Journal of Mathematical Physics

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Cite as: J. Math. Phys. **62**, 061509 (2021); https://doi.org/10.1063/5.0042767
Submitted: 03 January 2021 • Accepted: 03 June 2021 • Published Online: 23 June 2021

W. Schlag

Journal of Mathematical Physics | ARTICLE | scitation.org/journal/jmp

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Submitted: 3 January 2021 • Accepted: 3 June 2021 •
Published Online: 23 June 2021

W. Schlag[a)]

AFFILIATIONS

Yale University, Department of Mathematics, 10 Hillhouse Avenue, New Haven, Connecticut 06511, USA

Note: This paper is part of the Special Issue on Celebrating the work of Jean Bourgain.
[a)] **Author to whom correspondence should be addressed:** wilhelm.schlag@yale.edu

ABSTRACT

This paper introduces some of the basic mechanisms relating the behavior of the spectral measure of Schrödinger operators near zero energy to the long-term decay and dispersion of the associated Schrödinger and wave evolutions. These principles are illustrated by means of the author's work on decay of Schrödinger and wave equations under various types of perturbations, including those of the underlying metric. In particular, we consider local decay of solutions to the linear Schrödinger and wave equations on curved backgrounds that exhibit trapping. A particular application is waves on a Schwarzschild black hole spacetime. We elaborate on Price's law of local decay that accelerates with the angular momentum, which has recently been settled by Hintz, also in the much more difficult Kerr black hole setting. While the author's work on the same topic was conducted ten years ago, the global semiclassical representation techniques developed there have recently been applied by Krieger, Miao, and the author ["A stability theory beyond the co-rotational setting for critical wave maps blow up," arXiv:2009.08843 (2020)] to the nonlinear problem of stability of blowup solutions to critical wave maps under non-equivariant perturbations.

I. INTRODUCTION

This paper mainly serves as an introduction to the techniques used in the papers,[29,30] which are concerned with the local decay of waves on a Schwarzschild background. The decay estimates are obtained by separation of variables and the analysis of the flow for each angular momentum ℓ in Ref. 29. By means of a semiclassical WKB analysis in the parameter $\hbar := \ell^{-1}$ carried out by means of a global Liouville–Green transform, as well as semiclassical Mourre theory at energies near the top of the barrier,[30] these fibered estimates sum up over all angular momenta incurring the loss of finitely many angular derivatives. Note that Refs. 29 and 30 are not entirely self-contained and rely, in part, on Refs. 16, 17, 28, 74, and 75. As shown in these references, the Schrödinger flow can be analyzed analogously. The original motivation for Refs. 74 and 75 was to study the long-term dispersive behavior of solutions to Schrödinger and wave equations on specific non-compact manifolds exhibiting closed geodesics, such as the hyperboloid of one sheet. In analogy with the unique periodic geodesic on such a hyperboloid, which is exponentially unstable, the surface of closed geodesics around a Schwarzschild black hole is known as a *photon sphere* and corresponds to the collection of all periodic light rays. The photon sphere is also unstable.

Recently, in joint work with Krieger *et al.*,[56] the semiclassical techniques leading to a precise representation of the resolvent and the spectral measure for all energies and all small $\hbar$ developed in Refs. 16 and 17 played a crucial role in a nonlinear asymptotic stability question of blowup solutions to energy critical wave maps into the two-sphere. In stark contrast to the linear case, modes of fixed frequencies interact through the nonlinearities. Controlling these interactions naturally leads to a paradifferential calculus involving several simultaneous semiclassical parameters. The nonlinear work[56] served as the main motivation for writing this paper, which should not be mistaken for a general review. Numerous references are missing, which touch in one way or another on the ensuing discussion. A survey of dispersive decay of Schrödinger, wave, and Klein–Gordon evolutions involving electric, magnetic, and metric perturbations, including the semi-classical and gravitational literature, would require many hundreds of citations. The scope and purpose of this paper is much more limited. For example, magnetic and time-dependent potentials are not discussed in detail.

The author's investigations in this area were largely motivated by the book of Bourgain,[10] which states at the end of page 27: *On the other hand, it would be most interesting to prove that analogue of (1.99) in low dimensions* $d = 1, 2$. *This is certainly a project of independent importance.* Here, (1.99) refers to the pointwise decay of the Schrödinger evolution proved by Journé, Soffer, and Sogge[51] (see Sec. II).

II. LOWER ORDER PERTURBATIONS

The free Schrödinger evolution $\psi(t) = e^{-it\Delta}\psi_0$ in $\mathbb{R}^{d+1}_{t,x}$ satisfies the basic estimates

$$\|\psi(t)\|_{H^s} = \|\psi_0\|_{H^s}, \tag{1}$$

$$\|\psi(t)\|_\infty \le Ct^{-\frac{d}{2}}\|\psi_0\|_1, \tag{2}$$

as can be seen from the representation

$$\psi(t,x) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{i(t|\xi|^2 + x\cdot\xi)}\hat{f}(\xi)\,d\xi$$
$$= c(d)t^{-\frac{d}{2}}\int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{4t}} f(y)\,dy.$$

For the wave equation $\Box u = \partial_t^2 u - \Delta u = 0$ in $d+1$ dimensions, one has constancy of the energy

$$\mathscr{E}(u) = \|\nabla u\|_2^2 + \|\partial_t u\|_2^2 \tag{3}$$

as well as the dispersive decay

$$\|u(t)\|_\infty \lesssim t^{-\frac{d-1}{2}}\left(\|u(0)\|_{\dot{B}^{\frac{d+1}{2}}_{1,1}} + \|\partial_t u(0)\|_{\dot{B}^{\frac{d-1}{2}}_{1,1}}\right), \tag{4}$$

where $\dot{B}^\alpha_{1,1}$ stands for the usual Besov space: $\|f\|_{\dot{B}^\alpha_{1,1}} = \sum_{j\in\mathbb{Z}} 2^{\alpha j}\|P_j f\|_1$, where P_j is the Littlewood–Paley projection onto frequencies of size 2^j. In odd spatial dimensions, one can improve the right-hand side to

$$\|u(0)\|_{\dot{W}^{\frac{d+1}{2},1}} + \|\partial_t u(0)\|_{\dot{W}^{\frac{d-1}{2},1}},$$

where $\dot{W}^{\alpha,p}$ stands for the homogeneous Sobolev spaces. To obtain (4), one considers a fixed frequency shell $\{|\xi| \sim 2^j\}$ and rescales to $j = 0$. Then,

$$e^{it\sqrt{-\Delta}}P_0 f(x) = \int_{\mathbb{R}^{2d}} e^{i((x-y)\cdot\xi + t|\xi|)}\chi(\xi)\,d\xi\, f(y)\,dy,$$

where χ is a cutoff function corresponding to P_0. Passing to polar coordinates and applying stationary phase to integrals over spheres then yield the desired $t^{-\frac{d-1}{2}}$ decay.

While (1) and (3) are a result of the time-translation invariance of the underlying Lagrangians (via Noether's theorem) and therefore robust under perturbations that preserve this symmetry, (2) and (4) follow from the form of the fundamental solutions and are therefore less stable. In fact, much effort has been devoted to deriving similar dispersive estimates for perturbations of the free Schrödinger and wave equations in the past thirty years. The starting point in these investigations was to consider *local* decay estimates that are quite different from the global ones as in (2) and (4) (as we shall see below). *Local* here refers to the fact that the decay is measured only in weighted spaces rather than in a uniform sense.

A. Local decay for $-\Delta + V$

1. The Schrödinger evolution

In Ref. 50, Jensen and Kato showed that for $H = -\Delta + V$ in the three-dimensional case, with real-valued V that is bounded and decays at a sufficient polynomial, rate one has the local decay

$$\|\langle x\rangle^{-\sigma} e^{itH}P_c f\|_{L^2(\mathbb{R}^3)} \lesssim \langle t\rangle^{-\frac{3}{2}}\|\langle y\rangle^{\sigma} f\|_{L^2(\mathbb{R}^3)} \tag{5}$$

for some $\sigma > 0$ and with $P_c = \chi_{(0,\infty)}(H)$ being the projection onto the continuous spectrum. Moreover, one needs to assume that zero energy is neither an eigenvalue nor a resonance of H (which is also referred to as zero energy being regular, the other case being singular).

This latter property refers to the validity of the resolvent estimate

$$\sup_{\operatorname{Im} z>0}\|\langle x\rangle^{-\sigma}(-\Delta + V + z)^{-1}\langle x\rangle^{-\sigma}\|_{2\to 2} < \infty \tag{6}$$

with $\sigma > 0$ sufficiently large. Alternatively, it is the same as the nonexistence of $f \not\equiv 0$ with

$$Hf = 0, \quad f \in \bigcap_{\varepsilon>0} L^{2,-\frac{1}{2}-\varepsilon}(\mathbb{R}^3). \tag{7}$$

It was already observed by Rauch[70] for exponentially decaying potentials that a zero energy resonance or eigenvalue, i.e., in the case when (7) admits a nontrivial solution, destroys the dispersive estimate. More specifically, one loses one power of t in the decay law in that case.

To see the relevance of zero energy resonances, we expand the resolvent for $z \to 0$ in $\operatorname{Im} z > 0$ as follows:

$$\begin{aligned} R(z) &:= (-\Delta + V + z)^{-1} \\ &= z^{-1} B_{-1} + z^{-\frac{1}{2}} B_{-\frac{1}{2}} + B_0 + z^{\frac{1}{2}} B_{\frac{1}{2}} + \rho(z), \end{aligned} \tag{8}$$

where $B_{-1}, \ldots, B_1$ are bounded in weighted $L^2(\mathbb{R}^3)$-spaces, and with

$$\|\langle x\rangle^{-\sigma} \rho(z) f\|_2 \lesssim |z| \|\langle x\rangle^{\sigma} f\|_2$$

for small z. Clearly, B_{-1} is the orthogonal projection onto the zero eigenspace, and zero energy is regular for H iff $B_{-1} = B_{-\frac{1}{2}} = 0$. In general, $B_{-1}, B_{-\frac{1}{2}}$ are of finite rank. As an example, consider the case $V = 0$ in three dimensions, for which one has (with $z = -\zeta^2$)

$$(-\Delta - \zeta^2)^{-1}(x, y) = \frac{e^{i\zeta|x-y|}}{4\pi|x-y|}, \quad \operatorname{Im} \zeta > 0,$$

and the Laurent expansion (8) is now obtained by Taylor expanding the exponential on the right-hand side. It follows that zero energy is neither an eigenvalue nor a resonance in that case. In contrast, the one-dimensional case satisfies

$$(-\Delta - \zeta^2)^{-1}(x, y) = \frac{e^{i\zeta|x-y|}}{2i\zeta}, \quad \operatorname{Im} \zeta > 0,$$

and zero is a resonance (but not an eigenvalue). We used here that (8) remains correct in all *odd* dimensions, whereas in even dimensions, a logarithm appears. Indeed, the free resolvent in d-dimensions satisfies

$$(-\Delta - \zeta^2)^{-1}(x, y) = c_d\, \zeta^{\frac{d-2}{2}} |x-y|^{-\frac{d-2}{2}} H^{+}_{\frac{d-2}{2}}(\zeta|x-y|), \tag{9}$$

and the Hankel functions of integer order exhibit a logarithmic branch point at zero.

To pass to estimates on the evolution, one now uses the Laplace transform (as in the Hille–Yosida theorem) to conclude that

$$e^{itH} P_c = \frac{1}{2\pi} \int_{p_0 - i\infty}^{p_0 + i\infty} e^{tp} R(ip) P_c \, dp, \tag{10}$$

where $p_0 > 0$ is arbitrary. Assuming for simplicity that V is compactly supported, it follows from the resolvent identity that the Green function $R(ip)(x, y)$ admits a meromorphic continuation to the left-half plane. One now deforms the contour in (10) as shown in Fig. 1. The finitely many residues $\{\zeta_j\}$ of the resolvent in the left-half plane (which lie in $\mathbb{C}\backslash(-\infty, 0]$) contribute to the exponentially decaying expression

$$\sum_{\zeta_j} e^{\zeta_j t} P_{\phi_j},$$

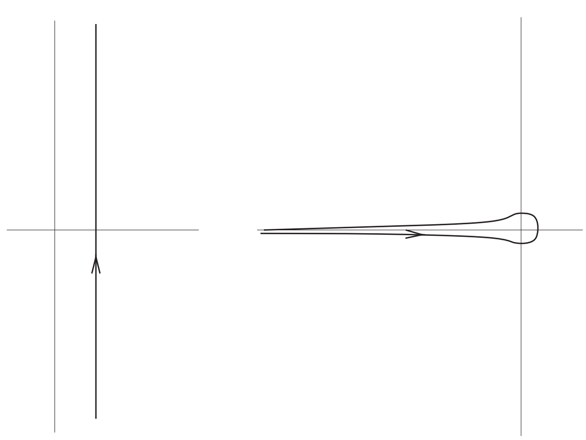

FIG. 1. Deforming the contour.

where P_{ϕ_j} is the projection onto the resonant states corresponding to the complex resonance at ζ_j (the resonant states are commonly referred to as *meta-stable states* or *quasinormal modes*). The more slowly decaying tail is a result of the branching of the resolvent at $p = 0$. More specifically, it can be read off from (8) via the following standard result, which is known as Watson's lemma (the notation ~ refers to asymptotic expansions in the sense of Poincaré):

Lemma II.1. Let f be a complex-valued function of a real variable x such that

- *f is continuous on $(0, \infty)$,*
- *$f(x) \sim \sum_{n=0}^{\infty} a_n x^{\lambda_n - 1}$ as $x \to 0+$ with $0 < \lambda_0 < \lambda_1 < \ldots$,*
- *$f(x) = O(e^{cx})$ as $x \to \infty$ for some $c > 0$.*

This condition can be removed since Watson's lemma is really local on some interval $(0, x_0)$, but we choose to state it in this global form. Then, for every small $\delta > 0$, one has

$$\int_0^\infty e^{-xp} f(x)\, dx \sim \sum_{n=0}^{\infty} \frac{a_n}{p^{\lambda_n}} \Gamma(\lambda_n)$$

as $|p| \to \infty$ in $|\arg(p)| \le \frac{\pi}{2} - \delta$.

Therefore, if $B_{-\frac{1}{2}} \neq 0$ in (8), then one obtains $t^{-\frac{1}{2}}$ *local* decay, whereas, otherwise, the rate is $t^{-\frac{3}{2}}$, which is the same as in (2). Evidently, the global (i.e., L^∞) decay can never be faster than the local one—whence the need to exclude zero energy resonance and eigenvalues to preserve (2). We remark that one can have $B_{-\frac{1}{2}} \neq 0$ even in case the only solutions to (7) are in L^2 (in other words, if zero energy is an eigenvalue but not a resonance). This implies that $t^{-\frac{3}{2}}$ does not result from applying P_c to the evolution even when zero is not a resonance but only an eigenvalue.

Starting from the spectral representation

$$e^{itH} P_c = \int_0^\infty e^{it\lambda} E(d\lambda) \tag{11}$$

instead of (10) with the spectral measure

$$E(d\lambda) = \frac{1}{2\pi i} [R(\lambda + i0) - R(\lambda - i0)] P_c\, d\lambda,$$

Jensen and Kato derived local decay estimates but under much less severe restrictions on the decay of V and also on the notion of *locality* in the decay estimate. However, it is clear from (11) that the main issue here is once again the contributions from $\lambda = 0$ coming from (8). Indeed, for energies $\lambda > \lambda_0 > 0$, where $\lambda_0 > 0$ is arbitrary but fixed, one has the so-called *limiting absorption* resolvent bounds

$$\sup_{\lambda > \lambda_0} \| \langle \cdot \rangle^{-\sigma} \partial_\lambda^k R(\lambda \pm i0) \langle \cdot \rangle^{-\sigma} \| < \infty$$

for all $0 \le k \le k_0$ and with $\sigma > 0$ depending on k (the value of k_0 here depends on the decay of V). These bounds allow one to integrate by parts in (11) in the range $\lambda > \lambda_0$, which leads to arbitrary decay in time.

