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GLOBAL CENTER STABLE MANIFOLD FOR THE DEFOCUSING ENERGY CRITICAL WAVE EQUATION WITH POTENTIAL

By Hao Jia, Baoping Liu, Wilhelm Schlag, and Guixiang Xu

Abstract. In this paper we consider the defocusing energy critical wave equation with a trapping potential in dimension 3. We prove that the set of initial data for which solutions scatter to an unstable excited state $(\phi,0)$ forms a finite co-dimensional path connected C^1 manifold in the energy space. This manifold is a global and unique center-stable manifold associated with $(\phi,0)$. It is constructed in a first step locally around *any solution scattering to* ϕ, which might be very far away from ϕ in the $\dot{H}^1\times L^2(\mathbb{R}^3)$ norm. In a second crucial step a no-return property is proved for any solution which starts near, but not on the local manifolds. This ensures that the local manifolds form a global one. Scattering to an unstable steady state is therefore a non-generic behavior, in a strong topological sense in the energy space. This extends a previous result of ours to the nonradial case. The new ingredients here are (i) application of the reversed Strichartz estimate from Beceanu-Goldberg to construct a local center stable manifold near any solution that scatters to $(\phi,0)$. This is needed since the endpoint of the standard Strichartz estimates fails nonradially. (ii) The nonradial channel of energy estimate introduced by Duyckaerts-Kenig-Merle, which is used to show that solutions that start off but near the local manifolds associated with ϕ emit some amount of energy into the far field in excess of the amount of energy beyond that of the steady state ϕ.

1. Introduction.

Fix $\beta>2$. Define

$$Y:=\left\{V\in C\big(\mathbb{R}^3\big):\ \sup_{x\in\mathbb{R}^3}\big(1+|x|\big)^{\beta}\big|V(x)\big|<\infty\right\}.$$

We study solutions to

$$\partial_{tt}u-\Delta u-Vu+u^5=0, \tag{1.1}$$

with initial data $\vec{u}(0)=(u_0,u_1)\in\dot{H}^1\times L^2(\mathbb{R}^3)$. Since for a short time the term Vu can be considered as a small perturbation, by adaptations of results in [2, 16, 17, 33] we know for any initial data $(u_0,u_1)\in\dot{H}^1\times L^2(\mathbb{R}^3)$, there exists a unique solution

$$u(t,x)\in C\big([0,\infty),\dot{H}^1\big)\cap L_t^5L_x^{10}\big([0,T)\times\mathbb{R}^3\big)$$

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for any $T < \infty$ to equation (1.1). Moreover, the energy

$$\mathcal{E}\big(\vec{u}(t)\big) = \mathcal{E}\big(u(t), \partial_t u(t)\big) := \int_{\mathbb{R}^3} \left[\frac{|\nabla u|^2}{2} + \frac{(\partial_t u)^2}{2} - \frac{Vu^2}{2} + \frac{u^6}{6}\right](x,t)\,dx$$

is conserved for all time.

If $V^+(x) = \max(V(x), 0)$ is large enough, then the operator $-\Delta - V$ may have negative eigenvalues. In this case, the equation admits a unique nontrivial ground state $Q > 0$ which is the global minimizer of

$$J(\phi) := \int_{\mathbb{R}^3} \left[\frac{|\nabla \phi|^2}{2} - \frac{V\phi^2}{2} + \frac{\phi^6}{6}\right] dx.$$

In addition to the ground states Q and $-Q$, there can be a number of "excited states" with higher energies (see Appendix A of [20]), which are changing sign steady states to equation (1.1) and decay like $O(\frac{1}{(1+|x|)})$. Small excited states are always unstable, but large excited states may be stable. These steady states play a fundamental role in understanding the long time dynamics for finite energy solutions to equation (1.1) with initial data of arbitrary energy. We say a steady state $(\phi, 0)$ is hyperbolic if the linearized operator $\mathcal{L}_\phi := \Delta - V + 5\phi^4$ around it has no zero eigenvalue nor zero resonance. We say a steady state $(\phi, 0)$ is stable if the linearized operator $\mathcal{L}_\phi$ has no negative eigenvalue. In the radial case we proved in [19, 20] that if we consider generic radial potential $V \in Y$ such that the equation admits only finitely many steady states, which are all hyperbolic, then generic data will lead to solutions that scatter to one of the stable steady states, while each unstable steady state will attract a finite codimensional C^1 manifold in the energy space. The result in [19] satisfactorily characterized the global dynamical behavior of all finite energy solutions to equation (1.1) in the radial case.

