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PHASE MIXING FOR SOLUTIONS TO 1D TRANSPORT EQUATION IN A CONFINING POTENTIAL

Sanchit Chaturvedi and Jonathan Luk

Department of Mathematics, Stanford University
450 Jane Stanford Way, Bldg 380
Stanford, CA 94305, USA

Abstract. Consider the linear transport equation in 1D under an external confining potential Φ:

$$\partial_t f + v\partial_x f - \partial_x \Phi \partial_v f = 0.$$

For $\Phi = \frac{x^2}{2} + \frac{\varepsilon x^4}{2}$ (with $\varepsilon > 0$ small), we prove phase mixing and quantitative decay estimates for $\partial_t \varphi := -\Delta^{-1} \int_{\mathbb{R}} \partial_t f \, \mathrm{d}v$, with an inverse polynomial decay rate $O(\langle t \rangle^{-2})$. In the proof, we develop a commuting vector field approach, suitably adapted to this setting. We will explain why we hope this is relevant for the nonlinear stability of the zero solution for the Vlasov–Poisson system in 1D under the external potential Φ.

1. Introduction.

Consider the linear transport equation in 1D

$$\partial_t f + v\partial_x f - \partial_x \Phi \partial_v f = 0, \tag{1}$$

for an unknown function $f : [0,\infty) \times \mathbb{R}_x \times \mathbb{R}_v \to \mathbb{R}_{\geq 0}$ with a smooth external confining potential $\Phi : \mathbb{R} \to \mathbb{R}$.

The following is the main result of this note:

Theorem 1.1. *Let $\varepsilon > 0$ and $\Phi(x) = \frac{x^2}{2} + \frac{\varepsilon x^4}{2}$. Consider the unique solution f to (1) with initial data $f \restriction_{t=0} = f_0$ such that*

- *$f_0 : \mathbb{R}_x \times \mathbb{R}_v \to \mathbb{R}_{\geq 0}$ is smooth, and*
- *there exists $c_s > 0$ such that $\mathrm{supp}(f_0) \subseteq \{(x,v) : c_s \leq \frac{v^2}{2} + \Phi(x) \leq c_s^{-1}\}$.*

Then, for ε sufficiently small, there exists $C > 0$ depending on ε and c_s such that the following estimate holds:

$$\sup_{x\in\mathbb{R}} |\partial_t \varphi|(t,x) \leq C\langle t \rangle^{-2} \sup_{(x,v)\in\mathbb{R}\times\mathbb{R}} \sum_{|\alpha|+|\beta|\leq 2} |\partial_x^\alpha \partial_v^\beta f_0|(x,v),$$

where φ is defined by

$$\partial_{xx}^2 \varphi(t,x) = \int_{\mathbb{R}} f(t,x,v) \, \mathrm{d}v, \quad \varphi(t,0) = \partial_x \varphi(t,0) = 0. \tag{2}$$

A few remarks of the theorem are in order.

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* Corresponding author: Jonathan Luk.

Remark 1.2 (Nonlinear Vlasov–Poisson system)**.** The reason that we are particularly concerned with $\partial_t\varphi$ is that it appears to be the quantity relevant for the stability of the zero solution for the nonlinear Vlasov–Poisson system in 1D; see Section 2.

It should be noted that φ itself is not expect to decay to 0 (since $\int_{\mathbb{R}} f\,\mathrm{d}v \geq 0$ and the x-support remains bounded). Thus the decay for $\partial_t\varphi$ can be viewed as a measure of the rate that φ approaches the limit $\lim_{t\to+\infty}\varphi(t,x)$.

Remark 1.3 (Derivatives of $\partial_t\varphi$)**.** For the applications on the Vlasov–Poisson system, one may also wish to obtain estimates for the derivatives of $\partial_t\varphi$. It is easy to extend our methods to obtain

$$|\partial_x\partial_t\varphi| \lesssim \langle t\rangle^{-1}, \quad |\partial_x^2\partial_t\varphi| \lesssim 1.$$

Notice that these decay rates, at least by themselves, do not seem sufficient for a global nonlinear result.

Remark 1.4 (Phase mixing and the choice of Φ)**.** The result in Theorem 1.1 can be interpreted as a quantitative phase-mixing statement. It is well-known that for

$$\Phi(x) = \frac{x^2}{2},$$

the solution to (1) does <u>not</u> undergo phase mixing (see chapter 3 in [8]). It is therefore important that we added the $\frac{\varepsilon x^4}{2}$ term in the definition of the potential.

On the other hand, there are other choices of Φ for which analogues of Theorem 1.1 hold. We expect that as long as Φ is even and satisfies the non-degeneracy condition of [31], then a similar decay estimate holds. The particular example we used is only chosen for concreteness.

Remark 1.5 (Method of proof)**.** It is well-known that the linear transport equation (1) can be written in action-angle variables, say (Q,K), in which case (1) takes the form

$$\partial_t f - c(K)\partial_Q f = 0. \tag{3}$$

When $c'(K)$ is bounded away from 0, phase mixing in the sense that f converges *weakly* to a limit can be obtained after solving (3) with a Fourier series in Q; see [31]. The point here is that φ is a (weighted) integral of f over a region of phase space that is most conveniently defined with respect to the (x,v) (as opposed to the action-angle) variables.

We quantify the strong convergence of $\varphi_t \to 0$ by finding an appropriate commuting vector field Y that is adapted to the action-angle variables. The fact that φ is naturally defined as an integral over v in (x,v) coordinates makes it tricky to prove decay using this vector field. Furthermore, we are only able to prove $1/\langle t\rangle^2$ decay; this is for instance in contrast to the decay of the density for the free transport equation on a torus. (Notice that what limits our result to only $1/\langle t\rangle^2$ decay is that unlike for the density for the free transport equation, we can only integrate by parts in the action variable twice because of boundary terms that arise. See Proposition 5.7 for details.)

