Commun. Math. Phys. 327, 309–332 (2014)
Digital Object Identifier (DOI) 10.1007/s00220-014-1900-9

Communications in
**Mathematical
Physics**

Threshold Phenomenon for the Quintic Wave Equation in Three Dimensions

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Received: 13 September 2012 / Accepted: 17 April 2013
Published online: 27 February 2014 –

Abstract: For the critical focusing wave equation $\Box u = u^5$ on $\mathbb{R}^{3+1}$ in the radial case, we establish the role of the "center stable" manifold Σ constructed in Krieger and Schlag (Am J Math 129(3):843–913, 2007) near the ground state $(W, 0)$ as a threshold between blowup and scattering to zero, establishing a conjecture going back to numerical work by Bizoń et al. (Nonlinearity 17(6):2187–2201, 2004). The underlying topology is stronger than the energy norm.

1. Introduction

We consider the energy-critical focusing nonlinear wave equation

$$\Box u = u^5, \quad \Box = \partial_t^2 - \Delta_x, \quad u[0] = (u, u_t)_{t=0} = (u_0, u_1) \tag{1.1}$$

on the Minkowski space $\mathbb{R}^{3+1}$ with radial data. The conserved energy is

$$E(u, \dot{u}) = \int_{\mathbb{R}^3} \Big(\frac{1}{2}|\nabla_{t,x} u|^2 - \frac{1}{6}|u|^6\Big)\, dx.$$

In a remarkable series of papers, [5–8] Duyckaerts, Kenig, and Merle gave the following characterization of the long-time dynamics for radial data $u[0] \in \dot{H}^1 \times L^2(\mathbb{R}^3)$ of arbitrary energy: either one has type-I blowup, i.e., $\|u[t]\|_{\dot{H}^1 \times L^2} \to \infty$ in finite time, or the solution decomposes into a (possible empty) sum of time-dependent dilates of the ground state stationary solution

$$W(x) := (1 + |x|^2/3)^{-\frac{1}{2}}$$

Support of the National Science Foundation DMS-0617854, DMS-1160817 for the third author, and the Swiss National Fund for the first author are gratefully acknowledged. The latter would like to thank the University of Chicago for its hospitality in August 2012.

together with a radiation term that acts like a free wave, up to a $o(1)$ as $t \to T_* \in (0, \infty]$. Here $[0, T_*)$ is the existence interval of the solution. See [8] for the precise theorem. We remark that Kenig, Merle [15] had studied the case of energies $E(u_0, u_1) < E(W, 0)$ and established a finite-time blowup vs. scattering dichotomy depending on whether $\|\nabla u_0\|_2 > \|\nabla W\|_2$ or $\|\nabla u_0\|_2 < \|\nabla W\|_2$. For the subcritical case, Payne and Sattinger [26] had given such a criterion but with global existence, and the scattering remained unknown. The latter gap was closed only recently by Ibrahim et al. [12] using the Kenig–Merle method.

The dynamics for the case $E(u_0, u_1) = E(W, 0)$ was described by Duyckaerts, Merle [9,10] who constructed the one-dimensional stable and unstable manifolds associated with W. Finally, [5] allowed energies slightly larger than $E(W, 0)$, and it was shown there that general type-II blowup occurs by dynamical non-selfsimilar rescaling of W. The existence of such blowup solutions was established by Krieger, Schlag, Tataru in [20]. An analogous construction in infinite time was carried out by Donninger and Krieger in [4]. In this context we would also like to mention the type-II blowup construction by Hillairet and Raphaël [11] for the 4-dimensional semilinear wave equation.

From a different perspective, and motivated in part by the phenomenological work [3] of Bizoń, Chmaj, and Tabor, Krieger and Schlag investigated in [19] the question of *conditional stability* of the ground state W. This is a very delicate question, and remains unsolved in the energy topology. Note that the aforementioned blowup solutions can be chosen to lie arbitrarily close relative to the energy topology to the soliton curve $\mathcal{S} := \{W_\lambda\}_{\lambda>0}$ where $W_\lambda(x) = \sqrt{\lambda} W(\lambda x)$. However, in a much stronger topology, [19] established the existence of a codimension-1 Lipschitz manifold Σ near W so that data chosen from this manifold exhibit asymptotically stable dynamics. See [19] for the exact formulation.

The question remained as to the dynamics for data near Σ, but which do not fall on Σ. As a start in this direction we mention the work by Karageorgis–Strauss [13] for a related model equation of the same scaling class as (1.1) where they show blow up for certain data with energy above that of the ground state, which are in a sense 'above the tangent space' of Σ.

In the subcritical case, Nakanishi and Schlag had shown, see [21–24], that this hypersurface Σ divides a small ball into two halves which exhibit the finite-time blowup vs. scattering dichotomy in forward time. This was carried out in the energy class, and Σ was identified with the center-stable manifold associated with the hyperbolic dynamics generated by linearizing about the ground state. See the seminal work by Bates, Jones [2] for an invariant manifold theorem in infinite dimensions, with applications to a certain class of Klein–Gordon equations.

For the energy critical wave equation (1.1), the authors [16,17] had shown a somewhat weaker result, namely the existence of four pairwise disjoint sets $A_{\pm,\pm}$ in the energy space near the soliton curve such that: (1) each set has a nonempty interior (2) the long-term dynamics (in both positive and negative times) for data taken from each set is determined as either blowup or global existence and scattering.

However, the question of existence of a center-stable manifold near W in the energy space remains open and appears delicate. Therefore, the results of [16,17] are not as complete as those in [24], in the sense that no comprehensive description of the dynamics near the soliton curve is obtained. This is also explained by the fact that the dynamics of the energy critical equation appear more complex due to the scaling invariance which is not a feature of the Klein–Gordon equation considered in [24], as evidenced by the variety of exotic type-II solutions. Moreover, the construction of the "center-stable" manifold[1] in [19] is significantly more involved than the corresponding manifold for the subcritical Klein–Gordon equation.

In this paper, we return to the point of view of [19] in order to establish a description of all possible dynamics with data near $(W, 0)$ in the following main theorem, albeit in a stronger topology than that given by the energy. To formulate it, we need the linearized operator $H := -\Delta - 5W^4$. It exhibits a unique negative eigenvalue $-k_0^2$ with $Hg_0 = -k_0^2 g_0$, and $g_0 > 0$ is smooth, radial, and exponentially decaying.

Theorem 1.1. *Fix $R > 1$. There exists an $\varepsilon_* = \varepsilon_*(R) > 0$ with the following property. Consider all pairs of radial functions (f_1, f_2) supported in $B(0, R)$ with $\|f_1\|_{H^3} + \|f_2\|_{H^2} < \varepsilon_*$. Denote by Σ the hypersurface constructed in [19], parametrized by such pairs (f_1, f_2) satisfying the condition $\langle k_0 f_1 + f_2, g_0\rangle = 0$. Pick initial data $v[0] \in \Sigma$ with*

$$v(0,\cdot) = f_1 + h(f_1, f_2)g_0, \quad v_t(0,\cdot) = f_2$$

Then the following holds:

- *if $\varepsilon_* > \delta_0 > 0$, then initial data*

$$\tilde{u}(0,\cdot) = W + f_1 + (h(f_1, f_2) + \delta_0)g_0, \quad \tilde{u}_t(0,\cdot) = f_2$$

  lead to solutions blowing up in finite positive time.
- *if $-\varepsilon_* < \delta_0 < 0$, then initial data*

$$\tilde{u}(0,\cdot) = W + f_1 + (h(f_1, f_2) + \delta_0)g_0, \quad \tilde{u}_t(0,\cdot) = f_2$$

  lead to solutions existing globally in forward time and scattering to zero in the energy space.

The hyper-plane $\langle k_0 f_1 + f_2, g_0\rangle = 0$ is the tangent space to Σ at $(W, 0)$, and it is denoted by Σ_0 in [19]. The function h is constructed in [19] and for any $0 < \delta \leqslant \varepsilon_*(R)$ one has the following properties: define the space

$$X_R := \{(f_1, f_2) \in H^3_{\mathrm{rad}}(\mathbb{R}^3) \times H^2_{\mathrm{rad}}(\mathbb{R}^3) \mid \mathrm{supp}(f_j) \subset B(0, R)\}$$

Then $h : B_\delta(0) \subset \Sigma_0 \to \mathbb{R}$ where $B_\delta(0)$ is relative to X_R and one has the estimates

$$|h(f_1, f_2)| \lesssim \|(f_1, f_2)\|^2_{X_R}, \qquad \forall\, (f_1, f_2) \in B_\delta(0)$$
$$|h(f_1, f_2) - h(\tilde{f}_1, \tilde{f}_2)| \lesssim \delta \|(f_1, f_2) - (\tilde{f}_1, \tilde{f}_2)\|_{X_R} \qquad \forall\, (f_1, f_2), (\tilde{f}_1, \tilde{f}_2) \in B_\delta(0)$$

The Lipschitz graph Σ is given by $(f_1 + h(f_1, f_2)g_0, f_2)$ where $(f_1, f_2) \in B_\delta(0) \subset \Sigma_0$. It is a Lipschitz hypersurface in X_R which approaches Σ_0 quadratically near the point $(W, 0)$. It is thus clear that Σ_0 is the tangent space to Σ at $(W, 0)$.

Finally, we note that our choice of topology is not optimal for this type of theorem, and our approach can be extended to more general initial conditions. On the other hand, we emphasize that the distinction between the energy topology $\dot{H}^1 \times L^2$ on the one hand, and a stronger one such as ours, has very dramatic effects. Indeed, solutions starting on the manifold Σ as constructed in [19] are shown there to approach $W_{a(\infty)}$ up to a radiation part where $a(\infty) \in (0, \infty)$. If a center-stable manifold can be constructed in

[1] We place "center-stable" in quotation marks, since Σ cannot be interpreted as such an object. In fact, the space X_R is not invariant under the flow.

$\dot{H}^1 \times L^2$, then we cannot expect the same behavior for solutions associated with such an object. Indeed, from [20] and [4] we know that energy solutions exist arbitrarily close to $(W, 0)$ in the energy topology for which $a(t)$ can approach either 0 or ∞ in finite or infinite time.

The idea of the proof of the theorem is to combine the precise description of solutions with data on Σ contained in [19, Definition 3] with the exit characterization of solutions established in [16]. The latter work allows us to confine ourselves to the situation in which the solution is close to $\mathcal{S}$, the family of rescalings $W_\lambda = \lambda^{\frac{1}{2}} W(\lambda \cdot)$ of W, whence we can rely purely on perturbative methods. The key for the proof is the following result.

