ON SOFIC APPROXIMATIONS OF $\mathbb{F}_2 \times \mathbb{F}_2$

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Abstract. We construct a sofic approximation of $\mathbb{F}_2 \times \mathbb{F}_2$ that is not essentially a "branched cover" of a sofic approximation by homomorphisms. This answers a question of L. Bowen.

1. Introduction

A countable group Γ is called sofic if it admits a sequence of almost actions on finite sets which are asymptotically free. To make this precise, endow the symmetric group $\mathrm{Sym}(X)$ of any finite set X with the normalized Hamming distance: $\mathrm{d_H}(\sigma, \tau) = |X|^{-1} \cdot |\{x \in X \mid \sigma(x) \neq \tau(x)\}|$.

Definition 1.1. A sequence of maps $\sigma_n : \Gamma \to \mathrm{Sym}(X_n)$, for finite sets X_n, is called an *asymptotic homomorphism* if $\lim\limits_{n\to\infty} \mathrm{d_H}(\sigma_n(g)\sigma_n(h), \sigma_n(gh)) = 0$, for all $g, h \in \Gamma$. An asymptotic homomorphism $\sigma_n : \Gamma \to \mathrm{Sym}(X_n)$ is called a *sofic approximation* of Γ if it satisfies that $\lim\limits_{n\to\infty} \mathrm{d_H}(\sigma_n(g), \mathrm{Id}_X) = 1$, for all $g \in \Gamma \setminus \{e\}$. The group Γ is called *sofic* if it has a sofic approximation.

In recent years, the study of sofic groups has received a lot of attention. It is now understood that soficity has a number of important consequences (see, e.g., [Bo18, Th18]). This is particularly interesting because sofic groups form a broad class, which includes all amenable and all residually finite groups. Moreover, it is a longstanding open problem whether every countable group is sofic.

This note is motivated by the problem of classifying the sofic approximations of a given sofic group. For amenable groups Γ, a satisfactory classification of sofic approximations was found in [ES11]: any sofic approximation of Γ is equivalent to one constructed from a disjoint union of Følner sets. Here, we say that two sofic approximations $\sigma_n : \Gamma \to \mathrm{Sym}(X_n)$ and $\tau_n : \Gamma \to \mathrm{Sym}(X_n)$ are *equivalent* if $\lim\limits_{n\to\infty} \mathrm{d_H}(\sigma_n(g), \tau_n(g)) = 0$, for all $g \in \Gamma$ [Bo17]. If Γ is a residually finite group, then it admits a sofic approximation $\sigma_n : \Gamma \to \mathrm{Sym}(X_n)$, where each σ_n is a homomorphism. Conversely, given a residually finite group Γ, one would ideally like to show that any sofic approximation of Γ is equivalent to one consisting of homomorphisms, and thus arises from the finite quotients of Γ. In this case, Γ is called *weakly stable* [AP15] [1]. The class of weakly stable groups includes all residually finite amenable groups [AP15] and the free groups. As shown in [LLM19], surface groups satisfy a flexible variant of weak stability. On the other hand, we proved in [Io19b, Theorem D] that the product of two non-abelian free groups is not weakly stable. Consequently, $\mathbb{F}_2 \times \mathbb{F}_2$ admits a sofic approximation which does not essentially come from a sequence of homomorphisms.

Our goal here is to strengthen this result and show the failure of a more general possible classification of sofic approximations of $\mathbb{F}_2 \times \mathbb{F}_2$ proposed by L. Bowen. This is formulated using the following:

Definition 1.2. Let $\sigma_n : \Gamma \to \mathrm{Sym}(X_n)$ and $\tau_n : \Gamma \to \mathrm{Sym}(Y_n)$ be asymptotic homomorphisms of a countable group Γ. We say that (σ_n) is a *branched covering* of (τ_n) if there are onto maps $\theta_n : X_n \to Y_n$ such that $\theta_n \circ \sigma_n(g) = \tau_n(g) \circ \theta_n$, for all $g \in \Gamma$, and θ_n is a d_n-to-one, for some $d_n \in \mathbb{N}$.

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[1] This is a weakening of the notion of *stability in permutations* (or P-*stability*), requiring that any asymptotic homomorphism is equivalent to one given by homomorphisms. For a survey of recent progress on stability, see [Io19b].

Remark 1.3. Assume the setting from Definition 1.2. Then $\mathrm{d_H}(\sigma_n(g), \mathrm{Id}_{X_n}) \geq \mathrm{d_H}(\tau_n(g), \mathrm{Id}_{Y_n})$, for every $g \in \Gamma$. Thus, if (τ_n) is a sofic approximation of Γ, then so is (σ_n). The branched covering construction therefore provides a way of producing new sofic approximations from old ones.

We also remark that any branched covering (σ_n) of (τ_n) arises from a sequence of "almost cocycles" for (τ_n). Indeed, let $Z_n = \{1, 2, ..., d_n\}$ and identify $X_n = Y_n \times Z_n$ so that $\theta_n : X_n \to Y_n$ is the projection map. Then $\sigma_n(g)(y, z) = (\tau_n(g)y, c_n(g, y)z)$, where $c_n : \Gamma \times Y_n \to \mathrm{Sym}(Z_n)$ is a map satisfying $\lim_{n\to\infty} |Y_n|^{-1} \cdot |\{y \in Y_n \mid c_n(gh, y) \neq c_n(g, \tau_n(h)y)c_n(h, y)\}| = 0$, for all $g, h \in \Gamma$.

At an Oberwolfach workshop in May 2011, Bowen asked (see [OWR11, page 1463, Question 7]) if any sofic approximation $\sigma_n : \Gamma \to \mathrm{Sym}(X_n)$ of $\Gamma = \mathbb{F}_2 \times \mathbb{F}_2$ is essentially a branched covering of some sofic approximation $\tau_n : \Gamma \to \mathrm{Sym}(Y_n)$ by homomorphisms, in the following sense: there are sofic approximations (σ'_n) and (τ'_n) of Γ such that (σ_n) is equivalent to (σ'_n), (σ'_n) is branched covering of (τ'_n) and (τ'_n) is equivalent to (τ_n).

Remark 1.4. To give a better understanding of the notion of being essentially a branched covering, we record two equivalent formulations of it. Let $\sigma_n : \Gamma \to \mathrm{Sym}(X_n)$ and $\tau_n : \Gamma \to \mathrm{Sym}(Y_n)$ be asymptotic homomorphisms of a countable group Γ. Then the following conditions are equivalent:

(i) (σ_n) is essentially a branched covering of (τ_n).
(ii) there are onto, d_n-to-one maps $\theta_n : X_n \to Y_n$, for some $d_n \in \mathbb{N}$, such that we have $\lim_{n\to\infty} \mathrm{d_H}(\theta_n \circ \sigma_n(g), \tau_n(g) \circ \theta_n) = 0$, for every $g \in \Gamma$.
(iii) (σ_n) is equivalent to a branched covering (σ'_n) of (τ_n) (i.e., one can take $\tau'_n = \tau_n$ in (i)).

It is clear that (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i). That (ii) $\Rightarrow$ (iii) is a consequence of the following fact: if X, Y are finite sets, $\sigma \in \mathrm{Sym}(X)$, $\tau \in \mathrm{Sym}(Y)$, and $\theta : X \to Y$ an onto, d-to-one map, for $d \in \mathbb{N}$, then there is $\sigma' \in \mathrm{Sym}(X)$ such that $\theta \circ \sigma' = \tau \circ \theta$ and $\mathrm{d_H}(\sigma', \sigma) \leq \mathrm{d_H}(\theta \circ \sigma, \tau \circ \theta)$.

Our main result settles Bowen's question in the negative. More precisely, we prove the following:

Theorem 1.5. *Let $\Gamma = \mathbb{F}_m \times \mathbb{F}_k$, for some integers $m, k \geq 2$. Then Γ admits a sofic approximation $\sigma_n : \Gamma \to \mathrm{Sym}(X_n)$ with the following property: there are no homomorphisms $\tau_n : \Gamma \to \mathrm{Sym}(Y_n)$ and maps $\theta_n : X_n \to Y_n$, for some finite sets Y_n, such that*

(a) $\lim_{n\to\infty} \mathrm{d_H}(\theta_n \circ \sigma_n(g), \tau_n(g) \circ \theta_n) = 0$, *for all* $g \in \Gamma$, *and*
(b) $\lim_{n\to\infty} \mathrm{d_H}(\theta_n \circ \sigma_n(g), \theta_n) = 1$, *for all* $g \in \Gamma \setminus \{e\}$.

Here, for finite sets X, Y and $\sigma, \tau : X \to Y$, we denote $\mathrm{d_H}(\sigma, \tau) = |X|^{-1} \cdot |\{x \in X \mid \sigma(x) \neq \tau(x)\}|$.

Theorem 1.5 implies that (σ_n) is not essentially a branched covering of a sofic approximation by homomorphisms. This follows by using Remark 1.4 ((i) $\Rightarrow$ (ii)) and noting that if each $\theta_n : X_n \to Y_n$ is d_n-to-one, for some $d_n \in \mathbb{N}$, and (τ_n) is a sofic approximation of Γ, then (a) implies (b).

Theorem 1.5 strengthens part of [Io19b, Theorem D]. More precisely, [Io19b, Theorem D] shows that $\Gamma = \mathbb{F}_m \times \mathbb{F}_k$ is not weakly very flexibly stable, for any integers $m, k \geq 2$, in the sense of [Io19b, Definition 1.6]. This amounts to the existence of a sofic approximation $\sigma_n : \Gamma \to \mathrm{Sym}(X_n)$ with the following property: $(\star)$ there are no finite sets Y_n, homomorphisms $\tau_n : \Gamma \to \mathrm{Sym}(Y_n)$ and one-to-one maps $\theta_n : X_n \to Y_n$ such that $\lim_{n\to\infty} \mathrm{d_H}(\theta_n \circ \sigma_n(g), \tau_n(g) \circ \theta_n) = 0$, for every $g \in \Gamma$.

As we explain in the comments below, the sofic approximation (σ_n) of Γ from the hypothesis of Theorem 1.5 is constructed following the strategy introduced in [Io19b]. As such, results from [Io19b] readily imply that (σ_n) satisfies $(\star)$. The main novelty in the proof of Theorem 1.5 consists of showing that any maps $\theta_n : X_n \to Y_n$ as in its statement must be "asymptotically one-to-one".

Moreover, we prove that if $\tau_n : \Gamma \to \mathrm{Sym}(Y_n)$ are arbitrary maps, then any maps $\theta_n : X_n \to Y_n$ which satisfy conditions (a) and (b) from Theorem 1.5 must be asymptotically one-to-one. This implies that (σ_n) is a minimal sofic approximation of Γ, in the sense that it is not equivalent to a proper (i.e., one satisfying $d_n > 1$, for every n) branched covering of any sofic approximation of Γ.

