LARGE GLOBAL SOLUTIONS FOR ENERGY SUPERCRITICAL NONLINEAR WAVE EQUATIONS ON $\mathbb{R}^{3+1}$.

By

JOACHIM KRIEGER* AND WILHELM SCHLAG*

Abstract. For the radial energy-supercritical nonlinear wave equation

$$\Box u = -u_{tt} + \triangle u = \pm u^7$$

on $\mathbb{R}^{3+1}$, we prove the existence of a class of global in forward time C^∞-smooth solutions with infinite critical Sobolev norm $\dot{H}^{7/6}(\mathbb{R}^3) \times \dot{H}^{1/6}(\mathbb{R}^3)$. These solutions admit a precise asymptotic description and are stable under suitably small perturbations. We also show that for the defocussing energy supercritical wave equation, we can construct such solutions which moreover satisfy the size condition

$$\|u(0, \cdot)\|_{L_x^\infty(|x|\geq 1)} > M$$

for arbitrarily prescribed $M > 0$. These solutions are stable under suitably small perturbations and admit a precise asymptotic description. Also, these solutions experience infinite inflation of the critical $\dot{H}^{7/6}$-norm in any forward light cone. Our method proceeds by regularization of self-similar solutions which are smooth away from the light-cone but singular on the light-cone. The argument crucially depends on the supercritical nature of the equation. Our approach should be seen as part of the program initiated in [10], [11], [4].

1 Introduction

We consider in this paper the energy super-critical defocussing/focussing nonlinear wave equation on $\mathbb{R}^{3+1}$,

$$\Box u \pm u^7 = u_{tt} - \Delta u \pm u^7 = 0. \tag{1.1}$$

The precise power does not play a significant role in the sequel, except for the fact that the problem is energy super-critical. As far as we know, in spite of certain evidence from numerical experiments in the defocussing case that solutions to sufficiently regular but large data appear to stay globally regular, there is no unconditional result asserting global existence of smooth solutions belonging to

*Support of the National Science Foundation DMS-1160817 for the second author, and the Swiss National Fund for the first author are gratefully acknowledged.

JOURNAL D'ANALYSE MATHÉMATIQUE, Vol. 133 (2017)
DOI 10.1007/s11854-017-0029-0

any class of "large data," excepting the trivial time periodic solutions in the defocussing case that do not depend on the spatial variable[1]. By large data, we mean data which are large in the scaling invariant, hence critical Sobolev space $\dot H^{\frac{7}{6}} \times \dot H^{\frac{1}{6}}$, and which do not possess some "hidden" smallness assumption[2], such as the Besov norm condition on the data $u[0] = (u, u_t)|_{t=0}$

$$\|u[0]\|_{\dot B_{\infty}^{\frac{7}{6},2}(\mathbb{R}^3)\times \dot B_{\infty}^{\frac{1}{6},2}(\mathbb{R}^3)} < \varepsilon \tag{1.2}$$

with ε depending on the size of $\|u[0]\|_{\dot H^{7/6}\times \dot H^{1/6}}$. More precisely, one might consider data $u[0] = (u, u_t)|_{t=0}$ large provided[3]

$$\|u[0]\|_{\dot H^{\frac{7}{6}}(\mathbb{R}^3)\times \dot H^{\frac{1}{6}}(\mathbb{R}^3)} \gg 1, \qquad \|u[0]\|_{\dot B_{\infty}^{7/6,2}(\mathbb{R}^3)\times \dot B_{\infty}^{1/6,2}(\mathbb{R}^3)} \gtrsim 1, \tag{1.3}$$

or also

$$\|u[0]\|_{\dot H^{7/6}(\mathbb{R}^3)\times \dot H^{1/6}(\mathbb{R}^3)} = \infty \tag{1.4}$$

We are interested only in C^∞-smooth initial data of precisely this type, although our construction of such data proceeds by regularizing certain self-similar solutions which exhibit a singularity on the light-cone. Thus, if such smooth data satisfy (1.4), this is due to insufficient decay at infinity, and not to some singular behavior in finite space-time. We note here that very sharp global existence results for data satisfying a weak Besov smallness condition such as (1.2) were derived by F. Planchon in [14], [15].

Our purpose in this paper is to exhibit a class of C^∞-smooth, global in forward time solutions which obey (1.4) and are thus outside the scope of a standard perturbative argument around zero, using the Strichartz framework. Moreover, in the defocussing case, we show that these solutions can be forced to have arbitrarily large amplitude[4] on the set $\{|x| \geq 1\}$. Our argument for the first result hinges crucially on the energy super-critical nature of the equation, and for the second uses both the defocussing as well as the supercritical character. As a byproduct of our method, we also obtain the stability of our solutions with respect to suitably mild perturbations. The main results of this paper are the following theorems.

[1]These solutions are, however, most likely unstable under generic perturbations.

[2]Such a smallness assumption can be used to show smallness of suitable critical Strichartz norms for the free wave propagation of the data, which in turn forces the nonlinear solution essentially to behave like a free wave.

[3]More precisely, the first norm $\|u[0]\|_{\dot H^{7/6}(\mathbb{R}^3)\times \dot H^{1/6}(\mathbb{R}^3)}$ is assumed to be extremely large compared to the Besov norm $\|u[0]\|_{\dot B_{\infty}^{7/6,2}(\mathbb{R}^3)\times \dot B_{\infty}^{1/6,2}(\mathbb{R}^3)}$, and so that the free wave propagation of the data does not have small critical Strichartz norms.

[4]Observe that any nonzero solution can be forced to have large amplitude near the origin by rescaling it. However, large amplitude far away from the origin corresponds (in the radial case) in some sense to "large solutions".

Theorem 1.1. *For both the defocussing/focussing supercritical nonlinear wave equation* (1.1) *on* $\mathbb{R}^{3+1}$*, there exist smooth data sets* $(f, g) \in C^\infty \times C^\infty$ *decaying at infinity to zero and satisfying*

$$\|(f, g)\|_{\dot{H}^{7/6}(\mathbb{R}^3)\times\dot{H}^{1/6}(\mathbb{R}^3)} = \infty \ \ \textit{but} \ \ \|(f, g)\|_{\dot{H}^{s}(\mathbb{R}^3)\times\dot{H}^{s-1}(\mathbb{R}^3)} < \infty$$

for any $s > 7/6$*, and such that the corresponding evolution of* (1.1) *exists globally in forward time as a* C^∞*-smooth solution. These solutions are stable under a certain class of perturbations and admit a precise asymptotic description. Furthermore, these solutions satisfy*

$$\lim_{t\to\infty} \|u(t, \cdot)\|_{\dot{H}^{7/6}(K_t)} = \infty, \quad \|u(t, \cdot)\|_{\dot{H}^{1}(K_t)} \sim c\, t^{1/6}$$

$c \neq 0$*, where* $K_t = \{(t, r) | r \leq t + C\}$ *for any* $C \in \mathbb{R}$*, i.e., both the critical norm and the energy blow up asymptotically in any forward light cone.*

We note that the solutions established by this theorem satisfy

$$\|f\|_{L^\infty_x(|x|\geq 1)} \ll 1$$

Also, the weak Besov norm $\|u\|_{\dot{B}^{7/6}_{2,\infty}}$ of these solutions is small, so that they in principle fall under the abstract Cauchy theory developed in [14],[15] (there, the data are of finite critical Sobolev norm, but the global existence result asserted in the preceding theorem follows easily from these works). Nonetheless, the exact asymptotic description of the solutions possible with our method appears to go beyond the standard Cauchy theory.

In the following theorem, we find solutions which are "more nonlinear," as evidenced by a highly oscillatory character, and which no longer satisfy the type of smallness conditions required for [14],[15]; in particular, we no longer have $\|u\|_{\dot{B}^{7/6}_{2,\infty}} \ll 1$. This remark will become clearer in Section 5. In this sense, we call the solutions constructed here "large solutions."

Theorem 1.2. *Let* $M > 0$ *be given arbitrarily. For the defocussing supercritical nonlinear wave equation* (1.1) *on* $\mathbb{R}^{3+1}$*, there exist smooth data sets* $(f, g) \in C^\infty \times C^\infty$ *decaying at infinity to zero and satisfying*

$$\|(f, g)\|_{\dot{H}^{7/6}(\mathbb{R}^3)\times\dot{H}^{1/6}(\mathbb{R}^3)} = \infty \ \ \textit{but} \ \ \|(f, g)\|_{\dot{H}^{s}(\mathbb{R}^3)\times\dot{H}^{s-1}(\mathbb{R}^3)} < \infty$$

for all $s > 7/6$*, as well as*

$$\|f\|_{L^\infty_x(|x|\geq 1)} > M, \tag{1.5}$$

and such that the corresponding evolution of (1.1) *exists globally in forward time as a* C^∞*-smooth solution. These solutions are stable under a certain class of* *perturbations and are not small in the Besov sense* (1.2) *for large* M. *They also admit a precise asymptotic description. Furthermore, they satisfy*

$$\lim_{t\to\infty} \|u(t,\cdot)\|_{\dot H^{7/6}(K_t)} = \infty, \quad \|u(t,\cdot)\|_{\dot H^1(K_t)} \sim c\, t^{1/6}$$

$c \neq 0$, *where* $K_t = \{(t,r) | r \le t + C\}$ *for any* $C \in \mathbb{R}$.

Let us also formulate two of the statements that follow from the methods of this paper for the context of smooth compactly supported data.

Theorem 1.3. *Consider the defocussing equation* (1.1). *For any* $M > 0$, *there exist smooth compactly supported radial data* (f, g) *with support in* $B_2(0)$ *that satisfy*

$$\|(f,g)\|_{\dot H^{7/6}\times \dot H^{1/6}(\mathbb{R}^3)} > M, \tag{1.6}$$

so that (1.1) *admits a smooth solution u for all times* $0 \le t \le 1$, *which furthermore satisfies*

$$\inf_{\frac{1}{p}+\frac{1}{q}=\frac{1}{2},\, p>2} \| \, |\nabla_x|^{\alpha(q)} u\|_{L_t^p L_x^q([0,1]\times\mathbb{R}^3)} \ge 1, \tag{1.7}$$

where $\alpha(q) = \frac{2}{q} + \frac{1}{6}$. *Moreover, the data can be chosen from an open nonempty set relative to the norm in* (1.6).

The space-time norms in (1.7) are examples of Strichartz norms relevant in this context. In fact, we may include any other admissible Strichartz norms in the infimum in (1.7), as well as in the following theorem.

Theorem 1.4. *Consider the defocussing equation* (1.1) *with the* +*-sign. For any* $M_1, M_2 > 0$, *there exist smooth compactly supported radial data* (f, g) *with support in some* $B_K(0)$, $K \ge 1$, *such that*

$$\|(f,g)\|_{\dot H^{7/6}\times \dot H^{1/6}(\mathbb{R}^3)} > M_1$$

and the evolution of these data exists on $0 \le t \le K/2$ *as a smooth function. Moreover, with* $\alpha(q)$ *as in the previous theorem,*

$$\inf_{\frac{1}{p}+\frac{1}{q}=\frac{1}{2},\, p>2} \| \, |\nabla_x|^{\alpha(q)} u\|_{L_t^p L_x^q([0,K/2]\times\mathbb{R}^3)} \ge 1,$$

and

$$\|f\|_{L_x^\infty(|x|\ge 1)} > M_2.$$

Note that the inequality

$$\inf_{\frac{1}{p}+\frac{1}{q}=\frac{1}{2},\, p>2} \| \, |\nabla_x|^{\alpha(p,q)} u \|_{L_t^p L_x^q([0,K/2]\times\mathbb{R}^3)} \geq 1$$

means that all the scale invariant Strichartz norms of the solution are not small, precluding a simple perturbative argument around the free wave propagation of the initial data. Furthermore, the condition on the support precludes a simple construction piecing together small solutions in disjoint light cones. In fact, the philosophy of this work is to use a perturbative approach around suitably constructed elliptic nonlinear objects, in this case, approximate self-similar solutions. More precisely, as in the method employed in [4], the idea is to use special singular solutions, obtained by making a self-similar ansatz, to generate non-trivial global dynamics via a carefully chosen regularization and solution of a perturbative problem. In fact, the regularization destroys the scaling invariance, and this turns out to be important for the ensuing perturbative argument. We observe also that the method of [4] grew directly out of the methods introduced in [10], [11]. In our present context, however, we do not rely on the spectral methods and parametrix constructions used in these references, but rather rely on the standard Strichartz and energy estimates.

In the following section, we construct smooth self-similar solutions of the form

$$u_0(t,r) = t^{-1/3} Q\left(\frac{r}{t}\right), \quad \text{either } r < t \text{ or } r > t, \tag{1.8}$$

by a reduction to a nonlinear Sturm-Liouville problem; see (2.1). We solve this ODE by contraction off of the leading linear behavior, assuming smallness in L^∞. This smallness also allows us to solve a nonlinear connection problem at an intermediate point such as $a := r/t = 1/2$ by the Inverse Function Theorem.

As we shall see, starting with small data at $a = 0$, $Q(a)$ exhibits a singularity of the form $|1-a|^{2/3}$ near $a = 1$, which precisely fails logarithmically to belong to the scaling critical Sobolev space $\dot{H}^{7/6}(\mathbb{R}^3)$, and its time-derivative fails logarithmically to belong to $\dot{H}^{1/6}(\mathbb{R}^3)$. This part of the construction does not depend on super-criticality in any way. In fact, it can be carried out in other dimensions and for other powers. In each case, the singularity falls logarithmically outside of the scaling critical space. For example, in $\mathbb{R}^5$ for the $\dot{H}^2 \times \dot{H}^1$-critical u^5 equation, the singularity is of the form $|1-a|^{3/2}$, whereas for the same equation in $\mathbb{R}^3$ (the energy critical one), the singularity is $|1-a|^{1/2}$.

In the second part of the construction, we first glue together the two solutions residing inside and outside the light-cone, respectively, at $r = t$ to form a continuous function $u_0(t,r)$ which decays as $r \to \infty$ at the rate $r^{-1/3}$ (and thus fails to lie in $\dot H^{7/6}$ at $r=\infty$). The decay $r^{-1/3}$ is generic; we may also achieve $r^{-4/3}$, but then the time-derivative fails to belong to $\dot H^{1/6}$ at $r=\infty$.

We then multiply the singular components of $u_0(t,r)$ by a smooth cutoff function equal to 1 away from $|r-t|\le 2C$ and vanishing on $|r-t|\le C$, say. This smooth function $u_1(t,r)$ no longer solves (1.1), but we show that we may add a smooth correction $v(t,r)$ to $u_1(t,r)$ so that $u(t,r)=u_1(t,r)+v(t,r)$ solves (1.1). This part of the argument depends crucially on the energy supercritical nature of the problem (although neither the exact power nor the focussing/defocussing character is relevant). This perturbative argument relies on an interplay between the scaling critical norm and the standard energy. We remark that the latter restricted to $r<t$ grows like $t^{1/3}$ as $t\to\infty$ due to incoming waves.

In the final part of the paper, we reconsider the self-similar solutions on the outside of the light cone, but only in the defocussing case. We show that one of the parameters determining the solution near the singularity at $a=1$ can be chosen arbitrarily large, leading to rapid growth and oscillation of the solution on the set $a>1$ but near a. The defocussing character of the problem permits us to extend these solutions all the way to $a\to+\infty$, where they again decay asymptotically like $a^{-1/3}$. We show that such a "large self-similar solution" can be glued to a "small self-similar solution" inside the light cone. Truncating (parts of) this continuous function to make it C^∞-smooth just as before, we then show that we can construct an exact C^∞ solution with just the behavior detailed in Theorem 1.2. The key to obtaining the smallness gain for the nonlinear estimates comes from choosing the time $t\ge T$ large enough.