Proposition 1.2. *There exists* $1 \gg \varepsilon_0 \gg \varepsilon_*$ *with the following property: Let* $\tilde{u}[0]$ *be data as in Theorem 1.1. Then there exist* $\tilde{\delta}_0 \neq 0$ *of the same sign as* δ_0*, a constant* k_∞ *with* $|k_0 - k_\infty| \ll 1$*, and a finite time* $T = T(\tilde{u}[0])$ *with* $\varepsilon_0 = |\tilde{\delta}_0| e^{k_\infty T} \gg \varepsilon_*$ *and such that at time* $t = T$*, we have a decoupling*

$$\tilde{u}(t, \cdot) = W_{\alpha_T} + \tilde{v}_{\alpha_T}, \quad |1 - \alpha_T| \ll 1,$$

with

$$\langle \tilde{v}_{\alpha_T}, \Lambda^* g_{\alpha_T} \rangle = 0, \quad \Lambda = r\partial_r + \frac{1}{2} \tag{1.2}$$

and furthermore

$$\langle \tilde{v}_{\alpha_T}, g_{\alpha_T} \rangle \simeq \tilde{\delta}_0 e^{k_\infty T}. \tag{1.3}$$

Proposition 1.2 guarantees that data which are obtained by adding $\delta_0 g_0$ to a point on Σ diverge exponentially away from Σ. The trajectory moves away from the "tube" of rescaled ground states $\mathcal{S}$ in a specific direction, depending on the sign of δ_0. Note that the "excitation" of the unstable mode g_0 can be arbitrarily small in Theorem 1.1. This is the main distinction from our previous works [16,17]. Indeed, in those cases this excitation needed to be sufficiently large so as to dominate the evolution from the beginning (and for as long as the trajectory remained inside a small neighborhood of $(W, 0)$, since otherwise the linearized dynamics cannot be compared to the nonlinear one).

At least on a heuristic level, our construction in Proposition 1.2 is motivated by the generalizations of the well-known Hartman-Grobman linearization theorem which applies to ODEs of the form $\dot{x} = Ax + f(x)$ in $\mathbb{R}^n$ where $f(0) = Df(0) = 0$ provided A has no eigenvalues on the imaginary axis. In that case there exists a homeomorphism $y = y(x)$ near $x = 0$ which linearizes the ODE in the sense that $\dot{y} = Ay$. If A does have spectrum on the imaginary axis, then there is a result known as Shoshitaishvili's theorem [27,28]; see also Palmer [25], which ensures partial linearization of the ODE in the form

$$\dot{y} = By + \varphi(y), \quad \dot{z} = Cz, \tag{1.4}$$

after a change of variables near $x = 0$. Here B has its spectrum on the imaginary axis, and C is the hyperbolic part, and φ satisfies $\varphi(0) = D\varphi(0) = 0$ (the y-equation captures the center-dynamics). Note that in the formulation (1.4) the center-stable manifold is precisely given by $z_+ = 0$ where z_+ are the coordinates for which C is expanding. In addition, since the change of coordinates is in fact bi-Hölder it also follows from (1.4) that the center-stable manifold $\mathcal{M}_{cs}$ is *exponentially repulsive* in the sense that if a trajectory starts near but not on $\mathcal{M}_{cs}$, then it will move away exponentially from $\mathcal{M}_{cs}$.

However, in this paper we do not rely on a partial linearization as in (1.4) since such a result is not available in our context. Rather, we show that the coupling between the "center-stable" dynamics obtained in [19] and the unstable hyperbolic dynamics is of a higher order in a suitable sense, which implies the exponential push away from Σ.

We conclude this introduction by showing how to deduce the main theorem from the previous proposition.

Proof of Theorem 1.1 assuming Proposition 1.2. Picking ε_* sufficiently small, the theory of [16] applies. In particular, while the data $\tilde{u}[0] = \big(\tilde{u}(0,\cdot), \tilde{u}_t(0,\cdot)\big)$ satisfy

$$\mathrm{dist}_{\dot{H}^1\times L^2}(\tilde{u}[0], \mathcal{S}\cup -\mathcal{S}) \lesssim \varepsilon_*, \tag{1.5}$$

where we identify $\mathcal{S} := (W_\lambda, 0)_{\lambda>0}$, we have

$$\mathrm{dist}_{\dot{H}^1\times L^2}(\tilde{u}[T], \mathcal{S}\cup -\mathcal{S}) \simeq |\tilde{\delta}_0| e^{k_\infty T} \tag{1.6}$$

provided we choose $|\tilde{\delta}_0| e^{k_\infty T}$ (and thus ε_0) sufficiently large in relation to ε_*. Indeed, this is a direct consequence of (1.3) combined with [16, Lemma 2.2]. But then equation (3.44) as well as Proposition 5.1, Proposition 6.2 in [16] imply that data with $\delta_0 > 0$ result in finite time blow up, while data with $\delta_0 < 0$ scatter to zero as $t \to +\infty$, with finite Strichartz norms. □

Inspection of this proof shows that we rely on several previous results. On the one hand, the proof of Proposition 1.2 depends crucially on the asymptotic analysis of the stable solutions constructed in [19], including all dispersive estimates of the radiative part. On the other hand, for the non-perturbative analysis we rely on key elements of our previous work [16], namely the one-pass theorem and the ejection mechanism in relation to the variational structure (see the K-functional in [16]). Note also that the latter paper requires the main theorem from [5] in order to preclude blowup in the regime $K \geqslant 0$ once the solution has excited the soliton tube. For a completely different construction in the energy space leading to a centerstable manifold for the same equation, see [18].

2. Proof of Proposition 1.2

It remains to prove Proposition 1.2, which we carry out via a bootstrap argument using suitable norms. The norms we use for the perturbation are adapted from those introduced in [19].

2.1. A modified representation of the data. Throughout we assume that (f_1, f_2) satisfy the conditions of Theorem 1.1. We start with data of the form

$$(f_1 + h(f_1, f_2)g_0, f_2) \in \Sigma$$

with the orthogonality condition $\langle k_0 f_1 + f_2, g_0\rangle = 0$. According to [19], these data can be evolved globally in forward time to a function $v(t,\cdot)$ so that $W_{a(t)} + v(t,\cdot)$ solves (1.1), with $|a(t) - a(0)| \ll 1$ for all $t \geqslant 0$. Let $g_\infty = g_\infty(f_1, f_2)$ be the unstable mode for the operator

$$\mathcal{H}(a(\infty)) = -\Delta - 5W^4_{a(\infty)} =: -\Delta + V$$

which is the reference Hamiltonian at $t = +\infty$. Writing

$$\Sigma_0 := \{\langle k_0 f_1 + f_2, g_0\rangle = 0\}$$

for the tangent plane to Σ, pick $\tilde{h}(f_1, f_2)$ such that

$$(f_1 + h(f_1, f_2)g_0 - \tilde{h}(f_1, f_2)g_\infty, f_2) \in \Sigma_0.$$

This is possible since $\|g_0 - g_\infty\|_2 \ll 1$. The map

$$(f_1, f_2) \mapsto (f_1 + h(f_1, f_2)g_0 - \tilde{h}(f_1, f_2)g_\infty, f_2) =: (\tilde{f}_1, f_2)$$

is Lipschitz continuous[2] and one-to-one on a small neighborhood $U \subset \Sigma_0$ of 0 (within the admissible data set as in Theorem 1.1). Lipschitz here refers to the X_R topology in the domain, and the radial $H^3 \times H^2$ topology in the target. In fact, this map equals the identity plus a Lipschitz map with very small Lip constant. This follows from the estimates (see [19], Section 4)

$$\begin{aligned} |\tilde{h}(f_1, f_2)| \simeq |h(f_1, f_2)| &\lesssim \|(f_1, f_2)\|^2, \\ \big|h(f_1, f_2) - h(g_1, g_2)\big| &\ll \|f_1 - g_1\|_{H^3} + \|f_2 - g_2\|_{H^2}. \end{aligned}$$

Committing abuse of notation, we write $\tilde{h} = \tilde{h}(\tilde{f}_1, f_2)$, $g_\infty = g_\infty(\tilde{f}_1, f_2)$, where it is to be kept in mind that g_∞ is associated with the asymptotic operator determined by the data $(f_1 + h(f_1, f_2)g_0, f_2)$. Then we have the identity

$$f_1 + h(f_1, f_2)g_0 = \tilde{f}_1 + \tilde{h}(\tilde{f}_1, f_2)g_\infty$$

and furthermore

$$(\tilde{f}_1 + \tilde{h}(\tilde{f}_1, f_2)g_\infty, f_2) \in \Sigma.$$

We next need to find an analogous representation for the shifted initial data

$$(f_1 + (h(f_1, f_2) + \delta_0)g_0, f_2).$$

Observe that the map

$$(\tilde{f}_1, f_2, \tilde{\delta}_0) \mapsto \tilde{f}_1 + (\tilde{\delta}_0 + \tilde{h}(\tilde{f}_1, f_2))g_\infty$$

is again Lipschitz and a homeomorphism for small values of the arguments. In particular, we can write

$$f_1 + (h(f_1, f_2) + \delta_0)g_0 = \tilde{f}_1 + (\tilde{\delta}_0 + \tilde{h}(\tilde{f}_1, f_2))g_\infty$$

where $\tilde{\delta}_0$ is a Lipschitz-function of (f_1, f_2, δ_0). Also, observe that Σ divides the data space into two connected components, which can be characterized by $\tilde{\delta}_0 > 0$, $\tilde{\delta}_0 < 0$. The same comment applies to δ_0, and necessarily $\delta_0 > 0$ corresponds to $\tilde{\delta}_0 > 0$.

[2] In fact, this map is smoother but we do not make this explicit in [19].