Comments on the proof of Theorem 1.5. We end the introduction with an informal outline of the proof of our main result. If a group Γ satisfies the conclusion of Theorem 1.5 then any group containing it as a finite index subgroup also does (see Lemma 4.1). Therefore, it suffices to prove Theorem 1.5 when $m \geq 5$ and $k \geq 3$. Fix a free decomposition $\mathbb{F}_m = \mathbb{F}_{m-1} * \mathbb{Z}$. Thus, we have $\Gamma = \mathbb{F}_m \times \mathbb{F}_k = (\mathbb{F}_{m-1} * \mathbb{Z}) \times \mathbb{F}_k$. The proof of Theorem 1.5 is divided between Sections 3 and 4:

(1) In Section 3, we use the work [ALW01] of Alon, Lubotzky and Widgerson who proved that expansion is not a group property. This allows us to define a sequence of finite groups G_p (indexed over primes $p \equiv 1 \pmod 3$) together with onto homomorphisms $\varphi_p : \mathbb{F}_{m-1} \to G_p$ and $\rho_p : \mathbb{F}_k \to G_p$ such that $\mathbb{F}_{m-1}$ has property (τ) with respect to $\{\ker(\varphi_p)\}$, while $\mathbb{F}_k$ does not have property (τ) with respect to $\{\ker(\rho_p)\}$. (A key property of G_p is that it has only one non-trivial normal subgroup. To the best of our knowledge, it is unknown if one can find such groups G_p which are simple; if this were the case, then the proof could be simplified considerably.) Following closely [Io19b] we then construct an asymptotic homomorphism $\sigma_p : \mathbb{F}_m \times \mathbb{F}_k \to \mathrm{Sym}(G_p)$ which satisfies $(\star)$ and that $\sigma_p(g,h)x = \varphi_p(g)x\rho_p(h)^{-1}$, for every $g \in \mathbb{F}_{m-1}, h \in \mathbb{F}_k$ and $x \in G_p$. Note, however, that the asymptotic homomorphism (σ_p) is not a sofic approximation.

(2) We begin Section 4 by augmenting the construction of (σ_p) to get a sofic approximation $\widetilde{\sigma}_p : \mathbb{F}_m \times \mathbb{F}_k \to \mathrm{Sym}(\widetilde{G}_p)$ which inherits the properties of (σ_p) listed above. The rest of Section 4 is devoted to proving that $(\widetilde{\sigma}_p)$ verifies the conclusion of Theorem 1.5. Assume by contradiction that there are homomorphisms $\tau_p : \mathbb{F}_m \times \mathbb{F}_k \to \mathrm{Sym}(Y_p)$, for some finite sets Y_p, and maps $\theta_p : \widetilde{G_p} \to Y_p$ which satisfy conditions (a) and (b) from Theorem 1.5. Condition (b) implies that the partition $\{\theta_p^{-1}(\{y\}) \mid y \in Y_p\}$ of $\widetilde{G}_p$ is $\widetilde{\sigma}_p(\mathbb{F}_m \times \mathbb{F}_k)$-asymptotically invariant. By combining the property (τ) assumption with a result from [Io19a] (see Section 2), we deduce that the partition $\{\theta_p^{-1}(\{y\}) \mid y \in Y_p\}$ is asymptotically equal to the coset partition $\{gN_p \mid g \in \widetilde{G}_p\}$ of $\widetilde{G}_p$, for some normal subgroup $N_p \trianglelefteq \widetilde{G}_p$. Some additional work, which uses condition (b), allows us to conclude that $N_p = \{e\}$, and thus θ_p is asymptotically one-to-one. This however contradicts the fact that $(\widetilde{\sigma}_p)$ satisfies $(\star)$.

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2. Property (τ) and almost invariant partitions

This section is devoted to a technical lemma which will be needed in the proof of our main theorem. Let Γ be a finitely generated group, S be a finite set of generators of Γ and $\{\Gamma_n\}_{n=1}^\infty$ be a sequence of finite index normal subgroups. Denote $G_n = \Gamma/\Gamma_n$ and let $p_n : \Gamma \to G_n$ be the quotient homomorphism. The following lemma asserts that if Γ has property (τ) with respect to $\{\Gamma_n\}_{n=1}^\infty$, then any partition of G_n which is almost invariant under the left multiplication action of Γ must essentially come from the left cosets of a subgroup of G_n.

Recall that Γ is said to have *property* (τ) with respect to $\{\Gamma_n\}_{n=1}^\infty$ if $\inf_n \kappa(G_n, p_n(S)) > 0$ [Lu94]. Here, given a finite group G and a set of generators $T \subset G$, the *Kazhdan constant* $\kappa(G,T)$ denotes the largest constant $\kappa > 0$ such that $\kappa \cdot \|\xi\| \leq \max_{g\in T} \|\pi(g)\xi - \xi\|$, for every $\xi \in \mathcal{H}$ and unitary representation $\pi : G \to \mathrm{U}(\mathcal{H})$ of G on a Hilbert space $\mathcal{H}$ which has no non-zero invariant vectors. We record the following remark which will be needed in the proof of Lemma 3.4.

Remark 2.1. Let $\pi : G \to \mathrm{U}(\mathcal{H})$ be a unitary representation of a finite group G. Let P be the orthogonal projection from $\mathcal{H}$ onto the closed subspace $\mathcal{H}^G$ of $\pi(G)$-invariant vectors and $\xi \in \mathcal{H}$. Then $\max_{g\in G} \|\pi(g)\xi - \xi\| \leq 2 \cdot \|\xi - P(\xi)\|$. Since the restriction of π to $\mathcal{H} \ominus \mathcal{H}^G$ has no non-zero invariant vectors, we get that $\kappa(G,T) \cdot \|\xi - P(\xi)\| \leq \max_{g\in T} \|\pi(g)\xi - \xi\|$ and further that

$$\frac{\kappa(G,T)}{2} \cdot \max_{g\in G} \|\pi(g)\xi - \xi\| \leq \max_{g\in T} \|\pi(g)\xi - \xi\|, \text{ for every } \xi \in \mathcal{H}. \tag{2.1}$$

Lemma 2.2. [Io19a] *In the above setting, assume that Γ has property (τ) with respect to $\{\Gamma_n\}_{n=1}^\infty$. For every n, let $\{X_{n,k}\}_{k=1}^{d_n}$ be a partition of G_n, for some $d_n \in \mathbb{N}$. Assume that for every n and $g \in \Gamma$, there exists a permutation $\sigma_{n,g}$ of $\{1,...,d_n\}$ such that $\lim_{n\to\infty} \frac{1}{|G_n|} \cdot \sum_{k=1}^{d_n} |gX_{n,k} \triangle X_{n,\sigma_{n,g}(k)}| = 0$.*

Then for every n we can find a subgroup $H_n < G_n$, a set $S_n \subset \{1,...,d_n\}$ and a one-to-one map $\omega_n : S_n \to G_n/H_n$ such that

$$\lim_{n\to\infty} \frac{1}{|G_n|} \cdot \sum_{k\in S_n} |X_{n,k} \triangle \omega_n(k)H_n| = 0 \quad \textit{and} \quad \lim_{n\to\infty} \frac{1}{|G_n|} \sum_{k\notin S_n} |X_{n,k}| = 0.$$

This result is a consequence of the proof of [Io19a, Theorem A]. For the reader's convenience, we indicate briefly how the proof of [Io19a, Theorem A] can be adapted to prove Lemma 2.2.

Proof. For every n, let $\pi_n : \Gamma \to \mathrm{U}(\ell^2(G_n \times G_n))$ be the unitary representation associated to the action $\Gamma \curvearrowright G_n \times G_n$ given by $g \cdot (x,y) = (gx, gy)$, and define the unit vector

$$\eta_n = \frac{1}{\sqrt{|G_n|}} \cdot \sum_{k=1}^{d_n} \frac{1}{\sqrt{|X_{n,k}|}} \mathbf{1}_{X_{n,k}\times X_{n,k}} \in \ell^2(G_n \times G_n).$$

We claim that $\|\pi_n(g)\eta_n - \eta_n\|_2 \to 0$, for every $g \in \Gamma$. To this end, fix $g \in \Gamma$. Then the hypothesis implies that $\frac{1}{|G_n|} \cdot \sum_{k=1}^{d_n} |gX_{n,k} \cap X_{n,\sigma_{n,g}(k)}| \to 1$ and thus $\frac{1}{|G_n|} \cdot \sum_{k=1}^{d_n} \sqrt{|X_{n,k}| \cdot |X_{n,\sigma_{n,g}(k)}|} \to 1$. Using a direct computation and the Cauchy-Schwarz inequality we derive that

$$\begin{aligned}
\langle \pi_n(g)\eta_n, \eta_n\rangle &= \frac{1}{|G_n|} \cdot \sum_{k,l=1}^{d_n} \frac{1}{\sqrt{|X_{n,k}| \cdot |X_{n,l}|}} |gX_{n,k} \cap X_{n,l}|^2 \\
&\geq \frac{1}{|G_n|} \cdot \sum_{k=1}^{d_n} \frac{1}{\sqrt{|X_{n,k}| \cdot |X_{n,\sigma_{n,g}(k)}|}} |gX_{n,k} \cap X_{n,\sigma_{n,g}(k)}|^2 \\
&\geq \frac{1}{|G_n|} \cdot \frac{\left(\sum_{k=1}^{d_n} |gX_{n,k} \cap X_{n,\sigma_{n,g}(k)}|\right)^2}{\sum_{k=1}^{d_n} \sqrt{|X_{n,k}| \cdot |X_{n,\sigma_{n,g}(k)}|}}.
\end{aligned}$$

Thus, $\liminf_{n\to\infty} \langle \pi_n(g)\eta_n, \eta_n\rangle \geq 1$ and since $\|\eta_n\|_2 = 1$, we conclude that $\|\pi_n(g)\eta_n - \eta_n\|_2 \to 0$.

Since Γ has property (τ) with respect to $\{\Gamma_n\}_{n=1}^\infty$, we get that $\kappa := \inf_n \kappa(G_n, p_n(S)) > 0$. By [Io19b, Lemma 2.5] we deduce that $\sup_{g\in\Gamma} \|\pi_n(g)\eta_n - \eta_n\|_2 \leq (2/\kappa) \cdot \max_{g\in S} \|\pi_n(g)\eta_n - \eta_n\|_2$, for every n. In combination with the above it follows that $\sup_{g\in\Gamma} \|\pi_n(g)\eta_n - \eta_n\|_2 \to 0$. Thus, we can find positive real numbers δ_n such that $\delta_n \to 0$ and $\sup_{g\in\Gamma} \|\pi_n(g)\eta_n - \eta_n\|_2^2 < 2\delta_n$, for every n.

