# Exclusion statistics for particles with a discrete spectrum

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**Abstract**

We formulate and study the microscopic statistical mechanics of systems of particles with exclusion statistics in a discrete one-body spectrum. The statistical mechanics of these systems can be expressed in terms of effective single-level grand partition functions obeying a generalization of the standard thermodynamic exclusion statistics equation of state. We derive explicit expressions for the thermodynamic potential in terms of microscopic cluster coefficients and show that the mean occupation numbers of levels satisfy a nesting relation involving a number of adjacent levels determined by the exclusion parameter. We apply the formalism to the harmonic Calogero model and point out a relation with the Ramanujan continued fraction identity and appropriate generalizations.

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# 1 Introduction

Anyons statistics [1], and the related topics of fractional and exclusion statistics [2], are enjoying renewed popularity since the 2020 announcement of experimental confirmations [3] of lowest Landau level (LLL) anyon excitations with statistics 1/3 in fractional quantum Hall samples at filling 1/3. The statistics relevant to these results are abelian, which is indeed the simplest and, in principle, easiest to observe among non conventional statistics, leaving aside possible more elaborate nonabelian extensions. In this context, it is of interest to study the manifestation of exclusion statistics in its microscopic setting, and this is the aim of the present work.

The concept of exclusions statistics, as introduced by Haldane [2], essentially holds only at the thermodynamic limit, or in situations where the Hilbert space consists of degenerate states as in the LLL of an external magnetic field. Its *ab initio* microscopic formulation (i.e., starting from a 1-body spectrum and filling it with particles obeying exclusion statistics, as can be done in the standard Bose and Fermi cases) is, in general, impossible for non integer (fractional) exclusion parameter $g$, since exact many-body states cannot be defined. Microscopic concepts become, in principle, accessible when the exclusion parameter $g$ is constrained to be an integer, but are again to a large extent ambiguous as they are not invariant under Hilbert space state reparametrizations.

This is to be contrasted to anyon statistics, defined in 2 dimensions in terms of a microscopic $N$-anyon quantum Hamiltonian [4], with a statistical (exchange) anyon parameter $\alpha$ taking continuous values in $[0, 2)$. Physics is periodic in $\alpha$ with period 2: $\alpha = 0$ corresponds to bosons, $\alpha = 1$ to fermions, and $\alpha = 2$ again to bosons. It is well known that Haldane/exclusion statistics and anyon statistics are intimately related when one considers anyons projected onto the LLL of an external magnetic field, the so-called LLL-anyon model [5]. Here exclusion statistics manifests itself for a system of particles with a continuous degenerate 1-body density of states in the LLL.

Our present aim is to focus on exclusion statistics in its most general microscopic setting, i.e., for particles with a discrete 1-body spectrum $\epsilon_1, \epsilon_2, \ldots, \epsilon_q$. In this situation, exclusion statistics requires a natural (and dynamically relevant) ordering of the 1-body energy eigenstates, as, for example, when a single quantum number $k$ indexes them. This is typically the case for 1-dimensional systems, but exclusion statistics can be relevant in more general settings provided that a principal quantum number induces a natural ordering of the spectrum, as it happens for example in the case of the 2-dimensional LLL-anyon model properly regularized at long distances by a harmonic well.

In the next section we will give a summary of anyon statistics in the context of the LLL-anyon model [5] and will review the intimate relation of Haldane exclusion statistics to LLL-anyon statistics. We will also consider the thermodynamics of the Calogero model [6, 7] as yet another example of a microscopic realization of exclusion statistics. This will allow for a general definition of the thermodynamics relations governing a gas of particles with exclusion statistics and a continuous 1-body density of states.

We will then move to the situation of present interest, the statistical mechanics of a gas of particles in $q$ discrete 1-body energy levels $\epsilon_1, \epsilon_2, \ldots, \epsilon_q$ in a specific ordering with an integer exclusion parameter $g$. Bosons correspond to $g = 0$, with no exclusion, while $g = 1$ is the Fermi case where no more than one particle per quantum state is allowed. For $g$-exclusion, levels can again be occupied by at most one particle, but in addition at least $g - 1$ unoccupied levels must exist between any two occupied states.

Our main results, presented in section **3**, are that the statistical mechanics of these systems can be written as a generalization of the thermodynamics relations governing LLL-anyon or Calogero particles, in terms of effective single-state grand partition functions obeying a generalization of the standard exclusion statistics equation of state. Interestingly, two distinct such effective partition functions can be defined, termed "forward" and "backward", obeying different equations of state but leading to identical statistical mechanics. We also define appropriate thermodynamic potentials and give their explicit expressions in terms of cluster coefficients that involve sums over generalized partitions of the particle number. The mean occupation number of each 1-body level is then expressed in terms of the effective grand partition functions and shown to satisfy nesting relations that involve $g$ nearby levels. Finally, in section **4** the formalism is applied to the specific case of the harmonic Calogero system, and a relation to the Ramanujan continued fraction identity is pointed out for $g = 2$ and related generalizations for $g > 2$. We conclude with some directions for future research.

## 2 LLL-anyons and exclusion thermodynamics

We start with a review of LLL-anyons in an isotropic harmonic trap of frequency $\omega$ and their intimate relation with exclusions statistics. The spectrum of the $N$-anyon system is

$$E_N = (\omega_t - \omega_c)\Big[\sum_{i=1}^{N} l_i + \alpha\frac{N(N-1)}{2}\Big] + N\omega_t \ , \quad 0 \le l_1 \le l_2 \le \ldots \le l_N \tag{1}$$

where $\omega_c$ is half the cyclotron frequency, $\omega_t = \sqrt{\omega_c^2 + \omega^2}$, the $l_i$ are 1-body LLL angular momentum quantum numbers in 2 dimensions, and $\alpha$ is understood to be in the interval $[0, 2)$. We stress that the harmonic well is introduced as a long distance regulator to split the degeneracy of the LLL (when $\omega = 0$ the LLL-anyon spectrum (1) reduces trivially to $E_N = N\omega_c$) and will be taken to vanish in the thermodynamic limit. When $\alpha : 0 \to 1$ the spectrum (1) interpolates continuously between the harmonic LLL-Bose and LLL-Fermi spectra. Going beyond $\alpha = 1$, we note that due to the presence of the magnetic field the Bose limit $\alpha \to 2^-$ differs from the standard Bose case $\alpha = 0$ because of some missing states. At $\alpha = 2$ these missing states are restored thanks to excited states merging into the LLL ground state.

The LLL-anyon spectrum (1) is, in fact, identical in form to the spectrum of the 1-dimensional harmonic Calogero model with interaction strength $\alpha(\alpha - 1)$. The relation between the two models is well established (see [9] for an explicit mapping) and the exclusion statistics interpretation of their statistical mechanics is a common feature. We will come back later in this section to the well-known connection of the Calogero model to exclusion statistics in the thermodynamic limit $\beta\omega \to 0$, and in section **4** we will examine in more detail its microscopic statistical mechanics in the dicrete case ($\beta\omega \neq 0$) using the results of section **3**.