2.2. The perturbative ansatz. Now given f_1, f_2, δ_0, let u be the solution of (1.1) corresponding to the data

$$(W + \tilde{f}_1 + \tilde{h}(\tilde{f}_1, f_2)g_\infty, f_2),\ (\tilde{f}_1 + \tilde{h}(\tilde{f}_1, f_2)g_\infty, f_2) \in \Sigma.$$

These are of course in general different from $(f_1 + h(f_1, f_2)g_0, f_2)$. Note that g_∞ is the unstable eigenmode corresponding to the evolution of u at $t = +\infty$. Also, denote by $\tilde{u}$ the solution corresponding to the data

$$(W + f_1 + (h(f_1, f_2) + \delta_0)g_0, f_2) = (W + \tilde{f}_1 + (\tilde{\delta}_0 + \tilde{h}(\tilde{f}_1, f_2))g_\infty, f_2).$$

We shall first make the simple perturbative ansatz

$$\tilde{u} = u + \eta = W_{a(t)} + u_* + \eta, \tag{2.1}$$

where we use the decoupling

$$u(t, \cdot) = W_{a(t)} + u_*(t, \cdot)$$

given in [19] with the bounds

$$\|u_*(t, \cdot)\|_{L^\infty_x} \leqslant \delta\langle t\rangle^{-1}, \quad \|\nabla_x u_*(t, \cdot)\|_{L^2_x + L^\infty_x} \leqslant \delta\langle t\rangle^{-\varepsilon} \tag{2.2}$$

$$\|\nabla u_*(t, \cdot)\|_{L^2_x} + \|\nabla^2 u_*(t, \cdot)\|_{L^2_x} \leqslant \delta, \quad |u_*(x, t)| \lesssim \delta\langle x\rangle^{-1} \tag{2.3}$$

for suitable $\delta = \delta(\varepsilon_*, R) \ll 1$; in fact, $\delta = C_0\varepsilon_*$ where C_0 is a big constant (depending on R). For the dilation parameter one has the bounds

$$|a(t) - a_\infty| \leqslant \delta\langle t\rangle^{-1}, \quad |\dot{a}(t)| \leqslant \delta\langle t\rangle^{-2} \tag{2.4}$$

and in particular $|a(t) - a_\infty| \ll 1$. In view of (2.1), we obtain the following equation for η:

$$\begin{aligned}\partial_{tt}\eta + \mathcal{H}(a(\infty))\eta &= N(u_* + \eta, W_{a(t)}) - N(u_*, W_{a(t)}) \\ &\quad + \big(\mathcal{H}(a(\infty)) - \mathcal{H}(a(t))\big)\eta =: F(t).\end{aligned} \tag{2.5}$$

Here we set $\mathcal{H}(a) = -\Delta_x - 5W_a^4$, and borrowing notation from [19], we have

$$N(v, W_a) = (v + W_a)^5 - W_a^5 - 5W_a^4 v. \tag{2.6}$$

The right-hand side in (2.5) further equals

$$\begin{aligned}F(t) &= 5(u^4 - W_{a(t)}^4)\eta + 10u^3\eta^2 + 10u^2\eta^3 + 5u\eta^4 + \eta^5 \\ &\quad + 5(W_{a(t)}^4 - W_{a(\infty)}^4)\eta \\ (u^4 - W_{a(t)}^4)\eta &= (u_*^4 + 4u_*^3 W_{a(t)} + 6u_*^2 W_{a(t)}^2 + 4u_* W_{a(t)}^3)\eta.\end{aligned} \tag{2.7}$$

Note that all terms linear in η are of the form $o(\eta)$, and they are also localized in space due to the decay of u_* and W. We shall write $\mathcal{H}(a(\infty)) = \mathcal{H}_\infty$ from now on, and denote the corresponding unstable mode by g_∞, with $\mathcal{H}_\infty g_\infty = -k_\infty^2 g_\infty$. It is natural to decompose

$$\eta = P_{g_\infty^\perp}\eta + \delta(t)g_\infty =: \tilde{\eta}(t, \cdot) + \delta(t)g_\infty. \tag{2.8}$$

The key to proving Proposition 1.2 is the following result.

Proposition 2.1. *Let $T > 0$ be such that $|\tilde{\delta}_0| e^{k_\infty T} \leqslant \varepsilon_0$. Then for any $t \in [0, T]$, we have the bounds*

$$|\delta(t)| \simeq |\tilde{\delta}_0| e^{k_\infty t}, \ \|\tilde{\eta}(t,\cdot)\|_{L^2_x} + \|\nabla_x \tilde{\eta}(t,\cdot)\|_{L^2_x} + \|\nabla^2_x \tilde{\eta}(t,\cdot)\|_{L^2_x} \ll |\tilde{\delta}_0| e^{k_\infty t} \quad (2.9)$$

for some fixed large M. Also, $\delta(t)$ has the same sign as $\tilde{\delta}_0$.

Proof of Proposition 2.1. Recall that

$$F(t) = N(u_* + \eta, W_{a(t)}) - N(u_*, W_{a(t)}) + \big(\mathcal{H}_\infty - \mathcal{H}(a(t))\big)\eta.$$

Then according to Section 3 in [19], we can write

$$\begin{aligned} \delta(t) &= (2k_\infty)^{-\frac{1}{2}}[n_+(t) + n_-(t)], \\ n_\pm(t) &= (\frac{k_\infty}{2})^{\frac{1}{2}} \tilde{\delta}_0 e^{\pm k_\infty t} + \int_0^t e^{\pm k_\infty (t-s)} \langle F(s), g_\infty \rangle \, ds. \end{aligned} \quad (2.10)$$

Moreover, we have the Duhamel-type formula

$$\tilde{\eta}(t,\cdot) = -\int_0^t \frac{\sin[(t-s)\sqrt{\mathcal{H}_\infty}]}{\sqrt{\mathcal{H}_\infty}} P_{g_\infty^\perp} F(s) \, ds. \quad (2.11)$$

Assume that the solution exists on some interval $[0, \tilde{T})$, $\tilde{T} \leqslant T$, and that it satisfies the following estimates, which we refer to as **bootstrap assumptions**:

$$\begin{aligned} |\delta(t)| &\leqslant 10 |\tilde{\delta}_0| e^{k_\infty t} \\ \|\tilde{\eta}(t,\cdot)\|_{L^2_x} + \|\nabla_x \tilde{\eta}(t,\cdot)\|_{L^2_x} + \|\nabla^2_x \tilde{\eta}(t,\cdot)\|_{L^2_x} &\leqslant \frac{2}{K} |\tilde{\delta}_0| e^{k_\infty t} \end{aligned} \quad (2.12)$$

for some large K, which will be chosen to depend on ε_0.

We shall now infer that $|\delta(t)| \simeq |\tilde{\delta}_0| e^{k_\infty t}$ with a proportionality factor in $[\frac{1}{4}, 4]$ and we will improve the second inequality by replacing $\frac{2}{K}$ by $\frac{1}{K}$. A standard continuity argument then implies Proposition 2.1.

(A) Improving the bound on $\tilde{\eta}$. We start with the L^2_x-norm. To control it, we use the simple bound

$$\begin{aligned} \Big\| \frac{\sin(t\sqrt{\mathcal{H}_\infty})}{\sqrt{\mathcal{H}_\infty}} P_{g_\infty^\perp} f \Big\|_{L^2_x} &= \Big\| \int_0^t \cos(s\sqrt{\mathcal{H}_\infty}) \, ds \, P_{g_\infty^\perp} f \Big\|_{L^2_x} \\ &\lesssim |t| \|f\|_{L^2_x}. \end{aligned} \quad (2.13)$$

Assume that we have the bound

$$\|F(s,\cdot)\|_{L^2_x} \ll \frac{|\tilde{\delta}_0|}{K} e^{k_\infty s}. \quad (2.14)$$

Then (2.13) implies

$$\begin{aligned} &\Big\| \int_0^t \frac{\sin[(t-s)\sqrt{\mathcal{H}_\infty}]}{\sqrt{\mathcal{H}_\infty}} P_{g_\infty^\perp} F(s) \, ds \Big\|_{L^2_x} \\ &\ll \frac{|\tilde{\delta}_0|}{K} \int_0^t (t-s) e^{k_\infty s} ds \lesssim \frac{|\tilde{\delta}_0|}{K} e^{k_\infty t} \end{aligned}$$

which recovers the dispersive type bound for $\tilde{\eta}$. The above bound (2.14) for F can be easily proved: for the difference

$$N(u_* + \eta, W_{a(t)}) - N(u_*, W_{a(t)})$$

it suffices to consider the "extreme" terms

$$u_* W_{a(t)}^3 \eta, \quad u_*^4 \eta, \quad u^3 \eta^2, \quad \eta^5, \tag{2.15}$$

see (2.7). We now check (2.14) for each of these expressions, bounding η as in (2.8) via (2.12) as follows:

$$\|\eta(t,\cdot)\|_{L_x^2} + \|\nabla_x \eta(t,\cdot)\|_{L_x^2} + \|\nabla_x^2 \eta(t,\cdot)\|_{L_x^2} \leqslant C_1 |\tilde{\delta}_0| e^{k_\infty t}$$

with an absolute constant C_1. In what follows, we will need to ensure that $\varepsilon_0 \ll K^{-1}$ (so that also $\delta \ll K^{-1}$).

For the first term in (2.15), we get

$$\left\| u_* W_{a(t)}^3 \eta \right\|_{L_x^2} \lesssim \|u_*\|_{L_x^\infty} \|W_{a(t)}^3\|_{L_x^\infty} \|\eta\|_{L_x^2} \ll \frac{|\tilde{\delta}_0|}{K} \langle t \rangle^{-1} e^{k_\infty t}.$$

For the second term in (2.15), we get

$$\left\| u_*^4 \eta \right\|_{L_x^2} \lesssim \|u_*\|_{L_x^\infty}^4 \|\eta\|_{L_x^2} \ll \frac{|\tilde{\delta}_0|}{K} \langle t \rangle^{-4} e^{k_\infty t}.$$

For the third term in (2.15), use that $H^2(\mathbb{R}^3) \subset L^\infty$ to obtain the bound

$$\left\| u^3 \eta^2 \right\|_{L_x^2} \lesssim \|u^3\|_{L_x^\infty} \|\eta\|_{L_x^\infty} \|\eta\|_{L_x^2} \lesssim \tilde{\delta}_0^2 e^{2k_\infty t} \ll \frac{|\tilde{\delta}_0|}{K} e^{k_\infty t}.$$

For the last term in (2.15), we similarly obtain

$$\left\| \eta^5 \right\|_{L_x^2} \lesssim \|\eta\|_{L_x^\infty}^4 \|\eta\|_{L_x^2} \lesssim |\tilde{\delta}_0|^5 e^{5k_\infty t} \ll \frac{|\tilde{\delta}_0|}{K} e^{k_\infty t}.$$

In order to complete the proof of the bound (2.14), it remains to control the term

$$\big(\mathcal{H}_\infty - \mathcal{H}(a(t))\big)\eta.$$

Due to the fast decay rate ($\simeq \langle x \rangle^{-4}$) of the potential $V = -5W_{a(t)}^4$, one easily infers

$$\|\big(\mathcal{H}_\infty - \mathcal{H}(a(t))\big)\eta\|_{L_x^2} \lesssim |a(\infty) - a(t)| |\tilde{\delta}_0| e^{k_\infty t} \ll \frac{|\tilde{\delta}_0|}{K} \langle t \rangle^{-1} e^{k_\infty t}.$$

This completes the bootstrap for the norm $\|\tilde{\eta}\|_{L_x^2}$.