Let n large enough such that $\delta_n < 10^{-12}$. Then

$$\frac{1}{|G_n|} \cdot \sum_{k,l=1}^{d_n} \frac{1}{\sqrt{|X_{n,k}| \cdot |X_{n,l}|}} \, |gX_{n,k} \cap X_{n,l}|^2 = \langle \pi_n(g)\eta_n, \eta_n \rangle > 1 - \delta_n, \text{ for every } g \in \Gamma. \tag{2.2}$$

Note that the Haar measure m_n of G_n is given by $m_n(X) = \frac{|X|}{|G_n|}$, for every subset $X \subset G_n$. By using this fact and (2.2) and applying verbatim the second part of the proof of [Io19a, Theorem A], we can find a subgroup $H_n < G_n$, a nonempty subset $S_n \subset \{1, ..., d_n\}$ and a map $\omega_n : S_n \to G_n/H_n$ such that $\sum_{k \in S_n} |X_{n,k}| \geq (1 - \sqrt{\delta_n}) \cdot |G_n|$, and $|X_{n,k} \triangle \omega_n(k)H_n| \leq 506\sqrt[4]{\delta_n} \cdot |X_{n,k}|$, for every $k \in S_n$.

If $k, l \in S_n$ and $k \neq l$, then using that $X_{n,k} \cap X_{n,l} = \emptyset$, we get that

$$\begin{aligned} |\omega_n(k)H_n \triangle \omega_n(l)H_n| &\geq |X_{n,k} \triangle X_{l,n}| - |X_{n,k} \triangle \omega_n(k)H_n| - |X_{n,l} \triangle \omega_n(l)H_n| \\ &\geq (1 - 506\sqrt[4]{\delta_n}) \cdot (|X_{n,k}| + |X_{n,l}|). \end{aligned}$$

Since $506\sqrt[4]{\delta_n} < 1$, we derive that $\omega_n(k)H_n \triangle \omega_n(l)H_n \neq \emptyset$. This implies that the map ω_n is one-to-one and the conclusion follows. ■

3. Construction of asymptotic homomorphisms

In this section, we establish two ingredients that will be needed in the proof of our main theorem. To explain this, fix integers $m \geq 5$ and $k \geq 3$, and denote $\Gamma = \mathbb{F}_{m-1}$ and $\Lambda = \mathbb{F}_k$. In the first part of this section, given a prime p with $p \equiv 1 \pmod 3$, we construct a finite group G_p and homomorphisms $\varphi_p : \Gamma \to G_p$, $\rho_p : \Lambda \to G_p$ with various special properties. In the second part of this section, we follow closely [Io19b, Section 6] to construct an asymptotic homomorphism $\sigma_p : (\Gamma * \mathbb{Z}) \times \Lambda \to \mathrm{Sym}(G_p)$ such that $\sigma_p(g,h)x = \varphi_p(g)x\rho_p(h)^{-1}$, for all $g \in \Gamma, h \in \Lambda$ and $x \in G_p$.

3.1. **A group theoretic construction.** In [ALW01], Alon, Lubotzky and Widgerson showed that expansion is not a group property. Thus, they introduced a method of constructing sequences of finite groups $\{G_n\}_{n=1}^{\infty}$ and generating sets S_n, T_n of fixed cardinality ($|S_n| = m$, $|T_n| = k$) such that the Cayley graphs of G_n are expanders with respect to S_n but not with respect to T_n. Equivalently, there are onto homomorphisms $p_n : \mathbb{F}_m \to G_n$, $q_n : \mathbb{F}_k \to G_n$ such that $\mathbb{F}_m$ has property (τ) with respect to $\{\ker(p_n)\}_{n=1}^{\infty}$ but $\mathbb{F}_k$ does not have property (τ) with respect to $\{\ker(q_n)\}_{n=1}^{\infty}$.

The proof of our main theorem relies on a particular case of the construction of [ALW01]. Let p be a prime with $p \equiv 1 \pmod 3$. Denote by $\mathrm{P}^1(F_p) = F_p \cup \{\infty\}$ the projective line over the field F_p with p elements. Consider the action of $\mathrm{PSL}_2(F_p) = \mathrm{SL}_2(F_p)/\{\pm \mathrm{I}\}$ on $\mathrm{P}^1(F_p)$ by linear fractional transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = \frac{ax+b}{cx+d}.$$

Further, we consider the vector space $F_3^{\mathrm{P}^1(F_p)}$ over F_3, and the permutational representation of $\mathrm{PSL}_2(F_p)$ on $F_3^{\mathrm{P}^1(F_p)}$ given by $g \cdot x = (x_{g^{-1} \cdot i})_{i \in \mathrm{P}^1(F_p)}$, for every $g \in \mathrm{PSL}_2(F_p)$ and $x = (x_i)_{i \in \mathrm{P}^1(F_p)}$. We identify $F_3^{\mathrm{P}^1(F_p)}$ with F_3^{p+1} using a fixed bijection $\mathrm{P}^1(F_p) \mapsto \{1, ..., p+1\}$ which sends ∞ to $p+1$. We continue by introducing the following:

Notation 3.1. We denote $A_p = \{(x_i) \in F_3^{p+1} \mid \sum_{i=1}^{p+1} x_i = 0\}$ and $H_p = \mathrm{PSL}_2(F_p)$. Then A_p is an H_p-invariant subspace of F_3^{p+1} with $|A_p| = |F_3^{p+1}|/3 = 3^p$. We denote $G_p = A_p \rtimes H_p$.

We next record the following elementary result, whose proof we include for completeness.

Lemma 3.2. *Let K be a group and $N < G_p \times K$ be a subgroup which is normalized by $G_p \times \{e\}$. Then N is equal to $\{e\} \times L$, $A_p \times L$ or $G_p \times L$, for some subgroup $L < K$.*

Proof. First, we claim that if $N < A_p$ is a subgroup which is normalized by H_p, then $N = \{e\}$ or $N = A_p$. To this end, suppose that N contains an element $x = (x_i)_{i=1}^{p+1}$ not equal to $(0,..,0)$. Since N is H_p-invariant and H_p acts transitively on $\mathrm{P}^1(F_p)$, we may assume that $y := x_{p+1} \neq 0$. Since the subgroup $U_p = \{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in F_p\}$ of H_p fixes $\infty \in \mathrm{P}^1(F_p)$ and acts transitively on F_p, we get

$$\sum_{g \in U_p} g \cdot x = (\sum_{i=1}^{p} x_i, ..., \sum_{i=1}^{p} x_i, p x_{p+1}) = (-y, ..., -y, py) \in N.$$

Since $y \neq 0$, we get that $(-1, ..., -1, p) \in N$. Let $e_k = (x_{k,i})_{i=1}^{p+1}$, where $x_{k,i} = -1$ for $i \neq k$ and $x_{k,k} = p$. Since H_p acts transitively on $\mathrm{P}^1(F_p)$, we get that $e_k \in N$, for all $1 \leq k \leq p$. Since $3 \nmid p+1$, the vectors $(e_k)_{k=1}^{p} \in A_p$ are linearly independent over F_3. As $\dim A_p = p$, we get that $N = A_p$.

Second, we claim that if $N < G_p$ is a normal subgroup, then $N = \{e\}$, $N = A_p$ or $N = G_p$. Let $\rho : G_p \to H_p$ be the quotient homomorphism. Then $\rho(N) < H_p$ is a normal subgroup and since H_p is a simple group, $\rho(N) = \{e\}$ or $\rho(N) = H_p$. If $\rho(N) = \{e\}$, then $N < A_p$ and the first claim implies that $N = \{e\}$ or $N = A_p$. It remains to analyze the case when $\rho(N) = H_p$. We first show that $N \cap A_p \neq \{e\}$. Assume by contradiction that $N \cap A_p = \{e\}$ and let $a \in A_p \setminus \{e\}$. Since $\rho(N) = H_p$, for any $h \in H_p$, there is $b \in A_p$ such that $bh \in N$. Since $N < G_p$ is normal, $abha^{-1} \in N$ and A_p is normal and abelian, we get that $aha^{-1}h^{-1} = ab(ha^{-1}h^{-1})b^{-1} = (abha^{-1})(bh)^{-1} \in N$. Since A_p is normal, we also have that $aha^{-1}h^{-1} = a(ha^{-1}h^{-1}) \in A_p$. Thus, $aha^{-1}h^{-1} \in N \cap A_p$ and hence $aha^{-1}h^{-1} = e$, for every $h \in H_p$, which contradicts that $a \neq e$. Finally, if $N \cap A_p \neq \{e\}$, then since $N \cap A_p < A_p$ is normalized by H_p, the first claim implies that $N \cap A_p = A_p$ and hence $N \supset A_p$. Since $\rho(N) = H_p$, it follows that $N = G_p$.

Let $N < G_p \times K$ be a subgroup which is normalized by $G_p \times \{e\}$. The second claim implies that $N \cap (G_p \times \{e\})$ is equal to $\{e\}$, $A_p \times \{e\}$ or $G_p \times \{e\}$. Note that if $(g, k) \in N$, for some $g \in G_p$ and $k \in K$, then $(ghg^{-1}h^{-1}, e) = (g, k)(h, e)(g, k)^{-1}(h, e)^{-1} \in N \cap (G_p \times \{e\})$, for every $h \in G_p$. If $N \cap (G_p \times \{e\}) = \{e\}$, it follows that $N \subset \{e\} \times K$, thus $N = \{e\} \times L$, for some subgroup $L < K$. If $N \cap (G_p \times \{e\}) = A_p \times \{e\}$, we get that if $(g, k) \in N$, then $ghg^{-1}h^{-1} \in A_p$, for every $h \in G_p$, and thus $g \in A_p$. Hence $A_p \times \{e\} \subset N \subset A_p \times K$, which implies that $N = A_p \times L$, for some subgroup $L < K$. Finally, if $N \cap (G_p \times \{e\}) = G_p \times \{e\}$, then $N = G_p \times L$, for some subgroup $L < K$. ■

In addition to the notation from 3.1, throughout the rest of this paper we will use the following:

Notation 3.3. Given a prime p with $p \equiv 1 \pmod 3$, we fix a prime $r_p > p$, denote $K_p = \mathrm{PSL}_2(F_{r_p})$ and let $\psi_p : \mathrm{PSL}_2(\mathbb{Z}) \to K_p$ be the quotient homomorphism. We denote $\widetilde{G}_p = G_p \times K_p$.