We cannot possibly do justice to the large body of work that has been devoted to studying the equation

$$\Box u\pm|u|^{p-1}u=u_{tt}-\Delta u\pm|u|^{p-1}u=0$$

in $\mathbb{R}^{3+1}$ (or other dimensions) for smooth, compactly supported data over the past fifty years. In the defocussing case, Jörgens [7] showed global existence for $p<5$, the subcritical regime. Struwe [17] then settled the energy critical case $p=5$ radially, and Grillakis [6] nonradially; see the book by Shatah and Struwe [16] for an account of these developments. A very general method to attack energy critical problems and, in particular, recover the result of Struwe and Grillakis was developed recently by Kenig and Merle in [8]. A much more quantitative approach, implying scattering and global space-time bounds explicitly in terms of the energy, but more contingent on the specific structure of the equation, was established in the work [2] by Bourgain in the context of the energy-critical defocussing radial nonlinear Schrödinger equation. These methods were then further developed by

Tao in [20] to treat a "slightly super-critical wave equation" (where the critical nonlinearity is multiplied by a logarithmic factor). In this context, we also mention Struwe's recent work on energy super-critical wave equations on $\mathbb{R}^{2+1}$ with exponential type nonlinearities, [18], [19]. Observe that all pure power nonlinear wave equations on $\mathbb{R}^{2+1}$ of the form $\Box u = \pm|u|^{p-1}u$, $p > 1$, are energy-subcritical. Lebeau [13, 12] studies instability of solutions to semi-linear equations including the supercritical equations such as (1.1), again in the defocussing case, relative to weaker norms than the scaling critical ones. We remark that the self-similar solutions constructed in the following section belong to all spaces of the form

$$\dot{H}^{\frac{7}{6}-\varepsilon}(\mathbb{R}^3) \times \dot{H}^{\frac{1}{6}-\varepsilon}(\mathbb{R}^3)$$

with $\varepsilon > 0$, provided we restrict them to the interior of the light-cone. It is conceivable that this might allow one to obtain aspects of the supercritical ill-posedness results as in Lebeau's work by solving backward from $t = 1$ to $t = 0$ inside of the cone. However, we do not pursue such matters here.

By Strichartz theory, cf. Lemma 4.2, the equation (1.1) is globally well-posed for smooth compactly supported data with small critical norm (in both the focussing and defocussing cases). It is also locally well-posed for any data in that norm, and the solutions preserve regularity and obey the finite propagation speed. Kenig and Merle [9] proved for (1.1) and the defocussing case that breakdown of smooth solutions in finite time T can only occur provided

$$\sup_{0<t<T} \|(u(t), u_t(t))\|_{\dot{H}^{7/6}(\mathbb{R}^3)\times\dot{H}^{1/6}(\mathbb{R}^3)} = \infty.$$

This work has generated many further developments of a similar character; see, for example, the recent work [3]. Bizoń, Maison, and Wasserman [1] established an infinite family of smooth solutions for the focussing supercritical equation (1.1) which are obtained by rescaling of a fixed profile. In essence, these authors observed via an ODE analysis that in addition to the ODE blowup $c(T-t)^{-1/3}$ present in the focussing equation (1.1), this equation also allows for infinitely many solutions obtained from this one by multiplication by a time-dependent non-constant profile of the form $U(r/(T-t))$. It is shown in [1] that there exists an infinite sequence of values $U(0)$ and $U(1)$ which give rise to a smooth solution of (1.1). We also mention here the works by Donninger and Schörkhuber [5] for the focussing supercritical wave equation, where they establish stability of the explicit ODE blow up solutions.

However, the investigations of this paper go in a very different direction since we are mainly concerned with the defocussing equation and global smooth solutions, as opposed to finite time blow up.

2 Self-similar solutions

2.1 The interior light-cone. We seek a solution of (1.1) of the form $u_0(t, r) = t^{-1/3}Q(r/t)$ for $0 \leq r < t$. In general, we expect these solutions to be singular at least on the light-cone, i.e., at $r/t = a = 1$, and a precise description of this failure of regularity plays a key role later on. To begin with, Q satisfies the ODE on $0 \leq a < 1$

$$(a^2 - 1)Q''(a) + \Big(\frac{8}{3}a - \frac{2}{a}\Big)Q'(a) + \frac{4}{9}Q(a) \pm Q(a)^7 = 0. \tag{2.1}$$

The natural initial conditions at $a = 0$ are

$$Q(0) = q_0 > 0, \quad Q'(0) = 0. \tag{2.2}$$

We first solve this initial value problem on the interval $0 \leq a \leq 1/2$; this leads to a 1-parameter family of solutions. We then solve the nonlinear connection problem at $a = 1/2$ with a 2-parameter family of solutions on the interval $(1/2, 1)$. The two parameters are important, since they allow us to apply the Inverse Function Theorem.

Lemma 2.1. *There exists small $\varepsilon > 0$ such that for each $0 \leq q_0 \leq \varepsilon$, equation* (2.1) *with initial conditions* (2.2) *admits a unique smooth solution on* $[0, 1/2]$. *Moreover, with Q_0 defined below just after* (2.8)*,*

$$\begin{aligned} Q(1/2) &= q_0 Q_0(1/2) + O(q_0^7), \\ Q'(1/2) &= q_0 Q_0'(1/2) + O(q_0^7), \end{aligned} \tag{2.3}$$

and the solution extends as a smooth even function the the interval $[-1/2, 1/2]$.

Proof. The associated homogeneous linear equation is

$$(a^2 - 1)Q''(a) + \Big(\frac{8}{3}a - \frac{2}{a}\Big)Q'(a) + \frac{4}{9}Q(a) = 0 \tag{2.4}$$

with fundamental system

$$\varphi_1(a) = a^{-1}(1 - a)^{2/3}, \quad \varphi_2(a) = a^{-1}(1 + a)^{\frac{2}{3}}. \tag{2.5}$$

The Green function for $0 < b < a < 1$

$$\begin{aligned} G(a, b) &:= \frac{\varphi_1(a)\varphi_2(b) - \varphi_1(b)\varphi_2(a)}{W(b)(b^2 - 1)}, \\ W(a) &:= \varphi_1(a)\varphi_2'(a) - \varphi_1'(a)\varphi_2(a) \end{aligned} \tag{2.6}$$

has the property that the inhomogeneous equation

$$(a^2-1)\varphi''(a)+\Big(\frac{8}{3}a-\frac{2}{a}\Big)\varphi'(a)+\frac{4}{9}\varphi(a)=f(a),\qquad \varphi(0)=0,\ \varphi'(0)=0 \tag{2.7}$$

is solved by

$$\varphi(a)=-\int_0^a G(a,b)f(b)\,db.$$

We therefore seek a solution of (2.1) with initial conditions (2.2) on $0\le a\le 1/2$ of the form

$$Q(a)=\frac{3}{4}q_0(\varphi_2(a)-\varphi_1(a))\pm\int_0^a G(a,b)Q(b)^7\,db. \tag{2.8}$$

Note that $Q_0(a):=\frac{3}{4}(\varphi_2(a)-\varphi_1(a))$ is even around $a=0$ and analytic . Moreover, $Q_0(0)=1$. Assume $0\le q_0\le\varepsilon$ and define the space

$$X_{q_0}:=q_0Q_0+\{h(a)\,|\,h\in C^2([0,1/2]),\ \|h\|_{C^2}\le q_0^6,\ |h(a)|\le q_0^6a^2\}$$

We equip the linear space defined by the set on the right-hand side with the norm

$$\|h\|_{C^2}+\sup_{0<a<\frac{1}{2}}a^{-2}|h(a)|$$

Our main claim is as follows.

Claim. *There exists* $\varepsilon>0$ *small such that for each* $0\le q_0\le\varepsilon$, *equation* (2.8) *has a unique solution in* X_{q_0}.

By explicit calculation,

$$W(a)=\frac{4}{3a^2}(1-a^2)^{-1/3} \tag{2.9}$$

and $(W(b)(b^2-1))^{-1}$ is analytic on $(-1,1)$ with expansion

$$-\frac{3}{4}b^2-\frac{1}{2}b^4+O(b^6)$$

as $b\to 0$. Second, for $0<b<a$,

$$G(a,b)=\frac{b}{a}\Big(-\frac{3}{4}+O(b^2)\Big)\big[(1-a)^{2/3}(1+b)^{2/3}-(1-b)^{2/3}(1+a)^{2/3}\big]$$

whence, in particular, $|G(a,b)|\le Cb$ for all $0<a\le 1/2$. Moreover, setting $b=ua$ with $0<u<1$ shows that

$$\tilde{G}(a,u):=G(a,au)$$

is smooth in $|a| < 1$ and $|u| < 1$ and satisfies the bound

$$\max_{|u|\le 1} \tilde{G}(a,u) \le C|a|.$$

Therefore,

$$\int_0^a |G(a,b)|\,db = a\int_0^1 |\tilde{G}(a,u)|\,du \le Ca^2. \tag{2.10}$$

Define

$$(Tf)(a) := q_0 Q_0(a) \pm \int_0^a G(a,b) f(b)^7\,db = q_0 Q_0(a) \pm a\int_0^1 \tilde{G}(a,u) f(au)^7\,du.$$

We claim that T is a contraction in X_{q_0} and therefore has a fixed point $f \in X_{q_0}$. Each $f \in X_{q_0}$ satisfies $|f(a)| \le Mq_0$ for all $0 \le a \le \frac{1}{2}$, where M is some absolute constant. Thus

$$h(a) := \int_0^a G(a,b) f(b)^7\,db$$

satisfies by (2.10)

$$|h(a)| \le CM^7 a^2 q_0^7 \ll q_0^6 a^2,\ \ |h'(a)| \le CM^7 a q_0^7 \ll q_0^6 a$$

as well as

$$|h''(a)| \le CM^7 q_0^7 \ll q_0^6,$$

provided q_0 is small. Hence $T : X_{q_0} \to X_{q_0}$. For the contraction, we estimate

$$\|Tf - Tg\|_{X_{q_0}} \le C(\|f\|_\infty + \|g\|_\infty)^6 \|f-g\|_\infty \le CM^6 q_0^6 \|f-g\|_{X_{q_0}}.$$

For small q_0, this implies that T is a contraction, and we are done with our main claim. We note from the integral equation that f is even on $[-1/2, 1/2]$.

As for the higher regularity, this of course follows from standard regularity results. We proceed by induction on the number of derivatives. Starting from the integral equation

$$f(a) = q_0 Q_0(a) \pm a\int_0^1 \tilde{G}(a,u) f(au)^7\,du,$$

we observe that

$$f^{(k)}(a) = q_0 Q_0^{(k)}(a) + H_k(a) \pm 7a\int_0^1 \tilde{G}(a,u) f(au)^6 f^{(k)}(au) u^k\,du \tag{2.11}$$

for every integer $k \ge 0$, where H_k is smooth; one has $H_0 = 0$ and

$$\pm H_1(a) = \int_0^1 \tilde{G}(a,u) f(au)^7\,du + a\int_0^1 \tilde{G}_a(a,u) f(au)^7\,du,$$

and so forth. Clearly, H_k only involves $k-1$ derivatives of f and is therefore small in the norm of continuous functions on the interval $[0, 1/2]$ by the inductive assumption. We can therefore contract (2.11) to produce a continuous small solution $f^{(k)}(a)$ on $[0, 1/2]$. This shows that f possesses any number of derivatives. $\square$

The solution is in fact analytic. We remark that one can also solve (2.1) near $a = 0$ (and thus also on $[0, 1/2)$) by power series. Writing the usual iteration for the coefficients shows that they are all positive. This is a reflection of the defocussing nature of (1.1). Thus, the solution, together with all derivatives, is monotone increasing. We have chosen to use the Green function since the nonlinear recursion is not entirely elementary.

Next, we solve backwards starting from $a = 1$.

Lemma 2.2. *For $q_1, q_2 \in (-\varepsilon, \varepsilon)$, there exists a unique solution $Q(a)$ of* (2.1) *on* $[1/2, 1)$ *of the form*

$$Q(a) = (1-a)^{2/3} Q_1(a) + Q_2(a) + (1-a)^{7/3} Q_3(a) \tag{2.12}$$

with $Q_1, Q_2, Q_3 \in C^\infty([1/2, 1])$ and

$$Q_1(a) = q_1(1 + O(1-a)), \quad Q_2(a) = q_2(1 + O(1-a)),$$
$$Q_3(a) = (|q_1|^7 + |q_2|^7) O(1),$$

where the $O(\cdot)$ terms are smooth functions in $a \in [1/2, 1]$. Finally,

$$\begin{aligned} Q(1/2) &= q_1\varphi_1(1/2) + q_2 2^{-\frac{2}{3}} \varphi_2(1/2) + O(|q_1|^7 + |q_2|^7), \\ Q'(1/2) &= q_1\varphi_1'(1/2) + q_2 2^{-\frac{2}{3}} \varphi_2'(1/2) + O(|q_1|^7 + |q_2|^7), \end{aligned} \tag{2.13}$$

where φ_1, φ_2 are the functions from (2.5).

Proof. We convert the ODE (2.1) into the integral equation

$$Q(a) = q_1\varphi_1(a) + q_2 2^{-2/3} \varphi_2(a) \mp \int_a^1 G(a, b) Q(b)^7 \, db, \tag{2.14}$$

where G is the Green function from (2.6). Since in this case $a < b < 1$, the integral comes with a negative sign.

By inspection, we see that $(1-a)^{-2/3}\varphi_1(a) = 1/a$, $2^{-2/3}\varphi_2(a) = ((1+a)/2)^{2/3}/a$ are analytic on $a > 0$, and equal 1 at $a = 1$. Furthermore, taking the Wronskian (2.9) into account, we find that the Green function (2.6) is of the form

$$G(a, b) = g_1(a, b) + (1-b)^{2/3}(1-a)^{-2/3} g_2(a, b), \tag{2.15}$$

where g_1, g_2 are smooth for $a, b \in [1/2, 1]$. If $\omega(b)$ is smooth on $[1/2, 1]$, then

$$\int_a^1 G(a,b)\omega(b)\,db = O(1-a) \tag{2.16}$$

is smooth on $[1/2, 1]$, and

$$\int_a^1 G(a,b)\omega(b)(1-b)^{2k/3}\,db = (1-a)^{2k/3}O(1-a), \quad k = 1, 2, \tag{2.17}$$

where $O(1-a)$ is a smooth function of $a \in [1/2, 1]$. This allows one to convert (2.14) into a system for Q_1, Q_2, Q_3 which we again solve by contraction. To be specific,

$$[(1-a)^{2/3}Q_1(a) + Q_2(a) + (1-a)^{7/3}Q_3(a)]^7 = \sum_{j=0}^{2}(1-a)^{2j/3}N_j(a, Q_1, Q_2, Q_3).$$

Here, each $N_j(a, Q_1, Q_2, Q_3)$ is a linear combination of terms of the form

$$(1-a)^m Q_1^{k_1}(a)Q_2^{k_2}(a)Q_3^{k_3}(a),$$

where $k_1 + k_2 + k_3 = 7$. In particular, if all Q_i are smooth, then N_j is, too. For example, N_2 contains the term $21Q_1^2(a)Q_2^5(a)$. We remark that in each N_j, the function Q_3 appears with a factor of at least $(1-a)$. For example, the term

$$7Q_2^6(a)(1-a)^{7/3}Q_3(a) = (1-a)^{4/3}7(1-a)Q_2^6(a)Q_3(a) \tag{2.18}$$

contributes $7(1-a)Q_2^6(a)Q_3(a)$ to N_2. We now solve for Q_j in the following form

$$\begin{aligned} Q_1(a) &= q_1a^{-1} \mp (1-a)^{-2/3}\int_a^1 G(a,b)(1-b)^{2/3}N_1(b, Q_1, Q_2, Q_3)\,db, \\ Q_2(a) &= q_2 2^{-2/3}\varphi_2(a) \mp \int_a^1 G(a,b)N_0(b, Q_1, Q_2, Q_3)\,db, \\ Q_3(a) &= \mp(1-a)^{-7/3}\int_a^1 G(a,b)(1-b)^{4/3}N_2(b, Q_1, Q_2, Q_3)\,db. \end{aligned} \tag{2.19}$$

By (2.16), (2.17), the right-hand sides are smooth if the Q_j are. We write the system (2.19) in the fixed-point form $\vec{Q} = T(\vec{Q})$, where T denotes the column vector of the right-hand sides and $\vec{Q} := (Q_1, Q_2, Q_3)$.