Next, consider the norm $\|\nabla \tilde{\eta}\|_{L_x^2}$. To control it, we use [19, eq. (36)] with $V = -5W(a(\infty))^4$:

$$\begin{aligned}
\|\nabla \tilde{\eta}\|_{L_x^2} &\leqslant \|\sqrt{\mathcal{H}_\infty}\, \tilde{\eta}\|_{L_x^2} + \||V|^{\frac{1}{2}} \tilde{\eta}\|_{L_x^2} \\
&\leqslant \int_0^t \|F(s,\cdot)\|_{L_x^2}\, ds + \||V|^{\frac{1}{2}}\|_{L_x^\infty} \|\tilde{\eta}\|_{L_x^2} \\
&\ll \frac{|\tilde{\delta}_0|}{K} e^{k_\infty t}.
\end{aligned}$$

Finally, we consider $\|\nabla_x^2\tilde{\eta}\|_{L_x^2}$:

$$\begin{aligned}\|\nabla^2\tilde{\eta}\|_{L_x^2} &\leqslant \|\mathcal{H}_\infty\,\tilde{\eta}\|_{L_x^2} + \|V\tilde{\eta}\|_{L_x^2}\\ &\leqslant \int_0^t \|\sqrt{\mathcal{H}_\infty}P_{g_\infty^\perp}F(s,\cdot)\|_{L_x^2}\,ds + \|V\|_{L_x^\infty}\|\tilde{\eta}\|_{L_x^2}.\end{aligned}$$

The final term here is $\ll \frac{|\tilde{\delta}_0|}{K}e^{k_\infty t}$ as desired, and for the integral we continue using [19, Eq. (35)]:

$$\int_0^t \|\sqrt{\mathcal{H}_\infty}P_{g_\infty^\perp}F(s,\cdot)\|_{L_x^2}\,ds \lesssim \int_0^t \|\nabla F(s,\cdot)\|_{L_x^2}\,ds. \tag{2.16}$$

To bound the integral on the right, we again consider the terms in (2.15). For the first of these, we have

$$\begin{aligned}&\left\|\nabla_x\left(u_* W_{a(t)}^3\eta\right)\right\|_{L_x^2}\\ &\lesssim \|\nabla_x u_*\|_{L_x^2+L_x^\infty}\|W_{a(t)}^3\|_{L_x^\infty\cap L_x^2}\|\eta\|_{L_x^\infty} + \|u_*\|_{L_x^\infty}\|\nabla_x(W_{a(t)}^3)\|_{L_x^\infty}\|\eta\|_{L_x^2}\\ &\quad+\|u_*\|_{L_x^\infty}\|W_{a(t)}^3\|_{L_x^\infty}\|\nabla_x\eta\|_{L_x^2}\\ &\ll \frac{|\tilde{\delta}_0|}{K}\langle t\rangle^{-\varepsilon}e^{k_\infty t} + \frac{|\tilde{\delta}_0|}{K}\langle t\rangle^{-1}e^{k_\infty t}.\end{aligned}$$

For the second term in (2.15), we obtain the contribution

$$\left\|\nabla_x\left(u_*^4\eta\right)\right\|_{L_x^2} \lesssim \|\nabla_x u_*\|_{L_x^2}\|u_*^3\|_{L_x^\infty}\|\eta\|_{L_x^\infty} + \|u_*^4\|_{L_x^\infty}\|\nabla_x\eta\|_{L_x^2} \ll \frac{|\tilde{\delta}_0|}{K}\langle t\rangle^{-3}e^{k_\infty t}.$$

For the last two terms of (2.15), we have the bounds

$$\begin{aligned}\left\|\nabla_x\left(u^3\eta^2\right)\right\|_{L_x^2} &\lesssim \|\nabla_x(u^3)\|_{L_x^2}\|\eta\|_{L_x^\infty}^2 + \|u^3\|_{L_x^\infty}\|\nabla_x\eta\|_{L_x^2}\|\eta\|_{L_x^\infty}\\ &\lesssim \tilde{\delta}_0^2 e^{2k_\infty t} \ll \frac{|\tilde{\delta}_0|}{K}e^{k_\infty t}\\ \left\|\nabla_x(\eta^5)\right\|_{L_x^2} &\lesssim \|\nabla_x\eta\|_{L_x^2}\|\eta^4\|_{L_x^\infty} \lesssim |\tilde{\delta}_0|^5 e^{k_\infty t} \ll \frac{|\tilde{\delta}_0|}{K}e^{k_\infty t}.\end{aligned}$$

Finally, one also easily checks that

$$\left\|\nabla_x\left((\mathcal{H}_\infty - \mathcal{H}(a(t)))\eta\right)\right\|_{L_x^2} \lesssim |a(\infty)-a(t)||\tilde{\delta}_0|e^{k_\infty t} \ll \frac{|\tilde{\delta}_0|}{K}\langle t\rangle^{-1}e^{k_\infty t}.$$

Before continuing, we make the following important observation from the proof:

Corollary 2.2. *The bootstrap assumption implies that we can write for* $j = 0, 1, 2$

$$\nabla_x^j\tilde{\eta}(t) = \tilde{\eta}_1^{(j)} + \tilde{\eta}_2^{(j)},$$

where we have

$$\begin{aligned}\|\tilde{\eta}_1^{(j)}(t,\cdot)\|_{L_x^2} &\ll \frac{|\tilde{\delta}_0|}{K}\langle t\rangle^{-\varepsilon}e^{k_\infty t}\\ \|\tilde{\eta}_2^{(j)}(t,\cdot)\|_{L_x^2} &\ll |\tilde{\delta}_0|^2 e^{2k_\infty t}.\end{aligned} \tag{2.17}$$

This corollary is important since it shows that the interactions of $\tilde{\eta}$ with itself as well as with the driving term u_* are much weaker than the principal unstable component of η, i.e., $\delta(t)$. We will have to take advantage of this improved bound in order to control the evolution of $\delta(t)$.

(B) Improving the control over $\delta(t)$. In order to complete the bound on η, we next need to control the growth of the coefficients $n_\pm(t)$. This appears more difficult due to the quadratic interactions in $F(s,\cdot)$ of the form $u_*\eta W^3_{a(t)}$. The issue here is that the dispersive bound for u_* only gives $\langle t\rangle^{-1}$ decay, which just fails to be integrable.

We start by deducing an improved bound for $n_-(t)$ departing from our bootstrap assumption. In view of (2.10) we have

$$n_-(t) = (\frac{k_\infty}{2})^{\frac{1}{2}}\tilde{\delta}_0 e^{-k_\infty t} + \int_0^t e^{-k_\infty(t-s)}\langle F(s,\cdot), g_\infty\rangle\, ds.$$

Using the bound (2.14) with the improvement implied by Corollary 2.2, we get the bound

$$|n_-(t)| \lesssim |\tilde{\delta}_0|\langle t\rangle^{-\frac{\varepsilon}{2}} e^{k_\infty t} + \tilde{\delta}_0^2 e^{2k_\infty t}. \tag{2.18}$$

We now use this, together with Corollary 2.2 as well as the a priori bounds on u_*, to derive the improved control over $n_+(t)$. We depart from the differential equation

$$\begin{aligned}
&\dot{n}_+(t) - k_\infty n_+(t)\\
&= \frac{n_+(t)}{(2k_\infty)^{\frac{1}{2}}}\langle g_\infty\big(20u_* W^3_{a(\infty)} + (a(\infty)-a(t))\partial_\lambda V|_{\lambda=a(\infty)}\big), g_\infty\rangle + F_+(t),
\end{aligned} \tag{2.19}$$

where we use the notation $V_\lambda := -5W_\lambda^4$ and

$$\begin{aligned}
F_+(t) = &\frac{n_-(t)}{(2k_\infty)^{\frac{1}{2}}}\langle 20u_* g_\infty W^3_{a(t)}, g_\infty\rangle + \langle 20u_*\tilde{\eta} W^3_{a(t)}, g_\infty\rangle\\
&+\frac{n_+(t)}{(2k_\infty)^{\frac{1}{2}}}\langle g_\infty\big(V_{a(\infty)} - V_{a(t)} - (a(\infty)-a(t))\partial_\lambda V|_{\lambda=a(\infty)}\big), g_\infty\rangle\\
&+G_+(t)
\end{aligned}$$

with

$$\begin{aligned}
G_+(t) = &\frac{n_+(t)}{(2k_\infty)^{\frac{1}{2}}}\langle 20u_* g_\infty(W^3_{a(t)} - W^3_{a(\infty)}), g_\infty\rangle\\
&+\langle N(u_*+\eta, W_{a(t)}) - N(u_*, W_{a(t)}) - 20\delta(t)u_* g_\infty W^3_{a(t)}, g_\infty\rangle\\
&+\langle\big(\mathcal{H}_\infty - \mathcal{H}(a(t))\big)[\tilde{\eta} + (2k_\infty)^{-\frac{1}{2}} n_-(t) g_\infty], g_\infty\rangle.
\end{aligned}$$

We infer from (2.19) that

$$n_+(t) = (\frac{\tilde{\delta}_0}{2})^{\frac{1}{2}} e^{k_\infty t+\Gamma(0,t)} + \int_0^t e^{k_\infty(t-s)+\Gamma(s,t)} F_+(s)\, ds, \tag{2.20}$$

where we use the notation

$$\Gamma(s,t) := \int_s^t \langle g_\infty\big(u_*(s_1,\cdot)W^3_{a(\infty)} + (a(\infty)-a(s_1))\partial_\lambda V|_{\lambda=a(\infty)}\big), g_\infty\rangle\, ds_1.$$

In order to proceed, we shall obtain uniform bounds on the phase function $\Gamma(s,t)$. These hinge on Proposition 3.2, to be proved in the next section. This proposition implies that

$$\sup_{s,t>0}\Big|\int_s^t \langle g_\infty\big(u_*(s_1,\cdot)W^3_{a(\infty)}, g_\infty\rangle\, ds_1\Big| \lesssim \|u_*\|_{L^\infty_x L^1_t} \ll 1. \tag{2.21}$$

It remains to estimate

$$\sup_{s,t}\int_s^t (a(\infty)-a(s_1))\, ds_1. \tag{2.22}$$

Note that the integrand decays like s_1^{-1} from the bounds in [19], which is no integrable. Lemma 2.3 shows nevertheless that (2.22) is uniformly bounded. This again hinges on Proposition 3.2.