The proof in [19, 20] relies crucially on the channel of energy estimate for the linear wave equation which was first developed by Duyckaerts-Kenig-Merle [12, 14]. The channel of energy estimate works best for wave equation in dimension 3 with radial data. In this case for many nonlinear problems, it characterizes the steady states as the only solutions that do not radiate energy in either time direction. It is an essential ingredient in the work of Duyckaerts, Kenig and Merle [14] where they established the "soliton resolution" for all type II solutions (i.e., solutions that stay bounded in energy norm up to time infinity or finite blow up time.) for focusing energy critical wave equation with radial data in $\mathbb{R}^3$. In the nonradial case or other dimensions, there are only weaker versions of the channel of energy estimate available [9, 13, 21], and they have been used to establish similar resolution results for focusing energy critical wave equations either under size restriction for the initial data [13], or along a sequence of times [8, 10, 18, 31]. All the results mentioned here belong to a larger effort that aims to understand the long time dynamics for solutions of dispersive equations in the presence of nontrivial coherent structures. We recall that as defined in [32], coherent structures are solutions that are localized in space, uniformly in time. Examples are solitons, kinks, vortices, monopoles, breathers, etc. Due to the limitation of techniques to deal with problems beyond the perturbative regime, we are still at an early stage of understanding of this type of problem. Hence the current interest is to work on carefully chosen models in order to develop our intuition and technique.

We refer the reader to [5, 6, 7, 18, 22, 23] and references therein for the related results on equivariant wave maps, and to [34, 35] for results on nonlinear Schrödinger equations with potential.

In this paper, we consider nonradial solutions to (1.1) and construct the global center stable manifold for unstable excited states. This gives us a better understanding of the non-generic behavior of solutions. More precisely, our result shows that solutions that scatter to unstable excited states form a finite co-dimentional manifold in the energy space and hence such solutions are non-generic in a very precise, topological sense. Although such results are expected, it is often not easy to rigorously confirm them, in a non-perturbative setting.

More precisely, we say a solution $\vec{u}$ scatters to steady state $(\phi,0)$ as $t \to +\infty$ if there exists a finite energy free wave $\vec{u}^L$ (solution to the linear wave equation) such that

$$\left\|\vec{u}(t)-(\phi,0)-\vec{u}^L(t)\right\|_{\dot{H}^1\times L^2} \longrightarrow 0, \quad \text{as } t \longrightarrow +\infty.$$

We establish the following result.

THEOREM 1.1. *Let Ω be an open dense subset of Y such that equation* (1.1) *with $V \in \Omega$ has only finitely many steady states which are all hyperbolic. Let Σ be the set of steady states. Denote $\vec{u}(t) := \vec{S}(t)(u_0,u_1)$ as the solution to equation* (1.1) *with initial data $(u_0,u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^3)$. For each $(\phi,0) \in \Sigma$, define*

(1.2)
$$\mathcal{M}_\phi := \left\{\left(u_0,u_1\right) \in \dot{H}^1 \times L^2\left(\mathbb{R}^3\right) : \vec{S}(t)\left(u_0,u_1\right) \text{ scatters to } (\phi,0) \text{ as } t \longrightarrow +\infty\right\}.$$

Denote

$$\mathcal{L}_\phi := -\Delta - V + 5\phi^4 \tag{1.3}$$

as the linearized operator around ϕ. If $\mathcal{L}_\phi$ has no negative eigenvalues, then $\mathcal{M}_\phi$ is an open set $\subseteq \dot{H}^1 \times L^2(\mathbb{R}^3)$. If $\mathcal{L}_\phi$ has n negative eigenvalues, then $\mathcal{M}_\phi$ is a path connected C^1 manifold $\subset \dot{H}^1 \times L^2(\mathbb{R}^3)$ of co-dimension n.