We set up a contraction for T in the space of continuous functions on the interval $[1/2, 1]$. For $\varepsilon > 0$, we find a unique solution of the form

$$\begin{aligned} Q_1(a) &= q_1a^{-1} + (|q_1| + |q_2|)^7(1-a)R_1(a), \\ Q_2(a) &= q_2 2^{-2/3}\varphi_2(a) + (|q_1| + |q_2|)^7(1-a)R_2(a), \\ Q_3(a) &= (|q_1| + |q_2|)^7 R_3(a), \end{aligned}$$

where R_j are continuous and satisfy $|R_j(a)| \le M$ on $[1/2, 1]$ for some absolute constant M.

Inserting these representations into (2.19), we gain at least one degree of regularity at $a = 1$; in other words, one factor of $(1 - a)$. For the terms involving R_1, R_2, this is clear, since each application of the integration in (2.19) gains a factor of $(1 - a)$. On the other hand, for the Q_3 term, we need to use the observation (2.18), i.e., the fact that Q_3 carries at least a factor of $(1 - a)$ when reinserted into the nonlinearity N_2. Repeating this procedure produces more and more smoothness at $a = 1$. The smoothness for $1/2 \le a < 1$ is clear. □

We can now solve (2.1) and thus obtain the special self-similar solutions of (1.1). The following corollary shows that such solutions (nonzero of course), necessarily exhibit the $(1 - a)^{\frac{2}{3}}$ singularity at $a = 1$.

Corollary 2.3. *For each small q_0, the ODE* (2.1) *has a unique C^2 solution $Q(a)$ on* $[0, 1)$ *with $Q(0) = q_0$ and $Q'(0) = 0$. This solution is of the form* (2.12) *near $a = 1$. We can have neither $q_1 = 0$ nor $q_2 = 0$.*

Proof. For given small q_0, let Q be the solution generated by Lemma 2.1. By the Inverse Function Theorem, we may find q_1, q_2 small so that (2.13) matches the values given by (2.3). Application of the inverse function theorem is justified since the derivative in q_1, q_2 at $(q_1, q_2) = 0$ of (2.13) is the Wronskian of φ_1, φ_2, which does not vanish. The final claim is seen for the same reason: we cannot achieve linear dependence of the solutions generated by Lemmas 2.1 and 2.2 when either $q_1 = 0$ or $q_2 = 0$. □

In particular, these solutions logarithmically fail to belong to $\dot{H}^{7/6}(\mathbb{R}^3)$. In fact, the function $(1 - a)^{2/3}$ fails to be in $\dot{H}^{7/6}$.

2.2 The exterior light-cone.

We next carry out a similar construction in the region $r > t$. Here $a = r/t > 1$, but the analysis is essentially the same. We begin of the analogue of Lemma 2.2.

Lemma 2.4. *For $\tilde{q}_1, \tilde{q}_2 \in (-\varepsilon, \varepsilon)$ there exists a unique solution $Q(a)$ of* (2.1) *on* $(1, 2]$ *of the form*

$$Q(a) = (a-1)^{2/3}\tilde{Q}_1(a) + \tilde{Q}_2(a) + (a-1)^{7/3}\tilde{Q}_3(a) \tag{2.20}$$

with $\tilde{Q}_1, \tilde{Q}_2, \tilde{Q}_3 \in C^\infty((1, 2])$ and

$$\tilde{Q}_1(a) = \tilde{q}_1(1 + O(a-1)), \quad \tilde{Q}_2(a) = \tilde{q}_2(1 + O(a-1)),$$
$$\tilde{Q}_3(a) = (|\tilde{q}_1|^7 + |\tilde{q}_2|^7)O(1),$$

where the $O(\cdot)$ terms are smooth functions in $a \in [1,2]$. Finally,

$$
\begin{aligned}
Q(2) &= \tilde{q}_1 \tilde{\varphi}_1(2) + \tilde{q}_2 2^{-2/3} \varphi_2(2) + O(|\tilde{q}_1|^7 + |\tilde{q}_2|^7),\\
Q'(2) &= \tilde{q}_1 \tilde{\varphi}_1'(2) + \tilde{q}_2 2^{-2/3} \varphi_2'(2) + O(|\tilde{q}_1|^7 + |\tilde{q}_2|^7),
\end{aligned} \tag{2.21}
$$

where $\tilde{\varphi}_1(a) = a^{-1}(a-1)^{2/3}$ and φ_2 is as in (2.5).

The proof is analogous to that of Lemma 2.2, and we omit it.

Next, we glue this solution together with one on $2 \le a < \infty$.

Lemma 2.5. *There exists small $\varepsilon > 0$ such that for each $|m_1|, |m_2| \le \varepsilon$, equation (2.1) admits a unique smooth solution on $[2,\infty)$ such that as $a \to \infty$*

$$
Q(a) = m_1 \tilde{\varphi}_1(a) + m_2 \varphi_2(a) + O(a^{-\frac{7}{3}}) \tag{2.22}
$$

and

$$
\begin{aligned}
Q(2) &= m_1 \tilde{\varphi}_1(2) + m_2 \varphi_2(2) + O((|m_1| + |m_2|)^7),\\
Q'(2) &= m_1 \tilde{\varphi}_1'(2) + m_2 \varphi_2'(2) + O((|m_1| + |m_2|)^7).
\end{aligned} \tag{2.23}
$$

Here, $\tilde{\varphi}_1, \varphi_2$ are as in Lemma 2.4.

Proof. We use the Green function (2.6) but defined in terms of $\tilde{\varphi}_1, \varphi_2$:

$$
\begin{aligned}
G(a,b) &:= \frac{\tilde{\varphi}_1(a)\varphi_2(b) - \tilde{\varphi}_1(b)\varphi_2(a)}{W(b)(b^2-1)}\\
W(a) &:= \tilde{\varphi}_1(a)\varphi_2'(a) - \tilde{\varphi}_1'(a)\varphi_2(a).
\end{aligned} \tag{2.24}
$$

The denominator is $W(b)(b^2-1)$, which decays at the rate $b^{-2/3}$ as $b \to \infty$. The perturbative ansatz is

$$
Q(a) = m_1 \tilde{\varphi}_1(a) + m_2 \varphi_2(a) \pm \int_a^\infty G(a,b) Q(b)^7 \, db. \tag{2.25}
$$

This is solved by contraction, and the asymptotics (2.22) follows by inserting the two types of asymptotic behaviors exhibited by $G(a,b)$, i.e., $a^{-4/3}b^{1/3}$, and $a^{-1/3}b^{-2/3}$. Integrating these against $Q(b)^7$, which decays at least as fast as $b^{-7/3}$, then shows that the integral in (2.25) decays as $a^{-7/3}$. Moreover, we obtain (2.23) by setting $a = 2$. □

Finally, we glue the two solutions together to obtain one on the whole interval $a > 1$. The following corollary is an immediate application of Lemmas 2.5 and 2.4.

Corollary 2.6. *For each small* m_1, m_2*, there exists a smooth solution* $Q(a)$ *of the ODE* (2.1) *on* $1 < a < \infty$*, with the asymptotics* (2.22) *as* $a \to \infty$*. As* $a \to 1$*, the solution obeys the representation* (2.20)*. The map* $(m_1, m_2) \mapsto (\tilde{q}_1, \tilde{q}_2)$ *is a diffeomorphism from a small neighborhood of* $(0, 0)$ *to another. Finally, there exists a linear map* $m \mapsto (m_1, m_2)$ *such that for every small* m*, the corresponding solution decays as* $m\, a^{-4/3}$ *as* $a \to \infty$*.*

Proof. As for the interior light-cone, we solve the connection problem at $a = 2$ by means of the Inverse Function Theorem. This is legitimate, again by smallness as well as the non-vanishing of the Wronskian. In general, we obtain a 2-parameter family. But we may cancel the leading order $a^{-1/3}$ as $a \to \infty$ by means of a linear relation between m_1, m_2. This is the claim relating to a linear map $m \mapsto (m_1, m_2)$ and produces decay at the rate $a^{-\frac{4}{3}}$. The result is a 1-parameter family of solutions. □

2.3 Matching at the light-cone. Combining Corollaries 2.3, 2.6 leads to the following conclusion. For the meaning of the parameters q_1, q_2, m_1, etc., see these corollaries.

Corollary 2.7. *For each small* q_0*, the ODE* (2.1) *has a unique* C^2 *solution* $Q(a)$ *on* $[0, 1)$ *with* $Q(0) = q_0$ *and* $Q'(0) = 0$*. There exist infinitely many continuous extensions of* $Q(a)$ *to* $a \geq 1$ *which solve* (2.1) *on* $a > 1$ *and decay at least at the rate* $a^{-1/3}$ *as* $a \to \infty$*. These extensions are given by Corollary 2.6. The global solutions on* $a \geq 0$ *satisfy* $q_2 = \tilde{q}_2$*, in the notation of Lemma 2.2. We denote these functions on* $a \geq 0$ *by* $Q_0(a)$*, and we have the global representation*

$$Q_0(a) = |1 - a|^{2/3}\left[Q_1(a) + |1 - a|^{5/3} Q_3(a)\right] + Q_2(a) \tag{2.26}$$

for all $a \geq 0$*. Then* Q_1, Q_2, Q_3 *are smooth away from* $a = 1$*,* Q_2 *is continuous at* $a = 1$*,* $Q_1(a) = Q_3(a) = 0$ *for* $a \geq 2$*, and* $a^{1/3} Q_2(a)$ *is bounded as* $a \to \infty$*.*

Proof. For small q_0, we solve (2.1) on $[0, 1)$; this gives us q_1, q_2. We then select small (m_1, m_2) such that $\tilde{q}_2 = q_2$. In general, we cannot expect this solution to decay faster than $a^{-1/3}$, since we will not hit the linear relation between m_1 and m_2 needed for this to happen. □

Note that the solutions of Corollary 2.7 are still small, since the contraction arguments by means of which they were constructed require smallness. This is also reflected in the property that the nonlinearity can be both focussing and defocussing. The smallness is expressed by the estimate $|q_0| + |\tilde{q}_1| \ll 1$, since then also $|q_1| + |q_2| \ll 1$ and $|\tilde{q}_2| = |q_2| \ll 1$.

Later, we modify the construction to allow large (in some sense) solutions outside of the light-cone. For this, it is essential that we only match $q_2 = \tilde{q}_2$, since the parameter $\tilde{q}_1$ is taken to be large. This construction is only possible for the defocussing equation.

3 Removing the singularity on the light-cone

Departing from the singular self-similar solutions constructed above, we now attempt to build global smooth solutions of (1.1) which are large in a suitable sense. In effect, we expect them to have infinite critical norm. Consider the self-similar solutions constructed in the preceding section, in particular, Q_0 from Corollary 2.7 with the asymptotic behavior for $a \to +\infty$ specified in Corollary 2.6. Now set $u_0(t, r) := t^{-1/3} Q_0(r/t)$, which we may assume to be of class C^0 across the light-cone $a = r/t = 1$, but in general no better. By construction, u solves (1.1) away from $t = r$. Moreover, $|Q_0(a)| \lesssim a^{-1/3}$ implies that

$$|u_0(t, r)| \lesssim (|q_0| + |\tilde{q}_1|) r^{-1/3} \quad \text{for all } r > 0. \tag{3.1}$$

In view of (2.26), we have

$$u_0(t, r) := t^{-1/3} |1 - a|^{2/3} \big[Q_1(a) + |1 - a|^{\frac{5}{3}} Q_3(a) \big] + t^{-1/3} Q_2(a), \tag{3.2}$$

where the function Q_3 is expected to be discontinuous across $a = 1$ while the functions $Q_1(a)$, $Q_2(a)$ are continuous on $a \geq 0$. Writing

$$Q_2(a) = Q_2(1) + Q_2(a) - Q_2(1), \tag{3.3}$$

we have $|Q_2(a) - Q_2(1)| = O(|1 - a|)$, and it is natural to incorporate this term into the term

$$|1 - a|^{2/3} \big[Q_1(a) + |1 - a|^{5/3} Q_3(a) \big]$$

in our representation of $u(t, r)$. Thus, with

$$\tilde{Q}_3(a) := |1 - a|^{4/3} Q_3(a) + |1 - a|^{-1} (Q_2(a) - Q_2(1)),$$

we obtain

$$u_0(t, r) = t^{-1/3} |1 - a|^{2/3} \big[Q_1(a) + |1 - a|^{1/3} \tilde{Q}_3(a) \big] + t^{-1/3} Q_2(1),$$

where Q_3 is smooth away from $a = 1$ but possibly discontinuous across it. We now abuse notation and write Q_3 again instead of $\tilde{Q}_3$.

We have now incorporated all the singular behavior of this solution into the term

$$|1-a|^{2/3}\big[Q_1(a)+|1-a|^{1/3}Q_3(a)\big] =: |1-a|^{2/3}X(a).$$

In order to excise the singularity, we introduce a smooth cutoff $\chi(t-r)$, which localizes the expression smoothly to a fixed distance C from the light-cone, i.e., $|t-r| \geq C$; the constant C here plays no role. In other words, $\chi(v) = 1$ for $|v| \geq 2C$ and $\chi(v) = 0$ for $|v| \leq C$.

We introduce the approximate solution

$$u(t,r) = t^{-1/3}\chi(t-r)|1-a|^{2/3}X(a)+t^{-1/3}Q_2(1). \tag{3.4}$$

Note that $u(t,r) = u_0(t,r)$ for all $|t-r| \geq 2C$. By construction, we have the following smallness property which plays an important role in our argument:

$$\|u\|_{L_t^6([T,T+1],L_x^{18}(\mathbb{R}^3))} \ll 1 \tag{3.5}$$

uniformly in $T \geq 1$. The norm here is an example of a Strichartz norm; see Lemma 4.2.

We now need to understand the error associated with the ansatz $u(t,r)$ in (3.4), i.e., we need to estimate $-u_{tt}+\triangle u \mp u^7$. We compute

$$\begin{aligned}
-u_{tt}+\triangle u \mp u^7 &= \chi(t-r)\Big(-\partial_t^2+\partial_r^2+\frac{2}{r}\partial_r\Big)\big(t^{-1/3}|1-a|^{2/3}X(a)+t^{1/3}Q_2(1)\big)\\
&\qquad +(1-\chi(t-r))\Big(-\partial_t^2+\partial_r^2+\frac{2}{r}\partial_r\Big)\big(t^{-1/3}Q_2(1)\big)+e_3\\
&\qquad \mp \big(t^{-1/3}\chi(t-r)|1-a|^{2/3}X(a)+t^{-1/3}Q_2(1)\big)^7\\
&= (1-\chi(t-r))\big(-\partial_t^2\big)\big(t^{-1/3}Q_2(1)\big)\\
&\qquad \pm \big[\chi(t-r)\big(t^{-1/3}|1-a|^{\frac{2}{3}}X(a)+t^{-1/3}Q_2(1)\big)^7\\
&\qquad \mp \big(t^{-1/3}\chi(t-r)|1-a|^{2/3}X(a)+t^{-1/3}Q_2(1)\big)^7\big]+e_3\\
&=: e_1+e_2+e_3,
\end{aligned}$$

where e_3 denotes those terms for which at least one derivative falls on $\chi(t-r)$. By the definition of χ, we may include a cutoff $(1-\tilde{\chi}(t-r))$ in front of e_2, where $\tilde{\chi}$ localizes to $|t-r| \geq 2C$, i.e., we can write

$$\begin{aligned}
e_2 &= \pm\chi(t-r)\big(t^{-1/3}|1-a|^{2/3}X(a)+t^{-1/3}Q_2(1)\big)^7\\
&\qquad \mp \big(t^{-1/3}|1-a|^{2/3}X(a)\chi(t-r)+t^{-1/3}Q_2(1)\big)^7\\
&= (1-\tilde{\chi}(t-r))\big[\pm\chi(t-r)\big(t^{-1/3}|1-a|^{2/3}X(a)+t^{-1/3}Q_2(1)\big)^7\\
&\qquad \mp \big(t^{-1/3}|1-a|^{2/3}X(a)\chi(t-r)+t^{-1/3}Q_2(1)\big)^7\big].
\end{aligned}$$

We can also write this as $e_2 = (1-\tilde{\chi})\big[u_1^7 - u_2^7\big]$, where we have the pointwise bound

$$|u_1(t,r)| + |u_2(t,r)| \lesssim t^{-1/3}.$$

As for e_3, we begin by collecting all terms in which $X(a)$ is not differentiated. Then with $(\cdots)'$ denoting the operator for which at least one derivative falls on χ, we obtain

$$\begin{aligned}
&\Big(-\partial_t^2 + \partial_r^2 + \frac{2}{r}\partial_r\Big)'\big(t^{-1/3}|1-a|^{2/3}\chi(t-r)\big)\\
&= 2\cdot\frac{1}{3}t^{-4/3}|1-a|^{2/3}\chi'(t-r) - 2\cdot\frac{2}{3}\frac{r}{t^2}\operatorname{sgn}(1-a)|1-a|^{-1/3}\chi'(t-r)t^{-1/3}\\
&\quad + 2\cdot\frac{2}{3}\frac{1}{t}\operatorname{sgn}(1-a)|1-a|^{-1/3}\chi'(t-r)t^{-1/3} - \frac{2}{r}t^{-1/3}|1-a|^{2/3}\chi'(t-r).
\end{aligned}$$

The preceding sum is seen to simplify to

$$2\Big(\frac{1}{t}-\frac{1}{r}\Big)t^{-1/3}\frac{|t-r|^{2/3}}{t^{2/3}}\chi'(t-r) = -\frac{2(t-r)|t-r|^{2/3}\chi'(t-r)}{rt^2},$$

which is one power of t better than expected. For this gain, it is important that $\chi(t-r)$ solve the 1-dimensional wave equation.

