T} - \delta_{ij}\frac{|v - \mathfrak{u}|^2}{3T}\right\}\sqrt{\mu}, \\ \mathcal{B}_i &:= \frac{v_i - \mathfrak{u}_i}{2\sqrt{T}}\left(\frac{|v - \mathfrak{u}|^2}{T} - 5\right)\sqrt{\mu}.\end{aligned} \tag{1.25}$$

and denote

$$\mathcal{A}_{i,j}^0 := \mathcal{A}_{i,j}(t, x_{\shortparallel}, 0, v) \quad \text{and} \quad \mathcal{B}_i^0 := \mathcal{B}_i(t, x_{\shortparallel}, 0, v).$$

We define the viscosity and thermal conductivity coefficients $\mu(T^0)$, $\kappa(T^0)$ of viscous boundary layer

$$\begin{aligned} \mu(T^0) &:= T^0\langle \mathcal{A}_{31}^0,\ \mathbf{L}_0^{-1}\mathcal{A}_{31}^0\rangle \equiv T^0\langle \mathcal{A}_{ij}^0,\ \mathbf{L}_0^{-1}\mathcal{A}_{ij}^0\rangle, \quad \forall i \neq j, \\ \kappa(T^0) &:= \frac{2}{3}T^0\langle \mathcal{B}_3^0,\ \mathbf{L}_0^{-1}\mathcal{B}_3^0\rangle \equiv \frac{2}{3}T^0\langle \mathcal{B}_i^0,\ \mathbf{L}_0^{-1}\mathcal{B}_i^0\rangle, \end{aligned} \qquad i, j = 1, 2, 3. \tag{1.26}$$

By Lemma 4.4 in [3], it holds that $\langle T^0\mathcal{A}_{33}^0, \mathbf{L}_0^{-1}\mathcal{A}_{33}^0\rangle = \frac{4}{3}\mu(T^0)$. We denote

$$f_1^0 := f_1(t, x_{\shortparallel}, 0, v) \equiv \left\{\frac{\rho_1^0}{\rho^0} + u_1^0 \cdot \frac{v - \mathfrak{u}^0}{T^0} + \frac{\theta_1^0}{6T^0}\left(\frac{|v - \mathfrak{u}^0|^2}{T^0} - 3\right)\right\}\sqrt{\mu_0},$$

where $(\rho_1^0, u_1^0, \theta_1^0) := (\rho_1, u_1, \theta_1)(t, x_{\shortparallel}, 0)$.

$\bar{f}_k$ will be constructed inductively as follows:

Lemma 1.1. *Let $\bar{f}_k$ $(k \geqq 1)$ be the solution of (1.15). For $k \geqq 1$, $(\bar{u}_{k,\shortparallel}, \bar{\theta}_k)$ satisfies*

$$\begin{aligned} &\rho^0\partial_t\bar{u}_{k,i} + \rho^0(\mathfrak{u}_{\shortparallel}^0 \cdot \nabla_{\shortparallel})\bar{u}_{k,i} + \rho^0(\partial_3\mathfrak{u}_3^0 \cdot y + u_{1,3}^0)\partial_y\bar{u}_{k,i} \\ &\quad + \rho^0\bar{u}_{k,\shortparallel} \cdot \nabla_{\shortparallel}\mathfrak{u}_i^0 - \frac{\partial_i p^0}{3T^0}\bar{\theta}_k - \mu(T^0)\partial_{yy}\bar{u}_{k,i} \\ &= \bar{\mathfrak{f}}_{k-1} =: -\rho^0\partial_y[(\partial_3\mathfrak{u}_i^0 \cdot y + u_{1,i}^0 + \bar{u}_{1,i})\bar{u}_{k,3}] - \left[\partial_i - \frac{\partial_i p^0}{p^0}\right]\bar{p}_k \\ &\quad + \bar{W}_{k-1,i} - T^0\partial_y\langle \bar{J}_{k-1},\ \mathcal{A}_{3i}^0\rangle, \quad i = 1, 2, \end{aligned} \tag{1.27}$$

$$\begin{aligned} &\rho^0\partial_t\bar{\theta}_k + \rho^0(\mathfrak{u}_{\shortparallel}^0 \cdot \nabla_{\shortparallel})\bar{\theta}_k + \rho^0(\partial_3\mathfrak{u}_3^0 \cdot y + u_{1,3}^0)\partial_y\bar{\theta}_k + \frac{2}{3}\rho^0\mathrm{div}\mathfrak{u}^0\bar{\theta}_k - \frac{3}{5}\kappa(T^0)\partial_{yy}\bar{\theta}_k \\ &= \bar{\mathfrak{g}}_{k-1} =: -\rho^0\partial_y[(3\partial_3T^0 \cdot y + \theta_1^0 + \bar{\theta}_1)\bar{u}_{k,3}] + \frac{3}{5}\bar{H}_{k-1} - \frac{6}{5}(T^0)^{\frac{3}{2}}\partial_y\langle \bar{J}_{k-1},\ \mathcal{B}_3^0\rangle \\ &\quad + \frac{3}{5}\left\{2\partial_t + 2\mathfrak{u}_{\shortparallel}^0 \cdot \nabla_{\shortparallel} + \frac{10}{3}\mathrm{div}\mathfrak{u}^0\right\}\bar{p}_k, \end{aligned} \tag{1.28}$$

where $\mathrm{div}\mathfrak{u}^0 := (\mathrm{div}\mathfrak{u})(t, x_{\shortparallel}, 0)$. *Once one solves $\bar{u}_{k,\shortparallel}$, then $(\mathbf{I} - \mathbf{P_0})\bar{f}_{k+1}$, $\bar{u}_{k+1,3}$, $\bar{p}_{k+1}$ can be determined by the following equations:*

$$\begin{aligned} (\mathbf{I} - \mathbf{P_0})\bar{f}_{k+1} = \mathbf{L}_0^{-1}\Bigg\{ &- (\mathbf{I} - \mathbf{P_0})(v_3\partial_y\mathbf{P}_0\bar{f}_k) \\ &+ \frac{y}{\sqrt{\mu_0}}\Big[Q(\partial_3\mu_0, \sqrt{\mu_0}\mathbf{P_0}\bar{f}_k) + Q(\sqrt{\mu_0}\mathbf{P_0}\bar{f}_k, \partial_3\mu_0)\Big] \\ &+ \frac{1}{\sqrt{\mu_0}}\Big[Q(F_1^0, \sqrt{\mu_0}\mathbf{P_0}\bar{f}_k) + Q(\sqrt{\mu_0}\mathbf{P_0}\bar{f}_k, F_1^0)\Big] \\ &+ \frac{1}{\sqrt{\mu_0}}\Big[Q(\sqrt{\mu_0}\bar{f}_1, \sqrt{\mu_0}\mathbf{P_0}\bar{f}_k) + Q(\sqrt{\mu_0}\mathbf{P_0}\bar{f}_k, \sqrt{\mu_0}\bar{f}_1)\Big]\Bigg\} + \bar{J}_{k-1}, \end{aligned} \tag{1.29}$$

$$\partial_y \bar{u}_{k+1,3} = -\frac{1}{\rho^0}\Big\{\partial_t \bar{\rho}_k + \mathrm{div}_{\shortparallel}(\rho^0 \bar{u}_{k,\shortparallel} + \bar{\rho}_k \mathfrak{u}^0_{\shortparallel})\Big\}, \tag{1.30}$$

$$\begin{aligned}\partial_y \bar{p}_{k+1} = &-\rho^0 \partial_t \bar{u}_{k,3} - \rho^0 (\mathfrak{u}^0_{\shortparallel}\cdot\nabla_{\shortparallel})\bar{u}_{k,3} + \rho^0 \partial_3 \mathfrak{u}^0_3\, \bar{u}_{k,3} \\ &- \frac{4}{3}\rho^0 \partial_y[(\partial_3 \mathfrak{u}^0_3 \cdot y + u^0_{1,3})\bar{u}_{k,3}] + \frac{4}{3}\mu(T^0)\partial_{yy}\bar{u}_{k,3} \\ &+ \frac{2}{3}\rho^0 \sum_{l=1}^{2} \partial_y([\partial_3 \mathfrak{u}^0_l \cdot y + u^0_{1,l} + \bar{u}_{1,l}]\bar{u}_{k,l}) \\ &- T^0 \partial_y \langle \bar{J}_{k-1}, \mathcal{A}^0_{33}\rangle + \bar{W}_{k-1,3},\end{aligned} \tag{1.31}$$

and

$$\bar{W}_{k-1,i} = -\sum_{j=1}^{2} \partial_j \int_{\mathbb{R}^3} T^0 \{\mathbf{I}-\mathbf{P_0}\}\bar{f}_k \cdot \mathcal{A}^0_{i,j}\, dv, \quad \textit{for } i = 1,2,3, \tag{1.32}$$

$$\begin{aligned}\bar{H}_{k-1} = &-\sum_{j=1}^{2} \partial_j \Big\{ 2(T^0)^{\frac{3}{2}} \int_{\mathbb{R}^3} \{\mathbf{I}-\mathbf{P_0}\}\bar{f}_k \cdot \mathcal{B}^0_j\, dv \\ &+ \sum_{l=1}^{2} 2T^0 \mathfrak{u}^0_l \int_{\mathbb{R}^3} \{\mathbf{I}-\mathbf{P_0}\}\bar{f}_k \cdot \mathcal{A}^0_{l,j}\, dv \Big\} - 2\mathfrak{u}^0_{\shortparallel} \cdot \bar{W}_{k-1,\shortparallel}.\end{aligned} \tag{1.33}$$

and

$$\begin{aligned}\bar{J}_{k-1} = \mathbf{L}_0^{-1}\Big\{ &-(\mathbf{I}-\mathbf{P_0})\Big(\frac{(\partial_t + v_{\shortparallel}\cdot\nabla_{\shortparallel})\bar{F}_{k-1}}{\sqrt{\mu_0}}\Big) - (\mathbf{I}-\mathbf{P_0})[v_3 \partial_y (\mathbf{I}-\mathbf{P_0})\bar{f}_k] \\ &+ \sum_{\substack{j+l=k+1\\ 2\leqq l\leqq \mathfrak{b},\, j\geqq 1}} \frac{y^l}{l!}\frac{1}{\sqrt{\mu_0}}\Big[Q(\partial_3^l \mu_0, \sqrt{\mu_0}\bar{f}_j) + Q(\sqrt{\mu_0}\bar{f}_j, \partial_3^l \mu_0)\Big] \\ &+ \sum_{\substack{i+j=k+1\\ i\geqq 2,\, j\geqq 1}} \frac{1}{\sqrt{\mu_0}}\Big[Q(F^0_i, \sqrt{\mu_0}\bar{f}_j) + Q(\sqrt{\mu_0}\bar{f}_j, F^0_i)\Big] \\ &+ \sum_{\substack{i+j=k+1\\ i,j\geqq 2}} \frac{Q(\sqrt{\mu_0}\bar{f}_i, \sqrt{\mu_0}\bar{f}_j)}{\sqrt{\mu_0}} \\ &+ \sum_{\substack{i+j+l=k+1\\ 1\leqq l\leqq \mathfrak{b},\, i,j\geqq 1}} \frac{y^l}{l!}\frac{1}{\sqrt{\mu_0}}\Big[Q(\partial_3^l F^0_i, \sqrt{\mu_0}\bar{f}_j) + Q(\sqrt{\mu_0}\bar{f}_j, \partial_3^l F^0_i)\Big] \\ &+ \frac{y}{\sqrt{\mu_0}}\Big[Q(\partial_3 \mu_0, \sqrt{\mu_0}(\mathbf{I}-\mathbf{P_0})\bar{f}_k) + Q(\sqrt{\mu_0}(\mathbf{I}-\mathbf{P_0})\bar{f}_k, \partial_3 \mu_0)\Big] \\ &+ \frac{1}{\sqrt{\mu_0}}\Big[Q(F^0_1, \sqrt{\mu_0}(\mathbf{I}-\mathbf{P_0})\bar{f}_k) + Q(\sqrt{\mu_0}(\mathbf{I}-\mathbf{P_0})\bar{f}_k, F^0_1)\Big] \\ &+ \frac{1}{\sqrt{\mu_0}}\Big[Q(\sqrt{\mu_0}\bar{f}_1, \sqrt{\mu_0}(\mathbf{I}-\mathbf{P_0})\bar{f}_k) + Q(\sqrt{\mu_0}(\mathbf{I}-\mathbf{P_0})\bar{f}_k, \sqrt{\mu_0}\bar{f}_1)\Big]\Big\},\end{aligned} \tag{1.34}$$

We point out that $\bar{W}_{k-1}$, $\bar{H}_{k-1}$ *and* $\bar{J}_{k-1}$ *depend on* $\bar{f}_j$, $1 \leqq j \leqq k-1$.

Remark 1.2. Since we have not found a direct reference which gives Lemma 1.1, so we present details of calculation in the "Appendix A" for completeness though it is somehow routine.

Remark 1.3. We remark that the Taylor expansion of μ in the derivation of the viscous boundary layer equations is crucial for the control of large velocity v in our L^2-L^∞ framework. On the other hand, such an expansion creates a factor of y, which leads to only algebraic decay in y in general. This is in stark contrast to the other standard boundary layer theories which typically have exponential decay in the normal direction.

Remark 1.4. For $k = 1$, noting $\bar{J}_0 = \bar{W}_0 = \bar{H}_0 = 0$ and (1.24), then the system (1.27)–(1.28) for $[\bar{u}_{1,\shortparallel}, \bar{\theta}_1]$ becomes

$$\begin{aligned} &\rho^0\partial_t\bar{u}_{1,i} + \rho^0(\mathrm{u}^0_{\shortparallel}\cdot\nabla_{\shortparallel})\bar{u}_{1,i} + \rho^0(\partial_3\mathrm{u}^0_3\cdot y + u^0_{1,3})\partial_y\bar{u}_{1,i} \\ &\quad + \rho^0\bar{u}_{1,\shortparallel}\cdot\nabla_{\shortparallel}\mathrm{u}^0_i - \frac{\partial_i p^0}{3T^0}\bar{\theta}_1 - \mu(T^0)\partial_{yy}\bar{u}_{1,i} = 0,\ i = 1, 2, \end{aligned} \tag{1.35}$$

$$\begin{aligned} &\rho^0\partial_t\bar{\theta}_1 + \rho^0(\mathrm{u}^0_{\shortparallel}\cdot\nabla_{\shortparallel})\bar{\theta}_1 + \rho^0(\partial_3\mathrm{u}^0_3\cdot y + u^0_{1,3})\partial_y\bar{\theta}_1 \\ &\quad + \frac{2}{3}\rho^0\mathrm{div}\mathrm{u}^0\bar{\theta}_1 - \frac{3}{5}\kappa(T^0)\partial_{yy}\bar{\theta}_1 = 0, \end{aligned} \tag{1.36}$$

which is indeed a linear system for $(\bar{u}_{1,\shortparallel}, \bar{\theta}_1)$. As indicated later in Remark 1.9, the ε-order viscous boundary layer $\bar{F}_1$ will appear if one of $\partial_3\mathrm{u}^0_1(t, x_{\shortparallel}, 0)$, $\partial_3\mathrm{u}^0_2(t, x_{\shortparallel}, 0)$ and $\partial_3 T^0(t, x_{\shortparallel}, 0)$ is nonzero. That means whether the main viscous boundary layer $\bar{F}_1$ appears depends only on the boundary properties of compressible Euler solution, and has no relation with the interior expansion F_1.

Remark 1.5. We point out that the zero-order boundary layer does not appear in the case of specular reflection boundary conditions. On the other hand, for the case of diffuse reflection boundary conditions, the zero-order boundary layer (Prandtl type boundary layer) must appear, which is a nonlinear and nonlocal system, see (3.197)–(3.202) in [44, Section 3.4.2] for its equations and boundary conditions. Such Prandtl type boundary layer makes the hydrodynamic limit of Boltzmann equation much harder than the specular reflection boundary case, and the problem will be considered in the future.

1.2.3. Knudsen Boundary Layer Expansion To construct the solution that satisfies the boundary condition at higher orders, we have to introduce the Knudsen boundary layer. Firstly, we define the new scaled normal coordinate as:

$$\eta := \frac{x_3}{\varepsilon^2}. \tag{1.37}$$

The Knudsen boundary layer expansion is defined as

$$\hat{F}^\varepsilon(t, x_{\shortparallel}, \eta) \sim \sum_{k=1}^{\infty} \varepsilon^k \hat{F}_k(t, x_{\shortparallel}, \eta, v).$$

Plugging $F^\varepsilon + \bar{F}^\varepsilon + \hat{F}^\varepsilon$ into (1.5) and comparing the order of ε, then using (1.7), (1.15), one obtains

$$
\begin{aligned}
&\frac{1}{\varepsilon}: \quad v_3 \frac{\partial \hat{F}_1}{\partial \eta} - \left[Q(\mu_0, \hat{F}_1) + Q(\hat{F}_1, \mu_0) \right] = 0, \\
&\varepsilon^0: \quad v_3 \frac{\partial \hat{F}_2}{\partial \eta} - \left[Q(\mu_0, \hat{F}_2) + Q(\hat{F}_2, \mu_0) \right] \\
&\qquad = Q(F_1^0 + \bar{F}_1^0, \hat{F}_1) + Q(\hat{F}_1, F_1^0 + \bar{F}_1^0) + Q(\hat{F}_1, \hat{F}_1), \\
&\qquad\qquad \vdots \\
&\varepsilon^k: \quad v_3 \frac{\partial \hat{F}_{k+2}}{\partial \eta} - \left[Q(\mu_0, \hat{F}_{k+2}) + Q(\hat{F}_{k+2}, \mu_0) \right] \\
&\quad = -\{\partial_t + v_{\shortparallel} \cdot \nabla_{\shortparallel}\} \hat{F}_k + \sum_{\substack{j+2l=k+2 \\ 1 \leqq l \leqq \mathfrak{b},\, j \geqq 1}} \frac{\eta^l}{l!} \left[Q(\partial_3^l \mu_0, \hat{F}_j) + Q(\hat{F}_j, \partial_3^l \mu_0) \right] \\
&\qquad + \sum_{\substack{i+j=k+2 \\ i,j \geqq 1}} \left[Q(F_i^0 + \bar{F}_i^0, \hat{F}_j) + Q(\hat{F}_j, F_i^0 + \bar{F}_i^0) + Q(\hat{F}_i, \hat{F}_j) \right] \\
&\qquad + \sum_{\substack{i+j+2l=k+2 \\ i,j \geqq 1, 1 \leqq l \leqq \mathfrak{b}}} \frac{\eta^l}{l!} \left[Q(\partial_3^l F_i^0, \hat{F}_j) + Q(\hat{F}_j, \partial_3^l F_i^0) \right] \\
&\qquad + \sum_{\substack{i+j+l=k+2 \\ i,j \geqq 1, 1 \leqq l \leqq \mathfrak{b}}} \frac{\eta^l}{l!} \left[Q(\partial_y^l \bar{F}_i^0, \hat{F}_j) + Q(\hat{F}_j, \partial_y^l \bar{F}_i^0) \right], \quad \text{for } k \geqq 1,
\end{aligned}
\tag{1.38}
$$

where we have used (1.16)–(1.17) and the Taylor expansion of $\bar{F}_i$

$$
\bar{F}_i(t, x_1, x_2, y, v) = \bar{F}_i^0 + \sum_{l=1}^{\mathfrak{b}} \frac{1}{l!} \partial_y^l \bar{F}_i^0 \cdot y^l + \frac{y^{\mathfrak{b}+1}}{(\mathfrak{b}+1)!} \partial_3^{\mathfrak{b}+1} \bar{\mathfrak{F}}_i,
$$

with

$$
\begin{aligned}
\partial_3^l \bar{F}_i^0 &:= (\partial_3^l \bar{F}_i)(t, x_1, x_2, 0, v), \\
\partial_3^{\mathfrak{b}+1} \bar{\mathfrak{F}}_i &:= (\partial_3^{\mathfrak{b}+1} \bar{F}_i)(t, x_1, x_2, \bar{\xi}_i, v),
\end{aligned}
\qquad \text{for } 0 \leqq l \leqq \mathfrak{b}
\tag{1.39}
$$

for some $\bar{\xi}_i \in [0, y]$.

It is noted that the Knudsen boundary layer (1.38) is in fact a steady problem with $(t, x_{\shortparallel})$ as parameters, and the well-posedness has already been obtained in [17,28] under some conditions on the source term and boundary condition. However, we shall use the existence results in [28] since the continuity and uniform estimate in $L^\infty_{\eta,v}$ is needed in the present paper.

1.3. Hilbert Expansion

Motivated by (3.184) and (3.189) in [44, Section 3.4.1], we consider the Boltzmann solution with the following Hilbert expansion with multiscales

$$F^{\varepsilon} = \mu(t,x,v) + \sum_{i=1}^{N} \varepsilon^{i} F_i(t,x,v) + \sum_{i=1}^{N} \varepsilon^{i} \bar{F}_i(t, x_{\shortparallel}, \frac{x_3}{\varepsilon}, v) + \sum_{i=1}^{N} \varepsilon^{i} \hat{F}_i(t, x_{\shortparallel}, \frac{x_3}{\varepsilon^2}, v) + \varepsilon^5 F_R^{\varepsilon}, \tag{1.40}$$

which, together with (1.7), (1.15) and (1.38), yields that the equation for reminder F_R^{ε} as

$$\partial_t F_R^{\varepsilon} + v\cdot\nabla_x F_R^{\varepsilon} - \frac{1}{\varepsilon^2}\{Q(\mu, F_R^{\varepsilon}) + Q(F_R^{\varepsilon}, \mu)\} = \varepsilon^3 Q(F_R^{\varepsilon}, F_R^{\varepsilon}) + \sum_{i=1}^{N} \varepsilon^{i-2}\{Q(F_i + \bar{F}_i + \hat{F}_i, F_R^{\varepsilon}) + Q(F_R^{\varepsilon}, F_i + \bar{F}_i + \hat{F}_i)\} + R^{\varepsilon} + \bar{R}^{\varepsilon} + \hat{R}^{\varepsilon}, \tag{1.41}$$

where

$$R^{\varepsilon} = -\varepsilon^{N-6}\{\partial_t + v\cdot\nabla_x\}(F_{N-1} + \varepsilon F_N) + \varepsilon^{N-6} \sum_{\substack{i+j\geqq N+1\\ 1\leqq i,j\leqq N}} \varepsilon^{i+j-N-1} Q(F_i, F_j), \tag{1.42}$$

$$\begin{aligned}
\bar{R}^{\varepsilon} = &-\varepsilon^{N-6}\{\partial_t + v_{\shortparallel}\cdot\nabla_{\shortparallel}\}(\bar{F}_{N-1} + \varepsilon \bar{F}_N) - \varepsilon^{N-6} v_3 \partial_y \bar{F}_N \\
&+ \varepsilon^{N-6} \sum_{\substack{j+l\geqq N+1\\ 1\leqq j\leqq N,\, 1\leqq l\leqq \mathfrak{b}}} \varepsilon^{l+j-N-1}\cdot \frac{y^l}{l!}\left[Q(\partial_3^l \mu_0, \bar{F}_j) + Q(\bar{F}_j, \partial_3^l \mu_0)\right] \\
&+ \varepsilon^{N-6} \sum_{\substack{i+j\geqq N+1\\ 1\leqq i,j\leqq N}} \varepsilon^{i+j-N-1}\left[Q(F_i^0, \bar{F}_j) + Q(\bar{F}_j, F_i^0) + Q(\bar{F}_i, \bar{F}_j)\right] \\
&+ \varepsilon^{N-6} \sum_{\substack{i+j+l\geqq N+1\\ 1\leqq i,j\leqq N,\, 1\leqq l\leqq \mathfrak{b}}} \varepsilon^{i+j+l-N-1}\cdot \frac{y^l}{l!}\left[Q(\partial_3^l F_i^0, \bar{F}_j) + Q(\bar{F}_j, \partial_3^l F_i^0)\right] \\
&+ \varepsilon^{\mathfrak{b}-5} \frac{y^{\mathfrak{b}+1}}{(\mathfrak{b}+1)!} \sum_{j=1}^{N} \varepsilon^{j-1}[Q(\partial_3^{\mathfrak{b}+1}\tilde{\mu}, \bar{F}_j) + Q(\bar{F}_j, \partial_3^{\mathfrak{b}+1}\tilde{\mu})] \\
&+ \varepsilon^{\mathfrak{b}-4} \frac{y^{\mathfrak{b}+1}}{(\mathfrak{b}+1)!} \sum_{i,j=1}^{N} \varepsilon^{i+j-2}\left[Q(\partial_3^{\mathfrak{b}+1}\mathfrak{F}_i, \bar{F}_j) + Q(\bar{F}_j, \partial_3^{\mathfrak{b}+1}\mathfrak{F}_i)\right],
\end{aligned} \tag{1.43}$$

and

$$
\begin{aligned}
\hat{R}^{\varepsilon} = &-\varepsilon^{N-6}\{\partial_t + v_{\shortparallel}\cdot\nabla_{\shortparallel}\}(\hat{F}_{N-1}+\varepsilon\hat{F}_N)\\
&+\varepsilon^{N-6}\sum_{\substack{j+2l\geqq N+1\\ 1\leqq j\leqq N,1\leqq l\leqq \mathfrak{b}}}\varepsilon^{j+2l-N-1}\cdot\frac{\eta^l}{l!}\big[Q(\partial_3^l\mu_0,\hat{F}_j)+Q(\hat{F}_j,\partial_3^l\mu_0)\big]\\
&+\varepsilon^{N-6}\sum_{\substack{i+j\geqq N+1\\ 1\leqq i,j\leqq N}}\varepsilon^{i+j-N-1}\big[Q(F_i^0+\bar{F}_i^0,\hat{F}_j)+Q(\hat{F}_j,F_i^0+\bar{F}_i^0)+Q(\hat{F}_i,\hat{F}_j)\big]\\
&+\varepsilon^{N-6}\sum_{\substack{i+j+2l\geqq N+1\\ 1\leqq i,j\leqq N,1\leqq l\leqq \mathfrak{b}}}\varepsilon^{i+j+2l-N-1}\cdot\frac{\eta^l}{l!}\big[Q(\partial_3^l F_i^0,\hat{F}_j)+Q(\hat{F}_j,\partial_3^l F_i^0)\big]\\
&+\varepsilon^{N-6}\sum_{\substack{i+j+l\geqq N+1\\ 1\leqq i,j\leqq N,1\leqq l\leqq \mathfrak{b}}}\varepsilon^{i+j+l-N-1}\cdot\frac{\eta^l}{l!}\big[Q(\partial_y^l \bar{F}_i^0,\hat{F}_j)+Q(\hat{F}_j,\partial_y^l \bar{F}_i^0)\big]\\
&+\varepsilon^{2\mathfrak{b}-4}\frac{\eta^{\mathfrak{b}+1}}{(\mathfrak{b}+1)!}\sum_{j=1}^{N}\varepsilon^{j-1}\big[Q(\partial_3^{\mathfrak{b}+1}\tilde{\mu},\hat{F}_j)+Q(\hat{F}_j,\partial_3^{\mathfrak{b}+1}\tilde{\mu})\big]\\
&+\varepsilon^{2\mathfrak{b}-3}\frac{\eta^{\mathfrak{b}+1}}{(\mathfrak{b}+1)!}\sum_{i,j=1}^{N}\varepsilon^{i+j-2}\big[Q(\partial_3^{\mathfrak{b}+1}\mathfrak{F}_i,\hat{F}_j)+Q(\hat{F}_j,\partial_3^{\mathfrak{b}+1}\mathfrak{F}_i)\big]\\
&+\varepsilon^{\mathfrak{b}-4}\frac{\eta^{\mathfrak{b}+1}}{(\mathfrak{b}+1)!}\sum_{i,j=1}^{N}\varepsilon^{i+j-2}\big[Q(\partial_3^{\mathfrak{b}+1}\bar{\mathfrak{F}}_i,\hat{F}_j)+Q(\hat{F}_j,\partial_3^{\mathfrak{b}+1}\bar{\mathfrak{F}}_i)\big],
\end{aligned}
\tag{1.44}
$$

where $\partial_3^l\mu_0$, $\partial_3^{\mathfrak{b}+1}\tilde{\mu}$, $\partial_3^l F_i^0$, $\partial_3^{\mathfrak{b}+1}\mathfrak{F}_i$ and $\partial_y^l\bar{F}_i^0$, $\partial_y^{\mathfrak{b}+1}\bar{\mathfrak{F}}_i$ are defined in (1.18), (1.39).

The main aim of the present paper is to establish the validity of the Hilbert expansion for the Boltzmann equation around the local Maxwellian μ determined by the compressible Euler equations (1.8), so it is natural to rewrite the remainder as

$$F_R^{\varepsilon} = \sqrt{\mu} f_R^{\varepsilon}. \tag{1.45}$$

To use the L^2-L^∞ framework [22], we introduce a global Maxwellian

$$\mu_M := \frac{1}{(2\pi T_M)^{3/2}}\exp\left\{-\frac{|v|^2}{2T_M}\right\},$$

where $T_M > 0$ satisfies the condition

$$T_M < \min_{x\in\mathbb{R}^3_+} T(t,x) \leqq \max_{x\in\mathbb{R}^3_+} T(t,x) < 2T_M. \tag{1.46}$$

By the assumption (1.46), one can easily deduce that there exists positive constant $C > 0$ such that for some $\frac{1}{2} < \alpha < 1$, the following holds:

$$\frac{1}{C}\mu_M \leqq \mu(t,x,v) \leqq C\mu_M^{\alpha}. \tag{1.47}$$

We further define

$$F_R^\varepsilon = \{1+|v|^2\}^{-\frac{\kappa}{2}} \sqrt{\mu_M} h_R^\varepsilon \equiv \frac{1}{w_\kappa(v)} \sqrt{\mu_M} h_R^\varepsilon, \tag{1.48}$$

with the velocity weight function

$$w_\kappa(v) := \{1+|v|^2\}^{\frac{\kappa}{2}}, \quad \text{for} \quad \kappa \geqq 0.$$

Theorem 1.6. *Let $\tau > 0$ be the life-span of smooth solution of compressible Euler equations (1.8). Let $\kappa \geqq 7$, $N \geqq 6$ and $\mathfrak{b} \geqq 5$. We assume the initial data*

$$F^\varepsilon(0,x,v) = \mu(0,x,v) + \sum_{i=1}^N \varepsilon^i \left\{ F_i(0,x,v) + \bar{F}_i(0, x_{\shortparallel}, \frac{x_3}{\varepsilon}, v) + \hat{F}_i(0, x_{\shortparallel}, \frac{x_3}{\varepsilon^2}, v) \right\}$$
$$+ \varepsilon^5 F_R^\varepsilon(0,x,v) \geqq 0,$$

and $F_i(0)$, $\bar{F}_i(0)$, $i = 1, \ldots, N$ satisfy the regularity and compatibility conditions described in Proposition 5.1 (see Remark 5.2 for details on the compatibility conditions), and

$$\Big\| \Big(\frac{F_R^\varepsilon}{\sqrt{\mu}} \Big)(0) \Big\|_{L^2_{x,v}} + \varepsilon^3 \Big\| (w_\kappa \frac{F_R^\varepsilon}{\sqrt{\mu_M}})(0) \Big\|_{L^\infty_{x,v}} < \infty.$$

Then there exists a small positive constants $\varepsilon_0 > 0$ such that IBVP of Boltzmann equation (1.5), (1.3) has a unique solution for $\varepsilon \in (0, \varepsilon_0]$ over the time interval $t \in [0, \tau]$ in the following form of expansion:

$$F^\varepsilon(t,x,v) = \mu(t,x,v) + \sum_{i=1}^N \varepsilon^i \left\{ F_i(t,x,v) + \bar{F}_i\left(t, x_{\shortparallel}, \frac{x_3}{\varepsilon}, v\right) + \hat{F}_i\left(t, x_{\shortparallel}, \frac{x_3}{\varepsilon^2}, v\right) \right\}$$
$$+ \varepsilon^5 F_R^\varepsilon(t,x,v) \geqq 0 \tag{1.49}$$

with

$$\sup_{t\in[0,\tau]} \left\{ \Big\| \frac{F_R^\varepsilon(t)}{\sqrt{\mu}} \Big\|_{L^2_{x,v}} + \varepsilon^3 \Big\| w_\kappa(v) \frac{F_R^\varepsilon(t)}{\sqrt{\mu_M}} \Big\|_{L^\infty_{x,v}} \right\} \leqq C(\tau) < \infty. \tag{1.50}$$

Here the functions $F_i(t,x,v)$, $\bar{F}_i(t, x_{\shortparallel}, y, v)$ and $\hat{F}_i(t, x_{\shortparallel}, \eta, v)$ are respectively the interior expansion, viscous and Knudsen boundary layers constructed in Proposition 5.1.

Remark 1.7. From (1.49)–(1.50) and the uniform estimates in Proposition 5.1, it is direct to check that

$$\sup_{t\in[0,\tau]} \left\{ \Big\| \Big(\frac{F^\varepsilon - \mu}{\sqrt{\mu}} \Big)(t) \Big\|_{L^2(\mathbb{R}^3_+ \times \mathbb{R}^3)} + \Big\| w_\kappa \Big(\frac{F^\varepsilon - \mu}{\sqrt{\mu_M}} \Big)(t) \Big\|_{L^\infty(\mathbb{R}^3_+ \times \mathbb{R}^3)} \right\} \leqq C\varepsilon \to 0.$$

Hence we have established the hydrodynamic limit from the Boltzmann equation to the compressible Euler system for the half-space problem.

Remark 1.8. For the initial data $F_i(0,x,v)$, $\bar{F}_i(0,x_{\shortparallel},y,v)$ and $\hat{F}_i(0,x_{\shortparallel},\eta,v)$, we only need to impose data on the macroscopic part of $F_i(0,x,v)$, and the part of the macroscopic part of viscous boundary layer $\bar{F}_i(0,x_{\shortparallel},y,v)$, and no conditions are needed on $\hat{F}_i(0,x_{\shortparallel},\eta,v)$, see Proposition 5.1 for more details. Also, from Proposition 5.1, we know that $\bar{F}_i$ decay algebraically with respect to y, and $\hat{F}_i$ decay exponentially with respect to η; and the decay estimates are crucial for us to close the estimate for $\bar{R}^\varepsilon$, $\hat{R}^\varepsilon$.

Remark 1.9. For the first order of viscous boundary layer $\bar{F}_1$, its boundary condition is closely related to the boundary value of compressible Euler solution, i.e., (see (5.9) for details)

$$\begin{cases}\partial_y\bar{u}_{1,i}(t,x_{\shortparallel},y)|_{y=0}=-\partial_3\mathfrak{u}_i^0(t,x_{\shortparallel},0), \quad i=1,2,\\ \partial_y\bar{\theta}(t,x_{\shortparallel},y)|_{y=0}=-3\partial_3T^0(t,x_{\shortparallel},0).\end{cases}\tag{1.51}$$

It is also noted that $\bar{F}_1$ does not interplay with F_1, see (1.24), (1.35)–(1.36) and (2.26). Hence, if one of $\partial_3\mathfrak{u}_1^0(t,x_{\shortparallel},0)$, $\partial_3\mathfrak{u}_2^0(t,x_{\shortparallel},0)$ and $\partial_3T^0(t,x_{\shortparallel},0)$ is nonzero, then the viscous boundary layer $\bar{F}_1$ must be nonzero. That means the ε-order viscous boundary layer $\bar{F}_1$ must be included in the Hilbert expansion. On the other hand, the first order of Knudsen boundary layer $\hat{F}_1$ does not appear, i.e., $\hat{F}_1\equiv 0$ (see (5.11) for details), and this is reasonable since the Knudsen boundary layer is used to mend the boundary condition at higher orders. Therefore the interplay of interior expansion, viscous and Knudsen boundary layers start from ε^2-order.

Remark 1.10. For non-flat domains, as pointed out in [47], one needs to modify the equation of expansion for boundary layers due to the non-trivial geometry.

1.4. Acoustic Limit

The acoustic system is the linearization of compressible Euler equations around a uniform fluid state, for instance, (1, 0, 1). After a suitable choice of units, the fluid fluctuations $(\varphi,\Phi,\vartheta)=(\varphi,\Phi_1,\Phi_2,\Phi_3,\vartheta)$ satisfies

$$\begin{cases}\partial_t\varphi+\mathrm{div}\Phi=0,\\ \partial_t\Phi+\nabla(\varphi+\vartheta)=0, \quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^3_+.\\ \frac{3}{2}\partial_t\vartheta+\mathrm{div}\Phi=0,\end{cases}\tag{1.52}$$

We impose (1.52) with the following initial and boundary data:

$$(\varphi,\Phi,\vartheta)(0,x)=(\varphi_0,\Phi_0,\vartheta_0)(x)\in H^{s_0}(\mathbb{R}^3_+), \quad \text{and} \quad \Phi_3(t,x)|_{x_3=0}=0.\tag{1.53}$$

Clearly, the IBVP (1.52)–(1.53) is a linear hyperbolic system with constant coefficients and characteristic boundary, and there exists a unique global smooth solution $(\varphi,\Phi,\vartheta)(t)\in H^{s_0}(\mathbb{R}^3_+)$. In fact, we can still use Lemma 3.1 with the Euler solution $(\rho,\mathfrak{u},T)=(1,0,1)$ in (3.1) and slightly different coefficients to obtain the global existence of smooth solution to IBVP (1.52)–(1.53). Moreover it holds that

$$\sup_{s\in[0,t]}\|(\varphi,\Phi,\vartheta)(s)\|_{H^{s_0}(\mathbb{R}^3_+)}\leqq C\big(t,\|(\varphi_0,\Phi_0,\vartheta_0)\|_{H^{s_0}(\mathbb{R}^3_+)}\big), \quad \forall t>0.$$

On the other hand, the acoustic system (1.52) can also be formally derived from the Boltzmann equation (1.5) by letting

$$F^{\varepsilon} = \tilde{\mu}_M + \delta G^{\varepsilon}, \tag{1.54}$$

where $\tilde{\mu}_M$ is the global Maxwellian determined by the uniform state (1, 0, 1), i.e.,

$$\tilde{\mu}_M = \frac{1}{(2\pi)^{\frac{3}{2}}} \exp\left(-\frac{|v|^2}{2}\right).$$

The fluctuation amplitude δ is a function of ε satisfying

$$\delta \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

For instance we can take

$$\delta = \varepsilon^{\varpi} \quad \text{for} \quad \varpi > 0.$$

With the above scalings, G^{ε} formally converges to

$$G := \left\{\varphi + v \cdot \Phi + \frac{|v|^2 - 3}{2}\vartheta\right\} \tilde{\mu}_M, \quad \text{as } \varepsilon \to 0, \tag{1.55}$$

where $(\varphi, \Phi, \vartheta)$ is the solution of acoustic system (1.52), see [2,4] for detailed formal derivation.

One of the purpose of present paper is to establish the acoustic limit for initial boundary value problem of Boltzmann equation over half-space $\mathbb{R}^3_+$. We use δ to denote the fluctuation amplitude and assume that

$$\frac{\varepsilon}{\delta} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Theorem 1.11. *Let $\tau > 0$ be any given time. Leting $\mu^{\delta}(0, x, v)$ be the local Maxwellian with initial datum $1 + \delta\varphi_0$, $\delta\Phi_0$ and $1 + \delta\vartheta_0$,*

$$\mu^{\delta}(0, x, v) = \frac{1 + \delta\varphi_0(x)}{[2\pi(1 + \delta\vartheta_0(x))]^{\frac{3}{2}}} \exp\left\{-\frac{|v - \delta\Phi_0(x)|^2}{2(1 + \delta\vartheta_0(x))}\right\},$$

where $(\varphi_0, \Phi_0, \vartheta_0)$ is the initial data given in (1.53). We assume that the conditions in Theorem 1.6 hold, and rewrite the corresponding Hilbert expansion established in Theorem 1.6 as

$$F^{\varepsilon}(t, x, v) = \mu^{\delta}(t, x, v) + \sum_{i=1}^{N} \varepsilon^i \left\{F_i(t, x, v) + \bar{F}_i\left(t, x_{\shortparallel}, \frac{x_3}{\varepsilon}, v\right) + \hat{F}_i\left(t, x_{\shortparallel}, \frac{x_3}{\varepsilon^2}, v\right)\right\}$$
$$+ \varepsilon^5 F_R^{\varepsilon}(t, x, v) \geqq 0.$$

Then there exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0]$ and $\delta \in (0, \delta_0]$, there exists a constant $C > 0$ so that

$$\sup_{0 \leqq t \leqq \tau} \left\{\|G^{\varepsilon}(t) - G(t)\|_{L^{\infty}(\mathbb{R}^3_+ \times \mathbb{R}^3)} + \|G^{\varepsilon}(t) - G(t)\|_{L^2(\mathbb{R}^3_+ \times \mathbb{R}^3)}\right\} \leqq C\{\delta + \frac{\varepsilon}{\delta}\},$$

where $\frac{\varepsilon}{\delta} \to 0$ as $\varepsilon \to 0$, and G^{ε} and G are defined in (1.54) and (1.55) respectively. The constant $C > 0$ here depends only on τ and initial data $\|(\varphi_0, \Phi_0, \vartheta_0)\|_{H^{s_0}(\mathbb{R}^3_+)}$.

We now briefly comment on the analysis of the present paper. For the Hilbert expansion of Boltzmann equation over the half-space $x \in \mathbb{R}^3_+$ with specular reflection boundary conditions, in general, the viscous and Knudsen boundary layers will appear. To solve the interior expansion, viscous and Knudsen boundary layers, we need to determine the boundary conditions so that each of them is well-posed. We notice that the Knudsen boundary layer (1.38) is indeed a steady problem with $(t, x_{\shortparallel}) \in [0, \tau] \times \mathbb{R}^2$ as parameters. From [28], we know that Knudsen boundary layer is well-posed in weighted $L^\infty_{\eta,v}$-space under the conditions (2.11)–(2.12), see Lemma 2.5 for details. In particular the weight with respect to normal variable η can grow exponentially, which is important for us to close the remainder estimate. In general, the source term on the right hand side of (1.38) does not satisfy (2.11). We use the idea of [1] to introduce a correct function $\hat{f}_{i,1}$ to overcome this difficulty, see Lemma 2.9 for details. To determine the boundary conditions for $F_i, \bar{F}_i, \hat{F}_i$, $i = 1, \ldots, N$, we require that each $F_i + \bar{F}_i + \hat{F}_i$ satisfies the specular reflection boundary conditions, so that together with (2.12), we can finally obtain the boundary conditions for interior expansion and viscous boundary layer, see Section 2.3 for details. Here we point out that F_i, $\bar{F}_i$ and $\hat{F}_i$ may not satisfy the specular reflection boundary conditions alone.

For the existence of interior expansion, we have to consider a linear hyperbolic system with characteristic boundary, see Lemma 2.2. We are able to construct local in time solution with desired energy estimate in the presence of boundary conditions in Lemma 3.1. For the existence of viscous boundary layer, we note that it is involved to a linear hyperbolic system with partial viscosity (only in the normal direction) and linear growth coefficients, which is not a standard linear parabolic system, see Lemma 1.1. By using the energy estimate and several cutoff approximate arguments, we establish its well-posedness in a weighted Sobolev space with algebraically growth weight of y, see Lemma 4.1 for details.

