Algebraic area enumeration of random walks on the honeycomb lattice

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Abstract

We study the enumeration of closed walks of given length and algebraic area on the honeycomb lattice. Using an irreducible operator realization of honeycomb lattice moves, we map the problem to a Hofstadter-like Hamiltonian and show that the generating function of closed walks maps to the grand partition function of a system of particles with exclusion statistics of order $g = 2$ and an appropriate spectrum, along the lines of a connection previously established by two of the authors. Reinterpreting the results in terms of the standard Hofstadter spectrum calls for a mixture of $g = 1$ (fermion) and $g = 2$ exclusion whose physical meaning and properties require further elucidation. In this context we also obtain some unexpected Fibonacci sequences within the weights of the combinatorial factors appearing in the counting of walks.

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1 Introduction

The algebraic area enumeration of closed random walks on two-dimensional lattices is a topic with rich mathematical and physical implications since it has an intimate connection to discrete quantum models. The algebraic area is defined as the total oriented area spanned by the walk as it traces the lattice. A unit lattice cell enclosed in a counter-clockwise (positive) direction has an area $+1$, whereas when enclosed in a clockwise (negative) direction it has an area -1. The total algebraic area is the area enclosed by the walk weighted by its winding number: if the walk winds around more than once, the area is counted with multiplicity. Figure 1 represents examples of closed random walks on the square, triangular and honeycomb lattices.
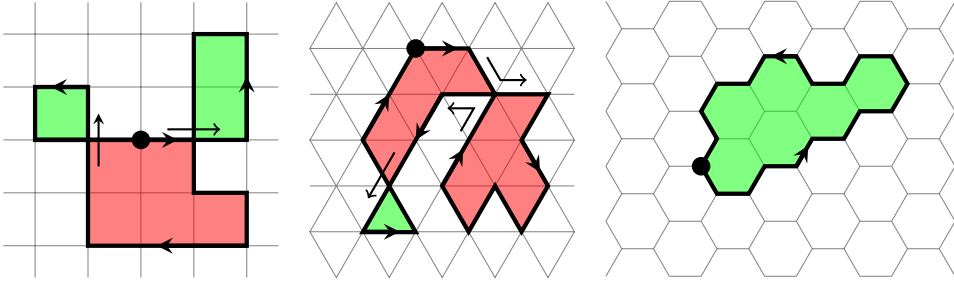

Figure 1: Closed random walks of length $\mathbf{n} = 20$ on the square, triangular and honeycomb lattice with algebraic area -2, -12 and 6, respectively.

In the case of the square lattice, the algebraic area enumeration is known to be embedded in the dynamics of the Hofstadter model [1] which describes the motion of an electron hopping on a square lattice in a uniform perpendicular magnetic field. The generating function for the number $C_{\mathbf{n}}(A)$ of closed walks of length $\mathbf{n} = 2n$ (necessarily even) enclosing an algebraic area A is given in terms of the trace of the Hofstadter Hamiltonian H_γ

$$\sum_A C_{\mathbf{n}}(A)Q^A = \ \mathrm{Tr}H_\gamma^{\mathbf{n}}, \tag{1}$$

where $\gamma = 2\pi\phi/\phi_0$ stands for the flux per plaquette in units of the flux quantum, $Q = e^{i\gamma}$, and H_γ is the Hofstadter Hamiltonian

$$H_\gamma = u + u^{-1} + v + v^{-1}.$$

The unitary operators u and v are unit magnetic translations (hopping operators) in the x and y directions of the square lattice and satisfy the "quantum torus" algebra

$$v\,u = Q\,u\,v \tag{2}$$

due to the perpendicular magnetic field piercing the lattice. Terms contributing to the trace in (1) must involve an equal number of u and u^{-1}, and of v and v^{-1}. Such terms represent closed paths, each power of H_γ representing one step. Because of the non

commuting u and v in (2) the total factor of Q for such paths can be seen to correspond to the algebraic area A of the path, $v^{-1}u^{-1}vu = \mathrm{Q}$ corresponding to a path around an elementary plaquette. In quantum mechanics the trace becomes a sum of the expectation value of H_γ over all quantum states, with an appropriate normalization.

In [2] an explicit algebraic area enumeration was obtained in terms of a sum over compositions of the integer n. In [3] and [4], an interpretation of this enumeration was given in terms of the statistical mechanics of particles obeying quantum *exclusion statistics* with exclusion parameter g ($g = 0$ for bosons, $g = 1$ for fermions, and higher g means a stronger exclusion beyond Fermi). The square lattice enumeration was found to be governed by $g = 2$ exclusion together with a Hofstadter-induced spectral function s_k accounting for the 1-body quantum spectrum, whereas different types of lattice walks were governed by higher values of g and, in general, other types of spectral functions. Explicit examples of such enumerations were given, in particular for Kreweras-like chiral walks on a triangular lattice [3], corresponding to yet another quantum Hofstadter-like model (chiral and non Hermitian, though) and $g = 3$ exclusion. This particular chiral model is to be distinguished from the triangular lattice Hofstadter-like model originally proposed in [5]. Its butterfly structure – among other Hosftadter-like models – has been studied in [6].

An interesting case is the honeycomb lattice. It arises naturally in the form of graphene and carbon nanotubes, and many of its quantum properties have been extensively studied (see, for example, [7, 8, 9]). The honeycomb lattice is also relevant in graph theory [10] and various physical models [11, 12, 13]. The quantum model for a particle hopping on the honeycomb lattice pierced by a perpendicular magnetic field was introduced in [14, 15]. The effect of lattice defects on its spectrum was investigated in [16] and its butterfly-like spectrum was obtained in [17].

In this work we address the question of the algebraic area enumeration of closed random walks on the honeycomb lattice: can this enumeration be explicitly obtained, and does it fall in the category described in [3] and [4], i.e., does it correspond to a particular exclusion statistics? We will show that, indeed, the honeycomb enumeration can be interpreted in terms of $g = 2$ exclusion provided that the Hofstadter spectral function s_k is "diluted" to a spectrum of alternating 1 and s_k. On the other hand, if we insist on using an undiluted s_k, then $g = 2$ exclusion has to be traded for a mixture of $g = 1$ and $g = 2$ exclusion whose physical meaning needs further clarification. In this process we will obtain some unexpected Fibonacci sequences, either for the number of compositions entering the enumeration or for the sum of the coefficients weighting particular compositions, the occurrence of which remains to be better understood.

The paper is structured as follows: In Section 2 we review the Hofstadter model on the square lattice, where the coefficients of the secular determinant of the Hofstadter Hamiltonian [18] are reinterpreted in terms of $g = 2$ exclusion partition functions. The algebraic area enumeration is then obtained in terms of the associated cluster coefficients. In Section 3 we study the honeycomb lattice and calculate the relevant partition functions

and cluster coefficients, arriving at an explicit algebraic area enumeration expression. Some open questions are exposed in the Conclusions.

2 Square lattice walks algebraic area enumeration

From now on we consider the flux γ per lattice cell to be rational, i.e., $\phi/\phi_0 = p/q$ with p and q co-prime, so $Q = \exp(2i\pi p/q)$.

2.1 Hofstadter Hamiltonian

When the magnetic flux is rational the quantum torus algebra has a finite-dimensional irreducible representation in which u and v are represented by the $q \times q$ "clock" and "shift" matrices [19]

$$
u = e^{ik_y}
\begin{pmatrix}
Q & 0 & 0 & \cdots & 0 & 0 \\
0 & Q^2 & 0 & \cdots & 0 & 0 \\
0 & 0 & Q^3 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & Q^{q-1} & 0 \\
0 & 0 & 0 & \cdots & 0 & Q^q
\end{pmatrix},
\quad
v = e^{ik_x}
\begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 \\
1 & 0 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}.
\tag{3}
$$

$k_x \in [0, 2\pi]$ and $k_y \in [0, 2\pi]$ are the quasimomenta in the x and y lattice directions and are related to the Casimirs of the u, v algebra

$$
u^q = e^{iqk_y}, \quad v^q = e^{iqk_x}.
$$

The Hofstadter Hamiltonian becomes the $q \times q$ matrix

$$
H_q =
\begin{pmatrix}
Qe^{ik_y} + Q^{-1}e^{-ik_y} & e^{ik_x} & 0 & \cdots & 0 & e^{-ik_x} \\
e^{-ik_x} & Q^2e^{ik_y} + Q^{-2}e^{-ik_y} & e^{ik_x} & \cdots & 0 & 0 \\
0 & e^{-ik_x} & () & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & () & e^{ik_x} \\
e^{ik_x} & 0 & 0 & \cdots & e^{-ik_x} & Q^qe^{ik_y} + Q^{-q}e^{-ik_y}
\end{pmatrix},
$$

whose spectrum follows from the zeros of the secular determinant $\det(1 - zH_q)$, where z denotes the inverse energy.