From the harmonic LLL-anyon spectrum (1) the $N$-body partition function $Z_N$, grand partition function $\mathcal{Z}$ and cluster coefficients $b_n$, defined as

$$\mathcal{Z} = \sum_{N=0}^{\infty} Z_N\, z^N \ , \quad \ln \mathcal{Z} = \sum_{n=1}^{\infty} b_n\, z^n$$

where $z$ is the fugacity, can be calculated. To probe the effect of the statistics, the cluster coefficients $b_n$ were studied in [5] and found to leading order in $\beta\omega$ (that is, for $\beta(\omega_t - \omega_c)$ small) to be

$$b_n = \frac{1}{\beta(\omega_t - \omega_c)} e^{-n\beta\omega_c}\, \frac{1}{n^2} \prod_{k=1}^{n-1} \frac{k - n\alpha}{k} \tag{2}$$

Taking then the thermodynamic limit $\beta\omega \to 0$, (which in the present case amounts to $1/(n\beta^2\omega^2) \to V/\lambda^2$, where $V$ is the macroscopically large 2-dimensional volume – here, area– of the system and $\lambda$ the thermal wavelength), the LLL-anyon thermodynamic potential follows

$$\ln \mathcal{Z} = N_L \ln y \tag{3}$$

where $N_L = BV/\Phi_0$ is the LLL degeneracy, i.e., the number of magnetic flux quanta in the volume – here, area – of the system. The function $y$ was found to satisfy

$$y - z e^{-\beta\omega_c} y^{1-\alpha} = 1 \ , \tag{4}$$

so that

$$y = \sum_{N=0}^{\infty} z^N e^{-\beta N\omega_c} \prod_{k=2}^{N} \frac{k - N\alpha}{k}$$

(by definition $\prod_{k=k_1}^{k_2} (\ldots) = 1$ when $k_2 < k_1$).

Using $\mathcal{Z} = y^{N_L}$, it was deduced [5]

$$Z_N = e^{-\beta N\omega_c} N_L \prod_{k=2}^{N} \frac{k + N_L - N\alpha - 1}{k}$$

$Z_N$ is the LLL-anyon $N$-body partition function for $N$ degenerate anyons at energy $N\omega_c$, thus identifying their degeneracy as

$$N_L \prod_{k=2}^{N} \frac{k + N_L - N\alpha - 1}{k} = \frac{N_L}{N!} \frac{(N + N_L - N\alpha - 1)!}{(N_L - N\alpha)!} \tag{5}$$

(factorials for fractional argument are defined in terms of the corresponding Γ-functions.)

Let us now allow $\alpha$ to take integer values beyond the interval $[0, 2)$: the degeneracy (5) counts the number of ways to put $N$ particles in $N_L$ degenerate quantum states in a circular configuration such that there are at least $\alpha - 1$ empty states between any two occupied states [8]. This is the hallmark of exclusion statistics with exclusion parameter $\alpha$. In particular, $\alpha = 2$ describes a Bose gas but with nontrivial $\alpha = 2$ exclusion.

Indeed, Haldane exclusion statistics postulates that given $G$ single-particle states already populated by $N - 1$ particles the number of quantum states available for an additional $N^{\text{th}}$ particle is $G - (N - 1)g$ (this is heuristic, and somewhat misleading [8]) where $g$ is the exclusion parameter. $g$ would need to be an integer for this to be meaningful, but this requirement can be dropped in the thermodynamic limit where $G$ and $N$ become large. Starting from the standard Bose degeneracy for $N$ bosons in $G$ quantum states

$$\frac{(N + G - 1)!}{N!(G - 1)!}$$

Haldane encoded exclusion by replacing $G$ by $G - (N - 1)g$ to propose the $N$-body exclusion degeneracy

$$\frac{(N + G - (N - 1)g - 1)!}{N!(G - (N - 1)g - 1)!} \tag{6}$$

When $g$ is a positive integer, this is the number of ways to put on a line $N$ particles in $G$ quantum states in such a way there are at least $g - 1$ empty states in between two occupied states. This is the same as the LLL-anyon counting (5) discussed above upon setting $g = \alpha$ and $G = N_L$ in the Haldane counting and placing the states on a line rather than a circle, which is irrelevant in the thermodynamic limit where $G = N_L \to \infty$. So Haldane exclusion statistics is identical to LLL-anyon statistics provided that the anyonic exchange statistical parameter $\alpha$ is allowed to take integer values beyond $[0, 2)$. Not surprisingly, in view of this intimate relation between LLL-anyon statistics and Haldane exclusion statistics, the LLL-anyon thermodynamic (3) and (4) can be directly recovered [10] from the Haldane Hilbert space counting (6).

In conclusion, exclusion/LLL-anyon thermodynamics amounts to

$$\ln \mathcal{Z} = N_L \ln y \quad , \quad y - z e^{-\beta\omega_c} y^{1-\alpha} = 1 \tag{7}$$

Let us focus on the mean particle number $\bar{N} = z\partial \ln Z/\partial z$ or, equivalently, on the LLL filling factor

$$\nu = \frac{\bar{N}}{N_L} = z\frac{\partial \ln y}{\partial z}$$

Using $y - z e^{-\beta\omega_c} y^{1-\alpha} = 1$ we can obtain

$$y = 1 + \frac{\nu}{1 - \alpha\,\nu}$$

and therefore also

$$z e^{-\beta\omega_c} = \frac{\nu}{(1 + (1 - \alpha)\nu)^{1-\alpha}(1 - \alpha\,\nu)^{\alpha}}$$

and from $\ln \mathcal{Z} = N_L \ln y$ we obtain the equation of state

$$\ln \mathcal{Z} = \beta PV = N_L \ln(1 + \frac{\nu}{1 - \alpha\,\nu})$$

with a critical filling $\nu = 1/\alpha$ where the pressure diverges [5]. Interestingly, at the critical filling the $N$-body LLL-anyon wave function is nondegenerate

$$\psi = \prod_{i<j}(z_i - z_i)^\alpha e^{-\omega_c \sum_{i=1}^N \bar{z}_i z_i/2} \tag{8}$$

which coincides with the Laughlin wavefunction when $\alpha = 2m+1$, $m = 1, 2, \ldots$, encoding $\alpha$-exclusion for an incompressible $N$-anyon state in the LLL.

The above thermodynamics readily generalizes to exclusion statistics systems with arbitrary 1-body density of states $\rho(\epsilon)$. In the thermodynamic limit we can apply relations (7) for states around energy $\epsilon$, with the number of states $N_L$ substituted by $\rho(\epsilon)d\epsilon$, obtaining

$$\ln \mathcal{Z} = \int_0^\infty \rho(\epsilon) \ln y(\epsilon)\; d\epsilon \quad , \quad y(\epsilon) - z e^{-\beta\epsilon} y(\epsilon)^{1-\alpha} = 1 \tag{9}$$

The LLL-anyon result (7) is recovered for $\rho(\epsilon) = N_L \delta(\epsilon - \omega_c)$. Similarly, for the mean occupation number $n(\epsilon)$ per level at energy $\epsilon$ and the mean particle number $\bar{N}$

$$n(\epsilon) = z\frac{\partial \ln y(\epsilon)}{\partial z} \quad , \quad \bar{N} = \int_0^\infty \rho(\epsilon) n(\epsilon)\; d\epsilon$$

we obtain, in view of (9),

$$\begin{aligned} y(\epsilon) &= 1 + \frac{n(\epsilon)}{1 - \alpha\; n(\epsilon)} \\ z e^{-\beta\epsilon} &= \frac{n(\epsilon)}{[1 + (1-\alpha) n(\epsilon)]^{1-\alpha} [1 - \alpha\; n(\epsilon)]^\alpha} \end{aligned} \tag{10}$$

and therefore

$$\ln \mathcal{Z} = \int_0^\infty \rho(\epsilon) \ln \left(1 + \frac{n(\epsilon)}{1 - \alpha\; n(\epsilon)}\right) d\epsilon \tag{11}$$