With above preparations, then we can establish the well-posedness of F_i, $\bar{F}_i$ and $\hat{F}_i$, $i = 1, \ldots, N$ and obtain the uniform estimate, see Proposition 5.1. Now with the help of uniform estimates in Proposition 5.1, we can use the L^2-L^∞ framework [20,23] to obtain the uniform estimate for remainder term F_R^ε, and hence obtain the solution of Boltzmann equation in the form of (1.49).

The paper is organized as follows: in Section 2, we reformulate the interior expansion and Knudsen boundary layers. Also we derive the corresponding boundary conditions so that the formulations of interior expansion, the viscous and Knudsen boundary layers are all well-posed. Section 3 is devoted to an existence theory for a linear hyperbolic system with characteristic boundary, which are used to construct interior expansion F_i. In Section 4, to construct the existence of viscous boundary layer $\bar{F}_i$, we establish an existence theory of IBVP for a linear parabolic system with degenerate viscosity and linear growth coefficients in a weighted Sobolev space. In Section 5, we construct solutions of interior expansion, the viscous and Knudsen boundary layers. Theorems 1.6 and 1.11 are proved in Sections 6 and 7, respectively. In "Appendix A" we give the proof of Lemma 1.1; we present a short proof of Lemma 2.5 in "Appendix B"; and we show some anisotropic trace estimates in "Appendix C".

Notations. Throughout this paper, C denotes a generic positive constant and vary from line to line, and $C(a), C(b), \dots$ denote the generic positive constants depending on $a,\ b, \dots$, respectively, which also may vary from line to line. We use $\langle\cdot,\cdot\rangle$ to denote the standard L^2 inner product in $\mathbb{R}^3_v$. $\|\cdot\|_{L^2}$ denotes the standard $L^2(\mathbb{R}^3_+\times\mathbb{R}^3_v)$-norm, and $\|\cdot\|_{L^\infty}$ denotes the $L^\infty(\mathbb{R}^3_+\times\mathbb{R}^3_v)$-norm.

2. Reformulations of Expansions and Boundary Conditions

2.1. Reformulation of Interior Expansion

Firstly we introduce the existence result on the compressible Euler equations.

Lemma 2.1. *Let $s_0 \geqq 3$ be some positive integer. Consider the IBVP of compressible Euler equations (1.8)–(1.10). Choose $\delta_1 > 0$ so that for any $\delta\in(0,\delta_1]$, the positivity of ρ_0 and T_0 is guaranteed. Then for $\delta\in(0,\delta_1]$, there is a family of classical solutions $(\rho^\delta,\mathfrak{u}^\delta,T^\delta)\in C([0,\tau^\delta];H^{s_0}(\mathbb{R}^3_+))\cap C^1([0,\tau^\delta];H^{s_0-1}(\mathbb{R}^3_+))$ of IBVP (1.8)–(1.10) such that $\rho^\delta>0$ and $T^\delta>0$, and the following estimate holds:*

$$\|(\rho^\delta-1,\mathfrak{u}^\delta,T^\delta-1)\|_{C([0,\tau^\delta];H^{s_0}(\mathbb{R}^3_+))\cap C^1([0,\tau^\delta];H^{s_0-1}(\mathbb{R}^3_+))}\leqq C_0. \tag{2.1}$$

The life-span τ^δ have the following lower bound

$$\tau^\delta\geqq\frac{C_1}{\delta}. \tag{2.2}$$

The constant C_0, C_1 are independent of δ, depending only on the H^{s_0}-norm of $(\varphi_0,\Phi_0,\vartheta_0)$.

We refer [42] for the local existence of the IBVP of compressible Euler equation (1.8)–(1.10); see also [10] and the references cited therein. We point out that the local existence result in [42] is for smooth bounded domain with C^∞ boundary, but the method can also be applied to our half-space problem. On the other hand, we can also obtain (2.1)–(2.2) by using similar arguments as to those in Lemma 3.1, below.

Throughout this paper, we will drop the superscript of $(\rho^\delta,\mathfrak{u}^\delta,T^\delta)$ when no confusion arises. To derive the estimates of interior expansion, i.e., $F_1(t,x,v),\dots,$ $F_N(t,x,v)$, we firstly present a useful lemma which will be used to estimate the bound of linear terms.

Lemma 2.2. [21] *Let $(\rho,\mathfrak{u},T)(t)$ be some smooth solution of compressible Euler equations (1.8). For each given nonnegative integer k, assume F_k's are found. Then the microscopic part of F_{k+1} is determined through the follow equation for F_k in (1.7):*

$$\{\mathbf{I}-\mathbf{P}\}\left(\frac{F_{k+1}}{\sqrt{\mu}}\right)=\mathbf{L}^{-1}\left(-\frac{\{\partial_t+v\cdot\nabla_x\}F_{k-1}-\sum_{\substack{i+j=k+1\\ i,j\geqq1}}Q(F_i,F_j)}{\sqrt{\mu}}\right) \tag{2.3}$$

for $k \geqq 0$, where we define $F_{-1} = 0$ for the consistency of notation. For the macroscopic part, $\rho_{k+1}, u_{k+1}, \theta_{k+1}$ satisfy the following:

$$
\begin{aligned}
&\partial_t \rho_{k+1} + \operatorname{div}_x(\rho u_{k+1} + \rho_{k+1}\mathfrak{u}) = 0,\\
&\rho\Big\{\partial_t u_{k+1} + u_{k+1}\cdot\nabla_x \mathfrak{u} + \mathfrak{u}\cdot\nabla_x u_{k+1}\Big\} - \frac{\rho_{k+1}}{\rho}\nabla_x(\rho T) + \nabla_x\left(\frac{\rho\theta_{k+1} + 3T\rho_{k+1}}{3}\right) = \mathfrak{f}_k,\\
&\rho\Big\{\partial_t\theta_{k+1} + \frac{2}{3}(\theta_{k+1}\operatorname{div}_x \mathfrak{u} + 3T\operatorname{div}_x u_{k+1}) + \mathfrak{u}\cdot\nabla_x\theta_{k+1} + 3u_{k+1}\cdot\nabla_x T\Big\} = \mathfrak{g}_k,
\end{aligned} \tag{2.4}
$$

with

$$
\begin{aligned}
\mathfrak{f}_{k,i} &= -\sum_{j=1}^{3}\partial_{x_j}\left(T\int_{\mathbb{R}^3}\mathcal{A}_{i,j}\frac{F_{k+1}}{\sqrt{\mu}}\,dv\right),\\
\mathfrak{g}_k &= -\sum_{i=1}^{3}\partial_{x_i}\left(2T^{\frac{3}{2}}\int_{\mathbb{R}^3}\mathcal{B}_i\frac{F_{k+1}}{\sqrt{\mu}}\,dv + \sum_{j=1}^{3}2\mathfrak{u}_j T\int_{\mathbb{R}^3}\mathcal{A}_{i,j}\frac{F_{k+1}}{\sqrt{\mu}}\,dv\right) - 2\mathfrak{u}\cdot\mathfrak{f}_k,
\end{aligned} \tag{2.5}
$$

where the $\mathcal{A}_{ij}$ and $\mathcal{B}_i$ are the Burnett functions defined (1.25), and we have used the subscript k for the source terms $\mathfrak{f}_k$ and $\mathfrak{g}_k$ in order to emphasize that the right hand side depends only on F_i's for $0 \leqq i \leqq k$.

Remark 2.3. To solve (1.7), it is equivalent to solve the linear hyperbolic system (2.4). Since it is an initial boundary value problem, we still need to impose a suitable boundary condition to (2.4). In fact, to ensure each $F_k + \bar{F}_k + \hat{F}_k$ satisfies the specular reflection boundary conditions and Knudsen layer are solvable, we can not impose boundary data of (2.4) arbitrarily. It is a very technical process to determine the boundary condition and we will show the details in Section 2.3 below.

Remark 2.4. The original version of Lemma 2.2 in [21] is for the Hilbert expansion of Vlasov-Poisson-Boltzmann equations, and one can obtain Lemma 2.2 by dropping the electric field. Lemma 2.2 is slightly different from the original version in [21] because we also consider the orders of ε^{2k-1}, but the proof is very similar, so we omit the details for simplicity of presentation. Noting $\mathcal{A}_{i,j}$ and $\mathcal{B}_i$ are microscopic functions, the source term $\mathfrak{f}_k$ and $\mathfrak{g}_k$ depend only on the microscopic part $(\mathbf{I}-\mathbf{P})(\frac{F_{k+1}}{\sqrt{\mu}})$ and hence depend only on F_i's for $0 \leqq i \leqq k$.

2.2. Reformulation of Knudsen Boundary Layer

Define $\hat{f}_k := \dfrac{\hat{F}_k}{\sqrt{\mu_0}}$, then we can rewrite (1.38) as

$$
v_3\frac{\partial \hat{f}_k}{\partial\eta} + \mathbf{L}_0\hat{f}_k = \hat{S}_k, \quad k \geqq 1, \tag{2.6}
$$

where $\hat{S}_k := \hat{S}_{k,1} + \hat{S}_{k,2}(k \geqq 1)$ with

$$
\hat{S}_{k,1} = -\mathbf{P_0}\left\{\frac{\{\partial_t + v_{\shortparallel}\cdot\nabla_{\shortparallel}\}\hat{F}_{k-2}}{\sqrt{\mu_0}}\right\}, \tag{2.7}
$$

$$
\begin{aligned}
\hat{S}_{k,2} = & \sum_{\substack{i+j=k\\ i,j\geqq 1}} \frac{1}{\sqrt{\mu_0}}\big[Q(F_i^0+\bar{F}_i^0,\sqrt{\mu_0}\hat{f}_j)+Q(\sqrt{\mu_0}\hat{f}_j,F_i^0+\bar{F}_i^0)\\
& +Q(\sqrt{\mu_0}\hat{f}_i,\sqrt{\mu_0}\hat{f}_j)\big]\\
& +\sum_{\substack{j+2l=k\\ 1\leqq l\leqq \mathfrak{b},\, j\geqq 1}} \frac{\eta^l}{l!}\frac{1}{\sqrt{\mu_0}}\big[Q(\partial_3^l\mu_0,\sqrt{\mu_0}\hat{f}_j)+Q(\sqrt{\mu_0}\hat{f}_j,\partial_3^l\mu_0)\big]\\
& +\sum_{\substack{i+j+2l=k\\ i,j\geqq 1, 1\leqq l\leqq \mathfrak{b}}} \frac{\eta^l}{l!}\frac{1}{\sqrt{\mu_0}}\big[Q(\partial_3^l F_i^0,\sqrt{\mu_0}\hat{f}_j)+Q(\sqrt{\mu_0}\hat{f}_j,\partial_3^l F_i^0)\big]\\
& +\sum_{\substack{i+j+l=k\\ i,j\geqq 1, 1\leqq l\leqq \mathfrak{b}}} \frac{\eta^l}{l!}\frac{1}{\sqrt{\mu_0}}\big[Q(\partial_y^l \bar{F}_i^0,\sqrt{\mu_0}\hat{f}_j)+Q(\sqrt{\mu_0}\hat{f}_j,\partial_y^l \bar{F}_i^0)\big]\\
& -(\mathbf{I}-\mathbf{P_0})\left\{\frac{\{\partial_t+v_{\shortparallel}\cdot\nabla_{\shortparallel}\}\hat{F}_{k-2}}{\sqrt{\mu_0}}\right\}.
\end{aligned} \tag{2.8}
$$

Here we have used the notation $\hat{F}_{-1}=\hat{F}_0=0$ for simplicity of presentation. It is easy to notice that $\hat{S}_{k,1}\in\mathcal{N}_0$, $\hat{S}_{k,2}\in\mathcal{N}_0^{\perp}$ and

$$
\hat{S}_1=\hat{S}_{1,1}=\hat{S}_{1,2}=\hat{S}_{2,1}=0. \tag{2.9}
$$

For later use, we introduce a result on the existence of solution to the Knudsen boundary layer problem with a perturbed specular reflection boundary conditions. Consider the following half-space linear problem

$$
\begin{cases}
v_3\partial_\eta f+\mathbf{L}_0 f=S(t,x_{\shortparallel},\eta,v),\\
f(t,x_{\shortparallel},0,v_{\shortparallel},v_3)|_{v_3>0}=f(t,x_{\shortparallel},0,v_{\shortparallel},-v_3)+f_b(t,x_{\shortparallel},v_{\shortparallel},-v_3),\\
\lim_{\eta\to\infty} f(t,x_{\shortparallel},\eta,v)=0,
\end{cases} \tag{2.10}
$$

where $\eta\in\mathbb{R}_+$ and $(t,x_{\shortparallel})\in[0,\tau]\times\mathbb{R}^2$. In fact, we think $(t,x_{\shortparallel})\in[0,\tau]\times\mathbb{R}^2$ to be parameters in (2.10). The function $f_b(t,x_{\shortparallel},v)$ is defined only for $v_3<0$, and we always assume that it is extended to be 0 for $v_3>0$.

Lemma 2.5. [28] *Let* $0\leqq\mathfrak{a}<\frac{1}{2}$ *and* $\kappa\geqq 3$. *For each* $(t,x_{\shortparallel})\in[0,\tau]\times\mathbb{R}^2$, *we assume that*

$$
S\in\mathcal{N}_0^{\perp} \text{ and } \|w_\kappa\mu_0^{-\mathfrak{a}}f_b(t,x_{\shortparallel},0,\cdot)\|_{L_v^\infty}+\|\nu^{-1}w_\kappa\mu_0^{-\mathfrak{a}}e^{\zeta_0\eta}S(t,x_{\shortparallel},\cdot,\cdot)\|_{L^\infty_{\eta,v}}<\infty, \tag{2.11}
$$

for some positive constant $\zeta_0 > 0$, and

$$\begin{cases} \int_{\mathbb{R}^3} v_3\, f_b(t, x_{\shortparallel}, v)\sqrt{\mu_0}\, dv & \equiv 0, \\ \int_{\mathbb{R}^3} (v_1 - \mathfrak{u}_1^0) v_3\, f_b(t, x_{\shortparallel}, v)\sqrt{\mu_0}\, dv & \equiv 0, \\ \int_{\mathbb{R}^3} (v_2 - \mathfrak{u}_2^0) v_3\, f_b(t, x_{\shortparallel}, v)\sqrt{\mu_0}\, dv & \equiv 0, \\ \int_{\mathbb{R}^3} |v - \mathfrak{u}^0|^2 v_3\, f_b(t, x_{\shortparallel}, v)\sqrt{\mu_0}\, dv & \equiv 0. \end{cases} \tag{2.12}$$

Then the boundary value problem (2.10)–(2.12) *has a unique solution f satisfying*

$$\begin{aligned} &\|w_\kappa \mu_0^{-\mathfrak{a}} e^{\zeta\eta} f(t, x_{\shortparallel}, \cdot, \cdot)\|_{L^\infty_{\eta,v}} + \|w_\kappa \mu_0^{-\mathfrak{a}} f(t, x_{\shortparallel}, 0, \cdot)\|_{L^\infty_v} \\ &\quad \leqq \frac{C}{\zeta_0 - \zeta}\Big(\|w_\kappa \mu_0^{-\mathfrak{a}} f_b(t, x_{\shortparallel}, 0, \cdot)\|_{L^\infty_v} + \|\nu^{-1} w_\kappa \mu_0^{-\mathfrak{a}} e^{\zeta_0\eta} S(t, x_{\shortparallel}, \cdot, \cdot)\|_{L^\infty_{\eta,v}}\Big), \end{aligned} \tag{2.13}$$

for all $(t, x_{\shortparallel}) \in [0, \tau] \times \mathbb{R}^2$, where $C > 0$ is a positive constant independent of $(t, x_{\shortparallel})$, and ζ is any positive constant such that $0 < \zeta < \zeta_0$. Moreover, if S is continuous in $(t, x_{\shortparallel}, \eta, v) \in [0, \tau] \times \mathbb{R}^2 \times \mathbb{R}_+ \times \mathbb{R}^3$ and f_b is continuous in $(t, x_{\shortparallel}, v_{\shortparallel}, -v_3) \in [0, \tau] \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}_+$, then the solution f is continuous away from the grazing set $[0, \tau] \times \gamma_0$.

Remark 2.6. A sketch proof of Lemma 2.5 is presented in "Appendix B", see [28, Section 3] for details.

Remark 2.7. As indicated in [24], in general, it is hard to obtain the normal derivatives estimates for the boundary value problem (2.10). Fortunately, it is easy to obtain the tangential and time derivatives estimates for the solution of (2.10), i.e.,

$$\begin{aligned} &\sum_{i+j\leqq r} \|w_\kappa \mu_0^{-\mathfrak{a}} e^{\zeta\eta} \partial_t^i \nabla_{\shortparallel}^j f(t, x_{\shortparallel}, \cdot, \cdot)\|_{L^\infty_{\eta,v}} + \|w_\kappa \mu_0^{-\mathfrak{a}} \partial_t^i \nabla_{\shortparallel}^j f(t, x_{\shortparallel}, 0, \cdot)\|_{L^\infty_v} \\ &\quad \leqq \frac{C}{\zeta_0 - \zeta} \sum_{i+j\leqq r} \Big\{\|w_\kappa \mu_0^{-\mathfrak{a}} \partial_t^i \nabla_{\shortparallel}^j f_b(t, x_{\shortparallel}, \cdot)\|_{L^\infty_v} + \|\nu^{-1} w_\kappa \mu_0^{-\mathfrak{a}} e^{\zeta_0\eta} \partial_t^i \nabla_{\shortparallel}^j S\|_{L^\infty_{\eta,v}}\Big\}, \end{aligned} \tag{2.14}$$

provided the right hand side of (2.14) is bounded. And such an estimate (2.14) is enough for us to establish the Hilbert expansion. To prove the estimate (2.14), we study the equation of $\partial_t^i \nabla_{\shortparallel}^j (\sqrt{\mu_0} f)$. It is direct to check that the new source term and boundary perturbation term satisfy the solvability conditions in Lemma 2.5, hence one can obtain the estimate for $\partial_t^i \nabla_{\shortparallel}^j (\sqrt{\mu_0} f)$ by applying Lemma 2.5, therefore (2.14) follows immediately.

Moreover, taking $L^\infty_{x_{\shortparallel}} \cap L^2_{x_{\shortparallel}}$ over (2.14), one obtains

$$\sum_{i+j\leqq r} \sup_{t\in[0,\tau]} \Big\{\|w_\kappa \mu_0^{-\mathfrak{a}} e^{\zeta\eta} \partial_t^i \nabla_{\shortparallel}^j f(t)\|_{L^\infty_{x_{\shortparallel},\eta,v} \cap L^2_{x_{\shortparallel}} L^\infty_{\eta,v}}$$

$$
\begin{aligned}
&+ \|w_\kappa \mu_0^{-\mathfrak{a}} \partial_t^i \nabla_{\shortparallel}^j f(t,\cdot,0,\cdot)\|_{L^\infty_{x_{\shortparallel},v} \cap L^2_{x_{\shortparallel}} L^\infty_v} \Big\} \\
&\leqq \frac{C}{\zeta_0 - \zeta} \sup_{t\in[0,\tau]} \Big\{ \sum_{i+j\leqq r} \Big\{ \|w_\kappa \mu_0^{-\mathfrak{a}} \partial_t^i \nabla_{\shortparallel}^j f_b(t)\|_{L^\infty_{x_{\shortparallel},v} \cap L^2_{x_{\shortparallel}} L^\infty_v} \\
&+ \sum_{i+j\leqq r} \|\nu^{-1} w_\kappa \mu_0^{-\mathfrak{a}} e^{\zeta_0 \eta} \partial_t^i \nabla_{\shortparallel}^j S(t)\|_{L^\infty_{x_{\shortparallel},\eta,v} \cap L^2_{x_{\shortparallel}} L^\infty_{\eta,v}} \Big\}.
\end{aligned}
\tag{2.15}
$$

Remark 2.8. Golse, Perthame and Sulem [17] have proved an existence result for (2.10) in the space $\int_{\mathbb{R}_+\times\mathbb{R}^3} (1+|v|)e^{2\zeta\eta} f^2 \,\mathrm{d}v\mathrm{d}\eta + \int_{\mathbb{R}^3} \|e^{\zeta\eta} f\|^2_{L^\infty_\eta} \mathrm{d}v$. In the present paper, since the continuity and weighted $L^\infty_{\eta,v}$ estimate are needed, so the second and third authors of present paper proved the well-posedness of (2.10) in the new functional space, see [28].

Since the source term $S \in \mathcal{N}_0^\perp$ in Lemma 2.5 is demanded, but $\hat{S}_k \notin \mathcal{N}_0^\perp (k \geqq 3)$ in general, i.e., $\hat{S}_{k,1} \neq 0$. Hence, to solve (2.6), we need to cancel the term $\hat{S}_{k,1}$. We assume that

$$
\hat{S}_{k,1} = \Big\{ \hat{a}_k + \hat{b}_k \cdot (v - \mathfrak{u}^0) + \hat{c}_k |v - \mathfrak{u}^0|^2 \Big\} \sqrt{\mu_0}, \tag{2.16}
$$

where $(\hat{a}_k, \hat{b}_k, \hat{c}_k) = (\hat{a}_k, \hat{b}_k, \hat{c}_k)(t, x_{\shortparallel}, \eta)$. By similar arguments as to those in [1], we have the next lemma. The details of proof are omitted for simplicity of presentation.

Lemma 2.9. *For $(\hat{a}_k, \hat{b}_k, \hat{c}_k)$ defined in (2.16), we assume that*

$$
\lim_{\eta\to\infty} e^{\zeta\eta} |(\hat{a}_k, \hat{b}_k, \hat{c}_k)(t, x_{\shortparallel}, \eta)| = 0
$$

for some positive constant $\zeta > 0$. Then there exists a function

$$
\begin{aligned}
\hat{f}_{k,1} = \Big\{ &\hat{A}_k v_3 + \hat{B}_{k,1} v_3 (v_1 - \mathfrak{u}_1^0) + \hat{B}_{k,2} v_3 (v_2 - \mathfrak{u}_2^0) \\
&+ \hat{B}_{k,3} + \hat{C}_k v_3 |v - \mathfrak{u}^0|^2 \Big\} \sqrt{\mu_0},
\end{aligned}
\tag{2.17}
$$

such that

$$
v_3 \partial_\eta \hat{f}_{k,1} - \hat{S}_{k,1} \in \mathcal{N}_0^\perp,
$$

where

$$
\begin{aligned}
\hat{A}_k(t, x_{\shortparallel}, \eta) &= -\int_\eta^\infty \Big(\frac{2}{T^0} \hat{a}_k + 3\hat{c}_k \Big)(t, x_{\shortparallel}, s) ds, \\
\hat{B}_{k,i}(t, x_{\shortparallel}, \eta) &= -\int_\eta^\infty \frac{1}{T^0} \hat{b}_{k,i}(t, x_{\shortparallel}, s) ds, \quad i = 1, 2, \\
\hat{B}_{k,3}(t, x_{\shortparallel}, \eta) &= -\int_\eta^\infty \hat{b}_{k,3}(t, x_{\shortparallel}, s) ds, \\
\hat{C}_k(t, x_{\shortparallel}, \eta) &= \frac{1}{5(T^0)^2} \int_\eta^\infty \hat{a}_k(t, x_{\shortparallel}, s) ds.
\end{aligned}
\tag{2.18}
$$

Moreover it holds that

$$|v_3\partial_\eta \hat{f}_{k,1} - \hat{S}_{k,1}| \leqq C|(\hat{a}_k, \hat{b}_k, \hat{c}_k)(t, x_{\shortparallel}, \eta)|(1+|v|)^4\sqrt{\mu_0},$$

and

$$|\hat{f}_{k,1}(t, x_{\shortparallel}, \eta, v)| \leqq C(1+|v|)^3\sqrt{\mu_0}\int_\eta^\infty |(\hat{a}_k, \hat{b}_k, \hat{c}_k)| \to 0 \text{ as } \eta \to \infty.$$

Remark 2.10. It is very important to note that $\hat{S}_{k,1}$ depends only on $\hat{f}_{k-2}$, which is already a known function when we consider the existence of $\hat{f}_k$. Thus $\hat{f}_{k,1}$ (or equivalent $(\hat{A}_k, \hat{B}_k, \hat{C}_k)$) is determined by $\hat{f}_{k-2}$. On the other hand, since $\hat{S}_{1,1} = \hat{S}_{2,1} = 0$, one has that

$$\hat{f}_{1,1} = \hat{f}_{2,1} \equiv 0, \tag{2.19}$$

which yields that $(\hat{A}_1, \hat{B}_1, \hat{C}_1) \equiv (0,0,0)$ and $(\hat{A}_2, \hat{B}_2, \hat{C}_2) \equiv (0,0,0)$.

Now we consider $\hat{f}_{k,2}$ satisfying

$$v_3\frac{\partial \hat{f}_{k,2}}{\partial\eta} + \mathbf{L}_0\hat{f}_{k,2} = \hat{S}_{k,2} - \mathbf{L}_0\hat{f}_{k,1} - \left(v_3\partial_\eta \hat{f}_{k,1} - \hat{S}_{k,1}\right) \in \mathcal{N}_0^\perp, \tag{2.20}$$

which can be solved by Lemma 2.5. Then it is easy to check that

$$\hat{f}_k := \hat{f}_{k,1} + \hat{f}_{k,2}, \tag{2.21}$$

is a solution of (2.6).

2.3. Boundary Conditions

To construct the solutions for interior expansion, viscous and Knudsen boundary layers, the remain problem is to determine suitable boundary conditions for well-posedness. As mentioned in Remark 2.3, we require that each $F_k + \bar{F}_k + \hat{F}_k$ satisfies the specular reflection boundary conditions, i.e.,

$$(f_k + \bar{f}_k + \hat{f}_k)(t, x_{\shortparallel}, 0, v_{\shortparallel}, v_3) = (f_k + \bar{f}_k + \hat{f}_k)(t, x_{\shortparallel}, 0, v_{\shortparallel}, -v_3),$$

which, together with (2.21), yields

$$\begin{aligned} &\hat{f}_{k,2}(t, x_{\shortparallel}, 0, v_{\shortparallel}, v_3)|_{v_3>0} \\ &\quad = \hat{f}_{k,2}(t, x_{\shortparallel}, 0, v_{\shortparallel}, -v_3) + [f_k + \bar{f}_k + \hat{f}_{k,1}](t, x_{\shortparallel}, 0, v_{\shortparallel}, -v_3) \\ &\qquad - [f_k + \bar{f}_k + \hat{f}_{k,1}](t, x_{\shortparallel}, 0, v_{\shortparallel}, v_3). \end{aligned} \tag{2.22}$$

For notational simplicity, we denote

$$\hat{g}_k(t, x_{\shortparallel}, v_{\shortparallel}, v_3) = \begin{cases} 0, \quad v_3 > 0, \\ [f_k + \bar{f}_k + \hat{f}_{k,1}](t, x_{\shortparallel}, 0, v_{\shortparallel}, v_3) \\ \qquad -[f_k + \bar{f}_k + \hat{f}_{k,1}](t, x_{\shortparallel}, 0, v_{\shortparallel}, -v_3), \quad v_3 < 0. \end{cases} \tag{2.23}$$

On the other hand, we impose the far field boundary condition

$$\lim_{\eta\to\infty} \hat{f}_{k,2}(t, x_{\shortparallel}, \eta, v) = 0. \tag{2.24}$$

Noting from Lemma 2.5, to solve (2.20), (2.22)–(2.23) and (2.24), we need $\hat{g}_k$ to satisfy (2.12), i.e.,

$$\begin{aligned}\int_{\mathbb{R}^3} v_3 \hat{g}_k \sqrt{\mu_0}\, \mathrm{d}v &= \int_{\mathbb{R}^3} (v_1 - \mathrm{u}_1^0) v_3 \hat{g}_k \sqrt{\mu_0}\, \mathrm{d}v = \int_{\mathbb{R}^3} (v_2 - \mathrm{u}_2^0) v_3 \hat{g}_k \sqrt{\mu_0}\, \mathrm{d}v \\ &= \int_{\mathbb{R}^3} v_3 |v - \mathrm{u}^0|^2 \hat{g}_k \sqrt{\mu_0}\, \mathrm{d}v = 0,\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\int_{\mathbb{R}^3} v_3 \sqrt{\mu_0} \mathbf{P}_0[f_k + \bar{f}_k](t, x_{\shortparallel}, 0, v)\, \mathrm{d}v = -\rho^0 T^0 (\hat{A}_k + 5T^0 \hat{C}_k)(t, x_{\shortparallel}, 0),\\ &\int_{\mathbb{R}^3} v_3 (v_i - \mathrm{u}_i^0) \sqrt{\mu_0} (\mathbf{I} - \mathbf{P_0})[f_k + \bar{f}_k](t, x_{\shortparallel}, 0, v)\, \mathrm{d}v \\ &\quad = -\rho^0 (T^0)^2 \hat{B}_{k,i}(t, x_{\shortparallel}, 0), \ \ i = 1, 2,\\ &\int_{\mathbb{R}^3} v_3 (|v - \mathrm{u}^0|^2 - 5T^0) \sqrt{\mu_0} (\mathbf{I} - \mathbf{P_0})[f_k + \bar{f}_k](t, x_{\shortparallel}, 0, v)\, \mathrm{d}v \\ &\quad = -10\rho^0 (T^0)^3 \hat{C}_k(t, x_{\shortparallel}, 0),\end{aligned} \tag{2.25}$$

where we have used (2.17).

For the case $k = 1$, from (1.21), (1.24) and (2.19), it is easy to know that $(2.25)_1$ is equivalent to

$$u_{1,3}(t, x_{\shortparallel}, 0) = 0. \tag{2.26}$$

Since $f_1(t, x_{\shortparallel}, 0, v)$, $\bar{f}_1(t, x_{\shortparallel}, 0, v) \in \mathcal{N}_0$, $(2.25)_{2,3}$ holds naturally for $k = 1$.

Now we consider (2.25) for the case $k \geqq 2$. From $(2.25)_1$, a direct calculation shows that

$$\begin{aligned}u_{k,3}(t, x_{\shortparallel}, 0) &= -\bar{u}_{k,3}(t, x_{\shortparallel}, 0) - T^0 (\hat{A}_k + 5T^0 \hat{C}_k)(t, x_{\shortparallel}, 0)\\ &= -\int_0^\infty \frac{1}{\rho^0} \Big\{ \partial_t \bar{\rho}_{k-1} + \mathrm{div}_{\shortparallel}(\rho^0 \bar{u}_{k-1,\shortparallel} + \bar{\rho}_{k-1} \mathrm{u}_{\shortparallel}^0) \Big\}(t, x_{\shortparallel}, y) dy \\ &\quad - T^0 (\hat{A}_k + 5T^0 \hat{C}_k)(t, x_{\shortparallel}, 0),\end{aligned} \tag{2.27}$$

where we have used (1.30). Clearly, the right hand side terms of (2.27) can be determined by f_i, $\bar{f}_i$ $(i \leqq k - 1)$ and $\hat{f}_j$ $(j \leqq k - 2)$.

Now we consider the rest terms of (2.25). By similar arguments as in (A.9)–(A.16), and utilizing (1.24) and (2.26), one can obtain, for $i = 1, 2$,

$$\int_{\mathbb{R}^3} v_3 (v_1 - \mathrm{u}_1^0) \sqrt{\mu_0} (\mathbf{I} - \mathbf{P_0}) \bar{f}_k(t, x_{\shortparallel}, 0, v)\, \mathrm{d}v$$

$$= -\Big[\mu(T^0)\partial_y \bar{u}_{k-1,i} - \rho^0(u_{1,i} + \bar{u}_{1,i})\bar{u}_{k-1,3} - \langle T^0 \mathcal{A}^0_{3i},\ \bar{J}_{k-2}\rangle\Big](t, x_{\shortparallel}, 0), \tag{2.28}$$

$$\int_{\mathbb{R}^3} v_3(|v - \mathfrak{u}^0|^2 - 5T^0)\sqrt{\mu_0}(\mathbf{I} - \mathbf{P_0})\bar{f}_k(t, x_{\shortparallel}, 0, v)\,\mathrm{d}v$$
$$= -\Big[\kappa(T^0)\partial_y \bar{\theta}_{k-1} - \frac{5}{3}\rho^0(\theta_1 + \bar{\theta}_1)\bar{u}_{k-1,3} - \langle 2(T^0)^{\frac{3}{2}}\mathcal{B}^0_3,\ \bar{J}_{k-2}\rangle\Big](t, x_{\shortparallel}, 0). \tag{2.29}$$

Using (2.28) and (2.29), we can rewrite $(2.25)_{2,3}$ as

$$\partial_y \bar{u}_{k-1,i}(t, x_{\shortparallel}, 0) = \frac{1}{\mu(T^0)}\Big\{[\rho^0(u_{1,i} + \bar{u}_{1,i})\bar{u}_{k-1,3}] + \langle T^0 \mathcal{A}^0_{3i},\ \bar{J}_{k-2}\rangle$$
$$+ \langle T^0 \mathcal{A}^0_{3i},\ (\mathbf{I} - \mathbf{P_0})f_k\rangle + \rho^0(T^0)^2\hat{B}_{k,i}\Big\}(t, x_{\shortparallel}, 0),\ i = 1, 2, \tag{2.30}$$

$$\partial_y \bar{\theta}_{k-1}(t, x_{\shortparallel}, 0) = \frac{1}{\kappa(T^0)}\Big\{\frac{5}{3}\rho^0[(\theta_1 + \bar{\theta}_1)\bar{u}_{k-1,3}] + \langle 2(T^0)^{\frac{3}{2}}\mathcal{B}^0_3,\ \bar{J}_{k-2}\rangle$$
$$+ \langle 2(T^0)^{\frac{3}{2}}\mathcal{B}^0_3,\ (\mathbf{I} - \mathbf{P_0})f_k\rangle + 10\rho^0(T^0)^3\hat{C}_k\Big\}(t, x_{\shortparallel}, 0). \tag{2.31}$$

Remark 2.11. It is easy to check that the terms on RHS of (2.30)–(2.31) depends on f_i $(i \leqq k-1)$, $\bar{f}_j$ $(j \leqq k-2)$ and $\hat{f}_l$ $(l \leqq k-2)$. Once we solve $(\bar{u}_{k-1}, \bar{\theta}_{k-1})$ with the boundary condition (2.30)–(2.31), then $\hat{f}_k$ will be solvable by using Lemma 2.5.

3. Existence of Solution for a Linear Hyperbolic System

To study existence of interior expansion, we first need to consider the following linear problem for $(\tilde{\rho}, \tilde{u}, \tilde{\theta})(t, x)$:

$$\begin{cases} \partial_t \tilde{\rho} + \mathrm{div}_x(\rho\tilde{u} + \tilde{\rho}\mathfrak{u}) = 0, \\ \rho\{\partial_t \tilde{u} + \tilde{u}\cdot\nabla_x \mathfrak{u} + \mathfrak{u}\cdot\nabla_x \tilde{u}\} - \dfrac{\nabla_x p}{\rho}\tilde{\rho} + \nabla_x\left(\dfrac{\rho\tilde{\theta} + 3T\tilde{\rho}}{3}\right) = \mathfrak{f}, \\ \rho\{\partial_t \tilde{\theta} + \dfrac{2}{3}(\tilde{\theta}\mathrm{div}_x \mathfrak{u} + 3T\mathrm{div}_x \tilde{u}) + \mathfrak{u}\cdot\nabla_x \tilde{\theta} + 3\tilde{u}\cdot\nabla_x T\} = \mathfrak{g}, \end{cases} \tag{3.1}$$

with $(t, x) \in (0, \tau) \times \mathbb{R}^3_+$. We impose (3.1) with a given boundary condition

$$\tilde{u}_3(t, x_{\shortparallel}, 0) = d(t, x_{\shortparallel}), \quad \forall (t, x) \in (0, \tau) \times \mathbb{R}^2, \tag{3.2}$$

and the initial condition

$$(\tilde{\rho}, \tilde{u}, \tilde{\theta})(0, x) = (\tilde{\rho}_0, \tilde{u}_0, \tilde{\theta}_0)(x). \tag{3.3}$$

For later use, we define

$$\partial^\alpha_{t,\shortparallel} = \partial^{\alpha_0}_t \partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2},$$

where α is a vector index which is different from the one defined in (1.47). Defining the nations $\|\cdot\|_{\mathcal{H}^k(\mathbb{R}^3_+)}$ and $\|\cdot\|_{\mathcal{H}^k(\mathbb{R}^2)}$,

$$\begin{aligned}\|f(t)\|^2_{\mathcal{H}^k(\mathbb{R}^3_+)} &= \sum_{|\alpha|+i\leqq k} \|\partial^\alpha_{t,\shortparallel}\partial^i_3 f(t)\|^2_{L^2(\mathbb{R}^3_+)},\\ \|g(t)\|^2_{\mathcal{H}^k(\mathbb{R}^2)} &= \sum_{|\alpha|\leqq k} \|\partial^\alpha_{t,\shortparallel} g(t)\|^2_{L^2(\mathbb{R}^2)}.\end{aligned}\tag{3.4}$$

Lemma 3.1. *Let $(\rho, \mathfrak{u}, T)$ be the smooth solution of compressible Euler system obtained in Lemma 2.1, and $\tau > 0$ be its lifespan. We assume that*

$$\|(\tilde\rho_0, \tilde u_0, \tilde\theta_0)\|^2_{\mathcal{H}^k(\mathbb{R}^3_+)} + \sup_{t\in(0,\tau)} \left[\|(\mathfrak{f}, \mathfrak{g})(t)\|^2_{\mathcal{H}^{k+1}(\mathbb{R}^3_+)} + \|d(t)\|^2_{\mathcal{H}^{k+2}}\right] < \infty, \tag{3.5}$$

with $k \geqq 3$, and the compatibility condition for initial data (3.3) is satisfied (Here the compatibility condition means that the initial data (3.3) satisfies the boundary condition (3.2), and the time-derivatives of initial data $(\tilde\rho_0, \tilde u_0, \tilde\theta_0)$ are defined through system (3.1) inductively). Then there exists a unique smooth solution to (3.1)–(3.3) for $t \in [0, \tau]$, which satisfies

$$\begin{aligned}\sup_{t\in[0,\tau]} \|(\tilde\rho, \tilde u, \tilde\theta)(t)\|^2_{\mathcal{H}^k(\mathbb{R}^3_+)} \leqq C(\tau, E_{k+2})\Big\{&\|(\tilde\rho_0, \tilde u_0, \tilde\theta_0)\|^2_{\mathcal{H}^k(\mathbb{R}^3_+)}\\ &+ \sup_{t\in[0,\tau]} \left[\|(\mathfrak{f}, \mathfrak{g})(t)\|^2_{\mathcal{H}^{k+1}(\mathbb{R}^3_+)} + \|d(t)\|^2_{\mathcal{H}^{k+2}(\mathbb{R}^2)}\right]\Big\},\end{aligned}\tag{3.6}$$

where $E_k := \sup_{t\in[0,\tau]} \|(\rho - 1, \mathfrak{u}, T - 1)(t)\|_{H^k}$.

Proof. We define

$$\tilde p := \frac{\rho\tilde\theta + 3T\tilde\rho}{3}.$$

To deal with the boundary terms, it is more convenient to use the variables $(\tilde p, \tilde u, \tilde\theta)$. Then (3.1) is equivalent to

$$\begin{cases}\partial_t\tilde p + \mathfrak{u}\cdot\nabla_x\tilde p + \dfrac{5}{3}p\,\mathrm{div}_x\tilde u + \dfrac{5}{3}\mathrm{div}_x\mathfrak{u}\,\tilde p + \nabla_x p\cdot\tilde u = \dfrac{1}{3}\mathfrak{g},\\ \rho\partial_t\tilde u + \rho\mathfrak{u}\cdot\nabla_x\tilde u + \nabla_x\tilde p - \dfrac{\nabla_x p}{p}\tilde p + \rho\tilde u\cdot\nabla_x\mathfrak{u} - \dfrac{\nabla_x p}{3T}\tilde\theta = \mathfrak{f},\\ \rho\partial_t\tilde\theta + \rho\mathfrak{u}\cdot\nabla_x\tilde\theta + 2p\,\mathrm{div}_x\tilde u + 3\rho\tilde u\cdot\nabla_x T + \dfrac{2}{3}\rho\mathrm{div}_x\mathfrak{u}\,\tilde\theta = \mathfrak{g}.\end{cases}\tag{3.7}$$

Let χ be a smooth monotonic cut-off function such that

$$\chi(s) = \begin{cases}1, & s\in[0,1],\\ 0, & s\in[2,\infty).\end{cases}\tag{3.8}$$

Then we define

$$u_d(t,x) := \big(0, 0, d(t, x_{\shortparallel})\chi(x_3)\big)^{\top} \quad \text{and} \quad \tilde{w} := \tilde{u} - u_d. \tag{3.9}$$

Now we can rewrite (3.7) to be

$$\begin{cases} \partial_t \tilde{p} + \mathfrak{u}\cdot\nabla_x \tilde{p} + \frac{5}{3} p \,\mathrm{div}_x \tilde{w} + \frac{5}{3}\mathrm{div}_x \mathfrak{u}\, \tilde{p} + \nabla_x p \cdot \tilde{w} \\ \qquad = G_0 := \frac{1}{3}\mathfrak{g} - \frac{5}{3} p\,\mathrm{div} u_d - \nabla_x p \cdot u_d, \\ \rho\partial_t \tilde{w} + \rho\mathfrak{u}\cdot\nabla_x \tilde{w} + \nabla_x \tilde{p} - \frac{\nabla_x p}{p}\tilde{p} + \rho\tilde{w}\cdot\nabla_x \mathfrak{u} - \frac{\nabla_x p}{3T}\tilde{\theta} \\ \qquad = G_1 := \mathfrak{f} - \rho\partial_t u_d - \rho\mathfrak{u}\cdot\nabla_x u_d - \rho u_d\cdot\nabla_x \mathfrak{u}, \\ \rho\partial_t\tilde{\theta} + \rho\mathfrak{u}\cdot\nabla_x\tilde{\theta} + 2p\,\mathrm{div}_x\tilde{w} + 3\rho\tilde{w}\cdot\nabla_x T + \frac{2}{3}\rho\mathrm{div}_x\mathfrak{u}\,\tilde{\theta} \\ \qquad = G_2 := \mathfrak{g} - 2p\,\mathrm{div}_x u_d - 3\rho u_d\cdot\nabla_x T. \end{cases} \tag{3.10}$$

From (3.2), the boundary condition now becomes

$$\tilde{w}_3(t, x_{\shortparallel}, 0) \equiv 0. \tag{3.11}$$

We can write the linear system (3.10) as a symmetric hyperbolic equations

$$A_0\partial_t U + \sum_{i=1}^{3} A_i\partial_i U + A_4 U = G, \tag{3.12}$$

where

$$U = \begin{pmatrix} \tilde{p} \\ \tilde{w} \\ \tilde{\theta} \end{pmatrix},\ A_0 = \begin{pmatrix} \frac{9}{5} & 0 & 0 & 0 & -\rho \\ 0 & \rho p & 0 & 0 & 0 \\ 0 & 0 & \rho p & 0 & 0 \\ 0 & 0 & 0 & \rho p & 0 \\ -\rho & 0 & 0 & 0 & \frac{5}{6} \end{pmatrix},\ A_1 = \begin{pmatrix} \frac{9}{5}\mathfrak{u}_1 & p & 0 & 0 & -\rho\mathfrak{u}_1 \\ p & \rho p\mathfrak{u}_1 & 0 & 0 & 0 \\ 0 & 0 & \rho p\mathfrak{u}_1 & 0 & 0 \\ 0 & 0 & 0 & \rho p\mathfrak{u}_1 & 0 \\ -\rho\mathfrak{u}_1 & 0 & 0 & 0 & \frac{5}{6}\rho^2\mathfrak{u}_1 \end{pmatrix},$$

and

$$A_2 = \begin{pmatrix} \frac{9}{5}\mathfrak{u}_2 & 0 & p & 0 & -\rho\mathfrak{u}_2 \\ 0 & \rho p\mathfrak{u}_1 & 0 & 0 & 0 \\ p & 0 & \rho p\mathfrak{u}_2 & 0 & 0 \\ 0 & 0 & 0 & \rho p\mathfrak{u}_2 & 0 \\ -\rho\mathfrak{u}_2 & 0 & 0 & 0 & \frac{5}{6}\rho^2\mathfrak{u}_2 \end{pmatrix},\ A_3 = \begin{pmatrix} \frac{9}{5}\mathfrak{u}_3 & 0 & 0 & p & -\rho\mathfrak{u}_3 \\ 0 & \rho p\mathfrak{u}_3 & 0 & 0 & 0 \\ 0 & 0 & \rho p\mathfrak{u}_3 & 0 & 0 \\ p & 0 & 0 & \rho p\mathfrak{u}_3 & 0 \\ -\rho\mathfrak{u}_3 & 0 & 0 & 0 & \frac{5}{6}\rho^2\mathfrak{u}_3 \end{pmatrix}. \tag{3.13}$$

The matrix A_4 and column vector G can be easily write down, and we do not give the details here. It is easy to check that A_0 is positive.