This secular determinant has been shown [18] to rewrite as

$$
\det(1 - zH_q) = \sum_{n=0}^{\lfloor q/2 \rfloor} (-1)^n Z(n) z^{2n} - 2\big(\cos(qk_x) + \cos(qk_y)\big) z^q,
\tag{4}
$$

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where the $Z(n)$'s are given by the nested trigonometric sums

$$Z(n) = \sum_{k_1=0}^{q-2n} \sum_{k_2=0}^{k_1} \cdots \sum_{k_n=0}^{k_{n-1}} 4\sin^2\left(\frac{\pi(k_1+2n-1)p}{q}\right) 4\sin^2\left(\frac{\pi(k_2+2n-3)p}{q}\right)\cdots$$
$$4\sin^2\left(\frac{\pi(k_{n-1}+3)p}{q}\right) 4\sin^2\left(\frac{\pi(k_n+1)p}{q}\right) \tag{5}$$

with $Z(0) = 1$.

As we shall see, $Z(n)$ in (5) is at the core of the lattice walks algebraic area enumeration. To recover (5) let us use an alternative form of the Hofstadter Hamiltonian involving a different but equivalent representation of the operators u and v, namely $-uv$ and v. They still satisfy the same quantum torus algebra

$$v\,(-uv) = Q\,(-uv)\,v,$$

albeit with a different Casimir $(-uv)^q = -e^{iq(k_x+k_y)}$, and lead to the new Hamiltonian

$$H_q' = -uv - (uv)^{-1} + v + v^{-1},$$

i.e.,

$$H_q' = \begin{pmatrix} 0 & (1-Qe^{ik_y})e^{ik_x} & 0 & \cdots & 0 & (1-Q^{-q}e^{-ik_y})e^{-ik_x} \\ (1-Q^{-1}e^{-ik_y})e^{-ik_x} & 0 & (1-Q^2e^{ik_y})e^{ik_x} & \cdots & 0 & 0 \\ 0 & (1-Q^{-2}e^{-ik_y})e^{-ik_x} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & (1-Q^{(q-1)}e^{ik_y})e^{ik_x} \\ (1-Q^qe^{ik_y})e^{ik_x} & 0 & 0 & \cdots & (1-Q^{-(q-1)}e^{-ik_y})e^{-ik_x} & 0 \end{pmatrix},$$

or, denoting $\omega(k) = (1-Q^k e^{ik_y})e^{ik_x}$,

$$H_q' = \begin{pmatrix} 0 & \omega(1) & 0 & \cdots & 0 & \bar\omega(q) \\ \bar\omega(1) & 0 & \omega(2) & \cdots & 0 & 0 \\ 0 & \bar\omega(2) & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \omega(q-1) \\ \omega(q) & 0 & 0 & \cdots & \bar\omega(q-1) & 0 \end{pmatrix}.$$

Its secular determinant is the same as that of H_q given in (4) but for the new Casimirs, that is,

$$\det(1-zH_q') = \sum_{n=0}^{\lfloor q/2 \rfloor} (-1)^n Z(n) z^{2n} - \left(\prod_{j=1}^{q}\omega(j) + \prod_{j=1}^{q}\bar\omega(j)\right) z^q$$
$$= \sum_{n=0}^{\lfloor q/2 \rfloor} (-1)^n Z(n) z^{2n} - 2\big(\cos(qk_x) - \cos(qk_x + qk_y)\big) z^q. \tag{6}$$

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Let us set $\omega(q) = 0$, which makes the cosine term in (6) vanish and the matrix H'_q tridiagonal

$$H'_q|_{\omega(q)=0} = \begin{pmatrix} 0 & (1-Q^{1-q})e^{ik_x} & 0 & \cdots & 0 & 0 \\ (1-Q^{q-1})e^{-ik_x} & 0 & (1-Q^{2-q})e^{ik_x} & \cdots & 0 & 0 \\ 0 & (1-Q^{q-2})e^{-ik_x} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & (1-Q^{-1})e^{ik_x} \\ 0 & 0 & 0 & \cdots & (1-Q)e^{-ik_x} & 0 \end{pmatrix}.$$

This form provides an iterative procedure for calculating the $Z(n)$'s. Putting aside for a moment that $Q = \exp(2i\pi p/q)$ and leaving it as a free parameter, we introduce the spectral function

$$s_k = (1 - Q^k)(1 - Q^{-k}). \tag{7}$$

Denoting the secular determinant $\det(1 - zH'_q|_{\omega(q)=0}) = d_q$, its expansion in terms of the first row yields

$$d_q = d_{q-1} - z^2 s_{q-1} d_{q-2}, \quad q \geq 2, \tag{8}$$

where, by convention, $d_0 = d_1 = 1$. Expanding d_q as a polynomial in z and solving the corresponding recursion relation for its coefficients, we obtain (see Appendix A)

$$Z(n) = \sum_{k_1=1}^{q-2n+1} \sum_{k_2=1}^{k_1} \cdots \sum_{k_n=1}^{k_{n-1}} s_{k_1+2n-2} s_{k_2+2n-4} \cdots s_{k_{n-1}+2} s_{k_n}, \tag{9}$$

which, upon restoring Q to its actual value $\exp(2i\pi p/q)$, i.e., the spectral function s_k to its actual form $s_k = 4\sin^2(\pi kp/q)$, gives (5).

The recursion (8) is at the root of the connection between square lattice walks and $g = 2$ exclusion statistics. Interpreting the spectral function s_k as the Boltzmann factor for a 1-body level $e^{-\beta\epsilon_k}$ and $-z^2$ as the fugacity z', (8) can be interpreted as an expansion of a grand partition function $\mathcal{Z}_{q-1}$ – here identified with d_q – of noninteracting particles in $q - 1$ quantum levels $\epsilon_1, \ldots, \epsilon_{q-1}$, obeying the exclusion principle that no two particles can occupy adjacent levels, namely

$$\mathcal{Z}_{q-1} = \mathcal{Z}_{q-2} + z' s_{q-1} \mathcal{Z}_{q-3}$$

in terms of the last level ϵ_{q-1} being empty (first term) or occupied (second term). Then (6) identifies $Z(n)$ as the n-body partition function for particles occupying these $q - 1$ quantum states, with gaps of 2 between successive terms reproducing $g = 2$ exclusion.

2.2 Algebraic area enumeration on the square lattice

As already stressed, when $Q = \exp(2i\pi p/q)$ the algebraic area counting (1)

$$\sum_A C_{\mathbf{n}}(A)Q^A = \frac{1}{q}\text{Tr}H_q^{\mathbf{n}} \tag{10}$$

involves a trace over a finite number q of quantum states. To normalize the contribution of each path to Q^A and reproduce the left-hand side of (10), a factor of $1/q$ must be included in the normalization. Also, when $\mathbf{n} \geq q$ the trace involves extra terms arising from the Casimirs k_x, k_y similar to the cosine terms in (4), corresponding to open paths that close only up to periods (q, q) on the lattice ("umklapp" terms on the quantum torus). These spurious contributions can be eliminated by integrating the Casimirs k_x and k_y over $[0, 2\pi]$ which makes all factors of e^{iqk_x} and e^{iqk_y} vanish. So the definition of the trace in (10) is

$$\text{Tr}H_q^{\mathbf{n}} = \frac{1}{(2\pi)^2} \int_0^{2\pi} dk_x \int_0^{2\pi} dk_y \, \text{tr}H_q^{\mathbf{n}},$$

which corresponds to summing over the q bands of the spectrum and over the scattering states labeled by k_x, k_y, in a continuum normalization.

To relate this trace to the $Z(n)$'s in (5) or, equivalently, in (9), one has to use

$$\log \det(1 - zH_q) = \text{tr} \log(1 - zH_q) = -\sum_{n=1}^{\infty} \frac{z^n}{n} \, \text{tr}H_q^n,$$

and the fact that, in statistical mechanics, the $Z(n)$ are viewed as n-body partition functions with cluster coefficients $b(n)$ defined via the grand partition function $\sum_{n=0}^{\infty} Z(n)z^n$

$$\log \left(\sum_{n=0}^{\infty} Z(n)z^n \right) = \sum_{n=1}^{\infty} b(n)z^n \tag{11}$$

with z playing the role of the fugacity. Trading z for $-z^2$ in (11), keeping in mind that trivially $\text{tr}H_q^{2n+1} = 0$, and putting everything together we reach the conclusion [2, 3] that the trace in (10) for $\mathbf{n} = 2n$ is nothing but the cluster coefficient $b(n)$ up to a trivial factor

$$\text{Tr}H_q^{\mathbf{n}} = 2n(-1)^{n+1}b(n). \tag{12}$$

The cluster coefficients can in turn be directly read from the $Z(n)$'s in (9): one gets

$$b(n) = (-1)^{n+1} \sum_{\substack{l_1, l_2, \ldots, l_j \\ \text{composition of } n}} c(l_1, l_2, \ldots, l_j) \sum_{k=1}^{q-j} s_{k+j-1}^{l_j} \cdots s_{k+1}^{l_2} s_k^{l_1}, \tag{13}$$

where the $c(l_1, l_2, \ldots, l_j)$'s are labeled by the compositions of the integer n with

$$c(l_1, l_2, \ldots, l_j) = \frac{\binom{l_1+l_2}{l_1}}{l_1 + l_2} \, l_2 \frac{\binom{l_2+l_3}{l_2}}{l_2 + l_3} \cdots l_{j-1} \frac{\binom{l_{j-1}+l_j}{l_{j-1}}}{l_{j-1} + l_j}. \tag{14}$$