Notice that the existence of the set Ω follows from [20, Theorem 6.1] and its proof. We note that there is no smallness assumption in the theorem, and the manifold can extend arbitrarily far away from the unstable steady state relative to the norm in $\dot{H}^1 \times L^2(\mathbb{R}^3)$.

Along the proof of Theorem 1.1, we also obtain completeness of scattering operator on the center manifold, i.e., for a fixed unstable steady state ϕ, given any linear wave u^L with finite energy, we can find a solution u to equation (1.1) such that u scatters to ϕ with the scattering profile u^L. See more details in Proposition 2.5.

Theorem 1.1 characterizes all solutions that scatter to a steady state. We expect that generically all solutions scatter to steady states. In the radial case, it was proved that for generic potential all finite energy solutions scatter to one of the steady states, but the proof depends on a particular form of the channel of energy inequality which is valid only in three dimensions and in the radial case. In the nonradial case, it remains an open problem how to characterize the generic behavior. It is perhaps worth pointing out that all nonradial large data results in the study of dynamics of nonlinear dispersive equations depend crucially on monotonicity formulae which are sensitive to algebraic features of the equation. There are currently no effective monotonicity formulae known for equation (1.1) in the nonradial case.

Compared with the radial case [19], we have two main difficulties in constructing the manifold:

(i) Consider any solution $\vec{U}$ that scatters to unstable excited states $(\phi,0)$. When we perturb around U, i.e., we write the solution as $U+\eta$, the resulting nonlinearity contains quadratic terms like $U(t)^3\eta^2$ which have a component that behaves like $\phi^3\eta^2$. Standard Strichartz estimate requires control of the nonlinearity in spaces such as $L^1_tL^2_x$, which forces us to estimate η in the endpoint Strichartz norm $L^2_tL^\infty_x$. However, the endpoint Strichartz estimate for free waves was shown to be false for general data in [24]. To overcome this technical obstacle, we use the reversed Strichartz estimate due to Beceanu and Goldberg [3]. By reverse Strichartz estimates, we mean estimates in the space $\|\cdot\|_{L^p_xL^q_t}$. That is, we first integrate in time and then in space, which is the reverse order of integration for the usual Strichartz estimates. This order of integration arises naturally in the context of KdV and derivative nonlinear Schrödinger equations, where the local smoothing effect needs to be exploited. For the wave equation the advantages of space-time reversal are less well known, see however Proposition 3.1 in [25] for an example of an $L^\infty_xL^1_t$ estimate which fails for $L^1_tL^\infty_x$. In that reference as well as in our case, the main feature is that the fundamental solution for linear wave equation in three dimensions is nonnegative and is integrable in time:

$$\int_0^\infty \frac{1}{t}\delta\big(|x|-t\big)\,dt=\frac{1}{|x|}. \tag{1.4}$$

This property can be used to trade decay in space for decay in time. For the $\phi^3\eta^2$ term, which is only quadratic in η, there is not enough decay in time to use the standard Strichartz estimates. On the other hand, there is enough decay in space thanks to the ϕ^3 term. This is exactly the right kind of problem where the reverse Strichartz estimates are more effective.

Using the reverse Strichartz estimates, we can follow the same techniques in [19] to construct a local, finite co-dimensional center stable manifold $\mathcal{M}$ near $\vec{U}(0)$ with the property that if a solution u starts on the manifold, i.e., $\vec{u}(0) \in \mathcal{M}$, then $\vec{u}(t)$ stays close to $\vec{U}(t)$ for all $t \geq 0$ and scatters to $(\phi, 0)$ as $t \to \infty$; if on the other hand, $\vec{u}(0)$ is close to $\vec{U}(0)$ but not on the manifold, then no matter how small $\|\vec{u}(0) - \vec{U}(0)\|_{\dot{H}^1 \times L^2}$ is, $\vec{u}(t)$ will deviate from $\vec{U}(t)$ by a fixed amount at a future time.