Since the matrix A_3 is singular on the boundary, hence the IBVP (3.10)–(3.11) is a linear hyperbolic system with characteristic boundary. We refer [9,41] for the local existence of smooth solutions. To close our lemma, one needs only to establish the *a priori* energy estimates.

It follows from Newtonian–Leibnitz formula that

$$\|U(t)\|_{\mathcal{H}^{k-1}}^2 \leqq \|U_0\|_{\mathcal{H}^{k-1}}^2 + 2\int_0^t \|\partial_t U(s)\|_{\mathcal{H}^{k-1}} \|U(s)\|_{\mathcal{H}^{k-1}} ds$$
$$\leqq \|U_0\|_{\mathcal{H}^{k-1}}^2 + \int_0^t \|U(s)\|_{\mathcal{H}^k}^2 ds. \tag{3.14}$$

Hence we need only to close the highest order derivatives estimates. Let $|\alpha|+i = k$, and applying $\partial_{t,\shortparallel}^\alpha \partial_3^i$ to (3.12), we obtain

$$A_0\partial_t\partial_{t,\shortparallel}^\alpha\partial_3^i U + \sum_{j=1}^3 A_j\partial_i\partial_{t,\shortparallel}^\alpha\partial_3^i U$$
$$= \partial_{t,\shortparallel}^\alpha\partial_3^i G - \partial_{t,\shortparallel}^\alpha\partial_3^i(A_4U) - [\partial_{t,\shortparallel}^\alpha\partial_3^i,\, A_0]\partial_t U - \sum_{j=1}^3[\partial_{t,\shortparallel}^\alpha\partial_3^i,\, A_j]\partial_j U, \tag{3.15}$$

where and whereafter the notation $[\cdot,\ \cdot]$ denote the commutator operator, i.e.,

$$[\partial^\alpha, f]g = \partial^\alpha(fg) - f\partial^\alpha g = \sum_{\beta+\gamma=\alpha, |\beta|\geqq 1} C_{\beta,\gamma}\partial^\beta f\cdot\partial^\gamma g.$$

Multiplying (3.15) by $\partial_{t,\shortparallel}^\alpha\partial_3^i U^\top$ and integrating the resultant equation over $[0, t]\times\mathbb{R}_+^3$, we obtain

$$\|\partial_{t,\shortparallel}^\alpha\partial_3^i U(t)\|_{L^2}^2 \leqq C\|\partial_{t,\shortparallel}^\alpha\partial_3^i U(0)\|_{L^2}^2 + C(E_{k+1})\int_0^t \|(U, G)(s)\|_{\mathcal{H}^k}^2\,\mathrm{d}s$$
$$+ C\left|\int_0^t\int_{\mathbb{R}^2}(\partial_{t,\shortparallel}^\alpha\partial_3^i U^\top A_3\partial_{t,\shortparallel}^\alpha\partial_3^i U)(s, x_\shortparallel, 0)\,\mathrm{d}x_\shortparallel\mathrm{d}s\right|. \tag{3.16}$$

For the boundary term on RHS of (3.16), noting (3.13) and $\mathfrak{u}_3(t, x_\shortparallel, 0) \equiv 0$, it holds that

$$\int_0^t\int_{\mathbb{R}^2}(\partial_{t,\shortparallel}^\alpha\partial_3^i U^\top A_3\partial_{t,\shortparallel}^\alpha\partial_3^i U)(s, x_\shortparallel, 0)\,\mathrm{d}x_\shortparallel\mathrm{d}s$$
$$= 2\int_0^t\int_{\mathbb{R}^2}(p\,\partial_{t,\shortparallel}^\alpha\partial_3^i\tilde{p}\cdot\partial_{t,\shortparallel}^\alpha\partial_3^i\tilde{w}_3)(s, x_\shortparallel, 0)\,\mathrm{d}x_\shortparallel\mathrm{d}s. \tag{3.17}$$

To close the above estimates, we use an induction argument on the number of normal derivatives ∂_3^i. For $i = 0$, it follows from (3.11) that

$$\partial_{t,\shortparallel}^\alpha\tilde{w}_3(t, x_\shortparallel, 0) \equiv 0, \tag{3.18}$$

which, together with (3.16) and (3.17), yields that

$$\sum_{|\alpha|=k}\|\partial_{t,\shortparallel}^\alpha U(t)\|_{L^2}^2 \leqq C\|U(0)\|_{\mathcal{H}^k}^2 + C(E_{k+1})\int_0^t\|(U, G)(s)\|_{\mathcal{H}^k}^2\mathrm{d}s.$$

Assume that we have already obtained

$$\sum_{|\alpha|+i=k, i\leqq l-1} \|\partial_{t,\shortparallel}^{\alpha}\partial_3^i U(t)\|_{L^2}^2 \leqq C(E_{k+1})\int_0^t \|U(s)\|_{\mathcal{H}^k}^2 + \|G(s)\|_{\mathcal{H}^{k+1}}^2 \mathrm{d}s$$

$$+ C\|(U,G)(0)\|_{\mathcal{H}^k}^2. \tag{3.19}$$

Next, we shall consider the case for $\|\partial_{t,\shortparallel}^{\alpha}\partial_3^l U(t)\|_{L^2}^2$ with $|\alpha|+l=k$. Noting (3.16) and (3.17), we need only to control the boundary term

$$\int_0^t\int_{\mathbb{R}^2} (p\,\partial_{t,\shortparallel}^{\alpha}\partial_3^l\tilde{p}\cdot\partial_{t,\shortparallel}^{\alpha}\partial_3^l\tilde{w}_3)(s,x_\shortparallel,0)\,\mathrm{d}x_\shortparallel\mathrm{d}s. \tag{3.20}$$

It follows from (3.10) that

$$\partial_3\tilde{p} = -\rho(\partial_t + \mathfrak{u}\cdot\nabla_x)\tilde{w}_3 + \frac{\partial_3 p}{p}\tilde{p} - \rho\tilde{w}\cdot\nabla_x\mathfrak{u}_3 + \frac{\partial_3 p}{3T}\tilde{\theta} + G_{1,3}, \tag{3.21}$$

$$p\partial_3\tilde{w}_3 = -\frac{3}{5}(\partial_t + \mathfrak{u}\cdot\nabla_x)\tilde{p} - p\sum_{i=1}^2\partial_i\tilde{w}_i - \mathrm{div}_x\mathfrak{u}\,\tilde{p} - \frac{3}{5}\nabla_x p\cdot\tilde{w} + \frac{3}{5}G_0. \tag{3.22}$$

By utilizing (1.8) and (1.9), we have

$$\partial_3 p(t,x_\shortparallel,0)\equiv 0 \quad\text{and}\quad \mathfrak{u}_3(t,x_\shortparallel,0)\equiv 0. \tag{3.23}$$

Substituting (3.23) into (3.21) and using (3.18), one obtains

$$\partial_3\tilde{p}(t,x_\shortparallel,0) = G_{1,3}(t,x_\shortparallel,0) \tag{3.24}$$

Applying ∂_3^{l-1} to (3.21) and (3.22), and using (3.23), then we have

$$\begin{aligned}\partial_3^l\tilde{p}(t,x_\shortparallel,0) &\cong (\partial_t+\mathfrak{u}\cdot\nabla_x)\partial_3^{l-1}\tilde{w}_3 + \partial_3^{l-1}\tilde{w}_3 + \partial_t\partial_3^{l-2}\tilde{w}_3\\ &\quad+\sum_{i=1}^2\sum_{|\beta|+j\leqq l-2}\partial_i\partial_{t,\shortparallel}^{\beta}\partial_3^j\tilde{w}_3\\ &\quad+\sum_{|\beta|+j\leqq l-2}\partial_{t,\shortparallel}^{\beta}\partial_3^j(\tilde{p},\tilde{w},\tilde{\theta}) + \partial_3^{l-1}G_{1,3},\end{aligned} \tag{3.25}$$

$$\begin{aligned}\partial_3^l\tilde{w}_3(t,x_\shortparallel,0) &\cong (\partial_t+\mathfrak{u}\cdot\nabla_x)\partial_3^{l-1}\tilde{p} + \sum_{i=1}^2\{\partial_i\partial_3^{l-1}\tilde{w}_i + \partial_3^{l-1}\tilde{w}_i\} + \partial_3^{l-1}\tilde{p}\\ &\quad+\sum_{i=1}^2\partial_i\partial_3^{l-2}(\tilde{p},\tilde{w}) + \sum_{|\beta|+j\leqq l-2}\partial_{t,\shortparallel}^{\beta}\partial_3^j(\tilde{p},\tilde{w}) + \partial_3^{l-1}G_0.\end{aligned} \tag{3.26}$$

where and whereafter we use "$\cong$" to ignore the exact coefficients which depends only on the Euler solution. Substituting (3.25) into (3.26), one can obtain

$$\partial_3^l\tilde{w}_3(t,x_\shortparallel,0) \cong (\partial_t+\mathfrak{u}\cdot\nabla_x)\partial_{t,\shortparallel}\partial_3^{l-2}\tilde{w}_3 + \sum_{i=1}^2\{\partial_i\partial_3^{l-1}\tilde{w}_i + \partial_3^{l-1}\tilde{w}_i\}$$

$$+ \partial_t \partial_3^{l-2} \tilde{w}_3 + \partial_t^2 \partial_3^{l-3} \tilde{w}_3 + \sum_{i=1}^{2} \sum_{|\beta|+j \leqq l-2} \partial_i \partial_{t,\shortparallel}^{\beta} \partial_3^j (\tilde{p}, \tilde{w})$$

$$+ \sum_{|\beta|+j \leqq l-2} \partial_{t,\shortparallel}^{\beta} \partial_3^j (\tilde{p}, \tilde{w}, \tilde{\theta}) + \partial^{l-1} G. \tag{3.27}$$

If l is even, using (3.18) and (3.27), step by step, one can get

$$\partial_3^l \tilde{w}_3(t, x_\shortparallel, 0) \cong \sum_{i=1}^{2} \sum_{j=0}^{\frac{l}{2}-1} \{\partial_i \partial_{t,\shortparallel}^{2j} \partial_3^{l-1-2j} \tilde{w}_i + \partial_{t,\shortparallel}^{2j} \partial_3^{l-1-2j} \tilde{w}_i\} + \partial_t^{l-1} \tilde{p}$$

$$+ \sum_{i=1}^{2} \sum_{|\beta|+j \leqq l-2} \partial_i \partial_{t,\shortparallel}^{\beta} \partial_3^j (\tilde{p}, \tilde{w})$$

$$+ \sum_{|\beta|+j \leqq l-2} \partial_{t,\shortparallel}^{\beta} \partial_3^j (\tilde{p}, \tilde{w}, \tilde{\theta}) + \sum_{j=0}^{l-1} \partial^j G. \tag{3.28}$$

Similarly, if l is odd, step by step, we have

$$\partial_3^l \tilde{w}_3(t, x_\shortparallel, 0) \cong (\partial_t + \mathfrak{u} \cdot \nabla_x) \partial_{t,\shortparallel}^{l-2} \partial_3 \tilde{w}_3 + \sum_{i=1}^{2} \sum_{j=0}^{\frac{l-3}{2}} \Big\{ \partial_i \partial_{t,\shortparallel}^{2j} \partial_3^{l-1-2j} \tilde{w}_i$$

$$+ \partial_{t,\shortparallel}^{2j} \partial_3^{l-1-2j} \tilde{w}_i \Big\} + \partial_t^{l-1} \tilde{p} + \sum_{i=1}^{2} \sum_{|\beta|+j \leqq l-2} \partial_i \partial_{t,\shortparallel}^{\beta} \partial_3^j (\tilde{p}, \tilde{w})$$

$$+ \sum_{|\beta|+j \leqq l-2} \partial_{t,\shortparallel}^{\beta} \partial_3^j (\tilde{p}, \tilde{w}, \tilde{\theta}) + \sum_{j=0}^{l-1} \partial^j G. \tag{3.29}$$

Substituting (3.22) into (3.29), one obtains, for l being odd, that

$$\partial_3^l \tilde{w}_3(t, x_\shortparallel, 0) \cong (\partial_t + \mathfrak{u} \cdot \nabla_x) \partial_{t,\shortparallel}^{l-1} \tilde{p} + \sum_{i=1}^{2} \sum_{j=0}^{\frac{l-1}{2}} \Big\{ \partial_i \partial_{t,\shortparallel}^{2j} \partial_3^{l-1-2j} \tilde{w}_i$$

$$+ \partial_{t,\shortparallel}^{2j} \partial_3^{l-1-2j} \tilde{w}_i \Big\} + \partial_t^{l-1} \tilde{p} + \sum_{i=1}^{2} \sum_{|\beta|+j \leqq l-2} \partial_i \partial_{t,\shortparallel}^{\beta} \partial_3^j (\tilde{p}, \tilde{w})$$

$$+ \sum_{|\beta|+j \leqq l-2} \partial_{t,\shortparallel}^{\beta} \partial_3^j (\tilde{p}, \tilde{w}, \tilde{\theta}) + \sum_{j=0}^{l-1} \partial^j G. \tag{3.30}$$

To estimate $\partial_3^l \tilde{p}$, we have to be careful. Let l be even. Substituting (3.28) and (3.30) into (3.25), we can get

$$\partial_3^l \tilde{p}(t, x_\shortparallel, 0)$$

$$
\begin{aligned}
&\cong (\partial_t + \mathfrak{u}\cdot\nabla_x)\partial_{t,\shortparallel}^{l-1}\tilde{p} + (\partial_t + \mathfrak{u}\cdot\nabla_x)\sum_{i=1}^{2}\sum_{j=0}^{\frac{l}{2}-1}\Big\{\partial_i\partial_{t,\shortparallel}^{2j}\partial_3^{l-2-2j}\tilde{w}_i\\
&\quad + \partial_{t,\shortparallel}^{2j}\partial_3^{l-2-2j}\tilde{w}_i\Big\} + \partial_t^{l-1}\tilde{p} + \sum_{i=1}^{2}\sum_{|\beta|+j\leqq l-2}\partial_i\partial_{t,\shortparallel}^{\beta}\partial_3^{j}(\tilde{p},\tilde{w})\\
&\quad + \sum_{|\beta|+j\leqq l-2}\partial_{t,\shortparallel}^{\beta}\partial_3^{j}(\tilde{p},\tilde{w},\tilde{\theta}) + \sum_{j=0}^{l-1}\partial^j G\\
&\cong (\partial_t + \mathfrak{u}\cdot\nabla_x)\partial_{t,\shortparallel}^{l-1}\tilde{p} + \sum_{i=1}^{2}\sum_{j=0}^{\frac{l}{2}-1}\partial_i^2\partial_{t,\shortparallel}^{2j}\partial_3^{l-2-2j}\tilde{p} + \partial_t^{l-1}\tilde{p}\\
&\quad + \sum_{i=1}^{2}\sum_{|\beta|+j\leqq l-2}\partial_i\partial_{t,\shortparallel}^{\beta}\partial_3^{j}(\tilde{p},\tilde{w},\tilde{\theta}) + \sum_{|\beta|+j\leqq l-2}\partial_{t,\shortparallel}^{\beta}\partial_3^{j}(\tilde{p},\tilde{w},\tilde{\theta}) + \sum_{j=0}^{l-1}\partial^j G,
\end{aligned}
\tag{3.31}
$$

where we have used the facts that

$$
(\partial_t + \mathfrak{u}\cdot\nabla_x)\tilde{w}_i \cong \partial_i\tilde{p} + O(1)(\tilde{p},\tilde{w},\tilde{\theta}) + G_{1,i}, \tag{3.32}
$$

which can be derived from (3.10).

Iterating (3.31) again, step by step, we have, for l being even, that

$$
\begin{aligned}
\partial_3^l\tilde{p}(t,x_\shortparallel,0) &\cong (\partial_t + \mathfrak{u}\cdot\nabla_x)\partial_{t,\shortparallel}^{l-1}\tilde{p} + \partial_t^{l-1}\tilde{p} + \sum_{i=1}^{2}\sum_{|\beta|+j\leqq l-2}\partial_i\partial_{t,\shortparallel}^{\beta}\partial_3^{j}(\tilde{p},\tilde{w},\tilde{\theta})\\
&\quad + \sum_{|\beta|+j\leqq l-2}\partial_{t,\shortparallel}^{\beta}\partial_3^{j}(\tilde{p},\tilde{w},\tilde{\theta}) + \sum_{j=0}^{l-1}\partial^j G,
\end{aligned}
$$

For l being odd, substituting (3.28) and (3.30) into (3.25), and using (3.32), we get

$$
\begin{aligned}
&\partial_3^l\tilde{p}(t,x_\shortparallel,0)\\
&\quad\cong (\partial_t + \mathfrak{u}\cdot\nabla_x)\sum_{i=1}^{2}\sum_{j=0}^{\frac{l-3}{2}}\{\partial_i\partial_{t,\shortparallel}^{2j}\partial_3^{l-2-2j}\tilde{w}_i + \partial_{t,\shortparallel}^{2j}\partial_3^{l-2-2j}\tilde{w}_i\} + \partial_t^{l-1}\tilde{p}\\
&\qquad + \sum_{i=1}^{2}\sum_{|\beta|+j\leqq l-2}\partial_i\partial_{t,\shortparallel}^{\beta}\partial_3^{j}(\tilde{p},\tilde{w},\tilde{\theta}) + \sum_{|\beta|+j\leqq l-2}\partial_{t,\shortparallel}^{\beta}\partial_3^{j}(\tilde{p},\tilde{w},\tilde{\theta}) + \sum_{j=0}^{l-1}\partial^j G\\
&\quad\cong \sum_{i=1}^{2}\sum_{j=0}^{\frac{l-3}{2}}\partial_i^2\partial_{t,\shortparallel}^{2j}\partial_3^{l-2-2j}\tilde{p} + \partial_t^{l-1}\tilde{p} + \sum_{i=1}^{2}\sum_{|\beta|+j\leqq l-2}\partial_i\partial_{t,\shortparallel}^{\beta}\partial_3^{j}(\tilde{p},\tilde{w},\tilde{\theta})
\end{aligned}
$$

$$+\sum_{|\beta|+j\leqq l-2}\partial_{t,\shortparallel}^{\beta}\partial_3^j(\tilde{p},\tilde{w},\tilde{\theta})+\sum_{j=0}^{l-1}\partial^j G. \tag{3.33}$$

Then, iterating (3.33) again, step by step, one obtains, for l being odd, that

$$\begin{aligned}\partial_3^l\tilde{p}(t,x_{\shortparallel},0)&\cong\partial_{t,\shortparallel}^{l-1}\partial_3\tilde{p}+\partial_t^{l-1}\tilde{p}+\sum_{i=1}^{2}\sum_{|\beta|+j\leqq l-2}\partial_i\partial_{t,\shortparallel}^{\beta}\partial_3^j(\tilde{p},\tilde{w},\tilde{\theta})\\&\quad+\sum_{|\beta|+j\leqq l-2}\partial_{t,\shortparallel}^{\beta}\partial_3^j(\tilde{p},\tilde{w},\tilde{\theta})+\sum_{j=0}^{l-1}\partial^jG\\&\cong\partial_t^{l-1}\tilde{p}+\sum_{i=1}^{2}\sum_{|\beta|+j\leqq l-2}\partial_i\partial_{t,\shortparallel}^{\beta}\partial_3^j(\tilde{p},\tilde{w},\tilde{\theta})+\sum_{|\beta|+j\leqq l-2}\partial_{t,\shortparallel}^{\beta}\partial_3^j(\tilde{p},\tilde{w},\tilde{\theta})\\&\quad+\sum_{j=0}^{l-1}\partial^jG,\end{aligned} \tag{3.34}$$

where we have used (3.24) in the last step.

Now we estimate the boundary term (3.20) when l is even. Using integration by parts, (3.25), (3.28) and Lemma C.1, it holds that

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^2}(p\partial_{t,\shortparallel}^{\alpha}\partial_3^l\tilde{p}\cdot\partial_{t,\shortparallel}^{\alpha}\partial_3^l\tilde{w}_3)(s,x_{\shortparallel},0)\,\mathrm{d}x_{\shortparallel}\mathrm{d}s\\&\cong\int_0^t\int_{\mathbb{R}^2}\Big\{\sum_{i=1}^{2}\sum_{j=0}^{\frac{l}{2}-1}\{\partial_i\partial_{t,\shortparallel}^{\alpha+2j}\partial_3^{l-1-2j}\tilde{w}_i+\partial_{t,\shortparallel}^{\alpha+2j}\partial_3^{l-1-2j}\tilde{w}_i\}+\partial_{t,\shortparallel}^{\alpha}\partial_t^{l-1}\tilde{p}\\&\quad+\sum_{i=1}^{2}\sum_{|\beta|+j\leqq l-2}\partial_i\partial_{t,\shortparallel}^{\alpha+\beta}\partial_3^j(\tilde{p},\tilde{w})+\sum_{|\beta|+j\leqq l-2}\partial_{t,\shortparallel}^{\alpha+\beta}\partial_3^j(\tilde{p},\tilde{w},\tilde{\theta})+\sum_{j=0}^{l-1}\partial_{t,\shortparallel}^{\alpha}\partial^jG\Big\}\\&\quad\times\Big\{(\partial_t+\mathfrak{u}\cdot\nabla_x)\partial_{t,\shortparallel}^{\alpha}\partial_3^{l-1}\tilde{w}_3+\partial_{t,\shortparallel}^{\alpha}\partial_3^{l-1}\tilde{w}_3+\partial_t\partial_{t,\shortparallel}^{\alpha}\partial_3^{l-2}\tilde{w}_3\\&\quad+\sum_{i=1}^{2}\sum_{|\beta|+j\leqq l-2}\partial_i\partial_{t,\shortparallel}^{\alpha+\beta}\partial_3^j\tilde{w}_3+\sum_{|\beta|+j\leqq l-2}\partial_{t,\shortparallel}^{\alpha+\beta}\partial_3^j(\tilde{p},\tilde{w},\tilde{\theta})+\partial_{t,\shortparallel}^{\alpha}\partial_3^{l-1}G\Big\}\,\mathrm{d}x_{\shortparallel}\mathrm{d}s\\&\cong\int_0^t\int_{\mathbb{R}^2}\partial_{t,\shortparallel}^{\alpha}\partial_3^{l-1}\tilde{w}_3\times(\partial_t+\mathfrak{u}\cdot\nabla_x)\Big\{\sum_{i=1}^{2}\sum_{j=0}^{\frac{l}{2}-1}[\partial_i\partial_{t,\shortparallel}^{\alpha+2j}\partial_3^{l-1-2j}\tilde{w}_i\\&\quad+\partial_{t,\shortparallel}^{\alpha+2j}\partial_3^{l-1-2j}\tilde{w}_i]+\partial_{t,\shortparallel}^{\alpha}\partial_t^{l-1}\tilde{p}+\sum_{i=1}^{2}\sum_{|\beta|+j\leqq l-2}\partial_i\partial_{t,\shortparallel}^{\alpha+\beta}\partial_3^j(\tilde{p},\tilde{w})\\&\quad+\sum_{|\beta|+j\leqq l-2}\partial_{t,\shortparallel}^{\alpha+\beta}\partial_3^j(\tilde{p},\tilde{w},\tilde{\theta})+\sum_{j=0}^{l-1}\partial_{t,\shortparallel}^{\alpha}\partial^jG_0\Big\}\,\mathrm{d}x_{\shortparallel}\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^2} \partial_{t,\shortparallel}^{\alpha}\partial_3^{l-1}\tilde{w}_3 \times \Big\{ \sum_{i=1}^{2}\sum_{j=0}^{\frac{l}{2}-1}\{\partial_i \partial_{t,\shortparallel}^{\alpha+2j}\partial_3^{l-1-2j}\tilde{w}_i + \partial_{t,\shortparallel}^{\alpha+2j}\partial_3^{l-1-2j}\tilde{w}_i\} \\
&+\partial_{t,\shortparallel}^{\alpha}\partial_t^{l-1}\tilde{p} + \sum_{i=1}^{2}\sum_{|\beta|+j\leqq l-2}\partial_i\partial_{t,\shortparallel}^{\alpha+\beta}\partial_3^{j}(\tilde{p},\tilde{w}) + \sum_{|\beta|+j\leqq l-2}\partial_{t,\shortparallel}^{\alpha+\beta}\partial_3^{j}(\tilde{p},\tilde{w},\tilde{\theta}) \\
&+\sum_{j=0}^{l-1}\partial_{t,\shortparallel}^{\alpha}\partial^{j}G\Big\}\,\mathrm{d}x_{\shortparallel}\Big|_0^t + C(E_{k+1})\int_0^t \|(U,G)(s)\|_{\mathcal{H}^k}^2\,\mathrm{d}s. \qquad (3.35)
\end{aligned}
$$

From (3.14), (3.19) and Lemma C.1, the second term on RHS of (3.35) can be bounded by

$$
\begin{aligned}
&\lambda\sum_{|\alpha|=k-l}\|\partial_{t,\shortparallel}^{\alpha}\partial_3^{l}U(t)\|_{L^2}^2 + C_\lambda\sum_{|\alpha|+i=k,\,i\leqq l-1}\|\partial_{t,\shortparallel}^{\alpha}\partial_3^{i}U(t)\|_{L^2}^2 \\
&\quad+\|U(t)\|_{\mathcal{H}^{k-1}}^2 + \|(U,G)(0)\|_{\mathcal{H}^k}^2 + \|G(t)\|_{\mathcal{H}^k}^2 \\
&\leqq \lambda\sum_{|\alpha|=k-l}\|\partial_{t,\shortparallel}^{\alpha}\partial_3^{l}U(t)\|_{L^2}^2 + C(E_{k+1})\|(U,G)(0)\|_{\mathcal{H}^k}^2 \\
&\quad+C(E_{k+1})\int_0^t \|U(s)\|_{\mathcal{H}^k}^2 + \|G(s)\|_{\mathcal{H}^{k+1}}^2\,\mathrm{d}s. \qquad (3.36)
\end{aligned}
$$

Integrating by parts with respect to x_1 or x_2 for the highest order terms, using (3.32) and Lemma C.1, we bound the first term on RHS of (3.35) by

$$
\begin{aligned}
&\sum_{i=1}^{2}\int_0^t\int_{\mathbb{R}^2}\partial_i\partial_{t,\shortparallel}^{\alpha}\partial_3^{l-1}\tilde{w}_3 \times \sum_{j=0}^{\frac{l}{2}-1}\{\partial_i\partial_{t,\shortparallel}^{\alpha+2j}\partial_3^{l-1-2j}\tilde{p} + \partial_{t,\shortparallel}^{\alpha+2j}\partial_3^{l-1-2j}\tilde{p}\} \\
&\quad+\int_0^t\int_{\mathbb{R}^2}\partial_{t,\shortparallel}^{\alpha}\partial_3^{l-1}\tilde{w}_3\cdot\partial_{t,\shortparallel}^{\alpha}\partial_t^{l}\tilde{p}\,\mathrm{d}x_{\shortparallel}\mathrm{d}s + C(E_{k+1})\int_0^t \|U\|_{\mathcal{H}^k}^2 + \|G\|_{\mathcal{H}^{k+1}}^2\mathrm{d}s \\
&\cong \sum_{i=1}^{2}\int_0^t\int_{\mathbb{R}^2}\partial_i\partial_{t,\shortparallel}^{\alpha}\partial_3^{l-1}\tilde{w}_3 \times \sum_{j=0}^{\frac{l}{2}-1}\partial_i\partial_{t,\shortparallel}^{\alpha+2j}\partial_t^{l-2-2j}\tilde{p}\,\mathrm{d}x_{\shortparallel}\mathrm{d}s \\
&\quad+\int_0^t\int_{\mathbb{R}^2}\partial_{t,\shortparallel}^{\alpha}\partial_3^{l-1}\tilde{w}_3\cdot\partial_{t,\shortparallel}^{\alpha}\partial_t^{l}\tilde{p}\,\mathrm{d}x_{\shortparallel}\mathrm{d}s + C(E_{k+1})\int_0^t \|U\|_{\mathcal{H}^k}^2 + \|G\|_{\mathcal{H}^{k+1}}^2\mathrm{d}s \\
&\cong \int_0^t\int_{\mathbb{R}^2}\partial_{t,\shortparallel}^{\alpha}\partial_3^{l-1}\tilde{w}_3\cdot\partial_{t,\shortparallel}^{\alpha}\partial_t^{l}\tilde{p}\,\mathrm{d}x_{\shortparallel}\mathrm{d}s + C(E_{k+1})\int_0^t \|U\|_{\mathcal{H}^k}^2 + \|G\|_{\mathcal{H}^{k+1}}^2\mathrm{d}s, \qquad (3.37)
\end{aligned}
$$

where we have used (3.34) for $\partial_3^{l-1-2j}\tilde{p}$.

If $\partial_{t,\shortparallel}^{\alpha} = \partial_t^{\alpha_0}\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}$ with $\alpha_1+\alpha_2 \geqq 1$, then by using Lemma C.1, the first term on RHS of (3.37) is controlled by

$$
\int_0^t\int_{\mathbb{R}^2}\partial_{t,\shortparallel}^{\alpha}\partial_3^{l-1}\tilde{w}_3\cdot\partial_{t,\shortparallel}^{\alpha}\partial_t^{l}\tilde{p}\,\mathrm{d}x_{\shortparallel}\mathrm{d}s \leqq \int_0^t \|U(s)\|_{\mathcal{H}^k}^2\mathrm{d}s.
$$

For the remaining case $\partial_{t,\shortparallel}^{\alpha}\tilde{p} = \partial_t^{k-l}\tilde{p}$, it follows from (3.30) and Lemma C.1 that

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}^2} \partial_t^{k-l}\partial_3^{l-1}\tilde{w}_3\cdot\partial_t^k\tilde{p}\,\mathrm{d}x_{\shortparallel}\mathrm{d}s\\
&\cong C\int_0^t \|U(s)\|_{\mathcal{H}^k}^2\mathrm{d}s + \int_0^t\int_{\mathbb{R}^2}\Big[\partial_t^{k-1}\tilde{p} + \sum_{i=1}^{2}\sum_{j\leqq l-2}\partial_i\partial_{t,\shortparallel}^{k-2-j}\partial_3^j(\tilde{p},\tilde{w})\\
&\quad + \sum_{|\beta|\leqq k-j-2,\, j\leqq l-2}\partial_{t,\shortparallel}^{\beta}\partial_3^j(\tilde{p},\tilde{w},\tilde{\theta}) + \sum_{j=0}^{k-2}\partial^j G\Big]\cdot\partial_t^k\tilde{p}\,\mathrm{d}x_{\shortparallel}\mathrm{d}s\\
&\cong \int_0^t\|(U,G)(s)\|_{\mathcal{H}^k}^2\mathrm{d}s + \int_{\mathbb{R}^2}|\partial_t^{k-1}\tilde{p}|^2\,\mathrm{d}x_{\shortparallel}\Big|_0^t\\
&\quad + \int_{\mathbb{R}^2}\Big[\sum_{i=1}^{2}\sum_{j\leqq l-2}\partial_i\partial_{t,\shortparallel}^{k-2-j}\partial_3^j(\tilde{p},\tilde{w})\\
&\quad + \sum_{|\beta|\leqq k-j-2,\, j\leqq l-2}\partial_{t,\shortparallel}^{\beta}\partial_3^j(\tilde{p},\tilde{w},\tilde{\theta}) + \sum_{j=0}^{k-2}\partial^j G\Big]\cdot\partial_t^{k-1}\tilde{p}\,\mathrm{d}x_{\shortparallel}\Big|_0^t\\
&\leqq C(E_{k+1})\Big\{\sum_{|\alpha|+j=k,\, j\leqq l-1}\|\partial_{t,\shortparallel}^{\alpha}\partial_3^j U(t)\|_{L^2}^2 + \|(U,G)(0)\|_{\mathcal{H}^k}^2\\
&\quad + \|G(t)\|_{\mathcal{H}^k}^2 + \int_0^t\|(U,G)(s)\|_{\mathcal{H}^k}^2\mathrm{d}s\Big\}.
\end{aligned}\tag{3.38}$$

Substituting (3.36)–(3.38) into (3.35), for l being even, we obtain

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}^2}(p\,\partial_{t,\shortparallel}^{\alpha}\partial_3^l\tilde{p}\cdot\partial_{t,\shortparallel}^{\alpha}\partial_3^l\tilde{w}_3)(s,x_{\shortparallel},0)\,\mathrm{d}x_{\shortparallel}\mathrm{d}s\\
&\quad\leqq C(E_{k+1})\Big\{\|(U,G)(0)\|_{\mathcal{H}^k}^2 + \int_0^t\|U(s)\|_{\mathcal{H}^k}^2 + \|G(s)\|_{\mathcal{H}^{k+1}}^2\mathrm{d}s\Big\}\\
&\quad\quad + \lambda\sum_{|\alpha|=k-l}\|\partial_{t,\shortparallel}^{\alpha}\partial_3^l U(t)\|_{L^2}^2.
\end{aligned}\tag{3.39}$$

For the case l being odd, by using (3.30) and (3.34), one can also prove (3.39). The proof is slightly easier than that for the even case, and we omit the details here for simplicity of presentation.

Combining (3.16), (3.17) and (3.39), and taking λ small, we get

$$\begin{aligned}
\sum_{|\alpha|=k-l}\|\partial_{t,\shortparallel}^{\alpha}\partial_3^l U(t)\|_{L^2}^2 &\leqq C(E_{k+1})\int_0^t\|U(s)\|_{\mathcal{H}^k}^2 + \|G(s)\|_{\mathcal{H}^{k+1}}^2\mathrm{d}s\\
&\quad + C(E_{k+1})\|(U,G)(0)\|_{\mathcal{H}^k}^2.
\end{aligned}$$

This completes the induction argument. Therefore we can obtain

$$\|U(t)\|_{\mathcal{H}^k}^2 \leqq C(E_{k+1})\Big\{\|(U,G)(0)\|_{\mathcal{H}^k}^2 + \int_0^t\|U(s)\|_{\mathcal{H}^k}^2 + \|G(s)\|_{\mathcal{H}^{k+1}}^2\mathrm{d}s\Big\}.$$

which, together with Gronwall's inequality, yields that

$$\|U(t)\|_{\mathcal{H}^k}^2 \leqq C(E_{k+1})\left(\|(U,G)(0)\|_{\mathcal{H}^k}+\int_0^t \|G(s)\|_{\mathcal{H}^{k+1}}^2\mathrm{d}s\right). \tag{3.40}$$

Hence we conclude (3.6) by using (3.9), (3.10) and (3.40). □

4. Existence of Solution for a Linear Parabolic System

To construct the solution of viscous boundary layer, we consider the following linear parabolic system of $(u,\theta)=(u_1,u_2,\theta)$

$$\begin{aligned}
&\rho^0\partial_t u_i+\rho^0(\mathfrak{u}_{\shortparallel}^0\cdot\nabla_{\shortparallel})u_i+\rho^0\partial_3\mathfrak{u}_3^0\cdot y\partial_y u_i\\
&\quad+\rho^0 u\cdot\nabla_{\shortparallel}\mathfrak{u}_i^0-\frac{\partial_i p^0}{3T^0}\theta=\mu(T^0)\partial_{yy}u_i+\mathfrak{f}_i,\ i=1,2,\\
&\rho^0\partial_t\theta+\rho^0(\mathfrak{u}_{\shortparallel}^0\cdot\nabla_{\shortparallel})\theta+\rho^0\partial_3\mathfrak{u}_3^0\cdot y\partial_y\theta+\frac{2}{3}\rho^0\mathrm{div}\mathfrak{u}^0\theta=\frac{3}{5}\kappa(T^0)\partial_{yy}\theta+\mathfrak{g},
\end{aligned} \tag{4.1}$$

where $(t,x_{\shortparallel},y)\in[0,\tau]\times\mathbb{R}^2\times\mathbb{R}_+$, and $(\rho^0,\mathfrak{u}^0,T^0,\partial_3\mathfrak{u}_3^0,\mathrm{div}\mathfrak{u}^0,\partial_1 p^0,\partial_2 p^0)$ are the corresponding values of Euler solution on the boundary $x_3=0$, which is independent of $y\in\mathbb{R}_+$. We impose the non-homogenous Neumann boundary conditions for (4.1), i.e.,

$$\begin{cases}\partial_y u_i(t,x_{\shortparallel},y)|_{y=0}=b_i(t,x_{\shortparallel}),\quad \partial_y\theta(t,x_{\shortparallel},y)|_{y=0}=a(t,x_{\shortparallel}),\\ \lim_{y\to\infty}(u,\theta)(t,x_{\shortparallel},y)=0.\end{cases} \tag{4.2}$$

We also impose the initial data

$$u(t,x_{\shortparallel},y)|_{t=0}=u_0(x_{\shortparallel},y),\quad \theta(t,x_{\shortparallel},0)|_{t=0}=\theta_0(x_{\shortparallel},y). \tag{4.3}$$

The initial data (u_0,θ_0) should satisfies the corresponding compatibility condition.

Let $l\geqq 0$, we define the notations

$$\|f\|_{L_l^2}^2=\iint(1+y)^l|f(x_{\shortparallel},y)|^2\,\mathrm{d}x_{\shortparallel}\mathrm{d}y, \tag{4.4}$$

and

$$\bar{x}:=(x_{\shortparallel},y),\quad\text{and}\quad \nabla_{\bar{x}}:=(\nabla_{\shortparallel},\partial_y)\equiv(\partial_{x_1},\partial_{x_2},\partial_y). \tag{4.5}$$

Lemma 4.1. *Let $l\geqq 0$, $k\geqq 3$, and the compatibility condition for the initial data (4.3) is satisfied (Here the compatibility condition means that the initial data (4.3) satisfies the boundary condition (4.2), and the time-derivatives of initial (u_0,θ_0) are defined through system (4.1) inductively). Assume that*

$$\sup_{t\in[0,\tau]}\left\{\sum_{\beta+2\gamma\leqq k+2}\|\nabla_{\shortparallel}^{\beta}\partial_t^{\gamma}(a,b)(t)\|_{L^2(\mathbb{R}^2)}^2+\sum_{j=0}^{k}\sum_{\beta+2\gamma=j}\|\nabla_{\bar{x}}^{\beta}\partial_t^{\gamma}(\mathfrak{f},\ \mathfrak{g})(t)\|_{L_{l_j}^2}^2\right\}<\infty.$$

with $l_j := l + 2(k-j)$, $0 \leqq j \leqq k$. *Then there exists a unique smooth solution* (u,θ) *of* (4.1)–(4.3) *over* $t \in [0,\tau]$, *which satisfies*

$$\sum_{j=0}^{k} \sum_{\beta+2\gamma=j} \sup_{t\in[0,\tau]} \left\{ \|\partial_t^\gamma \nabla_{\bar{x}}^\beta (u,\theta)(t)\|_{L^2_{l_j}}^2 + \int_0^t \|\partial_t^\gamma \nabla_{\bar{x}}^\beta \partial_y (u,\theta)\|_{L^2_{l_j}}^2 ds \right\}$$
$$\leqq C(\tau, E_{k+3}) \Bigg\{ \sum_{j=0}^{k} \sum_{\beta+2\gamma=j} \|\partial_t^\gamma \nabla_{\bar{x}}^\beta (u,\theta)(0)\|_{L^2_{l_j}}^2$$
$$+ \sup_{t\in[0,\tau]} \Bigg[\sum_{\beta+2\gamma\leqq k+2} \|\nabla_{\shortparallel}^\beta \partial_t^\gamma (a,b)(t)\|_{L^2(\mathbb{R}^2)}^2 + \sum_{j=0}^{k} \sum_{\beta+2\gamma=j} \|\nabla_{\bar{x}}^\beta \partial_t^\gamma (\mathfrak{f}, \mathfrak{g})(t)\|_{L^2_{l_j}}^2 \Bigg] \Bigg\}, \tag{4.6}$$

where the notation E_{k+3} *is defined in Lemma* 3.1.

Proof. We define the following background functions

$$u_b := y b(t, x_{\shortparallel}) \chi(y) \quad \text{and} \quad \theta_a := y a(t, x_{\shortparallel}) \chi(y), \tag{4.7}$$

where χ is the smooth monotonic cut-off function defined in (3.8). Clearly, both u_b and θ_a are smooth with compact support in y.