1.1. **Related result.**

Linear phase mixing results. In the particular context of Theorem 1.1, decay of $\partial_t \varphi$, but without a quantitative rate, can be inferred from the work [31].

There are many linear phase mixing result, the simplest setting for this is the linear free transport equation. This is well-known; see for instance notes [35] by Villani.

One of the most influential work on phase mixing is the groundbreaking paper [27] of Landau wherein he proposes a linear mechanism for damping for plasmas that does not involve dispersion or change in entropy. In the case of $\mathbb{T}^d$, this is even understood in a nonlinear setting; see the section on nonlinear results below. The situation is more subtle in $\mathbb{R}^d$, see [4, 22, 23, 26].

See also [5, 20, 34] for linear results on related models. In particular, we note that [20] also rely on action-angle variables in their analysis.

Relation with other phase-mixing problems with integrable underlying dynamics. As pointed out in [31], phase space mixing is relevant for the dynamics of kinetic models in many physical phenomena from stellar systems and dark matter halos to mixing of relativistic gas surrounding a black hole. See [15] for related discussions on dark matter halos. We also refer the interested reader to [8] for further background and discussions of phase mixing in other models, including the stability of galaxies.

We hope that the present work would also be a model problem and aid in understanding more complicated systems such as those described in [31]. One particularly interesting problem is the stability of the Schwarzschild solution to the Einstein–Vlasov system in spherical symmetry.

Nonlinear phase mixing results. Nonlinear Landau damping for Vlasov–Poisson on $\mathbb{T}^d$ was first proven in analytic regularity by Mouhot–Villani in their landmark paper [30]. Since then their work has been extended and simplified in [2] and [24].

See also other nonlinear results, e.g. in [1, 3, 12, 21, 25, 37].

Collisional problems with confining potentials. Confining potentials for kinetic equations have been well-studied, particularly for collisional models. Linear stability results can be found in [9, 13, 14, 16, 17].

In this connection, it would also be of interest to understand how phase mixing effects (studied in the present paper) interact with collisional effects (cf. [1, 12, 34]).

Commuting vector field method for kinetic models. As mentioned in Remark 1.5, our proof is based on a commutating vector field method. In the context of kinetic theory, the commutating vector field method has been most successful in capturing dispersion, see [6, 7, 18, 19, 28, 32, 33, 36] for some results for collisionless models and [10, 11, 29] for some results on collisional models.

Concerning phase mixing, the use of commuting vector fields to prove homogenization for the density for solutions to the linear transport equation on $\mathbb{T}^d$ seems to be a known folklore technique. The commuting vector field method is also useful for capturing phase mixing in a nonlinear and weakly collisional setting; see our previous joint work with Nguyen [12].

2. **The Vlasov–Poisson system.** The motivation of our result is the Vlasov–Poisson system:

$$\begin{cases} \partial_t f + v\partial_x f - (\partial_x \Phi + \partial_x \varphi)\partial_v f = 0, \\ \partial_x^2 \varphi = \int_{\mathbb{R}} f \, \mathrm{d}v. \end{cases} \tag{4}$$

Note that (4) can be rewritten as

$$\partial_t f + \{H, f\} = 0, \tag{5}$$

where H is the Hamiltonian given by

$$H(x, v) = \frac{v^2}{2} + \Phi(x) + \varphi(t, x). \tag{6}$$

Notice that $f \equiv 0$ is a solution to (4), and the transport equation (1) is the linearization of (4) near the zero solution.

One cannot hope that the term $\partial_x \varphi$ in the nonlinear term decays as $t \to +\infty$. (To see this, first note that $\int_{\mathbb{R}} f \, \mathrm{d}v \geq 0$ pointwise. Moreover, due to the confinement, one expects the x-support of f to be uniformly bounded for all $t \geq 0$.) At best one can hope that $\partial_x \varphi$ converges to some (non-trivial) limiting profile as $t \to +\infty$. For f satisfying the linear equation (1), such convergence (without a quantitative rate) has been shown in [31].

In anticipation of the nonlinear problem, it is important to understand the quantitative rate of convergence. Since $\partial_x \varphi$ does not converge to 0, it is natural to understand the decay rate of $\partial_t \partial_x \varphi$.

As a first step to understand (4), we look at the linearized problem (1) around the zero solution and prove that we get integrable decay for φ_t in the linearized dynamics.

Remark 2.1. Note that the Poisson's equation above reads

$$\partial_x^2 \varphi = \rho.$$

In particular, φ is only defined up to a harmonic function, i.e. a linear function a x. In Theorem 1.1, we remove this ambiguity by setting $\varphi(0) = (\partial_x \varphi)(0) = 0$. Notice that other normalization, e.g., $\varphi(-\infty) = (\partial_x \varphi)(-\infty) = 0$ would not change the function $\varphi_t = \partial_t \varphi$.

3. **The action-angle variables.**

3.1. **First change of variables.** From now on we will consider the Hamiltonian

$$H = \frac{v^2}{2} + \Phi(x).$$

This is the Hamiltonian for the equations (1), which is also (6) without the φ (the self-interaction term). As an intermediate step to getting the action-angle variables we use the change of coordinates

$$\begin{aligned} (t, x, v) &\mapsto (t, \chi, H) && \text{when } x > 0 \\ (t, x, v) &\mapsto (t, \pi - \chi, H) && \text{when } x \leq 0, \end{aligned}$$

where $\chi := \arcsin\left(\frac{v}{\sqrt{2H}}\right)$. (Note that this map is a diffeomorphism $[0, \infty)_t \times (\mathbb{R}^2_{x,v} \setminus \{(0, 0)\}) \to [0, \infty)_t \times \mathbb{S}^1_\chi \times (0, \infty)_H$, where we have identified $\mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$.)