The terms in e_3 where one derivative falls on $X(a)$ contribute

$$2t^{-1/3}|1-a|^{2/3}X'(a)(-\partial_t a - \partial_r a)\chi'(t-r) = 2t^{-7/3}|1-a|^{2/3}X'(a)(r-t)\chi'(t-r).$$

This term is localized to the region $|r-t| \lesssim 1$; and since

$$|1-a|^{2/3}X'(a) = Q_1'(a) + |1-a|Q_3'(a) - \frac{1}{3}\operatorname{sign}(1-a)Q_3(a),$$

it is of size $t^{-7/3}$ on that region. The remaining errors $e_{1,2}$ have the same properties, i.e., they are also localized to the region $|r-t| \lesssim 1$ and are of size $t^{-7/3}$.

Hence all these errors are seen to belong to $L_t^1 L_x^2$ for $t \geq 1$, since

$$\|t^{-7/3}(1-\chi(t-r))\|_{L^2(\mathbb{R}^3)} \lesssim t^{-4/3} \in L^1(1,\infty)$$

Thus all these errors beat the scaling. This is an essential feature of our construction.

4 Completing the approximate solution to an exact one

We now attempt to construct an exact solution of the form

$$\tilde{u}(t,r) := u(t,r) + v(t,r),$$

where u is defined in (3.4). The precise theorem is as follows.

Theorem 4.1. *Let u_0 be sufficiently small in the sense of Corollary 2.7, and let $u(t, r)$ be as in* (3.4). *Then for any compactly supported radial initial data*

$$v[1] = (v_0, v_1) \in \dot H^{7/6} \cap \dot H^1(\mathbb{R}^3) \times \dot H^{1/6} \cap L^2(\mathbb{R}^3)$$

sufficiently small with respect to the natural norm, there exists

$$v \in L_t^\infty \dot H^{7/6}(\mathbb{R}^3) \cap L_{t,loc}^\infty \dot H^1(\mathbb{R}^3) \cap S$$

with S any of the Strichartz spaces in Lemma 4.2, and

$$v_t \in L_t^\infty \dot H^{1/6}(\mathbb{R}^3) \cap L_{t,loc}^\infty L^2(\mathbb{R}^3)$$

on $[1, \infty) \times \mathbb{R}^3$ *such that* $\tilde u(t, r) := u(t, r) + v(t, r)$ *solves* (1.1). *Moreover, if*

$$v[1] \in \dot H^s(\mathbb{R}^3) \times \dot H^{s-1}(\mathbb{R}^3), \quad s > \frac{7}{6},$$

then

$$v[t] \in \dot H^s(\mathbb{R}^3) \times \dot H^{s-1}(\mathbb{R}^3) \quad \textit{for all } t \geq 1.$$

The proof of Theorem 4.1 proceeds via a bootstrap argument on the norm $\|v\|_{\dot H^{7/6} \cap \dot H^1}$. More precisely, assuming the solution to exist on an interval $[1, T]$ of regularity $\dot H^{\frac{7}{6}} \cap \dot H^1(\mathbb{R}^3)$, we deduce an a priori bound on a slightly time-weighted version of the preceding norm, where the weight depends on the data but is independent of T. Using a local well-posedness result, one can then let $T \to \infty$. The equation for v is simply the linearized one

$$-v_{tt} + \triangle v \mp 7u^6 v \mp \cdots \mp 7uv^6 \mp v^7 = \sum_{j=1}^{3} e_j. \tag{4.1}$$

At first sight, the natural space to iterate this in seems to be the Strichatz space $\|\cdot\|_S$ at the scaling of $\dot H^{\frac{7}{6}}$, which corresponds for example to the space-time norm $\|\cdot\|_{L_t^6 L_x^{18}}$.

For the sake of completeness, let us recall a class of Strichartz estimates relevant in this context.

Lemma 4.2. *Let u be the free wave propagation of the equation in $\mathbb{R}^{1+3}_{t,x}$*

$$\Box u = h, \quad u[0] = (f, g),$$

where (f, g) are smooth and compactly supported, and h is smooth with compact support on fixed-time slices. Then

$$\|u\|_{L_t^r L_x^s} + \sup_t \|(u, u_t)(t)\|_{\dot H^{7/6}(\mathbb{R}^3) \times \dot H^{1/6}(\mathbb{R}^3)} + \||\nabla|^\alpha u\|_{L_t^p L_x^q} \lesssim \|(f, g)\|_{\dot H^{7/6}(\mathbb{R}^3) \times \dot H^{1/6}(\mathbb{R}^3)} + \||\nabla|^{1/6} h\|_{L_t^1 L_x^2}, \tag{4.2}$$

where $3 < r \le \infty$ *and* $\frac{1}{3r} + \frac{1}{s} = \frac{1}{9}$ *(such as* $r = 6$ *and* $s = 18$*), and* $2 < p \le \infty$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, $\alpha(q) = \frac{2}{q} + \frac{1}{6}$. *By approximation, this extends to solutions in the Duhamel sense for which the right-hand side is finite.*

However, we observe that u is not bounded in $L_t^6 L_x^{18}$ due to a logarithmic divergence in infinite time. Thus a simple minded procedure using Strichartz and Hölder does not apply, and we are required to exploit the fine structure of the function u. In fact, this function lives at lower and lower frequencies as $t \to \infty$. One may then hope to exploit some additional low-frequency control on v coming from energy conservation to gain better control. The above theorem is a consequence of combining the following Proposition 4.3 on local existence with Proposition 4.4, which establishes a priori control of any local solution to (4.1) via a bootstrap argument.

Proposition 4.3. *Let* $T \ge 1$. *Assume that* $v[T]$ *is compactly supported with*

$$\|v[T]\|_{\dot H^{7/6} \times \dot H^{1/6}(\mathbb{R}^3)} \ll 1.$$

Then there exists a solution $v(t)$ *of* (4.1) *on the time-interval* $[T, T+1]$ *with the property that*

$$v \in L_t^\infty \dot H^{7/6}([T, T+1] \times \mathbb{R}^3), \quad v_t \in L_t^\infty \dot H^{1/6}([T, T+1] \times \mathbb{R}^3)$$

with compact support on every time slice $t \times \mathbb{R}^3$, $t \in [T, T+1]$. *Also, if* $v[T] \in \dot H^s(\mathbb{R}^3) \times \dot H^{s-1}(\mathbb{R}^3)$, *then* $v[t] \in \dot H^s(\mathbb{R}^3) \times \dot H^{s-1}(\mathbb{R}^3)$ *for all* $s > 7/6$.

The proof proceeds by a standard iteration; see Section 7. With Proposition 4.3 taken for granted, the main work is then encapsulated in the following result.

Proposition 4.4. *Let* u_0 *be sufficiently small in the sense of Corollary 2.7, and let* $u(t, r)$ *be as in* (3.4). *To be specific, in the notation of Corollary 2.7, we require that* $|q_0| + |\tilde q_1| \le \delta_2^3 \le \delta_1$ *be small. Let* (v, v_t) *be radial. Assume that*

$$v \in L_t^\infty \dot H^{7/6} \cap L_{t,loc}^\infty \dot H^1(\mathbb{R}^3), \quad v_t \in L_t^\infty \dot H^{1/6} \cap L_{t,loc}^\infty L^2(\mathbb{R}^3)$$

solves (4.1) *on* $[1, T] \times \mathbb{R}^3$ *(in the Duhamel sense). Assume further that*

$$\|v[1]\|_{\dot H^{7/6} \cap \dot H^1(\mathbb{R}^3) \times \dot H^{1/6} \cap L^2(\mathbb{R}^3)} \le \delta_1 \ll 1$$

is sufficiently small. Then for any $C > 1$ *sufficiently large (in an absolute sense, independently of* T*) with* $C\delta_1 \ll 1$, *as well as* $\varepsilon = \varepsilon(\delta_2) \ll 1$ *such that*

$$\|v\|_{L_t^6 L_x^{18}([1,T] \times \mathbb{R}^3)} + \sup_{t \in [1,T]} \|v[t]\|_{\dot H^{7/6} \cap t^\varepsilon \dot H^1(\mathbb{R}^3) \times \dot H^{1/6} \cap t^\varepsilon L^2(\mathbb{R}^3)} \le C\delta_1,$$

we have

$$\|v\|_{L_t^6 L_x^{18}([1,T]\times\mathbb{R}^3)} + \sup_{t\in[1,T]} \|v[t]\|_{\dot{H}^{7/6}\cap t^{\varepsilon}\dot{H}^1(\mathbb{R}^3)\times \dot{H}^{1/6}\cap t^{\varepsilon}L^2(\mathbb{R}^3)} \le \frac{C}{2}\delta_1.$$

The proof of Proposition 4.4 is accomplished in the following two subsections. We henceforth assume that $v(t,\cdot)$ satisfies the assumptions of the proposition.

4.1 Energy control.

We note that

$$\frac{d}{dt}\int_{\mathbb{R}^3}\Big[\frac{1}{2}\big(v_t^2 + |\nabla v|^2\big) \pm \frac{7}{2}u^6 v^2 \pm \cdots \pm uv^7 \pm \frac{1}{8}v^8\Big]\,dx = -\int_{\mathbb{R}^3}\Big[\sum_{j=1}^{3} e_j v_t \mp 21 u_t u^5 v^2 \mp \cdots \mp u_t v^7\Big]\,dx$$

Integrating from time 1 to time t, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^3}\Big[\frac{1}{2}\big(v_t^2 + |\nabla v|^2\big) \pm \frac{7}{2}u^6 v^2 \pm \cdots \pm uv^7 \pm \frac{1}{8}v^8\Big](t,\cdot)\,dx \\ &\qquad - \int_{\mathbb{R}^3}\Big[\frac{1}{2}\big(v_t^2 + |\nabla v|^2\big) \pm \frac{7}{2}u^6 v^2 \pm \cdots + uv^7 \pm \frac{1}{8}v^8\Big](1,\cdot)\,dx \\ &= \int_1^t \int_{\mathbb{R}^3}\Big[\sum_{j=1}^{3} -e_j v_t \pm 21 u_t u^5 v^2 \pm \cdots \pm u_t v^7\Big]\,dxdt. \end{aligned} \tag{4.3}$$

Our goal is to deduce the bound

$$\sup_{t\in[1,T]} t^{-\varepsilon}\|\nabla_{t,x}v(t)\|_{L_x^2} \ll C\delta_1$$

From the estimate $\sup_{t\ge 1}|u(t,r)| \lesssim \delta_2^3\, r^{-1/3}$ (cf. (3.1)), we conclude that

$$\int_{\mathbb{R}^3} u^6 v^2\,dx \lesssim \delta_2^{18}\int_{\mathbb{R}^3} r^{-2}v^2\,dx \lesssim \delta_2^{18}\,\|v\|_{\dot{H}^1}^2.$$

Since $\delta_2 \ll 1$, this term is absorbed by the principal term $\int_{\mathbb{R}^3}\frac{1}{2}\big(v_t^2 + |\nabla v|^2\big)\,dx$. In the defocussing case, this term can be removed by positivity. Further, observe that for $j\in[1,5]$, the pointwise bound $\sup_{r>0}|u(t,r)| \lesssim \delta_2^3\, t^{-1/3}$ implies

$$\|u^j v^{8-j}\|_{L_x^1} \lesssim \delta_2\, t^{-\frac{1}{9}}\,\|u\|_{L_x^{9+}}^{j-\frac{1}{3}}\,\|v\|_{L_x^p}^{8-j}, \tag{4.4}$$

where

$$1 = \frac{j-\frac{1}{3}}{9+} + \frac{8-j}{p}$$

This implies that

$$\frac{81}{13}- \leq p \leq \frac{189}{25}- \ \Rightarrow\ 6 < p < 9.$$

Recall the embeddings $\dot H^1(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$, $\dot H^{7/6}(\mathbb{R}^3) \subset L^9(\mathbb{R}^3)$. With $0 < \alpha < 1$ determined by

$$\frac{1}{p} = \frac{\alpha}{6} + \frac{1-\alpha}{9},$$

Sobolev's embedding and Hölder's inequality applied to (4.4) yield

$$\|u^j v^{8-j}\|_{L^1_x} \lesssim \delta_2\, t^{-\frac{1}{9}} \|u\|_{L_x^{9+}}^{j-\frac{1}{3}} \|v\|_{\dot H^1}^{\alpha(8-j)} \|v\|_{\dot H^{7/6}}^{(1-\alpha)(8-j)},$$

which we rewrite in the form

$$\|u^j v^{8-j}(t,\cdot)\|_{L^1_x} \lesssim \delta_2\, t^{\varepsilon\alpha(8-j)-\frac{1}{9}} \big(t^{-\varepsilon}\|v(t,\cdot)\|_{\dot H^1}\big)^{\alpha(8-j)} \|v(t,\cdot)\|_{\dot H^{7/6}}^{(1-\alpha)(8-j)}.$$

If we now choose ε so small that $\varepsilon\alpha(8-j) - \frac{1}{9} < 0$, we conclude that

$$\begin{aligned}\sup_{t\in[1,T]} \|u^j v^{8-j}(t,\cdot)\|_{L^1_x} &\lesssim \delta_2 \big(\sup_{t\in[1,T]} t^{-\varepsilon}\|v(t,\cdot)\|_{\dot H^1}\big)^{\alpha(8-j)} \|v(t,\cdot)\|_{\dot H^{7/6}}^{(1-\alpha)(8-j)} \\ &\lesssim \delta_2 \big(C\delta_1\big)^{8-j}\end{aligned}$$

We also note that

$$\|v^8(t,\cdot)\|_{L^1_x} \leq \|v(t,\cdot)\|_{\dot H^1}^2 \big(\sup_{t\in[1,T]} \|v(t,\cdot)\|_{\dot H^{7/6}}^6\big) \lesssim \|v(t,\cdot)\|_{\dot H^1}^2 (C\delta_1)^6,$$

where we have again used Hölder's inequality as well as the Sobolev embedding; so this term can again be absorbed by the principal term $\int_{\mathbb{R}^3} \frac{1}{2}\big(v_t^2 + |\nabla v|^2\big)\, dx$.