We define

$$\Psi = u - u_b \quad \text{and} \quad \Theta = \theta - \theta_a, \tag{4.8}$$

then (4.1) is reduced to

$$\begin{cases} \partial_t \Psi_i + (\mathfrak{u}_{\shortparallel}^0 \cdot \nabla_{\shortparallel}) \Psi_i + \partial_3 \mathfrak{u}_3^0 \cdot y \partial_y \Psi_i + \Psi \cdot \nabla_{\shortparallel} \mathfrak{u}_i^0 - \dfrac{\partial_i p^0}{3p^0} \Theta \\ \qquad\qquad = \tilde{\mu} \partial_{yy} \Psi_i + \tilde{\mathfrak{f}}_i, \; i = 1, 2, \\ \partial_t \Theta + (\mathfrak{u}_\sigma^0 \cdot \nabla_{\shortparallel}) \Theta + \partial_3 \mathfrak{u}_3^0 \cdot y \partial_y \Theta + \dfrac{2}{3} \mathrm{div} \mathfrak{u}^0 \, \Theta = \tilde{\kappa} \partial_{yy} \Theta + \tilde{\mathfrak{g}}, \end{cases} \tag{4.9}$$

where $\tilde{\mu} := \frac{1}{\rho^0} \mu(T^0)$, $\tilde{\kappa} := \frac{3}{5\rho^0} \kappa(T^0)$, and

$$\tilde{\mathfrak{f}}_i := \frac{1}{\rho^0} \mathfrak{f}_i - \partial_t u_{b,i} - (\mathfrak{u}_{\shortparallel}^0 \cdot \nabla_{\shortparallel}) u_{b,i} - \partial_3 \mathfrak{u}_3^0 \cdot y \partial_y u_{b,i} - u_b \cdot \nabla_{\shortparallel} \mathfrak{u}_i^0$$
$$+ \frac{\partial_i p^0}{3p^0} \theta_a + \tilde{\mu} \partial_{yy} u_{b,i}, \tag{4.10}$$
$$\tilde{\mathfrak{g}} := \frac{1}{\rho^0} \mathfrak{g} - \partial_t \theta_a - (\mathfrak{u}_{\shortparallel}^0 \cdot \nabla_{\shortparallel}) \theta_a - \partial_3 \mathfrak{u}_3^0 \cdot y \partial_y \theta_a - \frac{2}{3} \mathrm{div} \mathfrak{u}^0 \, \theta_a + \tilde{\kappa} \partial_{yy} \theta_a.$$

The boundary conditions (4.2) becomes

$$\begin{cases} \partial_y \Psi_i(t, x_{\shortparallel}, y)|_{y=0} = 0, \quad \partial_y \Theta(t, x_{\shortparallel}, y)|_{y=0} = 0, \; i = 1, 2, \\ \lim_{y\to\infty} (\Psi, \Theta)(t, x_{\shortparallel}, y) = 0. \end{cases} \tag{4.11}$$

Noting the coefficient $\partial_3 \mathfrak{u}_3^0 \cdot y$ in (4.9) is singular as $y \to \infty$ and there are no horizontal viscous terms $\Delta_{\shortparallel}\Psi$ and $\Delta_{\shortparallel}\Theta$, we can not directly apply the standard linear parabolic theory. To prove the existence of smooth solution to (4.9)–(4.11), we divide the proof into several steps.

Step 1. Approximate problem. We consider the following approximate problem

$$\begin{cases} \partial_t \Psi_i + (\mathfrak{u}_{\shortparallel}^0 \cdot \nabla_{\shortparallel})\Psi_i + \partial_3 \mathfrak{u}_3^0 \cdot y\chi_\sigma(y)\partial_y \Psi_i \\ \qquad\qquad +\Psi \cdot \nabla_{\shortparallel}\mathfrak{u}_i^0 - \dfrac{\partial_i p^0}{3p^0}\Theta = \tilde{\mu}\,\partial_{yy}\Psi_i + \xi\Delta_{\shortparallel}\Psi_i + \tilde{\mathfrak{f}}_i^\sigma, \; i = 1, 2, \\ \partial_t \Theta + (\mathfrak{u}_{\shortparallel}^0 \cdot \nabla_{\shortparallel})\Theta + \partial_3 \mathfrak{u}_3^0 \cdot y\partial_y\Theta + \dfrac{2}{3}\mathrm{div}\mathfrak{u}^0\Theta = \tilde{\kappa}\,\partial_{yy}\Theta + \xi\Delta_{\shortparallel}\Theta + \tilde{\mathfrak{g}}^\sigma, \end{cases} \tag{4.12}$$

where $(t, x_{\shortparallel}, y) \in [0, \tau] \times \mathbb{R}^2 \times [0, 3/\sigma]$, $\chi_\sigma(y) = \chi(\sigma y)$ with $0 < \xi, \sigma \ll 1$, and $\tilde{\mathfrak{f}}_i^\sigma := \chi_\sigma(y)\tilde{\mathfrak{f}}_i$, $\tilde{\mathfrak{g}}^\sigma := \chi_\sigma(y)\tilde{\mathfrak{g}}$. Here $\chi(\cdot)$ is defined in (3.8). We impose the following boundary conditions for (4.12):

$$\begin{cases} \partial_y \Psi_i(t, x_{\shortparallel}, y)\big|_{y=0} = 0, \quad \partial_y\Theta(t, x_{\shortparallel}, y)\big|_{y=0} = 0, \\ \Psi(t, x_{\shortparallel}, y)\big|_{y=\frac{3}{\sigma}} = 0, \quad \Theta(t, x_{\shortparallel}, y)\big|_{y=\frac{3}{\sigma}} = 0. \end{cases} \tag{4.13}$$

Imposing the cut-off initial data

$$\Psi(0, x_{\shortparallel}, y) = (u_0 - u_b)\chi_\sigma(y), \qquad \Theta(0, x_{\shortparallel}, y) = (\theta_0 - \theta_a)\chi_\sigma(y),$$

the compatibility condition of initial data at $y = \frac{3}{\sigma}$ is also satisfied due to the property of $\chi(s)$. For the approximate problem (4.12)–(4.13), now we can use the standard linear parabolic theory to obtain the existence of smooth solution in Sobolev space provided the initial data and $(\rho^0, \mathfrak{u}^0, T^0)$ are suitably smooth. To prove the lemma, we need only to obtain some uniform estimates of (Ψ, Θ) with respect to σ and ξ, then take the limit $\sigma, \xi \to 0+$.

Step 2. Uniform energy estimates. We use an induction argument to prove the uniform estimates. Firstly we consider the zero-order derivatives estimation. Multiplying $(4.12)_1$ by $(1 + y)^{l_0}\Psi_i$ and integrating the resultant equation over $[0, t] \times \mathbb{R}^2 \times [0, \frac{3}{\sigma}]$, we have

$$\begin{aligned} &\iint \frac{1}{2}(1+y)^{l_0}|\Psi_i(t, x_{\shortparallel}, y)|^2\, \mathrm{d}x_{\shortparallel}\mathrm{d}y + \frac{1}{2}\int_0^t \iint \partial_3\mathfrak{u}_3^0 y(1+y)^{l_0}\chi_\sigma(y)\partial_y(|\Psi_i|^2) \\ &\quad \leqq \int_0^t \iint \tilde{\mu}\,(1+y)^{l_0}\partial_{yy}\Psi_i\Psi_i\, \mathrm{d}x_{\shortparallel}\mathrm{d}y\mathrm{d}s + \xi\int_0^t \iint (1+y)^{l_0}\Delta_{\shortparallel}\Psi\cdot\Psi_i\, \mathrm{d}x_{\shortparallel}\mathrm{d}y\mathrm{d}s \\ &\qquad + C\|(\rho, \mathfrak{u}, T)\|_{W^{1,\infty}}\int_0^t \|(\Psi, \Theta)\|_{L^2_{l_0}}^2\, \mathrm{d}s + C\int_0^t \|\tilde{\mathfrak{f}}_i\|_{L^2_{l_0}}^2\, \mathrm{d}s + C\|\Psi_i(0)\|_{L^2_{l_0}}^2. \end{aligned} \tag{4.14}$$

Clearly, we have

$$\begin{cases} \chi_\sigma(y) \equiv 0, \ \forall\, y \geqq \frac{2}{\sigma}, \\ |y\partial_y\chi_\sigma(y)| = |y\sigma\chi'(\sigma y)| \leqq C, \ \forall\, y \in \mathbb{R}_+. \end{cases} \tag{4.15}$$

For the second term on LHS of (4.14), integrating by part w.r.t. y and using (4.15), we obtain

$$\left|\int_0^t \iint \partial_3 \mathfrak{u}_3^0 y(1+y)^{l_0}\chi_\sigma(y)\partial_y(|\Psi_i|^2)\,\mathrm{d}x_{\shortparallel}\mathrm{d}y\mathrm{d}s\right| \leqq C\|\mathfrak{u}\|_{W^{1,\infty}}\int_0^t \|\Psi_i(s)\|^2_{L^2_{l_0}}\,\mathrm{d}s. \tag{4.16}$$

For the viscous terms, integrating by parts yields

$$\iint (1+y)^{l_0}\Delta_{\shortparallel}\Psi_i\cdot\Psi_i\,\mathrm{d}x_{\shortparallel}\mathrm{d}y = -\|\nabla_{\shortparallel}\Psi_i\|^2_{L^2_{l_0}},$$

and

$$\begin{aligned}
&\iint \tilde{\mu}\,(1+y)^{l_0}\partial_{yy}\Psi_i\Psi_i\,\mathrm{d}x_{\shortparallel}\mathrm{d}y\\
&= -\iint \tilde{\mu}\,(1+y)^{l_0}|\partial_y\Psi_i|^2\,\mathrm{d}x_{\shortparallel}\mathrm{d}y - \iint l_0\tilde{\mu}\,(1+y)^{l_0-1}\partial_y\Psi_i\Psi_i\,\mathrm{d}x_{\shortparallel}\mathrm{d}y\\
&\quad + \int_{\mathbb{R}^2}\tilde{\mu}\,(1+y)^{l_0}\partial_y\Psi_i\Psi_i\,\mathrm{d}x_{\shortparallel}\Big|_{y=0}^{y=\frac{3}{\sigma}}\\
&\leqq -\frac{1}{2}\iint \tilde{\mu}\,(1+y)^{l_0}|\partial_y\Psi_i|^2\,\mathrm{d}x_{\shortparallel}dy + C\|\Psi_i\|^2_{L^2_{l_0}},
\end{aligned} \tag{4.17}$$

where we have used (4.13) in (4.17).

Substituting (4.16)–(4.17) into (4.14), we get that, for some positive constant $c_0 > 0$

$$\begin{aligned}
&\|\Psi(t)\|^2_{L^2_{l_0}} + \int_0^t \|\partial_y\Psi(s)\|^2_{L^2_{l_0}} + \xi\|\nabla_{\shortparallel}\Psi(s)\|^2_{L^2_{l_0}}\,\mathrm{d}s\\
&\leqq C\Big(\|\Psi(0)\|^2_{L^2_{l_0}} + \int_0^t \|\tilde{\mathfrak{f}}\|^2_{L^2_{l_0}}\,\mathrm{d}s\Big) + C(\|(\rho,\mathfrak{u},T)\|_{W^{1,\infty}})\int_0^t \|(\Psi,\Theta)(s)\|^2_{L^2_{l_0}}\,\mathrm{d}s.
\end{aligned} \tag{4.18}$$

Similarly, we can prove

$$\begin{aligned}
&\|\Theta(t)\|^2_{L^2_{l_0}} + \int_0^t \|\partial_y\Theta(s)\|^2_{L^2_{l_0}} + \xi\|\nabla_{\shortparallel}\Theta(s)\|^2_{L^2_{l_0}}\,\mathrm{d}s\\
&\leqq C\Big(\|\Theta(0)\|^2_{L^2_{l_0}} + \int_0^t \|\tilde{\mathfrak{g}}\|^2_{L^2_{l_0}}\,\mathrm{d}s\Big) + C(\|(\rho,\mathfrak{u},T)\|_{W^{1,\infty}})\int_0^t \|\Theta(s)\|^2_{L^2_{l_0}}\,\mathrm{d}s,
\end{aligned}$$

which, together with (4.18), yields

$$\begin{aligned}
&\|(\Psi,\Theta)(t)\|^2_{L^2_{l_0}} + \int_0^t \|\partial_y(\Psi,\Theta)(s)\|^2_{L^2_{l_0}} + \xi\|\nabla_{\shortparallel}(\Psi,\Theta)(s)\|^2_{L^2_{l_0}}\,\mathrm{d}s\\
&\leqq C\Big(\|(\Psi,\Theta)(0)\|^2_{L^2_{l_0}} + \int_0^t \|(\tilde{\mathfrak{f}},\tilde{\mathfrak{g}})(s)\|^2_{L^2_{l_0}}\,\mathrm{d}s\Big)\\
&\quad + C(\|(\rho,\mathfrak{u},T)\|_{W^{1,\infty}})\int_0^t \|(\Psi,\Theta)(s)\|^2_{L^2_{l_0}}\,\mathrm{d}s.
\end{aligned} \tag{4.19}$$

Now applying the Gronwall's inequality to (4.19), we have

$$\|(\Psi,\Theta)(t)\|^2_{L^2_{l_0}}+\int_0^t\|\partial_y(\Psi,\Theta)(s)\|^2_{L^2_{l_0}}+\xi\|\nabla_{\shortparallel}(\Psi,\Theta)(s)\|^2_{L^2_{l_0}}\,ds$$
$$\leqq C(t,\|(\rho,\mathfrak{u},T)\|_{W^{1,\infty}})\Big\{\|(\Psi,\Theta)(0)\|^2_{L^2_{l_0}}+\int_0^t\|(\tilde{\mathfrak{f}},\tilde{\mathfrak{g}})(s)\|^2_{L^2_{l_0}}\,ds\Big\}. \tag{4.20}$$

We shall use induction arguments to close the uniform energy estimates. We assume, for $0\leqq r\leqq k-1$ ($r\geqq 1$), that

$$\sum_{j=0}^{r-1}\sum_{2\alpha+|\beta|=j}\Big\{\int_0^t\|\partial_t^\alpha\nabla_{\bar{x}}^\beta\partial_y(\Psi,\Theta)(t)\|^2_{L^2_{l_j}}+\xi\|\nabla_{\shortparallel}\partial_t^\alpha\nabla_{\bar{x}}^\beta(\Psi,\Theta)\|^2_{L^2_{l_j}}\,ds$$
$$+\|\partial_t^\alpha\nabla_{\bar{x}}^\beta(\Psi,\Theta)(t)\|^2_{L^2_{l_j}}\Big\}$$
$$\leqq C(t,\|(\rho,\mathfrak{u},T)\|_{W^{r,\infty}})\Big\{\sum_{j=0}^{r-3}\sum_{\beta+2\gamma=j}\|\partial_t^\gamma\nabla_{\bar{x}}^\beta(\tilde{\mathfrak{f}},\tilde{\mathfrak{g}})(0)\|^2_{L^2_{l_j}}$$
$$+\sum_{j=0}^{r-1}\sum_{2\alpha+|\beta|=j}\Big[\|\partial_t^\alpha\nabla_{\bar{x}}^\beta(\Psi,\Theta)(0)\|^2_{L^2_{l_j}}+\int_0^t\|\partial_t^\alpha\nabla_{\bar{x}}^\beta(\tilde{\mathfrak{f}},\tilde{\mathfrak{g}})(s)\|^2_{L^2_{l_j}}\,ds\Big]\Big\}. \tag{4.21}$$

Here we point out that one order of time derivative ∂_t is equal to two orders of space derivatives.

Now we consider the r-order derivative estimates. Let $\partial_{\shortparallel}^\beta=\partial_{x_1}^{\beta_1}\partial_{x_2}^{\beta_2}$. Applying $\partial_t^\alpha\partial_{\shortparallel}^\beta$ to $(4.12)_1$, we have

$$\partial_t\partial_t^\alpha\partial_{\shortparallel}^\beta\Psi_i+\mathfrak{u}^0_{\shortparallel}\cdot\nabla_{\shortparallel}\partial_t^\alpha\partial_{\shortparallel}^\beta\Psi_i+\partial_3\mathfrak{u}^0_3\cdot y\chi_\sigma(y)\partial_y\partial_t^\alpha\partial_{\shortparallel}^\beta\Psi_i$$
$$=\partial_t^\alpha\partial_{\shortparallel}^\beta\big(\tilde{\mu}\partial_y^2\Psi_i\big)+\xi\Delta_{\shortparallel}\partial_t^\alpha\partial_{\shortparallel}^\beta\Psi_i-[\partial_t^\alpha\partial_{\shortparallel}^\beta,\,\mathfrak{u}^0_{\shortparallel}\cdot\nabla_{\shortparallel}]\Psi_i$$
$$-y\chi_\sigma(y)[\partial_t^\alpha\partial_{\shortparallel}^\beta,\,\partial_3\mathfrak{u}^0_3\partial_y]\Psi_i-\partial_t^\alpha\partial_{\shortparallel}^\beta\Big\{\Psi\cdot\nabla_{\shortparallel}\mathfrak{u}^0_i-\frac{\partial_i p^0}{3p^0}\Theta\Big\}+\partial_t^\alpha\partial_{\shortparallel}^\beta\tilde{\mathfrak{f}}_i^\sigma. \tag{4.22}$$

Let $2\alpha+|\beta|=r$. Multiplying (4.22) by $(1+y)^{l_r}\partial_t^\alpha\partial_{\shortparallel}^\beta\Psi_i$, and integrating over $[0,t]\times\mathbb{R}^2\times[0,\frac{3}{\sigma}]$, we have

$$\frac{1}{2}\|\partial_t^\alpha\partial_{\shortparallel}^\beta\Psi_i(t)\|^2_{L^2_{l_r}}+\xi\int_0^t\|\nabla_{\shortparallel}\partial_t^\alpha\partial_{\shortparallel}^\beta\Psi_i(s)\|^2_{L^2_{l_r}}\,ds$$
$$\leqq\frac{1}{2}\|\partial_t^\alpha\partial_{\shortparallel}^\beta\Psi_i(0)\|^2_{L^2_{l_r}}+\int_0^t\iint(1+y)^{l_r}\partial_t^\alpha\partial_{\shortparallel}^\beta\Psi_i\cdot\partial_t^\alpha\partial_{\shortparallel}^\beta\big(\tilde{\mu}\partial_y^2\Psi_i\big)\,dx_{\shortparallel}dyds$$
$$+\Big|\int_0^t\iint y\chi_\sigma(y)(1+y)^{l_r}[\partial_t^\alpha\partial_{\shortparallel}^\beta,\,\partial_3\mathfrak{u}^0_3\partial_y]\Psi_i\cdot\partial_t^\alpha\partial_{\shortparallel}^\beta\Psi_i\,dx_{\shortparallel}dyds\Big|$$
$$+C(t,\|(\rho,\mathfrak{u},T)\|_{W^{r+1,\infty}})\sum_{2\tilde{\alpha}+|\tilde{\beta}|\leqq r}\int_0^t\|\partial_t^{\tilde{\alpha}}\nabla_{\bar{x}}^{\tilde{\beta}}(\Psi,\Theta)(s)\|^2_{L^2_{l_r}}\,ds$$

$$+ C \sum_{2\tilde{\alpha}+|\tilde{\beta}|\leqq r} \int_0^t \|\partial_t^{\tilde{\alpha}} \partial_{\shortparallel}^{\tilde{\beta}} \tilde{\mathfrak{f}}_i(s)\|_{L^2_{l_r}}^2 \,\mathrm{d}s. \tag{4.23}$$

Using (4.13), we have that

$$\partial_t^\alpha \partial_{\shortparallel}^\beta \Psi_i \Big|_{y=\frac{3}{\sigma}} = 0 \quad \text{and} \quad \partial_t^\alpha \partial_{\shortparallel}^\beta \partial_y \Psi_i \Big|_{y=0} = 0,$$

which, together with integrating by parts, yields that

$$\begin{aligned}
&\int_0^t \iint (1+y)^{l_r} \partial_t^\alpha \partial_{\shortparallel}^\beta \Psi_i \cdot \partial_t^\alpha \partial_{\shortparallel}^\beta \left(\tilde{\mu} \partial_y^2 \Psi_i\right) \mathrm{d}x_{\shortparallel} \mathrm{d}y \mathrm{d}s \\
&= -\int_0^t \iint (1+y)^{l_r} \partial_t^\alpha \partial_{\shortparallel}^\beta \partial_y \Psi_i \cdot \partial_t^\alpha \partial_{\shortparallel}^\beta \left(\tilde{\mu} \partial_y \Psi_i\right) \mathrm{d}x_{\shortparallel} \mathrm{d}y \mathrm{d}s \\
&\quad - l_r \int_0^t \iint (1+y)^{l_r-1} \partial_t^\alpha \partial_{\shortparallel}^\beta \Psi_i \cdot \partial_t^\alpha \partial_{\shortparallel}^\beta \left(\tilde{\mu} \partial_y \Psi_i\right) \mathrm{d}x_{\shortparallel} \mathrm{d}y \mathrm{d}s \\
&\leqq -\frac{1}{2} \int_0^t \iint \tilde{\mu}\, (1+y)^{l_r} |\partial_t^\alpha \partial_{\shortparallel}^\beta \partial_y \Psi_i|^2 \,\mathrm{d}x_{\shortparallel} \mathrm{d}y \mathrm{d}s \\
&\quad + C(t, \|(\rho, \mathfrak{u}, T)\|_{W^{r+1,\infty}}) \sum_{j=0}^r \sum_{2\tilde{\alpha}+|\tilde{\beta}|=j} \int_0^t \|\partial_t^{\tilde{\alpha}} \nabla_{\bar{x}}^{\tilde{\beta}} \Psi(s)\|_{L^2_{l_j}}^2 \,\mathrm{d}s.
\end{aligned} \tag{4.24}$$

The third term on RHS of (4.23) is bounded by

$$\begin{aligned}
&\left| \int_0^t \iint y \chi_\sigma(y)(1+y)^{l_r} [\partial_t^\alpha \partial_{\shortparallel}^\beta, \partial_3 \mathfrak{u}_3^0 \partial_y] \Psi_i \cdot \partial_t^\alpha \partial_{\shortparallel}^\beta \Psi_i \,\mathrm{d}x_{\shortparallel} \mathrm{d}y \mathrm{d}s \right| \\
&\leqq C(t, \|(\rho, \mathfrak{u}, T)\|_{W^{r+1,\infty}}) \sum_{2\tilde{\alpha}+|\tilde{\beta}|\leqq r} \int_0^t \|\partial_t^{\tilde{\alpha}} \nabla_{\bar{x}}^{\tilde{\beta}} \Psi(s)\|_{L^2_{l_r}}^2 \,\mathrm{d}s \\
&\quad + \sum_{j=0}^{r-1} \int_0^t \sum_{2\tilde{\alpha}+|\tilde{\beta}|=j} \|\partial_y \partial_t^{\tilde{\alpha}} \nabla_{\bar{x}}^{\tilde{\beta}} \Psi(s)\|_{L^2_{l_j}}^2 \,\mathrm{d}s,
\end{aligned} \tag{4.25}$$

where we have used the fact $l_r + 2 = l_{r-1} \leqq l_j$ for $0 \leqq j \leqq r-1$.

Combining (4.23)–(4.25), and taking $\lambda > 0$ suitably small, we obtain

$$\begin{aligned}
&\sum_{2\alpha+|\beta|=r} \left\{ \|\partial_t^\alpha \partial_{\shortparallel}^\beta \Psi(t)\|_{L^2_{l_r}}^2 + \int_0^t \|\partial_y \partial_t^\alpha \partial_{\shortparallel}^\beta \Psi(s)\|_{L^2_{l_r}}^2 + \xi \|\nabla_{\shortparallel} \partial_t^\alpha \partial_{\shortparallel}^\beta \Psi(s)\|_{L^2_{l_r}}^2 \,\mathrm{d}s \right\} \\
&\leqq C \left\{ \sum_{2\alpha+|\beta|=r} \|\partial_t^\alpha \partial_{\shortparallel}^\beta \Psi(0)\|_{L^2_{l_r}}^2 + \sum_{j=0}^r \sum_{2\tilde{\alpha}+|\tilde{\beta}|=j} \int_0^t \|\partial_t^{\tilde{\alpha}} \partial_{\shortparallel}^{\tilde{\beta}} \tilde{\mathfrak{f}}(s)\|_{L^2_{l_j}}^2 \,\mathrm{d}s \right\} \\
&\quad + C(t, \|(\rho, \mathfrak{u}, T)\|_{W^{r+1,\infty}}) \sum_{j=0}^r \sum_{2\tilde{\alpha}+|\tilde{\beta}|=j} \int_0^t \|\partial_t^{\tilde{\alpha}} \nabla_{\bar{x}}^{\tilde{\beta}} (\Psi, \Theta)(s)\|_{L^2_{l_j}}^2 \,\mathrm{d}s \\
&\quad + C \sum_{j=0}^{r-1} \sum_{2\tilde{\alpha}+|\tilde{\beta}|=j} \int_0^t \|\partial_y \partial_t^{\tilde{\alpha}} \nabla_{\bar{x}}^{\tilde{\beta}} \Psi(s)\|_{L^2_{l_j}}^2 \,\mathrm{d}s.
\end{aligned} \tag{4.26}$$

For the normal derivative estimate, applying ∂_y to (4.22), we have

$$\begin{aligned}
&\partial_t\partial_t^\alpha\partial_{\shortparallel}^\beta\partial_y\Psi_i+\mathfrak{u}_{\shortparallel}^0\cdot\nabla_{\shortparallel}\partial_t^\alpha\partial_{\shortparallel}^\beta\partial_y\Psi_i+\partial_3\mathfrak{u}_3^0\cdot y\chi_\sigma(y)\partial_t^\alpha\partial_{\shortparallel}^\beta\partial_y^2\Psi_i\\
&\quad=\partial_t^\alpha\partial_{\shortparallel}^\beta\big(\tilde{\mu}\partial_y^3\Psi_i\big)+\xi\Delta_{\shortparallel}\partial_t^\alpha\partial_{\shortparallel}^\beta\partial_y\Psi_i-y\chi_\sigma(y)[\partial_t^\alpha\partial_{\shortparallel}^\beta,\,\partial_3\mathfrak{u}_3^0\partial_y^2]\Psi_i\\
&\qquad-\partial_3\mathfrak{u}_3^0\cdot\partial_y\big(y\chi_\sigma(y)\big)\partial_t^\alpha\partial_{\shortparallel}^\beta\partial_y\Psi_i-[\partial_t^\alpha\partial_{\shortparallel}^\beta,\ \mathfrak{u}_{\shortparallel}^0\cdot\nabla_{\shortparallel}]\partial_y\Psi_i\\
&\qquad-\partial_y\big(y\chi_\sigma(y)\big)[\partial_t^\alpha\partial_{\shortparallel}^\beta,\,\partial_3\mathfrak{u}_3^0\partial_y]\Psi_i-\partial_t^\alpha\partial_{\shortparallel}^\beta\partial_y\Big\{\Psi\cdot\nabla_{\shortparallel}\mathfrak{u}_i^0-\frac{\partial_i p^0}{3p^0}\Theta\Big\}\\
&\qquad+\partial_t^\alpha\partial_{\shortparallel}^\beta\partial_y\tilde{\mathfrak{f}}_i^\sigma.
\end{aligned}\tag{4.27}$$

Multiplying (4.27) by $(1+y)^{l_r}\partial_t^\alpha\partial_{\shortparallel}^\beta\partial_y\Psi_i$, and integrating over $[0,t]\times\mathbb{R}^2\times[0,\frac{3}{\sigma}]$, we have

$$\begin{aligned}
&\sum_{2\alpha+|\beta|=r-1}\Big\{\frac12\|\partial_t^\alpha\partial_{\shortparallel}^\beta\partial_y\Psi(t)\|_{L^2_{l_r}}^2+\int_0^t\xi\|\nabla_{\shortparallel}\partial_t^\alpha\partial_{\shortparallel}^\beta\partial_y\Psi(s)\|_{L^2_{l_r}}^2\,\mathrm{d}s\Big\}\\
&\leqq\sum_{2\alpha+|\beta|=r-1}\int_0^t\iint(1+y)^{l_r}\partial_t^\alpha\partial_{\shortparallel}^\beta\partial_y\Psi_i\cdot\partial_t^\alpha\partial_{\shortparallel}^\beta\big(\tilde{\mu}\partial_y^3\Psi_i\big)\,\mathrm{d}x_{\shortparallel}\mathrm{d}y\mathrm{d}s\\
&\quad+\Big\{\sum_{2\alpha+|\beta|=r-1}\frac12\|\partial_t^\alpha\partial_{\shortparallel}^\beta\partial_y\Psi(0)\|_{L^2_{l_r}}^2+C\sum_{j=0}^{r-1}\sum_{2\tilde{\alpha}+|\tilde{\beta}|=j}\int_0^t\|\partial_t^{\tilde{\alpha}}\partial_{\shortparallel}^{\tilde{\beta}}\partial_y\tilde{\mathfrak{f}}\|_{L^2_{l_j}}^2\,\mathrm{d}s\Big\}\\
&\quad+C(t,\|(\rho,\mathfrak{u},T)\|_{W^{r,\infty}})\sum_{j=0}^{r}\sum_{2\tilde{\alpha}+|\tilde{\beta}|=j}\int_0^t\|\partial_t^{\tilde{\alpha}}\nabla_{\bar{x}}^{\tilde{\beta}}(\Psi,\Theta)(s)\|_{L^2_{l_j}}^2\,\mathrm{d}s\\
&\quad+C\sum_{j=0}^{r-1}\sum_{2\tilde{\alpha}+|\tilde{\beta}|=j}\int_0^t\|\partial_y\partial_t^{\tilde{\alpha}}\nabla_{\bar{x}}^{\tilde{\beta}}\Psi(s)\|_{L^2_{l_j}}^2\,\mathrm{d}s.
\end{aligned}\tag{4.28}$$

Using (4.13) and (4.22), we have

$$\begin{cases}\partial_t^\alpha\partial_{\shortparallel}^\beta\partial_y\Psi_i\Big|_{y=0}=0,\\ \partial_t^\alpha\partial_{\shortparallel}^\beta\big(\tilde{\mu}\partial_y^2\Psi_i\big)\Big|_{y=\frac{3}{\sigma}}=-\xi\Delta_{\shortparallel}\partial_t^\alpha\partial_{\shortparallel}^\beta\Psi_i\Big|_{y=\frac{3}{\sigma}}=0,\end{cases}$$

which, together with integrating by parts, yields that

$$\begin{aligned}
&\int_0^t\iint(1+y)^{l_r}\partial_t^\alpha\partial_{\shortparallel}^\beta\partial_y\Psi_i\cdot\partial_t^\alpha\partial_{\shortparallel}^\beta\big(\tilde{\mu}\partial_y^3\Psi_i\big)\,\mathrm{d}x_{\shortparallel}\mathrm{d}y\mathrm{d}s\\
&\quad=-\int_0^t\iint(1+y)^{l_r}\partial_t^\alpha\partial_{\shortparallel}^\beta\partial_y^2\Psi_i\cdot\partial_t^\alpha\partial_{\shortparallel}^\beta\big(\tilde{\mu}\partial_y^2\Psi_i\big)\,\mathrm{d}x_{\shortparallel}\mathrm{d}y\mathrm{d}s\\
&\qquad-l_r\int_0^t\iint(1+y)^{l_r-1}\partial_t^\alpha\partial_{\shortparallel}^\beta\partial_y\Psi_i\cdot\partial_t^\alpha\partial_{\shortparallel}^\beta\big(\tilde{\mu}\partial_y^2\Psi_i\big)\,\mathrm{d}x_{\shortparallel}\mathrm{d}y\mathrm{d}s\\
&\quad\leqq-\frac12\int_0^t\iint\tilde{\mu}\,(1+y)^{l_r}|\partial_t^\alpha\partial_{\shortparallel}^\beta\partial_y^2\Psi_i|^2\,\mathrm{d}x_{\shortparallel}\mathrm{d}y\mathrm{d}s
\end{aligned}$$

$$+ C(t, \|(\rho, \mathfrak{u}, T)\|_{W^{r+1,\infty}}) \sum_{j=0}^{r} \sum_{2\tilde{\alpha}+|\tilde{\beta}|=j} \int_0^t \|\partial_t^{\tilde{\alpha}} \nabla_{\bar{x}}^{\tilde{\beta}} \Psi(s)\|_{L^2_{l_j}}^2 \, ds. \tag{4.29}$$

Combining (4.28)–(4.29), we obtain

$$\begin{aligned}
&\sum_{2\alpha+|\beta|=r-1} \left\{ \int_0^t \|\partial_y^2 \partial_t^\alpha \partial_{\shortparallel}^\beta \Psi(s)\|_{L^2_{l_r}}^2 + \xi \|\nabla_{\shortparallel} \partial_t^\alpha \partial_{\shortparallel}^\beta \partial_y \Psi(s)\|_{L^2_{l_r}}^2 \, ds \right. \\
&\qquad \left. + \frac{1}{2} \|\partial_t^\alpha \partial_{\shortparallel}^\beta \partial_y \Psi(t)\|_{L^2_{l_r}}^2 \right\} \\
&\leqq C \left\{ \sum_{2\alpha+|\beta|=r-1} \frac{1}{2} \|\partial_t^\alpha \partial_{\shortparallel}^\beta \partial_y \Psi(0)\|_{L^2_{l_r}}^2 + C \sum_{j=0}^{r-1} \sum_{2\tilde{\alpha}+|\tilde{\beta}|=j} \int_0^t \|\partial_t^{\tilde{\alpha}} \partial_{\shortparallel}^{\tilde{\beta}} \partial_y \tilde{\mathfrak{f}}\|_{L^2_{l_j}}^2 \right\} \\
&\quad + C(t, \|(\rho, \mathfrak{u}, T)\|_{W^{r,\infty}}) \sum_{j=0}^{r} \sum_{2\tilde{\alpha}+|\tilde{\beta}|=j} \int_0^t \|\partial_t^{\tilde{\alpha}} \nabla_{\bar{x}}^{\tilde{\beta}} (\Psi, \Theta)(s)\|_{L^2_{l_j}}^2 \, ds \\
&\quad + C \sum_{j=0}^{r-1} \sum_{2\tilde{\alpha}+|\tilde{\beta}|=j} \int_0^t \|\partial_y \partial_t^{\tilde{\alpha}} \nabla_{\bar{x}}^{\tilde{\beta}} \Psi(s)\|_{L^2_{l_j}}^2 \, ds.
\end{aligned} \tag{4.30}$$

For the higher order normal derivatives estimate, we shall use the equation directly. In fact, applying $\partial_t^\alpha \partial_{\shortparallel}^\beta \partial_y^n$ with $n \geqq 0$ to $(4.12)_1$, then we have

$$\begin{aligned}
\tilde{\mu} \partial_t^\alpha \partial_{\shortparallel}^\beta \partial_y^{n+2} \Psi_i &= \partial_t \partial_t^\alpha \partial_{\shortparallel}^\beta \partial_y^n \Psi_i - \xi \Delta_{\shortparallel} \partial_t^\alpha \partial_{\shortparallel}^\beta \partial_y^n \Psi_i - [\partial_t^\alpha \partial_{\shortparallel}^\beta, \tilde{\mu}] \partial_y^{n+2} \Psi_i \\
&\quad + \partial_t^\alpha \partial_{\shortparallel}^\beta \left\{ \partial_3 \mathfrak{u}_3^0 \cdot y \chi_\sigma(y) \partial_y^{n+1} \Psi_i \right\} + \partial_t^\alpha \partial_{\shortparallel}^\beta \left\{ \partial_3 \mathfrak{u}_3^0 [\partial_y^n, \, y \chi_\sigma(y)] \partial_y \Psi_i \right\} \\
&\quad + \partial_t^\alpha \partial_{\shortparallel}^\beta \left\{ \mathfrak{u}_{\shortparallel}^0 \cdot \nabla_{\shortparallel} \partial_y^n \Psi_i + \partial_y^n \Psi \cdot \nabla_{\shortparallel} \mathfrak{u}_i^0 - \frac{\partial_i p^0}{3 p^0} \partial_y^n \Theta \right\} - \partial_t^\alpha \partial_{\shortparallel}^\beta \partial_y^n \tilde{\mathfrak{f}}_i^\sigma,
\end{aligned}$$

which yields that

$$\begin{aligned}
&\sum_{2\alpha+|\beta|=m} |\partial_t^\alpha \partial_{\shortparallel}^\beta \partial_y^{n+2} \Psi| \\
&\leqq C \sum_{2\tilde{\alpha}+|\tilde{\beta}|=m+2} |\partial_t^{\tilde{\alpha}} \partial_{\shortparallel}^{\tilde{\beta}} \partial_y^n \Psi| + C(\|(\rho, \mathfrak{u}, T)\|_{W^{r,\infty}}) \left\{ \sum_{2\tilde{\alpha}+|\tilde{\beta}|\leqq m+n+1} |\partial_t^{\tilde{\alpha}} \nabla_{\bar{x}}^{\tilde{\beta}} (\Psi, \Theta)| \right. \\
&\quad \left. + \sum_{2\tilde{\alpha}+|\tilde{\beta}|\leqq m+n} y |\partial_y \partial_t^{\tilde{\alpha}} \nabla_{\bar{x}}^{\tilde{\beta}} \Psi| \right\} + \sum_{2\tilde{\alpha}+|\tilde{\beta}|\leqq m+n} |\partial_t^{\tilde{\alpha}} \nabla_{\bar{x}}^{\tilde{\beta}} \tilde{\mathfrak{f}}|.
\end{aligned} \tag{4.31}$$

For any fixed $0 \leqq n \leqq r-2$, by (4.31), a direct calculation shows that

$$\begin{aligned}
&\sum_{2\alpha+|\beta|=r-n-2} \left\{ \|\partial_y^{n+2} \partial_t^\alpha \partial_{\shortparallel}^\beta \Psi(t)\|_{L^2_{l_r}}^2 + \int_0^t \|\partial_y^{n+3} \partial_t^\alpha \partial_{\shortparallel}^\beta \Psi(s)\|_{L^2_{l_r}}^2 \, ds \right\} \\
&\leqq C \sum_{2\tilde{\alpha}+|\tilde{\beta}|=r-n} \left\{ \|\partial_y^n \partial_t^{\tilde{\alpha}} \partial_{\shortparallel}^{\tilde{\beta}} \Psi(t)\|_{L^2_{l_r}}^2 + \int_0^t \|\partial_y^{n+1} \partial_t^{\tilde{\alpha}} \partial_{\shortparallel}^{\tilde{\beta}} \Psi(s)\|_{L^2_{l_r}}^2 \, ds \right\}
\end{aligned}$$

$$+C(\|(\rho,\mathfrak{u},T)\|_{W^{r,\infty}})\sum_{j=0}^{r-1}\sum_{2\tilde{\alpha}+|\tilde{\beta}|=j}\left\{\|\partial_t^{\tilde{\alpha}}\nabla_{\bar{x}}^{\tilde{\beta}}(\Psi,\Theta)(t)\|_{L^2_{l_j}}^2+\int_0^t\|\partial_y\partial_t^{\tilde{\alpha}}\nabla_{\bar{x}}^{\tilde{\beta}}\Psi\|_{L^2_{l_j}}^2\,\mathrm{d}s\right\}$$

$$+C\sum_{j=0}^{r-2}\sum_{2\tilde{\alpha}+|\tilde{\beta}|=j}\|\partial_t^{\tilde{\alpha}}\nabla_{\bar{x}}^{\tilde{\beta}}\tilde{\mathfrak{f}}(t)\|_{L^2_{l_j}}^2+C\sum_{j=0}^{r-1}\sum_{2\tilde{\alpha}+|\tilde{\beta}|=j}\int_0^t\|\partial_t^{\tilde{\alpha}}\nabla_{\bar{x}}^{\tilde{\beta}}\tilde{\mathfrak{f}}(s)\|_{L^2_{l_j}}^2\,\mathrm{d}s. \tag{4.32}$$

Noting (4.26) and (4.30), by using induction arguments on $0\leqq n\leqq r-2$ in (4.32), we can obtain

$$\sum_{2\alpha+|\beta|=r}\left\{\|\partial_t^{\alpha}\nabla_{\bar{x}}^{\beta}\Psi(t)\|_{L^2_{l_r}}^2+\int_0^t\|\partial_y\partial_t^{\alpha}\nabla_{\bar{x}}^{\beta}\Psi(s)\|_{L^2_{l_r}}^2+\xi\|\nabla_{\shortparallel}\partial_t^{\alpha}\nabla_{\bar{x}}^{\beta}\Psi(s)\|_{L^2_{l_r}}^2\,\mathrm{d}s\right\}$$

$$\leqq C(\|(\rho,\mathfrak{u},T)\|_{W^{r,\infty}})\sum_{j=0}^{r-1}\sum_{2\tilde{\alpha}+|\tilde{\beta}|=j}\left\{\|\partial_t^{\tilde{\alpha}}\nabla_{\bar{x}}^{\tilde{\beta}}(\Psi,\Theta)(t)\|_{L^2_{l_j}}^2+\int_0^t\|\partial_y\partial_t^{\tilde{\alpha}}\nabla_{\bar{x}}^{\tilde{\beta}}\Psi\|_{L^2_{l_j}}^2\right\}$$

$$+C(t,\|(\rho,\mathfrak{u},T)\|_{W^{r+1,\infty}})\sum_{j=0}^{r}\sum_{2\tilde{\alpha}+|\tilde{\beta}|=j}\int_0^t\|\partial_t^{\tilde{\alpha}}\nabla_{\bar{x}}^{\tilde{\beta}}(\Psi,\Theta)(s)\|_{L^2_{l_j}}^2\,\mathrm{d}s$$

$$+C\sum_{j=0}^{r-2}\sum_{2\tilde{\alpha}+|\tilde{\beta}|=j}\|\partial_t^{\tilde{\alpha}}\nabla_{\bar{x}}^{\tilde{\beta}}\tilde{\mathfrak{f}}(t)\|_{L^2_{l_j}}^2+C\sum_{j=0}^{r}\sum_{2\tilde{\alpha}+|\tilde{\beta}|=j}\int_0^t\|\partial_t^{\tilde{\alpha}}\nabla_{\bar{x}}^{\tilde{\beta}}\tilde{\mathfrak{f}}(s)\|_{L^2_{l_j}}^2\,\mathrm{d}s$$

$$+C\sum_{2\alpha+|\beta|=r}\|\partial_t^{\alpha}\nabla_{\bar{x}}^{\beta}\Psi(0)\|_{L^2_{l_r}}^2$$

$$\leqq C(t,\|(\rho,\mathfrak{u},T)\|_{W^{r+1,\infty}})\left\{\sum_{j=0}^{r}\sum_{2\tilde{\alpha}+|\tilde{\beta}|=j}\left[\|\partial_t^{\alpha}\nabla_{\bar{x}}^{\beta}\Psi(0)\|_{L^2_{l_j}}^2+\int_0^t\|\partial_t^{\tilde{\alpha}}\nabla_{\bar{x}}^{\tilde{\beta}}(\tilde{\mathfrak{f}},\tilde{\mathfrak{g}})\|_{L^2_{l_j}}^2\right]\right.$$

$$\left.+\int_0^t\sum_{2\alpha+|\beta|=r}\|\partial_t^{\alpha}\nabla_{\bar{x}}^{\beta}(\Psi,\Theta)\|_{L^2_{l_j}}^2\,\mathrm{d}s+C\sum_{j=0}^{r-2}\sum_{2\tilde{\alpha}+|\tilde{\beta}|=j}\|\partial_t^{\tilde{\alpha}}\nabla_{\bar{x}}^{\tilde{\beta}}(\tilde{\mathfrak{f}},\tilde{\mathfrak{g}})(0)\|_{L^2_{l_j}}^2\right\}, \tag{4.33}$$

where we have used (4.21) in the second inequality.