First we calculate the Jacobian for $x > 0$:

$$J = \begin{pmatrix} \partial_t t & \partial_x t & \partial_v t \\ \partial_t H & \partial_x H & \partial_v H \\ \partial_t \chi & \partial_x \chi & \partial_v \chi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \Phi_x & v \\ 0 & -\frac{v}{2H} \cdot \frac{\Phi_x}{\sqrt{2\Phi}} & \frac{\sqrt{2\Phi}}{H} \end{pmatrix}.$$

As a result,

$$\det(J) = \frac{\Phi_x}{\sqrt{2\Phi}} \frac{(\Phi + v^2/2)}{H} = \frac{\Phi_x}{\sqrt{2\Phi}}.$$

Similarly, for $x \leq 0$,

$$\det(J) = -\frac{\Phi_x}{\sqrt{2\Phi}}.$$

Hence,

$$\det(J) = \operatorname{sign} x \frac{\Phi_x}{\sqrt{2\Phi}}.$$

Next by chain rule and using that H is independent of t, we get

$$\begin{aligned} \partial_x &= \operatorname{sign} x \partial_x \chi \partial_\chi + \partial_x H \partial_H \\ &= -\operatorname{sign} x \frac{v}{2H} \cdot \frac{\Phi_x}{\sqrt{2\Phi}} \partial_\chi + \Phi_x \partial_H, \end{aligned}$$

$$\begin{aligned} \partial_v &= \operatorname{sign} x \partial_v \chi \partial_\chi + \partial_v H \partial_H \\ &= \operatorname{sign} x \frac{\sqrt{2\Phi}}{H} \partial_\chi + v \partial_H. \end{aligned}$$

Plugging this in (5), we get the equation

$$\partial_t f - \operatorname{sign} x \frac{\Phi_x}{\sqrt{2\Phi}} \partial_\chi f = 0. \tag{7}$$

3.2. **Second change of variables.** The coefficient in front of $\partial_\chi f$ in (7) depends on both χ and H. To take care of this, we reparametrize χ (in a manner depending on H). More precisely, for a fixed H, we define $Q(\chi, H)$ such that

$$\frac{\mathrm{d}Q}{\mathrm{d}\chi} = \frac{c(H)}{a(\chi, H)}, \quad Q(0, H) = 0,$$

where $a(\chi, H) = \operatorname{sign} x \frac{\Phi_x(x)}{\sqrt{2\Phi(x)}}$ such that $x = x(\chi, H)$. To fix $c(H)$, we require that for every H,

$$2\pi = \int_0^{2\pi} \mathrm{d}Q = c(H) \int_0^{2\pi} \frac{1}{a(\chi, H)} \,\mathrm{d}\chi. \tag{8}$$

Now we define the change of variables, $(\chi, H) \mapsto (Q, K)$ where $K = H$. Then note,

$$a(\chi, H)\partial_\chi = c(H)\partial_Q$$

and

$$\partial_H = \partial_K + \frac{\partial Q}{\partial H} \partial_Q.$$

Thus in these coordinates, we can rewrite (7) as

$$\partial_t f - c(K)\partial_Q f = 0. \tag{9}$$

Further, the Jacobian is

$$\begin{pmatrix} \partial_H K & \partial_\chi K \\ \partial_H Q & \partial_\chi Q \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \partial_H Q & \frac{c(H)}{a(\chi,H)} \end{pmatrix}.$$

Note that the determinant is $\frac{c(H)}{a(\chi,H)}$. Further, since

$$\begin{aligned} a(\chi, H) &= \operatorname{sign} x \frac{\Phi_x(x)}{\sqrt{2\Phi(x)}} \\ &= \frac{1+2\varepsilon x^2}{\sqrt{1+\varepsilon x^2}}, \end{aligned}$$

we have that $a(\chi, H) \approx 1$ when x is in a compact subset of $\mathbb{R}$. As a result the determinant is bounded away from zero. For more details see Lemma 4.3.

4. **The commuting vector field.**

We first define the vector field

$$Y = tc'(H)\partial_Q - \partial_K.$$

In this section we prove that this vector field commutes with the transport operator as in (9) and that $|c'(H)| > 0$.

4.1. **Commutation property.**

The following commutation formula is an easy computation and thus we leave out the details.

Lemma 4.1. *Let $Y = tc'(H)\partial_Q - \partial_K$. Then*

$$[\partial_t - c(H)\partial_Q, Y] = 0.$$

The following is an easy consequence of Lemma 4.1:

Lemma 4.2. *Let f be a solution to (9) with initial data satisfying assumptions of Theorem 1.1. Then*

$$\sup_{(t,Q,K)\in[0,\infty)\times\mathbb{T}^1\times[c_s,c_s^{-1}]} \sum_{\ell\le 2} |Y^\ell f|(t,Q,K) \lesssim \sup_{(x,v)\in\mathbb{R}\times\mathbb{R}} \sum_{|\alpha|+|\beta|\le 2} |\partial_x^\alpha \partial_v^\beta f_0|(x,v).$$

Proof. By Lemma 4.1, we have that $Y^\ell f$ satisfies the transport equation (9) for any $\ell \in \mathbb{N} \cap \{0\}$. Hence we get the estimate

$$\sup_{(t,Q,K)\in[0,\infty)\times\mathbb{T}^1\times[c_s,c_s^{-1}]} \sum_{\ell\le 2} |Y^\ell f|(t,Q,K) \lesssim \sup_{(Q,K)\in\mathbb{T}^1\times[c_s,c_s^{-1}]} \sum_{\ell\le 2} |\partial_K^\ell f_0|(Q,K).$$

Since the change of variables $(Q,K) \to (x,v)$ is well-defined and bounded away from zero, we get the required result. □

4.2. **Positivity of $|c'(K)|$.**

We prove that $|c'(K)|$ is uniformly bounded below on the support of f. This plays a key role in the next section ensuring phase mixing.