The most general results on local decay for the Schrödinger evolution were obtained by Murata.[65] He derived expansions in time for evolutions e^{itH} in *all dimensions* and with elliptic $H = -p(D) + V$, where V is a compact operator in suitable weighted Sobolev spaces. As a general rule, the coefficients in these expansions corresponding to nongeneric threshold behavior (i.e., slow decay resulting from threshold eigenvalues or resonances) are *finite rank* operators that can be computed in terms of the eigenfunctions and resonant states. As an example, the one-dimensional free evolution satisfies

$$e^{-it\partial_x^2} f(x) = ct^{-\frac{1}{2}} \int f(y)\, dy + \rho(t) f(x),$$
$$\| \langle \cdot \rangle^{-\sigma} \rho(t) f \|_2 \lesssim t^{-\frac{3}{2}} \| \langle \cdot \rangle^{\sigma} f \|_2.$$

The appearance of the projection $f \mapsto \int f(y) \mathrm{d}y$ onto the constant functions is natural in view of the fact that the resonant function of $-\partial_x^2$ at zero energy is $f \equiv 1$. This also shows that one should expect $t^{-\frac{3}{2}}$ local decay for one-dimensional operators without zero energy resonance (note that, however, the *global* decay as in (2) is never faster than $t^{-\frac{1}{2}}$ if $d = 1$), at least assuming sufficient decay of V. This is indeed the case (see Ref. 65). In two dimensions, Murata obtained the faster local $L^2(\mathbb{R}^2)$ decay $t^{-1} \log^{-2} t$ for operators without resonance. Erdogan and Green[34] established the more difficult sharp weighted $L^1 \to L^\infty$ version of these global bounds in $\mathbb{R}^2$, assuming that zero energy is regular. These faster local decays (as compared to the global L^∞ decay) play a crucial role in certain applications to nonlinear stability results (see the work of Buslaev and Perelman[13] Krieger and Schlag[57] for the one-dimensional case and Kirr and Zarnescu[55] for examples of two-dimensional applications. Loosely speaking, the point here is that in contrast to the global decay rates these, faster non-resonant local rates are *integrable in time*, which allows one to close certain bootstrap arguments involving the Duhamel formula.

2. The wave evolution

Similar considerations apply to the wave equation. Indeed, let $\square u = 0$, with $(u(0), \partial_t u(0)) = (0, g)$ [initial data $(f, 0)$ are then handled by differentiating in time]. Then, instead of (10), one has

$$u(t) = \frac{\sin(t\sqrt{H})}{\sqrt{H}} P_c g = \frac{1}{2\pi i}\int_{p_0 - i\infty}^{p_0 + i\infty} e^{tp} R(p^2) P_c g dp, \tag{12}$$

where $p_0 > 0$. In contrast to the Schrödinger case, the resolvent $R(p^2)$ in *odd dimensions* is now analytic around $p = 0$ (assuming that there is no zero energy resonance or eigenvalue), which results in *arbitrary local decay* of $u(t)$. More precisely, if V decays exponentially, thus allowing for analytic continuation of the Green function to the left-half plane, one obtains exponential decay in time relative to weighted L^2 in space. This is, of course, a consequence of the *sharp Huyghens principle* in odd dimensions, which states that the fundamental solution of the free wave equation is localized to a sphere with radius given by the time. We see from this informal discussion that this principle is robust under perturbations [at least in the sense that the perturbed wave $u(t)$ will decay very rapidly at distances $\ll t$ from the origin, which, of course, is far from being able to describe the fundamental solution]. Note the stark contrast between the strong local decay of the wave equation as compared to the specific global decay given by (4).

On the other hand, in even dimensions, the resolvent will exhibit a $\log p$ singularity [see (9)]. Due to this branching of the resolvent at $p = 0$, Watson's lemma implies an explicit power law depending on the dimension governing the tail of the wave near the origin. This is in agreement with the fact that there is no sharp Huyghens principle in even dimensions.

To summarize this section, one sees that the *local decay* for both the Schrödinger and the wave equation is entirely determined by the singularity (often but not necessarily by branching) of the resolvent $(-\Delta + V + z)^{-1}$ at $p = 0$, where $z = -ip$ in the former case and $z = p^2$ in the latter case.

B. Global decay for $-\Delta + V$

1. The Schrödinger evolution

The first result that proved (2) for $H = -\Delta + V$ in dimensions $d \geq 3$ was obtained by Journé, Soffer, and Sogge.[51] Following the unpublished work by Ginibre, we now give a short proof of a simpler estimate, namely,

$$\| e^{itH} P_c f \|_{L^\infty + L^2(\mathbb{R}^d)} \lesssim \langle t \rangle^{-\frac{d}{2}} \| f \|_{L^1 \cap L^2(\mathbb{R}^d)}, \tag{13}$$

assuming that V has sufficient decay and that H has no zero energy eigenvalue or resonance. The logic here is that the Duhamel formula allows one to upgrade local decay to global one. More precisely, if

$$\| \langle x \rangle^{-\sigma} e^{itH} P_c f \|_{L^2(\mathbb{R}^d)} \lesssim \langle t \rangle^{-\frac{d}{2}} \| \langle y \rangle^{\sigma} f \|_{L^2(\mathbb{R}^d)}$$

and if V decays sufficiently fast, then the same estimate holds without weights in the sense of (13) (provided $d > 2$). More precisely, applying the Duhamel formula twice yields

$$\begin{aligned} e^{itH} P_c &= e^{-it\Delta} P_c + i \int_0^t e^{-i(t-s)\Delta} V e^{isH} P_c \, ds \\ &= e^{-it\Delta} P_c + i \int_0^t e^{-i(t-s)\Delta} V P_c e^{-is\Delta} \, ds \\ &\quad + \int_0^t \int_0^s e^{-i(t-s)\Delta} V e^{i(s-s')H} P_c V e^{-is'\Delta} \, ds' \, ds. \end{aligned}$$

Applying the local decay for e^{isH} from Sec. II A (with $|V|^{\frac{1}{2}}$ acting as weight, say) as well as the bound

$$\| e^{-it\Delta} f \|_{L^2 + L^\infty(\mathbb{R}^d)} \lesssim \langle t \rangle^{-\frac{d}{2}} \| f \|_{L^1 \cap L^2(\mathbb{R}^d)}$$

to this expression yields for $\| f \|_{L^1 \cap L^2(\mathbb{R}^d)} = 1$,

$$\begin{aligned} \| e^{itH} P_c f \|_{L^\infty + L^2(\mathbb{R}^d)} &\lesssim \langle t \rangle^{-\frac{d}{2}} + \int_0^t \langle t - s \rangle^{-\frac{d}{2}} \langle s \rangle^{-\frac{d}{2}} \, ds \\ &+ \int_0^t \int_0^s \langle t - s \rangle^{-\frac{d}{2}} \langle s - s' \rangle^{-\frac{d}{2}} \langle s' \rangle^{-\frac{d}{2}} \, ds' \, ds \lesssim \langle t \rangle^{-\frac{d}{2}} \end{aligned}$$

as claimed, provided d ≥ 3. The main gist of Ref. 51 is now to remove the L^2-piece from this argument. This is subtle, as the free estimate involved $(t-s)^{-\frac{d}{2}}$, which is not integrable at $s = t$. To overcome this difficulty, Journé, Soffer, and Sogge used the bound

$$\sup_{1\le p\le\infty} \|e^{-it\Delta} V e^{it\Delta}\|_{p\to p} \le \|\hat{V}\|_1.$$

The point here is that the left-hand side for $V = e^{ix\eta}$ is a translation operator composed of a unimodular factor and therefore L^p bounded.

Rodnianski and the author[71] proved that for all $t > 0$,

$$\|e^{itH} f\|_{L^\infty(\mathbb{R}^3)} \le C(V) t^{-\frac{3}{2}} \|f\|_{L^1(\mathbb{R}^3)}, \tag{14}$$

assuming that

$$\sup_{x\in\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|V(y)|}{|x-y|}\, dy < 4\pi \tag{15}$$

as well as that the so-called Rollnick norm of V is less than 4π. The left-hand side in (15) is commonly referred to as the *Kato norm* $\|\cdot\|_K$. The Rollnick condition precludes any spectral problems, such as eigenvalues and a zero energy singularity. The approach of Ref. 71 to the pointwise bounds is based on an expansion into an infinite Born series followed by term-wise estimation of the resulting kernels. The smallness condition on V guarantees convergence.

Remarkably, Beceanu and Goldberg[5] were able to show that the finiteness of the Kato norm alone suffices. More precisely, they showed that (14) holds for $e^{itH} P_c$ in three dimensions assuming (15) with 4π replaced by ∞ and that there are no imbedded eigenvalues and resonances in the continuous spectrum. They accomplished this by means of Beceanu's Wiener algebra techniques (see Ref. 4). Recall that Wiener's classical theorem states that for any $f \in L^1(\mathbb{R})$, the equation $(\delta_0 + f) * (\delta_0 + g) = \delta_0$ has a (unique) solution with $g \in L^1(\mathbb{R})$ if and only if $1 + \hat{f} \neq 0$ on $\mathbb{R}$. The relevance of this to the decay of solutions to

$$(i\partial_t - \Delta + V)\psi = F, \quad \psi(0) = \psi_0,$$

can be seen as follows: let $V_1 V_2 = V$, $|V_1| = |V_2|$ and set

$$(T_{V_2,V_1} F)(t) = \int_0^t V_2 e^{i(t-s)H_0} V_1 F(s)\, ds,$$

with $H_0 = -\Delta$. Then, on one hand, one has

$$V_2 \psi(t) = (\delta_0 \mathrm{Id} - i T_{V_2,V_1})^{-1} V_2 \Big(e^{itH_0} \psi_0 - i \int_0^t e^{i(t-s)H_0} F(s)\, ds \Big),$$

which is to be interpreted in the convolution algebra $\mathscr{B}(L^2(\mathbb{R}^3), \mathscr{M}_t L^2(\mathbb{R}^3))$, where $\mathscr{M}_t$ are the complex measures on the line. On the other hand, $\widehat{T_{V_2,V_1}}(\lambda) = i V_2 R_0^-(\lambda) V_1$, with $R_0^-(\lambda) = (H_0 - (\lambda - i0))^{-1}$. Hence, the invertibility of $\delta_0 \mathrm{I}d - i T_{V_2,V_1}$ in $\mathscr{B}(L^2(\mathbb{R}^3), \mathscr{M}_t L^2(\mathbb{R}^3))$ is the same as the pointwise invertibility of the Birman–Schwinger operator $\mathrm{I}d + V_2 R_0^-(\lambda) V_1$. This equivalence is delicate and requires $V \in L^{\frac{3}{2},1}(\mathbb{R}^3)$ the Lorentz space, whence $V_1, V_2 \in L^{3,2}(\mathbb{R}^3)$, and also the Keel–Tao Strichartz endpoint.[54] For the abstract Wiener theorem in this context, see Theorem 1.1 of Ref. 4 and Theorem 3 of Ref. 5.

An alternative and very general approach to proving L^p bounds on both wave and Schrödinger evolutions was found by Yajima[81,82] who proved L^p boundedness of the wave operators, with the limit being taken in the strong L^2-sense,

$$W = \lim_{t\to\infty} e^{-itH} e^{-it\Delta} \tag{16}$$

for all $1 \le p \le \infty$ and d ≥ 3. The fact that these operators exist and are isometries $L^2 \to \mathrm{Ran}(P_c(H))$ is a classical fact (see Ref. 52). They intertwine the free evolution with that of H in the sense that (with $H_0 = -\Delta$)

$$f(H) P_c(H) = W f(H_0) W^*$$

for any Borel function f on $\mathbb{R}$. In particular, $e^{itH} P_c(H) = W e^{itH_0} W^*$, and (2) therefore implies the bound

$$\|e^{itH} P_c f\|_\infty \le C t^{-\frac{d}{2}} \|f\|_1$$

whenever $W : L^\infty \to L^\infty$, $W^* : L^1 \to L^1$. Yajima obtains similar results on $W^{k,p}$ assuming more regularity on V (the amount of regularity depends on k). In view of our discussions of the role of zero energy resonances for local decay, it follows that Yajima's result[81] can only hold under the assumption that zero energy is neither a resonance nor an eigenvalue. In three dimensions,[81] the result requires $|V(x)| \lesssim \langle x \rangle^{-\sigma}$ with $\sigma > 5$ and therefore improves on.[51]

Yajima derives his L^p bounds by means of a finite Born series expansion with a remainder term involving the perturbed resolvent. In the case of small potentials, one can sum up the infinite Born expansion, leading to more precise results in terms of conditions on V. In view of the preceding discussion of Wiener theorems as a means of summing divergent series, it is natural to ask if Yajima's theorem could be approached by means of a suitable Wiener algebra. Beceanu and the author[6] carried this out and proved that the wave operators given by (16) in $\mathbb{R}^3$ are superpositions of reflections and translations. In fact, assuming that $|V(x)| \lesssim C\langle x\rangle^{-\frac{5}{2}-\varepsilon}$ and that zero energy is neither an eigenvalue nor a resonance, they showed that there exists $g(x,y,\omega) \in L^1_\omega \mathscr{M}_y L^\infty_x$ (with $\mathscr{M}_y$ being finite Borel measures in y), i.e.,

$$\int_{\mathbb{S}^2} \|g(x,dy,\omega)\|_{\mathscr{M}_y L^\infty_x}\, d\omega < \infty,$$

such that for $f \in L^2(\mathbb{R}^3)$, one has the representation formula for the wave operator

$$(Wf)(x) = f(x) + \int_{\mathbb{S}^2}\int_{\mathbb{R}^3} g(x,dy,\omega) f(S_\omega x - y)\, d\omega,$$

where $S_\omega x = x - 2(x\cdot\omega)\omega$ is a reflection. This, of course, implies that $W : X \to X$ is bounded for any function space X on $\mathbb{R}^3$ with a norm that is invariant under translations and reflections. The proof of this representation formula in Ref. 6 is not entirely straightforward. On one hand, the algebra to which the Wiener theorem is applied is somewhat delicate and requires casting the finite order Born series terms in Yajima's work[81] (which involve only finitely many potentials and free resolvents) in some iterative algebraic framework. In other words, one needs to find the correct algebra $\mathscr{A}$ and composition law $\circledast$ as well as operator T to write the third Born term, say, in the form $T \circledast T \circledast T$ in $\mathscr{A}$. Furthermore, the classical scattering theory based on weighted L^2 spaces does not suffice, and it is necessary to invoke the author's work with the work of Ionescu,[49] which revisits the classical Agmon–Kato–Kuroda theorem in the context of Fourier restriction and the Stein–Tomas theorem, as well as the Keel–Tao endpoint.[54] This, in turn, relies on the Carleman theorems and absence of imbedded eigenvalues obtained in Ref. 48. It is not known whether a structure theorem holds under a scaling-invariant assumption on V; see, however,Ref. 7 for such a result, albeit involving small scaling-invariant potentials.

In higher dimensions, it turns out that one needs to assume some regularity of V in order for the expected $L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$ bounds to hold. Indeed, Goldberg and Visan[42] showed that the dispersive bound can fail in dimensions $d > 3$ for potentials that belong to the class $C^{\frac{d-3}{2}-}(\mathbb{R}^d)$. The logic here is that the free resolvent takes the form (in odd dimensions)

$$(-\Delta - \lambda^2 + i0)^{-1}(r) = \frac{e^{i\lambda r}}{r^{d-2}} \sum_{j=0}^{\frac{d-3}{2}} c_j\, (\lambda r)^j, \tag{17}$$

and the highest power $\lambda^{\frac{d-3}{2}}$ here corresponds to a $\frac{d-3}{2}$ derivative loss on V. In the positive direction, Erdoğan and Green[33] proved the dispersive bound in dimensions $d = 5, 7$, assuming that $V \in C^{\frac{d-3}{2}}(\mathbb{R}^d)$ (zero energy resonances cannot arise in dimensions d ≥ 5).