A question naturally arises about the existence of other microscopic quantum models with the same kind of statistics. One known example, as already mentioned, is the Calogero model. The harmonic Calogero model on the 1-dimensional line with inverse-square 2-body interactions and a confining harmonic potential is described by the Hamiltonian

$$H_N = -\frac{1}{2}\sum_{i=}^N \frac{\partial^2}{\partial x_i^2} - \alpha(1-\alpha)\sum_{i<j}\frac{1}{(x_i - x_j)^2} + \frac{1}{2}\omega^2 \sum_{i=1}^N x_i^2 \tag{12}$$

Its spectrum is given by

$$E_N = \omega\,[\sum_{i=1}^N l_i + \alpha\frac{N(N-1)}{2} + \frac{N}{2}] \quad 0 \le l_1 \le l_2 \le \ldots \le l_N \tag{13}$$

where the $l_i$ are now 1-dimensional "pseudo-excitation numbers" labeling the 1-body harmonic eigenstates. This spectrum directly follows from the the LLL-anyon spectrum (1) upon letting $\omega_c \to 0$, i.e. in the absence of the external magnetic field, up to a trivial $N$-body energy shift $N\omega/2$. Note that $\omega$ now plays the role of a 1-dimensional long-distance regulator in the Calogero case, with the difference that the limit $\omega \to 0$ does not lead anymore to an infinitely degenerate LLL but, rather, to free 1-dimensional particles with generalized statistics manifesting through their scattering phase shift [6].

All the thermodynamic considerations presented in this section in the context of LLL anyons apply equally to Calogero particles. In particular, when $\omega_c = 0$ the cluster coefficients (2) in the thermodynamic limit $\beta\omega \to 0$ lead to (9, 10, 11) with density of states

$$\rho(\epsilon) = \frac{L}{\pi\sqrt{2\epsilon}}$$

i.e., a free 1-dimensional density of states on a space of macroscopically large length $L$.

So the 2-dimensional LLL-anyon and the 1-dimensional Calogero models both share the same exclusion statistics/thermodynamics (9, 10, 11). This is of course not surprising: as already stressed above one can show [9] that the 1-dimensional Calogero model is a particular projection of the 2-dimensional anyon model with same exclusion statistics and a free 1-body density of states on the line stemming in the LLL from the dimensional reduction $\lim_{\omega_c \to 0} N_L \delta(\epsilon - \omega_c) = \frac{L}{\pi\sqrt{2\epsilon}}$ induced by the vanishing magnetic field.

All said and done, (9, 10, 11) can be viewed in full generality as the defining thermodynamic relations for particles with exclusion statistics $\alpha$ and a continuous 1-body density of states $\rho(\epsilon)$. However, specific dynamical systems manifesting these thermodynamics, where exclusion statistics is *microscopically* realized in terms of $N$-body quantum Hamiltonians, are limited to the two cases above – the LLL-anyon and the Calogero models. In the next section we consider in general such systems, defined through a set of 1-body energy levels and an exclusion rule in filling them for many-body states, and derive their exact *microscopic statistical mechanics*, rather than their thermodynamics.

## 3 Exclusion statistics for a discrete 1-body spectrum

We now turn to exclusion statistics for a discrete density of states, that is for a 1-body spectrum $\epsilon(k)$, $k = 1, \ldots, q$. As stressed in the Introduction, we assume a natural ordering of levels $\epsilon(1), \epsilon(2), \ldots, \epsilon(q)$ in terms of the principal quantum number $k$ in $\epsilon(k)$, which is relevant to the definition of exclusion. The Boltzmann factor for the energy level $\epsilon(k)$ is

$$s(k) = e^{-\beta\epsilon(k)}$$

We call $s(k)$ the spectral function. Our focus is to derive relations analogous to (9, 10, 11) for particles with exclusion statistics in the discrete spectrum above.

### 3.1 $g = 2$

As a warmup, we start with the simplest case beyond Fermi statistics, i.e., $g = 2$ exclusion.

The $N$-body partition function is, by definition,

$$Z_N = \sum_{k_1=1}^{q-2N+2} \sum_{k_2=1}^{k_1} \cdots \sum_{k_N=1}^{k_{N-1}} s(k_1 + 2N - 2)s(k_2 + 2N - 4) \cdots s(k_{N-1} + 2)s(k_N)$$

where the cumulative $+2$ shifts in the arguments of the spectral function enforce $g = 2$ exclusion: adjacent 1-body levels $k$ and $k + 1$ cannot be populated. The grand partition function follows as

$$\mathcal{Z}_{1,q} = 1 + \sum_{N=1}^{(q+1)/2} Z_N z^N$$

(the indices $1, q$ refer to the first and last levels in the spectrum). The cluster expansion of the grand potential is

$$\ln \mathcal{Z}_{1,q} = \sum_{n=1}^{\infty} b_n z^n$$

with the cluster coefficients $b_n$ calculated to be

$$b_n = (-1)^{n-1} \sum_{\substack{l_1, l_2, \ldots, l_j \\ \text{composition of } n \\ j \le q}} c_2(l_1, l_2, \ldots, l_j) \sum_{k=1}^{q-j+1} s^{l_j}(k + j - 1) \cdots s^{l_2}(k + 1)s^{l_1}(k) \tag{14}$$

In (14), the sum is over all compositions (i.e., ordered partitions) of the integer $n$, with the number of parts $j$ of a given composition, by definition smaller than or equal to $n$, also constrained to be smaller than or equal to $q$, the number of available 1-body quantum states. The combinatorial coefficients $c_2(l_1, l_2, \ldots, l_j)$ are [11]

$$c_2(l_1, l_2, \ldots, l_j) = \frac{\binom{l_1+l_2}{l_1}}{l_1 + l_2} \; l_2 \frac{\binom{l_2+l_3}{l_2}}{l_2 + l_3} \; \cdots \; l_{j-1} \frac{\binom{l_{j-1}+l_j}{l_{j-1}}}{l_{j-1} + l_j} \, .$$

By rearranging the sums in (14), the cluster coefficient can also take the alternative form

$$b_n = (-1)^{n-1} \sum_{k=1}^{q} \sum_{\substack{l_1, l_2, \ldots, l_j \\ \text{composition of } n \\ j \le q-k+1}} c_2(l_1, l_2, \ldots, l_j) s(k + j - 1)^{l_j} \cdots s(k + 1)^{l_2} s(k)^{l_1} \, .$$

A useful observation [12] is that $\mathcal{Z}_{1,q}$ can be expressed as the secular determinant of the off-diagonal $(q + 1) \times (q + 1)$ matrix

$$H_{1,q} = \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 & 0 \\ s(1) & 0 & -1 & \cdots & 0 & 0 \\ 0 & s(2) & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & \cdots & s(q) & 0 \end{pmatrix} \tag{15}$$