Further, the trigonometric sums $\frac{1}{q} \sum_{k=1}^{q-j} s_{k+j-1}^{l_j} \cdots s_{k+1}^{l_2} s_k^{l_1}$ can also be computed [2, 4]

$$\frac{1}{q} \sum_{k=1}^{q-j} s_{k+j-1}^{l_j} \cdots s_{k+1}^{l_2} s_k^{l_1} = \sum_{A=-\infty}^{+\infty} \cos\left(\frac{2A\pi p}{q}\right)$$

$$\sum_{k_3=-l_3}^{l_3} \sum_{k_4=-l_4}^{l_4} \cdots \sum_{k_j=-l_j}^{l_j} \binom{2l_1}{l_1 + A + \sum_{i=3}^{j}(i-2)k_i} \binom{2l_2}{l_2 - A - \sum_{i=3}^{j}(i-1)k_i} \prod_{i=3}^{j} \binom{2l_i}{l_i + k_i}. \tag{15}$$

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Using (12), (13), (14) and (15) and keeping in mind that $\mathbf{n} = 2n$, we deduce the desired algebraic area counting

$$\sum_A C_{\mathbf{n}}(A)Q^A = \frac{1}{q}\,\mathrm{Tr}H_q^{\mathbf{n}} = 2n \sum_{\substack{l_1,l_2,\ldots,l_j \\ \text{composition of } n}} c(l_1, l_2, \ldots, l_j)\frac{1}{q}\sum_{k=1}^{q-j} s_{k+j-1}^{l_j} \cdots s_{k+1}^{l_2} s_k^{l_1},$$

i.e.,

$$C_{\mathbf{n}}(A) = 2n \sum_{\substack{l_1,l_2,\ldots,l_j \\ \text{composition of } n}} c(l_1, l_2, \ldots, l_j)$$

$$\sum_{k_3=-l_3}^{l_3} \sum_{k_4=-l_4}^{l_4} \cdots \sum_{k_j=-l_j}^{l_j} \binom{2l_1}{l_1 + A + \sum_{i=3}^{j}(i-2)k_i}\binom{2l_2}{l_2 - A - \sum_{i=3}^{j}(i-1)k_i} \prod_{i=3}^{j} \binom{2l_i}{l_i + k_i}.$$
(16)

We also note that, since

$$\sum_{\substack{l_1,l_2,\ldots,l_j \\ \text{composition of } n}} c(l_1, l_2, \ldots, l_j) = \frac{\binom{2n}{n}}{2n},$$

and, when $q \to \infty$ [2, 3],

$$\frac{1}{q}\sum_{k=1}^{q-j} s_{k+j-1}^{l_j} \cdots s_{k+1}^{l_2} s_k^{l_1} \to \binom{2(l_1 + l_2 + \ldots + l_j)}{l_1 + l_2 + \ldots + l_j},$$
(17)

the overall closed square lattice walks counting

$$2n \sum_{\substack{l_1,l_2,\ldots,l_j \\ \text{composition of } n}} c(l_1, l_2, \ldots, l_j)\binom{2(l_1 + l_2 + \ldots + l_j)}{l_1 + l_2 + \ldots + l_j} = \binom{2n}{n}^2 = \binom{\mathbf{n}}{\mathbf{n}/2}^2$$

is recovered as it should (see Appendix B for some enumeration examples).

3   Honeycomb lattice walks algebraic area enumeration

We plan to follow the same route as above to obtain an explicit algebraic area enumeration for closed walks on the honeycomb lattice.
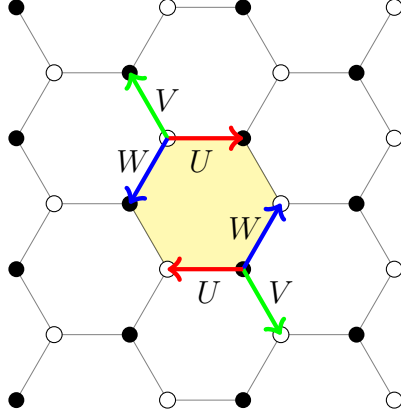
3.1 Honeycomb Hamiltonian

Figure 2: Hopping operators U, V, W on the honeycomb lattice with $U^2 = V^2 = W^2 = 1$ and $(UVW)^2 = Q$.

Consider a particle hopping on a honeycomb lattice pierced by a constant magnetic field (see Fig. 2). The lattice is bipartite with unitary operators U, V, W generating the hoppings in each direction and such that when the particle hops around a honeycomb cell it picks up a phase Q due to the magnetic field. They satisfy the honeycomb algebra

$$U^2 = V^2 = W^2 = 1, \quad (UVW)^2 = Q. \tag{18}$$

The Hofstadter-like Hamiltonian follows as

$$H_{\text{honeycomb}} = aU + bV + cW,$$

with $a, b, c \in \mathbb{R}^+$ transition amplitudes. The physical Hilbert space consists of the irreducible representations of the honeycomb algebra. As in the square lattice case, the quasimomenta are encoded in the Casimirs of the algebra.

In the case of an isotropic lattice, $a = b = c = 1$, and a rational flux, $Q = \exp(2i\pi p/q)$ with p and q co-prime, the irreducible representation of U, V and W for generic quasimomenta (Casimirs) becomes $2q$-dimensional (see Appendix C)

$$U = \begin{pmatrix} 0 & u \\ u^{-1} & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & v \\ v^{-1} & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & Q^{1/2}vu^{-1} \\ Q^{-1/2}uv^{-1} & 0 \end{pmatrix}$$

with u, v given in (3), and the honeycomb Hamiltonian reduces to the $2q \times 2q$ matrix

$$H_{2q} = \begin{pmatrix} 0 & u+v+Q^{1/2}vu^{-1} \\ u^{-1}+v^{-1}+Q^{-1/2}uv^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix}. \tag{19}$$

Its square is block-diagonal

$$H_{2q}^2 = \begin{pmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A \end{pmatrix} = \begin{pmatrix} H_q & 0 \\ 0 & \tilde{H}_q \end{pmatrix},$$

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where $H_q = AA^\dagger$ and $\tilde{H}_q = A^\dagger A$ have identical spectra equal to the square of the honeycomb Hamiltonian spectrum. Denoting

$$\omega(k) = Q^{-k}\left(1 + e^{-ik_y}Q^{\frac{1}{2}-k}\right)e^{-i(k_y-k_x)},$$

H_q can be rewritten as

$$H_q = \begin{pmatrix} 1+\omega(2)\bar{\omega}(2) & \omega(2) & 0 & \cdots & 0 & \bar{\omega}(1) \\ \bar{\omega}(2) & 1+\omega(3)\bar{\omega}(3) & \omega(3) & \cdots & 0 & 0 \\ 0 & \bar{\omega}(3) & () & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & () & \omega(q) \\ \omega(1) & 0 & 0 & \cdots & \bar{\omega}(q) & 1+\omega(1)\bar{\omega}(1) \end{pmatrix} \tag{20}$$

with secular determinant

$$\det(1 - zH_{2q}) = \det(1 - z^2 H_q)$$

$$= \sum_{n=0}^{q}(-1)^n Z(n)z^{2n} + \left((-1)^q\prod_{j=1}^{q}\omega(j)\bar{\omega}(j) - \prod_{j=1}^{q}\omega(j) - \prod_{j=1}^{q}\bar{\omega}(j)\right)z^{2q}$$

$$= \sum_{n=0}^{q}(-1)^n Z(n)z^{2n} + 2\left(-Q^{\frac{q}{2}}\Big(\cos(qk_x - 2qk_y) + \cos(qk_y)\Big) + (-1)^q\Big(\cos(qk_x - qk_y) + 1\Big)\right)z^{2q}. \tag{21}$$