The case of low dimensions $d = 1$ and $d = 2$ always requires a separate analysis since the free resolvent in those cases exhibits a zero energy singularity (more precisely, there is a zero energy resonance given by the constant state $f = 1$). We refer to the reader to Refs. 23, 41, and 80 for the one-dimensional case and Ref. 73 for dispersive estimates for the two-dimensional case, provided zero energy is regular. Erdogan and Green[38] carried out a more complete analysis of the dispersive decay in $\mathbb{R}^2$, allowing for s and p-wave resonances at zero energy. This classification refers to nonzero solutions ψ of $H\psi = 0$, which (i) are asymptotic to a nonzero constant at spatial ∞ for s-waves and (ii) are in $L^q(\mathbb{R}^2)$ for all $q > 2$ for p-waves. They showed that the s-wave resonance, which arises in the $V = 0$ case, leads to the same t^{-1} decay as in the free evolution, whereas the p-wave destroys this rate of decay. With Goldberg, these authors also obtained such a classification in $\mathbb{R}^4$. Finally, more recently, Erdoğan, Green, and Toprak applied spectral methods to analyze the delicate dispersive decay of the Dirac operator (see Ref. 35).

2. The wave equation

Starting with Beals and Strauss,[2,3] many authors considered the problem of proving the dispersive estimate (4) for equations $(\Box + V)u = 0$, $(u, \partial_t u)(0) = (f, g)$ (it will suffice to set $f = 0$). In Refs. 2 and 3, the potential is assumed to be either non-negative or small (which excludes any spectral problems), as well as rapidly decaying and smooth. The result is of the form (4) but with slightly more derivatives on the data. Georgiev and Visciglia[40] assumed that $0 \le V \le \langle x\rangle^{-2-\varepsilon}$ in three dimensions and obtained (4) for energies away from zero as well as Strichartz estimates for all energies. Cuccagna[19] proved Strichartz estimates in three dimensions, assuming that $|\partial^\alpha V(x)| \lesssim \langle x\rangle^{-3-\varepsilon}$ for $|\alpha| \le 2$ and that zero energy is regular. D'Ancona and Pierfelice[25] proved global dispersive (4) for $d = 3$, assuming that $\|\min(V,0)\|_K < 2\pi$ but with $\dot{B}^1_{1,1}$ on the right-hand side. Pierfelice[68] obtained the same result under the smallness assumption (15) (the arguments in Ref. 58 yield the same but with $\|\nabla g\|_1$ instead of the Besov norm). D'Ancona and Fanelli[24] considered the wave and Dirac equations in three dimensions,

$$u_{tt} - (\nabla + iA)^2 u + Vu = 0,$$
$$iU_t - \mathscr{D}U + MU = 0,$$

respectively. Assuming smallness of A, V, M but allowing nearly scaling-invariant singularities of these functions both at zero and infinity (which are $|x|^{-1}$, $|x|^{-2}$, and $|x|^{-\frac{1}{2}}$, respectively), the t^{-1} global decay is obtained but for data in weighted Soboloev and Besov spaces. By the aforementioned results of Yajima *et al.* on the $W^{k,p}$-boundedness of the wave operators, one can obtain L^p decay estimates for the wave equation from the free estimates (4). Note that the Besov spaces are then defined relative to H rather than the free Laplacian, but it is often possible to pass between the two. For a more recent reference on the integrated decay of waves, which also allows for magnetic perturbations, see the work of D'Ancona.[22]

3. The case of singular zero energy

Certain stability problems in physics lead to linear operators with a zero energy eigenvalue or resonance. Examples are the energy critical wave equation $\Box u - u^5 = 0$ in $\mathbb{R}^{1+3}$, which admits the stationary solutions $W_\lambda(x) := \lambda(1+\lambda^2|x|^2/3)^{-\frac{1}{2}}$ for $\lambda > 0$. Linearizing around W_λ leads to $H = -\Delta - 5W_\lambda^4$, which has $\partial_\lambda W_\lambda$ as a resonant mode of zero energy. Another example is the critical Yang–Mills problem in dimensions 4 + 1. It is therefore necessary to obtain dispersive bounds in this context as well. Note that the local decay of Sec. II A easily allows for this as the asymptotic expansions in time (as derived in Refs. 50 and 65, for example) isolate the contributions of the threshold singularities and identify them as being of finite rank. In the case of $L^1 \to L^\infty$, this required some additional work (see Refs. 36, 37, and 83 for the case of the Schrödinger evolution). Yajima[83] obtained explicit expressions for the term $Bt^{-\frac{1}{2}}$, which needs to be subtracted to obtain the $t^{-\frac{3}{2}}$ decay of the bulk (*explicit* here means that B can be computed from the zero energy and resonance states). The wave equation in three dimensions is analyzed in Ref. 58. We recall the main linear result from the latter reference.

Proposition II.2. Assume that V is a real-valued potential such that $|V(x)| \lesssim \langle x\rangle^{-\kappa}$, where $\kappa > 3$ is fixed but arbitrary. If zero energy is regular for H, then

$$\left\|\frac{\sin(t\sqrt{H})}{\sqrt{H}}P_c f\right\|_\infty \lesssim t^{-1}\|f\|_{W^{1,1}(\mathbb{R}^3)}$$

for all $t > 0$. Now assume that zero is a resonance but not an eigenvalue of $H = -\Delta + V$. Let ψ be the unique resonance function normalized so that $\int V\psi(x)dx = 1$. Then, there exists a constant $c_0 \neq 0$ such that

$$\left\|\frac{\sin(t\sqrt{H})}{\sqrt{H}}P_c f - c_0(\psi\otimes\psi)f\right\|_\infty \lesssim t^{-1}\|f\|_{W^{1,1}(\mathbb{R}^3)} \tag{18}$$

for all $t > 0$.

Several results exist on the boundedness of the wave operators on L^p in the case zero energy is singular. However, they are limited to a smaller range of p (in $d = 3$, one needs $\frac{3}{2} < p < 3$) and are less useful for nonlinear applications, at least in three dimensions. On the other hand, in $\mathbb{R}^2$, Erdogan, Goldberg, and Green[32] showed that the wave operators remain bounded in the full range $1 < p < \infty$ if zero energy exhibits only an s-wave resonance or only a zero energy eigenvalue.

For the Klein–Gordon equation on the line with a non-generic decaying potential (i.e., the associated Schrödinger operator exhibits a zero energy resonance), an analog of Proposition II.2 was obtained in Ref. 59, albeit for local decay. This is part of a larger body of work aiming at understanding kink stability.

III. METRIC PERTURBATIONS

If one replaces $-\Delta$ by the elliptic operator $H := -\sum_{j,k=1}^d \partial_j(a_{jk}(x)\partial_k)$, then one encounters a new obstruction to proving decay estimates in addition to the zero energy resonance or eigenvalue of Sec. II: the phenomenon of trapping, which is a *large energy problem*. Trapping refers to the possibility that the classical Hamiltonian

$$h(x,\xi) := \frac{1}{2}\sum_{j,k=1}^d a_{jk}(x)\xi^k\xi^j$$

exhibits closed trajectories. More precisely, assuming symmetry $a_{jk} = a_{kj}$, one has the Hamiltonian equations

$$\dot{x} := \sum_{j=1}^d a_{j\cdot}(x)\xi^j, \qquad \dot{\xi} = \frac{1}{2}\sum_{j,k=1}^d \nabla_x a_{jk}(x)\xi^k\xi^j,$$

which might exhibit time-periodic trajectories. To understand the crucial effect of the existence of closed geodesics, we consider the method of proving decay estimates using energy estimates,

$$\frac{d}{dt}\langle u, A(t)u\rangle = \langle u, i[H, A(t)]u\rangle + \langle u, \frac{\partial A(t)}{\partial t}u\rangle,$$

where $u = u(x,t)$ is the solution of the Schrödinger equation, with Hamiltonian H. A similar identity can be applied for the wave equation (see Ref. 8). Next, suppose that the expectation of $A(t)$ is bounded from above, uniformly in t, by $\|u\|_2$ and, moreover, that the commutator is positive in the sense that

$$i[H, A(t)] + \frac{\partial A(t)}{\partial t} \geq \theta B^\star B$$

for some $\theta > 0$ and some operator B. Upon integration over time, we obtain an integrated decay estimate for B,

$$\int_0^\infty \|Bu\|^2\, dt \leq c\|u(0)\|_2^2.$$

The operator family $A(t)$ is variably called a multiplier, a propagation observable, an escape function, or a conjugate operator.

To illustrate this further, let $h(x,\xi)$ be a classical Hamiltonian on $\mathbb{R}^{2d}$. If $[x(t), \xi(t)]$ is an orbit under the Hamiltonian flow of h, then

$$\frac{d}{dt} a(x(t), \xi(t)) = \{h, a\}(x(t), \xi(t)),$$

where the right-hand side is the Poisson bracket. For the Euclidean case, i.e., $h(x,\xi) = \frac{1}{2}\xi^2 + V(x)$, one can take $a(x,\xi) = x\cdot\xi = \{h, \frac{1}{2}|x|^2\}$, which gives $\{h, a\} = 2h - 2V - x\cdot\nabla V$. Now, suppose that $-2V(x) - x\cdot\nabla V(x) \geq 0$ for $|x| \geq R > 0$, say. Since h is conserved, we conclude that a trajectory with $h = \alpha > 0$, which remains in $|x| \geq R$, satisfies

$$\frac{d^2}{dt^2}\frac{1}{2}|x(t)|^2 \geq 2\alpha$$

and therefore $|x(t)|$ grows linearly in t. This indicates that $(x(t), \dot{x}(t))$ undergoes scattering like a free particle. Under a short-range condition on $V(x)$, i.e., $|V(x)| \leq C\langle x\rangle^{-1-\varepsilon}$, this is indeed the case; i.e., all trajectories that are not trapped are asymptotically free. See the book by Derezinski and Gerard[26] for a systematic development of these techniques in both classical and quantum mechanics.

Positive commutator methods are also used to prove refined average decay estimates that hold on subsets of the phase space. Such estimates for the wave and Schrödinger equations were first derived by Morawetz using the radial derivative operator and the generator of the conformal group as multipliers. These multipliers also work if repulsive interactions are added. However, modifications are needed if trapped geodesics are present and usually only lead to weaker estimates. A major step in this direction is the use of a *sharp localization of the energy* due to Mourre.[63] The energy estimate can be obtained by taking the derivative with respect to time of the expectation value of some operator, also called propagation observable as in Ref. 76. The remarkable paper by Hunziker, Sigal, and Soffer[46] presents a time-dependent approach to Mourre theory based on the *commutator expansion lemma* of Sigal and Soffer. The latter refers to expressing $[f(A), B]$ through a series of Taylor type involving higher-order commutators between A and B.

A parallel development to this approach was based on ΨDO methods. In this approach, one constructs a function on the phase space that has positive Poisson bracket with the principal symbol of the Hamiltonian. Then, one uses the quantized symbol of this function as a propagation observable and, by means of ΨDO theory, and in particular, Garding's inequality, passes to the desired smoothing (or limiting absorption) bound. Some of the earliest implementations of this approach are Refs. 15 and 27, and since then, a vast literature has developed in this direction.

The importance of a nontrapping condition is readily understood: it allows for the construction of monotonic propagation observables, globally in the phase space. In the presence of closed trajectories, this is not possible. However, when the trajectories are closed but (strongly) unstable, there is now substantial evidence that the decay estimates continue to hold in some sense.

On the level of the resolvents, one considers $(H - z)^{-1}$ with H being a variable coefficient operator as above, with a_{ij} being a short-range perturbation of a constant elliptic symbol. Furthermore, assume that all classical Hamiltonian orbits of large initial velocity are not trapped. Then, the limiting absorption principle

$$\sup_{\operatorname{Im} z>0, \operatorname{Re} z\geq N} \|\langle\cdot\rangle^{-\sigma}(H - z)^{-1}\chi_I(H)\langle\cdot\rangle^{-\sigma}\|_{2\to 2} < \infty \tag{19}$$

holds with $N > 0$ and $\sigma > 0$ sufficiently large (see Ref. 64). In fact, the nontrapping condition is necessary (see Theorem 2 in Ref. 64), and one also obtains (19) for derivatives in z of the resolvent. The latter property then clearly implies local decay on the time-evolution restricted to high energies.

In fact, while Doi[27] and Murata[65] showed that smoothing estimates and the usual decay estimates do not hold in the presence of trapping, Ikawa[47] shows that one still obtains local decay estimates for the Laplacian dynamics on $\mathbb{R}^n$ with several convex obstacles removed. In the meantime, the microlocal analysis on manifolds with unstable closed geodesics, of the resolvent of the Laplacian on one hand and the Schrödinger evolution on the other hand, has grown into a vast area in and of itself, which is intimately connected to the semiclassical analysis of scattering resonances. See, for example, the recent research monograph[9] on *Resonances for homoclinic trapped sets* or Dyatlov's introduction to the *fractal uncertainty principle*.[31]

In general relativity, unstable closed geodesics arise naturally in the study of the linear wave evolution on the background of both Schwarzschild and Kerr black holes. A substantial amount of work has accumulated around this topic (see, for example, the early works by Blue and Soffer; see Ref. 8 as well the very recent study of Price's law by Hintz[44]). The latter paper was preceded by the work of Tataru,[78] as well as the results by Donninger, Soffer, and the author[30] on the spatially local, but temporally global, decay of linear waves on Schwarzschild. Metcalfe, Tataru, and Tohaneanu[62] subsequently established Price's law on nonstationary spacetimes with sufficient decay in a suitable sense. Very recently, Angelopoulos, Aretakis, and Gajic[1] presented a "physical space" approach to Price's law on Kerr spacetimes in contrast to Hintz's microlocal technique. We now set out to describe the author's results in more detail.

A. Asymptotically conical surfaces of revolution

As a model case for the Schwarzschild manifold, Soffer, Staubach, and the author[74,75] studied wave evolutions on surfaces of revolution with conic ends. Let $\Omega \subset \mathbb{R}^N$ be an embedded compact d-dimensional Riemannian manifold with metric ds_Ω^2, and define the $(d+1)$-dimensional manifold

$$\mathcal{M} := \{(x, r(x)\omega) \mid x \in \mathbb{R},\ \omega \in \Omega\},$$
$$ds^2 = r^2(x) ds_\Omega^2 + (1 + r'(x)^2) dx^2,$$

where $r \in C^\infty(\mathbb{R})$ and $\inf_{x\in\mathbb{R}} r(x) > 0$. We say that there is a *conical end* at the right (or left) if

$$r(x) = |x|\,(1 + h(x)), \quad h^{(k)}(x) = O(x^{-2-k}) \quad \forall\ k \geq 0 \tag{20}$$

as $x \to \infty (x \to -\infty)$.

Of course, one can consider cones with arbitrary opening angles, but this adds nothing of substance. Examples of such manifolds are given by surfaces of revolution with $\Omega = S^1$ such as the one-sheeted hyperboloid that satisfies $r(x) = \sqrt{1+x^2}$. They have the property that the entire Hamiltonian flow on $\mathcal{M}$ is trapped on the set $[x_0, r(x_0)\Omega]$ when $r'(x_0) = 0$. From now, we will only consider S^1 as cross section Ω for the sake of simplicity. The only difference from the general case is that instead of $\{e^{\pm i\ell\theta}, \ell^2\}_{\ell=0}^\infty$, one has a complete system $\{Y_n, \mu_n\}_{n=0}^\infty$ of L^2-normalized eigenfunctions and eigenvalues, respectively, of Δ_Ω. In other words, $-\Delta_\Omega Y_n = \mu_n^2 Y_n$, where $0 = \mu_0^2 < \mu_1^2 \leq \mu_2^2 \leq \ldots$.