(ii) The local manifold construction ensures that any solution $\vec{u}(t)$ starting off the local manifold, i.e., $\vec{u}(0) \in B_\epsilon(\vec{U}(0)) \backslash \mathcal{M}_\phi$, will leave the time dependent neighborhood $B_\epsilon(\vec{U}(t))$ eventually. Up to this point, the argument is still essentially based on perturbative techniques. However, perturbative arguments alone are not sufficient to determine the dynamics when $\vec{u}(t)$ and $\vec{U}(t)$ separate from each other. In order to obtain information on the dynamics for all times, we use the channel of energy inequality introduced by Duyckaerts-Kenig-Merle [15] to show that the solution u necessarily radiates energy into the far field after it leaves $B_\epsilon(\vec{U}(t))$. This is the crucial global component in our paper. The channel of energy inequality we use here works for nonradial solutions and is not sensitive to the dimension. For another channel of energy inequality which applies in the nonradial case and in all dimensions, see the one for outgoing waves in [10, 11]. More precisely, since $\vec{U}$ scatters to ϕ, at large times we know that $\vec{U}(t)$ can essentially be identified as a free radiation at large distances and $(\phi, 0)$ in the finite region. If we take initial data $\vec{u}(0)$ and $\vec{U}(0)$ close enough so that at a given large time t the solutions are still sufficiently close, we can conclude that locally $\vec{u}(t)$ is essentially $(\phi, 0)$ plus a small but nontrivial perturbation. We will show that the perturbation contains a nontrivial unstable mode, which grows exponentially. Hence at a later time, when the unstable mode dominates all other modes, we use the channel of energy estimate in Lemma 3.6 to conclude that $\vec{u}$ will send out a fixed amount of energy into large distances and hence the energy left in the finite region is strictly less than that of $(\phi, 0)$. From this we know that $\vec{u}$ cannot scatter to $(\phi, 0)$. It is interesting to note that our argument shows that a solution, which starts close, but off of the manifold and far away from the unstable steady state, exhibits *two* types of radiation: a first radiation so that locally in space it is close to the unstable steady state at large times, and a second radiation which eventually pulls it off the steady state forever.

In effect, this second step is in the nature of a *one-pass theorem*, see [27, 28, 29]. While a virial identity is the key for the one-pass theorem in those references, here it is an exterior energy estimate.

Our paper is organized as follows. In Section 2, we construct the local center stable manifold for each solution that scatters to ϕ. In Section 3 we recall the perturbation lemma, prove the channel of energy estimate and also prove a result on the growth of the unstable modes. Lastly, in Section 4 we prove our main result Theorem 1.1.

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2. Construction of the local center-stable manifold. We begin with some notation. We use $c, C > 0$ to denote positive constants that may be different from line to line. For nonnegative quantities X and Y, we write $X \lesssim Y$ when $X \leq CY$ for some non-essential $C > 0$. When a given operator L has negative eigenvalues, we denote these as $-k^2$ with $k > 0$. Since we work with fixed potentials, we allow all constants to depend on the potential.

Let us first recall the definition of Lorentz spaces $L_x^{p,q}(\mathbb{R}^3)$ for $0 < p < \infty$ and $0 < q \leq \infty$

$$\|f\|_{L_x^{p,q}(\mathbb{R}^3)} := p^{\frac{1}{q}} \left\| \lambda \mu\{|f| \geq \lambda\}^{\frac{1}{p}} \right\|_{L^q(\mathbb{R}^+, \frac{d\lambda}{\lambda})}.$$

Here μ is the standard Lebesgue measure on $\mathbb{R}^3$. Clearly $L^{p,p} = L^p$ for any $0 < p < \infty$. We adopt the usual convention that $L^{\infty,\infty} = L^\infty$. Notice that $L^{p,q} \subset L^{p,r}$ whenever $q < r$. The Hölder inequality still holds for Lorentz spaces [30], viz.

$$\|fg\|_{L^{r,s}} \leq r' \|f\|_{L^{p_1,q_1}} \|g\|_{L^{p_2,q_2}} \text{ provided } \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r} < 1,\ \frac{1}{q_1} + \frac{1}{q_2} \geq \frac{1}{s} \tag{2.1}$$

and the endpoint

$$\|fg\|_{L^1} \leq \|f\|_{L^{p,q_1}} \|g\|_{L^{p',q_2}}, \quad \frac{1}{q_1} + \frac{1}{q_2} \geq 1. \tag{2.2}$$