The following result combines [ALW01] with a spectral result gap result from [BV12]. In its proof and later in the paper we will use the following consequence of strong approximation (see [LS03, Window 9]): if $\Gamma < \mathrm{PSL}_2(\mathbb{Z})$ is a non-amenable subgroup, then there is a finite set of primes S such that the natural homomorphism $\Gamma \mapsto \prod_{i=1}^{n} \mathrm{PSL}_2(F_{p_i})$ is onto, for any distinct primes $p_1, ..., p_n \notin S$.

Lemma 3.4. *Let $\Gamma = \mathbb{F}_{m-1}$, for $m \geq 5$. View Γ as a subgroup of $\mathrm{PSL}_2(\mathbb{Z})$. Then for any large enough prime p with $p \equiv 1 \pmod 3$, there is an onto homomorphism $\varphi_p : \Gamma \to G_p$ such that $\widetilde{\varphi}_p : \Gamma \to \widetilde{G}_p$ given by $\widetilde{\varphi}_p(g) = (\varphi_p(g), \psi_p(g))$ is onto and Γ has property (τ) with respect to $\{\ker(\widetilde{\varphi}_p)\}_p$.*

Proof. Let $a_1, ..., a_{m-1}$ be free generators of Γ and p be a prime with $p \equiv 1 \pmod 3$. Denote by $\xi_p : \mathrm{PSL}_2(\mathbb{Z}) \to H_p$ the quotient homomorphism and let $\eta_p : \mathrm{PSL}_2(\mathbb{Z}) \to H_p \times K_p$ be the homomorphism given by $\eta_p(g) = (\xi_p(g), \psi_p(g))$. Since $H_p \times K_p = \mathrm{PSL}_2(F_p) \times \mathrm{PSL}_2(F_{r_p}) \cong \mathrm{PSL}_2(\mathbb{Z}/pr_p\mathbb{Z})$ and $\langle a_1, ..., a_{m-3}\rangle \cong \mathbb{F}_{m-3}$ is a non-amenable subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ (as $m-3 \geq 2$), it follows that for large enough p we have $\eta_p(\langle a_1, ..., a_{m-3}\rangle) = H_p \times K_p$.

By applying [BV12, Theorem 1], we conclude that

$$\kappa_1 := \inf_p \kappa(H_p \times K_p, \{\eta_p(a_1), ..., \eta_p(a_{m-3})\}) > 0. \tag{3.1}$$

For $w \in A_p$, we denote by $w^{H_p} = \{h(w) = hwh^{-1} \mid h \in H_p\}$ the orbit of w under the action of H_p. By the proof of Lemma 3.2, the permutational representation of H_p on $A_p \subset F_3^{p+1}$ is irreducible. Thus, by applying [ALW01, Theorem 3.1], we can find $v_1(p), v_2(p) \in A_p \setminus \{e\}$ such that

$$\kappa_2 := \inf_p \kappa(A_p, v_1(p)^{H_p} \cup v_2(p)^{H_p}) > 0. \tag{3.2}$$

Define a homomorphism $\varphi_p : \Gamma \to G_p = A_p \rtimes H_p$ by letting $\varphi_p(a_i) = \xi_p(a_i)$, for $1 \leq i \leq m-3$, $\varphi_p(a_{m-2}) = v_1(p)\xi_p(a_{m-2})$ and $\varphi_p(a_{m-1}) = v_2(p)\xi_p(a_{m-1})$. Then $\widetilde{\varphi}_p : \Gamma \to \widetilde{G}_p$ given by $\widetilde{\varphi}_p(g) = (\varphi_p(g), \psi_p(g))$ is onto. Indeed, since $\widetilde{\varphi}_p(a_i) = \eta_p(a_i)$, for all $1 \leq i \leq m-3$, we get that $\widetilde{\varphi}_p(\Gamma)$ contains $\eta_p(\langle a_1, ..., a_{m-3}\rangle) = H_p \times K_p$. Thus, $\widetilde{\varphi}_p(\Gamma)$ also contains $(v_1(p), e) \in (A_p \setminus \{e\}) \times \{e\}$. Since A_p has no proper non-trivial H_p-invariant subgroup by the proof of Lemma 3.2, we derive that $\widetilde{\varphi}_p$ is onto. In particular, φ_p is onto.

Moreover, combining (3.1) and (3.2) as in [ALW01] implies that $\inf_p \kappa(\widetilde{G}_p, \{\widetilde{\varphi}_p(a_1), ..., \widetilde{\varphi}_p(a_{m-1})\}) > 0$. To justify this, put $\kappa := \frac{\min\{\kappa_1, \kappa_2\}}{2} > 0$. Let $\pi : \widetilde{G}_p \to \mathrm{U}(\mathcal{H})$ be a unitary representation with no non-zero invariant vectors and $\xi \in \mathcal{H}$. For $F \subset \widetilde{G}_p$, let $\Delta(F) = \max_{g \in F} \|\pi(g)\xi - \xi\|$ and note that $\Delta(F) \leq \Delta(F_1) + \Delta(F_2)$, whenever $F \subset F_1F_2$. Denote $\Delta := \Delta(\{\widetilde{\varphi}_p(a_i) \mid 1 \leq i \leq m-1\})$.

By combining (2.1) with (3.1) and (3.2) we get that

$$\kappa \cdot \Delta(H_p \times K_p) \leq \Delta(\{\widetilde{\varphi}_p(a_i) \mid 1 \leq i \leq m-3\}) \leq \Delta \quad \text{and} \tag{3.3}$$

$$\kappa \cdot \Delta(A_p \times \{e\}) \leq \Delta((v_1(p)^{H_p} \cup v_2(p)^{H_p}) \times \{e\}) = \max\{\Delta(v_1(p)^{H_p} \times \{e\}), \Delta(v_2(p)^{H_p} \times \{e\}\}. \tag{3.4}$$

Since $(v_1(p), e) \in \widetilde{\varphi}_p(a_{m-2})(H_p \times K_p)$ we derive that $v_1(p)^{H_p} \times \{e\} \subset (H_p \times K_p)\widetilde{\varphi}_p(a_{m-2})(H_p \times K_p)$. Thus, we have that

$$\Delta(v_1(p)^{H_p} \times \{e\}) \leq \Delta(\{\widetilde{\varphi}_p(a_{m-2})\}) + 2 \cdot \Delta(H_p \times K_p) \leq \Delta + 2 \cdot \Delta(H_p \times K_p). \tag{3.5}$$

Similarly, we get that

$$\Delta(v_2(p)^{H_p} \times \{e\}) \leq \Delta(\{\widetilde{\varphi}_p(a_{m-1})\}) + 2 \cdot \Delta(H_p \times K_p) \leq \Delta + 2 \cdot \Delta(H_p \times K_p). \tag{3.6}$$

By combining (3.3), (3.4), (3.5) and (3.6), we get that

$$\kappa \cdot \Delta(A_p \times \{e\}) \leq \Delta + 2 \cdot \Delta(H_p \times K_p) \leq \frac{\kappa + 2}{\kappa} \cdot \Delta. \tag{3.7}$$

Since $\widetilde{G}_p = (A_p \times \{e\})(H_p \times K_p)$, by combining (3.3) and (3.7) we derive that

$$\frac{\kappa^2}{2(\kappa+1)} \cdot \Delta(\widetilde{G}_p) \leq \frac{\kappa^2}{2(\kappa+1)}(\Delta(A_p \times \{e\}) + \Delta(H_p \times K_p)) \leq \Delta. \tag{3.8}$$

Since π has no non-zero invariant vectors, the element of minimal norm in the closure of the convex hull of $\{\pi(g)\xi \mid g \in \widetilde{G}_p\}$ is equal to 0. From this we get that $\|\xi\| \leq \Delta(\widetilde{G}_p)$. Together with (3.8), this implies that $\inf_p \kappa(\widetilde{G}_p, \{\widetilde{\varphi}_p(a_1), ..., \widetilde{\varphi}_p(a_{m-1})\}) \geq \frac{\kappa^2}{2(\kappa+1)} > 0$. Hence, Γ has property (τ) with respect to $\{\ker(\widetilde{\varphi}_p)\}_p$, which finishes the proof of the lemma. ■

Lemma 3.5. *Let $b_1, ..., b_k$ be free generators of $\Lambda = \mathbb{F}_k$, for $k \geq 3$. Then there is $C > 0$ such that for every large enough prime p with $p \equiv 1 \pmod 3$ there is an onto homomorphism $\rho_p : \Lambda \to G_p$ and a set $T_p \subset G_p$ satisfying that $\frac{1}{243} \leq \frac{|T_p|}{|G_p|} \leq \frac{1}{3}$ and $|T_p\rho_p(b_j)\triangle T_p| \leq \frac{C}{\sqrt{p}}|G_p|$, for every $1 \leq j \leq k$. Moreover, if $h \in \Lambda \setminus \{e\}$, then $\rho_p(h) \neq e$, for every large enough p.*

Proof. Let p be a prime with $p \equiv 1 \pmod 3$ and put $v(p) = (1, -1, 0, ..., 0) \in A_p$. For $x = (x_i) \in A_p$ and $j \in \{0, 1, 2\}$, denote $n_j(x) = |\{i \mid x_i = j \pmod 3\}|$. We define

$$S_p = \{x \in A_p \mid n_1(x) > n_0(x) + 2 \text{ and } n_1(x) > n_2(x) + 2\}. \tag{3.9}$$

First, we claim that

$$\frac{1}{243} \leq \frac{|S_p|}{|A_p|} \leq \frac{1}{3}. \tag{3.10}$$

Since $p \equiv 1 \pmod 3$, we have $a_1 = (2, 1, ..., 1) \in A_p$ and $a_2 = (1, 2, ..., 2) \in A_p$. Since the sets $S_p, a_1 + S_p, a_2 + S_p$ are pairwise disjoint, it follows that $|S_p| \leq \frac{|A_p|}{3}$. On the other hand, if we let $R_p = \{(x_i) \in F_3^{p-4} \mid n_1(x) \geq n_0(x) \text{ and } n_1(x) \geq n_2(x)\}$, then for every $x = (x_i) \in R_p$, there is $\tilde{x} = (\tilde{x}_i) \in S_p$ with $\tilde{x}_i = x_i$, for all $1 \leq i \leq p - 4$. Thus, $|S_p| \geq |R_p| \geq \frac{|F_3^{p-4}|}{3} = \frac{|A_p|}{243}$, proving (3.10).
Second, we claim that there is a constant $C > 0$ such that

$$|(v(p) + S_p)\triangle S_p| \leq \frac{C}{\sqrt{p}}|A_p|, \quad \text{for every } p. \tag{3.11}$$