Lemma 4.3. *For every $c_s < +\infty$, there exists $\varepsilon_0 > 0$ such that whenever $\varepsilon \in (0, \varepsilon_0]$, there is a small constant $\delta > 0$ (depending on c_s and ε) such that*

$$\inf_{K\in[c_s,c_s^{-1}]} |c'(K)| = \inf_{H\in[c_s,c_s^{-1}]} |c'(H)| \ge \delta.$$

Proof. By definition of $c(H)$, we have

$$\frac{2\pi}{c(H)} = \int_0^{2\pi} \left| \frac{\sqrt{2\Phi}}{\Phi'} \right| \mathrm{d}\chi,$$

so that using $\Phi = \frac{x^2}{2} + \frac{\varepsilon}{2}x^4$, we obtain

$$\frac{2\pi}{c(H)} = \int_0^{2\pi} \frac{\sqrt{1+\varepsilon x^2}}{(1+2\varepsilon x^2)} \mathrm{d}\chi.$$

Notice that for $H \in [c_s, c_s^{-1}]$, $|x|$ is bounded. It follows that $c(H) \approx 1$. Therefore, to prove strict positivity of $|c'(H)|$, it suffices to prove positivity of $\left|\frac{c'(H)}{c^2(H)}\right|$. Note that

$$\begin{aligned}
\frac{-2\pi c'(H)}{c^2(H)} &= \int_0^{2\pi} \partial_H \left(\frac{\sqrt{1+\varepsilon x^2}}{1+2\varepsilon x^2} \right) \mathrm{d}\chi \\
&= \int_0^{2\pi} \partial_H x \left[\frac{\varepsilon x}{\sqrt{1+\varepsilon x^2}(1+2\varepsilon x^2)} - \frac{4\varepsilon x\sqrt{1+\varepsilon x^2}}{(1+2\varepsilon x^2)^2} \right] \mathrm{d}\chi \qquad (10) \\
&= \int_0^{2\pi} \partial_H x \left[\frac{-3\varepsilon x - 2\varepsilon^2 x^3}{\sqrt{1+\varepsilon x^2}(1+2\varepsilon x^2)^2} \right] \mathrm{d}\chi.
\end{aligned}$$

Now we calculate $\partial_H x$. First we use the equation, $H = \frac{v^2}{2} + \Phi(x)$. Precisely, we have

$$1 = v\partial_H v + \Phi'(x)\partial_H x.$$

Thus

$$\partial_H x = \frac{1 - v\partial_H v}{\Phi'}. \qquad (11)$$

Next we use that $\frac{v}{\sqrt{2H}} = \sin\chi$,

$$0 = \frac{\partial_H v}{\sqrt{2H}} - \frac{v}{(2H)^{\frac{3}{2}}}.$$

Thus $\partial_H v = \frac{v}{2H}$. Plugging this into (11), we get that

$$\partial_H x = \frac{1 - \frac{v^2}{2H}}{\Phi'} = \frac{\cos^2\chi}{\Phi'(x)} = \frac{\cos^2\chi}{x + 2\varepsilon x^3}, \qquad (12)$$

where in the last equality we used $\Phi = \frac{x^2 + \varepsilon x^4}{2}$.

Plugging (12) back into (10), we get that

$$\begin{aligned}
\frac{-2\pi c'(H)}{c^2(H)} &= \int_0^{2\pi} \frac{\cos^2\chi}{x(1+2\varepsilon x^2)} \left[\frac{-x(3\varepsilon + 2\varepsilon^2 x^2)}{\sqrt{1+\varepsilon x^2}(1+2\varepsilon x^2)^2} \right] \mathrm{d}\chi \\
&= -\int_0^{2\pi} \cos^2\chi \left[\frac{(3\varepsilon + 2\varepsilon^2 x^2)}{\sqrt{1+\varepsilon x^2}(1+2\varepsilon x^2)^3} \right] \mathrm{d}\chi.
\end{aligned}$$

Finally note that since $|x|$ is bounded on the region of interest, after choosing ε_0 sufficiently small, we have

$$\frac{(3\varepsilon + 2\varepsilon^2 x^2)}{\sqrt{1+\varepsilon x^2}(1+2\varepsilon x^2)^3} \approx \varepsilon,$$

and thus $|c'(H)| > \delta$. $\square$

5. **Decay for φ_t.** In this section, we finally prove the decay for φ_t (recall Theorem 1.1).