(this $H_{1,q}$ is one of several equivalent choices). Specifically,

$$\mathcal{Z}_{1,q} = \det(1_{q+1} + z^{1/2} H_{1,q})$$

We also define the general truncated grand partition functions $\mathcal{Z}_{k,k'}$ $(1 \le k \le k' \le q)$ as

$$\mathcal{Z}_{k,k'} = \det(1 + z^{1/2} H_{k,k'}) \tag{16}$$

with $H_{k,k'}$ the truncated $(k'-k+2) \times (k'-k+2)$ matrix

$$H_{k,k'} = \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 & 0 \\ s(k) & 0 & -1 & \cdots & 0 & 0 \\ 0 & s(k+1) & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & \cdots & s(k') & 0 \end{pmatrix}$$

i.e., the matrix (15) for a truncated 1-body spectrum $\epsilon(k), \epsilon(k+1), \ldots, \epsilon(k')$ starting at level $k$ and ending at level $k'$. We also define that, trivially, $\mathcal{Z}_{k,k'} = 1$ when $k > k'$.

The grand partition function $\mathcal{Z}_{k,k'}$ can be expressed as

$$\mathcal{Z}_{k,k'} = 1 + \sum_{N=1}^{(k'-k+2)/2} Z_N(k,k') z^N$$

where $Z_N(k,k')$ stands for the $g=2$ exclusion $N$-body partition function for the truncated 1-body spectrum $\epsilon(k), \epsilon(k+1), \ldots, \epsilon(k')$, i.e.,

$$Z_N(k,k') = \sum_{k_1=k}^{k'-2N+2} \sum_{k_2=k}^{k_1} \cdots \sum_{k_N=k}^{k_{N-1}} s(k_1+2N-2) s(k_2+2N-4) \cdots s(k_{N-1}+2) s(k_N) \; .$$

It also follows that

$$\ln \mathcal{Z}_{k,k'} = -\sum_{n=1}^{\infty} (-z)^n \sum_{\substack{l_1, l_2, \ldots, l_j \\ \text{composition of } n \\ j \le k'-k+1}} c_2(l_1, l_2, \ldots, l_j) \sum_{l=k}^{k'-j+1} s^{l_j}(l+j-1) \cdots s^{l_2}(l+1) s^{l_1}(l) \tag{17}$$

where the number of parts $j$ is now bounded by the number $k'-k+1$ of available quantum states in the truncated spectrum (here, as well as in all other similar cluster expressions which will appear below).

Expanding the determinant (16) in terms of its first row we obtain

$$\mathcal{Z}_{1,q} = \mathcal{Z}_{2,q} + z s(1) \mathcal{Z}_{3,q} \tag{18}$$

The recursion (18) is self explanatory and could have been written directly: because of $g=2$ exclusion, the full grand partition function is the sum of a grand partition for a

1-body spectrum starting at level $\epsilon(2)$ with level $\epsilon(1)$ empty, and of $zs(1)$ (level $\epsilon(1)$ filled) times the one starting at level $\epsilon(3)$. This is the basis for a recursion scheme: expanding the determinants $\mathcal{Z}_{k,q}$ and $\mathcal{Z}_{1,k}$, as defined in (16), in terms of their first and last row, respectively, yields the recursion relations

$$\mathcal{Z}_{k,q} = \mathcal{Z}_{k+1,q} + zs(k)\mathcal{Z}_{k+2,q} \ , \quad \mathcal{Z}_{1,k} = \mathcal{Z}_{1,k-1} + zs(k)\mathcal{Z}_{1,k-2} \tag{19}$$

We now introduce "forward" and "backward" effective single-level grand partition functions $y_+(k)$ and $y_-(k)$, respectively, as

$$y_+(k) = \frac{\mathcal{Z}_{k,q}}{\mathcal{Z}_{k+1,q}} \ ; \quad y_-(k) = \frac{\mathcal{Z}_{1,k}}{\mathcal{Z}_{1,k-1}} \tag{20}$$

In terms of them, the full grand partition function achieves a product form

$$\mathcal{Z}_{1,q} = \prod_{k=1}^{q} y_+(k) = \prod_{k=1}^{q} y_-(k)$$

Moreover, the recursion relations (19) imply the nesting relations

$$y_\pm(k) - \frac{zs(k)}{y_\pm(k\pm 1)} = 1 \tag{21}$$

where $y_+(0) = y_-(0) = y_+(q+1) = y_-(q+1) = 1$ is understood. We also note that, as a consequence of (17, 20), the cluster expansions of $y_+(k), y_-(k)$ are

$$\ln y_+(k) = -\sum_{n=1}^{\infty}(-z)^n \sum_{\substack{l_1,l_2,\ldots,l_j \\ \text{composition of } n \\ j\le q-k+1}} c_2(l_1,l_2,\ldots,l_j)s(k)^{l_1}s(k+1)^{l_2}\cdots s(k+j-1)^{l_j}$$

$$\ln y_-(k) = -\sum_{n=1}^{\infty}(-z)^n \sum_{\substack{l_1,l_2,\ldots,l_j \\ \text{composition of } n \\ j\le k}} c_2(l_1,l_2,\ldots,l_j)s(k-j+1)^{l_j}\cdots s(k-1)^{l_2}s(k)^{l_1}$$

We reach the conclusion that for a discrete 1-body spectrum the statistical mechanics of $g=2$ exclusion particles amounts to

$$\ln \mathcal{Z}_{1,q} = \sum_{k=1}^{q} \ln y_\pm(k) \ , \quad y_\pm(k) - \frac{zs(k)}{y_\pm(k\pm 1)} = 1$$

These are indeed (9) when $\alpha = 2$ but with the proviso that, because of the discreteness of the spectrum, $\ln \mathcal{Z}_{1,q}$ is now a discrete sum instead of a continuous integral, and a discrete shift $k \to k\pm 1$ materializes in the argument of $y_+(k)$ or $y_-(k)$.

Turning to the mean particle number $\bar{N}$, it is given by

$$\bar{N} = z\frac{\partial \ln \mathcal{Z}_{1,q}}{\partial z} = \sum_{k=1}^{q} n_k$$

where the mean occupation number $n_k$ of the energy level $\epsilon(k)$ is, by definition,

$$n_k = s(k)\frac{\partial \ln \mathcal{Z}_{1,q}}{\partial s(k)}$$

Using the expansion relation

$$\mathcal{Z}_{1,q} = zs(k)\mathcal{Z}_{1,k-2}\mathcal{Z}_{k+2,q} + \mathcal{Z}_{1,k-1}\mathcal{Z}_{k+1,q} \ ,$$

(another self-explanatory identity of which both (18) and (19) are special cases) $n_k$ can be expressed directly in terms of truncated grand partition functions

$$n_k = zs(k)\,\frac{\mathcal{Z}_{1,k-2}\mathcal{Z}_{k+2,q}}{\mathcal{Z}_{1,q}} \quad \Leftrightarrow \quad 1-n_k = \frac{\mathcal{Z}_{1,k-1}\mathcal{Z}_{k+1,q}}{\mathcal{Z}_{1,q}} \ .$$