It remains to control the source terms on the right of (4.3). We start by estimating the contributions of the terms involving the errors e_j. First, we have

$$\int_1^t \int_{\mathbb{R}^3} e_1 v_t\, dxdt = \int_1^t \int_{\mathbb{R}^3} (1-\tilde\chi)[u_1^7 - u_2^7] v_t\, dxdt.$$

Recall that $1-\tilde\chi$ localizes to the strip $|t-r| \leq 2C$. Thus since $u_{1,2}^7 = O(t^{-7/3})$, we infer $\big\|(1-\tilde\chi)[u_1^7 - u_2^7](t,\cdot)\big\|_{L^2_x} \lesssim \delta_2^7\, t^{-4/3}$, and so

$$\Big|\int_1^t \int_{\mathbb{R}^3} e_1 v_t\, dxdt\Big| \lesssim \delta_2^7 \int_1^t s^{\varepsilon-\frac{4}{3}}\, ds \big(\sup_{t\in[1,T]} t^{-\varepsilon}\|v_t(t,\cdot)\|_{L^2_x}\big) \lesssim \delta_2^7\, C\delta_1.$$

The contributions of the terms involving $e_{2,3}$ are handled identically.

Next, consider the contributions of the terms

$$u_t u^5 v^2, \quad u_t v^7, \tag{4.5}$$

the intermediate terms in the space-time integral in (4.3) being handled similarly. The first of these terms is estimated as follows. Considering the region $|t-r| \le t/2$, from the formula for $u(t,r)$, we obtain

(4.6)
$$\begin{aligned} u_t &= \Big[-\frac{1}{3} t^{-4/3} \chi(t-r)|1-a|^{2/3} X(a) - \frac{1}{3} t^{-4/3} Q_2(1) \Big] + t^{-1/3} \chi'(t-r)|1-a|^{2/3} X(a) \\ &\quad + \frac{2}{3} \frac{r}{t^2} \chi(t-r)|t-r|^{-1/3} \mathrm{sign}(t-r) X(a) - t^{-1/3} \chi(t-r)|1-a|^{2/3} \frac{r}{t^2} X'(a) \\ &=: A_1 + A_2 + A_3 + A_4 \end{aligned}$$

We note (always restricting to $|r-t| \le t/2$) that $|A_1| + |A_4| \lesssim t^{-4/3}$, and so

$$\int_{|t-r| \le \frac{t}{2}} [|A_1| + |A_4|] u^5 v^2 \, dx \lesssim \delta_2^6 \, t^{-1} \int_{\mathbb{R}^3} r^{-2} v^2 \, dx \lesssim \delta_2^6 \, t^{-1} \| \nabla_x v \|_{L_x^2}^2 .$$

For the contribution of the term A_3, we use the fact that by radiality of v, $|v(t,r)| \lesssim r^{-1/2} \|v\|_{\dot{H}^1}$, which then gives

$$\begin{aligned} \int_{|t-r| \le \frac{t}{2}} |A_3| u^5 v^2 \, dx &\lesssim \delta_2^6 \|v\|_{\dot{H}^1}^2 t^{-1} \int_{|t-r| \le \frac{t}{2}} r^{-5/3} (t-r)^{-1/3} r^{-1} \, r^2 dr \\ &\lesssim \delta_2^6 \, \|v\|_{\dot{H}^1}^2 t^{-1} \int_{|t-r| \le \frac{t}{2}} r^{-2/3} (t-r)^{-1/3} dr \lesssim \delta_2^6 \, \|v\|_{\dot{H}^1}^2 t^{-1} \end{aligned}$$

Finally, for the contribution of the term A_2, we have

$$\int_{|t-r| \le \frac{t}{2}} |A_2| u^5 v^2 \, dx \lesssim \delta_2^6 \, t^{-8/3} \int_{|t-r| \le C} v^2 \, dx \lesssim \delta_2^6 \, t^{-4/3} \|v\|_{\dot{H}^1}^2 ,$$

where we have used Hölder's inequality and Sobolev's embedding to bound

$$\int_{|t-r| \le C} v^2 \, dx \lesssim t^{4/3} \|v\|_{\dot{H}^1}^2 .$$

It follows that

$$\int_{|t-r| \le \frac{t}{2}} |u_t u^5 v^2| \, dx \lesssim \delta_2^6 \, t^{-1} \|v\|_{\dot{H}^1}^2 .$$

On the other hand, for the region $|t-r| > t/2$ (assuming $t \gg 1$, as we may), we have $|u_t| \lesssim \delta_2 \, t^{-1} r^{-1/3}$; and so we conclude that

$$\big\| u_t u^5 v^2 \big\|_{L_x^1(|t-r| > t/2)} \lesssim \delta_2^6 \, t^{-1} \int_{\mathbb{R}^3} r^{-2} v^2 \, dx \lesssim \delta_2^6 \, t^{-1} \|\nabla_x v\|_{L_x^2}^2 ,$$

where we have used Hardy's inequality in dimension $n = 3$. It follows that

$$\int_1^t \int_{\mathbb{R}^3} |u_t u^5| v^2 \, dx dt \lesssim \delta_2^6 \Big(\int_1^t s^{2\varepsilon - 1} \, ds \Big) \big(\sup_{t \in [1,T]} t^{-\varepsilon} \|\nabla_{t,x} v\|_{L_x^2} \big)^2 \lesssim t^{2\varepsilon} \varepsilon^{-1} \delta_2^6 (C\delta_1)^2 .$$

For the second term in (4.5) above, we have in the region $|t-r| > t/2$

$$\big\|u_t v^7\big\|_{L^1_x(|t-r|>t/2)} \lesssim \delta_2\, t^{-4/3} \|\nabla_x v\|^4_{L^2_x} \|\,|\nabla|^{7/6} v\|^3_{L^2_x},$$

which gives

$$\int_1^t \int_{|s-r|>\frac{s}{2}} |u_t v^7|\, dxds \lesssim \delta_2 (C\delta_1)^3 \Big(\int_0^t s^{4\varepsilon-\frac{4}{3}}\, ds\Big) \big(\sup_{t\in[1,T]} t^{-\varepsilon} \|\nabla_x v(t,\cdot)\|_{L^2_x}\big)^4$$
$$\lesssim \delta_2 (C\delta_1)^7.$$

For the region $|t-r| \le t/2$, we invoke (4.6) as well as the inequality

$$|v(t,r)| \lesssim r^{-1/2} \|v\|_{\dot H^1}$$

to obtain

$$|u_t v^7| \lesssim t^{-3/2} \|v\|_{\dot H^1} v^6, \quad |t-r| \le \frac{t}{2},$$

whence

$$\int_1^t \int_{|s-r|\le s/2} |u_t v^7|\, dxds \lesssim \delta_2 \int_1^t s^{-3/2} s^{7\varepsilon}\, ds \big(\sup_{t\in[1,T]} t^{-\varepsilon} \|v(t,\cdot)\|_{\dot H^1}\big)^7 \lesssim \delta_2 (C\delta_1)^6.$$

Combining the preceding bounds used to estimate the right hand side of (4.3) and choosing $\delta_2 \le \delta_1$ sufficiently small (which can be done independently of ε), we get

$$\sup_{t\in[1,T]} t^{-\varepsilon} \|\nabla_{t,x} v\|_{L^2_x} \ll C\delta_1,$$

as required.

4.2 Critical norm control. Here, we return to (4.1), but this time to control the scaling invariant norm

$$\|v\|_S := \|v\|_{L^6_t L^{18}_x([1,T]\times\mathbb{R}^3)} + \sup_{t\in[1,T]} \|\,|\nabla|^{7/6} v\|_{L^2_x} + \sup_{t\in[1,T]} \|\,|\nabla|^{1/6} \partial_t v\|_{L^2_x}.$$

From Duhamel's principle, we have

$$\begin{aligned} \big\|v\big\|_S &\lesssim \big\|\,|\nabla|^{1/6}\big(u^6 v\big)\big\|_{L^1_t L^2_x} + \cdots + \big\|\,|\nabla|^{1/6}\big(u v^6\big)\big\|_{L^1_t L^2_x} + \|\,|\nabla|^{1/6}\big(v^7\big)\|_{L^1_t L^2_x} \\ &\quad + \sum_{j=1}^3 \big\|\,|\nabla|^{1/6} e_j\big\|_{L^1_t L^2_x} + \|v[1]\|_{\dot H^{7/6}\times \dot H^{1/6}}. \end{aligned} \tag{4.7}$$

By the explicit form of the errors e_j derived in Section 3,

$$\sum_{j=1}^3 \big\|\,|\nabla|^{1/6} e_j\big\|_{L^1_t L^2_x} \lesssim \delta_2^3;$$

and by assumption, $\|v[1]\|_{\dot H^{7/6}\times \dot H^{1/6}} \le \delta_1$. We now consider the more subtle terms

$$\big\|\,|\nabla|^{1/6}\big(u^6 v\big)\big\|_{L^1_t L^2_x}, \quad \big\|\,|\nabla|^{1/6}\big(uv^6\big)\big\|_{L^1_t L^2_x}, \tag{4.8}$$

the remaining intermediate power interactions being handled similarly (the term v^7 will be dealt with at the end). The ideas involved in estimating these products are as follows.

- The main contribution is expected to come from the *diagonal interactions*, i.e., the situation in which the frequencies of all factors are about the same.
- The factors u live essentially at low frequency.
- Due to energy control, the extra derivative $|\nabla|^{\frac{1}{6}}$ should help us gain from low frequencies.

We denote the "projection" onto frequencies $|\xi| \le \rho$ by $P_{\le\rho}$. As usual, this is not a true projection but rather effected by summing the Littlewood-Paley smooth frequency localizers up to that scale. In particular, $P_{\le\rho} f = f * \varphi_\rho$, where φ is a Schwartz function with $\int \varphi = 1$ and $\varphi_\rho(x) = \rho^3\varphi(\rho x)$. At the expense of allowing for rapidly decaying tails in the frequency localization (which is harmless), we may also assume that φ is compactly supported. Thus,

$$\begin{aligned} P_{\ge t^{-\sigma}}\, u(x) &= \int_{\mathbb{R}^3} (u(x) - u(x-y))\varphi_{t^{-\sigma}}(y)\, dy, \\ |P_{\ge t^{-\sigma}}\, u(x)| &\le \int_{\mathbb{R}^3} |u(x) - u(x - t^\sigma y)||\varphi(y)|\, dy \\ &\le t^\sigma \int_0^1 \int_{\mathbb{R}^3} |\nabla u(x - ht^\sigma y)||y||\varphi(y)|\, dy\, dh, \end{aligned}$$

which in particular implies $\big\|P_{\ge t^{-\sigma}}\, u(t,\cdot)\big\|_{L^{18}_x} \lesssim t^\sigma \|\partial_r u(t,\cdot)\|_{L^{18}_x}$. Since

$$\begin{aligned} u_r(t,r) = -\chi'(t-r)t^{-1/3}(1-a)^{2/3}X(a) - \frac{2}{3}\chi(t-r)t^{-1}(t-r)^{-1/3}X(a) \\ + \chi(t-r)t^{-4/3}(1-a)^{2/3}X'(a), \end{aligned}$$

it follows that $\big\|P_{\ge t^{-\sigma}}\, u(t,\cdot)\big\|_{L^{18}_x} \lesssim \delta_2^3\, t^{\sigma+\frac{1}{9}-\frac{1}{3}}$, which, for $\sigma > 0$ sufficiently small, is of course better than L^6_t.

Returning to (4.8), we now split

$$|\nabla|^{1/6}\big(u^6 v\big) = |\nabla|^{1/6}\big((P_{<t^{-\sigma}}\, u)^6 v\big) + |\nabla|^{1/6}\big(u^6 v\big)', \tag{4.9}$$

where the second term on the right-hand side is defined via this relation. We claim that this term can then be bounded in $L^1_t L^2_x$. In fact, by the fractional Leibnitz rule, we can schematically estimate it at a fixed time by

$$\begin{aligned}\big\| |\nabla|^{1/6}\big((P_{\geq t^{-\sigma}}u)u^5 v\big)(t,\cdot)\big\|_{L^2_x} &\lesssim \big\| |\nabla|^{1/6}P_{\geq t^{-\sigma}}u(t,\cdot)\big\|_{L^{18}_x}\big\|u(t,\cdot)\big\|^5_{L^{18}_x}\big\|v(t,\cdot)\big\|_{L^6_x}\\ &+\big\|P_{\geq t^{-\sigma}}u(t,\cdot)\big\|_{L^{18}_x}\big\| |\nabla|^{1/6}u(t,\cdot)\big\|_{L^{18}_x}\big\|u(t,\cdot)\big\|^4_{L^{18}_x}\big\|v(t,\cdot)\big\|_{L^6_x}\\ &+\big\|P_{\geq t^{-\sigma}}u(t,\cdot)\big\|_{L^{18}_x}\big\|u(t,\cdot)\big\|^5_{L^{18}_x}\big\| |\nabla|^{1/6}v(t,\cdot)\big\|_{L^6_x}.\end{aligned}\tag{4.10}$$

To estimate the L^1_t-norm of the right hand sides, we use the energy bound derived previously:

$$\begin{aligned}&\Big\|\big\| |\nabla|^{1/6}P_{\geq t^{-\sigma}}u(t,\cdot)\big\|_{L^{18}_x}\big\|u(t,\cdot)\big\|^5_{L^{18}_x}\big\|v(t,\cdot)\big\|_{L^6_x}\Big\|_{L^1_t[1,T]}\\ &\qquad\lesssim \delta_2^3\big\|t^{\sigma+\frac{1}{9}+\varepsilon-\frac{1}{3}}\big\|_{L^{6-}_t}\big\|u(t,\cdot)\big\|^5_{L^{6+}_tL^{18}_x}\big\|t^{-\varepsilon}v(t,\cdot)\big\|_{L^\infty_tL^6_x([1,T]\times\mathbb{R}^3)}\\ &\qquad\lesssim \delta_2^{18}C\delta_1,\end{aligned}$$

which is $\ll C\delta_1$. The remaining terms in (4.10) above are handled similarly, and so we have reduced the problem to that of estimating the first term in (4.9):

$$\big\| |\nabla|^{1/6}\big((P_{<t^{-\sigma}}u)^6 v\big)(t,\cdot)\big\|_{L^1_tL^2_x([1,T]\times\mathbb{R}^3)}$$

Using the fractional Leibnitz rule, we bound this by

$$\begin{aligned}&\big\| |\nabla|^{1/6}\big((P_{<t^{-\sigma}}u)^6 v\big)(t,\cdot)\big\|_{L^1_tL^2_x([1,T]\times\mathbb{R}^3)}\\ &\lesssim \big\|t^{\varepsilon}\| |\nabla|^{1/6}P_{<t^{-\sigma}}u(t,\cdot)\|_{L^{18}_x}\|P_{<t^{-\sigma}}u(t,\cdot)\|^5_{L^{18}_x}\big\|_{L^1_t}\big\|t^{-\varepsilon}\|v(t,\cdot)\|_{\dot H^1}\big\|_{L^\infty_t}\\ &\qquad\qquad+\big\|\|P_{<t^{-\sigma}}u(t,\cdot)\|^6_{L^{18+}_x}\| |\nabla|^{1/6}v(t,\cdot)\|_{L^{6-}_x}\big\|_{L^1_t}.\end{aligned}\tag{4.11}$$

To estimate the first term on the right, we use the fact that

$$\| |\nabla|^{1/6}P_{<t^{-\sigma}}u(t,\cdot)\|_{L^{18}_x}\lesssim t^{-\sigma/6}\|u(t,\cdot)\|_{L^{18}_x},$$

and obtain

$$\begin{aligned}&\big\|t^{\varepsilon}\| |\nabla|^{1/6}P_{<t^{-\sigma}}u(t,\cdot)\|_{L^{18}_x}\|P_{<t^{-\sigma}}u(t,\cdot)\|^5_{L^{18}_x}\big\|_{L^1_t}\big\|t^{-\varepsilon}\|v(t,\cdot)\|_{\dot H^1}\big\|_{L^\infty_t}\\ &\qquad\qquad\lesssim \big\|t^{\frac{\varepsilon}{6}-\frac{\sigma}{36}}\|u(t,\cdot)\|_{L^{18}_x}\big\|^6_{L^6_t}\big\|t^{-\varepsilon}\|v(t,\cdot)\|_{\dot H^1}\big\|_{L^\infty_t}\lesssim \delta_2^{18}C\delta_1\end{aligned}$$