3.2 Honeycomb coefficients $Z(n)$

Our aim is to find for the $Z(n)$ in (21) an expression analogous to the one in (5) or (9) obtained in the Hofstadter case. To this end, we reduce the honeycomb matrix (20) to a tridiagonal form by making both corners $\omega(1)$ and $\bar{\omega}(1)$ vanish, i.e., by setting $e^{-ik_y} = -Q^{\frac{1}{2}}$ so that $\omega(k)$ becomes

$$\omega(k)|_{\omega(1)=0} = -Q^{\frac{1}{2}-k}\left(1 - Q^{1-k}\right)e^{ik_x},$$

and

$$H_q\Big|_{\omega(1)=0} = \begin{pmatrix} 1+(1-Q^{-1})(1-Q) & -Q^{-\frac{3}{2}}(1-Q^{-1})e^{ik_x} & 0 & \cdots & 0 & 0 \\ -Q^{\frac{3}{2}}(1-Q)e^{-ik_x} & 1+(1-Q^{-2})(1-Q^2) & -Q^{-\frac{5}{2}}(1-Q^{-2})e^{ik_x} & \cdots & 0 & 0 \\ 0 & -Q^{\frac{5}{2}}(1-Q^2)e^{-ik_x} & () & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & () & -Q^{\frac{1}{2}-q}(1-Q^{-(q-1)})e^{ik_x} \\ 0 & 0 & 0 & \cdots & -Q^{q-\frac{1}{2}}(1-Q^{q-1})e^{-ik_x} & 1+(1-Q^{-q})(1-Q^q) \end{pmatrix}.$$

This also eliminates the z^{2q} umklapp term in (21), i.e., the secular determinant reduces to

$$\det\left(1 - z^2 H_q|_{\omega(1)=0}\right) = \sum_{n=0}^{q}(-1)^n Z(n)z^{2n}.$$

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Let us now consider Q as a free parameter and denote $d_q = \det(1 - z^2 H_q|_{\omega(1)=0})$. Then expanding d_q in terms of its bottom row we obtain the recursion relation

$$d_q = \left(1 - \left[1 + (1 - Q^q)(1 - Q^{-q})\right] z^2\right) d_{q-1} - z^4(1 - Q^{q-1})(1 - Q^{-(q-1)})d_{q-2}, \quad q \geq 1,$$

i.e.,

$$d_q = \left(1 - \left(1 + s_q\right)z^2\right)d_{q-1} - z^4 s_{q-1} d_{q-2}, \tag{22}$$

with $d_0 = 1$, $d_j = 0$ for $j < 0$, and s_k as in (7). From (22) we can iteratively derive the $Z(n)$ (see Appendix D).

The above recursion admits a simple $g = 2$ exclusion statistics interpretation. Consider a set of $2q$ energy levels with spectral parameters S_n, $n = 1, 2, \ldots, 2q$ given by

$$S_{2k-1} = 1, \quad S_{2k} = s_k,$$

that is, s_k "diluted" by unit insertions: $1, s_1, 1, s_2, \ldots, 1, s_q$, and consider the grand partition function of $g = 2$ exclusion particles in the above spectrum S_n with fugacity parameter z. Calling $\mathcal{Z}_{1,n}$ the truncated grand partition function for levels $S_1, S_2, \ldots, S_n$ and expanding it in terms of the last level n being empty or filled, we obtain the recursion relations

$$n = 2k: \qquad \mathcal{Z}_{1,2k} = \mathcal{Z}_{1,2k-1} + zs_k \mathcal{Z}_{1,2k-2},$$
$$n = 2k-1: \qquad \mathcal{Z}_{1,2k-1} = \mathcal{Z}_{1,2k-2} + z\mathcal{Z}_{1,2k-3}.$$

From the $n = 2k$ relation we can express the odd functions $\mathcal{Z}_{2k-1}$ in terms of even ones, $\mathcal{Z}_{1,2k-1} = \mathcal{Z}_{1,2k} - zs_k \mathcal{Z}_{1,2k-2}$. Substituting this expression in the $n = 2k - 1$ relation and rearranging we obtain

$$\mathcal{Z}_{1,2k} = (1 + z + zs_k)\mathcal{Z}_{1,2k-2} - z^2 s_{k-1} \mathcal{Z}_{1,2k-4}.$$

This is identical to the recursion (22) upon shifting $z \to -z^2$ and identifying $\mathcal{Z}_{1,2k} = d_k$. Moreover, $\mathcal{Z}_{2k}$ satisfies the same initial conditions as d_k, namely $\mathcal{Z}_{1,0} = 1$, $\mathcal{Z}_{1,2k} = 0$ for $k < 0$. Therefore, $d_q = \mathcal{Z}_{2q}$.

It follows that the expressions for the n-body partition functions $Z(n)$ and the cluster coefficients $b(n)$ are identical to the corresponding expressions (9) and (13) for square lattice walks but now, instead of the spectrum s_k, one has to consider the diluted spectrum S_k, $k = 1, \ldots, 2q$ (but note that $S_{2q} = s_q = 0$, so the levels effectively end at $S_{2q-1} = 1$)

$$Z(n) = \sum_{k_1=1}^{2q-2n+2} \sum_{k_2=1}^{k_1} \cdots \sum_{k_n=1}^{k_{n-1}} S_{k_1+2n-2} S_{k_2+2n-4} \cdots S_{k_{n-1}+2} S_{k_n},$$

$$b(n) = (-1)^{n+1} \sum_{\substack{l_1,l_2,\ldots,l_j \\ \text{composition of } n}} c(l_1, l_2, \ldots, l_j) \sum_{k=1}^{2q-j+1} S_{k+j-1}^{l_j} \cdots S_{k+1}^{l_2} S_k^{l_1}$$

with the same Hofstadter combinatorial factors $c(l_1, l_2, \ldots, l_j)$ given in (14). The corresponding diluted trigonometric sums $\frac{1}{q} \sum_{k=1}^{2q-j+1} S_{k+j-1}^{l_j} \cdots S_{k+1}^{l_2} S_k^{l_1}$ can be expressed as

$$\frac{1}{q} \sum_{k=1}^{2q-j+1} S_{k+j-1}^{l_j} \cdots S_{k+1}^{l_2} S_k^{l_1} = \sum_{A=-\infty}^{+\infty} \cos\left(\frac{2A\pi p}{q}\right)$$

$$\left(\sum_{k_5=-l_5}^{l_5} \sum_{k_7=-l_7}^{l_7} \cdots \sum_{k_{2\lfloor(j-1)/2\rfloor+1}=-l_{2\lfloor(j-1)/2\rfloor+1}}^{l_{2\lfloor(j-1)/2\rfloor+1}} \left({}_{l_1+A+\sum_{\substack{i=5 \\ i \text{ odd}}}^{2\lfloor(j-1)/2\rfloor+1}(i-3)k_i/2}^{2l_1} \right) \left({}_{l_3-A-\sum_{\substack{i=5 \\ i \text{ odd}}}^{2\lfloor(j-1)/2\rfloor+1}(i-1)k_i/2}^{2l_3} \right) \prod_{\substack{i=5 \\ i \text{ odd}}}^{2\lfloor(j-1)/2\rfloor+1} \binom{2l_i}{l_i+k_i} \right.$$

$$\left. + \sum_{k_6=-l_6}^{l_6} \sum_{k_8=-l_8}^{l_8} \cdots \sum_{k_{2\lfloor j/2\rfloor}=-l_{2\lfloor j/2\rfloor}}^{l_{2\lfloor j/2\rfloor}} \left({}_{l_2+A+\sum_{\substack{i=6 \\ i \text{ even}}}^{2\lfloor j/2\rfloor}(i-4)k_i/2}^{2l_2} \right) \left({}_{l_4-A-\sum_{\substack{i=6 \\ i \text{ even}}}^{2\lfloor j/2\rfloor}(i-2)k_i/2}^{2l_4} \right) \prod_{\substack{i=6 \\ i \text{ even}}}^{2\lfloor j/2\rfloor} \binom{2l_i}{l_i+k_i} \right).$$

Following the same steps as in Section (2.2) regarding the number $C_{\mathbf{n}}(A)$ of closed random walks of length $\mathbf{n} = 2n$ enclosing on the honeycomb lattice an algebraic area A, i.e., considering on the one hand

$$\sum_A C_{\mathbf{n}}(A) Q^A = \frac{1}{2q} \text{Tr} H_{2q}^{\mathbf{n}},$$

which is the anologous of (10) for the honeycomb Hamiltonian (19) (where the factor $1/q$ is replaced by $1/(2q)$ in view of a proper normalisation over the $2q$ states), and on the other hand

$$\text{Tr} H_{2q}^{\mathbf{n}} = 2n(-1)^{n+1} b(n),$$

which generalizes (12), the expressions above directly lead to an algebraic area enumeration similar to the square lattice walks enumeration (16).

In the sequel, we will consider d_q in terms of the original (undiluted) Hofstadter spectrum s_k. In that case, the $g = 2$ exclusion interpretation does not hold anymore and has to be traded for a mixture of $g = 2$ and $g = 1$ statistics, as we are going to show in detail.