Young's inequalities read as follows:

$$\|f * g\|_{L^{r,s}} \leq 3r \|f\|_{L^{p_1,q_1}} \|g\|_{L^{p_2,q_2}} \text{ provided } \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r} + 1 > 1,\ \frac{1}{q_1} + \frac{1}{q_2} \geq \frac{1}{s} \tag{2.3}$$

and the endpoint

$$\|f * g\|_{L^\infty} \leq \|f\|_{L^{p,q_1}} \|g\|_{L^{p',q_2}}, \quad \frac{1}{q_1} + \frac{1}{q_2} \geq 1. \tag{2.4}$$

Since $Y \subset L_x^{\frac{3}{2},1}(\mathbb{R}^3)$, Theorem 3, Theorem 1 and Corollary 2 of [3] imply the following *reversed Strichartz estimate* for wave equations with a potential $V \in Y$ in $\mathbb{R}^3$.

LEMMA 2.1. *Take $V \in Y$ such that the operator $-\Delta - V$ has no zero eigenvalues or zero resonance. Denote by $P^\perp$ the projection operator onto the continuous spectrum of $-\Delta - V$. Let*

$$\omega := \sqrt{P^\perp(-\Delta - V)}. \tag{2.5}$$

Let I be a time interval with $t_0 \in I$. Then for any $(f,g) \in \dot{H}^1 \times L^2(\mathbb{R}^3)$ and $F \in L_x^{6/5,2} L_t^\infty \cap L_x^{3/2,1} L_t^2(\mathbb{R}^3 \times I)$, the solution $\vec{\gamma}(t) = (\gamma(t), \partial_t\gamma(t))$ to the equation

$$\partial_{tt}\gamma + \omega^2\gamma = P^\perp F, \quad (t,x) \in I \times \mathbb{R}^3, \tag{2.6}$$

with $\vec{\gamma}(t_0) = P^\perp(f,g)$ satisfies

$$\begin{aligned} &\left\|(\gamma,\gamma_t)\right\|_{C_t^0(\dot{H}^1\times L^2)} + \|\gamma\|_{L_x^{6,2}L_t^\infty\cap L_x^\infty L_t^2(\mathbb{R}^3\times I)} \\ &\quad \le C\left(\left\|(f,g)\right\|_{\dot{H}^1\times L^2} + \|F\|_{L_x^{6/5,2}L_t^\infty\cap L_x^{3/2,1}L_t^2(\mathbb{R}^3\times I)}\right). \end{aligned} \tag{2.7}$$

The appearance of Lorentz spaces here is both natural and essential. Indeed, $|x|^{-1} \in L^{3,\infty}(\mathbb{R}^3)$, and by (2.2) or (2.4),

$$\sup_{y\in\mathbb{R}^3} \int_{\mathbb{R}^3} |f(x)||x-y|^{-1}\,dx \le C\|f\|_{L^{\frac{3}{2},1}(\mathbb{R}^3)},$$

cf. (1.4). Our main goal in this section is to prove the following result on the local center stable manifold.

THEOREM 2.2. *Let Ω be a dense open subset of Y such that equation* (1.1) *has only finitely many steady states, all of which are hyperbolic. Let $V \in \Omega \subset Y$. Suppose that $\vec{U}(t)$ is a finite energy solution to equation* (1.1) *which scatters to an unstable steady state* $(\phi, 0)$. *Let*

$$-k_1^2 \le -k_2^2 \le \cdots \le -k_n^2 < 0 \tag{2.8}$$

be the negative eigenvalues of $\mathcal{L}_\phi = -\Delta - V + 5\phi^4$ (counted with multiplicity) with orthonormal eigenfunctions $\rho_1, \rho_2, \dots, \rho_n$, respectively. We denote by P_i the projection operator onto the i-th eigenfunction and by $P^\perp$ the projection operator onto the continuous spectrum, i.e.,

$$P_i = \rho_i \otimes \rho_i, \quad P^\perp = I - \sum_{i=1}^n \rho_i \otimes \rho_i.$$

Decompose

$$\dot{H}^1 \times L^2(\mathbb{R}^3) = X_{cs} \oplus X_u, \tag{2.9}$$

where

$$X_{cs} = \left\{ (u_0, u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^3) : \left\langle k_j u_0 + u_1, \rho_j \right\rangle_{L^2} = 0, \text{ for all } 1 \le j \le n \right\}, \tag{2.10}$$

and

$$X_u = \operatorname{span}\left\{ (\rho_j, k_j\rho_j),\ 1 \le j \le n \right\}. \tag{2.11}$$