Lemma 5.1. *For f satisfying the assumptions of Theorem 1.1, and φ defined as in (2), we have the following formula*

$$-\varphi_t(t,x') = \int_0^{x'} \int_{\mathbb{R}} v[f(t,x,v) - f(t,0,v)]\,\mathrm{d}v\,\mathrm{d}x.$$

Proof. By the continuity equation (following directly from (1)), we have that

$$\rho_t = -\int_{\mathbb{R}} v\partial_x f\,\mathrm{d}v.$$

Thus

$$\partial_x^2 \varphi_t = \rho_t = -\int_{\mathbb{R}} v\partial_x f\,\mathrm{d}v.$$

Solving the Laplace's equation (with boundary conditions (2)), we get

$$-\varphi_t(t,x') = \int_0^{x'} \int_0^y \int_{\mathbb{R}} v\partial_x f(t,z,v)\,\mathrm{d}v\,\mathrm{d}z\,\mathrm{d}y.$$

Integrating by parts in z, we get

$$-\varphi_t(t,x') = \int_0^{x'} \int_{\mathbb{R}} v[f(t,y,v) - f(t,0,v)]\,\mathrm{d}v\,\mathrm{d}y.$$

Relabelling y as x yields the desired result. □

In view of Lemma 5.1, it suffices to bound $\int_0^{x'}\int_{\mathbb{R}} vf(t,0,v)\,\mathrm{d}v\,\mathrm{d}y$ and $\int_0^{x'}\int_{\mathbb{R}} vf(t,x,v)\,\mathrm{d}v\,\mathrm{d}x$, which will be achieved in the next two subsections respectively.

5.1. **Decay for the term involving $f(0,v)$.** We first prove decay for $\int_{\mathbb{R}} vf(t,0,v)$ $\mathrm{d}v$. Before proving the main estimate in Proposition 5.3, we first prove a lemma.

Lemma 5.2. *The level set $\{x=0\}$ corresponds to the level sets $\{Q = \frac{\pi}{2}\} \cap \{Q = -\frac{\pi}{2}\} \cup \{(x,v) = (0,0)\}$.*

Proof. First note that level set $\{x = 0\}$ corresponds to the level sets $\{\chi = \frac{\pi}{2}\} \cap \{\chi = -\frac{\pi}{2}\} \cup \{(x,v) = (0,0)\}$. This is because when $x = 0$, $\Phi(x) = 0$, and thus by definition (when $v \neq 0$) $\chi := \arcsin\left(\frac{v}{\sqrt{2H}}\right) = \arcsin(\pm 1) = \pm\frac{\pi}{2}$.

It thus remains to show that

$$\chi = \pm\frac{\pi}{2} \iff Q = \pm\frac{\pi}{2}. \tag{13}$$

Fix H. Since $a(\chi,H) = |\frac{\Phi_x}{\sqrt{\Phi}}|$ is independent of v, it is in particular even in v. Hence, we have

$$c(H)\int_0^{\pi} \frac{1}{a(\chi,H)}\,\mathrm{d}\chi = c(H)\int_{\pi}^{2\pi} \frac{1}{a(\chi,H)}\,\mathrm{d}\chi.$$

Further, by the evenness of Φ, we have

$$c(H)\int_0^{\pi/2} \frac{1}{a(\chi,H)}\,\mathrm{d}\chi = c(H)\int_{\pi/2}^{\pi} \frac{1}{a(\chi,H)}\,\mathrm{d}\chi.$$

Finally, since we have by construction,

$$c(H)\int_0^{2\pi}\frac{1}{a(\chi,H)}\,\mathrm{d}\chi = 2\pi,$$

we have that

$$Q(\chi=\pi/2,H) = c(H)\int_0^{\pi/2}\frac{1}{a(\chi,H)}\,\mathrm{d}\chi = \pi/2.$$

Similarly, $Q(\chi=-\pi/2,H)=-\pi/2$. Combining these, we obtain (13). □

Proposition 5.3. *For f satisfying the assumptions of Theorem 1.1, we have the following estimate:*

$$\Big|\int_{\mathbb{R}} vf(t,0,v)\,\mathrm{d}v\Big| \lesssim \langle t\rangle^{-2}\sup_{(x,v)\in\mathbb{R}\times\mathbb{R}}\sum_{|\alpha|+|\beta|\le 2}|\partial_x^\alpha\partial_v^\beta f_0|(x,v).$$

Proof. The transport equation preserves L^∞ bounds so that by the support properties, we obviously have

$$\Big|\int_{\mathbb{R}} vf(t,0,v)\,\mathrm{d}v\Big| \lesssim \sup_{(x,v)\in\mathbb{R}\times\mathbb{R}}|f_0|(x,v).$$

In other words, it suffices to prove the desired bound with t^{-2} instead of $\langle t\rangle^{-2}$.

Now note that

$$\int_{\mathbb{R}} vf(t,0,v)\,\mathrm{d}v = \int_0^\infty v[f(t,0,v)-f(t,0,-v)]\,\mathrm{d}v.$$

For clarity of notation, we let

$$\bar{f}(t,Q,K) = f(t,x,v).$$

Now writing in the (K,Q) variables, and using Lemma 5.2 together with the fact that $K=H=\frac{v^2}{2}$ when $x=0$, we have

$$\begin{aligned}\int_{\mathbb{R}} vf(t,0,v)\,\mathrm{d}v &= \int_0^\infty v[f(t,0,v)-f(t,0,-v)]\,\mathrm{d}v\\ &= \int_0^\infty [\bar{f}(t,\pi/2,K)-\bar{f}(t,-\pi/2,K)]\,\mathrm{d}K.\end{aligned}$$

By the fundamental theorem of calculus, we have

$$\int_0^\infty [\bar{f}(t,\pi/2,K)-\bar{f}(t,-\pi/2,K)]\,\mathrm{d}K = \int_{-\pi/2}^{\pi/2}\int_0^\infty \partial_Q\bar{f}(t,Q,K)\,\mathrm{d}K\,\mathrm{d}Q.$$