Lemma 2.3. *We have the averaged estimate*

$$\sup_{t>0}\Big|\int_0^t (a(\infty)-a(s))\, ds\Big| \ll 1.$$

Proof. Here we use the equation defining $a(t)$ in [19], given by (51) in loc. cit., which we copy here for $t \gtrsim 1$:

$$\dot a(t) = -c_0\Big(\frac{a(t)}{a(\infty)}\Big)^{\frac{5}{4}}\langle \partial_\lambda W_\lambda|_{\lambda=a(\infty)}, (V_{a(\infty)}-V_{a(t)})u_*(t,\cdot) + N(u_*(t,\cdot), W_{a(t)})\rangle.$$

We write this equation somewhat schematically in the form

$$\begin{aligned}\dot a(t) &= -c_0(a(\infty)-a(t))\langle \partial_\lambda W_\lambda|_{\lambda=a(\infty)}, u_*(t,\cdot)\partial_\lambda V_\lambda|_{\lambda=a(\infty)}\rangle\\ &\quad + O(|a(\infty)-a(t)|^2\langle |\partial_\lambda W_\lambda|_{\lambda=a(\infty)}|, |u_*(t,\cdot)|\langle x\rangle^{-4}\rangle)\\ &\quad -c_0\Big(\frac{a(t)}{a(\infty)}\Big)^{\frac{5}{4}}\langle \partial_\lambda W_\lambda|_{\lambda=a(\infty)}, N(u_*(t,\cdot), W_{a(t)})\rangle.\end{aligned}$$

Set $\alpha(t) := a(\infty)-a(t)$, and write this ODE in the form

$$\begin{aligned}\dot\alpha &= -\alpha\,\sigma - H\\ \sigma(t) &= -c_0\langle \partial_\lambda W_\lambda|_{\lambda=a(\infty)}, u_*(t,\cdot)\partial_\lambda V_\lambda|_{\lambda=a(\infty)}\rangle\\ H(t) &:= O(|a(\infty)-a(t)|^2\langle |\partial_\lambda W_\lambda|_{\lambda=a(\infty)}|, |u_*(t,\cdot)|\langle x\rangle^{-4}\rangle)\\ &\quad - c_0\Big(\frac{a(t)}{a(\infty)}\Big)^{\frac{5}{4}}\langle \partial_\lambda W_\lambda|_{\lambda=a(\infty)}, N(u_*(t,\cdot), W_{a(t)})\rangle.\end{aligned} \tag{2.23}$$

Solving from $t=\infty$ one obtains

$$\alpha(t) = \int_t^\infty e^{\int_t^s \sigma}\, H(s)\, ds. \tag{2.24}$$

Proposition 3.2 implies that

$$\sup_{s,t}\Big|\int_t^s \sigma\Big| \ll 1$$

which ensures that $e^{\int_t^s \sigma} = O(1)$ uniformly in s, t. We now claim that

$$\int_0^t (a(\infty) - a(\tilde{t}))\, d\tilde{t} = t \int_t^\infty e^{\int_t^s \sigma}\ H(s)\, ds + \int_0^t s\sigma(s) \int_s^\infty e^{\int_s^{\tilde{s}} \sigma}\ H(\tilde{s})\, d\tilde{s}\, ds. \quad (2.25)$$

To verify this, note first that both sides vanish at $t = 0$. Furthermore, taking a derivative in t reduces the equation to (2.24).

One has the bound

$$|H(t)| \lesssim \langle u_*^2, \langle x\rangle^{-4}\rangle + \delta\langle t\rangle^{-3} \quad (2.26)$$

with $0 < \delta \ll 1$. Therefore, on the one hand,

$$\sup_{t>0} \Big| t \int_t^\infty e^{\int_t^s \sigma}\ H(s)\, ds \Big| \ll 1.$$

On the other hand, $\sup_{s\geqslant 0} |s\sigma(s)| \ll 1$ whence

$$\Big| \int_0^t s\sigma(s) \int_s^\infty e^{\int_s^{\tilde{s}} \sigma}\ H(\tilde{s})\, d\tilde{s}\, ds \Big| \lesssim \int_0^t \int_s^\infty |H(\tilde{s})|\, d\tilde{s}\, ds = t \int_t^\infty |H(\tilde{s})|\, d\tilde{s} + \int_0^t s|H(s)|\, ds. \quad (2.27)$$

The first term is $\ll 1$ from (2.26), whereas the second integral is dominated by

$$\sup_{t>0} \Big| \int_0^t s|H(s)|\, ds \Big| \lesssim \sup_{s>0} \|s u_*(s, \cdot)\|_{L_x^\infty} \|u_*\|_{L_x^\infty L_s^1} + \delta \ll 1.$$

In conclusion (2.27) is $\ll 1$ which completes the proof of the lemma. □

In conjunction with (2.27) the lemma implies that the phase corrections $\Gamma(s, t)$ are uniformly small.

We next estimate the contributions of the various constituents of $F_+(s, \cdot)$ to the integral in (2.20). This will then lead to the completion of the proof of Proposition 2.1.

(1) *The contribution of* $\frac{n_-(t)}{(2k_\infty)^{\frac{1}{2}}} \langle u_* g_\infty W_{a(t)}^3, g_\infty\rangle + \langle u_* \tilde{\eta} W_{a(t)}^3, g_\infty\rangle$.

Using (2.18) as well as Corollary 2.2, we bound this by

$$\ll \int_0^t e^{k_\infty(t-s)+\Gamma(s,t)} \langle s\rangle^{-1} [|\tilde{\delta}_0| \langle s\rangle^{-\frac{\varepsilon}{2}} e^{k_\infty s} + \tilde{\delta}_0^2 e^{2k_\infty s}]\, ds$$
$$\lesssim |\tilde{\delta}_0| e^{k_\infty t} + \tilde{\delta}_0^2 e^{2k_\infty t}.$$

(2) *The contribution of* $\frac{n_+(t)}{(2k_\infty)^{\frac{1}{2}}} \langle g_\infty \big(V_{a(\infty)} - V_{a(t)} - (a(\infty) - a(t))\partial_\lambda V|_{\lambda=a(\infty)}\big), g_\infty\rangle$.

We can bound this by

$$\lesssim |\tilde{\delta}_0| \int_0^t e^{k_\infty(t-s)+\Gamma(s,t)} e^{k_\infty s} |a(\infty) - a(s)|^2\, ds \ll |\tilde{\delta}_0| e^{k_\infty t}.$$

We next consider the contributions of the constituents of $G_+(t)$:

(3) *The contribution of* $\frac{n_+(t)}{(2k_\infty)^{\frac{1}{2}}} \langle 20 u_* g_\infty (W_{a(t)}^3 - W_{a(0)}^3), g_\infty\rangle$.

Use the bound

$$\left|n_+(t)\langle u_* g_\infty (W^3_{a(t)} - W^3_{a(\infty)}), g_\infty\rangle\right| \ll \langle t\rangle^{-1}|a(\infty) - a(t)||n_+(t)|.$$

Hence the corresponding contribution is bounded by

$$\ll \int_0^t e^{k_\infty(t-s)+\Gamma(s,t)}\langle s\rangle^{-1}|a(\infty) - a(s)||n_+(s)|\,ds$$
$$\ll \int_0^t e^{k_\infty(t-s)+\Gamma(s,t)}\langle s\rangle^{-2}|\tilde{\delta}_0|e^{k_\infty s}\,ds \lesssim |\tilde{\delta}_0|e^{k_\infty t},$$

where we have used the bound (2.18) as well as the bootstrap assumption to control $n_+(t)$.

(4) *The contribution of* $\langle N(u_* + \eta, W_{a(t)}) - N(u_*, W_{a(t)}) - 20\delta(t)u_* g_\infty W^3_{a(t)}, g_\infty\rangle$.

Here we need to estimate the contributions of the following schematically written terms:

$$\langle u_*\tilde{\eta}W^3_{a(t)}, g_\infty\rangle,\ \langle \eta^2 W^3_{a(t)}, g_\infty\rangle,\ \langle \eta u^4_*, g_\infty\rangle,\ \langle \eta^5, g_\infty\rangle. \tag{2.28}$$

For the first term, we can bound the contribution by

$$\ll \int_0^t e^{k_\infty(t-s)+\Gamma(s,t)}(\langle s\rangle^{-1-\frac{\varepsilon}{2}}|\tilde{\delta}_0|e^{k_\infty s} + \tilde{\delta}_0^2\langle s\rangle^{-1}e^{2k_\infty s})\,ds \lesssim |\tilde{\delta}_0|e^{k_\infty t}.$$

The remaining terms are handled similarly.

(5) *The contribution of* $\langle\big(\mathcal{H}(a(\infty)) - \mathcal{H}(a(t))\big)[\tilde{\eta} + (2k_\infty)^{-\frac{1}{2}}n_-(t)g_\infty], g_\infty\rangle$.

Using (2.18) and Corollary 2.2, we bound the corresponding contribution by the exact same expression as in *(4)*.

This completes the proof of Proposition 2.1. □

It remains to prove Proposition 1.2. Thus fix a time T with $1 \gg |\tilde{\delta}_0|e^{k_\infty T} \gg \varepsilon_*$ where we can write

$$\tilde{u}(T,\cdot) = W_{a(T)} + u_* + \eta$$

as before. We need to pass to a representation

$$\tilde{u}(T,\cdot) = W_{\alpha_T} + \tilde{v}_{\alpha_T} \tag{2.29}$$

which satisfies $\langle \tilde{v}_{\alpha_T}, \Lambda^* g_{\alpha_T}\rangle = 0$. From [19] we can write

$$u_*(t,\cdot) = P_{g^\perp_\infty}u_* + \delta_*(t)g_\infty,\ |\delta_*(t)| \lesssim C(\varepsilon_*)\langle t\rangle^{-1}$$

In order to obtain the desired decomposition (2.29), we need to satisfy the relation

$$\langle P_{g^\perp_\infty}(u_* + \eta) + (\delta(T) + \delta_*(T))g_\infty + W_{a(T)} - W_{\alpha_T}, \Lambda^* g_{\alpha_T}\rangle = 0. \tag{2.30}$$

Observe that

$$W_{a(T)} - W_{\alpha_T} = (a(T) - \alpha_T)\partial_\lambda W_\lambda|_{\lambda=a(T)} + O(|a(T) - \alpha_T|^2)$$

and from (2.13) in [16] we have

$$|\langle \partial_\lambda W_\lambda|_{\lambda=a(T)}, \Lambda^* g_{a(T)}\rangle| \simeq 1$$

It follows that for $|a(T) - \alpha_T| \ll 1$ there is a unique solution of (2.30) which satisfies

$$|a(T) - \alpha_T| \lesssim |\tilde{\delta}_0|e^{k_\infty T} \ll 1.$$

To verify the condition (1.3), we need to compute

$$\langle P_{g_\infty^\perp}(u_* + \eta) + (\delta(T) + \delta_*(T))g_\infty + W_{a(T)} - W_{\alpha_T}, g_{\alpha_T}\rangle. \tag{2.31}$$

From Proposition 2.1 we have

$$|\delta(T)| \gg |\langle P_{g_\infty^\perp}(u_* + \eta), g_{\alpha_T}\rangle| + |\delta_*(T)|,$$

and furthermore

$$|\langle W_{a(T)} - W_{\alpha_T}, g_{\alpha_T}\rangle| = O(|a(T) - \alpha_T|^2) \ll |\tilde{\delta}_0| e^{k_\infty T} \simeq |\delta(T)|.$$

We have now proved the key growth condition

$$\langle \tilde{v}_{\alpha_T}, g_{\alpha_T}\rangle \simeq \tilde{\delta}_0 e^{k_\infty T}$$

which completes the proof of Proposition 1.2.