Then there exist $\epsilon_0 > 0$, T sufficiently large, a ball $B_{\epsilon_0}((0,0)) \subset \dot{H}^1 \times L^2(\mathbb{R}^3)$, and a smooth mapping

$$\Psi : \vec{U}(T) + \big(B_{\epsilon_0}((0,0)) \cap X_{cs}\big) \longrightarrow \dot{H}^1 \times L^2, \tag{2.12}$$

satisfying $\Psi(\vec{U}(T)) = \vec{U}(T)$, with the following property. Let $\widetilde{\mathcal{M}}$ be the graph of Ψ and set $\mathcal{M} = \vec{S}(-T)\widetilde{\mathcal{M}}$, where $\vec{S}(t)$ denotes the solution map for equation (1.1). *Then any solution to equation* (1.1) *with initial data $(u_0, u_1) \in \mathcal{M}$ scatters to $(\phi, 0)$. Moreover, there is an ϵ_1 with $0 < \epsilon_1 < \epsilon_0$, such that if a solution $\vec{u}(t)$ with initial data $(u_0, u_1) \in B_{\epsilon_1}(\vec{U}(0)) \subset \dot{H}^1 \times L^2(\mathbb{R}^3)$ satisfies*

$$\big\|\vec{u}(t) - \vec{U}(t)\big\|_{\dot{H}^1 \times L^2} < \epsilon_1 \quad \textit{for all } t \geq 0, \tag{2.13}$$

then $(u_0, u_1) \in \mathcal{M}$.

Remark. Ω as in the theorem exists, see [19]. The proof of Theorem 2.2 closely follows the argument for the local manifold in the radial case in [19]. However, there is an important additional technical difficulty: in order to control the quadratic nonlinear term $\phi^3\eta^2$ in η, we need to use reversed Strichartz estimates instead of the endpoint version of the standard Strichartz estimates—which do not hold in the nonradial case. We note that if ϵ_1 satisfies the theorem, then any smaller ϵ_1 will also suffice.

Proof. By the assumption that $\vec{U}$ scatters to ϕ, there exists a free radiation $\vec{U}^L \in \dot{H}^1 \times L^2(\mathbb{R}^3)$, such that

$$\lim_{t \to \infty} \big\|\vec{U}(t) - (\phi, 0) - \vec{U}^L(t)\big\|_{\dot{H}^1 \times L^2} = 0. \tag{2.14}$$

We now divide the construction of the center-stable manifold into the following four steps as those in [19].

Step 1: L^6 decay for free waves. We observe that for any finite energy free radiation $\vec{U}^L$, we have

$$\big\|U^L(t)\big\|_{L^6_x} \longrightarrow 0 \quad \text{as } t \longrightarrow \infty. \tag{2.15}$$

This is a simple consequence of the dispersive estimate for smooth free waves, and an approximation argument.

Step 2: Reversed space-time estimates for the radiation term $U - \phi$. Denote $h(t,x) = U(t,x) - \phi(x)$, then the radiation term h satisfies

$$h_{tt} - \Delta h - V(x)h + 5\phi^4 h + N(\phi, h) = 0, \tag{2.16}$$

where

$$N(\phi, h) = (\phi + h)^5 - \phi^5 - 5\phi^4 h = 10\phi^3 h^2 + 10\phi^2 h^3 + 5\phi h^4 + h^5.$$

In what follows, we will show that

$$\|h\|_{L_x^{6,2}L_t^\infty \cap L_x^\infty L_t^2(\mathbb{R}^3\times[T,\infty))} < \infty,$$

for sufficiently large T. From Agmon's estimate in [1], the eigenfunctions $\{\rho_i\}_i$ decay exponentially. Decomposing

$$h = \lambda_1(t)\rho_1 + \cdots + \lambda_n(t)\rho_n + \gamma,$$

with $\gamma \perp \rho_i$ for $i = 1,\ldots,n$, and plugging this into equation (2.16), we obtain