From this we can obtain the nesting relation for $n_k$ (see Appendix for the proof)

$$zs(k) = \frac{n_k(1-n_k)}{(1-n_{k-1}-n_k)(1-n_k-n_{k+1})} \tag{22}$$

and using (20) we can express $y_+(k)$ and $y_-(k)$ in terms of $n_k$

$$y_\pm(k) = 1 + \frac{n_k}{1-n_k-n_{k\mp 1}} \tag{23}$$

and finally the thermodynamic potential

$$\ln \mathcal{Z}_{1,q} = \sum_{k=1}^{q} \ln\left(1+\frac{n_k}{1-n_{k-1}-n_k}\right) = \sum_{k=1}^{q} \ln\left(1+\frac{n_k}{1-n_k-n_{k+1}}\right) \tag{24}$$

(22, 23, 24) are the generalization of the $\alpha = 2$ LLL-anyon/Calogero models (10, 11) thermodynamics relations for the discrete spectrum at hand, i.e., with the discrete shifts $k \to k \pm 1$ in the arguments of $n$. (24) is the equation of state of a gas of particles with exclusion $g = 2$ and populating discrete energy levels $\epsilon(k)$ whose occupation numbers $n_k$ are constrained by $n_k + n_{k+1} \leq 1$.

Note that the nesting relation (22) does not allow for finding the $n_k$ in an iterative way, starting either at $k = 1$ or at $k = q$, since already for $k = 1$ it involves $n_1$ *and* $n_2$ and similarly for $k = q$. On the other hand, (21) allows the calculation of $y_+(k)$ and $y_-(k)$ iteratively, starting from $k = 1$ for $y_-(k)$ and $k = q$ for $y_+(k)$:

$$y_-(1) = 1 + zs(1) \ , \quad y_-(2) = 1 + \frac{zs(2)}{1+zs(1)} \ , \ldots$$
$$y_+(q) = 1 + zs(q) \ , \quad y_+(q-1) = 1 + \frac{zs(q-1)}{1+zs(q)} \ , \ldots$$

From these and (22) we can express $n_k$ as

$$n_k = \frac{y_+(k)-1}{y_+(k)+y_-(k-1)-1} = \frac{y_-(k)-1}{y_-(k)+y_+(k+1)-1}$$

### 3.2 General $g$

For general integer exclusion parameter $g$ the $N$-body partition function reads

$$Z_N = \sum_{k_1=1}^{q-gN+g} \sum_{k_2=1}^{k_1} \cdots \sum_{k_N=1}^{k_{N-1}} s(k_1+gN-g)s(k_2+gN-2g)\cdots s(k_{N-1}+g)s(k_N) \quad (25)$$

where the $+g$ shift in the nested sum indices enforces $g$-exclusion. The grand partition function $\mathcal{Z}_{1,q}$ follows as

$$\mathcal{Z}_{1,q} = 1 + \sum_{N=1}^{(q+g-1)/g} Z_N z^N$$

and the cluster expansion as

$$\ln \mathcal{Z}_{1,q} = -\sum_{n=1}^{\infty} (-z)^n \sum_{\substack{l_1,l_2,\ldots,l_j \\ \text{g-composition of } n \\ j \le q}} c_g(l_1,l_2,\ldots,l_j) \sum_{k=1}^{q-j+1} s^{l_j}(k+j-1)\cdots s^{l_2}(k+1)s^{l_1}(k) \quad (26)$$

which is the $g$-generalization of (14). Here the sum is over all $g$-compositions [11] of the integer $n$ which are defined as the usual compositions but where now up to $g-2$ successive $l_i$ can be zero. The $c_g(l_1,l_2,\ldots,l_j)$ are given as

$$\begin{aligned} c_g(l_1,l_2,\ldots,l_j) &= \frac{(l_1+\cdots+l_{g-1}-1)!}{l_1!\cdots l_{g-1}!} \prod_{i=1}^{j-g+1} \binom{l_i+\cdots+l_{i+g-1}-1}{l_{i+g-1}} \\ &= \frac{\prod_{i=1}^{j-g+1}(l_i+\cdots+l_{i+g-1}-1)!}{\prod_{i=1}^{j-g}(l_{i+1}+\cdots+l_{i+g-1}-1)!} \prod_{i=1}^{j} \frac{1}{l_i!} \end{aligned}$$

As before we can express $\mathcal{Z}_{1,q}$ as the secular determinant of a $(q+g-1)\times(q+g-1)$ matrix

$$H_{1,q} = \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ s(1) & 0 & 0 & \ddots & -1 & 0 & 0 & \cdots & 0 \\ 0 & s(2) & 0 & \ddots & 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & s(3) & \ddots & 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & s(q-1) & 0 & 0 & \cdots & -1 \\ 0 & 0 & 0 & \cdots & 0 & s(q) & 0 & \cdots & 0 \end{pmatrix} \quad (27)$$

where there are now $g-1$ successive vanishing diagonals, realizing $g$-exclusion in the matrix representation. Then

$$\mathcal{Z}_{1,q} = \det(1 + z^{1/g} H_{1,q})$$

Proceeding as in the $g = 2$ case, we define the general truncated grand partition functions $\mathcal{Z}_{k,k'}$ $(1 \le k \le k' \le q)$ as

$$\mathcal{Z}_{k,k'} = \det(1 + z^{1/g} H_{k,k'})$$

with $H_{k,k'}$ the $(k' - k + g) \times (k' - k + g)$ matrix (27) for the truncated 1-body spectrum $\epsilon(k), \epsilon(k+1), \dots, \epsilon(k')$ starting at level $k$ and ending at level $k'$, and, as before, $\mathcal{Z}_{k,k'} = 1$ when $k > k'$. The grand partition function $\mathcal{Z}_{k,k'}$ can be expressed as

$$\mathcal{Z}_{k,k'} = 1 + \sum_{N=1}^{(k'-k+g)/g} Z_N(k,k') z^N$$

where $Z_N(k,k')$ stands for the truncated $g$-exclusion $N$-body partition function

$$Z_N(k,k') = \sum_{k_1=k}^{k'-gN+g} \sum_{k_2=k}^{k_1} \cdots \sum_{k_N=k}^{k_{N-1}} s(k_1 + gN - g) s(k_2 + gN - 2g) \cdots s(k_{N-1} + g) s(k_N)$$

We can again set up a recursion scheme by expanding the determinants $\mathcal{Z}_{k,q}$ and $\mathcal{Z}_{1,k}$ in terms of their first and last row, respectively, obtaining the recursion relations

$$\mathcal{Z}_{k,q} = \mathcal{Z}_{k+1,q} + z s(k) \mathcal{Z}_{k+g,q} \ , \quad \mathcal{Z}_{1,k} = \mathcal{Z}_{1,k-1} + z s(k) \mathcal{Z}_{1,k-g} \tag{28}$$

of clear $g$-exclusion statistics origin. We define forward and backward effective single-level grand partition functions $y_+(k)$ and $y_-(k)$ as in the $g = 2$ case (20), and the full grand partition function is again expressed as their product (21). The cluster expansion for $\mathcal{Z}_{k,k'}$ is as in (26) but for the truncated spectrum, that is

$$\ln \mathcal{Z}_{k,k'} = -\sum_{n=1}^{\infty} (-z)^n \sum_{\substack{l_1, l_2, \dots, l_j \\ \text{g-composition of } n \\ j \le k'-k+1}} c_g(l_1, l_2, \dots, l_j) \sum_{l=k}^{k'-j+1} s^{l_j}(l+j-1) \cdots s^{l_2}(l+1) s^{l_1}(l) \tag{29}$$

leading to expressions analogous to (22) for the logarithm $\ln y_+(k)$ and $\ln y_-(k)$.