To this end, note that Stirling's formula implies that there is a constant $c > 0$ such that

$$\binom{n}{k} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} \leq c \cdot \frac{2^n}{\sqrt{n}}, \quad \text{for every } n \geq k \geq 0. \tag{3.12}$$

Since $S_p \setminus (v(p) + S_p) \subset \{x \in F_3^{p+1} \mid n_0(x) + 5 \geq n_1(x) > n_0(x) + 2 \text{ or } n_2(x) + 5 \geq n_1(x) > n_2(x) + 2\}$ and $n_0(x), n_2(x) < \frac{p+1}{2}$, for every $x \in S_p$, by using (3.12) we get that

$$\begin{aligned}
|S_p \setminus (v(p) + S_p)| &\leq 2 \cdot \sum_{\substack{n_0+n_1+n_2=p+1\\ 0\leq n_0<\frac{p+1}{2}\\ n_2+2<n_1\leq n_2+5}} \frac{(p+1)!}{n_0!n_1!n_2!} \\
&= 2 \cdot \sum_{0\leq n_0<\frac{p+1}{2}} \binom{p+1}{n_0} \sum_{\substack{n_1+n_2=p+1-n_0\\ n_2+2<n_1\leq n_2+5}} \binom{p+1-n_0}{n_1} \\
&\leq 6c \cdot \sum_{0\leq n_0<\frac{p+1}{2}} \binom{p+1}{n_0} \frac{2^{p+1-n_0}}{\sqrt{p+1-n_0}} \\
&\leq 6c \cdot \sqrt{\frac{2}{p+1}} \cdot \sum_{0\leq n_0<\frac{p+1}{2}} \binom{p+1}{n_0} 2^{p+1-n_0} \\
&\leq 6c \cdot \sqrt{\frac{2}{p+1}} \cdot \frac{3^{p+1}}{2}
\end{aligned}$$

Since $|A_p| = 3^p$, this proves (3.11).

Let $\xi_p : \mathrm{PSL}_2(\mathbb{Z}) \to H_p = \mathrm{PSL}_2(F_p)$ be the quotient homomorphism. Since $\langle b_1, ..., b_{k-1}\rangle \cong \mathbb{F}_{k-1}$ is a non-amenable subgroup of $\mathrm{PSL}_2(\mathbb{Z})$, for every large enough p we have $\xi_p(\langle b_1, ..., b_{k-1}\rangle) = H_p$. We define a homomorphism $\rho_p : \Lambda \to G_p$ by $\rho_p(b_j) = \xi_p(b_j)$, for every $1 \leq j \leq k - 1$, and

$\rho_p(b_k) = \xi_p(b_k)v(p)$. Since A_p has no proper non-trivial H_p-invariant subgroup by the proof of Lemma 3.2, it follows that ρ_p is onto.

Next, note that S_p is H_p-invariant and let $T_p = H_p \cdot S_p = S_p \cdot H_p \subset G_p$. Then (3.10) and (3.11) imply that $\frac{|G_p|}{243} \leq |T_p| \leq \frac{|G_p|}{3}$ and $|T_p v(p) \triangle T_p| \leq \frac{C}{\sqrt{p}}|G_p|$. Since S_p is H_p-invariant, $T_p\rho_p(b_j) = T_p$, for all $1 \leq j \leq k-1$, and $T_p\rho_p(b_k) = T_p v(p)$. Hence, $|T_p\rho_p(b_j)\triangle T_p| \leq \frac{C}{\sqrt{p}}|G_p|$, for every $1 \leq j \leq k$.

Finally, since $\ker(\rho_p) \subset \ker(\xi_p) \cap \Lambda$, the moreover assertion follows. ∎

3.2. **Construction of asymptotic homomorphisms.** Assume the notation from 3.1 and 3.3, and let $\varphi_p : \Gamma \to G_p$ and $\rho_p : \Lambda \to G_p$ be the homomorphisms provided by Lemmas 3.4 and 3.5.

Lemma 3.6. *Let $t = \pm 1$ be a generator of $\mathbb{Z}$. Then there exists an asymptotic homomorphism $\sigma_p : (\Gamma * \mathbb{Z}) \times \Lambda \to \mathrm{Sym}(G_p)$, where p is a large enough prime with $p \equiv 1 \pmod 3$, so that*

(1) $\sigma_p(g, h)x = \varphi_p(g)x\rho_p(h)^{-1}$, *for every* $g \in \Gamma, h \in \Lambda, x \in G_p$,
(2) $|\{x \in G_p \mid \sigma_p(t, e)x = x\}| \geq \frac{|G_p|}{3}$,
(3) $\max\{\mathrm{d_H}(\sigma_p(t, e) \circ \sigma_p(e, h), \sigma_p(e, h) \circ \sigma_p(t, e)) \mid h \in \Lambda\} \geq \frac{1}{243}$, *and*
(4) $|\sigma_p(t, e)(A_p h)\triangle A_p h'| \geq \frac{1}{243}|A_p|$, *for every* $h, h' \in H_p$.

Proof. Define a homomorphism $\sigma_p : \Gamma\times\Lambda \to \mathrm{Sym}(G_p)$ using the formula from (1). In order to extend σ_p to an asymptotic homomorphism $\sigma_p : (\Gamma * \mathbb{Z}) \times \Lambda \to \mathrm{Sym}(G_p)$ we will define $\sigma_p(t, e) \in \mathrm{Sym}(G_p)$ such that $\lim_{p\to\infty} \mathrm{d_H}(\sigma_p(t, e)\sigma_p(e, h), \sigma_p(e, h)\sigma_p(t, e)) = 0$, for any $h \in \Lambda$.

To this end, let $a_p = (0, 0, 1..., 1) \in F_3^{p+1}$. Since $3 \mid p-1$, we get that $a_p \in A_p$. If $x \in F_3^{p+1}$ and $\tilde{x} = x - a_p$, then $n_1(\tilde{x}) - n_2(\tilde{x}) \leq n_0(x) - n_1(x) + 4$. This observation together with the definition (3.9) of S_p implies that $(a_p + S_p) \cap S_p = \emptyset$. Since $T_p = S_p \cdot H_p$, we get that $a_pT_p \cap T_p = \emptyset$.

Let $h_p \in H_p$ such that $h_p^2 \neq e$. Since $h_pT_p = T_p$, we get that $a_ph_pT_p = a_pT_p$. We can therefore define $\sigma_p(t, e) \in \mathrm{Sym}(G_p)$ by letting

$$\sigma_p(t, e)x = \begin{cases} a_ph_px, \text{ if } x \in T_p, \\ (a_ph_p)^{-1}x, \text{ if } x \in a_pT_p, \\ x, \text{ otherwise.} \end{cases}$$

Equivalently, for every $a \in A_p$ and $h \in H_p$ we have that

$$\sigma_p(t, e)(ah) = \begin{cases} a_ph_pah, \text{ if } a \in S_p, \\ (a_ph_p)^{-1}(ah), \text{ if } a \in a_pS_p, \\ ah, \text{ otherwise} \end{cases} \tag{3.13}$$

By Lemma 3.5 we have that $\lim_{p\to\infty} \frac{|T_p\rho_p(h)\triangle T_p|}{|G_p|} = 0$, for every $h \in \Lambda$.

To continue the proof we will need the following result extracted from [Io19b].

Lemma 3.7. [Io19b] *Assuming the above notation, we have*

(1) $\lim_{p\to\infty} \mathrm{d_H}(\sigma_p(t, e)\sigma_p(e, h), \sigma_p(e, h)\sigma_p(t, e)) = 0$, *for every* $h \in \Lambda$, *and*
(2) $\mathrm{d_H}(\sigma_p(t, e)\sigma_p(e, h), \sigma_p(e, h)\sigma_p(t, e)) \geq \frac{2|T_p\setminus T_p\rho_p(h)|}{|G_p|}$, *for every* $h \in \Lambda$.

The first part of Lemma 3.7 implies that (σ_p) is an asymptotic homomorphism which satisfies condition (1) by construction. Since $|T_p| \leq \frac{|G_p|}{3}$ by Lemma 3.5, condition (2) is satisfied. Since $\sum_{k\in G_p}|T_p \setminus T_pk| = \sum_{k\in G_p}(|T_p| - |T_p \cap T_pk|) = (|G_p| - |T_p|)\cdot|T_p|$, there is $k \in G_p$ such that $|T_p \setminus T_pk| \geq (1-\frac{|T_p|}{|G_p|})\cdot\frac{|T_p|}{|G_p|}\cdot|G_p|$. By Lemma 3.5 we get that $|T_p\setminus T_pk| \geq \frac{1}{2}\cdot\frac{1}{243}\cdot|G_p|$. Since ρ_p is onto, there is $h \in \Lambda$ such that $k = \rho_p(h)$ and hence $|T_p \setminus T_p\rho_p(h)| \geq \frac{1}{2}\cdot\frac{1}{243}\cdot|G_p|$. In combination with the second part of Lemma 3.7, we deduce condition (3).

To prove condition (4), fix $h, h' \in H_p$. Since $h_p^2 \neq e$, we have $h_ph \neq h'$ or $h_p^{-1}h \neq h'$. If $h_ph \neq h'$, then (3.13) implies that $S_p \subset \{a \in A_p \mid \sigma_p(t,e)(ah) \notin A_ph'\}$. If $h_p^{-1}h \neq h'$, then (3.13) implies that $a_pS_p \subset \{a \in A_p \mid \sigma_p(t,e)(ah) \notin A_ph'\}$. In either case, $|\{a \in A_p \mid \sigma_p(t,e)(ah) \notin A_ph'\}| \geq |S_p|$ and thus $|\sigma_p(t,e)(A_ph) \setminus A_ph'| \geq |S_p|$. By equation (3.10), this implies condition (4). ■

Proof of Lemma 3.7. The first part follows by arguing as in the proof of [Io19b, Lemma 6.1] using [Io19b, Lemma 2.2]. Since $(a_ph_p)^2 \neq e$, the second part follows from [Io19b, Lemma 2.2]. ■

Note that condition (2) of Lemma 3.6 implies that (σ_p) is not a sofic approximation of $(\Gamma * \mathbb{Z}) \times \Lambda$. In the next section, we will first build a sofic approximation $(\widetilde{\sigma}_p)$ of $(\Gamma * \mathbb{Z}) \times \Lambda$ out of (σ_p), which we will use to show that the conclusion of our main theorem holds for $(\Gamma * \mathbb{Z}) \times \Lambda = \mathbb{F}_m \times \mathbb{F}_k$.