For the second term on the right in (4.11), the idea is that we can place $|\nabla|^{1/6}v$ into L^{6-}_x while paying a small power of t, while placing the low frequency factors into L^{18+}_x, gaining a bit in t^{-1}. Specifically, from Sobolev's embedding $\dot H^{5/6}(\mathbb{R}^3)\subset L^{9/2}(\mathbb{R}^3)$, we infer that

$$\big\| |\nabla|^{1/6}v(t,\cdot)\big\|_{L^{9/2}_x}\lesssim t^{\varepsilon}\big(t^{-\varepsilon}\|\nabla_{t,x}v(t,\cdot)\|_{L^2_x}\big),$$

while Bernstein's inequality implies that

$$\big\|P_{<t^{-\sigma}}\, u(t,\cdot)\big\|_{L_x^{108/5}} \lesssim t^{-\sigma/36}\|u(t,\cdot)\|_{L_x^{18}}.$$

Since ε can be made small independently of σ above, and in particular we may assume $\varepsilon < \sigma/6$, we then get

$$\begin{aligned}\big\|(P_{<t^{-\sigma}}\, u)^6\, |\nabla|^{1/6} v\big\|_{L_t^1 L_x^2([1,T]\times\mathbb{R}^3)} &\lesssim \big\|t^{\varepsilon/6} P_{<t^{-\sigma}}\, u\big\|^6_{L_t^6 L_x^{108/5}} \big\|t^{-\varepsilon}\, |\nabla|^{1/6} v\big\|_{L_t^\infty L_x^{9/2}} \\ &\lesssim \delta_2^{18} \sup_t \big(t^{-\varepsilon}\|\nabla_{t,x} v(t,\cdot)\|_{L_x^2}\big) \lesssim C\delta_1\delta_2^{18}.\end{aligned}$$

The second term in (4.8) is handled similarly: we split

$$\begin{aligned}\|\,|\nabla|^{1/6}\big(uv^6\big)\|_{L_t^1 L_x^2([1,T]\times\mathbb{R}^3)} \le\ & \|\,|\nabla|^{1/6}\big((P_{\ge t^{-\sigma}}\, u)v^6\big)\|_{L_t^1 L_x^2([1,T]\times\mathbb{R}^3)} \\ &+ \|\,|\nabla|^{1/6}\big((P_{<t^{-\sigma}}\, u)v^6\big)\|_{L_t^1 L_x^2([1,T]\times\mathbb{R}^3)}\end{aligned} \tag{4.12}$$

The first term on the right-hand side is estimated by

$$\begin{aligned}&\|\,|\nabla|^{1/6}\big((P_{\ge t^{-\sigma}}\, u)v^6\big)\|_{L_t^1 L_x^2([1,T]\times\mathbb{R}^3)} \\ &\quad\lesssim \big\|t^{\varepsilon}\|\,|\nabla|^{1/6} P_{\ge t^{-\sigma}}\, u\|_{L_x^{18}} \|v\|^5_{L_x^{18}}\big\|_{L_t^1([1,T])} \sup_{t\in[1,T]} t^{-\varepsilon}\|v(t,\cdot)\|_{\dot H^1} \\ &\qquad + \big\|\|P_{\ge t^{-\sigma}} u\|_{L_x^{18}} \|v\|^5_{L_x^{18}}\big\|_{L_t^1([1,T])} \sup_{t\in[1,T]} \|\,|\nabla|^{1/6} v(t,\cdot)\|_{\dot H^1} \\ &\quad\lesssim \delta_2^3 \|t^{\varepsilon+\sigma+\frac19-\frac13}\|_{L_t^6([1,T])} \|v\|^5_{L_t^6 L_x^{18}} \sup_{t\in[1,T]} t^{-\varepsilon}\|v(t,\cdot)\|_{\dot H^1} \\ &\qquad + \delta_2^3 \|t^{\sigma+\frac19-\frac13}\|_{L_t^6([1,T])} \|v\|^5_{L_t^6 L_x^{18}} \sup_{t\in[1,T]} \|v(t,\cdot)\|_{\dot H^{7/6}},\end{aligned}$$

and so we can bound the last two terms by $\lesssim \delta_2^3(C\delta_1)^6$. To handle the second term on the right-hand side of (4.12), we observe that

$$\begin{aligned}&\|\,|\nabla|^{1/6}\big((P_{<t^{-\sigma}}\, u)v^6\big)\|_{L_t^1 L_x^2([1,T]\times\mathbb{R}^3)} \\ &\quad\lesssim \|t^{\varepsilon}\, |\nabla|^{1/6} P_{<t^{-\sigma}}\, u\|_{L_t^6 L_x^{18}([1,T]\times\mathbb{R}^3)} \|v\|^5_{L_t^6 L_x^{18}([1,T]\times\mathbb{R}^3)} \sup_{t\in[1,T]} t^{-\varepsilon}\|v(t,\cdot)\|_{\dot H^1} \\ &\qquad + \|t^{\varepsilon} P_{<t^{-\sigma}}\, u\|_{L_t^6 L_x^\infty([1,T]\times\mathbb{R}^3)} \|v\|^5_{L_t^6 L_x^{18}([1,T]\times\mathbb{R}^3)} \sup_{t\in[1,T]} t^{-\varepsilon}\|\,|\nabla|^{1/6} v(t,\cdot)\|_{L_x^{9/2}}.\end{aligned}$$

The first product on the right-hand side can be estimated by

$$\lesssim \delta_2^3 \|t^{\varepsilon-\frac{\sigma}{6}-\frac16}\|_{L_t^6([1,T]\times\mathbb{R}^3)} (C\delta_1)^6 \lesssim \delta_2^3 (C\delta_1)^6$$

For the second term above, we infer from Bernstein's inequality that

$$\|t^{\varepsilon} P_{<t^{-\sigma}}\, u\|_{L_t^6 L_x^\infty([1,T]\times\mathbb{R}^3)} \lesssim \delta_2^3 \|t^{\varepsilon-\frac{\sigma}{6}-\frac16}\|_{L_t^6([1,T])} \lesssim \delta_2^3,$$

and so we obtain

$$\|t^{\varepsilon}P_{<t^{-\sigma}}u\|_{L_t^6L_x^\infty([1,T]\times\mathbb{R}^3)}\|v\|^5_{L_t^6L_x^{18}([1,T]\times\mathbb{R}^3)}\sup_{t\in[1,T]} t^{-\varepsilon}\|\,|\nabla|^{\frac{1}{6}}v(t,\cdot)\|_{L_x^{9/2}} \lesssim \delta_2^3(C\delta_1)^6.$$

This concludes the estimate for the second term in (4.8).

To complete the bootstrap for the critical Strichartz norm, we also need to bound the contribution of the pure power term v^7 in (4.7). This we do by using

$$\|\,|\nabla|^{1/6}(v^7)\|_{L_t^1L_x^2([1,T]\times\mathbb{R}^3)} \lesssim \|v\|^6_{L_t^6L_x^{18}([1,T]\times\mathbb{R}^3)}\|v\|_{L_t^\infty\dot H^{7/6}([1,T]\times\mathbb{R}^3)} \lesssim (C\delta_1)^7$$

All of the preceding bounds are $\ll C\delta_1$, provided we pick $\delta_1 \geq \delta_2^3$ sufficiently small. This completes the bootstrap and hence the proof of the proposition.

4.3 The proofs of Theorems 4.1, 1.1, and 1.3. Theorem 4.1 follows from Proposition 4.4 by the standard bootstrap argument; indeed, we may initially take the constant C as large as we like, depending on the solution itself, the finiteness of the constant being guaranteed by the local well-posedness as in Proposition 4.3. Then the constant can be lowered until it reaches some large but absolute size independent of the time of existence.

Theorem 1.1 follows by taking the solution $u+v$ to (1.1) constructed in Theorem 4.1. The data (f,g) are equal to $((u+v)(1,\cdot),(u+v)_t(1,\cdot))$. The infinite critical norm is a consequence of the fact that (v,v_t) has finite critical norm, but (u,u_t) is given by (3.4). The finiteness of $\dot H^s\times\dot H^{s-1}$ for $s>7/6$ is a result of the asymptotic decay of $|u(t,r)|\sim r^{-1/3}$ (or $|u(t,r)|\sim r^{-4/3}$ non-generically) and $|u_t(t,r)|\sim r^{-4/3}$ as $r\to\infty$, respectively. So u lies in these spaces, and the perturbation v does so by construction. Also, the facts that

$$\lim_{t\to\infty}\|(u+v)(t,\cdot)\|_{\dot H^{7/6}(K_t)}=\infty,\ \|(u+v)(t,\cdot)\|_{\dot H^1(K_t)}\sim ct^{1/6},\, c\neq 0,$$

follow from the corresponding asymptotics of u combined with the bounds on v.

The stability claimed by the theorem is a result of the fact that the perturbation v belongs to an open set in the norms of Theorem 4.1.

Theorem 1.3 follows from Theorem 4.1 by truncation. Indeed, given M as in (1.6), we choose R so large that the data (f,g) as in Theorem 1.1 when restricted to $\{|x|<R\}$ have critical norm exceeding M or any other large constant. The theorem then follows by finite propagation speed, rescaling, and the fact that we may make the Strichartz norms of u large provided we integrate over a sufficiently large time-interval. For this theorem, it is essential to note that blowup for (1.1) can only occur at the origin, since we are dealing with the radial problem and there is a pointwise a priori bound for $r>0$ for all times $t>0$ in the defocussing case as a result of the Strauss estimate and the positive definite conserved energy for the defocussing equation (1.1). This is the reason that we restrict to the defocussing equation here.

5 Larger global solutions in the defocussing case

In this section, we revisit the ODE theory from Section 2 in the defocussing case. More precisely, we wish to exploit the flexibility of Corollary 2.7 with regard to the choice of the parameter $\tilde{q}_1$. While we match the outside solution with the inside one through the connection condition $\tilde{q}_2 = q_2$, which ensures continuity, the choice of $\tilde{q}_1$ is arbitrary. In Corollary 2.7, we still require $\tilde{q}_1$ to be small, since at that point we had only constructed solutions in $a > 1$ assuming both $\tilde{q}_1$ and $\tilde{q}_2$ to be small.

We now proceed to show that solutions of the ODE (2.1) exist in $a > 1$ for small $\tilde{q}_2$, but large $\tilde{q}_1$. We start by proving an analogue of Lemma 2.4 near $a = 1$. We then extend the solution to all of $a > 1$, which depends crucially on the defocussing character of the equation. For technical reasons, the expression in (5.1) differs from the one in Lemma 2.4. To be specific, instead of the power $(a-1)^{7/3}$, we use $(a-1)^{4/3}$, absorbing the factor $(a-1)$ into $\tilde{Q}_3$.

Lemma 5.1. *There exists $\varepsilon > 0$ small such that given $\tilde{q}_2 \in (-\varepsilon, 0) \cup (0, \varepsilon)$ and $\tilde{q}_1 \geq 1$, for some absolute sufficiently small constant $c > 0$, there exists a unique solution $Q(a)$ of (2.1) on $[1, 1+\ell]$ for $\ell = c|\tilde{q}_2|\tilde{q}_1^{-4/3}$ of the form*

$$Q(a) = (a-1)^{\frac{2}{3}}\tilde{Q}_1(a) + \tilde{Q}_2(a) + (a-1)^{4/3}\tilde{Q}_3(a) \tag{5.1}$$

with $\tilde{Q}_1, \tilde{Q}_2, \tilde{Q}_3 \in C^\infty([1, 1+\ell])$ and

$$\begin{aligned} &\tilde{Q}_1(a) = \tilde{q}_1(1 + O(a-1)), \quad \tilde{Q}_2(a) = \tilde{q}_2(1 + O(a-1)), \\ &\tilde{Q}_3(a) = \tilde{q}_1 O(1), \end{aligned} \tag{5.2}$$

where the $O(\cdot)$ terms are C^∞ functions in $a \in [1, 1+\ell]$. Also,

$$|\tilde{Q}_1(a)| \geq \frac{\tilde{q}_1}{2}, \quad |\tilde{Q}_2(a)| \leq C|\tilde{q}_2|, \quad |\tilde{Q}_3(a)| \leq C\tilde{q}_1, \tag{5.3}$$

uniformly in $a \in [1, 1+\ell]$ and with some absolute constant C. Finally, there exists $a_* \in (1, 1+\ell]$ such that

$$|Q(a_*)| \simeq |\tilde{q}_2|^{2/3}\tilde{q}_1^{1/9} \tag{5.4}$$

In particular, this can be made arbitrarily large by making $\tilde{q}_1$ sufficiently large.

Proof. We refer to the proof of Lemma 2.2. We start from the representation

$$Q(a) = (a-1)^{\frac{2}{3}}\tilde{Q}_1(a) + \tilde{Q}_2(a) + (a-1)^{4/3}\tilde{Q}_3(a),$$

where we furthermore assume the structure

$$\begin{aligned} \tilde{Q}_1(a) &= \tilde{q}_1(1 + O(a-1)), \\ \tilde{Q}_2(a) &= \tilde{q}_2(1 + O(a-1)), \\ \tilde{Q}_3(a) &= \tilde{q}_1 O(1). \end{aligned} \tag{5.5}$$

We then obtain $\tilde{Q}_{1,2,3}(a)$ as fixed points of the following system (see (2.19)):

$$\begin{aligned} \tilde{Q}_1(a) &= \tilde{q}_1 a^{-1} + (a-1)^{-2/3}\int_1^a G(a,b)(b-1)^{2/3} N_1(b, \tilde{Q}_1, \tilde{Q}_2, \tilde{Q}_3)\,db, \\ \tilde{Q}_2(a) &= \tilde{q}_2 2^{-2/3}\varphi_2(a) + \int_1^a G(a,b) N_0(b, \tilde{Q}_1, \tilde{Q}_2, \tilde{Q}_3)\,db, \\ \tilde{Q}_3(a) &= (a-1)^{-4/3}\int_1^a G(a,b)(b-1)^{\frac{4}{3}} N_2(b, \tilde{Q}_1, \tilde{Q}_2, \tilde{Q}_3)\,db. \end{aligned} \tag{5.6}$$

The Green function is the one from (2.15), viz,

$$G(a,b) = g_1(a,b) + (b-1)^{\frac{2}{3}}(a-1)^{-2/3} g_2(a,b),$$

and the source functions $N_k(b, \tilde{Q}_1, \tilde{Q}_2, \tilde{Q}_3)$, $k = 0, 1, 2$, can be written schematically as

$$\begin{aligned} N_1(b, \tilde{Q}_1, \tilde{Q}_2, \tilde{Q}_3) &= \sum_{2m_1+4m_3 \equiv 2(3)} C_{m_1,m_2,m_3}\,(b-1)^{\frac{2m_1+4m_3-2}{3}}\tilde{Q}_1^{m_1}(b)\tilde{Q}_2^{m_2}(b)\tilde{Q}_3^{m_3}(b), \\ N_0(b, \tilde{Q}_1, \tilde{Q}_2, \tilde{Q}_3) &= \sum_{2m_1+4m_3 \equiv 0(3)} C_{m_1,m_2,m_3}\,(b-1)^{\frac{2m_1+4m_3}{3}}\tilde{Q}_1^{m_1}(b)\tilde{Q}_2^{m_2}(b)\tilde{Q}_3^{m_3}(b), \\ N_2(b, \tilde{Q}_1, \tilde{Q}_2, \tilde{Q}_3) &= \sum_{2m_1+4m_3 \equiv 1(3)} C_{m_1,m_2,m_3}\,(b-1)^{\frac{2m_1+4m_3-4}{3}}\tilde{Q}_1^{m_1}(b)\tilde{Q}_2^{m_2}(b)\tilde{Q}_3^{m_3}(b). \end{aligned}$$

In these sums, m_1, m_2, m_3 are nonnegative integers such that $m_1 + m_2 + m_3 = 7$. In the first sum, we require a further restriction in the form $m_1 + m_3 \geq 1$; in the third sum, $m_1 \geq 2$ or $m_3 \geq 1$.