3.3   Modified statistics for the spectral function s_k

If we insist on keeping s_k as the spectral function, the first few $Z(n)$ rewrite as

$$Z(1) = \sum_{i=1}^{q} \left(1 + s_i\right),$$

$$Z(2) = + \sum_{i=1}^{q-1} \sum_{j=1}^{i} \left(1 + s_{i+1}\right)\left(1 + s_j\right)$$
$$- \sum_{i=1}^{q-1} s_i,$$

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$$Z(3) = + \sum_{i=1}^{q-2} \sum_{j=1}^{i} \sum_{k=1}^{j} \left(1 + s_{i+2}\right)\left(1 + s_{j+1}\right)\left(1 + s_k\right)$$

$$- \sum_{i=1}^{q-2} \sum_{j=1}^{i} \left(1 + s_{i+2}\right) s_j$$

$$- \sum_{i=1}^{q-2} \sum_{j=1}^{i} s_{i+1}\left(1 + s_j\right),$$

$$Z(4) = + \sum_{i=1}^{q-3} \sum_{j=1}^{i} \sum_{k=1}^{j} \sum_{l=1}^{k} \left(1 + s_{i+3}\right)\left(1 + s_{j+2}\right)\left(1 + s_{k+1}\right)\left(1 + s_l\right)$$

$$- \sum_{i=1}^{q-3} \sum_{j=1}^{i} \sum_{k=1}^{j} \left(1 + s_{i+3}\right)\left(1 + s_{j+2}\right) s_k$$

$$- \sum_{i=1}^{q-3} \sum_{j=1}^{i} \sum_{k=1}^{j} \left(1 + s_{i+3}\right) s_{j+1}\left(1 + s_k\right)$$

$$- \sum_{i=1}^{q-3} \sum_{j=1}^{i} \sum_{k=1}^{j} s_{i+2}\left(1 + s_{j+1}\right)\left(1 + s_k\right)$$

$$+ \sum_{i=1}^{q-3} \sum_{j=1}^{i} s_{i+2} s_j,$$

$$Z(5) = + \sum_{i=1}^{q-4} \sum_{j=1}^{i} \sum_{k=1}^{j} \sum_{l=1}^{k} \sum_{m=1}^{l} \left(1 + s_{i+4}\right)\left(1 + s_{j+3}\right)\left(1 + s_{k+2}\right)\left(1 + s_{l+1}\right)\left(1 + s_m\right)$$

$$- \sum_{i=1}^{q-4} \sum_{j=1}^{i} \sum_{k=1}^{j} \sum_{l=1}^{k} \left(1 + s_{i+4}\right)\left(1 + s_{j+3}\right)\left(1 + s_{k+2}\right)s_l$$

$$- \sum_{i=1}^{q-4} \sum_{j=1}^{i} \sum_{k=1}^{j} \sum_{l=1}^{k} \left(1 + s_{i+4}\right)\left(1 + s_{j+3}\right)s_{k+1}\left(1 + s_l\right)$$

$$- \sum_{i=1}^{q-4} \sum_{j=1}^{i} \sum_{k=1}^{j} \sum_{l=1}^{k} \left(1 + s_{i+4}\right)s_{j+2}\left(1 + s_{k+1}\right)\left(1 + s_l\right)$$

$$+ \sum_{i=1}^{q-4} \sum_{j=1}^{i} \sum_{k=1}^{j} \left(1 + s_{i+4}\right)s_{j+2}s_k$$

$$- \sum_{i=1}^{q-4} \sum_{j=1}^{i} \sum_{k=1}^{j} \sum_{l=1}^{k} s_{i+3}\left(1 + s_{j+2}\right)\left(1 + s_{k+1}\right)\left(1 + s_l\right)$$

$$+ \sum_{i=1}^{q-4} \sum_{j=1}^{i} \sum_{k=1}^{j} s_{i+3}\left(1 + s_{j+2}\right)s_k$$

$$+ \sum_{i=1}^{q-4} \sum_{j=1}^{i} \sum_{k=1}^{j} s_{i+3}s_{j+1}\left(1 + s_k\right).$$

We infer that in general the $Z(n)$'s are combinations of nested multiple sums of products of $(1 + s_k)$ and s_k such that

- The rightmost factor is either s_k or $\left(1 + s_k\right)$.

- Any factor multiplying s_i immediately on its left obeys $g = 2$ exclusions, i.e., $\sum_i \sum_j s_i s_j$ or $\sum_i \sum_j \left(1 + s_i\right)s_j$ where $i - j \geq 2$.

- Any factor multiplying $\left(1 + s_i\right)$ immediately on its left obeys $g = 1$ exclusions, i.e., $\sum_i \sum_j s_i\left(1 + s_j\right)$ or $\sum_i \sum_j \left(1 + s_i\right)\left(1 + s_j\right)$ where $i - j \geq 1$.

- The leftmost factor is either s_{i+n-2} or $\left(1 + s_{i+n-1}\right)$ with summation range $\sum_{i=1}^{q-(n-1)}$.

- The sign is $\pm(-1)^n$ for even/odd number of factors.

From these rules and the very definition (11) we get the $b(n)$'s in terms of single sums of products of s_k (up to terms involving s_q which vanish anyway) with a form a bit more

complicated than in the Hofstadter case

$$b(1) = \sum_{k=1}^{q-1} s_k + \sum_{k=1}^{q} s_k^0,$$

$$-b(2) = \frac{1}{2}\sum_{k=1}^{q-1} s_k^2 + 2\sum_{k=1}^{q-1} s_k + \frac{1}{2}\sum_{k=1}^{q} s_k^0,$$

$$b(3) = \frac{1}{3}\sum_{k=1}^{q-1} s_k^3 + 2\sum_{k=1}^{q-1} s_k^2 + \sum_{k=1}^{q-2} s_{k+1}s_k + 3\sum_{k=1}^{q-1} s_k + \frac{1}{3}\sum_{k=1}^{q} s_k^0,$$

$$-b(4) = \frac{1}{4}\sum_{k=1}^{q-1} s_k^4 + 2\sum_{k=1}^{q-1} s_k^3 + \sum_{k=1}^{q-2} s_{k+1}^2 s_k + \sum_{k=1}^{q-2} s_{k+1}s_k^2$$
$$+ 5\sum_{k=1}^{q-1} s_k^2 + 4\sum_{k=1}^{q-2} s_{k+1}s_k + 4\sum_{k=1}^{q-1} s_k + \frac{1}{4}\sum_{k=1}^{q} s_k^0,$$

$$b(5) = \frac{1}{5}\sum_{k=1}^{q-1} s_k^5 + 2\sum_{k=1}^{q-1} s_k^4 + \sum_{k=1}^{q-2} s_{k+1}^3 s_k + \sum_{k=1}^{q-2} s_{k+1}^2 s_k^2 + \sum_{k=1}^{q-2} s_{k+1}s_k^3$$
$$+ 7\sum_{k=1}^{q-1} s_k^3 + 6\sum_{k=1}^{q-2} s_{k+1}^2 s_k + 6\sum_{k=1}^{q-2} s_{k+1}s_k^2 + \sum_{k=1}^{q-3} s_{k+2}s_{k+1}s_k$$
$$+ 10\sum_{k=1}^{q-1} s_k^2 + 10\sum_{k=1}^{q-2} s_{k+1}s_k + 5\sum_{k=1}^{q-1} s_k + \frac{1}{5}\sum_{k=1}^{q} s_k^0,$$

i.e.,

$$b(n) = (-1)^{n+1} \sum_{\substack{l_1,l_2,\ldots,l_j \\ \text{composition of } n'=0,1,2,\ldots,n \\ j\leq\min(n',n-n'+1)}} c_n(l_1,l_2,\ldots,l_j) \sum_{k=1}^{q-j} s_{k+j-1}^{l_j}\cdots s_{k+1}^{l_2}s_k^{l_1}.$$

The new combinatorial coefficients $c_n(l_1, l_2, \ldots, l_j)$ are labeled by the compositions of $n' = 0, 1, 2, \ldots, n$ with a number of parts $j \leq \min(n', n-n'+1)$ (by convention the unique composition of $n' = 0$ has only one part and the trigonometric sum becomes $\sum_{k=1}^{q} s_k^0$). Since the number of compositions of an integer n' with j parts is $\binom{n'-1}{j-1}$, the total number of such compositions is

$$1+\sum_{n'=1}^{n}\sum_{j=1}^{\min(n',n-n'+1)}\binom{n'-1}{j-1} = 1+\sum_{j=1}^{\lfloor(n+1)/2\rfloor}\sum_{n'=j}^{n-j+1}\binom{n'-1}{j-1} = \sum_{j=0}^{\lfloor(n+1)/2\rfloor}\binom{n-j+1}{j} = F_{n+2}.$$

Note that the Fibonacci number F_{n+2} is also the number of compositions of $(n+1)$ with