3. Proof of the Dispersive Estimate on $\|u_*\|_{L_x^\infty L_t^1}$.

This section is devoted to the one estimate, namely on $\|u_*\|_{L_x^\infty L_t^1}$, which is not contained in [19]. As evidenced by the previous section this norm is of crucial importance for the nonlinear argument.

This section is devoted to the proof of this estimate, starting with the linear case. We use the expansions for the linear evolution associated with $\Box + V$, $V = -5W^4$, as derived in [19]. In what follows, $H = -\Delta + V$ in $\mathbb{R}^3$ where $H\psi = 0$ and ψ is the unique zero energy resonance function, i.e., $|\psi(x)| \simeq |x|^{-1}$ for large $|x|$. We assume that H does not have zero energy eigenfunctions.

Proposition 3.1. *We have the bounds*

$$\left\| \Big(\frac{\sin(t\sqrt{H})}{\sqrt{H}} P_c - c_0 \psi \otimes \psi \Big) f \right\|_{L_x^\infty L_t^1} \lesssim \|f\|_{W^{1,1}} \tag{3.1}$$

$$\left\| \cos(t\sqrt{H}) P_c f \right\|_{L_x^\infty L_t^1} \lesssim \|f\|_{W^{2,1}} \tag{3.2}$$

Proof. We begin with $V = 0$. For the sine evolution, we get (putting the argument $x = 0$)

$$\begin{aligned} \int_0^\infty \frac{1}{t} \Big| \int_{[|y|=t]} f(y)\,\sigma(dy) \Big| \, dt &= \int_0^\infty t^{-2} \Big| \int_{[|y|\leqslant t]} \nabla(f(y)y)\, dy \Big| \, dt \\ &\lesssim \int_{\mathbb{R}^3} |\nabla f(y)|\, dy + \Big(\int_{\mathbb{R}^3} \frac{|f(y)|}{|y|}\, dy \Big) \lesssim \int_{\mathbb{R}^3} |\nabla f(y)|\, dy. \end{aligned} \tag{3.3}$$

The last step uses integration by parts in polar coordinates.

For the cosine evolution, one has

$$\begin{aligned} \cos(t\sqrt{H}) f(x) &= \partial_t\, t \int_{S^2} f(x + ty)\, \sigma(dy) \\ &= \int_{S^2} \big[f(x + ty) + t(\nabla f)(x + ty) \cdot y \big] \sigma(dy) \end{aligned}$$

and so

$$\|\cos(t\sqrt{H})f\|_{L_x^\infty L_t^1} \lesssim \left\|\frac{f}{|x|^2}\right\|_{L_x^1} + \left\|\frac{\nabla f}{|x|}\right\|_{L_x^1} \lesssim \|D^2 f\|_{L_x^1}.$$

In case $V \neq 0$ we write the $\frac{\sin(t\sqrt{H})}{\sqrt{H}}$ evolution in the form

$$\frac{1}{i\pi}\int_0^\infty \frac{\sin(t\lambda)}{\lambda}[R_V^+(\lambda^2) - R_V^-(\lambda^2)]\,\lambda d\lambda = \frac{1}{i\pi}\int_{-\infty}^\infty \sin(t\lambda)R(\lambda)d\lambda, \tag{3.4}$$

where we have set $R(\lambda) := R_V^+(\lambda^2)$ if $\lambda > 0$ and $R(\lambda) = \overline{R(-\lambda)}$ if $\lambda < 0$. For the free resolvent, we write this as R_0. Then, by the usual resolvent expansions,

$$R = \sum_{k=0}^{2n-1}(-1)^k R_0(VR_0)^k + (R_0V)^n R(VR_0)^n. \tag{3.5}$$

We distinguish between small energies and all other energies. For the latter, we use (3.5). Let $\chi_0(\lambda) = 0$ for all $|\lambda| \leqslant \lambda_0$ and $\chi_0(\lambda) = 1$ if $|\lambda| > 2\lambda_0$. Here $\lambda_0 > 0$ is some small parameter. Fix some k as in (3.5) and consider the contribution of the corresponding Born term (ignoring a factor of $(4\pi)^{-k-1}$):

$$\begin{aligned}
&\int_{\mathbb{R}^{3(k+2)}}\int_{-\infty}^\infty \chi_0(\lambda)\sin(t\lambda)e^{i\lambda\sum_{j=0}^k |x_j - x_{j+1}|}\frac{\prod_{j=1}^k V(x_j)}{\prod_{j=0}^k |x_j - x_{j+1}|}\, f(x_0)\,d\lambda\,dx_0\ldots dx_k \\
&= \frac{1}{2i}\sum_\pm \pm \int_{\mathbb{R}}\int_{\mathbb{R}^{3k}} \widehat{\chi_0}(\xi)\int_{[|x_0-x_1| = \pm t - \xi - \sum_{j=1}^k |x_j - x_{j+1}| > 0]} \frac{f(x_0)}{|x_0 - x_1|}\,\sigma(dx_0) \\
&\quad \frac{\prod_{j=1}^k V(x_j)}{\prod_{j=1}^k |x_j - x_{j+1}|}\,dx_1\ldots dx_k\,d\xi,
\end{aligned}\tag{3.6}$$

where x_{k+1} is fixed. Placing absolute values inside these integrals and integrating over $t \in \mathbb{R}$ yields an upper bound

$$\begin{aligned}
&\int_{\mathbb{R}} |\widehat{\chi_0}(\xi)| \int_{\mathbb{R}^3}\frac{|f(x_0)|}{|x_0 - x_1|}\,dx_0 \int_{\mathbb{R}^{3k}}\frac{\prod_{j=1}^k |V(x_j)|}{\prod_{j=1}^k |x_j - x_{j+1}|}\,dx_1\ldots dx_k\,d\xi \\
&\lesssim \|\nabla f\|_1 \|\nabla V\|_1^k.
\end{aligned}\tag{3.7}$$

It remains to bound the contribution by the final term in (3.5) which involves the resolvent $R(\lambda)$. Its kernel $K(x, y)$ can be reduced to the form

$$\begin{aligned}
&\int e^{\pm it\lambda}\chi_0(\lambda)\langle R(\lambda)(VR_0(\lambda))^n(\cdot, x), (VR_0(-\lambda))^n(\cdot, y)\rangle\,d\lambda \\
&= \int e^{i\lambda[\pm t + (|x| + |y|)]}\chi_0(\lambda)\langle R(\lambda)(VR_0(\lambda))^{n-1}VG_x(\lambda, \cdot), \qquad (3.8) \\
&\quad (VR_0(-\lambda))^{n-1}VG_y(-\lambda, \cdot)\rangle\,d\lambda, \qquad (3.9)
\end{aligned}$$

where

$$G_x(\lambda, u) := \frac{e^{i\lambda(|x-u| - |x|)}}{4\pi|x - u|}$$

and the scalar product appearing in (3.9) is just another way of writing the composition of the operators. One has the following elementary bounds, see for example Lemma 11 in [19]:

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^3}\left\|\frac{d^j}{d\lambda^j}G_x(\lambda,\cdot)\right\|_{L^{2,-\sigma}} < C_{j,\sigma} \ \text{ provided } \ \sigma > \frac{1}{2}+j\\
&\sup_{x\in\mathbb{R}^3}\left\|\frac{d^j}{d\lambda^j}G_x(\lambda,\cdot)\right\|_{L^{2,-\sigma}} < \frac{C_{j,\sigma}}{\langle x\rangle} \ \text{ provided } \ \sigma > \frac{3}{2}+j
\end{aligned} \tag{3.10}$$

for all $j \geqslant 0$. Let for some large n (say $n = 10$)

$$a_{x,y}(\lambda) := \chi_0(\lambda)\langle R(\lambda)(VR_0(\lambda))^{n-1}VG_x(\lambda,\cdot), (VR_0(-\lambda))^{n-1}VG_y(-\lambda,\cdot)\rangle$$

Then in view of the preceding one concludes that $a_{x,y}(\lambda)$ has two derivatives in λ and

$$\left|\frac{d^j}{d\lambda^j}a_{x,y}(\lambda)\right| \lesssim (1+\lambda)^{-2} \quad \text{for } \ j = 0,1,2 \ \text{ and all } \ \lambda > 1. \tag{3.11}$$

Moreover,

$$\left|\frac{d^j}{d\lambda^j}a_{x,y}(\lambda)\right| \lesssim (1+\lambda)^{-2}(\langle x\rangle\langle y\rangle)^{-1} \quad \text{for } \ j = 0,1, \ \text{ and all } \ \lambda > 1. \tag{3.12}$$

The decay in λ here comes from the limiting absorption principle which refers to the following standard bounds for the free and perturbed resolvents:

$$\begin{aligned}
&\|R_V(\lambda^2 \pm i0)\|_{L^{2,\sigma}\to L^{2,-\sigma}} \lesssim \lambda^{-1}, \qquad \sigma > \frac{1}{2}\\
&\|\partial_\lambda^\ell R_V(\lambda^2 \pm i0)\|_{L^{2,\sigma}\to L^{2,-\sigma}} \lesssim 1, \qquad \sigma > \frac{1}{2}+\ell, \qquad \ell \geqslant 1
\end{aligned} \tag{3.13}$$

for λ separated from zero. The estimates (3.11) and (3.12) only require $|V(x)| \lesssim \langle x\rangle^{-\kappa}$ with $\kappa > 3$.

Let us assume first that $t > 1$. To estimate (3.9) we distinguish between $|t - (|x| + |y|)| < t/10$ and the opposite case. In the former case, we conclude that

$$\max(|x|,|y|) \gtrsim t$$

so that due to (3.11) we obtain

$$\left|\int e^{i\lambda[\pm t+(|x|+|y|)]}a_{x,y}(\lambda)\,d\lambda\right| \lesssim \chi_{[|x|+|y|>t]}(\langle x\rangle\langle y\rangle)^{-1} \tag{3.14}$$

Integrating (3.14) over $t \in \mathbb{R}$ yields a bound $O(1)$ which implies an $L^1_x \to L^\infty_y L^1_t$ estimate.

In the latter case we integrate by parts twice which gains t^{-2} for $|t| > 1$ from (3.11):

$$\left|\int e^{i\lambda[\pm t+(|x|+|y|)]}a_{x,y}(\lambda)\,d\lambda\right| \lesssim |t|^{-2}.$$

For $|t| \lesssim 1$ one has $O(1)$. We can again integrate this over $t \in \mathbb{R}$ as before.