4. Proof of the main theorem

This section is devoted to the proof of the main theorem. We will first prove the conclusion when $m \geq 5$ and $k \geq 3$. To this end, put $\Gamma = \mathbb{F}_{m-1}, \Sigma = \mathbb{F}_m$ and $\Lambda = \mathbb{F}_k$. We assume the notation from (3.1) and (3.3): $H_p = \mathrm{PSL}_2(F_p)$, $G_p = A_p \rtimes H_p$, $K_p = \mathrm{PSL}_2(F_{r_p})$, $\widetilde{G}_p = G_p \times K_p$, $\psi_p : \mathrm{PSL}_2(\mathbb{Z}) \to K_p$ is the quotient homomorphism, where $p < r_p$ are primes and $p \equiv 1 \pmod 3$.

In the first part of the proof, we construct a sofic approximation $\widetilde{\sigma}_p : \Sigma \times \Lambda \to \mathrm{Sym}(\widetilde{G}_p)$ of $\Sigma \times \Lambda$. View Σ, and thus Γ, as a subgroup of $\mathrm{PSL}_2(\mathbb{Z})$. Let $\varphi_p : \Gamma \to G_p$, $\widetilde{\varphi}_p : \Gamma \to \widetilde{G}_p$ and $\rho_p : \Lambda \to G_p$ be the onto homomorphisms given by Lemmas 3.4 and 3.5. Recall that $\widetilde{\varphi}_p(g) = (\varphi_p(g), \psi_p(g))$, for every $g \in \Gamma$. Let $\sigma_p : \Sigma \times \Lambda = (\Gamma * \mathbb{Z}) \times \Lambda \to \mathrm{Sym}(G_p)$ be the asymptotic homomorphism provided by Lemma 3.6. As therein, we denote by $t = \pm 1$ a generator of $\mathbb{Z}$.

Next, for any large enough prime p with $p \equiv 1 \pmod 3$, we define onto homomorphisms $\zeta_p : \Lambda \to K_p$ and $\widetilde{\rho}_p : \Lambda \to \widetilde{G}_p$. Fix a decomposition $\Lambda = \Delta * \mathbb{Z}$, where $\Delta = \mathbb{F}_{k-1}$ and view Δ as a subgroup of $\mathrm{PSL}_2(\mathbb{Z})$. Define $\zeta_p : \Lambda \to K_p$ by $\zeta_p(h) = \psi_p(h)$, if $h \in \Delta$, and $\zeta_p(h) = e$, if $h \in \mathbb{Z}$. Define $\widetilde{\rho}_p : \Lambda \to \widetilde{G}_p$ by $\widetilde{\rho}_p(h) = (\rho_p(h), \zeta_p(h))$, for all $h \in \Lambda$. Since Δ is non-amenable, ζ_p is onto, for large enough p. By Lemma 3.2, the only quotient groups of G_p are G_p, H_p and $\{e\}$. Since K_p is a simple group which is not isomorphic to neither G_p nor H_p, and ρ_p is onto, Goursat's lemma implies that $\widetilde{\rho}_p$ is onto, for large enough p.

We are now ready to define $\widetilde{\sigma}_p : \Sigma \times \Lambda \to \mathrm{Sym}(\widetilde{G}_p)$ by letting for $g \in \Sigma, h \in \Lambda, x \in G_p$ and $y \in K_p$

$$\widetilde{\sigma}_p(g,h)(x,y) = (\sigma_p(g,h)x, \psi_p(g)y\zeta_p(h)^{-1}) \tag{4.1}$$

Since Lemma 3.6 gives that $\sigma_p(g,h)x = \varphi_p(g)x\rho_p(h)^{-1}$, for all $g \in \Gamma, h \in \Lambda, x \in G_p$, we derive that

$$\widetilde{\sigma}_p(g,h)x = \widetilde{\varphi}_p(g)x\widetilde{\rho}_p(h)^{-1}, \text{ for all } g \in \Gamma, h \in \Lambda, x \in \widetilde{G}_p. \tag{4.2}$$

Claim 1. *$(\widetilde{\sigma}_p)$ is a sofic approximation of $\Sigma \times \Lambda$.*

Proof of Claim 1. It is clear that $(\widetilde{\sigma}_p)$ is an asymptotic homomorphism of $\Sigma \times \Lambda$. To see that it is a sofic approximation, let $C_L(g) = \{y \in L \mid gy = yg\}$ be the centralizer of an element y of a group

L. For every prime q and $g \in \mathrm{SL}_2(F_q) \setminus \{\pm I\}$, we have $|C_{\mathrm{SL}_2(F_q)}(g)| \leq \frac{|\mathrm{SL}_2(F_q)|}{q-1}$. This implies that

$$|C_{\mathrm{PSL}_2(F_q)}(g)| \leq \frac{|\mathrm{PSL}_2(F_q)|}{2(q-1)}, \text{ for every } g \in \mathrm{PSL}_2(F_q) \setminus \{e\}. \tag{4.3}$$

On the other hand, for all $(g,h) \in \Sigma \times \Lambda$ we have

$$\frac{|\{x \in \widetilde{G}_p \mid \widetilde{\sigma}_p(g,h)x = x\}|}{|\widetilde{G}_p|} \leq \frac{|\{y \in K_p \mid \psi_p(g)y\zeta_p(h)^{-1} = y\}|}{|K_p|} \leq \frac{|C_{K_p}(\psi_p(g))|}{|K_p|}. \tag{4.4}$$

If $g \neq e$, then $\psi_p(g) \neq e$, for large enough p. Therefore, by combining (4.3) and (4.4) we get that $\frac{|\{x \in \widetilde{G}_p | \widetilde{\sigma}_p(g,h)x = x\}|}{|\widetilde{G}_p|} \leq \frac{1}{2(r_p-1)}$, for large enough p. Thus, $\lim_{p\to\infty} \mathrm{d_H}(\widetilde{\sigma}_p(g,h), \mathrm{Id}_{\widetilde{G}_p}) = 1$, for all $(g,h) \in \Sigma \times \Lambda$ with $g \neq e$. If $h \in \Lambda \setminus \{e\}$, then $\widetilde{\sigma}_p(e,h)(x,y) = (x\rho_p(h)^{-1}, y\zeta_p(h)^{-1})$. Lemma 3.5 gives that $\rho_p(h) \neq e$ and thus $\mathrm{d_H}(\widetilde{\sigma}_p(e,h), \mathrm{Id}_{\widetilde{G}_p}) = 1$, for large enough p. This altogether proves that $(\widetilde{\sigma}_p)$ is a sofic approximation of $\Sigma \times \Lambda$. □

In the rest of the proof, we will show that the sofic approximation $(\widetilde{\sigma}_p)$ of $\Sigma \times \Lambda$ satisfies the conclusion of the main theorem. Towards this goal, let $\tau_p : \Sigma \times \Lambda \to \mathrm{Sym}(Y_p)$ be a sequence homomorphisms, for some finite sets Y_p, for which there exist maps $\theta_p : \widetilde{G}_p \to Y_p$ such that

(i) $\mathrm{d_H}(\theta_p \circ \widetilde{\sigma}_p(g), \tau_p(g) \circ \theta_p) \to 0$, for every $g \in \Sigma \times \Lambda$, and
(ii) $\mathrm{d_H}(\theta_p \circ \widetilde{\sigma}_p(g), \theta_p) \to 1$, for every $g \in (\Sigma \times \Lambda) \setminus \{e\}$.

For every p and $y \in Y_p$, we denote $X_p^y = \theta_p^{-1}(\{y\})$. We continue with the following:

Claim 2. *For every p there exist a normal subgroup $N_p < \widetilde{G}_p$, a subset $S_p \subset Y_p$ and a map $\omega_p : S_p \to \widetilde{G}_p$ such that*

$$\lim_{p\to\infty} \frac{1}{|\widetilde{G}_p|} \cdot \sum_{y \in S_p} |X_p^y \triangle \omega_p(y) N_p| = 0 \qquad \textit{and} \tag{4.5}$$

$$\lim_{p\to\infty} \frac{1}{|\widetilde{G}_p|} \cdot \sum_{y \notin S_p} |X_p^y| = 0. \tag{4.6}$$

Moroever, for every $g \in \Sigma \times \Lambda$, we have that $\lim_{p\to\infty} \frac{|N_p| \cdot |S_p \cap \tau_p(g)^{-1} S_p|}{|\widetilde{G}_p|} = 1$.

Proof of Claim 2. If $g \in \Sigma \times \Lambda$, then

$$\bigcup_{y \in Y_p} (\widetilde{\sigma}_p(g) X_p^y \triangle X_p^{\tau_p(g)(y)}) \subset \{x \in \widetilde{G}_p \mid \theta_p(\widetilde{\sigma}_p(g)^{-1}x) \neq \tau_p(g)^{-1}(\theta_p(x))\}.$$

Note that every $x \in \widetilde{G}_p$ belongs to at most two sets of the form $\widetilde{\sigma}_p(g) X_p^y \triangle X_p^{\tau_p(g)(y)}$, with $y \in Y_p$. Since $\mathrm{d_H}(\theta_p \circ \widetilde{\sigma}_p(g)^{-1}, \tau_p(g)^{-1} \circ \theta_p) \to 0$ by (i), we deduce that the partition $\{X_p^y\}_{y \in Y_p}$ of $\widetilde{G}_p$ is almost invariant under $\widetilde{\sigma}_p$, in the following sense:

$$\lim_{p\to\infty} \frac{1}{|\widetilde{G}_p|} \cdot \sum_{y \in Y_p} |\widetilde{\sigma}_p(g) X_p^y \triangle X_p^{\tau_p(g)(y)}| = 0, \text{ for every } g \in \Sigma \times \Lambda. \tag{4.7}$$

By Lemma 3.4, $\widetilde{\varphi}_p : \Gamma \to \widetilde{G}_p$ is an onto homomorphism such that Γ has property (τ) with respect to $\{\ker(\widetilde{\varphi}_p)\}$. Moreover, (4.2) gives that $\widetilde{\sigma}_p(g,e)(x) = \widetilde{\varphi}_p(g)x$, for all $g \in \Gamma$ and $x \in \widetilde{G}_p$. Since equation (4.7) holds for all $g \in \Gamma$, we can apply Lemma 2.2 to deduce the existence of a subgroup $N_p < \widetilde{G}_p$, a subset $S_p \subset Y_p$ and a map $\omega_p : S_p \to \widetilde{G}_p$, for every p, such that (4.5) and (4.6) hold.