Note that we do not specify the local geometry of $\mathcal{M}$ but only the asymptotic one at the ends. This allows for very different behaviors of the geodesics. For the case of the one-sheeted hyperboloid, for example, the geodesic flow around the unique periodic geodesic is hyperbolic in the sense of dynamical systems, whereas if we place a section of S^2 in the middle of $\mathcal{M}$, then we encounter a set of positive measures in the cotangent bundle, leading to stable periodic geodesics. These two scenarios are depicted in Fig. 2. It is natural to ask to what extent this local geometry affects the dispersion of the flow. The following result summarizes what is proved in Refs. 29 and 30 for the case of $\Omega = S^1$ (see those references for general compact Ω):

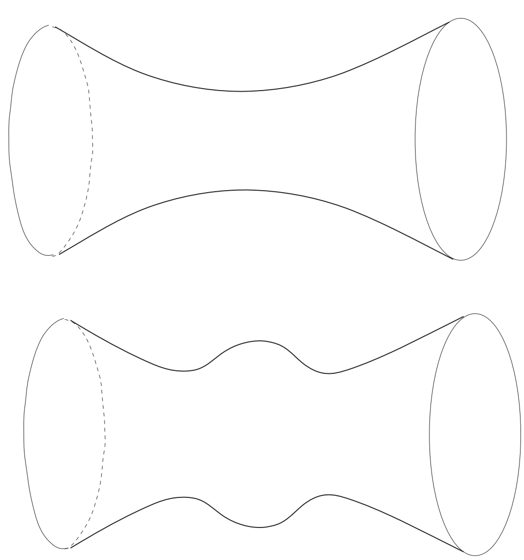

FIG. 2. Unstable vs stable geodesic flow.

Theorem III.1. *Let $\mathcal{M}$ be a surface that is asymptotically conical at both ends as defined above. For each $\ell \geq 0$ and all $0 \leq \sigma \leq \sqrt{2}\ell$, there exist constants $C(\ell, \mathcal{M}, \sigma)$ and $C_1(\ell, \mathcal{M}, \sigma)$ such that for all $t > 0$,*

$$\|w_\sigma e^{it\Delta_{\mathcal{M}}} f\|_{L^\infty(\mathcal{M})} \leq \frac{C(\ell, \mathcal{M}, \sigma)}{t^{1+\sigma}} \left\| \frac{f}{w_\sigma} \right\|_{L^1(\mathcal{M})}, \tag{21}$$

$$\|w_\sigma e^{it\sqrt{-\Delta_{\mathcal{M}}}} f\|_{L^\infty(\mathcal{M})} \leq \frac{C_1(\ell, \mathcal{M}, \sigma)}{t^{\frac{1}{2}+\sigma}} \left(\left\| \frac{\partial_x f}{w_\sigma} \right\|_{L^1(\mathcal{M})} + \left\| \frac{f}{w_\sigma} \right\|_{L^1(\mathcal{M})} \right), \tag{22}$$

provided $f = f(x, \theta) = e^{i\ell\theta}\tilde{f}(x)$, where $\tilde{f}$ does not depend on θ. Here, $w_\sigma(x) := \langle x \rangle^{-\sigma}$ are weights on $\mathcal{M}$.

In (22), one can obtain somewhat finer results by distinguishing between $\cos(t\sqrt{-\Delta_{\mathcal{M}}})$ and $\frac{\sin(\sqrt{-\Delta_{\mathcal{M}}})}{\sqrt{-\Delta_{\mathcal{M}}}}$ (see Ref. 29 for statements of that kind). Needless to say, $\sigma = 0$ is the analog of the usual dispersive decay estimate for the Schrödinger and wave evolutions on $\mathbb{R}^2$. We remark that as in the case of the plane $\mathbb{R}^2$, the free Laplacian $\Delta_{\mathcal{M}}$ exhibits a zero energy resonance, which is, however, only visible at $\ell = 0$ (this case is treated separately in Ref. 74, whereas Ref. 75 studies $\ell > 0$).

Clearly, the *local* decay given by $\sigma > 0$ has no analog in the Euclidean setting, and it also has no meaning for $\ell = 0$. The restriction $\sqrt{2}\ell$ is optimal in Theorem III.1, at least for the Schrödinger equation, and no faster decay can be obtained than the one stated in (21). The $\sqrt{2}$-factor comes from the opening angle of $\frac{\pi}{4}$ and changing that angle leads to different constants, namely, $\frac{1}{\cos(\theta)}$, where θ is the opening angle of the asymptotic cone.

A heuristic explanation for the existence of this accelerated *local decay* is given by the geodesic flow combined with the natural dispersion present in these equations. Indeed, the former will push any nontrapped geodesics into the ends (with ℓ playing the role of the velocity of the geodesics), whereas the latter will spread any data that are initially highly localized around a periodic geodesic away from it, thus making it susceptible to the mechanism we just described.

What is not clear from this heuristic is whether or not the localized decay law should depend on the local geometry (by which we mean the geometry that is not described by the asymptotic cones). Theorem III.1 shows that this is not so, since the local decay is *fixed* and given by a specific power. Therefore, one sees that the local geometry manifests itself exclusively through the constants $C(\ell, \mathcal{M}, \sigma)$. This is natural, as one would expect a much longer waiting time before the large t behavior of the theorem sets in if $\mathcal{M}$ exhibits stable geodesics. In fact, the constant $C(\ell)$ grows exponentially in that case as can be seen by solutions that are highly localized (microlocally) around a periodic geodesic (see Refs. 72 and 77).

In contrast, the methods of Ref. 30 show that this constant grows like ℓ^C if the manifold $\mathcal{M}$ has a unique periodic geodesic and is uniformly convex near it. This then allows one to sum up the estimates for each angular momentum as described by the following theorem:

Theorem III.2. *Let $\mathcal{M}$ be asymptotically conical at both ends as above, and suppose that $\mathcal{M}$ has a unique periodic geodesic and is uniformly convex near it. Then, for all $t > 0$ and any $\varepsilon > 0$ and with $\mathcal{D} := 1 - \partial_\theta^2$,*

$$\|w_{1+\varepsilon} e^{it\Delta_{\mathcal{M}}} w_{1+\varepsilon} f\|_{L^2(\mathcal{M})} \leq \frac{C(\mathcal{M}, \varepsilon)}{\langle t \rangle} \|\mathcal{D} f\|_{L^2(\mathcal{M})}, \tag{23}$$

$$\|w_1 e^{it\Delta_{\mathcal{M}}} w_1 f\|_{L^\infty(\mathcal{M})} \leq \frac{C(\mathcal{M}, \varepsilon)}{t} \|\mathcal{D}^{2+\varepsilon} f\|_{L^1(\mathcal{M})}, \tag{24}$$

provided $f = f(x, \theta)$ is Schwartz on $\mathcal{M}$, say. For the wave equation, one has

$$\|w_{\frac{1}{2}+\varepsilon} e^{\pm it\sqrt{-\Delta_{\mathcal{M}}}} w_{\frac{1}{2}+\varepsilon} f\|_{L^2(\mathcal{M})} \leq \frac{C_1(\mathcal{M}, \varepsilon)}{\langle t \rangle^{\frac{1}{2}}} \left(\|\mathcal{D}^{\frac{5}{4}} f'\|_{L^2(\mathcal{M})} + \|\mathcal{D}^{\frac{5}{4}} f\|_{L^2(\mathcal{M})} \right), \tag{25}$$

$$\|w_{\frac{1}{2}+\varepsilon} e^{\pm it\sqrt{-\Delta_{\mathcal{M}}}} w_{\frac{1}{2}+\varepsilon} f\|_{L^\infty(\mathcal{M})} \leq \frac{C_1(\mathcal{M}, \varepsilon)}{t^{\frac{1}{2}}} \left(\|\mathcal{D}^{\frac{9}{4}+\varepsilon} \partial_x f\|_{L^1(\mathcal{M})} + \|\mathcal{D}^{\frac{9}{4}+\varepsilon} f\|_{L^1(\mathcal{M})} \right). \tag{26}$$

The weights w_1 and $w_{\frac{1}{2}+\varepsilon}$ appearing in (24) and (26), respectively, are a by-product of our proof and can most likely be removed. The origin of the weights in our method will be explained in Sec. III B. One also obtains the accelerated decay rates that are better by $t^{-\sigma}$ as in

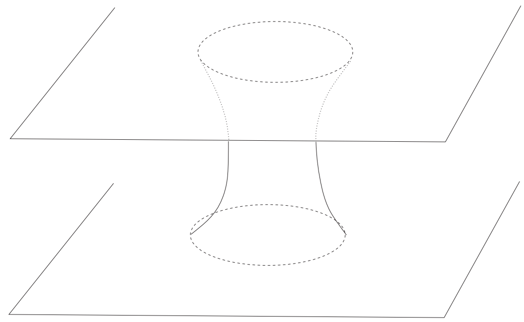

FIG. 3. Two planes joined by a neck.

Theorem III.1, provided one puts in the weights as before, makes the number of derivatives required on the right-hand side depend on σ, and provided the data are perpendicular to $e^{i\ell\theta}$ for $\sigma > \sqrt{2}\ell \geq 0$. We remark that one can think of the surfaces in Theorem III.1 as two planes joined by a neck (see Fig. 3). On the other hand, the methods that are currently used to prove Theorems III.1 and III.2 do not extend to the case of more necks, as then there is no clear way of separating variables.

There is no reason to expect that the number of derivatives required on the data in Theorem III.2 is optimal, in fact, it most certainly is not. Heuristically speaking, these derivatives measure the spreading or *non-concentration* of solutions near hyperbolic orbits in dependence on the angular momentum, which is a quantum effect. See, for example, the work of Christianson[14] on this topic.

Doi[27] proved that the presence of trapping destroys the so-called local smoothing estimate for the Schrödinger evolution. More precisely, he showed that one loses (even locally in time) the $\frac{1}{2}$-derivative gain present in $e^{it\Delta}$. Note that this does not constitute a contradiction to Theorem III.2 as the latter does not claim any gain of regularity (on the contrary, we lose angular derivatives). In a similar vein, Burq, Guillarmou, and Hassell[12] proved that Strichartz estimates may remain valid on metrics with trapping.

We now describe the method of proof leading to Theorem III.1. Later, we will discuss how to obtain Theorem III.2, which requires considerably more work. We will then also describe the result[30] for linear waves on Schwarzschild, which is very close to Theorem III.2.

To begin with, let ξ be the arclength along a generator of $\mathcal{M}$. Then, the Laplacian takes the form

$$\Delta_{\mathcal{M}} = \frac{1}{r(\xi)}\partial_\xi(r(\xi)\partial_\xi) + \frac{1}{r^2(\xi)}\Delta_{S^2}.$$

Now,

$$e^{-i\ell\theta}r^{\frac{1}{2}}(\xi)\Delta_{\mathcal{M}}(r^{-\frac{1}{2}}(\xi)e^{i\ell\theta}f(\xi)) = \mathcal{H}_\ell f,$$

with

$$\mathcal{H}_\ell = -\partial_\xi^2 + V_\ell, \qquad V_\ell(\xi) = \frac{2\ell^2 - \frac{1}{4}}{\langle\xi\rangle^2} + O(\langle\xi\rangle^{-3}), \tag{27}$$

where each ξ-derivative of the $O(\cdot)$-term gives one extra power of ξ as decay. We remark that the leading $\langle\xi\rangle^{-2}$ decay is *critical* for several reasons. For us, most relevant is the behavior of the Jost solutions as the energy λ^2 tends to zero; in fact, these Jost solutions are continuous in λ around $\lambda = 0$, provided the decay of the potential is at least $\langle\xi\rangle^{-2-\varepsilon}$ for some $\varepsilon > 0$. At $\varepsilon = 0$, this property is lost—which is precisely what allows for the accelerated decay of Theorem III.1. To be more specific, one first reduces Theorem III.1 [at least the Schrödinger bound (21), the wave equation being similar] via the spectral theorem to the point-wise bound

$$\sup_{\infty>\xi\geq\xi'>-\infty} (\langle\xi\rangle\langle\xi'\rangle)^{-\frac{1}{2}} \left| \int_0^\infty e^{it\lambda^2} \operatorname{Im}\left[\frac{f_{+,\ell}(\xi,\lambda)f_{-,\ell}(\xi',\lambda)}{W_\ell(\lambda)}\right] \lambda\, d\lambda \right| \leq C_\ell\, t^{-1-\sigma}, \tag{28}$$

where C_ℓ is a uniform constant and $\sigma = \sqrt{2}\ell$. Here, $f_{\pm,\ell}$ are the (outgoing) Jost solutions, which satisfy $\mathcal{H}_\ell f_{\pm,\ell} = \lambda^2 f_{\pm,\ell}$ and $f_{\pm,\ell} \sim e^{\pm i\lambda\xi}$ as $\xi \to \pm\infty$. Moreover, $W_\ell(\lambda)$ is the Wronskian of f_+, f_-. We remark that the quantity inside the absolute values in (28) is exactly

$$\int_0^\infty e^{it\lambda^2} E(d\lambda^2)(\xi,\xi'),$$

where $E(d\lambda^2)(\cdot,\cdot)$ is the kernel of the spectral resolution of $\mathscr{H}_\ell$. As usual,

$$f_+(\xi,\lambda) = e^{i\xi\lambda} + \int_\xi^\infty \frac{\sin(\lambda(\xi'-\xi))}{\lambda} V(\xi') f_+(\xi',\lambda)\, d\xi'.$$

From this formula, one immediately sees the aforementioned discontinuity at $\lambda = 0$ since $\xi V(\xi) \notin L^1(0,\infty)$. Setting $\xi = \xi' = 0$, (21) of Theorem III.1 reduces to the standard stationary phase type bound (with $\nu := \sqrt{2}\,\ell$)

$$\left|\int_0^\infty e^{it\lambda^2} \lambda^{1+2\nu} \chi(\lambda)\, d\lambda\right| \le C t^{-1-\nu},$$

where χ is a smooth cutoff function to the interval $[0,1]$, say. To see why the spectral measure should be as flat as $\lambda^{1+2\nu} d\lambda$, let us first give an informal proof of the fact that

$$W_\ell(\lambda) = c\lambda^{1-2\nu}(1 + o(1)) \qquad \lambda \to 0, \tag{29}$$

where $c \neq 0$. Since this Wronskian appears in the denominator of the resolvent, it at least serves as an indication that the spectral measure might be this small for small λ [one has to be very careful here, since the numerator is of the same size—however, the *imaginary part* of the resolvent has the desired size $O(\lambda^{2\nu})$]. To begin with, recall from basic scattering theory that the Wronskian is given by

$$W(\lambda) = \frac{-2i\lambda}{T(\lambda)}, \tag{30}$$

where $T(\lambda)$ is the transmission coefficient (see Fig. 4) (in that figure, the dashed line is supposed to indicate an energy level k^2, and the turning points are defined as the projections of the intersection of the graph with that line). By the so-called WKB approximation, one has to leading order that $T(\lambda) = e^{-S(\lambda)}$ with the action S given by

$$\begin{aligned} S(\lambda) &= \int_{x_0}^{x_1} \sqrt{\nu^2 \langle y\rangle^{-2} - \lambda^2}\, dy \\ &= \hbar^{-1} \int_{x_0}^{x_1} \sqrt{2\langle y\rangle^{-2} - \hbar^2\lambda^2}\, dy, \end{aligned}$$

with $x_0 < 0 < x_1$ being the turning points that are defined as $V(x_0) = V(x_1) = \lambda^2$. Note that we modified the potential by removing the cubic corrections as well as the $-\frac{1}{4}\langle\xi\rangle^{-2}$ part of the potential (the latter obviously requiring some justification). Furthermore, we used that $\nu = \sqrt{2}\ell$ and assumed $\ell > 0$. As a result,

$$S(\lambda) = 2\nu|\log\lambda|(1 + o(1)) \qquad \lambda \to 0,$$

which then gives (29) to leading order. To justify the removal of the $\frac{1}{4}\langle\xi\rangle^{-2}$-part of the potential V_ℓ, we simply note that the usual WKB ansatz for the zero energy solutions of $\mathscr{H}_\ell$, viz., $\mathscr{H}_\ell f = 0$ is the approximate equality

$$f(\xi) \simeq V_\ell^{-\frac{1}{4}}(\xi) e^{\pm\int_1^\xi \sqrt{V_\ell(\eta)}\, d\eta}.$$

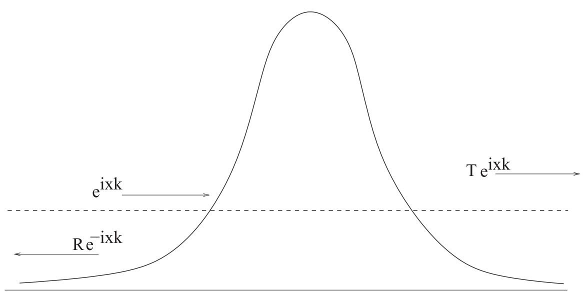

FIG. 4. Reflection and transmission coefficients.