Next, the Cauchy–Schwarz inequality implies

$$\begin{aligned}\int_{-\pi/2}^{\pi/2}\int_0^\infty \partial_Q\bar{f}(t,Q,K)\,\mathrm{d}K\,\mathrm{d}Q &= \sqrt{\pi}\left(\int_{-\pi/2}^{\pi/2}\left(\int_0^\infty \partial_Q\bar{f}(t,Q,K)\,\mathrm{d}K\right)^2\mathrm{d}Q\right)^{\frac{1}{2}}\\ &\lesssim \left(\int_0^{2\pi}\left(\int_0^\infty \partial_Q\bar{f}(t,Q,K)\,\mathrm{d}K\right)^2\mathrm{d}Q\right)^{\frac{1}{2}}.\end{aligned}$$

Now using Poincare's inequality we get that for any $\ell\ge 2$

$$\left(\int_0^{2\pi}\left(\int_0^\infty \partial_Q\bar{f}(t,Q,K)\,\mathrm{d}K\right)^2\mathrm{d}Q\right)^{\frac{1}{2}} \lesssim \left(\int_0^{2\pi}\left(\int_0^\infty \partial_Q^\ell\bar{f}(t,Q,K)\,\mathrm{d}K\right)^2\mathrm{d}Q\right)^{\frac{1}{2}}. \tag{14}$$

Now take $\ell = 2$. We write $\partial_Q = \frac{1}{c'(K)t}(Y + \partial_K)$ so that

$$\left(\int_0^{2\pi}\left(\int_0^\infty \partial_Q^2 \bar{f}(t,Q,K)\,\mathrm{d}K\right)^2 \mathrm{d}Q\right)^{\frac{1}{2}}$$

$$= \left(\int_0^{2\pi}\left(\int_0^\infty \frac{1}{|c'(K)|^2 t^2}(Y^2\bar{f} + 2\partial_K Y\bar{f} + \partial_K^2 \bar{f})(t,Q,K)\,\mathrm{d}K\right)^2 \mathrm{d}Q\right)^{\frac{1}{2}}$$

$$\lesssim \frac{1}{t^2}\left(\int_0^{2\pi}\left(\int_0^\infty (\sum_{k=0}^{2}|Y^k\bar{f}|)(t,Q,K)\,\mathrm{d}K\right)^2 \mathrm{d}Q\right)^{\frac{1}{2}}.$$

where in the last step we have integrated by parts in K and bounded $\frac{1}{|c'(K)|}$, $\frac{|c''(K)|}{|c'(K)|}$, etc. using Lemma 4.3 and the smoothness of c.

Finally, since $f(t,Q,K)$ is non-zero for $c_s \leq K \leq c_s^{-1}$ and $Q \in [0, 2\pi]$, we can take supremum in K and Q followed by Lemma 4.2 to get the required result. □

Remark 5.4. Notice that since we can take any $\ell \geq 2$ in (14), we can write each $\partial_Q = \frac{1}{c'(H)t}(Y + \partial_K)$ and integrate by parts in K many times to show that the term in Proposition 5.3 in fact decays faster than any inverse polynomial (depending on smoothness of f)!

In other words, the decay rate that we obtain in Theorem 1.1 is instead limited by the term treated in Proposition 5.7 below.

5.2. Decay for the term involving $f(y,v)$.

We now turn to the other term in Lemma 5.1. Before we obtain the main estimate in Proposition 5.7, we first prove two simple lemmas.

Lemma 5.5. *Under the change of variables $(x,v) \mapsto (Q,K)$ as in Section 3, the volume form transforms as follows:*

$$\mathrm{d}v\,\mathrm{d}x = c(K)\,\mathrm{d}Q\,\mathrm{d}K.$$

Proof. The Jacobian determinant for the change of variables $(x,v) \mapsto (\chi, H)$ is $a(\chi,H) = |\frac{\Phi_x}{\sqrt{\Phi}}|$. Further the Jacobian determinant for the change of variables $(\chi,H) \to (Q,K)$ is $\frac{c(H)}{a(\chi,H)}$ and hence the Jacobian determinant for $(x,v) \to (Q,K)$ is $c(H) = c(K)$. □

Lemma 5.6. *Let $\bar{f}(Q,K) = f(x,v)$ as above. There exists a function $\bar{g}(Q,K)$ such that*

$$\partial_Q^2 \bar{g} = \partial_Q \bar{f} \tag{15}$$

and

$$\max_{\ell\leq 2}\sup_K \|Y^\ell \bar{g}\|_{L^2_Q} \lesssim \max_{\ell \leq 2}\sup_{Q,K}|Y^\ell \bar{f}|. \tag{16}$$

Proof. We use the Fourier series of f in Q to get that

$$\bar{f}(Q,K) = \sum_{k=-\infty}^{k=\infty} \widehat{\bar{f}}_k(K)e^{ikQ}.$$

Now we define

$$\bar{g}(Q,K) := \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{ik}\widehat{\bar{f}}_k(K)e^{ikQ}.$$

Then we see that $\partial_Q^2 \bar{g} = \partial_Q \bar{f}$.

Using Plancherel's theorem and the above formulae, we can easily see that

$$\max_{\ell\leq 2}\sup_K \|Y^\ell \bar{g}\|_{L^2_Q} \lesssim \max_{\ell\leq 2}\sup_K \|Y^\ell \bar{f}\|_{L^2_Q}.$$

Finally, the result follows by taking supremum in Q and noting that the Q-range is bounded by 2π. □

Proposition 5.7. *For f satisfying the assumptions of Theorem 1.1, we have the following estimate:*

$$\Big|\int_0^{x'}\int_{\mathbb{R}} vf(t,x,v)\,\mathrm{d}v\,\mathrm{d}x\Big| \lesssim \langle t\rangle^{-2} \sup_{(x,v)\in\mathbb{R}\times\mathbb{R}} \sum_{|\alpha|+|\beta|\leq 2} |\partial_x^\alpha \partial_v^\beta f_0|(x,v).$$

Proof. As in the proof of Proposition 5.3, boundedness is obvious and thus it suffices to prove an estimate with $\langle t\rangle^{-2}$ replaced by t^{-2}.