To finish the proof of the claim, it remains to show that $N_p < \widetilde{G}_p$ is a normal subgroup and the moreover assertion holds. Combining (4.6) and (4.7) gives that $\lim_{p\to\infty} \frac{1}{|\widetilde{G}_p|} \cdot \sum_{y\notin S_p} |X_p^{\tau_p(g)(y)}| = 0$ and therefore

$$\lim_{p\to\infty} \frac{1}{|\widetilde{G}_p|} \cdot \sum_{y\notin \tau_p(g)^{-1}S_p} |X_p^y| = 0, \text{ for every } g \in \Sigma\times\Lambda.$$

Since $\sum_{y\in Y_p} |X_p^y| = |\widetilde{G}_p|$, this together with (4.6) implies that

$$\lim_{p\to\infty} \frac{1}{|\widetilde{G}_p|} \cdot \sum_{y\in S_p\cap\tau_p(g)^{-1}S_p} |X_p^y| = 1, \text{ for every } g \in \Sigma\times\Lambda. \tag{4.8}$$

By combining (4.5) and (4.8), the moreover assertion follows.

On the other hand, combining (4.5) and (4.7) gives that

$$\lim_{p\to\infty} \frac{1}{|\widetilde{G}_p|} \cdot \sum_{y\in S_p\cap\tau_p(g)^{-1}S_p} |\widetilde{\sigma}_p(g)(\omega_p(y)N_p)\triangle\omega_p(\tau_p(g)(y))N_p| = 0, \text{ for every } g \in \Sigma\times\Lambda. \tag{4.9}$$

Let $h \in \Lambda$. Then (4.9) and the moreover assertion imply that for every large enough p, there is $y \in S_p \cap \tau_p(e,h)^{-1}S_p$ such that $|\widetilde{\sigma}_p(e,h)(\omega_p(y)N_p)\triangle\omega_p(\tau_p(e,h)(y))N_p| < |N_p|$. Since by (4.2) we have $\widetilde{\sigma}_p(e,h)(x) = x\widetilde{\rho}_p(h)^{-1}$, for all $x \in \widetilde{G}_p$, we get $|\omega_p(y)N_p\widetilde{\rho}_p(h)^{-1}\triangle\omega_p(\tau_p(e,h)(y))N_p| < |N_p|$. Thus, if we put $a = \widetilde{\rho}_p(h)\omega_p(y)^{-1}\omega_p(\tau_p(e,h)(y)) \in \widetilde{G}_p$, then $|\widetilde{\rho}_p(h)N_p\widetilde{\rho}_p(h)^{-1}\triangle aN_p| < |N_p|$. Equivalently, we have that $|\widetilde{\rho}_p(h)N_p\widetilde{\rho}_p(h)^{-1}\cap aN_p| > \frac{|N_p|}{2}$. If $x \in \widetilde{\rho}_p(h)N_p\widetilde{\rho}_p(h)^{-1}\cap aN_p$, then $x^{-1}(\widetilde{\rho}_p(h)N_p\widetilde{\rho}_p(h)^{-1}\cap aN_p) \subset \widetilde{\rho}_p(h)N_p\widetilde{\rho}_p(h)^{-1}\cap N_p$. Thus, we get that $|\widetilde{\rho}_p(h)N_p\widetilde{\rho}_p(h)^{-1}\cap N_p| > \frac{|N_p|}{2}$ Since N_p is a finite group, it follows that $\widetilde{\rho}_p(h)N_p\widetilde{\rho}_p(h)^{-1} = N_p$, for every large enough prime p. Since this holds for every $h \in \Lambda$, Λ is finitely generated and $\widetilde{\rho}_p : \Lambda \to \widetilde{G}_p$ is onto, we derive that $N_p < \widetilde{G}_p$ is a normal subgroup. $\square$

Since $\widetilde{G}_p = G_p \times K_p$ and K_p is a simple group, Lemma 3.2 implies that N_p is one of the following six groups: $N_p^1 = \{e\}, N_p^2 = \{e\}\times K_p, N_p^3 = A_p\times K_p, N_p^4 = \widetilde{G}_p, N_p^5 = G_p\times\{e\}$ or $N_p^6 = A_p\times\{e\}$, for every large enough prime p. We continue with the following:

Claim 3. *$N_p = \{e\}$, for every large enough prime p.*

Proof of Claim 3. Assume that the claim is false. Then, after replacing $(\widetilde{\sigma}_p)$ with a subsequence, we may assume that there is $2 \leq i \leq 6$ such that $N_p = N_p^i$, for every p. We will prove that each of these five possibilities leads to a contradiction. To this end, note that if $g \in \Sigma\times\Lambda$, then $\cup_{y\in Y_p}(X_p^y \cap \widetilde{\sigma}_p(g)X_p^y) = \{x \in \widetilde{G}_p \mid (\theta_p\circ\widetilde{\sigma}_p(g)^{-1})(x) = \theta_p(x)\}$. Thus, by using (ii), we get that

$$\lim_{p\to\infty} \frac{1}{|\widetilde{G}_p|} \cdot \sum_{y\in Y_p} |X_p^y \cap \widetilde{\sigma}_p(g)X_p^y| = 0, \text{ for every } g \in (\Sigma\times\Lambda)\setminus\{e\}. \tag{4.10}$$

Case 1. *$N_p = L_p\times K_p$, where L_p is one of the groups $\{e\}, A_p$ or G_p.*

In this case, we will derive a contradiction by using condition (2) of Lemma 3.6. Note first that for every $y \in Y_p$, we have that $\omega_p(y)N_p = V_p^y \times K_p$, for some set $V_p^y \subset G_p$. By (4.1) we have that $\widetilde{\sigma}_p(t,e)(x,y) = (\sigma_p(t,e)x, \psi_p(t)y)$, for all $x \in G_p$ and $y \in K_p$. Thus, we get that

$$\widetilde{\sigma}_p(t,e)(\omega_p(y)N_p) = \sigma_p(t,e)V_p^y \times K_p = (\sigma_p(t,e)\times \mathrm{Id}_{K_p})(\omega_p(y)N_p).$$

This implies that $\omega_p(y)N_p \cap \widetilde{\sigma}_p(t,e)(\omega_p(y)N_p) \supset \{x \in \omega_p(y)N_p \mid (\sigma_p(t,e)\times \mathrm{Id}_{K_p})x = x\}$ and hence

$$\sum_{y\in S_p} |\omega_p(y)N_p \cap \widetilde{\sigma}_p(t,e)(\omega_p(y)N_p)| \geq |\{x \in \bigcup_{y\in S_p} \omega_p(y)N_p \mid (\sigma_p(t,e)\times \mathrm{Id}_{K_p})(x) = x\}|$$
$$\geq \{x \in \widetilde{G}_p \mid (\sigma_p(t,e)\times \mathrm{Id}_{K_p})(x) = x\}| - |\widetilde{G}_p \setminus (\bigcup_{y\in S_p} \omega_p(y)N_p))|.$$

Since $|\{x \in G_p \mid \sigma_p(t,e)x = x\}| \geq \frac{|G_p|}{3}$ by condition (2) of Lemma 3.6, while (4.5) and (4.6) imply that $\lim\limits_{p\to\infty} \frac{|\widetilde{G}_p \setminus (\cup_{y\in S_p}\omega_p(y)N_p)|}{|\widetilde{G}_p|} = 0$, we conclude that

$$\limsup_{p\to\infty} \frac{1}{|\widetilde{G}_p|}\cdot \sum_{y\in S_p} |\omega_p(y)N_p \cap \widetilde{\sigma}_p(t,e)(\omega_p(y)N_p)| \geq \frac{1}{3}.$$

In combination with (4.5) this gives that

$$\limsup_{p\to\infty} \frac{1}{|\widetilde{G}_p|}\cdot \sum_{y\in S_p} |X_p^y \cap \widetilde{\sigma}_p(t,e)X_p^y| \geq \frac{1}{3},$$

which contradicts (4.10).

Case 2. $N_p = G_p \times \{e\}$.

In this case, for every $y \in Y_p$ we have that $\omega_p(y)N_p = G_p \times W_p^y$, for some set $W_p^y \subset K_p$. Recall that by the construction of the homomorphism $\zeta_p : \Lambda \to K_p$ we can find $h \in \Lambda \setminus \{e\}$ such that $\zeta_p(h) = e$. Then $\widetilde{\rho}_p(h) = (\rho_p(h), e)$ and (4.2) gives that $\widetilde{\sigma}_p(e,h)(x,y) = (x\rho_p(h)^{-1}, y)$ for all $x \in G_p$ and $y \in K_p$. Hence $\widetilde{\sigma}_p(e,h)(\omega_p(y)N_p) = \omega_p(y)N_p$, for every $y \in Y_p$. In combination with (4.5), this implies that $\limsup_{p\to\infty} \frac{1}{|\widetilde{G}_p|}\cdot\sum_{y\in S_p} |X_p^y \cap \widetilde{\sigma}_p(e,h)X_p^y| \geq 1$, which contradicts (4.10).

Case 3. $N_p = A_p \times \{e\}$.

In this case, let $x, x' \in G_p$ and $y, y' \in K_p$. Then $(x,y)N_p = A_px \times \{y\}$ and $(x',y')N_p = A_px' \times \{y'\}$. By using the definition (4.1) of $\widetilde{\sigma}_p$ we derive that $\widetilde{\sigma}_p(t,e)((x,y)N_p) = \sigma_p(t,e)(A_px) \times \{\psi_p(t)y\}$. In combination with condition (4) from Lemma 3.6, we conclude that

$$|\widetilde{\sigma}_p(t,e)((x,y)N_p) \triangle (x',y')N_p| \geq |\sigma_p(t,e)(A_px)\triangle(A_px')| \geq \frac{|A_p|}{243} = \frac{|N_p|}{243}.$$

This inequality implies that

$$\frac{1}{|\widetilde{G}_p|}\cdot \sum_{y\in S_p\cap\tau_p(t,e)^{-1}S_p} |\widetilde{\sigma}_p(t,e)(\omega_p(y)N_p)\triangle\omega_p(\tau_p(t,e)(y))N_p| \geq \frac{|N_p|\cdot|S_p\cap\tau_p(t,e)^{-1}S_p|}{243\cdot|\widetilde{G}_p|}. \tag{4.11}$$

On the other hand, (4.9) and the moreover assertion of Lemma 2 imply that the left and right sides of (4.11) converge to 0 and $\frac{1}{243}$, as $p \to \infty$, respectively. This gives a contradiction.