We now turn to the contribution of small λ to the sin-evolution. We recall the following representation of the resolvent at small energies, see (105) in [19]:

$$R(\lambda) = i\tfrac{\beta}{\lambda}R_0(\lambda)vS_1vR_0(\lambda) + R_0(\lambda) - R_0(\lambda)vE(\lambda)vR_0(\lambda), \tag{3.15}$$

where with $w := \sqrt{|V|}$,

$$S_1 = \|w\psi\|_2^{-2} w\psi \otimes w\psi =: \tilde{\psi} \otimes \tilde{\psi}$$

and $\beta = 4\pi \Big(\int_{\mathbb{R}^3} V\psi\, dx \Big)^{-2} \|w\psi\|_2^2$. For the explicit form of $E(\lambda)$ see (104) in [19]. Next, we describe the contribution of each of the three terms in (3.15) to the sine-transform (3.4). We can ignore the second one, since it leads to the free case. The first term on the right-hand side of (3.15) yields the following expression in (3.4):

$$\begin{aligned}
\mathcal{S}_0(t)(x,y) &:= \frac{\beta}{\pi} \int \frac{\sin(t\lambda)}{\lambda} \chi_1(\lambda) \big[R_0(\lambda) v S_1 v R_0(\lambda) \big](x,y)\, d\lambda \\
&:= \|w\psi\|_2^{-2} \beta\, \psi(x)\psi(y) - \|w\psi\|_2^{-2} \frac{\beta}{2\pi} \int_{\mathbb{R}^6} \int_{[|\tau|>t]} \widehat{\chi}_1(\tau + |x-x'| + |y'-y|) \\
&\quad \frac{V(x')\psi(x')\, V(y')\psi(y')}{4\pi|x-x'|\, 4\pi|y'-y|}\, d\tau\, dx' dy'.
\end{aligned}$$

We need to verify that uniformly in $x, y \in \mathbb{R}^3$ the integral over $t \in \mathbb{R}$ of the last line is $O(1)$. Indeed

$$\begin{aligned}
&\int_{\mathbb{R}^6} \int_{\mathbb{R}^6} \int_{[|\tau|>t]} |\widehat{\chi}_1(\tau + |x-x'| + |y'-y|)| \frac{|V(x')\psi(x')\, V(y')\psi(y')|}{4\pi|x-x'|\, 4\pi|y'-y|}\, d\tau\, dx' dy' \Big| \\
&\lesssim \int_{[|x-x'|+|y-y'|<t/2]} \int_{[|\tau|>t]} |\widehat{\chi}_1(\tau + |x-x'| + |y'-y|)|\, d\tau \\
&\quad \frac{|V(x')\psi(x')|\, |V(y')\psi(y')|}{|x-x'|\, |y'-y|}\, dx' dy' \\
&\quad + \int_{[|x-x'|+|y-y'|>t/2]} \int |\widehat{\chi}_1(\tau + |x-x'| + |y'-y|)|\, d\tau \\
&\quad \frac{|V(x')\psi(x')|\, |V(y')\psi(y')|}{|x-x'|\, |y'-y|}\, dx' dy'.
\end{aligned}$$

The first integral in the final expression is rapidly decaying in t, and thus gives the desired bound, whereas the second one upon integration in t is bounded by

$$\int (|x-x'| + |y-y'|) \frac{|V(x')\psi(x')|\, |V(y')\psi(y')|}{|x-x'|\, |y'-y|}\, dx' dy' \lesssim 1. \tag{3.16}$$

Finally, we turn to the third term on the right-hand side of (3.15). The convergence of the Neumann series defining $E(\lambda)$ in L^2 for small λ was established in [19]. We analyze the contribution by the constant term, viz.

$$E(0) = (A_0 + S_1)^{-1} + E_1(0) S_1 m(0)^{-1} S_1 + S_1 E_2(0) S_1 + S_1 m(0)^{-1} S_1 E_1(0),$$

see (104) in [19]. From (108), (109) in [19] one has

$$\begin{aligned}
&\int_{\mathbb{R}^3} \int_{-\infty}^{\infty} \sin(t\lambda) \chi_1(\lambda) [R_0(\lambda) v E(0) v R_0(\lambda)](x,y)\, d\lambda\ f(x)\, dx \\
&\quad = \frac{1}{32 i \pi^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^6} \int_{-\infty}^{\infty} \delta(t + \xi + [|x-x'| + |y'-y|])\, \widehat{\chi}_1(\xi)\, d\xi
\end{aligned} \tag{3.17}$$

$$\frac{v(x')E(0)(x',y')v(y')}{|x-x'|\,|y-y'|}\,dx'dy'\; f(x)\,dx$$
$$-\frac{1}{32i\pi^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^6}\int_{-\infty}^{\infty}\delta(-t+\xi+[|x-x'|+|y'-y|])\;\widehat{\chi}_1(\xi)\,d\xi \tag{3.18}$$
$$\frac{v(x')E(0)(x',y')v(y')}{|x-x'|\,|y-y'|}\,dx'dy'\; f(x)\,dx.$$

Placing absolute values inside these expressions and integrating over $t \in \mathbb{R}$ yields an upper bound of the form (for y fixed)

$$\lesssim \int_{\mathbb{R}^3}\int_{\mathbb{R}^6}\int_{-\infty}^{\infty}|\widehat{\chi}_1(\xi)|\,d\xi\;\frac{|v(x')E(0)(x',y')v(y')|}{|x-x'|\,|y-y'|}\,dx'dy'\,|f(x)|\,dx$$
$$+\int_{\mathbb{R}^3}\int_{\mathbb{R}^6}\int_{-\infty}^{\infty}|\widehat{\chi}_1(\xi)|\,d\xi\;\frac{|v(x')E(0)(x',y')v(y')|}{|x-x'|\,|y-y'|}\,dx'dy'\;|f(x)|\,dx$$

which in turn is bounded by

$$\|\widehat{\chi}_1\|_1 \sup_x \left\|\frac{v(x')}{|x-x'|}\right\|^2_{L^2_{x'}} \|\,|E(0)(\cdot,\cdot)|\,\|_{2\to 2}\|f\|_1 \lesssim \|f\|_1 \tag{3.19}$$

since $E(0)$ is absolutely bounded on L^2, see [19].

To deal with $E(\lambda)$ we proceed as in [19] using the $F(\lambda)$-method. To be specific, we claim the bound

$$\int_{-\infty}^{\infty}\left|\int_{\mathbb{R}^6}\int_{-\infty}^{\infty}\sin(t\lambda)\chi_1(\lambda)[R_0(\lambda)vF(\lambda)vR_0(\lambda)](x,y)\,d\lambda\; f(x)\,dx\right| dt \lesssim \|f\|_1 \tag{3.20}$$

provided the operator-valued function $F(\lambda)$ satisfies

$$\int_{-\infty}^{\infty}\left\|\,|\widehat{\chi_1 F}(\xi)(\cdot,\cdot)|\,\right\|_{2\to 2} d\xi < \infty \tag{3.21}$$

The latter property holds for $E(\lambda)$, see (113), (116), (117) in [19]. To prove (3.20) we let $\chi_1\chi_2 = \chi_1$ for some bump function χ_2 and compute

$$\int_{\mathbb{R}^3}\int_{-\infty}^{\infty}\sin(t\lambda)\chi_1(\lambda)[R_0(\lambda)vF(\lambda)vR_0(\lambda)](x,y)\,d\lambda\; f(x)\,dx$$
$$=\frac{1}{32i\pi^2}\int_{\mathbb{R}^3}\int\int_{\mathbb{R}^6}\int_{-\infty}^{\infty}\delta(t+\xi+\eta+[|x-x'|+|y'-y|])\;\widehat{\chi}_1(\xi)\,d\xi$$
$$\frac{v(x')\widehat{\chi_2 F}(\eta)(x',y')v(y')}{|x-x'|\,|y-y'|}\,dx'dy'd\eta\; f(x)\,dx$$
$$-\frac{1}{32i\pi^2}\int_{\mathbb{R}^3}\int\int_{\mathbb{R}^6}\int_{-\infty}^{\infty}\delta(-t+\xi+\eta+[|x-x'|+|y'-y|])\;\widehat{\chi}_1(\xi)\,d\xi$$
$$\frac{v(x')\widehat{\chi_2 F}(\eta)(x',y')v(y')}{|x-x'|\,|y-y'|}\,dx'dy'd\eta\; f(x)\,dx.$$

Placing absolute values inside and integrating over $t \in \mathbb{R}$ yields the upper bound

$$\|\widehat{\chi}_1\|_1 \sup_x \Big\| \frac{v(x')}{|x-x'|} \Big\|^2_{L^2_{x'}} \int_{-\infty}^{\infty} \Big\| |\widehat{\chi_1 F}(\xi)(\cdot,\cdot)| \Big\|_{2\to 2} d\xi \; \|f\|_1 \lesssim \|f\|_1 \tag{3.22}$$

uniformly in $y \in \mathbb{R}^3$. This concludes the small λ argument for the sin-evolution, and in combination with the previous estimate for $\lambda > \lambda_0 > 0$ we have established (3.1).

It remains to estimate the cos-evolution, see (3.2). We base our analysis on the relation by

$$\cos(t\sqrt{H})P_c = \partial_t \frac{\sin(t\sqrt{H})}{\sqrt{H}} P_c, \tag{3.23}$$

The small frequencies present no problem, as (3.23) shows that the only difference in the oscillatory integrals is a factor of λ, which is small and thus immaterial. On the other hand, for large λ this extra factor accounts for the additional derivative on the data. To be more specific, the final term in the Born-series (3.5) does not present a problem either. This is due to the fact that in (3.11) and (3.12) we may obtain arbitrary decay in λ by taking n in (3.5) as large as wished (but of course fixed). In particular, we can absorb the extra power of λ coming from the ∂_t. It therefore just remains to treat the summands in (3.5) involving only the free resolvent. In analogy with (3.6) one has

$$\int_{\mathbb{R}^{3(k+2)}} \int_{-\infty}^{\infty} \chi_0(\lambda) \cos(t\lambda)\lambda\, e^{i\lambda \sum_{j=0}^k |x_j - x_{j+1}|} \frac{\prod_{j=1}^k V(x_j)}{\prod_{j=0}^k |x_j - x_{j+1}|} f(x_0)\, d\lambda\, dx_0 \dots dx_k$$
$$= \int_{\mathbb{R}^{3(k+2)}} \int_{-\infty}^{\infty} \chi_0(\lambda) \cos(t\lambda)\, \mathcal{L}^* \Big[e^{i\lambda \sum_{j=0}^k |x_j - x_{j+1}|} \frac{\prod_{j=1}^k V(x_j)}{\prod_{j=0}^k |x_j - x_{j+1}|} f(x_0) \Big] d\lambda\, dx_0 \dots dx_k,$$

where x_{k+1} is fixed and with

$$\mathcal{L} := \frac{1}{i\lambda} \frac{x_0 - x_1}{|x_0 - x_1|} \cdot \partial_{x_0}.$$