We first note that

$$\int_0^{x'}\int_{\mathbb{R}} vf(x,v)\,\mathrm{d}v\,\mathrm{d}x = \int_0^{x'}\int_0^\infty v[f(t,x,v)-f(t,x,-v)]\,\mathrm{d}v\,\mathrm{d}x.$$

Again let

$$\bar{f}(t,Q,K) = f(t,x,v).$$

Next we use the change of variables $(x,v)\mapsto (Q,K)$, that $v=\sqrt{2H}\sin\chi$ and Lemma 5.5 followed by the fundamental theorem of calculus to obtain

$$\begin{aligned}
&\int_0^{x'}\int_0^\infty v[f(t,x,v)-f(t,x,-v)]\,\mathrm{d}v\,\mathrm{d}x\\
&= \int_0^{\Phi(x')}\int_0^{\pi/2} c(K)\sqrt{2K}S(Q,K)[\bar{f}(t,Q,K)-\bar{f}(t,-Q,K)]\,\mathrm{d}Q\,\mathrm{d}K\\
&\quad+\int_{\Phi(x')}^\infty\int_{\mathfrak{Q}_K}^{\pi/2} c(K)\sqrt{2K}S(Q,K)[\bar{f}(t,Q,K)-\bar{f}(t,-Q,K)]\,\mathrm{d}Q\,\mathrm{d}K\\
&= \int_0^{\Phi(x')}\int_0^{\pi/2}\int_{-Q}^{Q} c(K)\sqrt{2K}S(Q,K)\partial_Q\bar{f}(t,Q',K)\,\mathrm{d}Q'\,\mathrm{d}Q\,\mathrm{d}K\\
&\quad+\int_{\Phi(x')}^\infty\int_{\mathfrak{Q}_K}^{\pi/2}\int_{-Q}^{Q} c(K)\sqrt{2K}S(Q,K)\partial_Q\bar{f}(t,Q',K)\,\mathrm{d}Q'\,\mathrm{d}Q\,\mathrm{d}K\\
&=: T_1+T_2,
\end{aligned}$$

where we have defined

- $S(Q,K) := \sin\chi$, and
- $\mathfrak{Q}_K$ to be the angle in (Q,K) coordinates which corresponds to the angle $\arccos\left(\sqrt{\frac{\Phi(x')}{H}}\right)$ in (χ,H) coordinates.

The reason that we split the integrals into T_1 and T_2 is as follows. Notice that

$$\{(x,v): x\in[0,x'],\, v\in[0,\infty)\} = \{(\chi,H): \sqrt{H}\cos\chi \leq \sqrt{\Phi(x')},\, \chi\in[0,\frac{\pi}{2}]\}.$$

Hence, if $K=H\leq\Phi(x')$, we integrate in the full range $\chi\in[0,\frac{\pi}{2}]$ (equivalently, $Q\in[0,\frac{\pi}{2}]$), while if $K=H>\Phi(x')$, then we restrict to $\chi\in\left[\arccos\left(\sqrt{\frac{\Phi(x')}{H}}\right),\frac{\pi}{2}\right]$ (equivalently, $Q\in[\mathfrak{Q}_K,\frac{\pi}{2}]$).

Now using Fubini's theorem, we have

$$T_1 = \int_{-\pi/2}^{\pi/2} \int_0^{\Phi(x')} \left(\int_{|Q'|}^{\pi/2} S(Q,K)\,\mathrm{d}Q \right) c(K)\sqrt{2K}\partial_Q \bar{f}(t,Q',K)\,\mathrm{d}K\,\mathrm{d}Q'$$

and

$$\begin{aligned} T_2 = &\int_{-\pi/2}^{\pi/2} \int_{\Phi(x')}^{\mathfrak{H}_{Q'}} \left(\int_{|Q'|}^{\pi/2} S(Q,K)\,\mathrm{d}Q \right) c(K)\sqrt{2K}\partial_Q \bar{f}(t,Q',K)\,\mathrm{d}K\,\mathrm{d}Q' \\ &+ \int_{-\pi/2}^{\pi/2} \int_{\mathfrak{H}_{Q'}}^{\infty} \left(\int_{\mathfrak{Q}_K}^{\pi/2} S(Q,K)\,\mathrm{d}Q \right) c(K)\sqrt{2K}\partial_Q \bar{f}(t,Q',K)\,\mathrm{d}K\,\mathrm{d}Q', \end{aligned}$$

where $\mathfrak{H}_{Q'}$ is such that $\left(\arccos\left(\sqrt{\frac{\Phi(x')}{\mathfrak{H}_{Q'}}}\right), \mathfrak{H}_{Q'}\right)$ in (χ, H) coordinates gets mapped to $(|Q'|, \mathfrak{H}_{Q'})$ in (Q,K) coordinates (such an $\mathfrak{H}_{Q'}$ exists because $\chi = \arccos\left(\sqrt{\frac{\Phi(x')}{H}}\right)$ increases as H does and Q is monotone[1] in χ.) In the application of Fubini's theorem above, we split T_2 into two terms. This is because on the domain of integration for T_2, we have $Q \geq \max\{|Q'|, \mathfrak{Q}_K\}$. We thus need to handle $\max\{|Q'|, \mathfrak{Q}_K\} = |Q'|$ and $\max\{|Q'|, \mathfrak{Q}_K\} = \mathfrak{Q}_K$ separately.