In view of (27), one obtains the asymptotic behavior $\xi^{\frac{1}{2}\pm\sqrt{\nu^2-\frac{1}{4}}}$ as $\xi\to\infty$. On the other hand, the exact solutions of

$$-f''(\xi)+\frac{\nu^2-\frac{1}{4}}{\xi^2}f(\xi)=0$$

are of the form $\xi^{\frac{1}{2}\pm\nu}$. The WKB approximation can therefore only be correct, provided the $-\frac{1}{4}\xi^{-2}$ term is removed from the potential V_ℓ [for a precise rendition—with control of error terms—of this heuristic discussion (see Sec. II of Ref. 75)]. Another important comment concerning V_ℓ is that (30), while true to leading order *semi-classically* as $\hbar=\ell^{-1}\to 0$, *provided* the energy $\lambda>\lambda_0>0$ (where the latter is fixed), does not necessarily hold as $\lambda\to 0$. The key property here is that $\mathscr{H}_\ell$ does not have *a zero energy resonance*, which means that there is no globally subordinate (or recessive) solution. This refers to solutions of the slowest allowed growth at both ends. For example, consider the operator $H=-\frac{d^2}{dx^2}+V$, where $V=\bar{V}$ satisfies $(1+|x|)V(x)\in L^1(\mathbb{R})$. Then, by the usual Jost/Volterra perturbation analysis, there is a fundamental system of solutions to $Hf=0$ consisting of $f_1(x)\sim x$, respectively, $f_2(x)\sim 1$ as $x\to\infty$. Thus, f_2 is the unique (up to nonzero factors) subordinate solution at $+\infty$. A resonance at zero energy therefore occurs if $Hf=0$ admits a solution $f\neq 0$, which is asymptotic to a constant for both $x\to\pm\infty$. Since the only other option would be some linear growth at either end, this is equivalent to $f\in L^\infty(\mathbb{R})$. This is a *universal* characterization of 0 energy resonances through solutions of $Hf=0$ even if V violates $(1+|x|)V(x)\in L^1(\mathbb{R})$ as in Bessel-type potentials arising in most problems discussed in this note, or for that matter, for V that are strongly singular. The latter means that V is locally bounded (for simplicity) but $\int_{-\infty}^{\infty}|V(x)|\,dx=\infty$, and we assume that H is limit point at both ends. Depending on the specific choice of V, one needs to find a fundamental system of $Hf=0$ at both $\pm\infty$ and then select the subordinate solution. A resonance is characterized by a solution, which is globally subordinate.

While our discussion has been largely heuristic, we emphasize that (29) is proved in Ref. 75 by means of an asymptotic description of the Jost solutions as $\lambda\to 0$. Moreover, it is shown there that the constant c in (29) vanishes in the case of a zero energy resonance, which indicates that the WKB approximation fails in that case as $\lambda\to 0$. Finally, we emphasize that the only natural small parameter in Ref. 75 for *fixed* $\ell\geq 1$ is the energy λ. This is in contrast to the summation problem in ℓ, where $\hbar:=\ell^{-1}$ represents another (and most important) small parameter. In fact, for large ℓ, the errors in the WKB approximations are controlled in terms of this small parameter rather than in terms of the small energy (we will return to this matter below). In order to be able to distinguish the two potentials in Fig. 5 or manifolds with distinct local geometries in Fig. 2, we therefore need to obtain precise asymptotics for the Jost solutions and the spectral measure for *both* small energies $0<\lambda<\lambda_0$ and all large $|\ell|$, *simultaneously*. This sets these problems apart from most of the semi-classical literature in several ways: (i) it is not enough to compute the limit $\hbar\to 0$. In fact, we need a precise asymptotic representation of the Jost solutions uniformly in small $\hbar$ and all energies. This will be explained in more detail in Sec. III B. (ii) The need for uniform control for all small energies is also in stark contrast to the literature, which typically restricts any semi-classical analysis to positive or large energies.

The rigorous proof of (29) proceeds by means of a classical matching method. To be more specific, consider the Schrödinger operator on the line (for notational convenience, we write x instead of ξ)

$$\mathscr{H}_\nu=-\partial_x^2+\left(\nu^2-\frac{1}{4}\right)\langle x\rangle^{-2}-U_\nu(x),$$
$$\frac{d^kU_\nu(x)}{dx^k}=O(x^{-3-k})$$

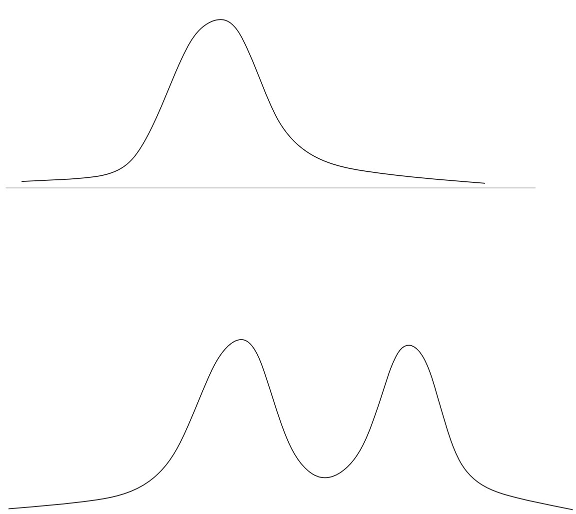

FIG. 5. Potentials corresponding to the surfaces in Fig. 2.

for all $k \geq 0$ as $x \to \pm\infty$ and with $\nu > 0$ fixed. To describe the Jost solution $f_{+,\nu}(x)$ on the interval $x \geq 1$, we start from the zero energy solutions

$$u_{0,\nu}^{+}(x) = x^{\frac{1}{2}+\nu}(1 + O(x^{-\alpha})),$$
$$u_{1,\nu}^{+}(x) = x^{\frac{1}{2}-\nu}(1 + O(x^{-1})) \text{ as } x \to \infty,$$

which form a fundamental system of $\mathscr{H}_\nu f = 0$ [and with $\alpha := \min(1, 2\nu)$]. Next, one perturbs these solutions with respect to the energy λ. More specifically, one shows via Volterra iteration that there is a basis $\{u_{0,\nu}^{+}(x,\lambda), u_{1,\nu}^{+}(x,\lambda)\}$ of solutions to the equation $\mathscr{H}_\nu f = \lambda^2 f$, which satisfy (at least for $\nu > 1$)

$$u_{j,\nu}^{+}(x,\lambda) = u_{j,\nu}^{+}(x)(1 + O(\lambda^2 x^2)) \tag{31}$$

on the interval $1 \leq x \ll \lambda^{-1}$ (we are only considering small λ for now). Clearly, one has

$$f_{+,\nu}(x,\lambda) = a_{+,\nu}(\lambda)u_{0,\nu}^{+}(x,\lambda) + b_{+,\nu}(\lambda)u_{1,\nu}^{+}(x,\lambda),$$

where the coefficients are given by

$$\begin{aligned} a_{\pm,\nu}(\lambda) &= -W(f_{\pm,\nu}(\cdot,\lambda), u_{1,\nu}^{\pm}(\cdot,\lambda)), \\ b_{\pm,\nu}(\lambda) &= W(f_{\pm,\nu}(\cdot,\lambda), u_{0,\nu}^{\pm}(\cdot,\lambda)). \end{aligned} \tag{32}$$

The aforementioned *matching* means nothing else than computing these Wronskians. The point where they are computed is chosen to be $\lambda^{-1+\varepsilon}$ with $\varepsilon > 0$ small and fixed. On one hand, this choice guarantees that the errors in (31) are $O(\lambda^{2\varepsilon})$, which is admissible. On the other hand, it requires that we obtain a sufficiently accurate description of the Jost solutions on $[\lambda^{-1+\varepsilon}, \infty)$. The latter is accomplished by comparing the outgoing Jost solution of the operator $\mathscr{H}_\nu$ to that of $\mathscr{H}_{0,\nu}$ given by

$$\mathscr{H}_{0,\nu} := -\partial_x^2 + \left(\nu^2 - \frac{1}{4}\right)x^{-2}.$$

The outgoing Jost solution of this operator on $\xi \geq 1$ equals

$$\sqrt{\frac{\pi}{2}}\, e^{i(2\nu+1)\pi/4}\sqrt{\xi\lambda}\, H_\nu^{(+)}(\xi\lambda),$$

which is asymptotic to $e^{i\xi\lambda}$ as $\xi \to \infty$. Here, $H_\nu^{(+)}(z) = J_\nu(z) + iY_\nu(z)$ is the usual Hankel function. Carrying out the perturbative analysis with $\mathscr{H}_{0,\nu}$ as giving the leading order allows one to approximate $f_+(\xi,\lambda)$ with small errors on the interval $(\lambda^{-1+\varepsilon}, \infty)$. With this asymptotic representation in hand, one now has the following result (see Proposition 3.12 in Ref. 75):

Proposition III.3. Let $\beta_\nu := \sqrt{\frac{\pi}{2}}\, e^{i(2\nu+1)\pi/4}$. With nonzero real constants $\alpha_{0,\nu}^{+}, \beta_{0,\nu}^{+}$ and some sufficiently small $\varepsilon > 0$,

$$\begin{aligned} a_{+,\nu}(\lambda) &= \lambda^{\frac{1}{2}+\nu}\beta_\nu(\alpha_{0,\nu}^{+} + O(\lambda^\varepsilon) + iO(\lambda^{(1-2\nu)\varepsilon})), \\ b_{+,\nu}(\lambda) &= i\lambda^{\frac{1}{2}-\nu}\beta_\nu(\beta_{0,\nu}^{+} + O(\lambda^\varepsilon) + iO(\lambda^{(1+2\nu)\varepsilon})) \end{aligned} \tag{33}$$

as $\lambda \to 0+$ with real-valued $O(\cdot)$ that behave like symbols under differentiation in λ. The asymptotics as $\lambda \to 0-$ follows from that as $\lambda \to 0+$ via the relations $a_{+,\nu}(-\lambda) = \overline{a_{+,\nu}(\lambda)}$, $b_{+,\nu}(-\lambda) = \overline{b_{+,\nu}(\lambda)}$.

Analogous expressions hold for $a_{-,\nu}$ and $b_{-,\nu}$, which, of course, refers to the solutions on $x \leq -1$. From these expansions, one then concludes the following statement for the Wronskian between $f_+(\cdot,\lambda)$ and $f_-(\cdot,\lambda)$:

$$W_\nu(\lambda) = ie^{i\nu\pi}\lambda^{1-2\nu}(W_{0,\nu} + O_{\mathbb{C}}(\lambda^\varepsilon)) \text{ as } \lambda \to 0+.$$

Here, $W_{0,\nu}$ is a real constant and $O_{\mathbb{C}}(\lambda^\varepsilon)$ is complex valued and of symbol type (meaning that each derivative loses one power). Most importantly, $W_{0,\nu} = 0$ if and only if zero is a resonance of $\mathscr{H}_\nu$. For the case of surfaces of revolutions, it is easy to exclude zero energy resonances of the associated Schrödinger operator, at least for $\ell \geq 1$. In fact, with $\mathscr{H}_\ell$ denoting the operator obtained for fixed angular momentum $\ell \geq 1$,

$$\mathscr{H}_\ell\left(r^{\frac{1}{2}}e^{\pm\ell y}\right) = 0, \quad y(\xi) = \int_0^\xi \frac{d\eta}{r(\eta)}.$$

Because y is odd, the smaller branch at $\xi = \infty$ has to be the larger one at $\xi = -\infty$, which places us in the nonresonant case. It is perhaps worth mentioning that the potentials arising from surfaces of revolution do not need to be non-negative (for positive potentials, it is evident that zero is not a resonance). In fact, if $\mathscr{M}$ has very large curvature, then the potential can be negative. We remark that for $\ell = 0$, it is proved in Ref. 74 that

$$W_0(\lambda) = 2\lambda\left(1 + ic_3 + i\frac{2}{\pi}\log\lambda\right) + O\left(\lambda^{\frac{3}{2}-\varepsilon}\right) \text{ as } \lambda \to 0+.$$

On a technical level, the logarithmic term in λ makes the $\ell = 0$ case somewhat harder to analyze than the cases $\ell \geq 1$. Not surprisingly, in proving dispersive estimates for $-\Delta_{\mathbb{R}^2} + V$, one encounters similar logarithmic issues (see Ref. 73).

In conclusion, we would like to stress that the estimates in Ref. 75 produce constants that grow very rapidly in ℓ, somewhat faster than e^{ℓ^2}, to be precise. This is due to a number of sources. First, for the small energy analysis we just described to work, one needs to choose the energy cutoff $\lambda_0 = \lambda_0(\ell)$ to depend on ℓ, which already introduces large constants into the proof. Second, for energies $\lambda > \lambda_0(\ell) > 0$, one uses a very crude method, namely, term-wise estimation of a Born series that cannot distinguish the sign of the potential. Even replacing the crude Born series by something more elaborate would not make much of a difference. Indeed, by the preceding discussion, the two manifolds in Fig. 2 behave very differently as far as the dependence of the constant on ℓ is concerned.

Since the *small energy matching* method outlined above cannot easily distinguish between these manifolds, we shall now discuss an approach that is capable of differentiating between them, albeit only for large ℓ. For this reason, the finite ℓ analysis of Refs. 74 and 75 is needed in the Proof of Theorem III.2.