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only parts 1 and 2. We obtain for the $c_n(l_1, l_2, \ldots, l_j)$'s

$$c_n(0) = \frac{1}{n},$$

$$c_n(l_1) = \frac{1}{l_1}\binom{n + l_1 - 1}{2l_1 - 1},$$

$$c_n(l_1, l_2) = \frac{1}{l_1 l_2}\sum_{m=0}^{\min(l_1, l_2)} m\binom{l_1}{m}\binom{l_2}{m}\binom{n + l_1 + l_2 - m - 1}{2(l_1 + l_2) - 1},$$

$$c_n(l_1, l_2, \ldots, l_j) = \frac{1}{l_1 l_2 \ldots l_j}\sum_{m_1=0}^{\min(l_1, l_2)}\sum_{m_2=0}^{\min(l_2, l_3)}\cdots\sum_{m_{j-1}=0}^{\min(l_{j-1}, l_j)}\left(\prod_{i=1}^{j-1} m_i\binom{l_i}{m_i}\binom{l_{i+1}}{m_i}\right)\binom{n + \sum_{i=1}^{j} l_i - \sum_{i=1}^{j-1} m_i - 1}{2\sum_{i=1}^{j} l_i - 1}.$$

(23)

We also note that by ignoring the n-dependant binomial $\binom{n + \sum_{i=1}^{j} l_i - \sum_{i=1}^{j-1} m_i - 1}{2\sum_{i=1}^{j} l_i - 1}$ in the sums (23) one recovers the $c(l_1, l_2, \ldots, l_j)$ in (14), that is,

$$c_n(l_1, l_2) \rightarrow \frac{1}{l_1 l_2}\sum_{m=0}^{\min(l_1, l_2)} m\binom{l_1}{m}\binom{l_2}{m} = \frac{\binom{l_1 + l_2}{l_1}}{l_1 + l_2},$$

and thus by factorization

$$c_n(l_1, l_2, \ldots, l_j) \rightarrow \frac{1}{l_1 l_2 \ldots l_j}\sum_{m_1=0}^{\min(l_1, l_2)}\sum_{m_2=0}^{\min(l_2, l_3)}\cdots\sum_{m_{j-1}=0}^{\min(l_{j-1}, l_j)}\prod_{i=1}^{j-1} m_i\binom{l_i}{m_i}\binom{l_{i+1}}{m_i} = \frac{\binom{l_1 + l_2}{l_1}}{l_1 + l_2}\; l_2\frac{\binom{l_2 + l_3}{l_2}}{l_2 + l_3}\cdots l_{j-1}\frac{\binom{l_{j-1} + l_j}{l_{j-1}}}{l_{j-1} + l_j}.$$

We also have

$$n\sum_{l=0}^{n} c_n(l) = F_{2n+1} + F_{2n-1} - 1,$$

where again a Fibonacci counting appears, and

$$n\sum_{\substack{l_1, l_2, \ldots, l_j \\ \text{composition of } n' \\ j \leq \min(n', n - n' + 1)}} c_n(l_1, l_2, \ldots, l_j) = \binom{n}{n'}^2,$$

from which we infer

$$n\sum_{\substack{l_1, l_2, \ldots, l_j \\ \text{composition of } n' = 0, 1, 2, \ldots, n \\ j \leq \min(n', n - n' + 1)}} c_n(l_1, l_2, \ldots, l_j) = \binom{2n}{n}.$$

Again using (17), the counting of closed honeycomb lattice walks of length $2n$ is recovered

$$n \sum_{\substack{l_1,l_2,\ldots,l_j \\ \text{composition of } n'=0,1,2,\ldots,n \\ j\leq \min(n',n-n'+1)}} c_n(l_1,l_2,\ldots,l_j)\binom{2(l_1+l_2+\ldots+l_j)}{l_1+l_2+\ldots+l_j}$$

$$= \sum_{n'=0}^{n}\left(n \sum_{\substack{l_1,l_2,\ldots,l_j \\ \text{composition of } n' \\ j\leq \min(n',n-n'+1)}} c_n(l_1,l_2,\ldots,l_j)\binom{2(l_1+l_2+\ldots+l_j)}{l_1+l_2+\ldots+l_j}\right)$$

$$= \sum_{n'=0}^{n}\binom{n}{n'}^2\binom{2n'}{n'}.$$

3.4   Algebraic area enumeration on the honeycomb lattice

Remembering that the spectrum of H_q is the square of that of the honeycomb Hamiltonian H_{2q}, the generating function for the number $C_{\mathbf{n}}(A)$ of closed walks of length $\mathbf{n} = 2n$ enclosing an algebraic area A can as well be given in terms of the trace of H_q^n weighted by $1/q$, i.e.,

$$\sum_A C_{\mathbf{n}}(A)\mathrm{Q}^A = \frac{1}{q}\mathrm{Tr}H_q^n,$$

where now, following again the steps of Section (2.2),

$$\mathrm{Tr}H_q^n = (-1)^{n+1}nb(n).$$

We arrive at the conclusion that on the honeycomb lattice the $C_{\mathbf{n}}(A)$'s are

$$C_{\mathbf{n}}(A) = n \sum_{\substack{l_1,l_2,\ldots,l_j \\ \text{composition of } n'=0,1,2,\ldots,n \\ j\leq \min(n',n-n'+1)}} c_n(l_1,l_2,\ldots,l_j)$$

$$\sum_{k_3=-l_3}^{l_3}\sum_{k_4=-l_4}^{l_4}\cdots\sum_{k_j=-l_j}^{l_j}\binom{2l_1}{l_1+A+\sum_{i=3}^{j}(i-2)k_i}\binom{2l_2}{l_2-A-\sum_{i=3}^{j}(i-1)k_i}\prod_{i=3}^{j}\binom{2l_i}{l_i+k_i}$$

with the $c_n(l_1,l_2,\ldots,l_j)$'s given in (23) and the algebraic area bounded[1] by $\lfloor (n^2+3)/12\rfloor$.

---

[1]The sequence OEIS A135711 states that the minimal perimeter of a polyhex with A cells is $2\lceil\sqrt{12A-3}\rceil$. The maximum A for walks of length $2n$ is then $\lfloor (n^2+3)/12\rfloor$.

A few examples of $\frac{1}{q}\mathrm{Tr}H_q^n$ are listed below, and the corresponding $C_\mathbf{n}(A)$ are listed in Table 1.

$$\frac{1}{q}\mathrm{Tr}H_q = 3,$$

$$\frac{1}{q}\mathrm{Tr}H_q^2 = 15,$$

$$\frac{1}{q}\mathrm{Tr}H_q^3 = 3\left(29 + 2\cos\frac{2\pi p}{q}\right),$$

$$\frac{1}{q}\mathrm{Tr}H_q^4 = 3\left(181 + 32\cos\frac{2\pi p}{q}\right),$$

$$\frac{1}{q}\mathrm{Tr}H_q^5 = 3\left(1181 + 360\cos\frac{2\pi p}{q} + 10\cos\frac{4\pi p}{q}\right),$$

$$\frac{1}{q}\mathrm{Tr}H_q^6 = 3\left(7953 + 3520\cos\frac{2\pi p}{q} + 242\cos\frac{4\pi p}{q} + 8\cos\frac{6\pi p}{q}\right),$$

$$\frac{1}{q}\mathrm{Tr}H_q^7 = 3\left(54923 + 32032\cos\frac{2\pi p}{q} + 3710\cos\frac{4\pi p}{q} + 266\cos\frac{6\pi p}{q} + 14\cos\frac{8\pi p}{q}\right).$$

	$\mathbf{n} = 2$	4	6	8	10	12	14
$A = \quad 0$	3	15	87	543	3543	23859	164769
± 1			6	96	1080	10560	96096
± 2					30	726	11130
± 3						24	798
± 4							42
total counting	3	15	93	639	4653	35169	272835

Table 1: $C_\mathbf{n}(A)$ up to $\mathbf{n} = 14$ for honeycomb lattice walks of length $\mathbf{n}$.

4   Conclusions

We demonstrated that the area counting of honeycomb walks derives from an exclusion statistics $g = 2$ system with a "diluted Hofstadter" spectrum. This fact calls for a more detailed justification: in previous works [3, 4], two of the authors had shown that lattice walks that map to exclusion statistics are of the general form

$$H = f(u)\, v + v^{1-g}\, g(u)$$

with u, v the quantum torus matrices and $f(u), g(u)$ scalar functions. The honeycomb Hamiltonian is apparently not of this form. However, the expression of a walk in terms of a Hamiltonian is not unique: alternative versions corresponding to modular transformations on the lattice, or, equivalently, alternative realizations of the quantum torus algebra, can exist. We expect that an alternative realization of the honeycomb Hamiltonian H_{2q} that makes its connection to $g = 2$ statistics and the diluted spectral function S_k manifest does exist, and is related to the form given in Section (3.1) by a unitary transformation. The identification of this transformation and the alternative form of H_{2q} is an interesting open question.

Further, the anisotropic honeycomb Hamiltonian with general transition amplitudes a, b, c, is of physical interest. The corresponding generating function of lattice walks would depend on these parameters and would "count" the number of moves in the three different lattice directions U, V, W separately. The calculation of this generalized generating function through traces of powers of the Hamiltonian appears to be within reach using the methods and techniques of this paper and constitutes a subject for further investigation.