Putting the above together we get,

$$\begin{aligned} T_1 + T_2 = &\int_{-\pi/2}^{\pi/2} \int_0^{\mathfrak{H}_{Q'}} \left(\int_{|Q'|}^{\pi/2} S(Q,K)\,\mathrm{d}Q \right) c(K)\sqrt{2K}\partial_Q \bar{f}(t,Q',K)\,\mathrm{d}K\,\mathrm{d}Q' \\ &+ \int_{-\pi/2}^{\pi/2} \int_{\mathfrak{H}_{Q'}}^{\infty} \left(\int_{\mathfrak{Q}_K}^{\pi/2} S(Q,K)\,\mathrm{d}Q \right) c(K)\sqrt{2K}\partial_Q \bar{f}(t,Q',K)\,\mathrm{d}K\,\mathrm{d}Q'. \end{aligned}$$

Now we use (15) from Lemma 5.6 and that $\partial_Q = \frac{1}{c'(K)t}(Y + \partial_K)$ to get that

$$\begin{aligned} &T_1 + T_2 \\ =&\frac{1}{t}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\mathfrak{H}_{Q'}} \frac{1}{c'(K)} \left(\int_{|Q'|}^{\frac{\pi}{2}} S(Q,K)\,\mathrm{d}Q \right) c(K)\sqrt{2K}(Y+\partial_K)\partial_Q \bar{g}(t,Q',K)\,\mathrm{d}K\,\mathrm{d}Q' \\ &+ \frac{1}{t}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\mathfrak{H}_{Q'}}^{\infty} \frac{1}{c'(K)} \left(\int_{\mathfrak{Q}_K}^{\frac{\pi}{2}} S(Q,K)\,\mathrm{d}Q \right) c(K)\sqrt{2K}(Y+\partial_K)\partial_Q \bar{g}(t,Q',K)\,\mathrm{d}K\,\mathrm{d}Q'. \end{aligned}$$

Next we integrate by parts in K. Since $\mathfrak{Q}_{\mathfrak{H}_{Q'}} = |Q'|$, we see that the boundary terms exactly cancel! Hence,

$$\begin{aligned} &\frac{1}{t}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\mathfrak{H}_{Q'}} \frac{1}{c'(K)} \left(\int_{|Q'|}^{\frac{\pi}{2}} S(Q,K)\,\mathrm{d}Q \right) c(K)\sqrt{2K}\partial_K\partial_Q \bar{g}(t,Q',K)\,\mathrm{d}K\,\mathrm{d}Q' \\ &+ \frac{1}{t}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\mathfrak{H}_{Q'}}^{\infty} \frac{1}{c'(K)} \left(\int_{\mathfrak{Q}_K}^{\frac{\pi}{2}} S(Q,K)\,\mathrm{d}Q \right) c(K)\sqrt{2K}\partial_K\partial_Q \bar{g}(t,Q',K)\,\mathrm{d}K\,\mathrm{d}Q' \\ =& -\frac{1}{t}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\mathfrak{H}_{Q'}} \partial_K\left(\frac{1}{c'(K)} \left(\int_{|Q'|}^{\frac{\pi}{2}} S(Q,K)\,\mathrm{d}Q \right) c(K)\sqrt{2K}\right)\partial_Q \bar{g}(t,Q',K)\,\mathrm{d}K\,\mathrm{d}Q' \\ &- \frac{1}{t}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\mathfrak{H}_{Q'}}^{\infty} \partial_K\left(\frac{1}{c'(K)} \left(\int_{\mathfrak{Q}_K}^{\frac{\pi}{2}} S(Q,K)\,\mathrm{d}Q \right) c(K)\sqrt{2K}\right)\partial_Q \bar{g}(t,Q',K)\,\mathrm{d}K\,\mathrm{d}Q'. \end{aligned}$$

Since there is no boundary term we can integrate by parts after writing $\partial_Q = \frac{1}{c'(K)t}(Y + \partial_K)$ once more. Next note that that $\bar{g}(Q,K)$ is nonzero only for $K \in$

[1]Since $a(\chi, H) > 0$, we have that Q is monotonically increasing as a function of χ and vice-versa.

$[c_s, c_s^{-1}]$ and that derivatives of $\frac{c(K)}{c'(K)}$ are bounded as $|c'(K)| \geq \delta$ by Lemma 4.3. Futher, $S(Q, K) = \sin\chi$ is smooth as a function of K. Thus

$$\sum_{\ell \leq 2} \partial_K^\ell \left(\frac{1}{c'(K)} \left(\int_{|Q'|}^{\pi/2} S(Q, K) \,\mathrm{d}Q \right) c(K)\sqrt{2K} \right) \lesssim 1.$$

(Notice that the second integration by parts generate boundary terms which no longer cancel, and we can no longer integrate by parts further.)

By Cauchy–Schwarz in Q' and K, we get that

$$T_1 + T_2 \lesssim \sum_{\ell \leq 2} \sup_K \|Y^\ell \bar{g}\|_{L^2_Q}.$$

Finally, an application of (16) from Lemma 5.6 followed by Lemma 4.2 gives us the required bound. □

Proof of Theorem 1.1. The proof follows by using Lemma 5.1 and combining the estimates from Proposition 5.3 and Proposition 5.7. □

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E-mail address: sanchat@stanford.edu
E-mail address: jluk@stanford.edu