As in (4.33), we can also get

$$\sum_{2\alpha+|\beta|=r}\left\{\|\partial_t^{\alpha}\nabla_{\bar{x}}^{\beta}\Theta(t)\|_{L^2_{l_r}}^2+\int_0^t\|\partial_y\partial_t^{\alpha}\nabla_{\bar{x}}^{\beta}\Theta(s)\|_{L^2_{l_r}}^2+\xi\|\nabla_{\shortparallel}\partial_t^{\alpha}\nabla_{\bar{x}}^{\beta}\Theta\|_{L^2_{l_r}}^2\,\mathrm{d}s\right\}$$

$$\leqq C(t,\|(\rho,\mathfrak{u},T)\|_{W^{r+1,\infty}})\left\{\sum_{j=0}^{r}\sum_{2\tilde{\alpha}+|\tilde{\beta}|=j}\left[\|\partial_t^{\alpha}\nabla_{\bar{x}}^{\beta}\Theta(0)\|_{L^2_{l_j}}^2+\int_0^t\|\partial_t^{\tilde{\alpha}}\nabla_{\bar{x}}^{\tilde{\beta}}(\tilde{\mathfrak{f}},\tilde{\mathfrak{g}})\|_{L^2_{l_j}}^2\right]\right.$$

$$\left.+\int_0^t\sum_{2\alpha+|\beta|=r}\|\partial_t^{\alpha}\nabla_{\bar{x}}^{\beta}\Theta(s)\|_{L^2_{l_j}}^2\,\mathrm{d}s+C\sum_{j=0}^{r-2}\sum_{2\tilde{\alpha}+|\tilde{\beta}|=j}\|\partial_t^{\tilde{\alpha}}\nabla_{\bar{x}}^{\tilde{\beta}}(\tilde{\mathfrak{f}},\tilde{\mathfrak{g}})(0)\|_{L^2_{l_j}}^2\right\}. \tag{4.34}$$

Now, combining (4.33) and (4.34) and then using the Gronwall's inequality, we obtain

$$\sum_{2\alpha+|\beta|=r}\left\{\int_0^t\|\partial_y\partial_t^{\alpha}\nabla_{\bar{x}}^{\beta}(\Psi,\Theta)(s)\|_{L^2_{l_r}}^2+\xi\|\nabla_{\shortparallel}\partial_t^{\alpha}\nabla_{\bar{x}}^{\beta}(\Psi,\Theta)(s)\|_{L^2_{l_r}}^2\,\mathrm{d}s\right.$$

$$+\|\partial_t^\alpha \nabla_{\bar{x}}^\beta(\Psi,\Theta)(t)\|^2_{L^2_{l_r}}\Bigg\}$$
$$\leqq C(t,\|(\rho,\mathfrak{u},T)\|_{W^{r+1,\infty}})\Bigg\{\sum_{j=0}^{r}\sum_{2\tilde{\alpha}+|\tilde{\beta}|=j}\Big[\|\partial_t^\alpha \nabla_{\bar{x}}^\beta(\Psi,\Theta)(0)\|^2_{L^2_{l_j}}+\int_0^t\|\partial_t^{\tilde{\alpha}}\nabla_{\bar{x}}^{\tilde{\beta}}(\tilde{\mathfrak{f}},\mathfrak{g})\|^2_{L^2_{l_j}}\Big]$$
$$+C\sum_{j=0}^{r-2}\sum_{2\tilde{\alpha}+|\tilde{\beta}|=j}\|\partial_t^{\tilde{\alpha}}\nabla_{\bar{x}}^{\tilde{\beta}}(\tilde{\mathfrak{f}},\mathfrak{g})(0)\|^2_{L^2_{l_j}}\Bigg\}.$$

Hence by the induction arguments, we have

$$\sum_{j=0}^{k}\sum_{2\alpha+|\beta|=j}\Bigg\{\int_0^t\|\partial_y\partial_t^\alpha\nabla_{\bar{x}}^\beta(\Psi,\Theta)(s)\|^2_{L^2_{l_j}}+\xi\|\nabla_{\shortparallel}\partial_t^\alpha\nabla_{\bar{x}}^\beta(\Psi,\Theta)(s)\|^2_{L^2_{l_j}}\,\mathrm{d}s$$
$$+\|\partial_t^\alpha\nabla_{\bar{x}}^\beta(\Psi,\Theta)(t)\|^2_{L^2_{l_j}}\Bigg\}$$
$$\leqq C(t,\|(\rho,\mathfrak{u},T)\|_{W^{k+1,\infty}})\Bigg\{\sum_{j=0}^{k}\sum_{2\tilde{\alpha}+|\tilde{\beta}|=j}\Big[\|\partial_t^\alpha\nabla_{\bar{x}}^\beta(\Psi,\Theta)(0)\|^2_{L^2_{l_j}}$$
$$+\int_0^t\|\partial_t^{\tilde{\alpha}}\nabla_{\bar{x}}^{\tilde{\beta}}(\tilde{\mathfrak{f}},\tilde{\mathfrak{g}})(s)\|^2_{L^2_{l_j}}\,\mathrm{d}s\Big]+C\sum_{j=0}^{k-2}\sum_{2\tilde{\alpha}+|\tilde{\beta}|=j}\|\partial_t^{\tilde{\alpha}}\nabla_{\bar{x}}^{\tilde{\beta}}(\tilde{\mathfrak{f}},\tilde{\mathfrak{g}})(0)\|^2_{L^2_{l_j}}\Bigg\}. \tag{4.35}$$

Step 3. Taking limits $\sigma,\xi\to 0+$. Based on the uniform estimates (4.35), we can first take the limit $\sigma\to 0+$, and then $\xi\to 0+$. Then, by using (4.7)–(4.10) and (4.35), Lemma 4.1 is proved, the details are omitted for simplicity of presentation. □

Remark 4.2. In the proof of Lemma 4.1, we can improve the polynomial decay of y to an exponential decay if $\partial_3\mathfrak{u}_3^0<0$ with a positive contribution in (4.16).

5. Construction on the Solutions of Expansions

We define the velocity weight functions

$$\tilde{w}_{\kappa_i}(v)=w_{\kappa_i}(v)\mu^{-\mathfrak{a}},\quad \mathfrak{w}_{\bar{\kappa}_i}(v)=w_{\bar{\kappa}_i}(v)\mu_0^{-\mathfrak{a}}\text{ and }\mathfrak{w}_{\hat{\kappa}_i}(v)=w_{\hat{\kappa}_i}(v)\mu_0^{-\mathfrak{a}} \tag{5.1}$$

for constants $\kappa_i,\bar{\kappa}_i,\hat{\kappa}_i\geqq 0$, $1\leqq i\leqq N$ and $0\leqq\mathfrak{a}<\frac{1}{2}$. Note that the weight function $\tilde{w}_{\kappa_i}$ depends on (t,x), while $\mathfrak{w}_{\bar{\kappa}_i}$ and $\mathfrak{w}_{\hat{\kappa}_i}$ depend on $(t,x_{\shortparallel})$. For later use, we define

$$\hat{x}=(x_{\shortparallel},\eta)\in\mathbb{R}^3_+,\quad \nabla_{\hat{x}}:=(\nabla_{\shortparallel},\partial_\eta),$$

and recall $\bar{x}=(x_{\shortparallel},y)\in\mathbb{R}^3_+$ and $\nabla_{\bar{x}}=(\nabla_{\shortparallel},\partial_y)$ in (4.5), and the weighted L^2_l-norm in (4.4).

Proposition 5.1. *Let* $0 \leqq \mathfrak{a} < \frac{1}{2}$ *in* (5.1)*. Let* $s_0, s_i, \bar{s}_i, \hat{s}_i \in \mathbb{N}_+$, $\kappa_i, \bar{\kappa}_i, \hat{\kappa}_i \in \mathbb{R}_+$ *for* $1 \leqq i \leqq N$*; and define* $l^i_j := \bar{l}_i + 2(\bar{s}_i - j)$ *for* $1 \leqq i \leqq N$, $0 \leqq j \leqq \bar{s}_i$*. For these parameters, we assume the restrictions* (5.42)–(5.44) *hold. Let the initial data* $(\rho_i, u_i, \theta_i)(0)$ *of IBVP* (2.4)*,* (2.26)*,* (2.27)*, and initial data* $(\bar{u}_{i,\shortparallel}, \bar{\theta}_i)(0)$ *of IBVP* (1.27)–(1.28)*,* (2.30)–(2.31) *satisfy*

$$\sum_{i=0}^{N}\Big\{\sum_{\gamma+\beta\leqq s_i}\|\partial_t^\gamma\nabla_x^\beta(\rho_i,u_i,\theta_i)(0)\|_{L^2_x}+\sum_{j=0}^{\bar{s}_i}\sum_{j=2\gamma+\beta}\|\partial_t^\gamma\nabla_{\bar{x}}^\beta(\bar{u}_{i,\shortparallel},\bar{\theta}_i)(0)\|^2_{L^2_{l^i_j}}\Big\}<\infty. \tag{5.2}$$

We also assume that the compatibility conditions for initial data $(\rho_i, u_i, \theta_i)(0)$ *and* $(\bar{u}_{i,\shortparallel}, \bar{\theta}_i)(0)$ *are satisfied (see Remark* 5.2 *for details). Then there exist solutions* $F_i = \sqrt{\mu} f_i$, $\bar{F}_i = \sqrt{\mu_0}\bar{f}_i$, $\hat{F}_i = \sqrt{\mu_0}\hat{f}_i$ *to* (1.7)*,* (1.15)*,* (1.38) *over the time interval* $t \in [0, \tau]$*, respectively. Moreover, we have the following uniform estimates*

$$\begin{aligned}
&\sup_{t\in[0,\tau]}\sum_{i=1}^{N}\Big\{\sum_{\gamma+\beta\leqq s_i}\|\tilde{w}_{\kappa_i}\partial_t^\gamma\nabla_x^\beta f_i(t)\|_{L^2_xL^\infty_v}+\sum_{j=0}^{\bar{s}_i}\sum_{j=2\gamma+\beta}\|\mathfrak{w}_{\bar{\kappa}_i}\partial_t^\gamma\nabla_{\bar{x}}^\beta\bar{f}_i(t)\|_{L^2_{l^i_j}L^\infty_v}\\
&\quad+\sum_{\gamma+\beta\leqq\hat{s}_i}\|e^{\zeta_i\cdot\eta}\mathfrak{w}_{\hat{\kappa}_i}\partial_t^\gamma\nabla_{\shortparallel}^\beta\hat{f}_i(t)\|_{L^\infty_{\hat{x},v}\cap L^2_{x_\shortparallel}L^\infty_{\eta,v}}\Big\}\\
&\leqq C\Big(\tau,\|(\varphi_0,\Phi_0,\vartheta_0)\|_{H^{s_0}}+\sum_{i=0}^{N}\sum_{\gamma+\beta\leqq s_i}\|\partial_t^\gamma\nabla_x^\beta(\rho_i,u_i,\theta_i)(0)\|_{L^2_x}\\
&\quad+\sum_{i=0}^{N}\sum_{j=0}^{\bar{s}_i}\sum_{j=2\gamma+\beta}\|\partial_t^\gamma\nabla_{\bar{x}}^\beta(\bar{u}_{i,\shortparallel},\bar{\theta}_i)(0)\|^2_{L^2_{l^i_j}}\Big),
\end{aligned} \tag{5.3}$$

where the positive constants $\zeta_i > 0\, (i = 1, \ldots, N)$ *satisfying* $\zeta_{i+1} = \frac{1}{2}\zeta_i$ *and* $\zeta_1 = 1$.

Remark 5.2. The compatibility conditions for $(\rho_k, u_k, \theta_k)(0)$ with $k = 1, \ldots, N$ mean that $u_{k,3}(0)$ satisfies the boundary conditions (2.26)–(2.27), and the time-derivatives of $(\rho_k, u_k, \theta_k)(0)$ are defined through system (2.4) inductively. The compatibility conditions for $(\bar{u}_{k,\shortparallel}, \bar{\theta}_k)(0)$ with $k = 1, \ldots, N$ mean that $(\bar{u}_{k,\shortparallel}, \bar{\theta}_k)(0)$ satisfies the boundary conditions (2.30)–(2.31), and the time-derivatives of $(\bar{u}_{k,\shortparallel}, \bar{\theta}_k)(0)$ are defined through equations (1.27)–(1.28) inductively.

Remark 5.3. Since the Knudsen boundary layer $\hat{f}_i$ is indeed a stationary problem with $(t, x_\shortparallel)$ as parameters, hence there is no necessary to give initial data for $\hat{f}_i$. Here the functions f_i, $\bar{f}_i$ are smooth, however $\partial_t^\gamma\nabla_\shortparallel^\beta\hat{f}_i$ is only continuous away from the grazing set $[0, \tau] \times \gamma_0$.

Proof. Since the proof is very complicate, we divide the proof into several steps.
Step 1. Construction of solutions f_1, $\bar{f}_1$ *and* $\hat{f}_1$.

Step 1.1. Construction of solution f_1. Noting $f_1 \in \mathcal{N}$, we need only to construct the macroscopic part (ρ_1, u_1, θ_1). Hence we consider (2.4) with $k = 0$ and the boundary condition (2.26). Then by using Lemma 3.1, we establish the existence of smooth solution of (2.4) (with $k = 0$), (2.26) with the following estimate

$$\sup_{t\in[0,\tau]} \sum_{\gamma+\beta\leqq s_1} \|\partial_t^\gamma \nabla_x^\beta (\rho_1, u_1, \theta_1)(t)\|_{L^2(\mathbb{R}^3_+)}$$
$$\leqq C(\tau, E_{2+s_1}) \sum_{\gamma+\beta\leqq s_1} \|\partial_t^\gamma \nabla_x^\beta (\rho_1, u_1, \theta_1)(0)\|_{L^2(\mathbb{R}^3_+)},$$

with $s_1 \gg 1$ such that $s_0 \geqq 2 + s_1 > 0$, and E_k is defined in Lemma 3.1. Therefore we have proved the existence of smooth solution f_1 over $[0, \tau]$ with

$$\sup_{t\in[0,\tau]} \|\tilde{w}_{\kappa_1} f_1(t)\|_{L^2_x L^\infty_v} \leqq C(\tau, E_{2+s_1}) \sum_{\gamma+\beta\leqq s_1} \|\partial_t^\gamma \nabla_x^\beta (\rho_1, u_1, \theta_1)(0)\|_{L^2(\mathbb{R}^3_+)},$$

for any $\kappa_1 > 0$.

Step 1.2. Construction of solution $\bar{f}_1$. Noting (1.22), (1.21) and (1.24), we need only to calculate $(\bar{u}_{1,\shortparallel}, \bar{\theta}_1)$. Taking $k = 2$ in (2.30)–(2.31), then using (1.24) and the facts that

$$\bar{J}_0 = 0, \quad (\hat{A}_2, \hat{B}_2, \hat{C}_2) = (0, 0, 0),$$

we have

$$\begin{cases} \partial_y \bar{u}_{1,i}(t, x_\shortparallel, 0) = \dfrac{1}{\mu(T^0)} \langle T^0 \mathcal{A}^0_{3i},\ (\mathbf{I} - \mathbf{P_0}) f_2 \rangle (t, x_\shortparallel, 0),\ i = 1, 2, \\ \partial_y \bar{\theta}_1(t, x_\shortparallel, 0) = \dfrac{1}{\kappa(T^0)} \langle 2(T^0)^{\frac{3}{2}} \mathcal{B}^0_3,\ (\mathbf{I} - \mathbf{P_0}) f_2 \rangle (t, x_\shortparallel, 0). \end{cases} \tag{5.4}$$

It follows from (2.3) that

$$\{\mathbf{I} - \mathbf{P}\} f_2 = \mathbf{L}^{-1} \left(-\frac{\{\partial_t + v \cdot \nabla_x\}\mu}{\sqrt{\mu}} + \frac{1}{\sqrt{\mu}} Q(\sqrt{\mu} f_1, \sqrt{\mu} f_1) \right).$$

From (A.13)

$$\{\mathbf{I} - \mathbf{P}\} \left(-\frac{\{\partial_t + v \cdot \nabla_x\}\mu}{\sqrt{\mu}} \right) = \sum_{j,l=1}^{3} \partial_j \mathfrak{u}_l \mathcal{A}_{jl} + \sum_{j=1}^{3} \frac{\partial_j T}{\sqrt{T}} \mathcal{B}_j. \tag{5.5}$$

Since $f_1 \in \mathcal{N}$, as in (A.11)–(A.12), it holds that

$$\mathbf{L}^{-1} \left(\frac{1}{\sqrt{\mu}} Q(\sqrt{\mu} f_1, \sqrt{\mu} f_1) \right)$$
$$= \sum_{j,l=1}^{3} \frac{1}{2T} u_{1,l} u_{1,j} \mathcal{A}_{lj} + \frac{\theta_1}{3T^{\frac{3}{2}}} u_1 \cdot \mathcal{B} + \frac{\theta_1^2}{72T^2} \{\mathbf{I} - \mathbf{P}\} \left\{ \left(\frac{|v - \mathfrak{u}|^2}{T} - 5 \right)^2 \sqrt{\mu} \right\},$$

which, together with (5.5), yields that

$$\{\mathbf{I} - \mathbf{P}\} f_2 = -\mathbf{L}^{-1} \left\{ \sum_{j,l=1}^{3} \partial_j \mathfrak{u}_l \mathcal{A}_{jl} + \sum_{j=1}^{3} \frac{\partial_j T}{T} \mathcal{B}_j \right\} + \sum_{j,l=1}^{3} \frac{1}{2T} u_{1,l} u_{1,j} \mathcal{A}_{lj}$$

$$+\frac{\theta_1}{3T^{\frac{3}{2}}}u_1\cdot\mathcal{B}+\frac{\theta_1^2}{72T^2}\{\mathbf{I}-\mathbf{P}\}\left\{\left(\frac{|v-\mathfrak{u}|^2}{T}-5\right)^2\sqrt{\mu}\right\}. \tag{5.6}$$

Since $\mathbf{L}^{-1}$ preserves the decay property of v, it is direct to check that

$$|\{\mathbf{I}-\mathbf{P}\}f_2(t,x,v)|\leqq C(|\nabla(\mathfrak{u},T)|+|(u_1,\theta_1)|^2)(t,x)\,(1+|v|)^4\sqrt{\mu}.$$

Noting (2.26) and $\partial_1\mathfrak{u}_3(t,x_{\shortparallel},0)=\partial_2\mathfrak{u}_3(t,x_{\shortparallel},0)=0$, and using (1.26), (5.6), we compute

$$\langle T^0\mathcal{A}_{3i}^0,\ (\mathbf{I}-\mathbf{P})f_2\rangle(t,x_{\shortparallel},0)=-\partial_3\mathfrak{u}_i^0\langle T^0\mathcal{A}_{3i}^0,\ \mathbf{L}_0^{-1}\mathcal{A}_{3i}^0\rangle=-\mu(T^0)\,\partial_3\mathfrak{u}_i^0, \tag{5.7}$$

for $i=1,2$, and

$$\begin{aligned}\langle 2(T^0)^{\frac{3}{2}}\mathcal{B}_3^0,\ (\mathbf{I}-\mathbf{P})f_2\rangle(t,x_{\shortparallel},0)&=-2T^0\partial_3T^0\langle\mathcal{B}_3^0,\ \mathbf{L}_0^{-1}\mathcal{B}_3^0\rangle\\&=-3\kappa(T^0)\,\partial_3T^0.\end{aligned} \tag{5.8}$$

Substituting (5.7) and (5.8) into (5.4), we get the exact expression of boundary condition for viscous boundary layer $(\bar{u}_{1,i},\bar{\theta}_1)$, i.e.,

$$\partial_y\bar{u}_{1,i}(t,x_{\shortparallel},0)=-\partial_3\mathfrak{u}_i^0,\quad \partial_y\bar{\theta}_1(t,x_{\shortparallel},0)=-3\partial_3T^0. \tag{5.9}$$

By using Lemma 4.1, we can obtain the existence of smooth solution of (1.35)–(1.36) and (5.9) over $[0,\tau]\times\mathbb{R}_+^3$ satisfying

$$\begin{aligned}&\sum_{j=0}^{\bar{s}_1}\sum_{\beta+2\gamma=j}\left\{\|\partial_t^\gamma\nabla_{\bar{x}}^\beta(\bar{u}_{1,\shortparallel},\bar{\theta}_1)(t)\|_{L^2_{l_j^1}}^2+\int_0^\tau\|\partial_t^\gamma\nabla_{\bar{x}}^\beta\partial_y(\bar{u}_{1,\shortparallel},\bar{\theta}_1)(t)\|_{L^2_{l_j^1}}^2\,\mathrm{d}t\right\}\\&\quad\leqq C(\tau,E_{3+\bar{s}_1})\left\{\sum_{j=0}^{\bar{s}_1}\sum_{\beta+2\gamma=j}\|\partial_t^\gamma\nabla_{\bar{x}}^\beta(\bar{u}_{1,\shortparallel},\bar{\theta}_1)(0)\|_{L^2_{l_j^1}}^2\right.\\&\qquad\left.+\sup_{t\in[0,\tau]}\sum_{\beta+2\gamma\leqq\bar{s}_1+2}\|\partial_{\shortparallel}^\beta\partial_t^\gamma(\partial_3\mathfrak{u}_1^0,\ \partial_3\mathfrak{u}_2^0,\ \partial_3T^0)(t)\|_{L^2(\mathbb{R}^2)}^2\right\}\\&\quad\leqq C\Big(\tau,E_{4+\bar{s}_1},\sum_{j=0}^{s_1}\sum_{\beta+2\gamma=j}\|\partial_t^\gamma\nabla_{\bar{x}}^\beta(\bar{u}_{1,\shortparallel},\bar{\theta}_1)(0)\|_{L^2_{l_j^1}}^2\Big),\end{aligned} \tag{5.10}$$

where $l_j^1=\bar{l}_1+2(\bar{s}_1-j)$ with $\bar{l}_1\gg 1$ and $s_0\geqq 4+\bar{s}_1$. Combining (5.10) with (1.24), we get

$$\begin{aligned}&\sum_{j=0}^{\bar{s}_1}\sum_{\beta+2\gamma=j}\|\mathfrak{w}_{\bar{\kappa}_1}\partial_t^\gamma\nabla_{\bar{x}}^\beta\bar{f}_1(t)\|_{L^2_{l_j^1}L^\infty_v}\\&\quad\leqq C\Big(\tau,E_{4+\bar{s}_1},\sum_{j=0}^{\bar{s}_1}\sum_{\beta+2\gamma=j}\|\partial_t^\gamma\nabla_{\bar{x}}^\beta(\bar{u}_{1,\shortparallel},\bar{\theta}_1)(0)\|_{L^2_{l_j^1}}^2\Big),\end{aligned}$$

for any $\bar{\kappa}_1>0$.

Step 1.3. Construction of solution $\hat{f}_1$. From (2.19), we know $\hat{f}_{1,1} \equiv 0$, we need only to consider $\hat{f}_{1,2}$. Noting (2.22)(with $k = 1$), using (2.26) and $(1.24)_1$, one has the boundary condition for $\hat{f}_{1,2}$

$$\hat{f}_{1,2}(t, x_{\shortparallel}, 0, v_{\shortparallel}, v_3)|_{v_3>0} = \hat{f}_{1,2}(t, x_{\shortparallel}, 0, v_{\shortparallel}, -v_3),$$

which, together with Lemma 2.5, yields the existence of $\hat{f}_{1,2}$ with $\hat{f}^{\varepsilon}_{1,2} \equiv 0$. Therefore we have proved the existence of $\hat{f}_1$ with

$$\hat{f}_1 \equiv 0. \tag{5.11}$$

Such an absence of the ε-th order Knudsen boundary layer is expected since the Knudsen boundary layer is used to mend the boundary conditions at higher orders.

Step 2. Construction of solutions f_k, $\bar{f}_k$ *and* $\hat{f}_k$. We shall use an induction argument. Suppose we have already proved the existence of f_i, $\bar{f}_i$ and $\hat{f}_i$ for $1 \leqq i \leqq k$ such that

$$D_k + \bar{D}_k + \hat{D}_k \leqq C\Big(\tau, E_{s_0}, \sum_{i=0}^{k}\sum_{j=0}^{\bar{s}_i}\sum_{j=2\gamma+\beta} \|\partial_t^\gamma \nabla_{\bar{x}}^\beta (\bar{u}_{i,\shortparallel}, \bar{\theta}_i)(0)\|^2_{L^2_{l^i_j}} + \sum_{i=0}^{k}\sum_{\gamma+\beta\leqq s_i} \|\partial_t^\gamma \nabla^\beta (\rho_i, u_i, \theta_i)(0)\|_{L^2_x}\Big), \tag{5.12}$$

where

$$D_k := \sup_{t\in[0,\tau]}\Big\{\sum_{i=1}^{k}\sum_{\gamma+\beta\leqq s_i} \|\tilde{w}_{\kappa_i}\partial_t^\gamma \nabla_x^\beta f_i(t)\|_{L^2_x L^\infty_v}\Big\} < \infty,$$

$$\bar{D}_k := \sup_{t\in[0,\tau]}\Big\{\sum_{i=1}^{k}\sum_{j=0}^{\bar{s}_i}\sum_{j=2\gamma+\beta} \|\mathfrak{w}_{\bar{\kappa}_i}\partial_t^\gamma \nabla_{\bar{x}}^\beta \bar{f}_i(t)\|_{L^2_{l^i_j} L^\infty_v}\Big\} < \infty,$$

$$\hat{D}_k := \sup_{t\in[0,\tau]}\Big\{\sum_{i=1}^{k}\sum_{\gamma+\beta\leqq \hat{s}_i} \|e^{\zeta_i\cdot\eta}\mathfrak{w}_{\hat{\kappa}_i}\partial_t^\gamma \nabla_{\shortparallel}^\beta \hat{f}_i(t)\|_{L^\infty_{\hat{x},v}\cap L^2_{x_{\shortparallel}} L^\infty_{\eta,v}}\Big\} < \infty$$

for some $s_i > \bar{s}_i > \hat{s}_i \geqq s_{i+1} > \bar{s}_{i+1} > \hat{s}_{i+1} \geqq 1$, $\kappa_i \gg \bar{\kappa}_i \gg \hat{\kappa}_i \gg \kappa_{i+1} \gg \bar{\kappa}_{i+1} \gg \hat{\kappa}_{i+1} \gg 1$ with $1 \leqq i \leqq k-1$, and $l^i_j = \bar{l}_i + 2(\bar{s}_i - j)$, $\bar{l}_i \gg 1$ with $1 \leqq i \leqq k$ and $0 \leqq j \leqq \bar{s}_i$. In the following, we consider the existence of f_{k+1}, $\bar{f}_{k+1}$ and $\hat{f}_{k+1}$.

Step 2.1. Construction of solution f_{k+1}. Let $r \geqq 3$. Since $\mathbf{L}^{-1}$ preserves the decay property of v, by using (2.3) and Sobolev inequality, we have that

$$\sum_{\gamma+|\beta|\leqq r} \|\tilde{w}_{\kappa_{k+1}}\partial_t^\gamma \nabla_x^\beta \{\mathbf{I}-\mathbf{P}\} f_{k+1}(t)\|_{L^2_x L^\infty_v} \leqq C\Big(E_{r+1}, \sum_{j=1}^{k}\sum_{\gamma+\beta\leqq r+1} \|\tilde{w}_{\kappa_j}\partial_t^\gamma \nabla_x^\beta f_j(t)\|_{L^2_x L^\infty_v}\Big). \tag{5.13}$$

To obtain f_{k+1}, we still need to obtain the estimate for macroscopic part. For the source terms on RHS of (2.4), it follows from (2.5), (5.13) that

$$\sum_{\gamma+|\beta|\leqq r+1} \|\partial_t^\gamma \nabla_x^\beta (\mathfrak{f}_k, \mathfrak{g}_k)(t)\|_{L_x^2} \leqq C\Big(E_{r+3}, \sum_{j=1}^{k} \sum_{\gamma+\beta\leqq r+3} \|\tilde{w}_{\kappa_j} \partial_t^\gamma \nabla_x^\beta f_j(t)\|_{L_x^2 L_v^\infty}\Big). \tag{5.14}$$

Before applying Lemma 3.1, we need to estimate the boundary condition. Noting (2.27) (with k replaced by $k+1$), we have that

$$\begin{aligned}
&\sum_{\gamma+\beta\leqq r+2} \|\partial_t^\gamma \partial_{\shortparallel}^\beta u_{k+1,3}(t,\cdot,0)\|_{L^2} \\
&\leqq C(E_{r+4})\Big\{ \sum_{\gamma+\beta\leqq r+3} \int_0^\infty \|\partial_t^\gamma \partial_{\shortparallel}^\beta (\bar{\rho}_k, \bar{u}_k)(t,\cdot,y)\|_{L^2(\mathbb{R}^2)} dy \\
&\quad + \sum_{\gamma+\beta\leqq r+2} \|\partial_t^\gamma \partial_{\shortparallel}^\beta (\hat{A}_{k+1}, \hat{C}_{k+1})(t,\cdot,0)\|_{L^2(\mathbb{R}^2)}\Big\}.
\end{aligned} \tag{5.15}$$

From $l_j^k > 1$, a direct calculation shows that

$$\begin{aligned}
&\sum_{\gamma+\beta\leqq r+3} \int_0^\infty \|\partial_t^\gamma \partial_{\shortparallel}^\beta (\bar{\rho}_k, \bar{u}_k)(t,\cdot,y)\|_{L^2(\mathbb{R}^2)} dy \\
&\leqq C \sum_{j=0}^{2(r+3)} \sum_{\beta+2\gamma=j} \|\partial_t^\gamma \partial_{\shortparallel}^\beta \bar{f}_k(t)\|_{L_{l_j^k}^2 L_v^\infty}.
\end{aligned} \tag{5.16}$$

By using (2.18), (2.16) and (2.7), one obtains

$$\begin{aligned}
&\sum_{\gamma+\beta\leqq r+2} \|\partial_t^\gamma \partial_{\shortparallel}^\beta (\hat{A}_{k+1}, \hat{B}_{k+1}, \hat{C}_{k+1})(t,\cdot,0)\|_{L^2(\mathbb{R}^2)} \\
&\leqq C(E_{r+4}) \sum_{\gamma+\beta\leqq r+2} \int_0^\infty \|\partial_t^\gamma \partial_{\shortparallel}^\beta (\hat{a}_{k+1}, \hat{b}_{k+1}, \hat{c}_{k+1})(t,\cdot,\eta)\|_{L_{x_{\shortparallel}}^2} d\eta \\
&\leqq C(E_{r+4}) \sum_{\gamma+\beta\leqq r+2} \|e^{\frac{1}{2}\zeta_{k-1}\cdot\eta} \partial_t^\gamma \partial_{\shortparallel}^\beta (\hat{a}_{k+1}, \hat{b}_{k+1}, \hat{c}_{k+1})(t)\|_{L_{x_{\shortparallel}}^2 L_\eta^\infty} \\
&\leqq C(E_{r+5}) \sum_{\gamma+\beta\leqq r+3} \|e^{\frac{1}{2}\zeta_{k-1}\cdot\eta}\, \mathfrak{w}_{\hat{\kappa}_{k-1}} \partial_t^\gamma \partial_{\shortparallel}^\beta \hat{f}_{k-1}(t)\|_{L_{x_{\shortparallel}}^2 L_{\eta,v}^\infty}.
\end{aligned} \tag{5.17}$$

Here we emphasize that although only time and tangential derivatives are available for the Knudsen boundary layer $\hat{f}_i$, but the trace $(\hat{A}_{k+1}, \hat{B}_{k+1}, \hat{C}_{k+1})(t,\cdot,0)$ (equivalent to $\hat{f}_{k+1,1}(t, x_{\shortparallel}, 0, v)$) is indeed well-defined from (2.18).

Substituting (5.16)–(5.17) into (5.15), we deduce

$$\sum_{\gamma+\beta\leqq r+2} \|\partial_t^\gamma \partial_{\shortparallel}^\beta u_{k+1,3}(t,\cdot,0)\|_{L^2} \leqq C(E_{r+5})\Big\{ \sum_{j=0}^{2(r+3)} \sum_{\beta+2\gamma=j} \|\partial_t^\gamma \partial_{\shortparallel}^\beta \bar{f}_k(t)\|_{L_{l_j^k}^2 L_v^\infty}$$

$$+ \sum_{\gamma+\beta \leqq r+3} \| e^{\frac{1}{2}\zeta_{k-1}\cdot\eta} \mathfrak{w}_{\hat{\kappa}_{k-1}} \partial_t^\gamma \partial_{\shortparallel}^\beta \hat{f}_{k-1}(t) \|_{L^2_{x_{\shortparallel}} L^\infty_{\eta,v}} \Big\}. \tag{5.18}$$

Therefore (3.5) is valid with time and tangential derivatives and we can apply Lemma 3.1.

Now applying Lemma 3.1, using (5.14) and (5.18), we obtain

$$\begin{aligned} &\sum_{\gamma+\beta \leqq r} \| \partial_t^\gamma \nabla_x^\beta (\rho_{k+1}, u_{k+1}, \theta_{k+1})(t) \|_{L^2_x} \\ &\leqq C\Big(\tau, E_{r+5}, \sum_{\gamma+\beta \leqq r} \| \partial_t^\gamma \nabla_x^\beta (\rho_{k+1}, u_{k+1}, \theta_{k+1})(0) \|_{L^2_x}, \\ &\quad \sup_{t\in[0,\tau]} \Big[\sum_{j=0}^{2(r+3)} \sum_{\beta+2\gamma=j} \| \partial_t^\gamma \partial_{\shortparallel}^\beta \bar{f}_k(t) \|_{L^2_{l^k_j} L^\infty_v} + \sum_{\gamma+\beta \leqq r+3} \| e^{\frac{1}{2}\zeta_{k-1}\cdot\eta} \mathfrak{w}_{\hat{\kappa}_{k-1}} \partial_t^\gamma \partial_{\shortparallel}^\beta \hat{f}_{k-1}(t) \|_{L^2_{x_{\shortparallel}} L^\infty_{\eta,v}} \\ &\quad + \sum_{j=1}^{k} \sum_{\gamma+\beta \leqq r+3} \| \tilde{w}_{\kappa_j} \partial_t^\gamma \nabla_x^\beta f_j(t) \|_{L^2_x L^\infty_v} \Big] \Big). \end{aligned} \tag{5.19}$$

Now, taking $r = s_{k+1}$ yields

$$s_0 \geqq 5 + s_{k+1}, \quad s_k \geqq s_{k+1} + 3, \quad \hat{s}_{k-1} \geqq s_{k+1} + 3 \quad \text{and } \bar{s}_k \geqq 2(s_{k+1} + 3).$$

Therefore, combining (5.19) with (5.13), we get

$$\begin{aligned} &\sum_{\gamma+\beta \leqq s_{k+1}} \| \tilde{w}_{\kappa_{k+1}} \partial_t^\gamma \nabla_x^\beta f_{k+1}(t) \|_{L^2_x L^\infty_v} \\ &\quad \leqq C\Big(\tau, E_{s_0}, D_k + \bar{D}_k + \hat{D}_{k-1}, \sum_{\gamma+\beta \leqq s_{k+1}} \| \partial_t^\gamma \nabla_x^\beta (\rho_{k+1}, u_{k+1}, \theta_{k+1})(0) \|_{L^2_x} \Big). \end{aligned} \tag{5.20}$$

Having constructed f_i with $1 \leqq i \leqq k+1$, by using (2.3), we deduce

$$\begin{aligned} &\sum_{\gamma+|\beta| \leqq s_{k+1}-1} \| \tilde{w}_{\kappa_{k+2}} \partial_t^\gamma \nabla_x^\beta \{\mathbf{I} - \mathbf{P}\} f_{k+2}(t) \|_{L^2_x L^\infty_v} \\ &\quad \leqq C\Big(E_{s_{k+1}}, \sum_{i=1}^{k+1} \sum_{\gamma+\beta \leqq s_{k+1}} \| \tilde{w}_{\kappa_i} \partial_t^\gamma \nabla_x^\beta f_i(t) \|_{L^2_x L^\infty_v} \Big) \\ &\quad \leqq C\Big(\tau, E_{s_0}, D_k + \bar{D}_k + \hat{D}_{k-1}, \sum_{\gamma+\beta \leqq s_{k+1}} \| \partial_t^\gamma \nabla_x^\beta (\rho_{k+1}, u_{k+1}, \theta_{k+1})(0) \|_{L^2_x} \Big), \end{aligned} \tag{5.21}$$

which will be used when consider the trace of $(\partial_y \bar{u}_{k+1,\shortparallel},\ \partial_y \bar{\theta}_{k+1})(t, x_{\shortparallel}, 0)$ in the following.

Step 2.2. Construction of solution for $\bar{f}_{k+1}$. Since $\mathbf{L}_0^{-1}$ preserves the decay property of v, then it follows from (1.29) that

$$\begin{aligned}&\sum_{2\gamma+\beta=j}\|\mathfrak{w}_{\bar{\kappa}_{k+1}}\partial_t^\gamma\nabla_{\bar{x}}^\beta\{\mathbf{I}-\mathbf{P_0}\}\bar{f}_{k+1}(t)\|_{L_l^2L_v^\infty}\\&\leqq C\Big(E_{j+3},\sum_{2\gamma+\beta\leqq j+3}\|\tilde{w}_{\kappa_1}\partial_t^\gamma\nabla_x^\beta f_1(t)\|_{L_x^2L_v^\infty}+\sum_{2\gamma+\beta\leqq j}\|\mathfrak{w}_{\bar{\kappa}_1}\partial_t^\gamma\nabla_{\bar{x}}^\beta\bar{f}_1(t)\|_{L_{\bar{x}}^2L_v^\infty}\\&\quad+\sum_{2\gamma+\beta\leqq j+1}\|\mathfrak{w}_{\bar{\kappa}_k}\partial_t^\gamma\nabla_{\bar{x}}^\beta\bar{f}_k(t)\|_{L_{2+l}^2L_v^\infty}\Big)+\sum_{2\gamma+\beta=j}\|\mathfrak{w}_{\bar{\kappa}_{k+1}}\partial_t^\gamma\nabla_{\bar{x}}^\beta\bar{J}_{k-1}(t)\|_{L_l^2L_v^\infty}.\end{aligned}\tag{5.22}$$

From (1.34), a direct calculation shows that

$$\begin{aligned}&\sum_{2\gamma+\beta=j}\|\mathfrak{w}_{\bar{\kappa}_{k+1}}\partial_t^\gamma\nabla_{\bar{x}}^\beta\bar{J}_{k-1}(t)\|_{L_l^2L_v^\infty}\\&\leqq C\Big(E_{j+\mathfrak{b}+2},\sum_{i=0}^{k-1}\sum_{2\gamma+\beta\leqq j+2}\|\mathfrak{w}_{\bar{\kappa}_i}\partial_t^\gamma\nabla_{\bar{x}}^\beta\bar{f}_i(t)\|_{L_{l+2\mathfrak{b}}^2L_v^\infty}\\&\quad+\sum_{i=0}^{k}\sum_{2\gamma+\beta\leqq j+2+\mathfrak{b}}\|\tilde{w}_{\kappa_i}\partial_t^\gamma\nabla_x^\beta f_i(t)\|_{L_x^2L_v^\infty}\Big)\\&\quad+C(E_{j+\mathfrak{b}+2})\sum_{2\gamma+\beta\leqq j+1}\|\mathfrak{w}_{\bar{\kappa}_k}\partial_t^\gamma\nabla_{\bar{x}}^\beta\{\mathbf{I}-\mathbf{P_0}\}\bar{f}_k(t)\|_{L_{l+2}^2L_v^\infty}.\end{aligned}\tag{5.23}$$

Substituting (5.23) into (5.22), and noting $l_j^{k+1}\ll l_j^i$ for $1\leqq i\leqq k$, we can obtain the estimate for microscopic part $\{\mathbf{I}-\mathbf{P_0}\}\bar{f}_{k+1}$

$$\begin{aligned}&\sum_{2\gamma+\beta=j}\|\tilde{w}_{\bar{\kappa}_{k+1}}\partial_t^\gamma\nabla_{\bar{x}}^\beta\{\mathbf{I}-\mathbf{P_0}\}\bar{f}_{k+1}(t)\|_{L_l^2L_v^\infty}\\&\leqq C\Big(E_{j+\mathfrak{b}+2},\sum_{i=0}^{k}\sum_{2\gamma+\beta\leqq j+\mathfrak{b}+2}\|\tilde{w}_{\kappa_i}\partial_t^\gamma\nabla_x^\beta f_i(t)\|_{L_x^2L_v^\infty}\\&\quad+\sum_{i=1}^{k}\sum_{2\gamma+\beta\leqq j+2}\|\mathfrak{w}_{\bar{\kappa}_i}\partial_t^\gamma\nabla_{\bar{x}}^\beta\bar{f}_i(t)\|_{L_{l+2\mathfrak{b}}^2L_v^\infty}\Big).\end{aligned}\tag{5.24}$$

On the other hand, substituting (5.24) (with k replaced by $k-1$) into (5.23), one can obtain a better estimate for $\bar{J}_{k-1}$

$$\begin{aligned}&\sum_{2\gamma+\beta=j}\|\mathfrak{w}_{\bar{\kappa}_{k+1}}\partial_t^\gamma\nabla_{\bar{x}}^\beta\bar{J}_{k-1}(t)\|_{L_l^2L_v^\infty}\\&\leqq C\Big(E_{j+\mathfrak{b}+3},\sum_{i=0}^{k-1}\sum_{2\gamma+\beta\leqq j+3}\|\mathfrak{w}_{\bar{\kappa}_i}\partial_t^\gamma\nabla_{\bar{x}}^\beta\bar{f}_i(t)\|_{L_{l+2\mathfrak{b}}^2L_v^\infty}\end{aligned}$$

$$+\sum_{i=0}^{k}\sum_{2\gamma+\beta\leqq j+3+\mathfrak{b}}\|\tilde{w}_{\kappa_i}\partial_t^{\gamma}\nabla_x^{\beta}f_i(t)\|_{L_x^2L_v^\infty}\Big). \tag{5.25}$$

For $\bar{W}_{k-1}$, $\bar{H}_{k-1}$, it follows from (1.32)–(1.33) and (5.24) that

$$\begin{aligned}
&\sum_{2\gamma+\beta=j}\|\partial_t^{\gamma}\nabla_{\bar{x}}^{\beta}(\bar{W}_{k-1},\bar{H}_{k-1})(t)\|_{L_l^2}\\
&\leqq C(E_{j+3})\sum_{2\gamma+\beta\leqq j+1}\|\partial_t^{\gamma}\nabla_{\bar{x}}^{\beta}\{\mathbf{I}-\mathbf{P_0}\}\bar{f}_k(t)\|_{L_l^2L_v^\infty}\\
&\leqq C\Big(E_{j+\mathfrak{b}+3},\sum_{i=0}^{k-1}\sum_{2\gamma+\beta\leqq j+3+\mathfrak{b}}\|\tilde{w}_{\kappa_i}\partial_t^{\gamma}\nabla_x^{\beta}f_i(t)\|_{L_x^2L_v^\infty}\\
&\quad+\sum_{i=1}^{k-1}\sum_{2\gamma+\beta\leqq j+3}\|\mathfrak{w}_{\bar{\kappa}_i}\partial_t^{\gamma}\nabla_{\bar{x}}^{\beta}\bar{f}_i(t)\|_{L_{l+2\mathfrak{b}}^2L_v^\infty}\Big).
\end{aligned}\tag{5.26}$$

For $\bar{u}_{k+1,3}(t,x_{\shortparallel},y)$, it follows from (1.30) and (1.20) that

$$\bar{u}_{k+1,3}(t,x_{\shortparallel},y)=-\int_y^{\infty}\frac{1}{\rho^0}\Big\{\partial_t\bar{\rho}_k+\mathrm{div}_{\shortparallel}(\rho^0\bar{u}_{k,\shortparallel}+\bar{\rho}_k\mathfrak{u}_{\shortparallel}^0)\Big\}(t,x_{\shortparallel},z)\,\mathrm{d}z, \tag{5.27}$$

which yields that

$$\begin{aligned}
&\sum_{2\gamma+\beta=j}\|\partial_t^{\gamma}\nabla_{\shortparallel}^{\beta}\bar{u}_{k+1,3}(t)\|_{L_l^2}\\
&\leqq C(E_{j+3})\int_0^{\infty}(1+y)^l\mathrm{d}y\int_y^{\infty}\sum_{2\gamma+\beta\leqq j+2}\|\partial_t^{\gamma}\nabla_{\shortparallel}^{\beta}(\bar{\rho}_k,\bar{u}_k)(t,\cdot,z)\|_{L^2(\mathbb{R}^2)}\,\mathrm{d}z\\
&\leqq C(E_{j+3})\int_0^{\infty}(1+y)^l\Big(\int_y^{\infty}(1+z)^{-2l-4}\,\mathrm{d}z\Big)^{\frac{1}{2}}dy\sum_{2\gamma+\beta\leqq j+2}\|\partial_t^{\gamma}\nabla_{\shortparallel}^{\beta}(\bar{\rho}_k,\bar{u}_k)(t)\|_{L_{2l+4}^2}\\
&\leqq C(E_{j+3})\sum_{2\gamma+\beta\leqq j+2}\|\partial_t^{\gamma}\nabla_{\shortparallel}^{\beta}(\bar{\rho}_k,\bar{u}_k)(t)\|_{L_{2l+4}^2}\\
&\leqq C(E_{j+3})\sum_{2\gamma+\beta\leqq j+2}\|\partial_t^{\gamma}\nabla_{\shortparallel}^{\beta}\bar{f}_k(t)\|_{L_{2l+4}^2L_v^\infty},\quad\text{for}\quad l\geqq 0.
\end{aligned}\tag{5.28}$$

On the other hand, we assume that $\nabla_{\bar{x}}^{\beta}$ contains at least one ∂_y, then it follows from (1.30) that

$$\begin{aligned}
\sum_{2\gamma+\beta=j}\|\partial_t^{\gamma}\nabla_{\bar{x}}^{\beta}\bar{u}_{k+1,3}^{\varepsilon}(t)\|_{L_l^2}&\leqq\sum_{2\gamma+\beta=j}\|\partial_t^{\gamma}\bar{\nabla}^{\beta-1}\partial_y\bar{u}_{k+1,3}^{\varepsilon}(t)\|_{L_l^2}\\
&\leqq C(E_{j+3})\sum_{2\gamma+\beta\leqq j+1}\|\partial_t^{\gamma}\nabla_{\bar{x}}^{\beta}\bar{f}_k^{\varepsilon}(t)\|_{L_l^2L_v^\infty},\ \text{for } l\geqq 0,
\end{aligned}$$

which, together with (5.28), yields that for $l \geqq 0$,

$$\sum_{2\gamma+\beta=j} \|\partial_t^\gamma \nabla_{\bar{x}}^\beta \bar{u}_{k+1,3}(t)\|_{L_l^2} \leqq C(E_{j+3}) \sum_{2\gamma+\beta\leqq j+2} \|\partial_t^\gamma \nabla_{\bar{x}}^\beta \bar{f}_k(t)\|_{L^2_{2l+4}L^\infty_v}. \tag{5.29}$$

By using (1.31) and the same arguments as (5.27)–(5.29), one can obtain the estimate for $\bar{p}^\varepsilon_{k+1}$

$$\begin{aligned}\sum_{2\gamma+\beta=j} \|\partial_t^\gamma \nabla_{\bar{x}}^\beta \bar{p}_{k+1,3}(t)\|_{L_l^2} &\leqq C(E_{j+3})\Big(\sum_{2\gamma+\beta\leqq j+2} \|\partial_t^\gamma \nabla_{\bar{x}}^\beta \bar{f}_k(t)\|_{L^2_{2l+6}L^\infty_v} \\ &\quad + \sum_{2\gamma+\beta\leqq j+1} \|\partial_t^\gamma \nabla_{\bar{x}}^\beta \bar{J}_{k-1}(t)\|_{L^2_{2l+4}L^\infty_v} + \sum_{2\gamma+\beta\leqq j} \|\partial_t^\gamma \nabla_{\bar{x}}^\beta \bar{W}_{k-1}(t)\|_{L^2_{2l+4}}\Big) \\ &\leqq C\Big(E_{j+4+\mathfrak{b}}, \sum_{i=1}^k \sum_{2\gamma+\beta\leqq j+4} \|\mathfrak{w}_{\bar{\kappa}_i} \partial_t^\gamma \nabla_{\bar{x}}^\beta \bar{f}_i(t)\|_{L^2_{2l+4+2\mathfrak{b}}L^\infty_v} \\ &\quad + \sum_{i=1}^k \sum_{2\gamma+\beta\leqq j+4+\mathfrak{b}} \|\tilde{w}_{\kappa_i} \partial_t^\gamma \nabla_x^\beta f_i(t)\|_{L^2_x L^\infty_v}\Big),\end{aligned} \tag{5.30}$$

where we have used (5.25)–(5.26) in the last inequality.