B. Summation over all angular momenta

We shall now prove Theorem III.2. We will follow Ref. 30 and sketch how to obtain (23) and (24), with the case of the wave equation being similar. With V_ℓ as in (27), we claim the following bound:

$$\int_{-\infty}^{\infty} \langle\xi\rangle^{-2} |e^{it\mathscr{H}_\ell} u(\xi)|^2\, d\xi \lesssim \langle t\rangle^{-2}\ell^4 \int_{-\infty}^{\infty} \langle\xi\rangle^2 |u(\xi)|^2\, d\xi. \tag{34}$$

The proof of (34) will be discussed below. Taking it for granted, suppose that f is a Schwartz function on $\mathscr{M}$ and write

$$f(\xi,\theta) = \sum_{\ell=-\infty}^{\infty} e^{i\ell\theta} f_\ell(\xi) = \sum_{\ell=-\infty}^{\infty} e^{i\ell\theta} r^{-\frac{1}{2}}(\xi) u_\ell(\xi).$$

Then,

$$\begin{aligned} e^{it\Delta_{\mathscr{M}}} f &= \sum_{\ell=-\infty}^{\infty} e^{it\Delta_{\mathscr{M}}}\left[e^{i\ell\theta} r^{-\frac{1}{2}}(\xi) u_\ell(\xi)\right] \\ &= \sum_{\ell=-\infty}^{\infty} e^{i\ell\theta} r^{-\frac{1}{2}}(\xi)\left[e^{it\mathscr{H}_\ell} u_\ell\right](\xi), \end{aligned}$$

whence

$$\begin{aligned} &\|w_1 e^{it\Delta_{\mathscr{M}}} f\|_{L^2(\mathscr{M})}^2 \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^{2\pi} w_1^2(\xi)\left|\sum_{\ell=-\infty}^{\infty} e^{i\ell\theta} r^{-\frac{1}{2}}(\xi)\left[e^{it\mathscr{H}_\ell} u_\ell\right](\xi)\right|^2 r(\xi)\, d\xi d\theta \\ &\lesssim \sum_{\ell=-\infty}^{\infty} \int_{-\infty}^{\infty} \langle\xi\rangle^{-2} |e^{it\mathscr{H}_\ell} u_\ell(\xi)|^2\, d\xi \\ &\lesssim \sum_{\ell=-\infty}^{\infty} \langle t\rangle^{-2}\langle\ell\rangle^4 \int_{-\infty}^{\infty} \langle\xi\rangle^2 |f_\ell(\xi)|^2 r(\xi)\, d\xi \\ &\lesssim \langle t\rangle^{-2} \sum_{\ell=-\infty}^{\infty} \int_{-\infty}^{\infty} \langle\ell\rangle^4 w_{-1}(\xi)^2 \left|\int_0^{2\pi} f(\xi,\theta) e^{-i\ell\theta}\, d\theta\right|^2 r(\xi)\, d\xi \\ &\lesssim \langle t\rangle^{-2} \|w_{-1}(1-\partial_\theta^2) f\|_{L^2(\mathscr{M})}^2, \end{aligned}$$

which is (23). To prove (34), it is clear from Theorem III.1 that it suffices to consider ℓ large, say, $|\ell| \geq \ell_0 \gg 1$. Fixing such an ℓ, one switches to a semi-classical representation via the identity

$$e^{it\mathscr{H}_\ell} = e^{i\frac{t}{\hbar^2}\mathscr{H}(\hbar)}, \qquad \mathscr{H}(\hbar) := -\hbar^2\partial_\xi^2 + \hbar^2 V_\ell,$$

where V_ℓ is as in (27) and with $\hbar := \ell^{-1}$. By construction, $V(\xi,\hbar) := \hbar^2 V_\ell(\xi)$ has the property that its maximal height is now essentially fixed at $V_{\max}(\hbar) = V_{\max}(0) + O(\hbar^2)$ with $V_{\max}(0) \simeq 1$. The essential property of the potential is that it has a unique nondegenerate maximum, i.e., it looks like the one on top in Fig. 4.

For the remainder of this section, $\hbar$ will be small. From the spectral representation, one has

$$e^{i\frac{t}{\hbar^2}\mathscr{H}(\hbar)} = \frac{2}{\pi}\hbar^{-2}\int_0^\infty e^{i\frac{t}{\hbar^2}E^2}\operatorname{Im}\left[\frac{f_+(x,E;\hbar)f_-(x',E;\hbar)}{W(f_+(\cdot,E;\hbar),f_-(\cdot,E;\hbar))}\right]E\,dE, \tag{35}$$

with $f_\pm$ being the outgoing Jost solutions for the semi-classical operator $\mathscr{H}(\hbar)$, which means that

$$(-\hbar^2\partial_x^2 + V(x;\hbar))f_\pm(x,E;\hbar) = E^2 f_\pm(x,E;\hbar)$$
$$f_\pm(x,E;\hbar) \sim e^{\pm\frac{i}{\hbar}Ex} \qquad x \to \pm\infty.$$

With $\varepsilon > 0$ fixed and small (independently of $\hbar$), one now considers energies $0 < E < \varepsilon$ (low), $\varepsilon < E < 100$ (intermediate), and $E > 100$ (large) separately. The middle interval is further split into energies $\varepsilon < E < V_{\max}(0) - \varepsilon$, $V_{\max}(0) - \varepsilon < E < 100$, respectively. The latter interval is to some extent the most important of all as it contains the nondegenerate maximum of the potential $V(\hbar)$. We shall see that it is precisely this maximum that determines the number of derivatives lost in the process of summing over ℓ.

The easiest region is $E > 100$. Indeed, for these energies, the potential is essentially negligible and a classical WKB approximation reduces matters to the free case. This means (again heuristically) that (23) is a consequence of the $L^1 \to L^\infty(\mathbb{R}^2)$ bound on $e^{it\Delta_{\mathbb{R}^2}}$, which explains the weights $w_{1+\varepsilon}$.

1. WKB in the doubly asymptotic limit $\hbar \to 0$ and $E \to 0$

The low-lying energies $0 < E < \varepsilon$ are also treated by means of WKB, but there one faces the difficulty that the WKB approximation of the generalized eigenfunctions needs to be accurate in the entire range $0 < E < \varepsilon$ and $0 < \hbar < \hbar_0$. There exists an extensive literature on the validity of the WKB approximation, provided the energy stays away from zero, i.e., $E > E_0 > 0$ uniformly in $\hbar$ (see, for example, Refs. 67 or 69. However, the issue of controlling all errors in the WKB method uniformly in small $\hbar$ *and small* E does not seem to have been considered before. For the problem of sending $E \to 0$, it is, of course, most relevant that the potential has the (critical) inverse square decay, as was already apparent in the discussion of the matching method in Sec. III A.

This lead Costin, Schlag, Staubach, and Tanveer[16] to carry out a systematic analysis of this two-parameter WKB problem for inverse square potentials. More specifically, they considered the scattering matrix

$$\Sigma(E;\hbar) = \begin{bmatrix} t(E;\hbar) & r_-(E;\hbar) \\ r_+(E;\hbar) & t(E;\hbar)\end{bmatrix} = \begin{bmatrix}\Sigma_{11}(E;\hbar) & \Sigma_{12}(E;\hbar) \\ \Sigma_{21}(E;\hbar) & \Sigma_{22}(E;\hbar)\end{bmatrix}$$

for the semiclassical operator

$$P(x,\hbar D) := -\hbar^2\frac{d^2}{dx^2} + V(x)$$

with inverse square V (asymptotically, as $|x| \to \infty$) and obtained the following result:

Theorem III.4. *Let $V \in C^\infty(\mathbb{R})$ with $V > 0$ and $V(x) = \mu_\pm^2 x^{-2} + O(x^{-3})$ as $x \to \pm\infty$, where $\mu_+ \neq 0$, $\mu_- \neq 0$, and $\partial_x^k O(x^{-3}) = O(x^{-3-k})$ for all $k \geq 0$. Denote*

$$V_0(x;\hbar) := V(x) + \frac{\hbar^2}{4}\langle x\rangle^{-2}, \tag{36}$$

and let $E_0 > 0$ be such that for all $0 < E < E_0$ and $0 < \hbar < 1$, $V_0(x;\hbar) = E$ has a unique pair of solutions, which we denote by $x_2(E;\hbar) < 0 < x_1(E;\hbar)$. Define

$$\begin{aligned} S(E;\hbar) &:= \int_{x_2(E;\hbar)}^{x_1(E;\hbar)}\sqrt{V_0(y;\hbar) - E}\,dy, \\ T_+(E;\hbar) &:= x_1(E;\hbar)\sqrt{E} - \int_{x_1(E;\hbar)}^\infty\left(\sqrt{E - V_0(y;\hbar)} - \sqrt{E}\right)dy, \\ T_-(E;\hbar) &:= -x_2(E;\hbar)\sqrt{E} - \int_{-\infty}^{x_2(E;\hbar)}\left(\sqrt{E - V_0(y;\hbar)} - \sqrt{E}\right)dy, \end{aligned} \tag{37}$$

as well as $T(E;\hbar) := T_+(E;\hbar) + T_-(E;\hbar)$. Then, for all $0 < \hbar < \hbar_0$, where $\hbar_0 = \hbar_0(V) > 0$ is small and $0 < E < E_0$,

$$\begin{aligned}\Sigma_{11}(E;\hbar) &= e^{-\frac{1}{\hbar}(S(E;\hbar)+iT(E;\hbar))}(1 + \hbar\,\sigma_{11}(E;\hbar)), \\ \Sigma_{12}(E;\hbar) &= -ie^{-\frac{2i}{\hbar}T_+(E;\hbar)}(1 + \hbar\,\sigma_{12}(E;\hbar))\end{aligned} \tag{38}$$

where the correction terms satisfy the bounds

$$|\partial_E^k \sigma_{11}(E;\hbar)| + |\partial_E^k \sigma_{12}(E;\hbar)| \le C_k E^{-k} \quad \forall\ k \ge 0, \tag{39}$$

with a constant C_k that only depends on k and V. The same conclusion holds if instead of (36) we were to define V_0 as $V_0 := V + \hbar^2 V_1$ with $V_1 \in C^\infty(\mathbb{R})$, $V_1(x;\hbar) = \frac{1}{4}\langle x\rangle^{-2} + O(x^{-3})$ as $x \to \pm\infty$ with $\partial_x^k O(x^{-3}) = O(x^{-3-k})$ for all $k \ge 0$ and uniformly in $0 < \hbar \ll 1$.

Note the correction of the original potential by $\frac{\hbar^2}{4}\langle x\rangle^{-2}$ in (36). Without this correction, the errors σ_{11}, etc., diverge as $E \to 0$. The proof of this result, of course, requires a careful analysis of the Jost solutions, which is then needed in the analysis of the stationary phase analysis of (35).

The analysis of the Jost solutions is based on the Liouville–Green transform, which we now recall (see Ref. 67). Given any second order equation $f''(x) = Q(x)f(x)$ on some interval I and any diffeomorphism $w : I \to J$ onto some interval J, define $g(w) := (w'(x))^{\frac{1}{2}} f(x)$, where $w = w(x)$. Then, by the chain rule, $f'' = Qf$ is the same as $g''(w) = \tilde{Q}(w)g(w)$, where

$$\begin{aligned}\tilde{Q}(w) &:= \frac{Q(x)}{(w'(x))^2} - (w'(x))^{-\frac{3}{2}}\partial_x^2 (w'(x))^{-\frac{1}{2}} \\ &= \frac{Q(x)}{(w'(x))^2} - \frac{3}{4}\frac{(w''(x))^2}{(w'(x))^4} + \frac{1}{2}\frac{w'''(x)}{(w'(x))^2}.\end{aligned}$$

To apply this transformation, one chooses w so that

$$\frac{Q(x)}{(w'(x))^2} = Q_0(w), \tag{40}$$

where Q_0 is some *normal form*. Then, the problem becomes

$$\begin{aligned} g''(w) &= Q_0(w)g(w) - V(w)g(w), \\ V(w) &:= \frac{3}{4}\frac{(w''(x))^2}{(w'(x))^4} - \frac{1}{2}\frac{w'''(x)}{(w'(x))^2},\end{aligned} \tag{41}$$

where V is treated as a perturbation. This is only admissible if Q_0 is in some suitable sense close to Q. The determination of Q_0 is done on a case by case basis. For example, if Q does not vanish on I, then one can take $Q_0 = \operatorname{sign}(Q)$, which leads to the classical WKB ansatz, i.e.,

$$Q^{-\frac{1}{4}}(x)e^{\pm\int_{x_0}^x \sqrt{Q}(y)\,dy} \text{ or } |Q|^{-\frac{1}{4}}(x)e^{\pm i\int_{x_0}^x \sqrt{|Q|}(y)\,dy},$$

depending on whether $Q > 0$ or $Q < 0$, respectively. If Q does vanish at $x_0 \in I$ with $Q'(x_0) \ne 0$, then one maps x_0 to $w = 0$ and chooses $Q_0(w) = w$. In other words, the comparison equation is the Airy equation. The equation for w in that case is $w(x)w'(x)^2 = Q(x)$, which yields

$$w(x) = \operatorname{sign}(x - x_0)\Big|\frac{3}{2}\int_{x_0}^x \sqrt{|Q(y)|}\,dy\Big|^{\frac{2}{3}}, \tag{42}$$

which is known as the Langer transform.[67] It is easy to check that w is (locally around x_0) smooth (or analytic), provided Q is smooth (or analytic). It is precisely this Langer transform that is used in Ref. 16, where it is written as follows for $x \ge 0$:

$$\begin{aligned}\zeta &= \zeta(x,E;\hbar) \\ &:= \operatorname{sign}(x - x_1(E;\hbar))\Big|\frac{3}{2}\int_{x_1(E;\hbar)}^x \sqrt{|V_0(x;\hbar) - E|}\,d\eta\Big|^{\frac{2}{3}},\end{aligned}$$

with $x_1(E;\hbar) > 0$ being the unique turning point (for E small). The equation transforms as follows:

Lemma III.5. There exists $E_0 = E_0(V) > 0$ so that for all $0 < E < E_0$, one has the following properties: the equation $V_0(x;\hbar) - E = 0$ has a unique (simple) solution on $x > 0$, which we denote by $x_1 = x_1(E;\hbar)$. With $Q_0 := V_0 - E$,

$$\begin{aligned}\zeta &= \zeta(x,E;\hbar) \\ &:= \operatorname{sign}(x - x_1(E;\hbar))\Big|\frac{3}{2}\int_{x_1(E;\hbar)}^x \sqrt{|Q_0(u,E;\hbar)|}\,du\Big|^{\frac{2}{3}}\end{aligned} \tag{43}$$

defines a smooth change of variables $x \mapsto \zeta$ for all $x \ge 0$. Let $q := -\frac{Q_0}{\zeta}$. Then, $q > 0$, $\frac{d\zeta}{dx} = \zeta' = \sqrt{q}$, and

$$-\hbar^2 f'' + (V - E)f = 0$$

transforms into

$$-\hbar^2 \ddot{w}(\zeta) = (\zeta + \hbar^2 \tilde{V}(\zeta, E; \hbar))w(\zeta) \tag{44}$$

under $w = \sqrt{\zeta'} f = q^{\frac{1}{4}} f$. *Here,* $\dot{} = \frac{d}{d\zeta}$ *and*

$$\tilde{V} := \frac{1}{4} q^{-1} \langle x \rangle^{-2} - q^{-\frac{1}{4}} \frac{d^2 q^{\frac{1}{4}}}{d\zeta^2}.$$

The asymptotic description of the Jost solutions is found by matching the Airy approximations at the turning point $w = 0$. A fundamental solution of the transformed equation (i.e., in the ζ variable) to the left of the turning point is described in terms of the Airy function Ai, Bi by the following result from Ref. 16:

Proposition III.6. Let $\hbar_0 > 0$ *be small. A fundamental system of solutions to (44) in the range* $\zeta \le 0$ *is given by*

$$\phi_1(\zeta, E, \hbar) = \mathrm{Ai}(\tau)[1 + \hbar a_1(\zeta, E, \hbar)],$$
$$\phi_2(\zeta, E, \hbar) = \mathrm{Bi}(\tau)[1 + \hbar a_2(\zeta, E, \hbar)]$$

with $\tau := -\hbar^{-\frac{2}{3}}\zeta$. *Here,* a_1, a_2 *are smooth, real-valued, and satisfy the bounds for all* $k \ge 0$ *and* $j = 1, 2$ *and with* $\zeta_0 := \zeta(0, E)$,

$$\begin{aligned} |\partial_E^k a_j(\zeta, E, \hbar)| &\lesssim E^{-k} \min\Big[\hbar^{\frac{1}{3}} \langle \hbar^{-\frac{2}{3}} \zeta \rangle^{\frac{1}{2}}, 1\Big] \\ |\partial_E^k \partial_\zeta a_j(\zeta, E, \hbar)| &\lesssim E^{-k} \Big[\hbar^{-\frac{1}{3}} \langle \hbar^{-\frac{2}{3}} \zeta \rangle^{-\frac{1}{2}} \chi_{[-1 \le \zeta \le 0]} + |\zeta|^{\frac{1}{2}} \chi_{[\zeta_0 \le \zeta \le -1]}\Big] \end{aligned} \tag{45}$$

uniformly in the parameters $0 < \hbar < \hbar_0$, $0 < E < E_0$.