To obtain the desired fixed point for (5.6), we show that the bounds

$$|\tilde{Q}_1(a)| + |\tilde{Q}_3(a)| \leq C\tilde{q}_1, \quad |\tilde{Q}_2(a)| \leq C|\tilde{q}_2| \tag{5.7}$$

improve upon themselves on the interval $a \in [1, 1+\ell]$ with ℓ, as in the statement of the lemma, once they are inserted into the system (5.6). To be precise, we prove that (5.7) implies the same bounds with $C/2$ instead of C, provided that constant is larger than some absolute one.

To accomplish this, we rely on the choice of $\ell = c|\tilde{q}_2|\tilde{q}_1^{-4/3}$. In the $\tilde{Q}_1$ integral, we estimate

$$\tilde{q}_1^{-1}\big|(a-1)^{-2/3}\int_1^a G(a,b)(b-1)^{2/3}N_1(b,\tilde{Q}_1,\tilde{Q}_2,\tilde{Q}_3)\,db\big| \lesssim \sum_{2m_1+4m_3\equiv 2(3)} \ell\,\ell^{\frac{2(m_1-1)}{3}}\tilde{q}_1^{m_1-1}\ell^{\frac{4m_3}{3}}\tilde{q}_1^{m_3}$$

Recall that $m_1+m_3\geq 1$ in this case. First note that $\ell^{4m_3/3}\tilde{q}_1^{m_3}\leq 1$ for all $m_3\geq 0$. If $m_1\geq 1$, we may estimate

$$\ell\,\ell^{2(m_1-1)/3}\tilde{q}_1^{m_1-1}\leq \ell+\ell^5\tilde{q}_1^6\ll 1$$

for all $m_1 = 1,2,\ldots,7$. On the other hand, if $m_1 = 0$, then either $m_3 = 2$ or $m_3 = 5$, whence

$$\ell\,\ell^{\frac{2(m_1-1)}{3}}\tilde{q}_1^{m_1-1}\ell^{\frac{4m_3}{3}}\tilde{q}_1^{m_3}\lesssim \ell^3\tilde{q}_1+\ell^7\tilde{q}_1^4\ll 1.$$

As for the source term involving N_0, we get

$$\Big|\int_1^a G(a,b)N_0(b,\tilde{Q}_1,\tilde{Q}_2,\tilde{Q}_3)\,db\Big|\lesssim \sum_{2m_1+4m_3\equiv 0(3)} \ell\,\ell^{\frac{2m_1+4m_3}{3}}\tilde{q}_1^{m_1+m_3},$$

where the sum runs over all integers $0\leq m_1,m_3\leq 7$. Note that here, we have $m_1\leq 6$. The general term of this finite sum is decreasing in m_3, irrespective of m_1; so it suffices to set $m_3 = 0$ and to evaluate at the endpoints $m_1 = 0$ and $m_1 = 6$, respectively. In summary, this yields the bound $\ell+(\ell^{5/6}\tilde{q}_1)^6\ll|\tilde{q}_2|$. Finally, for the contribution of N_3, we get in case $m_3\geq 1$

$$\tilde{q}_1^{-1}\big|(a-1)^{-4/3}\int_1^a G(a,b)(b-1)^{4/3}N_2(b,\tilde{Q}_1,\tilde{Q}_2,\tilde{Q}_3)\,db\big| \lesssim \sum_{2m_1+4m_3\equiv 1(3)} \ell\,\ell^{\frac{2m_1+4(m_3-1)}{3}}\tilde{q}_1^{m_1+m_3-1}$$

Once again, the general term is decreasing in m_3, so it suffices to consider the pairs (m_1,m_3) from the list

$$(0,1),\ (1,2),\ (2,0),\ (3,1),\ (4,2),\ (5,0),\ (6,1),$$

in which m_3 is always the smallest possible member, given the value of m_1. Hence the upper bound is of the form

$$\begin{aligned}&\lesssim \ell(1+\ell\tilde{q}_1)+\ell\tilde{q}_1\big[1+(\ell\tilde{q}_1)^2+(\ell\tilde{q}_1)^4\big]+\ell^3\tilde{q}_1^4\big[1+(\ell\tilde{q}_1)^2\big]\\ &\lesssim \ell+\ell\tilde{q}_1+\ell^3\tilde{q}_1^4\ll 1,\end{aligned}$$

where we have used the fact that $\ell\tilde{q}_1 = c|\tilde{q}_2|\tilde{q}_1^{-1/3} \ll 1$.

The existence of the desired fixed point on $[1, 1+\ell]$ now follows from this in standard fashion. Also, $\tilde{Q}_1(a) \geq \tilde{q}_1/2$, provided we pick the constant c small enough. To be specific, we define the space

$$X := \{(\tilde{Q}_1, \tilde{Q}_2, \tilde{Q}_3) \in (C^0([1, 1+\ell]))^3 \mid (5.7) \text{ holds}\},$$

where the constant in (5.7) is absolute. By the preceding analysis, we see that the complete metric space X is mapped onto itself by (5.6). Moreover, taking differences shows that the system is a contraction in X. It is a standard calculus exercise to verify that the integrals in (5.6) are $C^1([1, 1+\ell])$, and iterating this property shows that the left-hand side of (5.6) is in fact $C^\infty([1, 1+\ell])$. In particular, we obtain (5.2) and (5.3), the latter being implied by the integral estimates from above.

Finally, picking $a_* = 1 + \frac{\ell}{2}$, we obtain

$$Q(a_*) \simeq \left(c|\tilde{q}_2|\tilde{q}_1^{-\frac{4}{3}}\right)^{\frac{2}{3}}\tilde{q}_1 \simeq |\tilde{q}_2|^{2/3}\tilde{q}_1^{1/9}$$

which gives (5.4). □

Having solved the ODE (2.1) near the singularity, we now show that the solution may be extended to the region $a \geq 1+\ell$. For this we need the equation to be defocussing.

Lemma 5.2. *The solutions of* (2.1) *on* $(1, 1+\ell]$ *constructed in Lemma 5.1 can be extended to* $(1, \infty)$ *as smooth globally bounded functions* $Q(a)$. *For large values,*

$$|Q(a)| \lesssim a^{-1/3}, \qquad |Q'(a)| \lesssim a^{-4/3},$$

as $a \to \infty$ *with nonvanishing constants* c_1, c_2.

Proof. We construct an integrating factor to remove the first order derivative in (2.1). Thus introduce the auxiliary function

$$f(\tilde{a}) = \frac{1}{2}\frac{\frac{8}{3}\tilde{a} - \frac{2}{\tilde{a}}}{\tilde{a}^2 - 1}, \qquad \tilde{a} \in [1+\ell, \infty)$$

as well as the new dependent variable

$$X(a) := Q(a)w(a), \qquad w(a) := e^{\int_{1+\ell}^a f(\tilde{a})\,d\tilde{a}}.$$

Then the original ODE is equivalent to the following ODE for X:

$$X''(a) + g(a)X + \left(\frac{e^{-6\int_{1+\ell}^a f(\tilde{a})\,d\tilde{a}}}{a^2 - 1}\right)X^7 = 0, \tag{5.8}$$

where $g(a) := 5/9(a^2-1)^2$. To obtain global regularity, it suffices to exhibit an a priori L^∞-bound for X on any finite interval $[1+\ell, L]$.

In order to obtain such a bound, we multiply the equation by X' and integrate. This yields the "energy estimate"

$$\begin{aligned}\frac{1}{2}(X')^2(a) + \frac{1}{2}X^2(a)g(a) + \frac{1}{8}\Big(\frac{e^{-6\int_{1+\ell}^a f(\tilde a)\,d\tilde a}}{a^2-1}\Big)X^8(a) \\ = -\int_{1+\ell}^a \big(\frac{3}{4}f(\tilde a) + \frac{a}{4(a^2-1)^2}\big)e^{-6\int_{1+\ell}^{\tilde a} f(a_1)\,da_1}X(a)^8\,d\tilde a \\ -\int_{1+\ell}^a \frac{10a}{9(a^2-1)^3}X^2(\tilde a)\,d\tilde a + \frac{1}{2}(X')^2(1+\ell) + \frac{1}{2}X^2(1+\ell)g(1+\ell) \\ + \frac{1}{8(\ell^2+2\ell)}X^8(1+\ell) \\ \le \frac{1}{2}(X')^2(1+\ell) + \frac{1}{2}X^2(1+\ell)g(1+\ell) + \frac{1}{8(\ell^2+2\ell)}X^8(1+\ell).\end{aligned}$$

Here, we have used the fact that $f(a) > 0$ for all $a > 1$. Thus, one has an a priori bound

$$\Big(\frac{e^{-6\int_{1+\ell}^a f(\tilde a)\,d\tilde a}}{a^2-1}\Big)X^8(a) \le C(\ell), \quad a \in [1+\ell, \infty).$$

Since

$$\int_{1+\ell}^a f(\tilde a)\,d\tilde a \sim \frac{4}{3}\log a, \quad w(a) \sim a^{4/3}$$

as $a \to \infty$, we obtain

$$\frac{e^{-6\int_{1+\ell}^a f(\tilde a)\,d\tilde a}}{a^2-1} \sim a^{-10},$$

and hence

$$|X(a)| \le D(\ell)a^{5/4}, \quad a \in [1+\ell, \infty).$$

We now return to the original dependent variable. This last bound implies the decay

$$|Q(a)| = |X(a)|e^{-\int_{1+\ell}^a f(\tilde a)\,d\tilde a} \le E(\ell)a^{-1/12}$$

as $a \to \infty$, whence $Q(a) \to 0$ as $a \to \infty$. From the energy estimate, we furthermore infer that $|X'(a)| \le C(\ell)$ for all $a \ge 1+\ell$, whence

$$\begin{gathered}|Q'(a)w(a) + Q(a)w'(a)| \le C(\ell) \\ |Q'(a)| \le C(\ell)w^{-1}(a) + |Q(a)||w'(a)|w^{-1}(a) \lesssim a^{-13/12}\end{gathered}$$

But this implies that for each $\varepsilon > 0$, there exists a_ε sufficiently large such that $|Q(a_\varepsilon)| + |Q'(a_\varepsilon)| < \varepsilon$. This means that on the interval $[a_\varepsilon, \infty)$, we are in the small data case, and we may solve (2.1) by perturbing around the corresponding solution of the linear part.

To be specific, we are precisely in the regime of Lemma 2.5. For the reader's convenience, we sketch the details. With the linear fundamental system

$$\tilde{\varphi}_1(a) = a^{-1}(a-1)^{2/3}, \quad \varphi_1(a) = a^{-1}(1+a)^{2/3}$$

we define the Green function

$$G(a,b) := \frac{\tilde{\varphi}_1(a)\varphi_2(b) - \tilde{\varphi}_1(b)\varphi_2(a)}{W(b)(b^2-1)}$$
$$W(a) := \tilde{\varphi}_1(a)\varphi_2'(a) - \tilde{\varphi}_1'(a)\varphi_2(a)$$

The denominator is $W(b)(b^2-1)$, which decays at the rate $b^{-2/3}$ as $b \to \infty$. The perturbative approach is to seek a nonlinear solution in the form

$$\tilde{Q}(a) = m_1\tilde{\varphi}_1(a) + m_2\varphi_2(a) + \int_a^\infty G(a,b)\tilde{Q}(b)^7\,db \tag{5.9}$$

for all $a \geq a_\varepsilon$ where m_1, m_2 are small. As in Section 2, one shows that (5.9) admits a unique solution for any such choice of m_1, m_2 and that the map $(m_1, m_2) \mapsto (\tilde{Q}(a_\varepsilon), \tilde{Q}'(a_\varepsilon))$ is a diffeomorphism from one small neighborhood of the origin to another. So, in particular, we find (m_1, m_2) in (5.9) such that

$$(\tilde{Q}(a_\varepsilon), \tilde{Q}'(a_\varepsilon)) = (Q(a_\varepsilon), Q'(a_\varepsilon)),$$

and we see that $\tilde{Q}(a) = Q(a)$ for all $a \geq a_\varepsilon$. This means that generically, using the asymptotics of the fundamental system $\tilde{\varphi}_1(a), \varphi_2(a)$, we have

$$|Q(a)| \sim a^{-1/3}, \; |Q'(a)| \sim a^{-4/3}$$

as $a \to \infty$, But it is possible that we satisfy the linear relation between m_1, m_2 which cancels the leading order $a^{-1/3}$, leading to the faster decay $a^{-4/3}$. □

Because of the energy estimate, which played a pivotal role in the proof, the previous lemma essentially depends on the defocussing character of the problem. We remark that in contrast to the small solutions constructed in Section 2, the large solutions constructed in this section may oscillate wildly in the interval $(1, a_*)$ because equation (5.8) is a nonlinear oscillator equation.

6 Gluing the self-similar solutions, excision and completion to a global smooth solution

In this section, we follow the scheme that we deployed above in the small solution case to excise the singularity from the light-cone so as the obtain global smooth solution of the defocussing equation (1.1). Combining Corollary 2.7 with the results of the previous section, we arrive at the following conclusion.

Proposition 6.1. *Given q_0 small enough as well as $\tilde{q}_1$ arbitrary, there exists a continuous function $Q(a)$ which is smooth away from $a = 1$, solves* (2.1) *on* $[0, 1) \cup (1, \infty)$*, and satisfies*

$$Q(0) = q_0, \quad Q'(0) = 0,$$

as well as a representation (5.1) *(where $|\tilde{q}_2| \ll 1$ depends on q_0). We have the asymptotic behavior*

$$|Q(a)| \lesssim a^{-1/3}, \quad a \to \infty.$$

This function has the representation (2.12) *for $a \in [1/2, 1)$, as well as the representation* (5.1) *for $a \in (1, 1 + \ell]$ with $\ell = \ell(\tilde{q}_1, \tilde{q}_2)$.*

Proceeding in exact analogy to Section 3, we introduce the modified approximate solution

$$u(t, r) = t^{-1/3}\chi(t - r)|1 - a|^{2/3}X(a) + t^{-1/3}Q_2(1). \tag{6.1}$$

We attempt to turn this into an actual solution of (1.1) (in the defocussing case) by adding a correction term $v(t, r)$. Here v solves (4.1). In analogy with Section 4, we state two main propositions, the first one being local existence for the linearized equation about $u(t, r)$.

Proposition 6.2. *Let u be as in* (6.1) *above, and assume that $v[T]$ is compactly supported with*

$$\|v[T]\|_{\dot{H}^{7/6}(\mathbb{R}^3)\times\dot{H}^{1/6}(\mathbb{R}^3)} \ll 1.$$

Then there exist some time $T_1 > T$ and a solution

$$v \in L_t^\infty \dot{H}^{7/6}([T, T_1] \times \mathbb{R}^3), \quad v_t \in L_t^\infty \dot{H}^{1/6}([T, T_1] \times \mathbb{R}^3),$$

of (4.1) *with compact support on every time slice $t \times \mathbb{R}^3$, $t \in [T, T_1]$. Also, if $v[T] \in \dot{H}^s(\mathbb{R}^3) \times \dot{H}^{s-1}(\mathbb{R}^3)$, then $v[t] \in \dot{H}^s(\mathbb{R}^3) \times \dot{H}^{s-1}(\mathbb{R}^3)$ for all $s > 7/6$.*

This is proved in essentially the same fashion as Proposition 4.3; see the final section.