Note that $\mathcal{L} e^{i\lambda|x_0 - x_1|} = e^{i\lambda|x_0-x_1|}$. The x_0-derivative in (3.24) can fall on either $|x_0 - x_1|^{-1}$ or $f(x_0)$. In the latter case we proceed exactly as in (3.6) and obtain an upper bound for the $L^\infty_y L^1_t$-norm by $\|D^2 f\|_1$. In the former case one replaces f with $\frac{f(x_0)}{|x_0 - x_1|}$ and again proceeds as in (3.6). The resulting bound is

$$\sup_{x' \in \mathbb{R}^3} \Big\| \nabla_x \Big(\frac{f(x)}{|x-x'|} \Big) \Big\|_{L^1_x} \lesssim \|D^2 f\|_1$$

as desired. □

We use the preceding proposition to obtain the following key bound on u_*:

Proposition 3.2. *Let $W_a(t) + u_*$ be the solution of* (1.1) *with data $u_*[0] = (f_1 + h(f_1, f_2)g_0, f_2) \in \Sigma$, as given in [19]. Then we have the bound*

$$\|u_*\|_{L^\infty_x L^1_t} \ll 1. \tag{3.24}$$

Proof. We use formula (33) in [19] which gives the representation

$$\begin{aligned} u_*(t,\cdot) &= \cos(t\sqrt{\mathcal{H}_\infty})P_{g_\infty^\perp}w_1 + \mathcal{S}(t)P_{g_\infty^\perp}w_2 \\ &- \int_0^t \dot{a}(s)\cos([t-s]\sqrt{\mathcal{H}_\infty})P_{g_\infty^\perp}\big[\partial_\lambda W_\lambda|_{\lambda=a(s)} - \big(\frac{a(\infty)}{a(s)}\big)^{\frac{5}{4}}\partial_\lambda W_\lambda|_{\lambda=a(\infty)}\big]\,ds \\ &- \int_0^t \mathcal{S}(t-s)P_{g_\infty^\perp}\big[(V_{a(\infty)} - V_{a(s)})u_*(s,\cdot) + N(u_*, W_{a(s)})\big]\,ds \\ &- R(t,\cdot) \end{aligned}$$

with $R(t,\cdot)$ compactly supported in t and bounded, whence irrelevant for the proof. Also, we have

$$w_1 = f_1 + h(f_1, f_2)g_0, \quad w_2 = f_2$$

and we use the notation

$$\mathcal{S}(t) = \frac{\sin(t\sqrt{H_\infty})}{\sqrt{H_\infty}}P_c - c_0\psi\otimes\psi$$

with the same notation as in Proposition 3.1. Then the bound (3.24) is implied by Proposition 3.1 for the expression

$$\cos(t\sqrt{\mathcal{H}_\infty})P_{g_\infty^\perp}w_1 + \mathcal{S}(t)P_{g_\infty^\perp}w_2$$

and hence it remains to bound the Duhamel terms. We write

$$\begin{aligned} &\int_0^t \dot{a}(s)\cos([t-s]\sqrt{\mathcal{H}_\infty})P_{g_\infty^\perp}[\ldots]\,ds \\ &= \int_0^\infty \dot{a}(s)\cos([t-s]\sqrt{\mathcal{H}_\infty})P_{g_\infty^\perp}[\ldots]\,ds \\ &\quad - \int_t^\infty \dot{a}(s)\cos([t-s]\sqrt{\mathcal{H}_\infty})P_{g_\infty^\perp}[\ldots]\,ds \end{aligned}$$

and similarly for the expression

$$\int_0^t \mathcal{S}(t-s)P_{g_\infty^\perp}[\ldots]\,ds$$

(1) *Contribution of the cosine terms.*
From [19] we infer the bound

$$\big|\nabla_x^j\big(\partial_\lambda W_\lambda|_{\lambda=a(s)} - \big(\frac{a(\infty)}{a(s)}\big)^{\frac{5}{4}}\partial_\lambda W_\lambda|_{\lambda=a(\infty)}\big)\big| \lesssim |a(\infty)-a(s)|\langle x\rangle^{-3-j}$$

whence from (2.4) we infer

$$\big\|\dot{a}(s)P_{g_\infty^\perp}\big[\partial_\lambda W_\lambda|_{\lambda=a(s)} - \big(\frac{a(\infty)}{a(s)}\big)^{\frac{5}{4}}\partial_\lambda W_\lambda|_{\lambda=a(\infty)}\big]\big\|_{L_s^1 W^{2,1}} \ll \big\|\langle s\rangle^{-3}\big\|_{L_s^1} \lesssim 1.$$

Then Proposition 3.1 implies

$$\Big\|\int_0^\infty \dot{a}(s)\cos([t-s]\sqrt{\mathcal{H}_\infty})P_{g_\infty^\perp}[\ldots]\,ds\Big\|_{L_x^\infty L_t^1} \ll 1.$$

For the second Duhamel cosine term, $\int_t^\infty \dot{a}(s)\cos([t-s]\sqrt{\mathcal{H}_\infty})P_{g_\infty^\perp}\big[\dots\big]\,ds$, we can crudely use Sobolev embedding $H^2(\mathbb{R}^3) \subset L^\infty$:

$$\begin{aligned}
&\Big|\int_t^\infty \dot{a}(s)\cos([t-s]\sqrt{\mathcal{H}_\infty})P_{g_\infty^\perp}\big[\dots\big]\,ds\Big| \\
&\leqslant \Big\|\int_t^\infty \dot{a}(s)\cos([t-s]\sqrt{\mathcal{H}_\infty})P_{g_\infty^\perp}\big[\dots\big]\,ds\Big\|_{H^2} \\
&\lesssim \int_t^\infty s^{-2}\|P_{g_\infty^\perp}\big[\dots\big]\|_{H^2}\,ds \ll t^{-2}
\end{aligned}$$

which is integrable.

(2) *Contribution of the sine terms.*
First, consider the term

$$\int_0^\infty \mathcal{S}(t-s)P_{g_\infty^\perp}\big[(V_{a(\infty)}-V_{a(s)})u_*(s,\cdot)+N(u_*,W_{a(s)})\big]\,ds.$$

Using Proposition 3.1, it suffices to prove

$$\|P_{g_\infty^\perp}\big[(V_{a(\infty)}-V_{a(s)})u_*(s,\cdot)+N(u_*,W_{a(s)})\|_{L_s^1W^{1,1}} \ll 1.$$

Note that

$$\begin{aligned}
\|(V_{a(\infty)}-V_{a(s)})u_*(s,\cdot)\|_{W^{1,1}} &\lesssim |a(\infty)-a(s)|\big(\|u_*(s,\cdot)\|_{L_x^\infty}+\|\nabla_x u\|_{L_x^2+L_x^\infty}\big) \\
&\ll \langle s\rangle^{-1-\varepsilon}
\end{aligned}$$

thanks to (2.2), which is integrable. As for the term $N(u_*,W_{a(s)})$, we consider the contributions of $u_*^2W_{a(s)}^3$, u_*^5. For the first, we obtain

$$\big\|u_*^2(s,\cdot)W_{a(s)}^3\big\|_{W^{1,1}} \lesssim \|u_*(s,\cdot)\|_{L_x^\infty}\|u_*\|_{W^{1,2}+W^{1,M}}\|W_{a(s)}^3\|_{L_x^{1+}} \lesssim \langle s\rangle^{-1-\frac{\varepsilon}{2}},$$

where we have interpolated between the second bound of (2.2) and the first one of (2.3); this decay rate is again integrable.

For the pure power term, we get

$$\big\|u_*^5\big\|_{W^{1,1}} \lesssim \|u_*(s,\cdot)\|_{L_x^\infty}\|u_*(s,\cdot)\|_{W^{1,2}+W^{1,M}}\|u_*^3\|_{L_x^{1+}\cap L_x^2} \ll \langle s\rangle^{-1-\frac{\varepsilon}{2}}.$$

Here we have also used the strong spatial decay estimate for u_*, i. e. the second bound of (2.3). This completes the estimate for the contribution of the first sine Duhamel term.

It remains to consider the expression

$$\int_t^\infty \mathcal{S}(t-s)P_{g_\infty^\perp}\big[(V_{a(\infty)}-V_{a(s)})u_*(s,\cdot)+N(u_*,W_{a(s)})\big]\,ds,$$

where we will again use a pointwise decay bound. This time we have to combine the strong dispersive bound provided by the key Proposition 9 in [19] with Sobolev. We decompose

$$\begin{aligned}
&\int_t^\infty \mathcal{S}(t-s)P_{g_\infty^\perp}\big[(V_{a(\infty)}-V_{a(s)})u_*(s,\cdot)+N(u_*,W_{a(s)})\big]\,ds \\
&=\int_t^{t+1} \mathcal{S}(t-s)P_{g_\infty^\perp}\big[(V_{a(\infty)}-V_{a(s)})u_*(s,\cdot)+N(u_*,W_{a(s)})\big]\,ds \\
&\quad+\int_{t+1}^\infty \mathcal{S}(t-s)P_{g_\infty^\perp}\big[(V_{a(\infty)}-V_{a(s)})u_*(s,\cdot)+N(u_*,W_{a(s)})\big]\,ds.
\end{aligned}$$

For the first term, use

$$\left\| (V_{a(\infty)} - V_{a(s)}) u_*(s, \cdot) + N(u_*, W_{a(s)}) \right\|_{H^2} \ll \langle s \rangle^{-2}$$

whence we get, using $H^2(\mathbb{R}^3) \subset L^\infty$,

$$\begin{aligned} &\| \int_t^{t+1} \mathcal{S}(t-s) P_{g_\infty^\perp} \big[(V_{a(\infty)} - V_{a(s)}) u_*(s, \cdot) + N(u_*, W_{a(s)}) \big] \, ds \\ &\quad \ll \int_t^{t+1} \langle s \rangle^{-2} \, ds \leqslant \langle t \rangle^{-2}, \end{aligned}$$

an integrable bound.

For the second integral above, we bound it by

$$\begin{aligned} &\| \int_{t+1}^\infty \mathcal{S}(t-s) P_{g_\infty^\perp} \big[(V_{a(\infty)} - V_{a(s)}) u_*(s, \cdot) + N(u_*, W_{a(s)}) \big] \, ds \|_{L_x^\infty} \\ &\lesssim \int_{t+1}^\infty (t-s)^{-1} \big\| \big[(V_{a(\infty)} - V_{a(s)}) u_*(s, \cdot) + N(u_*, W_{a(s)}) \big] \big\|_{W^{1,1}} \, ds \\ &\ll \int_{t+1}^\infty (t-s)^{-1} \langle s \rangle^{-1-\frac{\varepsilon}{2}} \, ds \lesssim \log t \langle t \rangle^{-1-\frac{\varepsilon}{2}}, \end{aligned}$$

which is again integrable in t. This concludes the proof of Proposition 3.2. □

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Communicated by P. Constantin