Acknowledgements

References

[1] D. R. Hofstadter, Phys. Rev. B **14**, 2239 (1976).

[2] S. Ouvry and S. Wu, J. Phys. A: Math. Theor. **52**, 255201 (2019).

[3] S. Ouvry and A. P. Polychronakos, Nucl. Phys. B **948**, 114731 (2019).

[4] S. Ouvry and A. P. Polychronakos, Nucl. Phys. B **960**, 115174 (2020).

[5] F. H. Claro and G. H. Wannier, Phys. Rev. B **19**, 6068 (1979).

[6] F. Yılmaz and M. Ö. Oktel, Phys. Rev. A **95**, 063628 (2017).

[7] K. S. Novoselov, Z. Jiang, Y. Zhang, S. V. Morozov, H. L. Stormer, U. Zeitler, J. C. Maan, G. S. Boebinger, P. Kim, and A. K. Geim, Science **315**, 1379 (2007).

[8] K. I. Bolotin, F. Ghahari, M. D. Shulman, H. L. Stormer, and P. Kim, Nature **462**, 196 (2009).

[9] E. Perfetto, J. González, F. Guinea, S. Bellucci, and P. Onorato, Phys. Rev. B **76**, 125430 (2007).

[10] M.-H. Kang, Discrete Math. **311**, 2384 (2011).

[11] A. Kitaev, Ann. Phys. **321**, 2 (2006).

[12] M. Levin and X.-G. Wen, Phys. Rev. B **71**, 045110 (2005).

[13] I. F. Herbut, Phys. Rev. Lett. **97**, 146401 (2006).

[14] R. Rammal, J. Phys. **46**, 1345 (1985).

[15] Ch. Kreft and R. Seiler, J. Math. Phys. **37**, 5207 (1996).

[16] A. L. C. Pereira and P. A. Schulz, Phys. Rev. B **78**, 125402 (2008).

[17] A. Agazzi, J.-P. Eckmann, and G. M. Graf, J. Stat. Phys. **156**, 426 (2014).

[18] Ch. Kreft, SFB 288 Preprint No. 89, TU-Berlin, 1993 (to be published).

[19] H. Weyl, *The theory of groups and quantum mechanics* (Dover, New York, 1950), Chap. IV, p. 279.

Appendices

A $Z(n)$ for square lattice walks

We denote $Z(n)$ as $Z_q(n)$ to include its dependence on q.

Substituting $d_q = \displaystyle\sum_{n=0}^{\lfloor q/2 \rfloor} (-1)^n Z_q(n) z^{2n}$ into (8) and equating the coefficient of z^{2n} on both sides, we get

$$
\begin{aligned}
Z_q(n) &= Z_{q-1}(n) + s_{q-1} Z_{q-2}(n-1) \\
&= Z_{q-2}(n) + s_{q-2} Z_{q-3}(n-1) + s_{q-1} Z_{q-2}(n-1) \\
&= \cdots \\
&= Z_1(n) + \sum_{m=0}^{q-2} s_{m+1} Z_m(n-1).
\end{aligned}
$$

Since $Z_m(n-1) = 0$ for $n-1 > \lfloor m/2 \rfloor$, i.e., $m < 2n - 2$, we obtain

$$
Z_q(n) = \sum_{m=2n-2}^{q-2} s_{m+1} Z_m(n-1)
$$

with $Z_q(0) = 1$.

Thus,

$$
\begin{aligned}
Z_q(1) &= \sum_{m=0}^{q-2} s_{m+1} Z_m(0) \\
&= \sum_{k_1=1}^{q-1} s_{k_1},
\end{aligned}
$$

$$
\begin{aligned}
Z_q(2) &= \sum_{m=2}^{q-2} s_{m+1} Z_m(1) \\
&= \sum_{m=2}^{q-2} \sum_{k_1=1}^{m-1} s_{m+1} s_{k_1} \\
&= \sum_{k_1=1}^{q-3} \sum_{k_2=1}^{k_1} s_{k_1+2} s_{k_2},
\end{aligned}
$$

$$Z_q(3) = \sum_{m=4}^{q-2} s_{m+1} Z_m(2)$$

$$= \sum_{m=4}^{q-2} \sum_{k_1=1}^{m-3} \sum_{k_2=1}^{k_1} s_{m+1} s_{k_1+2} s_{k_2}$$

$$= \sum_{k_1=1}^{q-5} \sum_{k_2=1}^{k_1} \sum_{k_3=1}^{k_2} s_{k_1+4} s_{k_2+2} s_{k_3}.$$

The formula (9) can be then proven by mathematical induction, where we check

$$Z_q(n+1) = \sum_{m=2n}^{q-2} s_{m+1} Z_m(n)$$

$$= \sum_{m=2n}^{q-2} \sum_{k_1=1}^{m-2n+1} \sum_{k_2=1}^{k_1} \sum_{k_3=1}^{k_2} \cdots \sum_{k_n=1}^{k_{n-1}} s_{m+1} s_{k_1+2n-2} \cdots s_{k_{n-1}+2} s_{k_n}$$

$$= \sum_{k_1=1}^{q-2n-1} \sum_{k_2=1}^{k_1} \sum_{k_3=1}^{k_2} \sum_{k_4=1}^{k_3} \cdots \sum_{k_{n+1}=1}^{k_n} s_{k_1+2n} s_{k_2+2n-2} \cdots s_{k_n+2} s_{k_{n+1}}.$$

B Examples of algebraic area enumeration of random walks on the square lattice

A few examples of $\frac{1}{q}\mathrm{Tr}H_q^{\mathbf{n}}$ and the corresponding $C_{\mathbf{n}}(A)$'s are listed below and in Table 2.

$$\frac{1}{q}\mathrm{Tr}H_q^2 = 4,$$

$$\frac{1}{q}\mathrm{Tr}H_q^4 = 4\left(7 + 2\cos\frac{2\pi p}{q}\right),$$

$$\frac{1}{q}\mathrm{Tr}H_q^6 = 4\left(58 + 36\cos\frac{2\pi p}{q} + 6\cos\frac{4\pi p}{q}\right),$$

$$\frac{1}{q}\mathrm{Tr}H_q^8 = 4\left(539 + 504\cos\frac{2\pi p}{q} + 154\cos\frac{4\pi p}{q} + 24\cos\frac{6\pi p}{q} + 4\cos\frac{8\pi p}{q}\right),$$

$$\frac{1}{q}\mathrm{Tr}H_q^{10} = 4\left(5486 + 6580\cos\frac{2\pi p}{q} + 2770\cos\frac{4\pi p}{q} + 780\cos\frac{6\pi p}{q} + 210\cos\frac{8\pi p}{q} + 40\cos\frac{10\pi p}{q} + 10\cos\frac{12\pi p}{q}\right).$$

	n = 2	4	6	8	10
$A =$ 0	4	28	232	2156	21944
± 1		8	144	2016	26320
± 2			24	616	11080
± 3				96	3120
± 4				16	840
± 5					160
± 6					40
counting	4	36	400	4900	63504

Table 2: $C_{\mathbf{n}}(A)$ up to $\mathbf{n} = 10$ for square lattice walks of length $\mathbf{n}$.

C Representation of the honeycomb algebra

Define three new operators u, v, σ as

$$\sigma = Q^{-1/2}UVW, \ u = U\sigma, \ v = V\sigma$$
$$\Rightarrow \quad U = u\sigma, \ V = v\sigma, \ W = Q^{1/2}v\sigma u.$$

From the honeycomb algebra (18) we see that σ, u and v are all unitary and satisfy

$$vu = Quv, \ u\sigma = \sigma u^{-1}, \ \sigma v^{-1} = v\sigma, \ \sigma^2 = 1. \tag{24}$$

Since U, V and W can be uniquely expressed in terms of σ, u and v, it is sufficient to derive the irreducible representation ("irrep" for short) of u, v and σ.