Note that from the standard asymptotic behavior of the Airy functions, viz.,

$$\begin{aligned} &\mathrm{Bi}(x) = \pi^{-\frac{1}{2}} x^{-\frac{1}{4}} e^{\frac{2}{3}x^{\frac{3}{2}}} \Big[1 + O\Big(x^{-\frac{3}{2}}\Big)\Big] \text{ as } x \to \infty, \\ &\mathrm{Bi}(x) \ge \mathrm{Bi}(0) > 0 \quad \forall\, x \ge 0, \\ &\mathrm{Ai}(x) = \frac{1}{2} \pi^{-\frac{1}{2}} x^{-\frac{1}{4}} e^{-\frac{2}{3}x^{\frac{3}{2}}} \Big[1 + O\Big(x^{-\frac{3}{2}}\Big)\Big] \text{ as } x \to \infty, \\ &Ai(x) > 0 \quad \forall\, x \ge 0, \end{aligned}$$

the action integral appears naturally in this context [cf. (43)]. To the right of the turning point, one has the following oscillatory basis.

Proposition III.7. Let $\hbar_0 > 0$ *be small. In the range* $\zeta \ge 0$, *a basis of solutions to (44) is given by*

$$\psi_1(\zeta, E; \hbar) = (\mathrm{Ai}(\tau) + i\mathrm{Bi}(\tau))[1 + \hbar b_1(\zeta, E; \hbar)],$$
$$\psi_2(\zeta, E; \hbar) = (\mathrm{Ai}(\tau) - i\mathrm{Bi}(\tau))[1 + \hbar b_2(\zeta, E; \hbar)]$$

with $\tau := -\hbar^{-\frac{2}{3}}\zeta$ *and where* b_1, b_2 *are smooth, complex-valued, and satisfy the bounds for all* $k \ge 0$ *and* $j = 1, 2$,

$$\begin{aligned} |\partial_E^k b_j(\zeta, E; \hbar)| &\le C_k E^{-k} \langle \zeta \rangle^{-\frac{3}{2}}, \\ |\partial_\zeta \partial_E^k b_j(\zeta, E)| &\le C_k E^{-k} \hbar^{-\frac{1}{3}} \langle \hbar^{-\frac{2}{3}} \zeta \rangle^{-\frac{1}{2}} \langle \zeta \rangle^{-2} \end{aligned} \tag{46}$$

uniformly in the parameters $0 < \hbar < \hbar_0$, $0 < E < E_0$, $\zeta \ge 0$.

We remark that the Langer transform is not the only possibility here. In fact, in Ref. 17, an alternative approach is used, which reduces the potential to a Bessel normal form. This is again done by means of a suitable stretching, i.e., a Liouville–Green transform.

2. Intermediate energies and the top of the barrier

Intermediate energies, including the maximum energy of the potential, can be treated by means of an approximation of the generalized eigenfunctions. This was carried out in detail by Costin, Park, and the author by means of a Liouville–Green transformation that reduces the potential near the maximum to a purely quadratic normal form (see Ref. 18, Proposition 2). In this way, one arrives at a perturbed Weber equation instead of the Airy equation as above.

However, Ref. 30 follows a different route: a Mourre estimate followed by a semi-classical version of the propagation bounds in Ref. 46. Mourre[63] introduced the powerful idea that the quantum analog, i.e.,

$$\chi_I(H) i[H, A] \chi_I(H) \ge \theta \chi_I(H) > 0,$$

where $H = -\Delta + V$, $A = px + xp$, $p = -i\nabla$, and $\chi_I(H)$ localizes H to some compact interval I of positive energies, entails a limiting absorption bound on the resolvent localized to I (which is some form of scattering). Hunziker, Sigal, and Soffer[46] developed a time-dependent and abstract approach to Mourre theory by means of propagation estimates in the spirit of Sigal and Soffer.[76] The main result of Ref. 46 is the following theorem:

Theorem III.8. *Let A, H be self adjoint operators on some Hilbert space, and assume the Mourre estimate*

$$E_I i[H,A]E_I \geq \theta E_I, \tag{47}$$

where $\theta > 0$, $I \subset \mathbb{R}$ is some compact interval, and E_I is the spectral projector onto I relative to H. Assume, furthermore, that all iterated commutators of $f(H)$ with A are bounded where $f \in C_0^\infty(\mathbb{R})$. Let $\chi^\pm$ be the indicator functions of $\mathbb{R}^\pm$, respectively. Then, for any $m \geq 1$,

$$\|\chi^-(A - a - \theta' t)e^{iHt}g(H)\chi^+(A-a)\| \leq C(m,\theta,\theta')\, t^{-m}$$

for any $g \in C_0^\infty(I)$ and any $0 < \theta' < \theta$ uniformly in $a \in \mathbb{R}$.

As simple consequence of this result is the following propagation estimate, which is clearly most important in the context of Theorem III.2:

$$\|\langle A\rangle^{-\alpha}e^{iHt}g(H)\langle A\rangle^{-\alpha}\| \leq C(\alpha)\,\langle t\rangle^{-\alpha} \tag{48}$$

for any $\alpha > 0$. In application, one typically takes $A = \frac{1}{2}(px + xp)$, the generator of dilations, or some variant thereof. Taking $\alpha = 1$ shows that one needs at least w_1 in the Schrödinger case of Theorem III.2.

One needs to resolve the following two issues before applying this theory to Theorem III.2:

- We require a semi-classical version of Ref. 46.
- The top of the barrier energy is trapping in the classical sense.

While the first issue is a routine variant of Ref. 46, the second is not. In the nontrapping case, Graf[43] and Hislop and Nakamura[45] showed that the classical nontrapping condition $\{a,h\} > \alpha > 0$ on the entire energy level $\{h = E_0 > 0\}$ implies the Mourre estimate (47) for I some small interval around E_0 (in the semi-classical case with $\hbar$ sufficiently small). In the case of surfaces of revolution as in Theorem III.2, this fact, together with Theorem III.8, implies that one can handle energies in the range $\varepsilon < E < V_{\max}(0) - \varepsilon$ since they verify a classical nontrapping condition. On the other hand, for energies near $V_{\max}(0)$, this fails since the top energy is classically trapping. Nevertheless, the Heisenberg uncertainty principle (or the semiclassical harmonic oscillator) guarantees (47).

Indeed, with $V(x) = 1 - \frac{1}{2}\langle Qx, x\rangle + O(|x|^3)$ with Q positive definite,

$$\{h,a\} = \xi^2 - x\cdot\nabla V = \xi^2 + \langle Qx,x\rangle + O(|x|^3) \geq \theta(\xi^2 + x^2)$$

for small x. However, $p^2 + q^2 \geq c > 0$ by the uncertainty principle, which indicates that one should expect that (47) continues to hold at a non-degenerate maximum. For a rigorous rendition of this argument, see Refs. 11, 30, and 66.

Generally speaking, the problem of obtaining a representation of the resolvent and the spectral measure and of proving a limiting absorption principle for energies near a potential barrier has received much attention (see the monograph by Bony *et al.*[9] and the earlier literature cited there, such as the classical work by Helffer and Sjöstrand in the 1980s).

This concludes our informal sketch of the proof of (23). As for (24), one proceeds analogously by dividing energies into three regions, low, intermediate, and high. In the low and high cases, one obtains pointwise bounds without weights from the WKB arguments outlined above, followed by oscillatory integral estimates as in Ref. 75. For the intermediate regime, one uses the L^2 bound (from the Mourre–Hunziker–Sigal–Soffer estimates), which requires a weight w_1 followed by the Sobolev embedding theorem. Note that the latter costs one power of ℓ, whereas summation over ℓ requires another weight of the form $\ell^{1+\varepsilon}$, which explains the loss of $(1-\partial_\theta^2)^{1+\varepsilon}$ on the right-hand side of (24) as compared to (23).

As a final remark, we would like to emphasize that the sketch of Proof of Theorem III.2, which we just concluded, is an adaptation of the argument, which was developed for the Schwarzschild case in Ref. 30.

C. The Schwarzschild case

The results on surfaces of revolution are relevant to another problem, namely, the decay of linear waves on a Schwarzschild black hole background. To be more specific, choose coordinates such that the exterior region of the black hole can be written as $(t, r, (\theta,\phi)) \in \mathbb{R} \times (2M, \infty) \times S^2$ with the metric

$$g = -F(r)dt^2 + F(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

where $F(r) = 1 - \frac{2M}{r}$ and, as usual, $M > 0$ denotes the mass. We now introduce the well-known *Regge–Wheeler tortoise coordinate* r_*, which (up to an additive constant) is defined by the relation

$$F = \frac{dr}{dr_*}.$$

In this new coordinate system, the outer region is described by $(t, r_*, (\theta, \phi)) \in \mathbb{R} \times \mathbb{R} \times S^2$,

$$g = -F(r)dt^2 + F(r)dr_*^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{49}$$

with F as above and r is now interpreted as a function of r_*. Explicitly, r_* is computed as

$$r_* = r + 2M\log\left(\frac{r}{2M} - 1\right).$$

Generally, the Laplace–Beltrami operator on a manifold with metric g is given by

$$\Box_g = \frac{1}{\sqrt{|\det(g_{\mu\nu})|}}\partial_\mu\left(\sqrt{|\det(g_{\mu\nu})|}g^{\mu\nu}\partial_\nu\right),$$

and thus, for the metric g in (49), we obtain

$$\Box_g = F^{-1}\left(-\partial_t^2 + \frac{1}{r^2}\partial_{r_*}\left(r^2\partial_{r_*}\right)\right) + \frac{1}{r^2}\Delta_{S^2}.$$

By setting $\psi(t, r_*, \theta, \phi) = r(r_*)\tilde{\psi}(t, r_*, \theta, \phi)$ and writing $x = r_*$, the wave equation $\Box_g\tilde{\psi} = 0$ is equivalent to

$$-\partial_t^2\psi + \partial_x^2\psi - \frac{F}{r}\frac{dF}{dr}\psi + \frac{F}{r^2}\Delta_{S^2}\psi = 0. \tag{50}$$

The mathematically rigorous analysis of this equation goes back to Wald[79] and Kay,[53] who established uniform boundedness of solutions. In the spirit of the positive commutator methods outlined above, Dafermos and Rodniansk[21] found a robust approach based on carefully chosen vector fields and multipliers. See the work of Luk[60,61] that is in a similar spirit. As already noted, Blue and Soffer[8] proved local decay estimates using Morawetz estimates. Dafermos and Rodnianski[20] proved Price's t^{-3} decay law for a nonlinear problem but assuming spherical symmetry.

The purpose of this section is to discuss recent work of Donninger and the authors on pointwise decay for solutions to Eq. (50). Different types of decay estimates have been proved before. Our results differ from the above in certain respects: the methods we use are based on constructing the Green's function and deriving the needed estimates on it. Previous works in this direction include mainly the series of papers by Finster, Kamran, Smoller, and Yau (see, for example, Ref. 39, where the first pointwise decay result for Kerr black holes was proved).

As in the case for surfaces of revolution, we freeze the angular momentum ℓ or, in other words, we project onto a spherical harmonic. More precisely, let $Y_{\ell,m}$ be a spherical harmonic [that is, an eigenfunction of the Laplacian on S^2 with eigenvalue $-\ell(\ell+1)$], and insert the ansatz $\psi(t, x, \theta, \phi) = \psi_{\ell,m}(t, x)Y_{\ell,m}(\theta, \phi)$ in Eq. (50). This yields the *Regge–Wheeler equation*

$$\partial_t^2\psi_{\ell,m} - \partial_x^2\psi_{\ell,m} + V_{\ell,\sigma}(x)\psi_{\ell,m} = 0$$

with $\sigma = 1$, where

$$V_{\ell,\sigma}(x) = \left(1 - \frac{2M}{r(x)}\right)\left(\frac{\ell(\ell+1)}{r^2(x)} + \frac{2M\sigma}{r^3(x)}\right)$$

is known as the *Regge–Wheeler potential*. The other physically relevant values of the parameter σ are $\sigma = -3, 0$. For more background, we refer the reader to the introduction of Refs. 20 or 29.

We immediately note some crucial features of $V_{\ell,\sigma}$: it decays exponentially as $x \to -\infty$, it decays according to an inverse square law as $x \to +\infty$, provided $\ell > 0$, and like an inverse cube if $\ell = 0$. Moreover, it has a unique nondegenerate maximum that is located at the *photon sphere*. It consists of closed light rays and replaces the unique periodic geodesic, which we encountered in Theorem III.2.

Hence, we expect that at least some of the machinery that we described above in the surface case applies here as well. However, the Regge–Wheeller potential is considerably more difficult to deal with.

The main result of Ref. 29 is the following pointwise decay, which captures the so-called Price law for fixed angular momentum. Strictly speaking, it is still off by one power of t from the sharpest form of Price's law, which is $t^{-2\ell-3}$, whereas the following result proves $t^{-2\ell-2}$ (we shall comment on that issue below). Note how the accelerated decay for higher values of ℓ mirrors what we saw for the surfaces of revolution in Theorem III.1. Hintz[44] recently closed the gap of the missing power of t and thus finished the proof of Price's law.

Theorem III.9. *Let $(\ell,\sigma)\notin\{(0,0),(0,-3),(1,-3)\}$, $\alpha\in\mathbb{N}$ and $1\le\alpha\le 2\ell+3$. Then, the solution operators for the Regge–Wheeler equation satisfy the estimates*

$$\|w_\alpha \cos\Big(t\sqrt{\mathscr{H}_{\ell,\sigma}}\Big)f\|_{L^\infty(\mathbb{R})}\le C_{\ell,\alpha}\langle t\rangle^{-\alpha}\left(\left\|\frac{f'}{w_\alpha}\right\|_{L^1(\mathbb{R})}+\left\|\frac{f}{w_\alpha}\right\|_{L^1(\mathbb{R})}\right)$$

and

$$\left\|w_\alpha\frac{\sin(t\sqrt{\mathscr{H}_{\ell,\sigma}})}{\sqrt{\mathscr{H}_{\ell,\sigma}}}f\right\|_{L^\infty(\mathbb{R})}\le C_{\ell,\alpha}\langle t\rangle^{-\alpha+1}\left\|\frac{f}{w_\alpha}\right\|_{L^1(\mathbb{R})}$$

for all $t\ge 0$, where $w_\alpha(x):=\langle x\rangle^{-\alpha}$.

The values of (σ,ℓ) that we exclude here are precisely those where the Regge–Wheeler potential gives rise to zero energy resonances. Physically speaking, they correspond to a gauge invariance, such as changing the mass, and are therefore irrelevant.

The Proof of Theorem III.9 is based on representing the solution as an oscillatory integral in the energy variable λ; schematically, one may write

$$\psi(t,x)=\int U(t,\lambda)\mathrm{Im}\big[G_{\ell,\sigma}(x,x',\lambda)\big]f(x')\,dx'd\lambda,$$

where $U(t,\lambda)$ is a combination of $\cos(t\lambda)$ and $\sin(t\lambda)$ terms and $G_{\ell,\sigma}(x,x',\lambda)$ is the kernel (Green's function) of the resolvent of the operator $\mathscr{H}_{\ell,\sigma}$. In analogy with Theorem III.1, $G_{\ell,\sigma}(x,x',\lambda)$ is constructed in terms of the Jost solutions, and we obtain these functions in various domains of the (x,λ) plane by perturbative arguments: for $|x\lambda|$ small, we perturb in λ around $\lambda=0$, whereas for $|x\lambda|$ large, we perturb off of Hankel functions. This is done in such a way that there remains a small window where the two different perturbative solutions can be glued together. One of the main technical difficulties of the proof lies with the fact that we need good estimates for arbitrary derivatives of the perturbative solutions. This is necessary in order to control the oscillatory integrals. The most important contributions come from $\lambda\sim 0$, and we therefore need to derive the exact asymptotics of the Green's function and its derivatives in the limit $\lambda\to 0$. For instance, we prove that

$$\mathrm{Im}\left[G_{\ell,\sigma}(0,0,\lambda)\right]=\lambda P_\ell(\lambda^2)+O(\lambda^{2\ell+1})$$

as $\lambda\to 0+$, where P_ℓ is a polynomial of degree $\ell-1$ (we set $P_0\equiv 0$) and the O-term satisfies $O^{(k)}(\lambda^{2\ell+1})=O(\lambda^{2\ell+1-k})$ for all $k\in\mathbb{N}_0$.