The recursion relation (28) implies the nesting relations

$$y_\pm(k) - \frac{z s(k)}{\prod_{i=1}^{g-1} y_+(k \pm i)} = 1 \tag{30}$$

We reach the conclusion that the statistical mechanics for $g$-exclusion particles in a discrete 1-body spectrum $\epsilon(1), \epsilon(2), \dots, \epsilon(q)$ amounts to

$$\ln \mathcal{Z}_{1,q} = \sum_{k=1}^{q} \ln y_\pm(k) \ , \quad y_\pm(k) - \frac{z s(k)}{\prod_{i=1}^{g-1} y_\pm(k \pm i)} = 1$$

which is again basically (9) with $\alpha$ replaced by $g$, but with a discrete sum instead of a continuous integral and the discrete shifts $k \to k \pm 1 \ldots \to k \pm (g-1)$ in the argument of $y_+(k)$ and $y_-(k)$.

Likewise, because of the identity

$$\mathcal{Z}_{1,q} = \mathcal{Z}_{1,k-1}\mathcal{Z}_{k+g-1,q} + \sum_{i=k}^{k+g-2} zs(i)\mathcal{Z}_{1,i-g}\mathcal{Z}_{i+g,q} \ ,$$

of obvious exclusion statistics origin (which can also be obtained by expanding the secular determinant with respect to its $k^{\rm th}$ row), the mean occupation number $n_k$ defined in (22) becomes

$$n_k = zs(k)\frac{\mathcal{Z}_{1,k-g}\mathcal{Z}_{k+g,q}}{\mathcal{Z}_{1,q}}$$

and satisfies the nesting relation (see Appendix for the proof)

$$zs(k) = n_k \frac{\prod_{j=1}^{g-1}\left(1-\sum_{i=1}^{g-1} n_{k+j-i}\right)}{\prod_{j=0}^{g-1}\left(1-\sum_{i=0}^{g-1} n_{k+j-i}\right)}$$

and combining with (30) we obtain the expressions for $y_+(k)$ and $y_-(k)$

$$y_\pm(k) = 1 + \frac{n_k}{1-\sum_{j=0}^{g-1} n_{k\mp j}}$$

leading to the expression for the thermodynamic potential

$$\ln \mathcal{Z}_{1,q} = \sum_{k=1}^{q} \ln\left(1+\frac{n_k}{1-\sum_{i=0}^{g-1} n_{k-i}}\right) = \sum_{k=1}^{q} \ln\left(1+\frac{n_k}{1-\sum_{i=0}^{g-1} n_{k+i}}\right)$$

again in close similarity to (10, 11) for $\alpha$ replaced by $g$. When the occupation numbers are such that for any given $k$ their sum over $g$ neighboring levels $\sum_{i=0}^{g-1} n_{k-i} = 1$, the system has reached the maximal critical filling allowed by exclusion statistics.

# 4 The harmonic Calogero model

As an illustration of the formalism developed in the previous section, let us focus on the $N$-body Calogero harmonic spectrum (13), here again for convenience shifted by $N\omega/2$, not any more in the thermodynamic limit $\beta\omega \to 0$ as in Section (2), where the harmonic well played the role of a long-distance regulator, but keeping $\beta\omega$ physical and finite. This is an example of a exclusion-$g$ statistics system with a discrete 1-body spectrum, arising from a *microscopic* $N$-body Hamiltonian with spectral function $s(k) = x^k$, $k = 1, 2, \ldots, \infty$, with $x = e^{-\beta\omega}$, i.e., a harmonic linear 1-body spectrum. We could focus as well on the $N$-body LLL-anyon harmonic spectrum (1) with spectral function

$$s(k) = x^k x_c \tag{31}$$

where $x = e^{-\beta(\omega_t - \omega_c)}$ and $x_c = e^{-\beta\omega_c}$.

### 4.1 Number of states $q$ finite

Assume, now, that the number of 1-body states is finite, that is, $k = 1, 2, \ldots, q$. The integer $q$ is an effective high-energy cutoff which is not actually needed for the consistency of the results; later on, the $q \to \infty$ limit will be taken. The $N$-body spectrum (13), with $\alpha$ traded for $g$ and the $l_i$ bounded by $q - (N-1)g$, yields a partition function which is nothing but $Z_N$ in (25) for the spectral function $s(k) = x^k$ discussed above. The grand partition function follows as

$$\mathcal{Z}_{1,q} = 1 + \sum_{N=1}^{(q+g-1)/g} z^N x^{N+gN(N-1)/2} \prod_{j=1}^{N} \frac{1 - x^{j+q-1-(N-1)g}}{1 - x^j}$$

In this specific situation, where $s(k) = x^k$, we see that $\mathcal{Z}_{k,k'}$ is actually $\mathcal{Z}_{1,q}$ but with $q \to k' - k + 1$ and $z \to z x^{k-1}$

$$\mathcal{Z}_{k,k'} = 1 + \sum_{N=1}^{(k'-k+g)/g} z^N x^{N+gN(N-1)/2} \prod_{j=1}^{N} \frac{x^{k-1} - x^{j+k'-1-(N-1)g}}{1 - x^j}$$

From the cluster expansion (29) we obtain

$$\begin{aligned} \ln \mathcal{Z}_{k,q} &= -\sum_{n=1}^{\infty} (-z)^n \sum_{\substack{l_1,l_2,\ldots,l_j \\ \text{g-composition of } n \\ j \le q-k+1}} c_g(l_1, l_2, \ldots, l_j) \sum_{l=k}^{q-j+1} x^{nl} x^{\sum_{i=1}^{j}(i-1)l_i} \\ &= -\sum_{n=1}^{\infty} \frac{(-z)^n}{1 - x^n} \sum_{\substack{l_1,l_2,\ldots,l_j \\ \text{g-composition of } n \\ j \le q-k+1}} \left[x^{nk} - x^{n(q-j+2)}\right] c_g(l_1, l_2, \ldots, l_j)\, x^{\sum_{i=1}^{j}(i-1)l_i} \end{aligned} \tag{32}$$

and similarly

$$\ln \mathcal{Z}_{1,k} = -\sum_{n=1}^{\infty} \frac{(-z)^n}{1 - x^n} \sum_{\substack{l_1,l_2,\ldots,l_j \\ \text{g-composition of } n \\ j \le k}} \left[x^{n} - x^{n(k-j+2)}\right] c_g(l_1, l_2, \ldots, l_j)\, x^{\sum_{i=1}^{j}(i-1)l_i}$$

which imply the cluster expansion for the forward and backward effective single-level grand partition functions

$$\begin{aligned} \ln y_+(k) &= -\sum_{n=1}^{\infty} (-z x^k)^n \sum_{\substack{l_1,l_2,\ldots,l_j \\ \text{g-composition of } n \\ j \le q-k+1}} c_g(l_1, l_2, \ldots, l_j)\, x^{\sum_{i=1}^{j}(i-1)l_i} \\ \ln y_-(k) &= -\sum_{n=1}^{\infty} (-z x^k)^n \sum_{\substack{l_1,l_2,\ldots,l_j \\ \text{g-composition of } n \\ j \le k}} c_g(l_1, l_2, \ldots, l_j)\, x^{-\sum_{i=1}^{j}(i-1)l_i} \end{aligned} \tag{33}$$

The above expressions are remarkably similar, differing in the sign of the exponent and in the allowed $g$-compositions. Note that the expression for $y_-(k)$ does not involve $q$, since it looks "back" towards $k = 1$, and is, thus, universal.