By similar arguments as to those of (5.28), we can have the following trace estimate:

$$\begin{aligned}&\sum_{2\gamma+\beta=j} \|\partial_t^\gamma \nabla_{\shortparallel}^\beta \bar{u}_{k+1,3}(t,\cdot,0)\|_{L^2(\mathbb{R}^2)} \\ &\quad\leqq C(E_{j+3}) \int_0^\infty \sum_{2\gamma+\beta\leqq j+2} \|\partial_t^\gamma \nabla_{\shortparallel}^\beta (\bar{\rho}_k, \bar{u}_k)(t,\cdot,z)\|_{L^2(\mathbb{R}^2)}\,\mathrm{d}z \\ &\quad\leqq C(E_{j+3}) \sum_{2\gamma+\beta\leqq j+2} \|\partial_t^\gamma \nabla_{\shortparallel}^\beta (\bar{\rho}_k, \bar{u}_k)(t)\|_{L_l^2} \\ &\quad\leqq C(E_{j+3}) \sum_{2\gamma+\beta\leqq j+2} \|\partial_t^\gamma \nabla_{\shortparallel}^\beta \bar{f}_k(t)\|_{L^2_l L^\infty_v}, \quad \text{for} \quad l > 1.\end{aligned} \tag{5.31}$$

Taking $s_0 \geqq \bar{s}_{k+1} + \mathfrak{b} + 6$, $s_k > \bar{s}_k > \hat{s}_k > s_{k+1} \geqq \bar{s}_{k+1} + 7 + \mathfrak{b}$, and $l^i_j \geqq 2\mathfrak{b}$ for $i \leqq k$, $j \leqq \bar{s}_k$, we deduce from (2.30)–(2.31) that

$$\begin{aligned}&\sum_{2\gamma+\beta\leqq \bar{s}_{k+1}+2} \|\partial_t^\gamma \nabla_{\shortparallel}^\beta (\partial_y \bar{u}_{k+1,\shortparallel},\ \partial_y \bar{\theta}_{k+1})(t,\cdot,0)\|_{L^2(\mathbb{R}^2)} \\ &\leqq C\Big(E_{\bar{s}_{k+1}+4}, \sum_{2\gamma+\beta\leqq \bar{s}_{k+1}+4} \big[\|\tilde{w}_{\kappa_1} \partial_t^\gamma \nabla_x^\beta f_1(t)\|_{L^2} + \|\mathfrak{w}_{\bar{\kappa}_1} \partial_t^\gamma \nabla_{\bar{x}}^\beta \bar{f}_1(t)\|_{L^2}\big] \\ &\quad + \sum_{2\gamma+\beta\leqq \bar{s}_{k+1}+2} \Big[\|\partial_t^\gamma \nabla_{\shortparallel}^\beta \bar{u}_{k+1,3}(t,\cdot,0)\|_{L^2_{x_\shortparallel}} + \|\partial_t^\gamma \nabla_{\shortparallel}^\beta \{\mathbf{I}-\mathbf{P}\} f_{k+2}(t,\cdot,0,\cdot)\|_{L^2_{x_\shortparallel} L^\infty_v}\end{aligned}$$

$$
\begin{aligned}
&+ \|\partial_t^\gamma \nabla_{_{\shortparallel}}^\beta \bar{J}_k(t,\cdot,0,\cdot)\|_{L^2_{x_{\shortparallel}} L^\infty_v} + \|\partial_t^\gamma \nabla_{_{\shortparallel}}^\beta (\hat{A}_{k+2}, \hat{B}_{k+2}, \hat{C}_{k+2})(t,\cdot,0)\|_{L^2_{x_{\shortparallel}}} \Big]\Big) \\
&\leqq C\Big(\tau, E_{s_0}, D_k + \bar{D}_k + \hat{D}_k, \sum_{\gamma+\beta \leqq s_{k+1}} \|\partial_t^\gamma \nabla_x^\beta (\rho_{k+1}, u_{k+1}, \theta_{k+1})(0)\|_{L^2_x}\Big) < \infty,
\end{aligned}
\tag{5.32}
$$

where we have used (5.31),(5.20), (5.21), (5.25), (5.17) (with $k-1$ replaced by k in (5.25), (5.17)) and the trace theorem.

Similarly, for the source terms of (1.27)–(1.28), from (5.25)–(5.26) and (5.29)–(5.30), a direct calculation shows that

$$
\sum_{j=0}^{\bar{s}_{k+1}} \sum_{\beta+2\gamma=j} \|(\partial_y \partial_t^\gamma \nabla_{\bar{x}}^\beta (\bar{\mathfrak{f}}_k, \bar{\mathfrak{g}}_k), \partial_t^\gamma \nabla_{\bar{x}}^\beta (\bar{\mathfrak{f}}_k, \bar{\mathfrak{g}}_k))(t)\|^2_{L^2_{l_j^{k+1}}} \leqq C(\tau, E_{s_0}, D_k + \bar{D}_k),
\tag{5.33}
$$

where we have taken

$$
s_0 \geqq \bar{s}_{k+1} + \mathfrak{b} + 6, \;\; s_{k+1} \geqq \bar{s}_{k+1} + 8 + \mathfrak{b}, \;\; l_j^i \geqq 2 l_j^{k+1} + 18 + 2\mathfrak{b}, \;\; \text{for } 1 \leqq i \leqq k,
\tag{5.34}
$$

with $l_j^{k+1} = \bar{l}_{k+1} + 2(\bar{s}_{k+1} - j)$ and $\bar{l}_{k+1} \gg 1$.

Using Lemma 4.1 (the time and tangential derivatives estimate (5.32) for the boundary condition are used enough when using Lemma 4.1) and (5.32)–(5.33), and noting (5.34), one can obtain

$$
\begin{aligned}
&\sum_{j=0}^{\bar{s}_{k+1}} \sum_{j=2\gamma+\beta} \Big(\|\partial_t^\gamma \nabla_{\bar{x}}^\beta (\bar{u}_{k+1,\shortparallel}, \bar{\theta}_{k+1})(t)\|^2_{L^2_{l_j^{k+1}}} + \int_0^\tau \|\partial_t^\gamma \nabla_{\bar{x}}^\beta \partial_y (\bar{u}_{k+1,\shortparallel}, \bar{\theta}_{k+1})\|^2_{L^2_{l_j^{k+1}}} \, \mathrm{d}t \Big) \\
&\leqq C\Big(\tau, E_{s_0}, D_k + \bar{D}_k + \hat{D}_k, \sum_{j=0}^{\bar{s}_{k+1}} \sum_{j=2\gamma+\beta} \|\partial_t^\gamma \nabla_{\bar{x}}^\beta (\bar{u}_{k+1,\shortparallel}, \bar{\theta}_{k+1})(0)\|^2_{L^2_{l_j^{k+1}}} \\
&\quad + \sum_{\gamma+\beta \leqq s_{k+1}} \|\partial_t^\gamma \nabla_x^\beta (\rho_{k+1}, u_{k+1}, \theta_{k+1})(0)\|_{L^2_x}\Big).
\end{aligned}
\tag{5.35}
$$

Finally, combining (5.24), (5.29), (5.30) and (5.35), and noting (5.34), we have

$$
\begin{aligned}
&\sum_{j=0}^{\bar{s}_{k+1}} \sum_{j=2\gamma+\beta} \|\mathfrak{w}_{\hat{\kappa}_{k+1}} \partial_t^\gamma \nabla_{\bar{x}}^\beta \bar{f}_{k+1}(t)\|^2_{L^2_{l_j^{k+1}} L^\infty_v} \\
&\quad \leqq C\Big(\tau, E_{s_0}, D_k + \bar{D}_k + \hat{D}_k, \sum_{j=0}^{\bar{s}_{k+1}} \sum_{j=2\gamma+\beta} \|\partial_t^\gamma \nabla_{\bar{x}}^\beta (\bar{u}_{k+1,\shortparallel}, \bar{\theta}_{k+1})(0)\|^2_{L^2_{l_j^{k+1}}} \\
&\quad\quad + \sum_{\gamma+\beta \leqq s_{k+1}} \|\partial_t^\gamma \nabla_x^\beta (\rho_{k+1}, u_{k+1}, \theta_{k+1})(0)\|_{L^2_x}\Big) < \infty.
\end{aligned}
\tag{5.36}
$$

Step 2.3. Construction of solution $\hat{f}_{k+1}$. Letting $0 \leqq \zeta \leqq \zeta_{k-1}$, by using (2.18), (2.16) and (2.7), we have

$$\begin{aligned}
&\sum_{\gamma+\beta\leqq\hat{s}_{k+1}} |e^{\zeta\cdot\eta}\partial_t^\gamma \partial_{\shortparallel}^\beta (\hat{A}_{k+1}, \hat{B}_{k+1}, \hat{C}_{k+1})(t, x_{\shortparallel}, \eta)| \\
&\leqq C(E_{2+\hat{s}_{k+1}}) \sum_{\gamma+\beta\leqq\hat{s}_{k+1}} e^{\zeta\cdot\eta} \int_\eta^\infty e^{-\zeta_{k-1}\cdot z}\, \mathrm{d}z\, \|e^{\zeta_{k-1}\cdot z}\partial_t^\gamma \partial_{\shortparallel}^\beta (\hat{a}_{k+1}, \hat{b}_{k+1}, \hat{c}_{k+1})(t, x_{\shortparallel}, \cdot)\|_{L_z^\infty} \\
&\leqq C(E_{2+\hat{s}_{k+1}}) \sum_{\gamma+\beta\leqq\hat{s}_{k+1}} e^{-(\zeta_{k-1}-\zeta)\cdot\eta}\, \|e^{\zeta_{k-1}\cdot z}\partial_t^\gamma \partial_{\shortparallel}^\beta (\hat{a}_{k+1}, \hat{b}_{k+1}, \hat{c}_{k+1})(t, x_{\shortparallel}, \cdot)\|_{L_z^\infty} \\
&\leqq C(E_{3+\hat{s}_{k+1}}) \sum_{\gamma+\beta\leqq 1+\hat{s}_{k+1}} \|e^{\zeta_{k-1}\cdot\eta}\, \mathfrak{w}_{\hat{\kappa}_{k-1}} \partial_t^\gamma \partial_{\shortparallel}^\beta \hat{f}_{k-1}(t, x_{\shortparallel}, \cdot, \cdot)\|_{L^\infty_{\eta,v}},
\end{aligned} \tag{5.37}$$

and

$$\begin{aligned}
&\sum_{\gamma+\beta\leqq\hat{s}_{k+1}} |e^{\zeta\cdot\eta}\partial_t^\gamma \partial_{\shortparallel}^\beta \partial_\eta (\hat{A}_{k+1}, \hat{B}_{k+1}, \hat{C}_{k+1})(t, x_{\shortparallel}, \eta)| \\
&\leqq C(E_{2+\hat{s}_{k+1}}) \sum_{\gamma+\beta\leqq\hat{s}_{k+1}} \|e^{\zeta\cdot\eta}\partial_t^\gamma \partial_{\shortparallel}^\beta (\hat{a}_{k+1}, \hat{b}_{k+1}, \hat{c}_{k+1})(t, x_{\shortparallel}, \cdot)\|_{L_z^\infty} \\
&\leqq C(E_{3+\hat{s}_{k+1}}) \sum_{\gamma+\beta\leqq 1+\hat{s}_{k+1}} \|e^{\zeta_{k-1}\cdot\eta}\, \mathfrak{w}_{\hat{\kappa}_{k-1}} \partial_t^\gamma \partial_{\shortparallel}^\beta \hat{f}_{k-1}(t, x_{\shortparallel}, \cdot, \cdot)\|_{L^\infty_{\eta,v}}.
\end{aligned} \tag{5.38}$$

Then, combining (5.37), (5.38) with (2.17), we obtain the existence of $\hat{f}_{k+1,1}(t)$ with

$$\begin{aligned}
&\sum_{i=0,1} \sum_{\gamma+\beta\leqq\hat{s}_{k+1}} \|e^{\frac{3}{2}\zeta_{k+1}\cdot\eta}\mathfrak{w}_{\hat{\kappa}_{k+1,1}} \partial_t^\gamma \partial_{\shortparallel}^\beta \partial_\eta^i \hat{f}_{k+1,1}(t)\|_{L^\infty_{\hat{x},v}\cap L^2_{x_{\shortparallel}} L^\infty_{\eta,v}} \\
&\leqq C(E_{3+s_{k+1}}) \sum_{\gamma+\beta\leqq 1+\hat{s}_{k+1}} \|e^{\zeta_{k-1}\cdot\eta}\, \mathfrak{w}_{\hat{\kappa}_{k-1}} \partial_t^\gamma \partial_{\shortparallel}^\beta \hat{f}_{k-1}(t)\|_{L^\infty_{\hat{x},v}\cap L^2_{x_{\shortparallel}} L^\infty_{\eta,v}},
\end{aligned} \tag{5.39}$$

where we have used $\hat{\kappa}_{k+1,1} \ll \hat{\kappa}_{k-1}$, $1 + \hat{s}_{k+1} \leqq \hat{s}_{k-1}$, and $\zeta = \frac{3}{2}\zeta_{k+1}$ such that $0 < \frac{3}{2}\zeta_{k+1} \leqq \zeta_{k-1}$. Moreover, from (2.18) and (2.17), we conclude that $\hat{f}_{k+1,1}$ is a continuous function over $(t, x_{\shortparallel}, \eta, v) \in [0, \tau] \times \mathbb{R}^2 \times \mathbb{R}_+ \times \mathbb{R}^3$.

Using (5.39), (2.17) and the trace theorem, one can obtain

$$\begin{aligned}
&\sum_{\gamma+\beta\leqq\hat{s}_{k+1}} \|e^{\frac{3}{2}\zeta_{k+1}\cdot\eta}\mathfrak{w}_{\hat{\kappa}_{k+1,1}} \partial_t^\gamma \partial_{\shortparallel}^\beta \hat{f}_{k+1,1}(t, \cdot, 0, \cdot)\|_{L^\infty_{x_{\shortparallel},v}\cap L^2_{x_{\shortparallel}} L^\infty_v} \\
&\leqq C(E_{3+s_{k+1}}) \sum_{\gamma+\beta\leqq 1+\hat{s}_{k+1}} \|e^{\zeta_{k-1}\cdot\eta}\, \mathfrak{w}_{\hat{\kappa}_{k-1}} \partial_t^\gamma \partial_{\shortparallel}^\beta \hat{f}_{k-1}(t)\|_{L^\infty_{\hat{x},v}\cap L^2_{x_{\shortparallel}} L^\infty_{\eta,v}}.
\end{aligned} \tag{5.40}$$

We still need to construct $\hat{f}_{k+1,2}$. Firstly, it follows from (2.23), (5.20), (5.36), the trace theorem and (5.40) that

$$\sum_{\gamma+\beta\leqq\hat{s}_{k+1}} \|\mathfrak{w}_{\hat{\kappa}_{k+1,1}} \partial_t^\gamma \nabla_{\shortparallel}^\beta \hat{g}_{k+1}(t)\|_{L^\infty_{x_{\shortparallel},v}\cap L^2_{x_{\shortparallel}} L^\infty_v}$$

$$\leqq \sum_{\gamma+\beta\leqq\hat{s}_{k+1}} \|\mathfrak{w}_{\hat{\kappa}_{k+1,1}} \partial_t^\gamma \nabla_{\shortparallel}^\beta (f_{k+1}, \bar{f}_{k+1}, \hat{f}_{k+1,1})(t,\cdot,0,\cdot)\|_{L^\infty_{x_{\shortparallel},v}\cap L^2_{x_{\shortparallel}} L^\infty_v}$$

$$\leqq C\Big(\tau, E_{s_0}, D_k+\bar{D}_k+\hat{D}_k, \sum_{j=0}^{\bar{s}_{k+1}} \sum_{j=2\gamma+\beta} \|\partial_t^\gamma \nabla_{\bar{x}}^\beta (\bar{u}_{k+1,\shortparallel}, \bar{\theta}_{k+1})(0)\|^2_{L^2_{l_j^{k+1}}}$$

$$+ \sum_{\gamma+\beta\leqq s_{k+1}} \|\partial_t^\gamma \nabla_x^\beta (\rho_{k+1}, u_{k+1}, \theta_{k+1})(0)\|_{L^2_x}\Big),$$

provided that $2+\hat{s}_{k+1} \leqq \bar{s}_{k+1}$.

On the other hand, using (2.8) and Sobolev inequality, a direct calculation shows that

$$\sum_{\gamma+\beta\leqq\hat{s}_{k+1}} \|e^{\frac{3}{4}\zeta_k\cdot\eta} \mathfrak{w}_{\hat{\kappa}_{k+1,1}} \partial_t^\gamma \partial_{\shortparallel}^\beta \hat{S}_{k+1,2}(t)\|_{L^\infty_{\bar{x},v}\cap L^2_{x_{\shortparallel}} L^\infty_{\eta,v}}$$

$$\leqq C\Big(E_{\mathfrak{b}+2+\hat{s}_{k+1}}, \sum_{i=1}^{k} \sum_{\gamma+\beta\leqq\hat{s}_{k+1}+\mathfrak{b}} \big[\|\tilde{w}_{\kappa_i} \partial_t^\gamma \partial_{\shortparallel}^\beta f_i(t)\|_{L^\infty_{x,v}} + \|\mathfrak{w}_{\hat{\kappa}_i} \partial_t^\gamma \partial_{\shortparallel}^\beta \bar{f}_i(t)\|_{L^\infty_{\bar{x},v}}\big]$$

$$+ \sum_{i=1}^{k} \sum_{\gamma+\beta\leqq 1+\hat{s}_{k+1}} \|\eta^{\mathfrak{b}} e^{\frac{3}{4}\zeta_k\cdot\eta} \mathfrak{w}_{\hat{\kappa}_i} \partial_t^\gamma \partial_{\shortparallel}^\beta \hat{f}_i(t)\|_{L^\infty_{\bar{x},v}\cap L^2_{x_{\shortparallel}} L^\infty_{\eta,v}}\Big)$$

$$\leqq C\Big(\tau, E_{s_0}, D_k+\bar{D}_k+\hat{D}_k\Big),$$

provided that $s_0 \geqq m+2+\hat{s}_{k+1}$, $\hat{s}_k \geqq 2+\hat{s}_{k+1}$ and $\hat{s}_{k+1}+\mathfrak{b}+2 \leqq \frac{1}{2}\bar{s}_k$. Clearly for $\hat{S}_{k+1,1}$,

$$\sum_{\gamma+\beta\leqq\hat{s}_{k+1}} \|e^{\frac{3}{4}\zeta_k\cdot\eta} \mathfrak{w}_{\hat{\kappa}_{k+1,1}} \partial_t^\gamma \partial_{\shortparallel}^\beta \hat{S}_{k+1,1}(t)\|_{L^\infty_{\bar{x},v}\cap L^2_{x_{\shortparallel}} L^\infty_{\eta,v}}$$

$$\leqq C(E_{2+\hat{s}_{k+1}}) \sum_{\gamma+\beta\leqq\hat{s}_{k+1}} \|e^{\frac{3}{4}\zeta_k\cdot\eta} \partial_t^\gamma \partial_{\shortparallel}^\beta (\hat{a}_{k+1}, \hat{b}_{k+1}, \hat{c}_{k+1})(t, x_{\shortparallel}, \cdot)\|_{L^\infty_{\bar{x}}\cap L^2_{x_{\shortparallel}} L^\infty_{\eta}}$$

$$\leqq C(E_{s_0}, \hat{D}_{k-1}). \tag{5.41}$$

Let $0 < \zeta_{k+1} \leqq \frac{1}{2}\zeta_k$ and $1 \ll \hat{\kappa}_{k+1} \ll \hat{\kappa}_{k+1,1} \ll \bar{\kappa}_{k+1}$. Then, by using Lemma 2.5, (2.15), and (5.39)–(5.41), one establish the existence of solution $\hat{f}_{k+1,2}(t)$ over $t \in [0,\tau]$ with

$$\sum_{\beta+\gamma\leqq\hat{s}_{k+1}} \Big\{\|\mathfrak{w}_{\hat{\kappa}_{k+1}} e^{\zeta_{k+1}\eta} \partial_t^\gamma \nabla_{\shortparallel}^\beta \hat{f}_{k+1,2}(t)\|_{L^\infty_{x_{\shortparallel},\eta,v}\cap L^2_{x_{\shortparallel}} L^\infty_{\eta,v}}$$

$$+ \|\mathfrak{w}_{\hat{\kappa}_{k+1}} \partial_t^\gamma \nabla_{\shortparallel}^\beta \hat{f}_{k+1,2}(t,\cdot,0,\cdot)\|_{L^\infty_{x_{\shortparallel},v}\cap L^2_{x_{\shortparallel}} L^\infty_v}\Big\}$$

$$\leqq C\Big(\tau, E_{s_0}, D_k+\bar{D}_k+\hat{D}_k, \sum_{j=0}^{\bar{s}_{k+1}} \sum_{j=2\gamma+\beta} \|\partial_t^\gamma \nabla_{\bar{x}}^\beta (\bar{u}_{k+1,\shortparallel}, \bar{\theta}_{k+1})(0)\|^2_{L^2_{l_j^{k+1}}}$$

$$+\sum_{\gamma+\beta\leqq s_{k+1}}\|\partial_t^{\gamma}\nabla_x^{\beta}(\rho_{k+1},u_{k+1},\theta_{k+1})(0)\|_{L_x^2}\Big),$$

which, together with (5.39) and (5.40), yields the existence of solution $\hat{f}_{k+1}$ satisfying

$$\begin{aligned}&\sum_{\beta+\gamma\leqq\hat{s}_{k+1}}\Big\{\|\mathfrak{w}_{\hat{\kappa}_{k+1}}e^{\zeta_{k+1}\eta}\partial_t^{\gamma}\nabla_{\shortparallel}^{\beta}\hat{f}_{k+1}(t)\|_{L^\infty_{x_{\shortparallel},\eta,v}\cap L^2_{x_{\shortparallel}}L^\infty_{\eta,v}}\\&\quad+\|\mathfrak{w}_{\hat{\kappa}_{k+1}}\partial_t^{\gamma}\nabla_{\shortparallel}^{\beta}\hat{f}_{k+1}(t,\cdot,0,\cdot)\|_{L^\infty_{x_{\shortparallel},v}\cap L^2_{x_{\shortparallel}}L^\infty_v}\Big\}\\&\leqq C\Big(\tau,E_{s_0},D_k+\bar{D}_k+\hat{D}_k,\sum_{j=0}^{\bar{s}_{k+1}}\sum_{j=2\gamma+\beta}\|\partial_t^{\gamma}\nabla_{\bar{x}}^{\beta}(\bar{u}_{k+1,\shortparallel},\bar{\theta}_{k+1})(0)\|^2_{L^2_{l_j^{k+1}}}\\&\quad+\sum_{\gamma+\beta\leqq s_{k+1}}\|\partial_t^{\gamma}\nabla_x^{\beta}(\rho_{k+1},u_{k+1},\theta_{k+1})(0)\|_{L_x^2}\Big).\end{aligned}$$

Step 3. Combining all above estimates and the induction assumption (5.12), we have proved the existence of solutions $f_i,\ \bar{f}_i,\ \hat{f}_i,\ i=1,\dots,N$ with

$$\begin{aligned}D_N+\bar{D}_N+\hat{D}_N\leqq C\Big(\tau,E_{s_0},&\sum_{i=0}^{N}\sum_{j=0}^{\bar{s}_i}\sum_{j=2\gamma+\beta}\|\partial_t^{\gamma}\nabla_{\bar{x}}^{\beta}(\bar{u}_{i,\shortparallel},\bar{\theta}_i)(0)\|^2_{L^2_{l^i_j}}\\&+\sum_{i=0}^{N}\sum_{\gamma+\beta\leqq s_i}\|\partial_t^{\gamma}\nabla_x^{\beta}(\rho_i,u_i,\theta_i)(0)\|_{L_x^2}\Big),\end{aligned}$$

where we have chosen $s_i,\ \bar{s}_i,\ \hat{s}_i$ such that

$$\begin{aligned}&s_0\geqq s_1+\mathfrak{b}+6,\quad s_1=\bar{s}_1=\hat{s}_1\gg 1;\\&s_1>s_i>\bar{s}_i>\hat{s}_i\geqq s_{i+1}>\bar{s}_{i+1}>\hat{s}_{i+1}\geqq\cdots\gg 1,\ \text{for } i=2,\dots,N-1;\\&s_{i+1}\leqq\min\{\hat{s}_i,\ \tfrac{1}{2}\bar{s}_i-3\},\ \bar{s}_{i+1}\leqq s_{i+1}-8-\mathfrak{b},\ \hat{s}_{i+1}\leqq\tfrac{1}{2}\bar{s}_{i+1}-2-\mathfrak{b},\\&\quad\text{for } i=1,\dots,N-1,\end{aligned}\tag{5.42}$$

and taken $l^i_j=\bar{l}_j+2(\bar{s}_i-j)$ with $0\leqq j\leqq\bar{s}_i$ so that

$$l^N_j\gg 2\mathfrak{b}\quad\text{and}\quad l^i_j\geqq 2l^{i+1}_j+18+2\mathfrak{b},\ \text{for } 1\leqq i\leqq N-1.\tag{5.43}$$

Here we can taken $s_1=\bar{s}_1=\hat{s}_1$ because $f_1,\ \bar{f}_1$ and $\hat{f}_1$ depend only on the Euler solution, and do not depend on each other. We also point out that $f_i,\ \bar{f}_i$ are smooth, but $\hat{f}_i$ is only continuous away from the grazing set $\{(x_{\shortparallel},0,v)\,|\,x_{\shortparallel}\in\mathbb{R}^2,\ v_{\shortparallel}\in\mathbb{R}^2,\ v_3\neq 0\}$. For the velocity weight functions, we demand

$$\kappa_i\gg\bar{\kappa}_i\gg\hat{\kappa}_i\gg\kappa_{i+1}\gg\bar{\kappa}_{i+1}\gg\hat{\kappa}_{i+1}\gg 1\tag{5.44}$$

for $1 \leqq i \leqq N-1$, and we do not describe the precise relations between κ_i, $\bar{\kappa}_i$ and $\hat{\kappa}_i$ because the functions F_i, $\bar{F}_i$ and $\hat{F}_i$ indeed decay exponentially with respect to particle velocity v. This completes the proof. □

Remark 5.4. To establish the interior expansion F_k, viscous boundary layer $\bar{F}_k$ and Knudsen boundary layer $\hat{F}_k$, one should deal with the boundary interaction very carefully. In fact, due to the boundary effects, one can only obtain the uniform estimates of time and tangential derivatives for the Knudsen boundary layer $\hat{F}_k$. Fortunately, such time and tangential derivatives estimates of Knudsen layer are enough to control the boundary interplay, see (5.15)–(5.17) and (5.32) for details.

6. Hilbert Expansion: Proof of Theorem 1.6

In this section, with the uniform estimates in Proposition 5.1, we shall use L^2-L^∞ method to estimate the remainder term F_R^ε in (1.40) over half-space. Firstly, from the formulation of boundary condition in Section 2.3, it is easy to know that F_R^ε satisfies the specular reflection boundary conditions, i.e.,

$$F_R^\varepsilon(t,x,v)|_{\gamma_-} = F_R^\varepsilon(t,x_{\shortparallel},0,v_{\shortparallel},-v_3). \tag{6.1}$$

6.1. L^2-Energy Estimate

Recalling the definition of f_R^ε in (1.45), we rewrite the equation terms of f_R^ε as

$$\begin{aligned}
&\partial_t f_R^\varepsilon + v\cdot\nabla_x f_R^\varepsilon + \frac{1}{\varepsilon^2}\mathbf{L} f_R^\varepsilon \\
&\quad = -\frac{\{\partial_t + v\cdot\nabla_x\}\sqrt{\mu}}{\sqrt{\mu}} f_R^\varepsilon + \varepsilon^3\frac{1}{\sqrt{\mu}} Q(\sqrt{\mu} f_R^\varepsilon, \sqrt{\mu} f_R^\varepsilon) \\
&\qquad + \sum_{i=1}^N \varepsilon^{i-2}\frac{1}{\sqrt{\mu}}\Big\{Q(F_i+\bar{F}_i+\hat{F}_i, \sqrt{\mu} f_R^\varepsilon) + Q(\sqrt{\mu} f_R^\varepsilon, F_i+\bar{F}_i+\hat{F}_i)\Big\} \\
&\qquad + \frac{1}{\sqrt{\mu}} R^\varepsilon + \frac{1}{\sqrt{\mu}}\bar{R}^\varepsilon + \frac{1}{\sqrt{\mu}}\hat{R}^\varepsilon,
\end{aligned} \tag{6.2}$$

where R^ε, $\bar{R}^\varepsilon$, $\hat{R}^\varepsilon$ are defined in (1.42), (1.43) and (1.44), respectively. From (6.1), we know that f_R^ε satisfies specular reflection boundary conditions

$$f_R^\varepsilon(t,x_1,x_2,0,v_1,v_2,v_3)|_{v_3>0} = f_R^\varepsilon(t,x_1,x_2,0,v_1,v_2,-v_3). \tag{6.3}$$

Lemma 6.1. *Let* $0 < \frac{1}{2\alpha}(1-\alpha) < \mathfrak{a} < \frac{1}{2}$, $\kappa \geqq 7$, $N \geqq 6$ *and* $\mathfrak{b} \geqq 5$. *Let* $\tau > 0$ *be the life span of compressible Euler solution obtained in Lemma 2.1, then there exists a suitably small constant* $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in (0,\varepsilon_0)$, *it holds that*

$$\begin{aligned}
&\frac{d}{dt}\|f_R^\varepsilon(t)\|_{L^2}^2 + \frac{c_0}{2\varepsilon^2}\|\{\mathbf{I}-\mathbf{P}\} f_R^\varepsilon(t)\|_\nu^2 \\
&\quad \leqq C\Big\{1+\varepsilon^8\|h_R^\varepsilon(t)\|_{L^\infty}^2\Big\}\cdot(\|f_R^\varepsilon(t)\|_{L^2}^2+1), \ \textit{for } t\in[0,\tau].
\end{aligned} \tag{6.4}$$

Proof. Multiplying (6.2) by f_R^ε and integrating over $\mathbb{R}^3_+ \times \mathbb{R}^3$, one obtains that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f_R^\varepsilon\|_{L^2}^2 + \frac{c_0}{\varepsilon^2}\|\{\mathbf{I}-\mathbf{P}\}f_R^\varepsilon\|_\nu^2 - \frac{1}{2}\int_{\partial\mathbb{R}^3_+}\int_{\mathbb{R}^3} v_3|f_R^\varepsilon(t,x_1,x_2,0,v)|^2\,\mathrm{d}x_1\mathrm{d}x_2\,\mathrm{d}v\\
&\quad = -\int_{\mathbb{R}^3_+}\int_{\mathbb{R}^3}\frac{\{\partial_t + v\cdot\nabla_x\}\sqrt{\mu}}{\sqrt{\mu}}|f_R^\varepsilon|^2 + \varepsilon^3\int_{\mathbb{R}^3_+}\int_{\mathbb{R}^3}\frac{1}{\sqrt{\mu}}Q(\sqrt{\mu}f_R^\varepsilon,\sqrt{\mu}f_R^\varepsilon)f_R^\varepsilon\\
&\qquad + \int_{\mathbb{R}^3_+}\int_{\mathbb{R}^3}\sum_{i=1}^N \varepsilon^{i-2}\frac{1}{\sqrt{\mu}}\Big\{Q(F_i+\bar{F}_i+\hat{F}_i,\sqrt{\mu}f_R^\varepsilon) + Q(\sqrt{\mu}f_R^\varepsilon, F_i+\bar{F}_i+\hat{F}_i)\Big\}f_R^\varepsilon\\
&\qquad + \int_{\mathbb{R}^3_+}\int_{\mathbb{R}^3}\left\{\frac{1}{\sqrt{\mu}}R^\varepsilon + \frac{1}{\sqrt{\mu}}\bar{R}^\varepsilon + \frac{1}{\sqrt{\mu}}\hat{R}^\varepsilon\right\}f_R^\varepsilon. \qquad (6.5)
\end{aligned}
$$

Using the boundary condition (6.3) yields

$$
\int_{\partial\mathbb{R}^3_+}\int_{\mathbb{R}^3} v_3|f_R^\varepsilon(t,x_1,x_2,0,v)|^2\,\mathrm{d}x_1\mathrm{d}x_2\,\mathrm{d}v = 0.
$$

For any $\lambda > 0$, as in [23], taking $\kappa \geqq 7$, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^3_+}\int_{\mathbb{R}^3}\frac{\{\partial_t + (v\cdot\nabla_x)\}\sqrt{\mu}}{\sqrt{\mu}}|f_R^\varepsilon|^2\,\mathrm{d}v\,\mathrm{d}x\\
&\quad \leqq C\int_{\mathbb{R}^3_+}\int_{\mathbb{R}^3}|(\nabla_x\rho,\nabla_x\mathfrak{u},\nabla_x T)|(1+|v|)^3|f_R^\varepsilon|^2\,\mathrm{d}v\mathrm{d}x\\
&\quad \leqq C\left\{\int_{\mathbb{R}^3_+}\int_{|v|\leqq\frac{\lambda}{\varepsilon}} + \int_{\mathbb{R}^3_+}\int_{|v|\geqq\frac{\lambda}{\varepsilon}}\right\}(\ldots)\,\mathrm{d}v\mathrm{d}x\\
&\quad \leqq C\frac{\lambda}{\varepsilon^2}\|\{\mathbf{I}-\mathbf{P}\}f_R^\varepsilon\|_\nu^2 + C_\lambda(1+\varepsilon^4\|h_R^\varepsilon\|_{L^\infty})\|f_R^\varepsilon\|_{L^2},
\end{aligned}
$$

and

$$
\begin{aligned}
&\varepsilon^3\int_{\mathbb{R}^3_+}\int_{\mathbb{R}^3}\frac{1}{\sqrt{\mu}}Q(\sqrt{\mu}f_R^\varepsilon,\sqrt{\mu}f_R^\varepsilon)f_R^\varepsilon\,\mathrm{d}v\mathrm{d}x\\
&\quad = \varepsilon^3\int_{\mathbb{R}^3_+}\int_{\mathbb{R}^3}\frac{1}{\sqrt{\mu}}Q(\sqrt{\mu}f_R^\varepsilon,\sqrt{\mu}f_R^\varepsilon)\{\mathbf{I}-\mathbf{P}\}f_R^\varepsilon\,\mathrm{d}v\mathrm{d}x\\
&\quad \leqq \varepsilon^3\|\{\mathbf{I}-\mathbf{P}\}f_R^\varepsilon\|_\nu\|h_R^\varepsilon\|_{L^\infty}\|f_R^\varepsilon\|_{L^2}\\
&\quad \leqq \frac{\lambda}{\varepsilon^2}\|\{\mathbf{I}-\mathbf{P}\}f_R^\varepsilon\|_\nu^2 + C_\lambda\varepsilon^8\|h_R^\varepsilon\|_{L^\infty}^2\|f_R^\varepsilon\|_{L^2}^2.
\end{aligned}
$$

From (5.42),

$$
s_N > \bar{s}_N \geqq 2\mathfrak{b} + 4 + \hat{s}_N, \quad \hat{s}_N \geqq 1,
$$

which, together with (5.43), (5.44), (5.3) and Sobolev imbedding theorem, yields that, for $1 \leqq i \leqq N$ and $t \in [0, \tau]$,

$$\sum_{k=0}^{2\mathfrak{b}+2} \left\{ \left\| \tilde{w}_{\kappa_i}(v) \nabla_{t,x}^k f_i(t) \right\|_{L^2_{x,v}} + \left\| \tilde{w}_{\kappa_i} \nabla_{t,x}^k f_i(t) \right\|_{L^\infty_{x,v}} \right\} \leqq C_R(\tau),$$
$$\sum_{k=0}^{\mathfrak{b}+2} \left\{ \left\| \mathfrak{w}_{\bar{\kappa}_i} (1+y)^{\mathfrak{b}+9} \nabla_{t,\bar{x}}^k \bar{f}_i(t) \right\|_{L^2_{\bar{x},v}} + \left\| \mathfrak{w}_{\bar{\kappa}_i} (1+y)^{\mathfrak{b}+9} \nabla_{t,\bar{x}}^k \bar{f}_i(t) \right\|_{L^\infty_{\bar{x},v}} \right\} \leqq C_R(\tau),$$
$$\sum_{k=0,1} \left\{ \left\| \mathfrak{w}_{\bar{\kappa}_i} e^{\frac{1}{2N}\cdot\eta} \nabla_{t,x_{\shortparallel}} \hat{f}_i(t) \right\|_{L^2_{\hat{x},v}} + \left\| \mathfrak{w}_{\bar{\kappa}_i} e^{\frac{1}{2N}\cdot\eta} \nabla_{t,x_{\shortparallel}} \hat{f}_i(t) \right\|_{L^\infty_{\hat{x},v}} \right\} \leqq C_R(\tau), \tag{6.6}$$

where we have denoted

$$C_R(\tau) := C\Bigg(\tau, \|(\varphi_0, \Phi_0, \vartheta_0)\|_{H^{s_0}} + \sum_{i=0}^N \sum_{\gamma+\beta \leqq s_i} \|\partial_t^\gamma \nabla_x^\beta (\rho_i, u_i, \theta_i)(0)\|_{L^2_x}$$
$$+ \sum_{i=0}^N \sum_{j=0}^{\bar{s}_i} \sum_{j=2\gamma+\beta} \|\partial_t^\gamma \nabla_{\bar{x}}^\beta (\bar{u}_{i,\shortparallel}, \bar{\theta}_i)(0)\|^2_{L^2_{l^i_j}}\Bigg).$$