Proposition 6.3. *Let $C \geq 1$ and $\tilde{q}_1$ as in the preceding proposition be fixed, $T \geq 1$ sufficiently large, and q_0 (as in the preceding proposition) sufficiently small. For a sufficiently small δ_1, suppose that $v[T]$ with support on $r \in [T - C, T + C]$, satisfies*

$$\|v[T]\|_{\dot{H}^{7/6}\cap\dot{H}^1(\mathbb{R}^3)\times\dot{H}^{1/6}\cap L^2(\mathbb{R}^3)} \leq \delta_1 \ll 1.$$

Then there exist $C_1 > 1$ *with* $C_1\delta_1 \ll 1$ *and* $\varepsilon = \varepsilon(q_0) > 0$ *such that for any* $T_1 > T$,

$$\|v\|_{L_t^6L_x^{18}([T,T_1]\times\mathbb{R}^3)} + \sup_{t\in[T,T_1]} \|v[t]\|_{\dot H^{7/6}\cap(t-T)^\varepsilon \dot H^1(\mathbb{R}^3)\times \dot H^{1/6}\cap(t-T)^\varepsilon L^2(\mathbb{R}^3)} \le C_1\delta_1$$

implies

$$\|v\|_{L_t^6L_x^{18}([T,T_1]\times\mathbb{R}^3)} + \sup_{t\in[T,T_1]} \|v[t]\|_{\dot H^{7/6}\cap(t-T)^\varepsilon \dot H^1(\mathbb{R}^3)\times \dot H^{1/6}\cap(t-T)^\varepsilon L^2(\mathbb{R}^3)} \le \frac{C_1}{2}\delta_1.$$

We may also include any other Strichartz norm on the left-hand side; see Lemma 4.2.

To prove this proposition, we proceed in close analogy to the proof of Proposition 4.4, considering first the energy and then the scaling invariant norm. Notice carefully that the errors e_j are not small in a pointwise sense. However, due to the support condition on v and by taking the initial time T sufficiently large, we will see that the influence of the errors e_j on the solution can be made as small as we wish.

6.1 Energy control. Observe that for $t \in [T, T_1]$,

$$\begin{aligned}
&\int_{\mathbb{R}^3} \Big[\frac{1}{2}\Big(v_t^2 + |\nabla v|^2\Big) + \frac{7}{2}u^6v^2 + \cdots + uv^7 + \frac{1}{8}v^8\Big](t,\cdot)\,dx \\
&\qquad - \int_{\mathbb{R}^3} \Big[\frac{1}{2}\Big(v_t^2 + |\nabla v|^2\Big) + \frac{7}{2}u^6v^2 + \cdots + uv^7 + \frac{1}{8}v^8\Big](T,\cdot)\,dx \\
&\qquad = \int_T^t \int_{\mathbb{R}^3} \Big[\sum_{j=1}^{3} -e_jv_t + 21u_tu^5v^2 + \cdots + u_tv^7\Big]\,dxdt.
\end{aligned} \tag{6.2}$$

Note that

$$\int u^6(T,\cdot)v^2(T,\cdot)\,dx \lesssim \|v\|_{\dot H^1}^2 \int_{T-C}^{T+C} r^{-2}r^{-1}r^2\,dr \lesssim T^{-1}\|v\|_{\dot H^1}^2 \ll \delta_1^2,$$

provided we choose T large enough (depending on $\tilde q_1$, which now influences the size of u). Exactly as in Subsection 4.1, we obtain for $j \in [1,5]$ the bounds

$$\sup_{t\in[T,T_1]} \|u^jv^{8-j}(t,\cdot)\|_{L_x^1} \lesssim T^{\varepsilon\alpha(8-j)-\frac{1}{9}}\Big(\sup_{t\in[T,T_1]}(T-t)^{-\varepsilon}\|v(t,\cdot)\|_{\dot H^1}\Big)^{\alpha(8-j)}\|v(t,\cdot)\|_{\dot H^{7/6}}^{(1-\alpha)(8-j)},$$

where here and in the sequel, the implicit constant depends on $\tilde q_1$; and, by choosing T sufficiently large, we can make this $\ll \delta_1^2$. Moreover, the contribution of the pure power term v^8 is estimated as before by

$$\|v^8(T,\cdot)\|_{L^1_x} \le \|v(t,\cdot)\|_{\dot H^1}^2 \big(\sup_{t\in[1,T]} \|v(t,\cdot)\|_{\dot H^{\frac{7}{6}}}^6 \big) \lesssim \|v(t,\cdot)\|_{\dot H^1}^2 (C\delta_1)^6 \ll \delta_1^2.$$

It remains to control the terms on the right hand side of (6.2). The contribution of the terms involving the e_j is again straightforward. In fact, just as in Subsection 4.1, we get

$$\Big| \int_T^t e_j v_t \, dxdt \Big| \lesssim \Big(\int_T^t s^{\varepsilon-\frac{4}{3}} \, ds \Big) \big(\sup_{t\in[T,T_1]} (T-t)^{-\varepsilon} \|v_t(t,\cdot)\|_{L^2_x} \big) \ll \delta_1^2,$$

provided we choose T sufficiently large. As to the remaining source terms $21 u_t u^5 v_t, \dots, u_t v^7$, only the first one is delicate, as the others all result in gains in T, whence the required smallness gain. To handle the delicate term, we write (on a fixed time slice)

$$\int u_t u^5 v^2 \, dx = \int_{r<t} u_t u^5 v^2 \, dx + \int_{t<r<t+C} u_t u^5 v^2 \, dx.$$

Here we have exploited the fact that by Huygens' principle, the perturbation v is supported in the neighborhood $r < t + C$ of the forward light cone. Since the approximately self-similar $u(t,r)$ is given by the small-data ansatz in the interior of the light cone, we can repeat verbatim the estimates following (4.5) to conclude that

$$\int_T^{T_1} \int_{r<t} \big| u_t u^5 v^2 \big| \, dxdt \ll \delta_1^2,$$

provided q_0 is chosen sufficiently small. Thus, consider now the term

$$\int_{t<r<t+C} u_t u^5 v^2 \, dx.$$

Using the bound $|u_t| \lesssim t^{-1}$ (see (4.6)), we can bound this by

$$\begin{aligned}\Big| \int_{t<r<t+C} u_t u^5 v^2 \, dx \Big| &\lesssim \|v\|_{\dot H^1}^2 t^{-8/3} \int_{t<r<t+C} r^{-1} r^2 \, dr \\ &\lesssim t^{2\varepsilon-\frac{5}{3}} \big(\sup_{t\in[T,T_1]} (T-t)^{-\varepsilon} \|v(t,\cdot)\|_{\dot H^1} \big)^2;\end{aligned}$$

and so we infer

$$\begin{aligned}\int_T^{T_1} \int_{t<r<t+C} \big| u_t u^5 v^2 \big| \, dxdt &\lesssim \int_T^{T_1} t^{2\varepsilon-\frac{5}{3}} \, dt \big(\sup_{t\in[T,T_1]} (T-t)^{-\varepsilon} \|v(t,\cdot)\|_{\dot H^1} \big)^2 \\ &\ll \delta_1^2,\end{aligned}$$

provided T is sufficiently large.

The expression $\int_T^t \int u_t v^7 \, dxdt$, as well as all the "intermediate source terms" in (6.2), are again all estimated just as in Subsection 4.1, resulting in gains of a factor T^{-1}, which furnishes the required smallness. This completes the bootstrap for the energy norm

$$\sup_{t\in[T,T_1]} (T-t)^{-\varepsilon} \|v(t,\cdot)\|_{\dot{H}^1} + \sup_{t\in[T,T_1]} (T-t)^{-\varepsilon} \|v_t(t,\cdot)\|_{L^2}.$$

6.2 Critical norm control. We repeat the estimates from Subsection 4.2, which are all seen to result in a gain of a factor T^{-1} (for the nonlinear source terms), and so the bootstrap is immediate by choosing T large enough. This completes the proof of Proposition 6.3.

6.3 Proofs of Theorems 1.2 and 1.4. Invoking Propositions 6.2 and 6.3, we have shown that the approximate solution $u(t,r)$ can be completed to an exact global-in-forward time solution

$$\tilde{u}(t,r) = u(t,r) + v(t,r).$$

Moreover, this solution preserves any additional regularity of the data $v[T]$ above $\dot{H}^{7/6}(\mathbb{R}^3) \times \dot{H}^{1/6}(\mathbb{R}^3)$.

Translating the time $t = T$ to time $t = 0$, picking $\tilde{q}_1 \gg M^9$ (see Lemma 5.1), and re-scaling $\tilde{u}(t,r) \longrightarrow T^{\frac{1}{3}}\tilde{u}(Tt, Tr)$, we have now shown Theorem 1.2. The largeness condition (1.5) is an immediate consequence of the estimate (5.4) and the fact that we may choose the initial data so as not to destroy this pointwise property. Moreover, the largeness of the weak Besov norm follows from the fact that for radial functions on $\mathbb{R}^3$,

$$\|f\|_{L_x^\infty(|x|\geq 1)} \lesssim \|f\|_{\dot{B}_\infty^{\frac{7}{6},2}}.$$

The remaining assertions of Theorem 1.2 are proved just as for Theorem 1.1. Theorem 1.4 is proved by truncation, analogously to the way in which we obtained Theorem 1.3. Once again, finite time blowup can only occur at the origin due to the pointwise a priori bound for all $r > 0$ (fixed) uniformly in time as a result of the conserved positive definite energy.

7 Local solvability of the perturbative equation

Here, we prove Proposition 4.3. Let $v[T]$ be as in that proposition. We immediately observe from their definition that the errors e_j have compact support on fixed time slices; hence the compact (spatial) support of $v[T]$ implies that of $v[t]$ for any t. We construct v as a limit of the iterative process

$$-v_{tt}^{(j)} + \triangle v^{(j)} \mp 7u^6 v^{(j-1)} \mp \cdots \mp 7u(v^{(j-1)})^6 \mp (v^{(j-1)})^7 = \sum_{i=1}^{3} e_i, \quad j \geq 1,$$
$$v^{(0)}(\cdot) = S(\cdot)(v[T]),$$

where $S(\cdot)$ denotes the standard free wave propagator. We assume that

$$\|v[T]\|_{\dot{H}^{7/6}\times \dot{H}^{1/6}(\mathbb{R}^3)} \leq \delta,$$

where δ is some small but absolute constant, and then show that the sequence $v^{(j)}$ converges in $L_t^6 L_x^{18}(\mathbb{R}^3) \cap L_t^\infty \dot{H}^{7/6}(\mathbb{R}^3)$ on the time slice $[T, T+1] \times \mathbb{R}^3$. We may assume that

$$\|u\|_{L_t^6 L_x^{18}([T,T+1]\times\mathbb{R}^3)} \leq C_1\delta, \quad \sum_{i=1}^{3} \|e_i\|_{L_t^1 \dot{H}^{1/6}([T,T+1]\times\mathbb{R}^3)} \leq C_2\delta,$$

for some constants $C_{1,2}$, uniformly in $T \geq 1$; see Corollary 2.7. We conclude from Strichartz' inequality (see Lemma 4.2) that

$$\begin{aligned}\|v^{(j)}\|_{L_t^6 L_x^{18}\cap L_t^\infty \dot{H}^{7/6}(\mathbb{R}^3)} + \|\partial_t v^{(j)}\|_{L_t^\infty \dot{H}^{1/6}(\mathbb{R}^3)} \\ \leq C_3\Big[\sum_{k=0}^{6} \|\,|\nabla|^{1/6}(u^k (v^{(j-1)})^{7-k})\|_{L_t^1 L_x^2([T,T+1]\times\mathbb{R}^3)} + \sum_{i=1}^{3}\|e_i\|_{L_t^1 \dot{H}^{1/6}(\mathbb{R}^3)}\Big]\end{aligned}$$

We have

$$\|(v^{(j-1)})^7\|_{L_t^1 \dot{H}^{1/6}([T,T+1]\times\mathbb{R}^3)} \leq C_4 \|v^{(j-1)}\|^7_{(L_t^6 L_x^{18}\cap L_t^\infty \dot{H}^{7/6})([T,T+1]\times\mathbb{R}^3)}; \tag{7.1}$$

indeed, by the fractional Leibnitz rule,

$$\|f^7\|_{\dot{H}^{1/6}(\mathbb{R}^3)} \lesssim \|f\|_{\dot{W}^{\frac{1}{6},6}(\mathbb{R}^3)} \|f^6\|_{L^3(\mathbb{R}^3)} \lesssim \|f\|_{\dot{H}^{7/6}(\mathbb{R}^3)} \|f\|^6_{L^{18}(\mathbb{R}^3)},$$

and integrating this in time over $[T, T+1]$ yields (7.1). By the same type of reasoning, for $k = 1, 2, \ldots, 6$, we have

$$\begin{aligned}\|u^k(v^{(j-1)})^{7-k}\|_{L_t^1 \dot{H}^{1/6}([T,T+1]\times\mathbb{R}^3)} &\leq C_5 \|\,|\nabla|^{1/6} u\|_{L_t^6 L_x^9([T,T+1]\times\mathbb{R}^3)} \|u\|^{k-1}_{L_t^6 L_x^{18}([T,T+1]\times\mathbb{R}^3)} \\ &\quad \times \|v^{(j-1)}\|^{6-k}_{L_t^6 L_x^{18}([T,T+1]\times\mathbb{R}^3)} \|v^{(j-1)}\|_{L_t^\infty \dot{H}^{7/6}([T,T+1]\times\mathbb{R}^3)} \\ &+ C_6 \|u\|^k_{L_t^6 L_x^{18}([T,T+1]\times\mathbb{R}^3)} \|v^{(j-1)}\|^{6-k}_{L_t^6 L_x^{18}([T,T+1]\times\mathbb{R}^3)} \|\,|\nabla|^{1/6} v^{(j-1)}\|_{L_t^\infty \dot{H}^1([T,T+1]\times\mathbb{R}^3)},\end{aligned}$$

where we have also used the Sobolev embedding (in the context of functions vanishing at infinity on $\mathbb{R}^3$) $\dot{H}^1(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$. Note that $|\nabla|^{1/6} u(t, \cdot) \in L_x^9$ due to symbolic behavior with respect to r for $r \gg t$. It then follows that provided we make the inductive assumption

$$\|v^{(j-1)}\|_{L_t^6 L_x^{18} \cap L_t^\infty \dot{H}^{7/6}([T,T+1]\times\mathbb{R}^3)} \leq K\delta$$

for some sufficiently large constant K (independent of δ), we obtain

$$\|v^{(j)}\|_{L_t^6 L_x^{18} \cap L_t^\infty \dot{H}^{7/6}([T,T+1]\times\mathbb{R}^3)} \leq C_7 K^7 \delta^7 + C_8\,\delta,$$

where we have exploited the fact that $\sum_{i=1}^3 \|e_i\|_{L_t^1 \dot{H}^{1/6}} \leq C_8\,\delta$ as well as

$$\|u\|_{L_t^6 L_x^{18} \cap L_t^6 |\nabla|^{-1/6} L_x^9([T,T+1]\times\mathbb{R}^3)} \leq C_9\,\delta$$

from our choice of δ. Choosing $\delta > 0$ small enough in relation to C_7 and K large enough in relation to C_8, we obtain

$$\|v^{(j)}\|_{L_t^6 L_x^{18} \cap L_t^\infty \dot{H}^{7/6}([T,T+1]\times\mathbb{R}^3)} \leq K\delta,$$

and thus we get the desired a priori bound. Passing to the difference equation yields the convergence of the $v^{(j)}$. The higher derivative bounds follow in standard fashion by differentiating the equation for $v^{(j)}$. This completes the proof of Proposition 4.3.

As for Proposition 6.2, the main difference lies with the fact that the function u is no longer small. Thus in order to ensure convergence of the iteration, one needs to replace the interval $[T, T+1]$ by one of the form $[T, T+\kappa]$, where $\kappa = \kappa(u)$ depends on

$$\sup_t \|\,|\nabla|^{1/6} u\|_{L_x^9} + \sup_t \|u\|_{L_x^{18}}.$$

Otherwise, the argument is identical to the preceding one.

Acknowledgments. Joachim Krieger thanks the University of Chicago for its hospitality in July 2013, where a portion of this research was conducted.

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Joachim Krieger
Bâtiment des Mathématiques
EPFL, Station 8
CH-1015 Lausanne, Switzerland
email: joachim.krieger@epfl.ch

Wilhelm Schlag
Department of Mathematics
The University of Chicago
Chicago, IL 60615, USA
email: schlag@math.uchicago.edu

(Received May 1, 2014 and in revised form December 1, 2014)