## 4.2 Number of states $q \to \infty$: the harmonic Calogero model

Clearly in the $q \to \infty$ limit, i.e., the harmonic Calogero model, the constraint on the number of parts $j \le q-k+1$ in (32) or (33) disappears: (32) for $k=1$ is the harmonic Calogero thermodynamic potential

$$\ln \mathcal{Z}_{1,q} = -\sum_{n=1}^{\infty} (-z)^n \frac{x^n}{1-x^n} \sum_{\substack{l_1,l_2,\dots,l_j \\ \text{g-composition of } n}} c_g(l_1,l_2,\dots,l_j)\, x^{\sum_{i=1}^{j}(i-1)l_i}$$

with the harmonic Calogero cluster coefficients

$$b_n = (-1)^{n-1}\frac{1}{1-x^n} \sum_{\substack{l_1,l_2,\dots,l_j \\ \text{g-composition of } n}} c_g(l_1,l_2,\dots,l_j)\, x^{\sum_{i=1}^{j} i\, l_i} \ . \tag{34}$$

The $b_n$ in (34) encode $g$-exclusion statistics arising from the microscopic $N$-body harmonic Calogero model, i.e., from its Hamiltonian (12) and $N$-body spectrum (13).

Let us consider the small $\beta\omega$ limit to recover the cluster coefficients of Section 2. For small $\beta\omega$, i.e., $x = e^{-\beta\omega} \simeq 1-\beta\omega$, the cluster coefficient $b_n$ in (34) becomes

$$b_n = (-1)^{n-1}\frac{1}{n\beta\omega} \sum_{\substack{l_1,l_2,\dots,l_j \\ \text{g-composition of } n}} c_g(l_1,l_2,\dots,l_j)$$

Using [9]

$$\sum_{\substack{l_1,l_2,\dots,l_j \\ \text{g-composition of } n}} c_g(l_1,l_2,\dots,l_j) = \frac{\binom{gn}{n}}{gn}$$

it simplifies to

$$b_n = (-1)^{n-1}\frac{1}{\beta\omega}\frac{\binom{gn}{n}}{gn^2}$$

and indeed coincides, when $g$ is traded for $\alpha$, with (2) when $\omega_c = 0$. In order to recover the full cluster coefficient (2), i.e, with $\omega_c \neq 0$, we have to use the spectral function (31), that is, put $x = e^{-\beta(\omega_t - \omega_c)}$ in (34) and multiply by $x_c^n$.

We remark that in (34) the coefficients of the polynomial

$$\sum_{\substack{l_1,l_2,\dots,l_j \\ \text{g-composition of } n}} c_g(l_1,l_2,\dots,l_j)\, x^{\sum_{i=1}^{j}(i-1)l_i}$$

are, when multiplied by $n$, a $g$-generalization of the coefficients OEIS A227532 and the related A227543, in relation to the Ramanujan continuous fraction. Indeed, denoting

$$H_g(z,x) = \mathcal{Z}_{2,\infty}(-z/x,x) = 1 + \sum_{N=1}^{\infty} z^N x^{gN(N-1)/2} \prod_{j=1}^{N} \frac{x}{x^j - 1} \ ,$$

$$G_g(z,x) = \mathcal{Z}_{1,\infty}(-z/x,x) = 1 + \sum_{N=1}^{\infty} z^N x^{gN(N-1)/2} \prod_{j=1}^{N} \frac{1}{x^j - 1} \ ,$$

then $y_+(1) = \mathcal{Z}_{1,q}/\mathcal{Z}_{2,q}$ in (33) yields the $q \to \infty$ identity

$$\sum_{n=1}^{\infty} z^n \sum_{\substack{l_1,l_2,\ldots,l_j \\ \text{g-composition of } n}} c_g(l_1,l_2,\ldots,l_j)\ x^{\sum_{i=1}^{j}(i-1)l_i} = \ln \frac{H_g(z,x)}{G_g(z,x)} \ . \tag{35}$$

When $g = 1$, i.e., Fermi statistics and Pauli exclusion, the $c_1(l_1,l_2,\ldots,l_j)$ are non vanishing for the sole composition $l_1 = n$ with $c_1(n) = 1/n$. In this case (35) trivially reduces to $\sum_{n=1}^{\infty} z^n/n = -\ln(1-z)$.

When $g = 2$, writing (21) for $z \to -z/x$ and $s(k) = x^k$ in the form

$$\frac{1}{y_+(k)} = \frac{1}{1 - \frac{zx^{k-1}}{y_+(k+1)}}$$

and solving it iteratively for $y_+(1)$ it yields as a solution the Ramanujan continuous fraction. Exponentiating (35), then, yields the identity

$$\exp\left(\sum_{n=1}^{\infty} z^n \sum_{\substack{l_1,l_2,\ldots,l_j \\ \text{composition of } n}} c_2(l_1,l_2,\ldots,l_j)\ x^{\sum_{i=1}^{j}(i-1)l_i}\right) = \frac{H_2(z,x)}{G_2(z,x)} = \frac{1}{1+\frac{-z}{1+\frac{-zx}{1+\frac{-zx^2}{1+\frac{-zx^3}{1+\frac{-zx^4}{1+\ldots}}}}}}$$

The LHS expresses the log of the Ramanujan fraction in terms of $c_2(l_1,l_2,\ldots,l_j)$.

For a general $g$, exponentiating (35) gives an alternative expression of the ratio $H_g(z,x)/G_g(z,x)$ in terms of the $c_g(l_1,l_2,\ldots,l_j)$ as a $g \geq 2$-generalisation of the OEIS coefficients A227532 and A227543. Note, however, that in the case $g > 2$ the iterative solution of (30) for $1/y_+$ leads to a complicated expression and does not yield any simple continuous fraction as in the $g = 2$ case.

# 5 Conclusions

The analysis and results of this work could be applied to other systems, and in particular to the Calogero-Sutherland model of particles with an inverse sine squared two-body potential, which can be viewed as exclusion-$g$ particles on the circle. The spectral function for this model is (setting the length of the circle to $2\pi$)

$$s(k) = e^{-\beta k^2/2} \ , \quad k = 0, \pm 1, \pm 2, \ldots$$

with $k$ interpreted as the discrete momentum of exclusion particles (also referred to as "pseudomomentum"). The spectrum is symmetric and unbounded in both ends, so forward and backward effective functions $y_\pm(k)$ are essentially equivalent, satisfying $y_+(k) = y_-(-k)$, a relation that survives the introduction of a symmetric cutoff $|k| \leq q$. We could also impose an asymmetric restriction to the spectrum, such as $k \geq 0$, which actually corresponds to a symmetry-reduced Calogero-Sutherland model with inverse sine and inverse cosine squared potentials plus an additional one-body interaction with the point at the origin.