Noting (1.47), we have, for $1 \leqq i \leqq N$, that

$$\left| w_\kappa(v) \frac{\sqrt{\mu_0}}{\sqrt{\mu}} \bar{f}_i(t, x_{\shortparallel}, y, v) \right| \leqq C |w_\kappa(v) \mu_0^{-\mathfrak{a}} \bar{f}_i(t, x_{\shortparallel}, y, v)| \cdot \frac{\mu_0^{\frac{1}{2}+\mathfrak{a}}}{\mu^{\frac{1}{2}}}$$
$$\leqq C |w_\kappa(v) \mu_0^{-\mathfrak{a}} \bar{f}_i(t, x_{\shortparallel}, y, v)| \cdot (\mu_M)^{\left(\frac{1}{2}+\mathfrak{a}\right)\alpha - \frac{1}{2}},$$
$$\left| w_\kappa(v) \frac{\sqrt{\mu_0}}{\sqrt{\mu}} \hat{f}_i(t, x_{\shortparallel}, \eta, v) \right| \leqq C |w_\kappa(v) \mu_0^{-\mathfrak{a}} \hat{f}_i(t, x_{\shortparallel}, y, v)| \cdot (\mu_M)^{\left(\frac{1}{2}+\mathfrak{a}\right)\alpha - \frac{1}{2}}. \tag{6.7}$$

Taking $0 < \frac{1}{2\alpha}(1-\alpha) < \mathfrak{a} < \frac{1}{2}$, we have $(\frac{1}{2}+\mathfrak{a})\alpha - \frac{1}{2} > 0$ which, together with (5.1), (6.6) and (6.7), implies that the third term on RHS of (6.5) is bounded by

$$\|\{\mathbf{I}-\mathbf{P}\} f_R^\varepsilon\|_\nu \|f_R^\varepsilon\|_\nu \cdot \sum_{i=1}^N \varepsilon^{i-2} \Bigg\{ \|w_\kappa f_i\|_{L^\infty_{x,v}} + \|w_\kappa \frac{\sqrt{\mu_0}}{\sqrt{\mu}} \bar{f}_i\|_{L^\infty_{\bar{x},v}}$$
$$+ \|w_\kappa \frac{\sqrt{\mu_0}}{\sqrt{\mu}} \hat{f}_i\|_{L^\infty_{\hat{x},v}} \Bigg\}$$
$$\leqq C\frac{1}{\varepsilon} \|\{\mathbf{I}-\mathbf{P}\} f_R^\varepsilon\|_\nu \|f_R^\varepsilon\|_\nu \leqq \frac{\lambda}{\varepsilon^2} \|\{\mathbf{I}-\mathbf{P}\} f_R^\varepsilon\|_\nu^2 + C_\lambda \|f_R^\varepsilon\|_\nu^2$$
$$\leqq (\lambda + C_\lambda \varepsilon^2) \frac{1}{\varepsilon^2} \|\{\mathbf{I}-\mathbf{P}\} f_R^\varepsilon\|_\nu^2 + C_\lambda \|f_R^\varepsilon\|^2_{L^2}.$$

From (1.42) and $(6.6)_1$, a direct calculation shows that

$$\left(\int_{\mathbb{R}^3_+}\int_{\mathbb{R}^3}|\frac{1}{\sqrt{\mu}}R^\varepsilon|^2\,\mathrm{d}v\mathrm{d}x\right)^{\frac{1}{2}} \leqq C\varepsilon^{N-6}. \tag{6.8}$$

It follows from (1.43), (1.44) and (6.6) that

$$\begin{aligned}\left(\int_{\mathbb{R}^3_+}\int_{\mathbb{R}^3}|\frac{1}{\sqrt{\mu}}\bar{R}^\varepsilon|^2\,\mathrm{d}v\mathrm{d}x\right)^{\frac{1}{2}} &\leqq C(\varepsilon^{N-5.5}+\varepsilon^{\mathfrak{b}-4.5}),\\ \left(\int_{\mathbb{R}^3_+}\int_{\mathbb{R}^3}|\frac{1}{\sqrt{\mu}}\hat{R}^\varepsilon|^2\,\mathrm{d}v\mathrm{d}x\right)^{\frac{1}{2}} &\leqq C(\varepsilon^{N-5}+\varepsilon^{\mathfrak{b}-3}).\end{aligned} \tag{6.9}$$

Combining (6.8)–(6.9) and Cauchy inequality, we conclude that

$$\left|\int_{\mathbb{R}^3_+}\int_{\mathbb{R}^3}\left\{\frac{1}{\sqrt{\mu}}R^\varepsilon+\frac{1}{\sqrt{\mu}}\bar{R}^\varepsilon+\frac{1}{\sqrt{\mu}}\hat{R}^\varepsilon\right\}f_R^\varepsilon\,\mathrm{d}v\mathrm{d}x\right| \leqq C(\varepsilon^{N-6}+\varepsilon^{\mathfrak{b}-5})\|f_R^\varepsilon\|_{L^2}.$$

Hence (6.4) follows from above estimates. This completes the proof of Lemma 6.1. □

6.2. Weighted L^∞-Estimate

Given (t, x, v), let $[X(s), V(s)]$ be the backward bi-characteristics of the Boltzmann equation, which is determined by

$$\begin{cases}\dfrac{\mathrm{d}X(s)}{\mathrm{d}s}=V(s), \quad \dfrac{\mathrm{d}V(s)}{\mathrm{d}s}=0,\\ [X(t), V(t)]=[x, v].\end{cases}$$

The solution is then given by

$$[X(s), V(s)] = [X(s; t, x, v), V(s; t, x, v)] = [x-(t-s)v, v].$$

For each (x, v) with $x \in \overline{\mathbb{R}}^3_+$ and $v_3 \neq 0$, we define its *backward exit time* $t_{\mathbf{b}}(x, v) \geqq 0$ to be the last moment at which the back-time straight line

$$[X(s; 0, x, v), V(s; 0, x, v)]$$

remains in $\overline{\mathbb{R}}^3_+$:

$$t_{\mathbf{b}}(x, v) = \sup\{\tau \geqq 0 : x-\tau v \in \mathbb{R}^3_+\}.$$

We therefore have $x - t_{\mathbf{b}}v \in \partial\mathbb{R}^3_+$ and $x_3 - t_{\mathbf{b}}v_3 = 0$. We also define

$$x_{\mathbf{b}}(x, v) = x(t_{\mathbf{b}}) = x - t_{\mathbf{b}}v \in \partial\mathbb{R}^3_+.$$

Note that $v\cdot\vec{n}(x_{\mathbf{b}}) = v\cdot\vec{n}(x_{\mathbf{b}}(x, v)) < 0$ always holds true.

For half space problem the back-time trajectory is very simple, and the particle hit the boundary at most one time. More precisely, for the case $v_3 < 0$, the back-time cycle is a straight line and does not hit the boundary; on the other hand, for $v_3 > 0$, the back-time cycle will hit the boundary for one time. Now let $x \in \overline{\mathbb{R}}^3_+$, $(x, v) \notin \gamma_0 \cup \gamma_-$ and $(t_0, x_0, v_0) = (t, x, v)$, the back-time cycle is defined as

$$\begin{cases} X_{cl}(s; t, x, v) = \mathbf{1}_{[t_1,t_0)}(s)\{x - v(t-s)\} + \mathbf{1}_{(-\infty,t_1)}(s)\{x - R_{x_\mathbf{b}} v(t-s)\}, \\ V_{cl}(s; t, x, v) = \mathbf{1}_{[t_1,t_0)}(s) v + \mathbf{1}_{(-\infty,t_1)}(s) R_{x_\mathbf{b}} v, \end{cases}$$

with

$$(t_1, x_\mathbf{b}) = (t - t_\mathbf{b}(x, v), x_\mathbf{b}(x, v)).$$

The explicit formula is

$$\begin{aligned} & t_\mathbf{b}(x, v) = \frac{x_3}{v_3}, \quad \text{for } v_3 > 0 \quad \text{and} \quad t_\mathbf{b}(x, v) = \infty, \quad \text{for } v_3 < 0; \\ & V_{cl}(s) = \begin{cases} (v_1, v_2, v_3), & \text{if } s \in [t_1, t] \\ (v_1, v_2, -v_3), & \text{if } s \in (-\infty, t_1), \end{cases} \\ & X_{cl}(s) = x - v(t-s), \quad \text{if } s \in [t_1, t], \\ & X_{cl}(s) = \begin{cases} x_1 - v_1(t-s), \\ x_2 - v_2(t-s), \\ -x_3 + v_3(t-s), \end{cases} \quad \text{if } s \in (-\infty, t_1). \end{aligned} \tag{6.10}$$

As in [22,23], we denote

$$L_M g = -\frac{1}{\sqrt{\mu_M}} \Big\{ Q(\mu, \sqrt{\mu_M} g) + Q(\sqrt{\mu_M} g, \mu) \Big\} = \nu(\mu) g - K g,$$

where the frequency $\nu(\mu)$ has been defined in (1.11) and $Kg = K_2 g - K_1 g$ with

$$\begin{aligned} K_1 g &= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |(v-u) \cdot \omega| \sqrt{\mu_M(u)} \frac{\mu(v)}{\sqrt{\mu_M(v)}} g(u) \, \mathrm{d}u \mathrm{d}\omega, \\ K_2 g &= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |(v-u) \cdot \omega| \mu(u') \frac{\sqrt{\mu_M(v')}}{\sqrt{\mu_M(v)}} g(v') \, \mathrm{d}u \mathrm{d}\omega \\ &\quad + \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |(v-u) \cdot \omega| \mu(v') \frac{\sqrt{\mu_M(u')}}{\sqrt{\mu_M(v)}} g(u') \, \mathrm{d}u \mathrm{d}\omega. \end{aligned}$$

Lemma 6.2. [23] *It holds that* $Kg(v) = \int_{\mathbb{R}^3} l(v, v') g(v') dv'$ *where the kernel* $l(v, v')$ *satisfies*

$$|l(v, v')| \leqq C \frac{\exp\{-c|v - v'|^2\}}{|v - v'|}. \tag{6.11}$$

Letting $K_w g \equiv w_\kappa K(\frac{g}{w_\kappa})$, we deduce from (1.41) and (1.48) that

$$\begin{aligned}
&\partial_t h_R^\varepsilon + v\cdot\nabla_x h_R^\varepsilon + \frac{\nu(\mu)}{\varepsilon^2}h_R^\varepsilon - \frac{1}{\varepsilon^2}K_w h_R^\varepsilon\\
&= \sum_{i=1}^N \varepsilon^{i-2}\frac{w_\kappa(v)}{\sqrt{\mu_M(v)}}\Big\{Q(F_i+\bar F_i+\hat F_i, \frac{\sqrt{\mu_M}h_R^\varepsilon}{w_\kappa}) + Q\Big(\frac{\sqrt{\mu_M}h_R^\varepsilon}{w_\kappa}, F_i+\bar F_i+\hat F_i\Big)\Big\}\\
&\quad + \varepsilon^3\frac{w_\kappa}{\sqrt{\mu_M}}Q\Big(\frac{\sqrt{\mu_M}h_R^\varepsilon}{w_\kappa}, \frac{\sqrt{\mu_M}h_R^\varepsilon}{w_\kappa}\Big) + \frac{w_\kappa}{\sqrt{\mu_M}}\big[R^\varepsilon+\bar R^\varepsilon+\hat R^\varepsilon\big]. \qquad (6.12)
\end{aligned}$$

Lemma 6.3. *For $t\in[0,\tau]$, it holds that*

$$\sup_{0\leqq s\leqq t}\|\varepsilon^3 h_R^\varepsilon(s)\|_{L^\infty} \leqq C(t)\{\|\varepsilon^3 h_R^\varepsilon(0)\|_{L^\infty} + \varepsilon^{N-1} + \varepsilon^{\mathfrak{b}}\} + \sup_{0\leqq s\leqq t}\|f_R^\varepsilon(s)\|_{L^2}.$$

Proof. For any (t,x,v), integrating (6.12) along the backward trajectory, one has that

$$\begin{aligned}
h_R^\varepsilon(t,x,v) &= \exp\Big\{-\frac{1}{\varepsilon^2}\int_0^t \nu(\xi)\,\mathrm{d}\xi\Big\}h_R^\varepsilon(0, X_{cl}(0), V_{cl}(0))\\
&\quad + \frac{1}{\varepsilon^2}\int_0^t \exp\Big\{-\frac{1}{\varepsilon^2}\int_s^t \nu(\xi)\,\mathrm{d}\xi\Big\}(K_w h_R^\varepsilon)(s, X_{cl}(s), V_{cl}(s))\mathrm{d}s,\\
&\quad + \varepsilon^3\int_0^t \exp\Big\{-\frac{1}{\varepsilon^2}\int_s^t \nu(\xi)\,\mathrm{d}\xi\Big\}\left(\frac{w_\kappa}{\sqrt{\mu_M}}Q\left(\frac{h_R^\varepsilon\sqrt{\mu_M}}{w_\kappa}, \frac{h_R^\varepsilon\sqrt{\mu_M}}{w_\kappa}\right)\right)(s, X_{cl}(s), V_{cl}(s))\mathrm{d}s\\
&\quad + \int_0^t \exp\Big\{-\frac{1}{\varepsilon^2}\int_s^t \nu(\xi)d\xi\Big\}\cdot\left\{\sum_{i=1}^N \varepsilon^{i-2}\frac{w_\kappa}{\sqrt{\mu_M}}Q\left(F_i+\bar F_i+\hat F_i, \frac{h_R^\varepsilon\sqrt{\mu_M}}{w_\kappa}\right)\right.\\
&\quad \left.+ \sum_{i=1}^N \varepsilon^{i-2}\frac{w_\kappa}{\sqrt{\mu_M}}Q\left(\frac{h_R^\varepsilon\sqrt{\mu_M}}{w_\kappa}, F_i+\bar F_i+\hat F_i\right)\right\}(s, X_{cl}(s), V_{cl}(s))\mathrm{d}s\\
&\quad + \int_0^t \exp\Big\{-\frac{1}{\varepsilon^2}\int_s^t \nu(\xi)\,\mathrm{d}\xi\Big\}\left(\frac{w_\kappa}{\sqrt{\mu_M}}[R^\varepsilon+\bar R^\varepsilon+\hat R^\varepsilon]\right)(s, X_{cl}(s), V_{cl}(s))\mathrm{d}s, \quad (6.13)
\end{aligned}$$

where we have denoted

$$\nu(\xi) = \nu(\mu)(\xi, V_{cl}(\xi), X_{cl}(\xi)).$$

It follows from (6.10) that $|V_{cl}(s)|\equiv|v|$. Then a direct calculation shows that

$$\nu(\mu)\sim\nu_M(v) := \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u,\theta)\mu_M(u)\mathrm{d}\omega\mathrm{d}u \cong (1+|v|),$$

and

$$\int_0^t \exp\Big\{-\frac{1}{\varepsilon^2}\int_s^t \nu(\xi)\,\mathrm{d}\xi\Big\}\nu(\mu)\mathrm{d}s \lesssim \int_0^t \exp\Big\{-\frac{\nu_M(v)(t-s)}{C\varepsilon^2}\Big\}\nu_M(v)\mathrm{d}s \leqq O(\varepsilon^2).$$

For the first term on RHS of (6.13), it is easy to know that

$$\left|\exp\left\{-\frac{1}{\varepsilon^2}\int_0^t \nu(\xi)\,\mathrm{d}\xi\right\}h_R^\varepsilon(0, X_{cl}(0), V_{cl}(0))\right| \leqq C\exp\Big(-\frac{\nu_M(v)t}{C\varepsilon^2}\Big)\|h_R^\varepsilon(0)\|_{L^\infty}.$$

We note that

$$\left|\frac{w_\kappa(v)}{\sqrt{\mu_M}}Q\left(\frac{h_R^\varepsilon\sqrt{\mu_M}}{w_\kappa}, \frac{h_R^\varepsilon\sqrt{\mu_M}}{w_\kappa}\right)(s)\right| \leqq C\nu_M(v)\|h_R^\varepsilon(s)\|_{L^\infty}^2 \leqq C\nu(s)\|h_R^\varepsilon(s)\|_{L^\infty}^2,$$

then the third term on RHS of (6.13) is bounded by

$$C\varepsilon^3\int_0^t \exp\left\{-\frac{1}{\varepsilon^2}\int_s^t \nu(\xi)\,\mathrm{d}\xi\right\}\nu(s)\|h_R^\varepsilon(s)\|_{L^\infty}^2\mathrm{d}s \leqq C\varepsilon^5\sup_{0\leqq s\leqq t}\|h_R^\varepsilon(s)\|_{L^\infty}^2.$$

From (1.47), (5.1) and (6.6), a direct calculation shows that

$$\begin{aligned}
&\left|\sum_{i=1}^N \varepsilon^{i-2}\frac{w_\kappa}{\sqrt{\mu_M}}\left\{Q\left(F_i+\bar{F}_i+\hat{F}_i, \frac{h_R^\varepsilon\sqrt{\mu_M}}{w_\kappa}\right)+Q\left(\frac{h_R^\varepsilon\sqrt{\mu_M}}{w_\kappa}, F_i+\bar{F}_i+\hat{F}_i\right)\right\}(s)\right| \\
&\leqq C\nu_M(v)\|h_R^\varepsilon(s)\|_{L^\infty}\cdot\sum_{i=1}^N\varepsilon^{i-2}\left\|w_\kappa\left[\frac{\sqrt{\mu}}{\sqrt{\mu_M}}f_i(s)+\frac{\sqrt{\mu_0}}{\sqrt{\mu_M}}\bar{f}_i(s)+\frac{\sqrt{\mu_0}}{\sqrt{\mu_M}}\hat{f}_i(s)\right]\right\|_{L^\infty} \\
&\leqq C\nu_M(v)\|h_R^\varepsilon(s)\|_{L^\infty}\cdot\sum_{i=1}^N\varepsilon^{i-2}\Big[\|\tilde{w}_{k_i}f_i(s)\|_{L^\infty}+\|\mathfrak{w}_{\bar{k}_i}\bar{f}_i(s)\|_{L^\infty}+\|\mathfrak{w}_{\hat{k}_i}\hat{f}_i(s)\|_{L^\infty}\Big] \\
&\leqq C_R(s)\frac{\nu_M(v)}{\varepsilon}\|h_R^\varepsilon(s)\|_{L^\infty},
\end{aligned}$$

where we have used $\frac{1-\alpha}{2\alpha} < \mathfrak{a} < \frac{1}{2}$. Then the fourth term on RHS of (6.13) is bounded by

$$C_R(t)\frac{1}{\varepsilon}\int_0^t \exp\left\{-\frac{1}{\varepsilon^2}\int_s^t \nu(\xi)\,\mathrm{d}\xi\right\}\nu_M(v)\|h_R^\varepsilon(s)\|_{L^\infty}\mathrm{d}s \leqq C_R(t)\varepsilon\sup_{0\leqq s\leqq t}\|h_R^\varepsilon(s)\|_{L^\infty}.$$

Similarly, it follows from (1.42)–(1.44), (1.47) and (6.6) that

$$\left|\left(\frac{w_\kappa}{\sqrt{\mu_M}}[R^\varepsilon+\bar{R}^\varepsilon+\hat{R}^\varepsilon]\right)(s)\right| \leqq C_R(s)(\varepsilon^{N-6}+\varepsilon^{\mathfrak{b}-5}),$$

which implies that the last term on RHS of (6.13) is bounded by $C_R(t)[\varepsilon^{N-4}+\varepsilon^{\mathfrak{b}-3}]$.

Let $l_w(v,v')$ be the corresponding kernel associated with K_w. Recalling (6.11) we have

$$|l_w(v,v')| \leqq C\frac{w_\kappa(v')\exp\{-c|v-v'|^2\}}{w_\kappa(v)|v-v'|} \leqq C\frac{\exp\{-\frac{3}{4}c|v-v'|^2\}}{|v-v'|}. \tag{6.14}$$

Now we can bound the second term on RHS of (6.13) by

$$\frac{1}{\varepsilon^2}\int_0^t \exp\left\{-\frac{1}{\varepsilon^2}\int_s^t \nu(\xi)\,\mathrm{d}\xi\right\}\int_{\mathbb{R}^3}|l_w(V_{cl}(s),v')h_R^\varepsilon(s,X_{cl}(s),v')|\,\mathrm{d}v'\mathrm{d}s. \tag{6.15}$$

We denote $V'_{cl}(s_1) = V_{cl}(s_1; s, X_{cl}(s), v')$ and $X'_{cl}(s_1) = X_{cl}(s_1; s, X_{cl}(s), v')$. Using (6.13) again to (6.15), then we can bound (6.15) by

$$\begin{aligned}
&\frac{1}{\varepsilon^4}\int_0^t \exp\left\{-\frac{1}{\varepsilon^2}\int_s^t \nu(\xi)\,\mathrm{d}\xi\right\}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |l_w(V_{cl}(s), v')l_w(V'_{cl}(s_1), v'')| \\
&\quad\times \int_0^s \exp\left\{-\frac{1}{\varepsilon^2}\int_{s_1}^s \nu(v')(\xi)\mathrm{d}\xi\right\}|h_R^\varepsilon(s_1, X'_{cl}(s_1), v'')|\,\mathrm{d}v'\mathrm{d}v''\mathrm{d}s_1\mathrm{d}s \\
&\quad + C\|h_R^\varepsilon(0)\|_{L^\infty} + C_R(t)\left\{\varepsilon \sup_{0\leqq s\leqq t}\|h_R^\varepsilon(s)\|_{L^\infty} + \varepsilon^{N-4} + \varepsilon^{\mathfrak{b}-3}\right\} \\
&\quad + C(t)\varepsilon^5 \sup_{0\leqq s\leqq t}\|h_R^\varepsilon(s)\|_{L^\infty}^2,
\end{aligned} \tag{6.16}$$

where we have denoted $\nu(v')(s) = \nu(\mu)(s, X'_{cl}(s), V'_{cl}(s))$ for simplicity of presentation. And we also used the following fact

$$\int_{\mathbb{R}^3} |l_w(v, v')|\mathrm{d}v' \leqq C(1+|v|)^{-1}, \tag{6.17}$$

which follows from (6.14).

We now concentrate on the first term in (6.16). As in [23], we divide things into several cases.

Case 1 For $|v| \geqq \mathfrak{N}$, by using (6.17), one deduces the following bound:

$$\begin{aligned}
&\frac{C}{\varepsilon^4}\sup_{0\leqq s\leqq t}\|h_R^\varepsilon(s)\|_{L^\infty}\int_0^t \exp\left\{-\frac{\nu_M(v)(t-s)}{C\varepsilon^2}\right\}\int_{\mathbb{R}^3}|l_w(V_{cl}(s), v')| \\
&\quad\times\int_0^s \exp\left\{-\frac{\nu_M(v')(s-s_1)}{C\varepsilon^2}\right\}\int_{\mathbb{R}^3}|l_w(V'_{cl}(s_1), v'')|\mathrm{d}v''\mathrm{d}s_1\mathrm{d}v'\mathrm{d}s \\
&\leqq \frac{C}{\mathfrak{N}}\sup_{0\leqq s\leqq t}\|h_R^\varepsilon(s)\|_{L^\infty}.
\end{aligned}$$

Case 2 For either $|v| \leqq \mathfrak{N}$, $|v'| \geqq 2\mathfrak{N}$ or $|v'| \leqq 2\mathfrak{N}$, $|v''| \geqq 3\mathfrak{N}$, noting $|V_{cl}(s)| = |v|$ and $|V'_{cl}(s_1)| = |v'|$ we get either $|V_{cl}(s) - v'| \geqq \mathfrak{N}$ or $|V'_{cl}(s_1) - v''| \geqq \mathfrak{N}$, then either one of the following is valid for some small positive constant $0 < c_1 \leqq \frac{c}{32}$ (where $c > 0$ is defined in Lemma 6.2):

$$\begin{aligned}
|l_w(V_{cl}(s), v')| &\leqq e^{-c_1\mathfrak{N}^2}|l_w(V_{cl}(s), v')\exp\left(c_1|V_{cl}(s) - v'|^2\right)|, \\
|l_w(V'_{cl}(s), v'')| &\leqq e^{-c_1\mathfrak{N}^2}|l_w(V'_{cl}(s_1), v'')\exp\left(c_1|V'_{cl}(s_1) - v'|^2\right)|,
\end{aligned}$$

which, together with (6.14), yields that

$$\begin{aligned}
\int_{\mathbb{R}^3}|l_w(v, v')e^{c_1|v-v'|^2}|\mathrm{d}v' &\leqq \frac{C}{1+|v|}, \\
\int_{\mathbb{R}^3}|l_w(v', v'')e^{c_1|v'-v''|^2}|\mathrm{d}v'' &\leqq \frac{C}{1+|v'|}.
\end{aligned} \tag{6.18}$$

Hence, for the case of $|v-v'| \geqq \mathfrak{N}$ or $|v'-v''| \geqq \mathfrak{N}$, it follows from (6.18) that

$$\begin{aligned}&\int_0^t\int_0^s\left\{\int\int_{|v|\leqq\mathfrak{N},|v'|\geqq 2\mathfrak{N}}+\int\int_{|v'|\leqq 2\mathfrak{N},|v''|\geqq 3\mathfrak{N}}\right\}(\cdots)\mathrm{d}v''\mathrm{d}v'\mathrm{d}s_1\mathrm{d}s\\&\leqq \frac{C}{\varepsilon^4}e^{-c_1\mathfrak{N}^2}\sup_{0\leqq s\leqq t}\|h_R^\varepsilon(s)\|_{L^\infty}\int_0^t\int_0^s\int|l_w(v,v')|\exp\left\{-\frac{\nu_M(v)(t-s)}{C\varepsilon^2}\right\}\\&\quad\times\exp\left\{-\frac{\nu_M(v')(s-s_1)}{C\varepsilon^2}\right\}\mathrm{d}v'\mathrm{d}s_1\mathrm{d}s\\&\leqq Ce^{-c_1\mathfrak{N}^2}\sup_{0\leqq s\leqq t}\|h_R^\varepsilon(s)\|_{L^\infty}.\end{aligned}$$

Case 3a $|v|\leqq\mathfrak{N}$, $|v'|\leqq 2\mathfrak{N}$, $|v''|\leqq 3\mathfrak{N}$. We note $\nu_M(v)\geqq\nu_0>0$ where ν_0 is a positive constant independent of v. Furthermore, we assume that $s-s_1\leqq\lambda\varepsilon^2$ for some small $\lambda>0$ determined later. Then the corresponding part of the first term in (6.16) is bounded by

$$\begin{aligned}&\frac{C}{\varepsilon^4}\int_0^t\int_{s-\lambda\varepsilon^2}^s\exp\left\{-\frac{\nu_0(t-s)}{\varepsilon^2}\right\}\exp\left\{-\frac{\nu_0(s-s_1)}{\varepsilon^2}\right\}\|h_R^\varepsilon(s_1)\|_{L^\infty}\mathrm{d}s_1\mathrm{d}s\\&\quad\leqq C\sup_{0\leqq s\leqq t}\{\|h_R^\varepsilon(s)\|_{L^\infty}\}\cdot\frac{1}{\varepsilon^4}\int_0^t\exp\left\{-\frac{\nu_0(t-s)}{\varepsilon^2}\right\}\mathrm{d}s\cdot\int_{s-\lambda\varepsilon^2}^s\mathrm{d}s_1\\&\quad\leqq C\lambda\sup_{0\leqq s\leqq t}\{\|h_R^\varepsilon(s)\|_{L^\infty}\}.\end{aligned}$$

Case 3b $|v|\leqq\mathfrak{N}$, $|v'|\leqq 2\mathfrak{N}$, $|v''|\leqq 3\mathfrak{N}$ and $s-s_1\geqq\lambda\varepsilon^2$. This is the last remaining case. We can bound the corresponding part of the first term in (6.16) by

$$\begin{aligned}&\frac{C}{\varepsilon^4}\int_0^t\int_D\int_0^{s-\lambda\varepsilon^2}\exp\left\{-\frac{\nu_0(t-s)}{C\varepsilon^2}\right\}\exp\left\{-\frac{\nu_0(s-s_1)}{C\varepsilon^2}\right\}\\&\quad\times|l_w(V_{cl}(s),v')l_w(V'_{cl}(s_1),v'')\cdot h_R^\varepsilon(s_1,X'_{cl}(s_1),v'')|\mathrm{d}s_1\mathrm{d}v'\mathrm{d}v''\mathrm{d}s,\end{aligned}\tag{6.19}$$

where $D=\{|v'|\leqq 2\mathfrak{N},|v''|\leqq 3\mathfrak{N}\}$. It follows from (6.14) that

$$\int_{\mathbb{R}^3}|l_w(v,v')|^2\mathrm{d}v'\leqq C,$$

which, together with Cauchy inequality, yields that (6.19) is bounded by

$$\begin{aligned}&\frac{C}{\varepsilon^4}\left\{\int_0^t\int_D\int_0^{s-\lambda\varepsilon^2}\exp\left\{-\frac{\nu_0(t-s_1)}{C\varepsilon^2}\right\}|h_R^\varepsilon(s_1,X'_{cl}(s_1),v'')|^2\mathrm{d}v''\mathrm{d}v'\mathrm{d}s_1\mathrm{d}s\right\}^{\frac12}\\&\quad\times\left\{\int_0^t\int_D\int_0^{s-\lambda\varepsilon^2}\exp\left\{-\frac{\nu_0(t-s_1)}{C\varepsilon^2}\right\}|l_w(V_{cl}(s),v')l_w(V'_{cl}(s_1),v'')|^2\mathrm{d}v''\mathrm{d}v'\mathrm{d}s_1\mathrm{d}s\right\}^{\frac12}\\&\leqq\frac{C_{\mathfrak{N}}}{\varepsilon^2}\left\{\int_0^t\int_D\int_0^{s-\lambda\varepsilon^2}\exp\left\{-\frac{\nu_0(t-s_1)}{C\varepsilon^2}\right\}|f_R^\varepsilon(s_1,X'_{cl}(s_1),v'')|^2\mathrm{d}v''\mathrm{d}v'\mathrm{d}s_1\mathrm{d}s\right\}^{\frac12}\end{aligned}$$

$$
\begin{aligned}
&= \frac{C_{\mathfrak{N}}}{\varepsilon^2}\left\{\int_0^t \int_D \int_{\max\{t_1',0\}}^{s-\lambda\varepsilon^2} \exp\left\{-\frac{\nu_0(t-s_1)}{\varepsilon^2}\right\}|f_R^\varepsilon(s_1, X_{cl}'(s_1), v'')|^2 \mathrm{d}s_1 \mathrm{d}v' \mathrm{d}v'' \mathrm{d}s\right\}^{\frac{1}{2}} \\
&\quad + \frac{C_{\mathfrak{N}}}{\varepsilon^2}\left\{\int_0^t \int_D \int_0^{t_1'} \mathbf{I}_{\{t_1'>0\}} \exp\left\{-\frac{\nu_0(t-s_1)}{\varepsilon^2}\right\}|f_R^\varepsilon(s_1, X_{cl}'(s_1), v'')|^2 \mathrm{d}s_1 \mathrm{d}v' \mathrm{d}v'' \mathrm{d}s\right\}^{\frac{1}{2}},
\end{aligned} \tag{6.20}
$$

where $t_1' := s - t_{\mathbf{b}}(X_{cl}'(s), v')$. To estimate (6.20), we integrate over v', and make a change of variable $v' \longmapsto z := X_{cl}'(s_1)$. From the explicit formula $(6.10)_{3,4}$, we have that

$$
\frac{\partial z}{\partial v'} = \frac{\partial X_{cl}'(s_1)}{\partial v'} = \begin{pmatrix} -(s-s_1) & 0 & 0 \\ 0 & -(s-s_1) & 0 \\ 0 & 0 & -(s-s_1) \end{pmatrix}, \tag{6.21}
$$

if $\max\{0, t_1'\} \leqq s_1 \leqq s - \lambda\varepsilon^2$, and

$$
\frac{\partial z}{\partial v'} = \frac{\partial X_{cl}'(s_1)}{\partial v'} = \begin{pmatrix} -(s-s_1) & 0 & 0 \\ 0 & -(s-s_1) & 0 \\ 0 & 0 & s-s_1 \end{pmatrix}, \text{ if } 0 \leqq s_1 \leqq t_1'. \tag{6.22}
$$

From (6.21) and (6.22), for both cases, it holds that

$$
\left|\det\left(\frac{\partial z}{\partial v'}\right)(s_1)\right| = (s-s_1)^3 \geqq (\lambda\varepsilon^2)^3 > 0, \text{ for } s_1 \in [0, s-\lambda\varepsilon^2],
$$

which yields that, for $s_1 \in [0, s-\lambda\varepsilon^2]$,

$$
\int_{|v'|\leqq 2\mathfrak{N}} |f_R^\varepsilon(s_1, X_{cl}'(s_1), v'')|^2 \mathrm{d}v' \leqq \frac{C}{\lambda^3\varepsilon^6}\int_{\mathbb{R}^3_+} |f^\varepsilon(s_1, z, v'')|^2 \,\mathrm{d}z. \tag{6.23}
$$

Using (6.23), we can further bound the two terms in RHS of (6.20) by

$$
\frac{C_{\mathfrak{N},\lambda}}{\varepsilon^3} \sup_{0\leqq s\leqq t} \|f_R^\varepsilon(s)\|_{L^2}.
$$

Collecting all the above terms and multiplying them with ε^3, for any small $\lambda > 0$ and large $\mathfrak{N} > 0$, one obtains that

$$
\begin{aligned}
\sup_{0\leqq s\leqq t} \{\|\varepsilon^3 h_R^\varepsilon(s)\|_{L^\infty}\} &\leqq C_R(t)\left\{\|\varepsilon^3 h_R^\varepsilon(0)\|_{L^\infty} + \varepsilon^{N-1} + \varepsilon^{\mathfrak{b}}\right\} \\
&\quad + C(t)\varepsilon^2 \sup_{0\leqq s\leqq t} \|\varepsilon^3 h_R^\varepsilon(s)\|_{L^\infty}^2 + C\left\{\lambda + \frac{1}{\mathfrak{N}} + C_R(t)\varepsilon\right\} \sup_{0\leqq s\leqq t} \|\varepsilon^3 h_R^\varepsilon(s)\|_{L^\infty} \\
&\quad + C_{\mathfrak{N},\lambda} \sup_{0\leqq s\leqq t} \|f_R^\varepsilon(s)\|_{L^2}.
\end{aligned}
$$

Noting $t \in [0, \tau]$, first taking $\mathfrak{N} \gg 1$ large enough and $\lambda > 0$ small, and finally choosing $0 < \varepsilon \leqq \varepsilon_0$ with ε_0 small enough, we deduce

$$
\sup_{0\leqq s\leqq t} \{\|\varepsilon^3 h_R^\varepsilon(s)\|_{L^\infty}\} \leqq C_R(t)\left\{\|\varepsilon^3 h_R^\varepsilon(0)\|_{L^\infty} + \varepsilon^{N-1} + \varepsilon^{\mathfrak{b}}\right\} + C \sup_{0\leqq s\leqq t} \|f_R^\varepsilon(s)\|_{L^2}.
$$

Therefore the proof of Lemma 6.3 is completed. □

6.3. Proof of Theorem 1.6

With Lemmas 6.1 and 6.3 in hand, the rest proof is the same as [23]. We omit the details for simplicity of presentation. Therefore we complete the proof of Theorem 1.6.

7. Acoustic Limit: Proof of Theorem 1.11

To prove Theorem 1.11, we first derive the estimate for two solutions to compressible Euler equations (1.8)–(1.10) and acoustic systems (1.52)–(1.53). We define $(\varphi_d^\delta, \Phi_d^\delta, \vartheta_d^\delta)$ as

$$\varphi_d^\delta := \frac{1}{\delta^2}(\rho^\delta - 1 - \delta\varphi), \quad \Phi_d^\delta := \frac{1}{\delta^2}(\mathrm{u}^\delta - \delta\Phi), \quad \vartheta_d^\delta := \frac{1}{\delta^2}(T^\delta - 1 - \delta\vartheta).$$

As in [23], a direct calculation shows that

$$\begin{cases} \partial_t\varphi_d^\delta + (\mathrm{u}^\delta\cdot\nabla)\varphi_d^\delta + \rho^\delta\,\mathrm{div}\Phi_d^\delta + \delta[\nabla\varphi\cdot\Phi_d^\delta + \mathrm{div}\Phi\,\varphi_d^\delta] = -\mathrm{div}(\varphi\Phi), \\ \rho^\delta\partial_t\Phi_d^\delta + \rho^\delta(\mathrm{u}^\delta\cdot\nabla)\Phi_d^\delta + \nabla\big(\rho^\delta\vartheta_d^\delta + T^\delta\varphi_d^\delta\big) \\ \quad -\vartheta_d^\delta\nabla\rho^\delta - \varphi_d^\delta\nabla T^\delta + \delta[\partial_t\Phi\,\varphi_d^\delta + \rho^\delta(\Phi_d^\delta\cdot\nabla)\Phi + \vartheta_d^\delta\nabla\varphi + \varphi_d^\delta\nabla\vartheta] \\ \quad = -\varphi\partial_t\Phi - \rho^\delta(\Phi\cdot\nabla)\Phi - \nabla(\varphi\vartheta), \\ \partial_t\vartheta_d^\delta + (\mathrm{u}^\delta\cdot\nabla)\vartheta_d^\delta + \frac{2}{3}T^\delta\,\mathrm{div}\Phi_d^\delta + \delta[\nabla\vartheta\cdot\Phi_d^\delta + \frac{2}{3}\mathrm{div}\Phi\,\vartheta_d^\delta] \\ \quad = -\Phi\cdot\nabla\vartheta - \frac{2}{3}\vartheta\,\mathrm{div}\Phi, \end{cases} \tag{7.1}$$

with $(t, x) \in [0, \tau] \times \mathbb{R}^3_+$, and the initial and boundary conditions

$$(\varphi_d^\delta, \Phi_d^\delta, \vartheta_d^\delta)|_{t=0} = 0 \quad \text{and} \quad \Phi_{d,3}^\delta(t, x_{\shortparallel}, 0) \equiv 0. \tag{7.2}$$

Clearly, (7.1) is a linear hyperbolic system with characteristic boundary. Then we apply Lemma 3.1 to (7.1)–(7.2) (the coefficients of (7.1) is slightly different, but Lemma 3.1 is still applicable) to obtain

$$\sup_{t\in[0,\tau]} \|(\varphi_d^\delta, \Phi_d^\delta, \vartheta_d^\delta)(t)\|_{\mathcal{H}^k(\mathbb{R}^3_+)} \leqq C(\tau, \|(\varphi_0, \Phi_0, \vartheta_0)\|_{H^{s_0}(\mathbb{R}^3_+)}) \tag{7.3}$$

provided $k \leqq s_0 - 2$ where $\mathcal{H}^s(\mathbb{R}^3_+)$ is defined in (3.4).