Operators u and v satisfy the quantum torus algebra and have a q-dimensional irrep if $Q = \exp(2i\pi p/q)$. However, σ can be embedded within this irrep only for specific values of the Casimirs $u^q = e^{i\phi}$ and $v^q = e^{i\theta}$. Indeed, assuming σ acts within this irrep,

$$u^q = \sigma u^q \sigma = u^{-q} \quad \Rightarrow \quad e^{i\phi} = e^{-i\phi}.$$

So ϕ can only be 0 or π (mod 2π), and similarly for θ. For $\theta, \phi \in \{0, \pi\}$ we can show that the irrep of (24) is unique up to unitary transformations, and up to the algebra automorphism $\sigma \to -\sigma$, and is given by the action on basis states $|n\rangle$

$$u\,|n\rangle = e^{i(\phi + 2\pi pn)/q}\,|n\rangle \quad, \quad n = 0, 1, \ldots, q-1,$$
$$v\,|n\rangle = e^{i\theta/q}\,|n-1\rangle \quad, \quad |-1\rangle \equiv |q-1\rangle,$$
$$\sigma\,|n\rangle = e^{i\theta(2n-r)/q}\,|r-n\rangle, \quad rp + \phi/\pi = 0 \ (\text{mod } q).$$

The "pivot" r in the inversion action of σ is $r = 0$, if $\phi = 0$, and the primary solution of the Diophantine equation $kq - rp = 1$, if $\phi = \pi$. The momenta $qk_x = \theta$ and $qk_y = \phi$ in this irrep are quantized as

$$k_x = \frac{\pi n_x}{q}, \quad k_y = \frac{\pi n_y}{q}, \quad n_x, n_y \in \mathbb{Z}. \tag{25}$$

For either θ or $\phi \notin \{0, \pi\}$ the irrep of (24) must decompose into more than one q-dimensional irreps of the quantum torus algebra u, v with σ mixing the irreps. The minimal irrep of the full algebra (24) involves 2 irreps of the torus algebra, all other situations being reducible. Representing all operators in block diagonal form in the space of the two irreps u_i, v_i, $i = 1, 2$, with Casimirs $u_i^q = e^{i\phi_i}$, $v_i^q = e^{i\theta_i}$,

$$u = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}, \quad v = \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}, \quad \sigma = \begin{pmatrix} A & B \\ B^\dagger & C \end{pmatrix},$$

and implementing the relations $\sigma u^q \sigma = u^{-q}$, $\sigma v^q \sigma = v^{-q}$ leads to

$$\left(e^{i\phi_1} - e^{-i\phi_1}\right) A = \left(e^{i\phi_2} - e^{-i\phi_2}\right) C = \left(e^{i\phi_1} - e^{-i\phi_2}\right) B = 0,$$
$$\left(e^{i\theta_1} - e^{-i\theta_1}\right) A = \left(e^{i\theta_2} - e^{-i\theta_2}\right) C = \left(e^{i\theta_1} - e^{-i\theta_2}\right) B = 0.$$

Since not both of ϕ_1, ϕ_2 and of θ_1, θ_2 can be 0 or π, the above relations imply $A = C = 0$. $\sigma^2 = 1$ then implies $B^\dagger B = 1$, and the last equalities above require $\phi_1 = -\phi_2$, $\theta_1 = -\theta_2$. Further, a unitary transformation

$$S = \begin{pmatrix} B^\dagger & 0 \\ 0 & 1 \end{pmatrix}, \quad u \to SuS^{-1}, \quad v \to SvS^{-1}, \quad \sigma \to S\sigma S^{-1}$$

eliminates B in σ, and $\sigma u \sigma = u^{-1}$, $\sigma v \sigma = v^{-1}$ imply $u_1 = u_2^{-1}$, $v_1 = v_2^{-1}$. Altogether, the irrep of (24) for two arbitrary Casimirs $\phi = \phi_1 = -\phi_2$, $\theta = \theta_1 = -\theta_2$, is given by the $2q$-dimensional matrices

$$u = \begin{pmatrix} u_o & 0 \\ 0 & u_o^{-1} \end{pmatrix}, \quad v = \begin{pmatrix} v_o & 0 \\ 0 & v_o^{-1} \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{26}$$

where u_o and v_o are the basic q-dimensional quantum torus irrep with Casimirs $e^{i\phi}$ and $e^{i\theta}$. Finally, from (24) we obtain the corresponding irreducible forms for U, V, W

$$U = \begin{pmatrix} 0 & u_o \\ u_o^{-1} & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & v_o \\ v_o^{-1} & 0 \end{pmatrix}, \quad W = Q^{1/2} \begin{pmatrix} 0 & v_o u_o^{-1} \\ v_o^{-1} u_o & 0 \end{pmatrix}.$$

We conclude with a demonstration that the above representation becomes reducible if $\phi, \theta \in \{0, \pi\}$. In that case, as we demonstrated before in (25), there is a $q \times q$ matrix σ_o (to be distinguished from the $2q \times 2q$ matrix σ in (26) above) satisfying (24) for the matrices u_o and v_o. Performing the unitary transformation

$$S_o = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -\sigma_o \\ \sigma_o & 1 \end{pmatrix}$$

on all matrices, and using $\sigma_o u_o \sigma_o = u_o^{-1}$ etc., we obtain

$$u = \begin{pmatrix} u_o & 0 \\ 0 & u_o^{-1} \end{pmatrix}, \quad v = \begin{pmatrix} v_o & 0 \\ 0 & v_o^{-1} \end{pmatrix}, \quad \sigma = \begin{pmatrix} \sigma_o & 0 \\ 0 & -\sigma_o \end{pmatrix},$$

or

$$U = \begin{pmatrix} u_o \sigma_o & 0 \\ 0 & -\sigma_o u_o \end{pmatrix}, \quad V = \begin{pmatrix} v_o \sigma_o & 0 \\ 0 & -\sigma_o v_o \end{pmatrix}, \quad W = Q^{1/2} \begin{pmatrix} v_o u_o^{-1} \sigma_o & 0 \\ 0 & -\sigma_o v_o u_o^{-1} \end{pmatrix}$$

reducing to the direct sum of two q-dimensional irreps.

D $Z(n)$ for honeycomb lattice walks

We denote $Z(n)$ as $Z_q(n)$ to include its dependence on q.

Substituting $d_q = \sum_{n=0}^{q}(-1)^n Z_q(n) z^{2n}$ into (22) and equating the coefficient of z^{2n} on both sides, we get

$$
\begin{aligned}
Z_q(n) &= Z_{q-1}(n) + \left(1+s_q\right)Z_{q-1}(n-1) - s_{q-1}Z_{q-2}(n-2) \\
&= Z_{q-2}(n) + \left(1+s_{q-1}\right)Z_{q-2}(n-1) + \left(1+s_q\right)Z_{q-1}(n-1) - s_{q-2}Z_{q-3}(n-2) - s_{q-1}Z_{q-2}(n-2) \\
&= \cdots \\
&= Z_1(n) + \sum_{m=1}^{q-1}\left(1+s_{m+1}\right)Z_m(n-1) - \sum_{m=0}^{q-2} s_{m+1}Z_m(n-2).
\end{aligned}
$$

Since $Z_m(n) = 0$ for $n > m$, we obtain

$$
Z_q(n) = \sum_{m=n-1}^{q-1}\left(1+s_{m+1}\right)Z_m(n-1) - \sum_{m=n-2}^{q-2} s_{m+1}Z_m(n-2)
$$

with $Z_q(0) = 1$ and $Z_q(j) = 0$ for $j < 0$.

Thus,

$$
\begin{aligned}
Z_q(1) &= \sum_{m=0}^{q-1}\left(1+s_{m+1}\right)Z_m(0) \\
&= \sum_{k_1=1}^{q}\left(1+s_{k_1}\right),
\end{aligned}
$$

$$
\begin{aligned}
Z_q(2) &= \sum_{m=1}^{q-1}\left(1+s_{m+1}\right)Z_m(1) - \sum_{m=0}^{q-2} s_{m+1}Z_m(0) \\
&= \sum_{m=1}^{q-1}\sum_{k_1=1}^{m}\left(1+s_{m+1}\right)\left(1+s_{k_1}\right) - \sum_{m=0}^{q-2} s_{m+1} \\
&= \sum_{k_1=1}^{q-1}\sum_{k_2=1}^{k_1}\left(1+s_{k_1+1}\right)\left(1+s_{k_2}\right) - \sum_{k_1=1}^{q-1} s_{k_1},
\end{aligned}
$$

$$
\begin{aligned}
Z_q(3) &= \sum_{m=2}^{q-1}\left(1+s_{m+1}\right)Z_m(2) - \sum_{m=1}^{q-2} s_{m+1}Z_m(1) \\
&= \sum_{m=2}^{q-1}\sum_{k_1=1}^{m-1}\sum_{k_2=1}^{k_1}\left(1+s_{m+1}\right)\left(1+s_{k_1+1}\right)\left(1+s_{k_2}\right) - \sum_{m=2}^{q-1}\sum_{k_1=1}^{m-1}\left(1+s_{m+1}\right)s_{k_1} - \sum_{m=1}^{q-2}\sum_{k_1=1}^{m} s_{m+1}\left(1+s_{k_1}\right) \\
&= \sum_{k_1=1}^{q-2}\sum_{k_2=1}^{k_1}\sum_{k_3=1}^{k_2}\left(1+s_{k_1+2}\right)\left(1+s_{k_2+1}\right)\left(1+s_{k_3}\right) - \sum_{k_1=1}^{q-2}\sum_{k_2=1}^{k_1}\left(1+s_{k_1+2}\right)s_{k_2} - \sum_{k_1=1}^{q-2}\sum_{k_2=1}^{k_1} s_{k_1+1}\left(1+s_{k_2}\right).
\end{aligned}
$$