The full Calogero-Sutherland spectrum can be treated by separating in the $N$-body partition function the (unrestricted) sum over the smallest momentum $k_1$. The sum of the remaining momenta can be cast as the grand partition function of $N-1$ particles with spectral function $s(k) = e^{-\beta(k_1+g+k)^2}$ with $k \geq 0$. This makes the spectrum bounded to the left, but creates the added complication of having to sum at the end over $k_1$. We defer any further discussion of the microscopic statistics of the Calogero-Sutherland model to future work.

In both the Calogero and LLL-anyon systems we can define "quasihole" excitations, corresponding to minimal "gaps" in a completely filled state, that behave as particles with exclusion statistics $1/g$. We can define similar excitations in the microscopic (discrete) case, starting from a maximally filled many-body state and moving all particles to the right of a marked particle by one level, or to the left by minus one level, therefore creating a quasihole. All the standard features of quasiholes emerge in this picture: the removal of one particle creates $g$ quasiholes, identifying them as $-1/g$ particle each, they "move" in the energy spectrum by steps of $g$, therefore corresponding to an $1/g$ dilution of the density of states, and in a macroscopic span of $K$ successive levels a number $K$ of quasiholes can be"packed" together. Since the effective density of states has been decreased by a factor of $1/g$, this last property identifies quasiholes as particles with $1/g$ exclusion statistics.

Several qualitatively new features arise, however, in the discrete setting: there are, now, $g$ distinct "fully filled" states, related to each other through a common shift of all particles by $1, 2, \dots g-1$ levels, so these states are not perturbatively connected. Further, each quasihole's energy becomes a nontrivial combination of several energy levels. The exact properties of quasiholes, and a potential reformulation of the statistical mechanics of the system in terms of them, remain fascinating objects of further study.

Exclusion statistics are related to the generating functions for the algebraic area counting of lattice walks [11], a connection that arises from the matrix determinant representation as exposed in section **3**. The Calogero model results, in particular, are related to a set of directed walks termed Dyck or Lukasiewicz paths. This connection has been exploited to derive results for such walks [12] but it can also be used in reverse: the generating functions of generalized types of walks can be considered as effective descriptions of particular exclusion-type statistics. The identification of such statistics, or of the walks corresponding to other known types of statistics, is an interesting topic for investigation.

The most interesting possible application of our results is, however, in situations of physical significance, such as fractional quantum Hall and related systems. This would require measurements that probe the microscopic quantities studied in this work, such as, e.g., the single-level mean occupation numbers $n_k$, looking for possible signatures within recent experimental results. Also LLL-anyon statistics at maximal filling is encoded in Laughlin type wavefuctions (8) describing bulk particles with exclusion $\alpha$, while experimentalists [3] observe quasiholes with statistics $1/\alpha$. Identifying possible measurements allowing for an experimental confirmation of bulk particles with exclusion $\alpha$ remains an important subject for further theoretical research and experimental developments.

# 6 Appendix

To derive (22), we use the recursion (19) to rewrite $\mathcal{Z}_{k+1,q} = \mathcal{Z}_{k+2,q} + zs(k+1)\mathcal{Z}_{k+3,q}$ so that (22) becomes

$$1 - n_k - n_{k+1} = \frac{\mathcal{Z}_{1,k-1}\mathcal{Z}_{k+2,q}}{\mathcal{Z}_{1,q}} \ . \tag{36}$$

Then from (22) and (36) get

$$\frac{\mathcal{Z}_{k+2,q}}{\mathcal{Z}_{k+1,q}} = \frac{1 - n_k - n_{k+1}}{1 - n_k} \tag{37}$$

Likewise from (36) get

$$1 - n_{k-1} - n_k = \frac{\mathcal{Z}_{1,k-2}\mathcal{Z}_{k+1,q}}{\mathcal{Z}_{1,q}} \tag{38}$$

or, again using (22),

$$\frac{1 - n_{k-1} - n_k}{n_k} = \frac{\mathcal{Z}_{k+1,q}}{zs(k)\mathcal{Z}_{k+2,q}} \ . \tag{39}$$

Multiplying (37) and (39) we obtain (22).

The proof of the corresponding relation for $g > 2$ proceeds along similar lines, requiring now a "telescoping" product relation to eliminate the various intermediate $\mathcal{Z}_{k,k'}$, and we leave it as an exercise to the reader.

# References


[1] J.M. Leinaas, J. Myrheim, On the theory of identical particles, Nuovo Cimento 37B (1977) 1–23. For an earlier work on the subject see: M.G.G. Laidlaw, C.M. de Witt, Feynman Functional Integrals for Systems of Indistinguishable Particles, Phys. Rev D3 (1971) 1375.

[2] F.D.M. Haldane, "Fractional statistics in arbitrary dimensions: A generalization of the Pauli principle", Phys. Rev. Lett. **67** (1991) 937–940.

[3] H. Bartolomei *et al.*, "Fractional statistics in anyon collisions", arXiv:2006.13157v1

[4] F. Wilczek, "Magnetic flux, angular momentum, and statistics", Phys. Rev. Lett. 48 (1982) 1144–1146.

[5] A. Dasnières de Veigy and S. Ouvry, "Equation of State of an Anyon gas in a Strong Magnetic Field", Phys. Rev. Lett. **72** (1994) 600; "One-dimensional Statistical Mechanics for Identical Particles: the Calogero and Anyon Cases", Mod. Phys. Lett. A 10 (1995) 1; S. Ouvry, "Anyons and Lowest Landau Level Anyons", Séminaire Poincaré "Le Spin!" (2007) arXiv:0712.2174, Birkhuser Verlag AG

[6] A.P. Polychronakos, "Nonrelativistic Bosonization and Fractional Statistics," Nucl. Phys. B **324** (1989) 597.

[7] S.B. Isakov, "Fractional statistics in one dimension: modeling by means of $1/x^2$ interaction and statistical mechanics", Int. Jour. Mod. Phys. A **09** V.15 (1994) 2563.

[8] A.P. Polychronakos, "Probabilities and path-integral realization of exclusion statistics," Phys. Lett. B **365** (1996) 202.

[9] S. Ouvry, "On the Relation between the Anyon and the Calogero Models", Phys. Lett. B510 (2001) 335; S. Ouvry and A.P. Polychronakos, "Mapping the Calogero model on the Anyon model", Nuclear Physics, Section B 936C (2018) 189-205

[10] Y.S. Wu, "Statistical distribution for generalized ideal gas of fractional-statistics particles", Phys. Rev. Lett. 73 (1994) 922–925.

[11] S. Ouvry and S. Wu, "The algebraic area of closed lattice random walks", J. Phys. A: Math. Theor. **52** (2019) 255201 ; S. Ouvry and A.P. Polychronakos, "Exclusion statistics and lattice random walks", NPB[FS] 948 (2019) 114731; "Lattice walk area combinatorics, some remarkable trigonometric sums and Apéry-like numbers", NPB[FS] 960 (2020) 115174

[12] S. Ouvry and A.P. Polychronakos, "Hamiltonian and exclusion statistics approach to Dyck-like paths", arXiv: (2021)