With (7.3) in hand, we can prove Theorem 1.11 by using the same arguments as in section 3.2 of [23]. The details are omitted for simplicity of presentation. Therefore this completes the proof of Theorem 1.11. □

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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Appendix A: Proof of Lemma 1.1

In "Appendix A", we prove Lemma 1.1. For the macroscopic variables $\bar{\rho}_k$, $\bar{u}_k$ and $\bar{\theta}_k$ of $\bar{F}_k$, a direct calculation shows that

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \bar{F}_k \mathrm{d}v = \bar{\rho}_k, \quad \int_{\mathbb{R}^3} (v_i - \mathfrak{u}_i^0)\bar{F}_k \mathrm{d}v = \rho^0 \bar{u}_{k,i}, \quad \int_{\mathbb{R}^3} v_i \bar{F}_k \mathrm{d}v = \rho^0 \bar{u}_{k,i} + \bar{\rho}_k \mathfrak{u}_i^0,\\
&\int_{\mathbb{R}^3} |v|^2 \bar{F}_k \mathrm{d}v = \rho^0 \bar{\theta}_k + 3T^0 \bar{\rho}_k + 2\rho^0 \mathfrak{u}^0 \cdot \bar{u}_k + \bar{\rho}_k |\mathfrak{u}^0|^2,\\
&\int_{\mathbb{R}^3} |v - \mathfrak{u}^0|^2 \bar{F}_k \mathrm{d}v = \rho^0 \bar{\theta}_k + 3T^0 \bar{\rho}_k,\\
&\int_{\mathbb{R}^3} v_i^2 \bar{F}_k \mathrm{d}v = 2\rho^0 \mathfrak{u}_i^0 \bar{u}_{k,i} + \bar{\rho}_k |\mathfrak{u}_i^0|^2 + \frac{\rho^0 \bar{\theta}_k + 3T^0 \bar{\rho}_k}{3} + \int_{\mathbb{R}^3} T^0 (\mathbf{I} - \mathbf{P}_0)\bar{f}_k \cdot \mathcal{A}_{ii}^0 \mathrm{d}v,\\
&\int_{\mathbb{R}^3} v_i v_j \bar{F}_k \mathrm{d}v = \rho^0 \mathfrak{u}_i^0 \bar{u}_{k,j} + \rho^0 \mathfrak{u}_j^0 \bar{u}_{k,i} + \bar{\rho}_k \mathfrak{u}_i^0 \mathfrak{u}_j^0 + \int_{\mathbb{R}^3} T^0 (\mathbf{I} - \mathbf{P}_0)\bar{f}_k \cdot \mathcal{A}_{ij}^0 \mathrm{d}v, \ i \neq j,\\
&\int_{\mathbb{R}^3} v_i |v|^2 \bar{F}_k \mathrm{d}v\\
&\quad = (5T^0 + |\mathfrak{u}^0|^2)\rho^0 \bar{u}_{k,i} + \frac{5}{3}\mathfrak{u}_i^0 (\rho^0 \bar{\theta}_k + 3T^0 \bar{\rho}_k) + \mathfrak{u}_i^0 (2\rho^0 \mathfrak{u}^0 \cdot \bar{u}_k + \bar{\rho}_k |\mathfrak{u}^0|^2)\\
&\qquad + \sum_{l=1}^{2} \int_{\mathbb{R}^3} 2T^0 \mathfrak{u}_l^0 \mathcal{A}_{il}^0 \cdot (\mathbf{I} - \mathbf{P}_0)\bar{f}_k \mathrm{d}v + \int_{\mathbb{R}^3} 2(T^0)^{\frac{3}{2}} \mathcal{B}_i^0 \cdot (\mathbf{I} - \mathbf{P}_0)\bar{f}_k \mathrm{d}v.
\end{aligned}
\tag{A.1}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^3} v_3 \sqrt{\mu_0} \cdot \partial_y \mathbf{P}_0 \bar{f}_k \mathrm{d}v = \rho^0 \partial_y \bar{u}_{k,3},\\
&\int_{\mathbb{R}^3} v_i v_3 \sqrt{\mu_0} \cdot \partial_y \mathbf{P}_0 \bar{f}_k \mathrm{d}v = \rho^0 \mathfrak{u}_i^0 \partial_y \bar{u}_{k,3}, \ i = 1, 2,\\
&\int_{\mathbb{R}^3} v_3^2 \sqrt{\mu_0} \cdot \partial_y \mathbf{P}_0 \bar{f}_k \mathrm{d}v = \partial_y \left(\frac{\rho^0 \bar{\theta}_k + 3T^0 \bar{\rho}_k}{3} \right),\\
&\int_{\mathbb{R}^3} v_3 |v|^2 \sqrt{\mu_0} \cdot \partial_y \mathbf{P}_0 \bar{f}_k \mathrm{d}v = \rho^0 (5T^0 + |\mathfrak{u}^0|^2) \partial_y \bar{u}_{k,3},\\
&\int_{\mathbb{R}^3} v_3 (|v - \mathfrak{u}^0|^2 - 5T^0) \sqrt{\mu_0} \cdot \partial_y \mathbf{P}_0 \bar{f}_k \, \mathrm{d}v = 0.
\end{aligned}
\tag{A.2}
$$

Multiplying (1.15)$_3$ by 1, v, $|v|^2$, integrating over $\mathbb{R}^3$ and using (A.1)–(A.2), we obtain

$$\partial_t \bar{\rho}_k + \sum_{j=1}^{2} \partial_j (\rho^0 \bar{u}_{k,j} + \bar{\rho}_k \mathfrak{u}_j^0) + \rho^0 \partial_y \bar{u}_{k+1,3} = 0, \tag{A.3}$$

$$\begin{aligned}
&\partial_t (\rho^0 \bar{u}_{k,i} + \bar{\rho}_k \mathfrak{u}_i^0) + \sum_{j=1}^{2} \partial_j \Big(\rho^0 \mathfrak{u}_j^0 \bar{u}_{k,i} + \rho^0 \mathfrak{u}_i^0 \bar{u}_{k,j} + \bar{\rho}_k \mathfrak{u}_i^0 \mathfrak{u}_j^0 + \delta_{ij} \frac{\rho^0 \bar{\theta}_k + 3T^0 \bar{\rho}_k}{3}\Big) \\
&\quad + \rho^0 \mathfrak{u}_i^0 \partial_y \bar{u}_{k+1,3} + \delta_{3i}\, \partial_y \left(\frac{\rho^0 \bar{\theta}_{k+1} + 3T^0 \bar{\rho}_{k+1}}{3}\right) \\
&\quad + \int_{\mathbb{R}^3} v_i v_3 \sqrt{\mu_0} \cdot (\mathbf{I} - \mathbf{P}_0) \partial_y \bar{f}_{k+1} \mathrm{d}v = \bar{W}_{k-1,i},
\end{aligned} \tag{A.4}$$

and

$$\begin{aligned}
&\partial_t (\rho^0 \bar{\theta}_k + 3T^0 \bar{\rho}_k + 2\rho^0 \mathfrak{u}^0 \cdot \bar{u}_k + \bar{\rho}_k |\mathfrak{u}^0|^2) \\
&\quad + \sum_{j=0}^{2} \partial_j \{(5T^0 + |\mathfrak{u}^0|^2) \rho^0 \bar{u}_{k,i} + \frac{5}{3} \mathfrak{u}_i^0 (\rho^0 \bar{\theta}_k + 3T^0 \bar{\rho}_k) + \mathfrak{u}_i^0 (2\rho^0 \mathfrak{u}^0 \cdot \bar{u}_k + \bar{\rho}_k |\mathfrak{u}^0|^2)\} \\
&\quad + \rho^0 (5T^0 + |\mathfrak{u}^0|^2) \partial_y \bar{u}_{k+1,3} + \int_{\mathbb{R}^3} v_3 |v|^2 \sqrt{\mu_0} \cdot (\mathbf{I} - \mathbf{P}_0) \partial_y \bar{f}_{k+1} \mathrm{d}v \\
&\quad = -\sum_{j=1}^{2} \partial_j \left\{ \sum_{l=1}^{2} 2T^0 \mathfrak{u}_l^0 \int_{\mathbb{R}^3} \mathcal{A}_{jl}^0 \cdot (\mathbf{I} - \mathbf{P}_0) \bar{f}_k \mathrm{d}v + 2(T^0)^{\frac{3}{2}} \int_{\mathbb{R}^3} \mathcal{B}_j^0 \cdot (\mathbf{I} - \mathbf{P}_0) \bar{f}_k \mathrm{d}v \right\}.
\end{aligned} \tag{A.5}$$

Then (1.30) follow directly from (A.3).

Substituting (A.3) into (A.4) and by tedious calculations, one can obtain

$$\begin{aligned}
&\rho^0 \partial_t \bar{u}_{k,i} + \rho^0 (\mathfrak{u}_{\shortparallel}^0 \cdot \nabla_{\shortparallel}) \bar{u}_{k,i} - \rho^0 \partial_3 \mathfrak{u}_3^0\, \bar{u}_{k,i} - \bar{\rho}_k \frac{\partial_i p^0}{\rho^0} + \rho^0 \bar{u}_{k,\shortparallel} \cdot \nabla_{\shortparallel} \mathfrak{u}_i^0 \\
&\quad + \partial_y \langle T^0 \mathcal{A}_{3i}^0,\ (\mathbf{I} - \mathbf{P}_0) \bar{f}_{k+1} \rangle = \bar{W}_{k-1,i} - \partial_i \bar{p}_k,\ i = 1, 2,
\end{aligned} \tag{A.6}$$

$$\begin{aligned}
&\rho^0 \partial_t \bar{u}_{k,3} + \rho^0 (\mathfrak{u}_{\shortparallel}^0 \cdot \nabla_{\shortparallel}) \bar{u}_{k,3} - \rho^0 \partial_3 \mathfrak{u}_3^0\, \bar{u}_{k,3} + \partial_y \bar{p}_{k+1} \\
&\quad + \partial_y \langle T^0 \mathcal{A}_{33}^0,\ (\mathbf{I} - \mathbf{P}_0) \bar{f}_{k+1} \rangle = \bar{W}_{k-1,3}.
\end{aligned} \tag{A.7}$$

Similarly, from (A.3)–(A.4), the equation (A.5) for $\bar{\theta}_k$ can be reduced to be

$$\begin{aligned}
&\frac{5}{3} \rho^0 \partial_t \bar{\theta}_k + \frac{5}{3} \rho^0 \mathfrak{u}_{\shortparallel}^0 \cdot \nabla_{\shortparallel} \bar{\theta}_k + \Big(\frac{10}{9} \rho^0 \mathrm{div} \mathfrak{u}^0 - \frac{5}{3} \rho^0 \partial_3 \mathfrak{u}_3^0\Big) \bar{\theta}_k \\
&\quad + (3\rho^0 \nabla_{\shortparallel} T^0 - 2T^0 \nabla_{\shortparallel} \rho^0) \bar{u}_{k,\shortparallel} + \partial_y \langle 2(T^0)^{\frac{3}{2}} \mathcal{B}_3^0,\ (\mathbf{I} - \mathbf{P}_0) \bar{f}_{k+1} \rangle \\
&\quad = \bar{H}_{k-1} + \{2\partial_t + 2\mathfrak{u}_{\shortparallel}^0 \cdot \nabla_{\shortparallel} + \frac{10}{3} \mathrm{div} \mathfrak{u}^0\} \bar{p}_k.
\end{aligned} \tag{A.8}$$

We still need to deal with the microscopic parts in (A.6)–(A.8). In fact, by using (1.15), we obtain (1.29). Then by using (1.29), we have

$$\Big\langle (\mathbf{I} - \mathbf{P_0}) \bar{f}_{k+1}, \mathcal{A}_{3i}^0 \Big\rangle = \Big\langle -\mathbf{L}_0^{-1} \{(\mathbf{I} - \mathbf{P_0})(v_3 \partial_y \mathbf{P}_0 \bar{f}_k)\},\ \mathcal{A}_{3i}^0 \Big\rangle$$

$$
\begin{aligned}
&+\Big\langle \mathbf{L}_0^{-1}\Big\{\frac{y}{\sqrt{\mu_0}}[Q(\partial_3\mu_0, \sqrt{\mu_0}\mathbf{P_0}\bar{f}_k) + Q(\sqrt{\mu_0}\mathbf{P_0}\bar{f}_k, \partial_3\mu_0)]\\
&+\frac{1}{\sqrt{\mu_0}}[Q(\sqrt{\mu_0}f_1^0, \sqrt{\mu_0}\mathbf{P_0}\bar{f}_k) + Q(\sqrt{\mu_0}\mathbf{P_0}\bar{f}_k, \sqrt{\mu_0}f_1^0)]\\
&+\frac{1}{\sqrt{\mu_0}}[Q(\sqrt{\mu_0}\bar{f}_1, \sqrt{\mu_0}\mathbf{P_0}\bar{f}_k) + Q(\sqrt{\mu_0}\mathbf{P_0}\bar{f}_k, \sqrt{\mu_0}\bar{f}_1)]\Big\},\ \mathcal{A}_{3i}^0\Big\rangle\\
&+\Big\langle \bar{J}_{k-1},\ \mathcal{A}_{3i}^0\Big\rangle. \qquad \text{(A.9)}
\end{aligned}
$$

A direct calculation shows that

$$
\begin{aligned}
&\Big\langle -\mathbf{L}_0^{-1}\{(\mathbf{I}-\mathbf{P_0})(v_3\partial_y\mathbf{P}_0\bar{f}_k)\},\ \mathcal{A}_{3i}^0\Big\rangle = \Big\langle -v_3\partial_y\mathbf{P}_0\bar{f}_k,\ \mathbf{L}_0^{-1}\mathcal{A}_{3i}^0\Big\rangle\\
&= -\Big\langle\Big\{\frac{\partial_y\bar{\rho}_k}{\rho^0} + \partial_y\bar{u}_k\cdot\frac{v-\mathfrak{u}^0}{T^0} + \frac{\partial_y\bar{\theta}_k}{6T^0}\Big(\frac{|v-\mathfrak{u}^0|^2}{T^0}-3\Big)\Big\}v_3\sqrt{\mu_0},\ \mathbf{L}_0^{-1}\mathcal{A}_{3i}^0\Big\rangle\\
&= -\partial_y\bar{u}_{k,i}\Big\langle\mathcal{A}_{3i}^0,\ \mathbf{L}_0^{-1}\mathcal{A}_{3i}^0\Big\rangle = \begin{cases} -\dfrac{\mu(T^0)}{T^0}\partial_y\bar{u}_{k,i},\ i=1,2,\\ -\dfrac{4}{3}\dfrac{\mu(T^0)}{T^0}\partial_y\bar{u}_{k,3},\ i=3,\end{cases} \qquad \text{(A.10)}
\end{aligned}
$$

where we have used (1.26) in the last equality.

From [19, p.649], we know that

$$
\begin{aligned}
&\mathbf{L}_0^{-1}\Big\{\frac{1}{\sqrt{\mu_0}}Q(\sqrt{\mu_0}\mathbf{P}_0g, \sqrt{\mu_0}\mathbf{P}_0\bar{f}_k) + \frac{1}{\sqrt{\mu_0}}Q(\sqrt{\mu_0}\mathbf{P}_0\bar{f}_k, \sqrt{\mu_0}\mathbf{P}_0g)\Big\}\\
&= (\mathbf{I}-\mathbf{P}_0)\left\{\frac{\mathbf{P}_0g\cdot\mathbf{P}_0\bar{f}_k}{\sqrt{\mu_0}}\right\}. \qquad \text{(A.11)}
\end{aligned}
$$

We assume

$$
\mathbf{P}_0g = \left\{\frac{a}{\rho^0} + b\cdot\frac{v-\mathfrak{u}^0}{T^0} + \frac{c}{6T^0}\left(\frac{|v-\mathfrak{u}^0|^2}{T^0}-3\right)\right\}\sqrt{\mu_0}.
$$

Then a direct calculation shows that

$$
\begin{aligned}
(\mathbf{I}-\mathbf{P}_0)\left\{\frac{\mathbf{P}_0g\cdot\mathbf{P}_0\bar{f}_k}{\sqrt{\mu_0}}\right\} &= \sum_{l,j=1}^{3}\frac{b_l\,\bar{u}_{k,j}}{T^0}\mathcal{A}_{lj}^0 + \frac{1}{\sqrt{T^0}}\Big(\frac{\bar{\theta}_k}{3T^0}b + \frac{c}{3T^0}\bar{u}_k\Big)\cdot\mathcal{B}^0\\
&\quad + \frac{c\cdot\bar{\theta}_k}{36(T^0)^2}(\mathbf{I}-\mathbf{P}_0)\left\{\Big(\frac{|v-\mathfrak{u}^0|^2}{T^0}-5\Big)^2\sqrt{\mu_0}\right\}. \qquad \text{(A.12)}
\end{aligned}
$$

Noting

$$
\frac{\partial\mu}{\sqrt{\mu}} = \left\{\frac{\partial\rho}{\rho} + \partial\mathfrak{u}\cdot\frac{v-\mathfrak{u}}{T} + \frac{3\partial T}{6T}\left(\frac{|v-\mathfrak{u}|^2}{T}-3\right)\right\}\sqrt{\mu}, \qquad \text{(A.13)}
$$

which, together with (A.11) and (A.12), implies that the second term on RHS of (A.9) is expressed as

$$\sum_{l,j=1}^{3} \frac{\bar{u}_{k,j}}{T^0}[y\partial_3 \mathfrak{u}_l^0 + u_{1,l}^0 + \bar{u}_{1,l}]\Big\langle \mathcal{A}_{lj}^0,\ \mathcal{A}_{3i}^0 \Big\rangle$$
$$= \begin{cases} \frac{\rho^0}{T^0}\{[\partial_3 \mathfrak{u}_3^0 \cdot y + u_{1,3}^0 + \bar{u}_{1,3}]\bar{u}_{k,i} + [\partial_3 \mathfrak{u}_i^0 \cdot y + u_{1,i}^0 + \bar{u}_{1,i}]\bar{u}_{k,3}\},\ i = 1, 2, \\ \frac{4}{3}\frac{\rho^0}{T^0}[\partial_3 \mathfrak{u}_3^0 \cdot y + u_{1,3}^0 + \bar{u}_{1,3}]\bar{u}_{k,3} \\ \qquad -\frac{2}{3}\frac{\rho^0}{T^0}\sum_{l=1}^{2}[\partial_3 \mathfrak{u}_l^0 \cdot y + u_{1,l}^0 + \bar{u}_{1,l}]\bar{u}_{k,l},\ i = 3, \end{cases} \tag{A.14}$$

where we have used

$$\Big\langle \mathcal{A}_{ii},\ \mathcal{A}_{jj} \Big\rangle = -\frac{2}{3}\rho,\quad \Big\langle \mathcal{A}_{ij},\ \mathcal{A}_{ij} \Big\rangle = \rho,\ \text{for } i \neq j,\quad \text{and} \Big\langle \mathcal{A}_{ii},\ \mathcal{A}_{ii} \Big\rangle = \frac{4}{3}\rho.$$

Combining (A.9), (A.10) and (A.14), we obtain

$$\partial_y \langle T^0 \mathcal{A}_{3i}^0,\ (\mathbf{I} - \mathbf{P}_0)\bar{f}_{k+1} \rangle$$
$$= \begin{cases} -\mu(T^0)\partial_{yy}\bar{u}_{k,i} + \rho^0 \partial_y\{[\partial_3 \mathfrak{u}_3^0 \cdot y + u_{1,3}^0 + \bar{u}_{1,3}]\bar{u}_{k,i}\} \\ \ +\rho^0 \partial_y\{[\partial_3 \mathfrak{u}_i^0 \cdot y + u_{1,i}^0 + \bar{u}_{1,i}]\bar{u}_{k,3}\} + T^0 \partial_y \langle \bar{J}_{k-1},\ \mathcal{A}_{3i}^0 \rangle,\ i = 1, 2, \\ -\frac{4}{3}\mu(T^0)\partial_{yy}\bar{u}_{k,3} + \frac{4}{3}\rho^0 \partial_y\{[\partial_3 \mathfrak{u}_3^0 \cdot y + u_{1,3}^0 + \bar{u}_{1,3}]\bar{u}_{k,3}\} \\ \ -\frac{2}{3}\rho^0 \sum_{l=1}^{2}[\partial_3 \mathfrak{u}_l^0 \cdot y + u_{1,l}^0 + \bar{u}_{1,l}]\bar{u}_{k,l} + T^0 \partial_y \langle \bar{J}_{k-1},\ \mathcal{A}_{33}^0 \rangle,\ i = 3. \end{cases} \tag{A.15}$$

Substituting (A.15) into (A.6) and (A.7), then by tedious calculations, one can get (1.27) and (1.31).

As in (A.9)–(A.15), we can obtain

$$\begin{aligned} &\partial_y \langle 2(T^0)^{\frac{3}{2}} \mathcal{B}_3^0,\ (\mathbf{I} - \mathbf{P}_0)\bar{f}_{k+1} \rangle \\ &\quad = -\kappa(T^0)\partial_{yy}\bar{\theta}_k + \frac{5}{3}\rho^0 \partial_y\{(\partial_3 \mathfrak{u}_3^0 \cdot y + u_{1,3}^0 + \bar{u}_{1,3})\bar{\theta}_k\} \\ &\qquad + \frac{5}{3}\rho^0 \partial_y\{(3\partial_3 T_3^0 \cdot y + \theta_1^0 + \bar{\theta}_1)\bar{u}_{k,3}\} + 2(T^0)^{\frac{3}{2}}\partial_y \langle \bar{J}_{k-1},\ \mathcal{B}_3^0 \rangle, \end{aligned} \tag{A.16}$$

which, together with (A.8) and (1.24), yields (1.28). Therefore the proof of Lemma 1.1 is completed. □

Appendix B: Sketch Proof of Lemma 2.5

Here we sketch the key steps of proof for Lemma 2.5, and we refer the reader to [28, Section 3] for the details. Here $(t, x_{\shortparallel}) \in [0, \tau] \times \mathbb{R}^2$ are the parameters in (2.10), so we shall not write them down explicitly in the following.

Recall the monotonic smooth cut-off function $\chi(\cdot)$ defined in (3.8). Similar as in [17], we define

$$\mathfrak{f}(\eta, v) := f(\eta, v) + \chi(\eta) f_b(v), \tag{B.1}$$

then (2.10) is equivalent

$$\begin{cases} v_3\partial_\eta \mathfrak{f} + \mathbf{L}_0\mathfrak{f} = g := S + v_3\partial_\eta\chi(\eta)f_b(v) + \chi(\eta)\mathbf{L}_0 f_b, \\ \mathfrak{f}(0,v)|_{v_3>0} = \mathfrak{f}(0,Rv), & \eta > 0, \quad v \in \mathbb{R}^3, \\ \lim_{\eta\to\infty}\mathfrak{f}(\eta,v) = 0, \end{cases} \tag{B.2}$$

with $Rv = (v_1, v_2, -v_3)$. Hence we only need to prove the existence of solution to (B.2). And we divide the proof into several steps.

Step 1. As in [12, Section 3.2], using the L^2-L^∞ method developed in [20], we can construct a unique solution for the following truncated problem with penalized term

$$\begin{cases} \varepsilon\mathfrak{f}^\varepsilon + v_3\partial_\eta\mathfrak{f}^\varepsilon + \mathbf{L}_0\mathfrak{f}^\varepsilon = g, \\ \mathfrak{f}^\varepsilon(\eta,v)|_{\gamma_-} = \mathfrak{f}^\varepsilon(\eta,R_\eta v), \end{cases} \quad (\eta,v)\in(0,d)\times\mathbb{R}^3, \tag{B.3}$$

where $d \in [1,+\infty)$, $\varepsilon \in (0,1]$, $R_\eta v := v - 2(v\cdot\vec{n}(\eta))\vec{n}(\eta)$, $\vec{n}(0) = (0,0,-1)$ and $\vec{n}(d) = (0,0,1)$. Moreover it holds that

$$\|w_\kappa\mu_0^{-\mathfrak{a}}\mathfrak{f}^\varepsilon\|_{L^\infty_{\eta,v}} + |w_\kappa\mu_0^{-\mathfrak{a}}\mathfrak{f}^\varepsilon|_{L^\infty(\gamma)} \leqq C_{\varepsilon,d}\|\nu^{-1}w_\kappa\mu_0^{-\mathfrak{a}}g\|_{L^\infty_{\eta,v}}, \tag{B.4}$$

where the positive constant $C_{\varepsilon,d} > 0$ depends only on ε and d. Hereafter we take $0 \leqq \mathfrak{a} < \frac{1}{2}$ and $\kappa \geqq 3$.

Step 2. Taking the limit $\varepsilon \to 0+$. Noting the conditions (2.11)–(2.12) and using (B.3), a direct calculations shows that

$$\int_0^d a^\varepsilon(\eta)d\eta = \int_0^d b_1^\varepsilon(\eta)d\eta = \int_0^d b_2^\varepsilon(\eta)d\eta = \int_0^d c^\varepsilon(\eta)d\eta = 0, \tag{B.5}$$

where we have used the notation

$$\mathbf{P}_0\mathfrak{f}^\varepsilon(\eta,v) = \{a^\varepsilon(\eta) + b^\varepsilon\cdot(v-\mathfrak{u}^0) + c^\varepsilon(x)(|v-\mathfrak{u}^0|^2-3)\}\sqrt{\mu_0}.$$

Furthermore, by choosing suitable test function and noting (B.5), for the solutions $\mathfrak{f}^\varepsilon$ constructed in Step 1, we deduce

$$\|\mathbf{P}_0\mathfrak{f}^\varepsilon\|^2_{L^2_{\eta,v}} \leqq Cd^6\Big\{\|(\mathbf{I}-\mathbf{P}_0)\mathfrak{f}^\varepsilon\|^2_\nu + \|g\|^2_{L^2_{\eta,v}}\Big\}. \tag{B.6}$$

Applying the energy estimate to (B.3) and using (B.6), we can have

$$\|\mathfrak{f}^\varepsilon\|^2_{L^2_{\eta,v}} \leqq C_d\|g\|^2_{L^2_{\eta,v}}, \tag{B.7}$$

which, together with the L^2-L^∞ method, yields the uniform estimate (uniform in $\varepsilon \in (0,1]$

$$\|w_\kappa\mu_0^{-\mathfrak{a}}\mathfrak{f}^\varepsilon\|_{L^\infty_{\eta,v}} + |w_\kappa\mu_0^{-\mathfrak{a}}\mathfrak{f}^\varepsilon|_{L^\infty(\gamma)} \leqq C_d\|\nu^{-1}w_\kappa\mu_0^{-\mathfrak{a}}g\|_{L^\infty_{\eta,v}}. \tag{B.8}$$

With the help of uniform estimates (B.7)–(B.8), we can take the limit $\varepsilon \to 0+$, and obtain the unique solution $\mathfrak{f}_d$ to the linearized steady Boltzmann equation

$$\begin{cases} v_3\partial_\eta \mathfrak{f}_d + \mathbf{L}_0\mathfrak{f}_d = g, \quad (\eta, v) \in (0, d) \times \mathbb{R}^3, \\ \mathfrak{f}_d(\eta, v)|_{\gamma_-} = \mathfrak{f}_d(\eta, R_\eta v), \end{cases} \tag{B.9}$$

with

$$\|w_\kappa \mu_0^{-\mathfrak{a}} \mathfrak{f}_d\|_{L^\infty_{\eta,v}} + |w_\kappa \mu_0^{-\mathfrak{a}} \mathfrak{f}_d|_{L^\infty(\gamma)} \leqq C_d \|\nu^{-1} w_\kappa \mu_0^{-\mathfrak{a}} g\|_{L^\infty_{\eta,v}}. \tag{B.10}$$

Step 3. Taking the limit $d \to +\infty$. To obtain the solution for half-space problem, we need some uniform estimate independent of d, then we can take the limit $d \to \infty$. Let $\mathfrak{f}_d$ be the solution of (B.9), we denote

$$\mathbf{P}_0\mathfrak{f}_d(\eta, v) = \big[a(\eta) + b(\eta)\cdot(v - \mathfrak{u}^0) + c(\eta)(|v - \mathfrak{u}^0|^2 - 3)\big]\sqrt{\mu_0}.$$

Multiplying (B.9) by $\sqrt{\mu_0}$ and using (2.11)–(2.12), we have

$$0 = \frac{\mathrm{d}}{\mathrm{d}\eta}\int_{\mathbb{R}^3} v_3\sqrt{\mu_0} f(\eta, v)\mathrm{d}v = \frac{\mathrm{d}}{\mathrm{d}\eta} b_3(\eta) \equiv 0.$$

Since $\mathfrak{f}_d$ satisfies the specular boundary, it holds that $b_3(x)|_{x=0} = b_3(x)|_{x=d} = 0$, which yields $b_3(\eta) = 0, \quad \text{for } \eta \in [0, d]$. Let $(\phi_0, \phi_1, \phi_2, \phi_3)$ be some constants chosen later, we define

$$\begin{aligned} \bar{\mathfrak{f}}_d(\eta, v) = {} & [\bar{a}(\eta) + \bar{b}_1(\eta)\cdot(v_1 - \mathfrak{u}_1^0) + \bar{b}_2(\eta)\cdot(v_2 - \mathfrak{u}_2^0) + \bar{c}(\eta)(|v - \mathfrak{u}^0|^2 - 3)]\sqrt{\mu_0} \\ & + (\mathbf{I} - \mathbf{P}_0)\mathfrak{f}_d, \end{aligned}$$

where $\bar{a}(\eta) = a(\eta) + \phi_0, \quad \bar{b}_i(\eta) = b_i(\eta) + \phi_i, \quad i = 1, 2, \quad \text{and} \quad \bar{c}(\eta) = c(\eta) + \phi_3$. In fact, we can prove that there exist constants $(\phi_0, \phi_1, \phi_2, \phi_3)$ such that

$$\int_{\mathbb{R}^3} v_3\bar{\mathfrak{f}}_d(d, v)(v_3\sqrt{\mu_0}, \mathbf{L}_0^{-1}(\mathcal{A}_{31}^0), \mathbf{L}_0^{-1}(\mathcal{A}_{32}^0), \mathbf{L}_0^{-1}(\mathcal{B}_3^0))\mathrm{d}v = (0, 0, 0, 0). \tag{B.11}$$

With the above chosen constants $(\phi_0, \phi_1, \phi_2, \phi_3)$, then we can obtain

$$\|e^{\zeta\eta}\bar{\mathfrak{f}}_d\|_{L^2_{\eta,v}} \leqq \frac{C}{\zeta_1 - \zeta}\|e^{\zeta_1\eta} g\|_{L^2_{\eta,v}} \tag{B.12}$$

with $0 < \zeta < \zeta_1 \leqq \zeta_0$, and the constant C is independent of d. Now combining (B.12) and the L^2-L^∞ method, one can get

$$\|w_\kappa \mu_0^{-\mathfrak{a}} e^{\zeta\eta}\bar{\mathfrak{f}}_d\|_{L^\infty_{\eta,v}} + |w_\kappa \mu_0^{-\mathfrak{a}} e^{\zeta\eta}\bar{\mathfrak{f}}_d|_{L^\infty(\gamma)} \leqq \frac{C}{\zeta_0 - \zeta}\|w_\kappa \mu_0^{-\mathfrak{a}} \nu^{-1} e^{\zeta_0\eta} g\|_{L^\infty_{\eta,v}}. \tag{B.13}$$

Noting the above uniform estimates (B.12)–(B.13) and using the L^2-L^∞ method, we can take the limit $d \to +\infty$ to obtain the unique solution $\mathfrak{f}$ of (B.2). The uniform estimate (2.13) follows directly from (B.13) and (B.1). Therefore the proof of Lemma 2.5 is completed. □

Appendix C: Some Trace Inequalities

Lemma C.1. (1). *Let* $\Omega_b := \{(x_{\shortparallel}, x_3) : x_{\shortparallel} \in \mathbb{R}^2,\ x_3 \in [0, b]\}$ *with* $b \geqq 1$. *We assume* $f, g \in H^1(\Omega_b)$, *it holds, for any* $x_3 \in [0, b]$, *that*

$$\left|\int_{\mathbb{R}^2} (fg)(x_{\shortparallel}, x_3)dx_{\shortparallel}\right| \leqq \|\partial_{x_3}(f, g)\|_{L^2(\Omega_b)}\|(f, g)\|_{L^2(\Omega_b)} + \frac{1}{b}\|f\|_{L^2(\Omega_b)}\|g\|_{L^2(\Omega_b)}. \tag{C.1}$$

For $i = 1, 2$, *we have*

$$\left|\int_{\mathbb{R}^2} (\partial_{x_i} f \cdot g)(x_{\shortparallel}, x_3)dx_{\shortparallel}\right| \leqq \|\partial_{x_3}(f, g)\|_{L^2(\Omega_b)}\|\partial_{x_i}(f, g)\|_{L^2(\Omega_b)} + \frac{1}{b}\|\partial_{x_i} f\|_{L^2(\Omega_b)}\|g\|_{L^2(\Omega_b)}. \tag{C.2}$$

(2). *Let* $f, g \in H^1(\mathbb{R}^3_+)$, *and* $x_3 \in \mathbb{R}_+$, *it holds that*

$$\left|\int_{\mathbb{R}^2} (\partial_{x_i} f \cdot g)(x_{\shortparallel}, x_3)dx_{\shortparallel}\right| \leqq \|\partial_{x_3}(f, g)\|_{L^2(\mathbb{R}^3_+)}\|\partial_{x_i}(f, g)\|_{L^2(\mathbb{R}^3_+)}, \text{ for } i = 1, 2. \tag{C.3}$$

Proof. We only prove (C.1) and (C.2) since (C.3) can be obtained by taking the limit $b \to \infty$ in (C.2).

Without loss of generality, we assume that $f, g \in C^2(\Omega_b) \cap H^1(\Omega_b)$. There exists a point $z_b \in [0, b]$ so that

$$\int_{\mathbb{R}^2} (fg)(x_{\shortparallel}, z_b)\, dx_{\shortparallel} = \frac{1}{b}\int_0^b \int_{\mathbb{R}^2} (fg)(x_{\shortparallel}, x_3)\, dx_{\shortparallel}dx_3. \tag{C.4}$$

For any given $x_3 \in [0, b]$, by using (C.4), we have

$$\begin{aligned}\left|\int_{\mathbb{R}^2} (fg)(x_{\shortparallel}, x_3)\, dx_{\shortparallel}\right| &\leqq \left|\int_{z_b}^{x_3} \int_{\mathbb{R}^2} \partial_z(fg)\, dx_{\shortparallel}\, dz\right| + \frac{1}{b}\left|\int_0^b \int_{\mathbb{R}^2} (fg)(x_{\shortparallel}, x_3)\, dx_{\shortparallel}dx_3\right| \\ &\leqq \int_0^b \int_{\mathbb{R}^2} |\partial_z f \cdot g| + |f \cdot \partial_z g|\, dx_{\shortparallel}\, dz + \frac{1}{b}\|f\|_{L^2(\Omega_b)}\|g\|_{L^2(\Omega_b)} \\ &\leqq \|\partial_{x_3}(f, g)\|_{L^2(\Omega_b)}\|(f, g)\|_{L^2(\Omega_b)} + \frac{1}{b}\|f\|_{L^2(\Omega_b)}\|g\|_{L^2(\Omega_b)}.\end{aligned}$$

Hence we conclude (C.1).

Similar as (C.4), for $i = 1, 2$, there exists a point $z_b \in [0, b]$ so that

$$\int_{\mathbb{R}^2} (\partial_{x_i} f \cdot g)(x_{\shortparallel}, z_b)\, dx_{\shortparallel} = \frac{1}{b}\int_0^b \int_{\mathbb{R}^2} (\partial_{x_i} f \cdot g)(x_{\shortparallel}, x_3)\, dx_{\shortparallel}dx_3.$$

For $x_3 \in [0, b]$, integrating by parts w.r.t x_i, we obtain

$$\left|\int_{\mathbb{R}^2} (\partial_{x_i} f \cdot g)(x_{\shortparallel}, x_3)\, dx_{\shortparallel}\right|$$

$$
\begin{aligned}
&\leqq \left| \int_{z_b}^{x_3} \int_{\mathbb{R}^2} \partial_z (\partial_{x_i} f \cdot g) \,\mathrm{d}x_{\shortparallel} \,\mathrm{d}z \right| + \frac{1}{b} \left| \int_0^b \int_{\mathbb{R}^2} (\partial_{x_i} f \cdot g)(x_{\shortparallel}, x_3) \,\mathrm{d}x_{\shortparallel} \mathrm{d}x_3 \right| \\
&\leqq \left| \int_0^b \int_{\mathbb{R}^2} \partial_z f \cdot \partial_{x_i} g \,\mathrm{d}x_{\shortparallel} \,\mathrm{d}z \right| + \left| \int_0^b \int_{\mathbb{R}^2} \partial_{x_i} f \cdot \partial_z g \,\mathrm{d}x_{\shortparallel} \,\mathrm{d}z \right| \\
&\quad + \frac{1}{b} \|\partial_{x_i} f\|_{L^2(\Omega_b)} \|g\|_{L^2(\Omega_b)} \\
&\leqq \|\partial_{x_3}(f, g)\|_{L^2(\Omega_b)} \|\partial_{x_i}(f, g)\|_{L^2(\Omega_b)} + \frac{1}{b} \|\partial_{x_i} f\|_{L^2(\Omega_b)} \|g\|_{L^2(\Omega_b)}.
\end{aligned}
$$

This completes (C.2). □

References

1. Bardos, C., Caflisch, R.E., Nicolaenko, B.: The Milne and Kramers problems for the Boltzmann equation of a hard sphere gas. *Commun. Pure Appl. Math.* **39**, 323–352, 1986
2. Bardos, C., Golse, F., Levermore, C.D.: Fluid dynamic limits of kinetic equations. I. Formal derivations. *J. Stat. Phys.* **63**(1–2), 323–344, 1991
3. Bardos, C., Golse, F., Levermore, C.D.: Fluid dynamic limits of kinetic equations. II. Convergence proofs for the Boltzmann equation. *Commun. Pure Appl. Math.* **46**(5), 667–753, 1993
4. Bardos, C., Golse, F., Levermore, C.D.: The acoustic limit for the Boltzamnn equation. *Arch. Ration. Mech. Anal.* **153**, 177–204, 2000
5. Bardos, C., Ukai, S.: The classical incompressible Navier–Stokes limit of the Boltzmann equation. *Math. Models Methods Appl. Sci.* **1**(2), 235–257, 1991
6. Boltzmann, L.: Weitere Studien über das Wärmegleichgewicht unter Gasmolekülen. *Sitzungs. Akad. Wiss. Wien* **66**, 275–370, 1872; translated as: Further studies on the thermal equilibrium of gas molecules. Kinetic theory, vol. 2, 88-174. Pergamon, London, 1966.
7. Caflisch, R.E.: The fluid dynamic limit of the nonlinear Boltzmann equation. *Commun. Pure Appl. Math.* **33**(5), 651–666, 1980
8. Caprino, S., Esposito, R., Marra, R., Pulvirenti, M.: Hydrodynamic limits of the Vlasov equation. *Commun. Partial Differ. Equ.* **18**(5–6), 805–820, 1993
9. Chen, S.X.: On the initial-boundary value problems for quasilinear symmetric hyperbolic systems and their applications. *Chin. Ann. Math.* **1**(3–4), 511–521, 1980
10. Chen, S.X.: Initial boundary value problems for quasilinear symmetric hyperbolic systems with characteristic boundary. *Front. Math. China* **2**(1), 87–102, 2007
11. DiPerna, R.J., Lions, P.-L.: On the Cauchy problem for Boltzmann equations: global existence and weak stability. *Ann. Math. (2)* **130**(2), 321–366, 1989
12. Duan, R.J., Huang, F.M., Wang, Y., Zhang, Z.: Effects of soft interaction and non-isothermal boundary upon long-time dynamics of rarefied gas. *Arch. Ration. Mech. Anal.* **234**(2), 925–1006, 2019
13. Esposito, R., Lebowitz, J.L., Marra, R.: Hydrodynamic limit of the stationary Boltzmann equation in a slab. *Commun. Math. Phys.* **160**(1), 49–80, 1994
14. Esposito, R., Guo, Y., Marra, R.: Hydrodynamic limit of a kinetic gas flow past an obstacle. *Commun. Math. Phys.* **364**(2), 765–823, 2018
15. Esposito, R., Kim, C., Guo, Y., Marra, R.: Stationary solutions to the Boltzmann equation in the hydrodynamic limit. *Ann. PDE* **4**(1), 1–119, 2018
16. Golse, F., Saint-Raymond, L.: The Navier–Stokes limit of the Boltzmann equation for bounded collision kernels. *Invent. Math.* **155**(1), 81–161, 2004
17. Golse, F., Perthame, B., Sulem, C.: On a boundary layer problem for the nonlinear Boltzmann equation. *Arch. Ration. Mech. Anal.* **103**(1), 81–96, 1988

18. Grad, H.: Asymptotic equivalence of the Navier–Stokes and nonlinear Boltzmann equations. In: *Proc. Sympos. Applied Mathematics*, vol. 17 (Amer. Math. Soc., 1965), pp. 154–183
19. Guo, Y.: Boltzmann diffusive limit beyond the Navier–Stokes approximation. *Commun. Pure Appl. Math.* **59**(5), 626–687, 2006
20. Guo, Y.: Decay and continuity of the Boltzmann equation in bounded domains. *Arch. Ration. Mech. Anal.* **197**(3), 713–809, 2010
21. Guo, Y., Jang, J.: Global Hilbert expansion for the Vlasov–Poisson–Boltzmann system. *Commun. Math. Phys.* **299**(2), 469–501, 2010
22. Guo, Y., Jang, J., Jiang, N.: Local Hilbert expansion for the Boltzmann equation. *Kinet. Relat. Models* **2**(1), 205–214, 2009
23. Guo, Y., Jang, J., Jiang, N.: Acoustic limit for the Boltzmann equation in optimal scaling. *Commun. Pure Appl. Math.* **63**(3), 337–361, 2010
24. Guo, Y., Kim, C., Tonon, D., Trescases, A.: Regularity of the Boltzmann equation in convex domains. *Invent. Math.* **207**, 115–290, 2017
25. Guo, Y., Liu, S.Q.: Incompressible hydrodynamic approximation with viscous heating to the Boltzmann equation. *Math. Models Methods Appl. Sci.* **27**(12), 2261–2296, 2017
26. Hilbert, D.: Mathematical problems. *Bull. Am. Math. Soc. (N.S.)* **37**(4), 407–436, 2000
27. Huang, F.M., Jiang, S., Wang, Y.: Zero dissipation limit of full compressible Navier–Stokes equations with a Riemann initial data. *Commun. Inf. Syst.* **13**(2), 211–246, 2013
28. Huang, F.M., Jiang, Z.H., Wang, Y.: Boundary layer solution of the Boltzmann equation for specular boundary condition. arXiv: 2008.06621.
29. Huang, F.M., Wang, Y., Yang, T.: Hydrodynamic limit of the Boltzmann equation with contact discontinuities. *Commun. Math. Phys.* **295**, 293–326, 2010
30. Huang, F.M., Wang, Y., Yang, T.: Fluid dynamic limit to the Riemann solutions of Euler equations: I. Superposition of rarefaction waves and contact discontinuity. *Kinet. Relat. Models* **3**, 685–728, 2010
31. Huang, F.M., Wang, Y., Wang, Y., Yang, T.: The limit of the Boltzmann equation to the Euler equations for Riemann problems. *SIAM J. Math. Anal.* **45**(3), 1741–1811, 2013
32. Jang, J., Jiang, N.: Acoustic limit of the Boltzmann equation: classical solutions. *Discrete Contin. Dyn. Syst.* **25**(3), 869–882, 2009
33. Jang, J., Kim, C.: Incompressible Euler limit from Boltzmann equation with diffuse boundary condition for analytic data. arXiv: 2005.12192
34. Jiang, N., Levermore, C.D., Masmoudi, N.: Remarks on the acoustic limit for the Boltzmann equation. *Commun. Partial Differ. Eqs.* **35**(9), 1590–1609, 2010
35. Jiang, N., Masmoudi, N.: Boundary layers and incompressible Navier–Stokes–Fourier limit of the Boltzmann equation in bounded domain I. *Commun. Pure Appl. Math.* **70**(1), 90–171, 2017
36. Lachowicz, M.: On the initial layer and the existence theorem for the nonlinear Boltzmann equation. *Math. Methods Appl. Sci.* **9**(3), 342–366, 1987
37. Masmoudi, N., Saint-Raymond, L.: From the Boltzmann equation to the Stokes–Fourier system in a bounded domain. *Commun. Pure Appl. Math.* **56**(9), 1263–1293, 2003
38. Maxwell, J.C.: On the dynamical theory of gases. Philos. Trans. Roy. Soc. London Ser. A **157**, 49-88, 1867. Reprinted in The scientific letters and papers of James Clerk Maxwell, vol. II, 1862-1873, 26-78. Dover, New York, 1965
39. Nishida, T.: Fluid dynamical limit of the nonlinear Boltzmann equation to the level of the compressible Euler equation. *Commun. Math. Phys.* **61**(2), 119–148, 1978
40. Saint-Raymond, L.: *Hydrodynamic limits of the Boltzmann equation*. Lecture Notes in Mathe- matics, 1971. Springer, Berlin (2009). https://doi.org/10.1007/978-3-540-92847-8
41. Secchi, P.: Linear symmetric hyperbolic systems with characteristic boundary. *Math. Methods Appl. Sci.* **18**(11), 855–870, 1995

42. Schochet, S.: The compressible Euler equations in a bounded domain: existence of solutions and the incompressible limit. *Commun. Math. Phys*. **104**, 49–75, 1986
43. Sone, Y.: *Kinetic Theory and Fluid Dynamics*. Birkhäuser Boston Inc, Boston 2002
44. Sone, Y.: *Molecular Gas Dynamics. Theory, Techniques, and Applications*. Birkhäuser Boston Inc, Boston 2007
45. Ukai, S., Asano, K.: The Euler limit and the initial layer of the nonlinear Boltzmann equation. *Hokkaido Math. J*. **12**, 303–324, 1983
46. Wang, G.F., Wang, Y., Zhou, J.W.: Hydrodynamic limit of the Boltzmann equation to the planar rarefaction wave in three dimensional space. arXiv:1910.05890.
47. Wu, L.: Hydrodynamic limit with geometric correction of stationary Boltzmann equation. *J. Differ. Equ*. **260**, 7152–7249, 2016
48. Xin, Z.P., Zeng, H.H.: Convergence to the rarefaction waves for the nonlinear Boltzmann equation and compressible Navier–Stokes equations. *J. Differ. Equ*. **249**, 827–871, 2010
49. Yu, S.H.: Hydrodynamic limits with shock waves of the Boltzmann equations. *Commun. Pure Appl. Math* **58**, 409–443, 2005

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