As already noted before, for $\ell=0$, the Regge–Wheeler potential decays like an inverse cube as $x\to\infty$. This case is covered by the following result of Donninger and the author:[28]

Theorem III.10. *Let $V\in C^{[\alpha]+1}(\mathbb{R})$ with $V(x)=|x|^{-\alpha}[c_\pm+O(|x|^{-\beta})]$ as $x\to\pm\infty$, where $2<\alpha\le 4$, $\beta=\frac{1}{2}(\alpha-2)^2$, $c_\pm\in\mathbb{R}$, and $|O^{(k)}(|x|^{-\beta})|\lesssim|x|^{-\beta-k}$ for $k=1,2,\ldots,[\alpha]+1$. Denote by A the self-adjoint Schrödinger operator $Af:=-f''+Vf$ in $L^2(\mathbb{R})$, and assume that A has no bound states and no resonance at zero energy. Then, the following decay bounds hold:*

$$\|\langle\cdot\rangle^{-\alpha-1}\cos(t\sqrt{A})f\|_{L^\infty(\mathbb{R})}\lesssim\langle t\rangle^{-\alpha}\big(\|\langle\cdot\rangle^{\alpha+1}f'\|_{L^1(\mathbb{R})}+\|\langle\cdot\rangle^{\alpha+1}f\|_{L^1(\mathbb{R})}\big)$$

and

$$\left\|\langle\cdot\rangle^{-\alpha-1}\frac{\sin(t\sqrt{A})}{\sqrt{A}}f\right\|_{L^\infty(\mathbb{R})}\lesssim\langle t\rangle^{-\alpha}\|\langle\cdot\rangle^{\alpha+1}f\|_{L^1(\mathbb{R})}$$

for all $t\ge 0$.

In particular, this gives t^{-3} for $\alpha = 3$, which is the sharp form of Price's law for $\ell = 0$. It is important to realize that the decay of the waves in Theorems III.9 and III.10 is really a manifestation of transport rather than of dispersion. Indeed, D'Alembert's formula shows that any solution of

$$\partial_{tt} u - \partial_{xx} u = 0, \qquad u(0) = f,\ \partial_t u(0) = g$$

with Schwartz data (say) and $\int g(x)dx = 0$ satisfies

$$\|\langle x\rangle^{-\alpha} u(t)\|_\infty \le C(\alpha)\, t^{-\alpha}$$

for *any* $\alpha \ge 0$. This vanishing mean condition can be attributed to the zero energy resonance for the free Laplacian in one dimension. Needless to say, the one-dimensional problem does not exhibit any sort of dispersion but is governed by linear transport that leads to this arbitrary local decay of the waves. It is very interesting to note (but perhaps not immediately clear) that the sharp Huyghens principle in three dimensions is still visible in the local decay law of Theorem III.9. In fact, we claim that the sharp $t^{-2\ell-3}$ Price law (at least for $\ell \ge 1$) is a result of the correction term of the form $\frac{\log x}{x^3}$ in the Regge–Wheeler potential rather than the leading inverse square decay as $x \to +\infty$.

To clarify this point, we now present a simple model case from Ref. 17. With $a > 0$,

$$\mathscr{H} := -\partial_x^2 + V, \quad V(x) = \begin{cases} 0 & \text{if } x \le -1 \\ \dfrac{a^2 - \frac14}{x^2} & \text{if } x \ge 1. \end{cases}$$

Moreover, $V \in C^\infty(\mathbb{R})$ is such that $\mathscr{H}$ has *no zero energy resonance,* which means that there does not exist a globally subordinate (or recessive) solution $\mathscr{H} f = 0$ other than $f \equiv 0$. Recall that this refers to solutions of the slowest allowed growth at both ends, which means here that $f(x) = O(1)$ as $x \to -\infty$ and $f(x) = O(x^{\frac12 - a})$ as $x \to +\infty$. Then, one has the following local decay estimates for the wave equation with potential V:

Proposition III.11. Under the above assumptions on $\mathscr{H}$*,*

$$\|\langle x\rangle^{-\sigma} \frac{\sin(t\sqrt{\mathscr{H}})}{\sqrt{\mathscr{H}}} P_{(0,\infty)}(\mathscr{H}) g\|_\infty \le C\langle t\rangle^{-2a-1} \|\langle x\rangle^\sigma g\|_1,$$
$$\|\langle x\rangle^{-\sigma} \cos(t\sqrt{\mathscr{H}}) P_{(0,\infty)}(\mathscr{H}) f\|_\infty \le C\langle t\rangle^{-2a-2}\big(\|\langle x\rangle^\sigma f\|_1 + \|\langle x\rangle^\sigma f'\|_1\big),$$

where $\sigma > 0$ is sufficiently large depending on a. These decay rates are optimal, provided $a \notin \mathbb{Z}_0^+ + \frac12$. In the latter case, one obtains decay t^{-N} for any N (provided σ is taken sufficiently large depending on N).

Proof. We prove the first bound, the second one being very similar. Thus, let $\psi(t,x)$ be a solution of the problem

$$\partial_t^2 \psi - \partial_x^2 \psi + V\psi = 0, \quad \psi(0,x) = 0,\ \partial_t \psi(0,x) = g,$$

where g is Schwartz, say, and set for $\mathrm{Re}(p) > 0$,

$$\hat\psi(p,x) := \int_0^\infty e^{-tp} \psi(t,x)\, dt.$$

Then,

$$(\mathscr{H} + p^2)\hat\psi(p,\cdot) = g,$$

which has a unique bounded solution

$$\begin{aligned} \hat\psi(p,x) &= \int_{-\infty}^\infty G(p;x,y)\, g(y)\, dy \\ &= \int_{-\infty}^x \frac{f_+(x,p) f_-(y,p)}{W(p)} g(y)\, dy \\ &\quad - \int_x^\infty \frac{f_+(y,p) f_-(x,p)}{W(p)} g(y)\, dy \end{aligned}$$

with constant Wronskian $W(p) := f_+(x,p) f_-'(x,p) - f_+'(x,p) f_-(x,p)$. Here, $f_\pm(x,p)$ are the Jost solutions

$$(\mathscr{H} + p^2) f_\pm(\cdot,p) = 0, \qquad f_\pm(x,p) \sim e^{\mp xp} \text{ as } x \to \pm\infty.$$

The goal is now to obtain the expansion of $f_\pm(x,p)$ in small p, as this then yields the large time asymptotics of, with arbitrary $p_0 > 0$,

$$\psi(t,x) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\int_{p_0-i\infty}^{p_0+i\infty} e^{tp}\, G(p;x,y)\,dp\,g(y)\,dy \tag{51}$$

via contour deformation and Watson's lemma. By the choice of potential V,

$$f_-(x,p) = e^{px} \text{ for } x \le -1,$$
$$f_+(x,p) = \frac{\pi i}{2} e^{a\pi i/2} H_a^{(1)}(ipx)\Big(\frac{2px}{\pi}\Big)^{\frac{1}{2}} \text{ for } x \ge 1.$$

One can continue $f_-(x,p)$ to the right of $x = -1$, which yields an entire function in p for each fixed x. The nonresonance condition for $p = 0$ means that $f_-(\cdot,0)$ and $x^{\frac{1}{2}-a}$ are linearly independent at $x = 1$. Since $H_a^{(1)} = J_a + iY_a$ and, up to constant factors,

$$J_a(u) \sim u^a(1 + O(u^2)), \quad Y_a(u) \sim u^{-a}(1 + O(u^2))$$

as $u \to 0$ with analytic $O(u^2)$ (at least provided a is not an integer), we conclude that

$$W(p) = c(V)\,p^{\frac{1}{2}-a}\big[1 + O(p^2) + \tilde{c}(V)\,p^{2a}(1 + O(p^2))\big] \text{ as } p \to 0,$$

with $O(p^2)$ analytic in a neighborhood of $p = 0$ and with $c(V) \ne 0$. This is obtained by computing $W(p)$ at $x = 1$, say, and by noting that the most singular contribution to $W(p)$ around $p = 0$ is $c(V)\,p^{\frac{1}{2}-a}$. By inspection, $c(V) = 0$ is the same as a zero energy resonance, which is excluded. If a is a positive integer, then as $p \to 0$,

$$W(p) = c(V)\,p^{\frac{1}{2}-a}\big[1 + O(p^2) + \tilde{c}(V)\,p^{2a}\log(p)(1 + O(p^2))\big].$$

For simplicity, let us first freeze x, y, say, $x = y = 1$. Then, one concludes from the preceding that

$$G(p;1,1) = C(V)\,p^{2a}\big[1 + O(p^2) + \tilde{c}(V)\,p^{2a}(1 + O(p^2))\big] \tag{52}$$

for small $p \in \mathbb{C}\backslash(-\infty,0]$ and analytic $O(p^2)$ around $p = 0$, whereas for the case of $a \in \mathbb{Z}$,

$$G(p;1,1) = C(V)\,p^{2a}\log(p)\big[1 + O(p^2) + \tilde{c}(V)\,p^{2a}\log(p)(1 + O(p^2))\big].$$

The stated decay law now follows via Watson's lemma in a standard fashion. Note the special role of integer but odd $2a$ (which is the exceptional case in the statement of the proposition): in that case, (52) is analytic in small p, whence one can push the contour in (51) through $p = 0$, leading to exponential decay (at least as far as the contribution of small p is concerned).

We now discuss the Watson lemma in more detail. First, we move the contour in (51) onto the imaginary axis,

$$\psi(t,1) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{tp}\,G(p;1,1)\,dp.$$

The contribution due to $1 - \chi(p)$ is shown via integration by parts to decay faster than any power of t [use that $G(iE;x,y) = O(E^{-1})$ for large E, uniformly in x, y]. On the other hand, for the contribution of χ, we retain only finitely many terms from $G(p;1,1)$ with a remainder that is smooth enough around $p = 0$ so as to yield the desired decay again by integration by parts. Finally, the first remaining term is of the form [up to a constant factor $C(V)$]

$$\int_{-i\varepsilon}^{i\varepsilon} e^{tp}\,p^{2a}\chi(p)\,dp.$$

We also have a $\log p$ factor if $a \in \mathbb{Z}$. One now extends this to

$$\int_\gamma e^{tp}\,p^{2a}\,dp, \tag{53}$$

where γ is a curve that contains $[-i\varepsilon, i\varepsilon]$ and is asymptotic to $[0, e^{i\theta}\infty]$ and $[-e^{i\theta}\infty, 0]$, respectively, and the ends. Note that the integrals we inserted here decrease like t^{-N} for any N by integration by parts. By Cauchy's theorem, this is the same as

$$2\,\sin(2a\pi)\int_0^\infty e^{-tp}\,p^{2a}\,dp = 2\,\sin(2a\pi)t^{-2a-1}\Gamma(2a+1),$$

which is the decay rate stated in the proposition. Note that if $a = \ell + \frac{1}{2}$, then this term vanishes, leading to the exceptional behavior stated above. On the other hand, if $a \in \mathbb{Z}$, then this contribution does not vanish due to the $\log(p)$ factor. Finally, we need to remove the restriction $x = y = 1$. However, we have set up our argument in such a way that this modification is easy. First, the contribution of $|p| > \varepsilon$ is again shown to decay at an arbitrary rate via integration by parts. Now, this procedure brings down as many powers of x, y as given by the desired power of t^{-1}. Next, the contribution of the finitely many terms involving p^{2a}, etc., is similar to before, and each one of these terms comes with a corresponding weight in x and y. Finally, the remainder in $G(p; x, y)$ after subtracting that initial segment is again sufficiently smooth in p, and therefore, integration by parts yields the desired decay leading to another instance of requiring large σ. □

The significance of this proposition lies with proximity of V to the Regge–Wheeler potential. Indeed, we replaced the exponential tails on the left by zero and retained the inverse square tails on the right (ignoring the higher-order corrections). In the case of the Regge–Wheeler potential, one has $a^2 - \frac{1}{4} = \ell(\ell+1)$, which implies that $a = \ell + \frac{1}{2}$, which is the *exceptional case of Proposition III.11*. Formally speaking, $2a + 1 = 2\ell + 2$ corresponds exactly to the decay rate of Theorem III.9, whereas the Price law $t^{-2\ell-3}$ is therefore seen to be a result of the $\frac{\log x}{x^3}$ correction to the far field in $V_{\ell,\sigma}$. In fact, it is shown in Ref. 17 that the Price law is due to the nonanalytic term $p^{2a+1}\log p$ instead of p^{2a} in (53). To accomplish this, one derives an expansion of $f_+(x, p)$ in small p, taking into account as many terms from $V_{\ell,\sigma}$ as required for obtaining Price's law and the next few corrections to it. The route taken in Ref. 17 consists of a reduction of the Regge–Wheeler potential to a normal form by means of a Liouville–Green transform. The normal form here consists of the potential without any corrections to the leading $\frac{\ell(\ell+1)}{x^2}$ decay. The branching around $p = 0$ then results from the change of independent variable. Arguing as in the previous proof then yields the sharp $t^{-2\ell-3}$ Price law.

To conclude this survey, let us state the main local decay result from Ref. 30.

Theorem III.12. *The following decay estimates hold for solutions ψ of (50) with data $\psi[0] = (\psi_0, \psi_1)$:*

$$\|\langle x\rangle^{-\frac{9}{2}-}\psi(t)\|_{L^2} \lesssim \langle t\rangle^{-3}\|\langle x\rangle^{\frac{9}{2}+}(\not\nabla^5\partial_x\psi_0, \not\nabla^5\psi_0, \not\nabla^4\psi_1)\|_{L^2}, \tag{54}$$

$$\|\langle x\rangle^{-4}\psi(t)\|_{L^\infty} \lesssim \langle t\rangle^{-3}\|\langle x\rangle^{4}(\not\nabla^{10}\partial_x\psi_0, \not\nabla^{10}\psi_0, \not\nabla^9\psi_1)\|_{L^1}, \tag{55}$$

where $\not\nabla$ stands for the angular derivatives. The notation $a\pm$ stands for $a \pm \varepsilon$ where $\varepsilon > 0$ is arbitrary (the choice determines the constants involved). In addition, instead of $(\not\nabla^{10}, \not\nabla^9)$ in (55), one needs less, namely, $(\not\nabla^{\sigma+1}, \not\nabla^{\sigma})$, where $\sigma > 8$ is arbitrary. Here, $L^2 := L^2_x(\mathbb{R}; L^2(S^2))$, $L^1 := L^1_x(\mathbb{R}; L^1(S^2))$, and $L^\infty := L^\infty_x(\mathbb{R}; L^\infty(S^2))$.

It is obtained by summation in ℓ following the same line of reasoning that lead to Theorem III.2 above. The most significant complication is due to the asymmetry of the Regge–Wheeler potential: while the inverse square potential for $x \to \infty$ is covered by Ref. 16 as before, the exponentially decaying part on the left requires another WKB analysis. We refer the reader to Ref. 30 for the details.

DEDICATION

This article is dedicated to the memory of Jean Bourgain.

ACKNOWLEDGMENTS

The author was partly supported by the National Science Foundation (Grant No. DMS-1902691). The author thanks an anonymous referee for numerous helpful comments that improved the presentation.

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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