This altogether finishes the proof of Claim 3. $\square$

We next claim that θ_p is "asymptotically one-to-one": there exists a set $X_p \subset \widetilde{G}_p$ such that $\theta_{p|X_p}$ is one-to-one and $\lim\limits_{p\to\infty}\frac{|X_p|}{|\widetilde{G}_p|} = 1$. To see this, let $S_p' = \{y \in S_p \mid |X_p^y| = 1\}$. If $y \in S_p \setminus S_p'$, then $|X_p^y| \neq 1$. Since $N_p = \{e\}$ we have that $|\omega_p(y)N_p| = 1$ and thus $|X_p^y \triangle \omega_p(y)N_p| \geq \frac{|X_p^y|}{2}$. In combination with (4.5) we derive that $\lim\limits_{p\to\infty} \frac{1}{|\widetilde{G}_p|}\cdot\sum_{y\in S_p\setminus S_p'} |X_p^y| = 0$. Together with (4.6) this further implies that $\lim_{p\to\infty}\frac{1}{|\widetilde{G}_p|}\cdot\sum_{y\notin S'_p}|X_p^y|=0$. Thus, if we let $X_p=\cup_{y\in S'_p}X_p^y$, then $\lim_{p\to\infty}\frac{|X_p|}{|\widetilde{G}_p|}=1$. Since $|X_p^y|=1$, for every $y\in S'_p$, we also have that $\theta_{p|X_p}$ is one-to-one, which proves our claim.

Finally, we will use that $\tau_p:\Sigma\times\Lambda\to\mathrm{Sym}(Y_p)$ is a homomorphism, for every p, in combination with [Io19b, Theorem 5.1] to derive a contradiction. Consider the disjoint union $Z_p=Y_p\sqcup(\widetilde{G}_p\setminus X_p)$ and extend τ_p to a homomorphism $\tau_p:\Sigma\times\Lambda\to\mathrm{Sym}(Z_p)$ by letting $\tau_p(g)_{|\widetilde{G}_p\setminus X_p}=\mathrm{Id}_{\widetilde{G}_p\setminus X_p}$, for every $g\in\Sigma\times\Lambda$. We also define a one-to-one map $\Theta_p:\widetilde{G}_p\to Z_p$ by letting

$$\Theta_p(x)=\begin{cases}\theta_p(x), \text{ if } x\in X_p, \text{ and}\\ x, \text{ if } x\in\widetilde{G}_p\setminus X_p.\end{cases}$$

If $g\in\Sigma\times\Lambda$, then $\Theta_p(x)=\theta_p(x)$ and $\Theta_p(\widetilde{\sigma}_p(g)x)=\theta_p(\widetilde{\sigma}_p(g)x)$, for every $x\in X_p\cap\widetilde{\sigma}_p(g)^{-1}X_p$. From this it follows that

$$\mathrm{d_H}(\Theta_p\circ\widetilde{\sigma}_p(g),\tau_p(g)\circ\Theta_p)\leq\mathrm{d_H}(\theta_p\circ\widetilde{\sigma}_p(g),\tau_p(g)\circ\theta_p)+\frac{|\widetilde{G}_p\setminus(X_p\cap\widetilde{\sigma}_p(g)^{-1}X_p)|}{|\widetilde{G}_p|}.$$

By using (i), that $|\widetilde{G}_p\setminus(X_p\cap\widetilde{\sigma}_p(g)^{-1}X_p)|\leq 2\cdot|\widetilde{G}_p\setminus X_p|$ and that $\lim_{p\to\infty}\frac{|X_p|}{|\widetilde{G}_p|}=1$, we deduce that

$$\lim_{p\to\infty}\mathrm{d_H}(\Theta_p\circ\widetilde{\sigma}_p(g),\tau_p(g)\circ\Theta_p)=0, \text{ for every } g\in\Sigma\times\Lambda. \tag{4.12}$$

Note that $\widetilde{\sigma}_p(g,h)x=\widetilde{\varphi}_p(g)x\widetilde{\rho}_p(h)^{-1}$, for all $g\in\Gamma,h\in\Lambda,x\in\widetilde{G}_p$ by (4.2), $\widetilde{\varphi}_p$ is onto and Γ has property (τ) with respect to $\{\ker(\widetilde{\varphi}_p)\}_p$ by Lemma 3.4. Since τ_p is a homomorphism and Θ_p is one-to-one, for every p, we can apply [Io19b, Theorem 5.1] to conclude that

$$\lim_{p\to\infty}\Big(\max\{\mathrm{d_H}(\widetilde{\sigma}_p(t,e)\circ\widetilde{\sigma}_p(e,h),\widetilde{\sigma}_p(e,h)\circ\widetilde{\sigma}_p(t,e))\mid h\in\Lambda\}\Big)=0.$$

Using the definition 4.1 of $\widetilde{\sigma}_p$, this can be equivalently written as

$$\lim_{p\to\infty}\Big(\max\{\mathrm{d_H}(\sigma_p(t,e)\circ\sigma_p(e,h),\sigma_p(e,h)\circ\sigma_p(t,e))\mid h\in\Lambda\}\Big)=0,$$

which contradicts condition (3) from Lemma 3.6.

This finishes the proof of our main theorem in the case $m\geq 5$ and $k\geq 3$. In the general case, when $m,k\geq 2$, note first that $\mathbb{F}_{m+3}\times\mathbb{F}_{k+1}$ satisfies the conclusion by the above. Since $\mathbb{F}_{m+3}\times\mathbb{F}_{k+1}$ can be realized as a finite index subgroup of $\mathbb{F}_m\times\mathbb{F}_k$, the following lemma implies that the conclusion also holds for $\mathbb{F}_m\times\mathbb{F}_k$. ■

Consider the following property for a sofic approximation $\sigma_n:\Gamma_0\to\mathrm{Sym}(X_n)$ of a countable group Γ_0: $(\diamond)$ there are a sofic approximation $\tau_n:\Gamma_0\to\mathrm{Sym}(Y_n)$ and maps $\theta_n:X_n\to Y_n$ such that

(a) τ_n is a homomorphism, for every $n\in\mathbb{N}$,
(b) $\mathrm{d_H}(\theta_n\circ\sigma_n(g),\tau_n(g)\circ\theta_n)\to 0$, for every $g\in\Gamma_0$, and
(c) $\mathrm{d_H}(\theta_n\circ\sigma_n(g),\theta_n)\to 1$, for every $g\in\Gamma_0\setminus\{e\}$.

Let $\Gamma_0<\Gamma$ be a finite index inclusion of countable groups. The next lemma shows that if Γ_0 has a sofic approximation which fails $(\diamond)$, then so does Γ. The proof of this fact relies on an induction argument, following closely [Io19b, Section 3.3]. Let $s:\Gamma/\Gamma_0\to\Gamma$ be a map such that $s(e\Gamma_0)=e$ and $s(g\Gamma_0)\in g\Gamma_0$, for every $g\in\Gamma$. Then $c:\Gamma\times\Gamma/\Gamma_0\to\Gamma_0$ given by $c(g,h\Gamma_0)=s(gh\Gamma_0)^{-1}gs(h\Gamma_0)$ is a cocycle for the left multiplication action $\Gamma\curvearrowright\Gamma/\Gamma_0$.

Let $\sigma_n:\Gamma_0\to\mathrm{Sym}(X_n)$ be a sofic approximation of Γ_0 and define $\mathrm{Ind}_{\Gamma_0}^{\Gamma}(\sigma_n):\Gamma\to\mathrm{Sym}(\Gamma/\Gamma_0\times X_n)$ by letting $\mathrm{Ind}_{\Gamma_0}^{\Gamma}(\sigma_n)(g)(h\Gamma_0,x)=(gh\Gamma_0,\sigma_n(c(g,h\Gamma_0))x)$, for every $g\in\Gamma,h\Gamma_0\in\Gamma/\Gamma_0$ and $x\in X_n$. The proof of [Io19b, Lemma 3.3] shows that $\mathrm{Ind}_{\Gamma_0}^{\Gamma}(\sigma_n)$ is a sofic approximation of Γ.

Lemma 4.1. *If the induced sofic approximation* $(\mathrm{Ind}_{\Gamma_0}^{\Gamma}(\sigma_n))$ *satisfies* $(\diamond)$*, then* (σ_n) *satisfies* $(\diamond)$.

Proof. Assume that $\widetilde{\sigma}_n := \mathrm{Ind}_{\Gamma_0}^{\Gamma}(\sigma_n) : \Gamma \to \mathrm{Sym}(\widetilde{X}_n)$ satisfies $(\diamond)$, where $\widetilde{X}_n = \Gamma/\Gamma_0 \times X_n$. Let $\tau_n : \Gamma \to \mathrm{Sym}(Y_n)$ be a sofic approximation by homomorphisms and $\widetilde{\theta}_n : \widetilde{X}_n \to Y_n$ be maps such that $\mathrm{d_H}(\widetilde{\theta}_n \circ \widetilde{\sigma}_n(g), \tau_n(g) \circ \widetilde{\theta}_n) \to 0$, for every $g \in \Gamma$, and $\mathrm{d_H}(\widetilde{\theta}_n \circ \widetilde{\sigma}_n(g), \widetilde{\theta}_n) \to 1$, for every $g \in \Gamma \setminus \{e\}$. If $g \in \Gamma_0$, then $\widetilde{\sigma}_n(g)$ leaves $e\Gamma_0 \times X_n$ invariant and $\widetilde{\sigma}_n(g)(e\Gamma_0, x) = (e\Gamma_0, \sigma_n(g)x)$, for every $x \in X_n$. Thus, the restriction of $\widetilde{\sigma_n}_{|\Gamma_0}$ to $e\Gamma_0 \times X_n$ can be identified to σ_n. Denote by $\theta_n : e\Gamma_0 \times X_n \to Y_n$ the restriction of $\widetilde{\theta}_n$ to $e\Gamma_0 \times X_n$. Then it is clear that for every $g \in \Gamma_0$ we have that

$$\mathrm{d_H}(\theta_n \circ \widetilde{\sigma_n}(g)_{|e\Gamma_0 \times X_n}, \tau_n(g) \circ \theta_n) \leq [\Gamma : \Gamma_0] \cdot \mathrm{d_H}(\widetilde{\theta}_n \circ \widetilde{\sigma}_n(g), \tau_n(g) \circ \widetilde{\theta}_n) \text{ and}$$

$$1 - \mathrm{d_H}(\theta_n \circ \widetilde{\sigma_n}(g)_{|e\Gamma_0 \times X_n}, \theta_n) \leq [\Gamma : \Gamma_0] \cdot \big(1 - \mathrm{d_H}(\widetilde{\theta}_n \circ \widetilde{\sigma}_n(g), \widetilde{\theta}_n)\big).$$

These inequalities imply that the maps θ_n witness that (σ_n) satisfies $(\diamond)$. ■

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