$$\sum_{i=1}^{n}\big(\ddot{\lambda}_i(t) - k_i^2\lambda_i(t)\big)\rho_i + \ddot{\gamma} + \mathcal{L}_\phi\gamma = N(\phi,h), \tag{2.17}$$

where $\mathcal{L}_\phi = -\Delta - V + 5\phi^4$. By orthogonality between $\gamma(t)$ and ρ_i, $i = 1,\ldots,n$, we derive the following equations for $\lambda_i(t)$ and $\gamma(t,x)$:

$$\begin{cases} \ddot{\lambda}_i(t) - k_i^2\lambda_i(t) = P_iN(\phi,h) := N_{\rho_i}, & i = 1,\ldots,n \\ \ddot{\gamma} + \omega^2\gamma = P^\perp N(\phi,h) := N_c, & \omega := \sqrt{P^\perp\mathcal{L}_\phi}. \end{cases} \tag{2.18}$$

By the decay of the potential V and the steady state ϕ, we know that $-V+5\phi^4$ in the linearized operator $\mathcal{L}_\phi$ decays like $O(\frac{1}{(1+|x|)^{\min\{\beta,4\}}})$, which is better than the critical rate $O(\frac{1}{|x|^2})$ as $|x| \to \infty$. Hence we can apply the result of Proposition 6 in [3] and conclude that the reversed Strichartz estimates as in Lemma 2.1 hold for solutions to the equation

$$\gamma_{tt} + \omega^2\gamma = F, \tag{2.19}$$

where F satisfies the compatibility condition $P^\perp F = F$.

From (2.14) and (2.15), we know that

$$\lim_{T\to\infty}\big\|h(t,x)\big\|_{L_t^\infty L_x^6([T,\infty)\times\mathbb{R}^3)} = 0.$$

Also by the exponential decay of ρ_i, we have

$$\big|\lambda_i(t)\big| = \big|\langle\rho_i|h\rangle\big| \le \big\|\rho_i\big\|_{L^{\frac{6}{5}}}\big\|h(t,x)\big\|_{L_x^6(\mathbb{R}^3)} \longrightarrow 0 \quad \text{as } t \longrightarrow \infty.$$

Let $\Gamma(t)$ be the solution operator for the equation $\gamma_{tt} + \omega^2\gamma = 0$, i.e.,

$$\Gamma(t-t_0)\big(\gamma(t_0),\dot{\gamma}(t_0)\big) = \cos\big(\omega(t-t_0)\big)\gamma(t_0) + \frac{1}{\omega}\sin\big(\omega(t-t_0)\big)\dot{\gamma}(t_0).$$

We claim:

Claim 2.2.1.

$$\lim_{T\to+\infty}\big\|\Gamma(t-T)\big(\gamma(T),\dot{\gamma}(T)\big)\big\|_{L_x^{6,2}L_t^\infty\cap L_x^\infty L_t^2(\mathbb{R}^3\times[T,\infty))} = 0. \tag{2.20}$$

We postpone the proof of Claim 2.2.1 to the end of this section.

Hence given a small positive number $\epsilon \ll 1$, which will be chosen later, we can pick a large time $T = T(\epsilon, U)$, such that

$$\|h\|_{L_t^\infty L_x^6([T,\infty)\times\mathbb{R}^3)} \le \epsilon \tag{2.21}$$

$$\big\|\lambda_i(t)\big\|_{L_t^\infty([T,\infty))} \le \epsilon \tag{2.22}$$

$$\big\|\Gamma(t-T)\big(\gamma(T),\dot\gamma(T)\big)\big\|_{L_x^{6,2}L_t^\infty \cap L_x^\infty L_t^2(\mathbb{R}^3\times[T,\infty))} \le \epsilon. \tag{2.23}$$

From (2.23) and (2.22), by the reverse Strichartz estimates in Lemma 2.1, it follows that the linear solution h^L to

$$\partial_{tt}h^L - \Delta h^L + 5\phi^4 h^L - V h^L = 0,$$

with initial data $\vec{h}^L(T) = \vec{h}(T)$ satisfies that

$$\big\|h^L\big\|_{L_x^{6,2}L_t^\infty \cap L_x^\infty L_t^2(\mathbb{R}^3\times[T,\widetilde{T}))} \le \frac{K}{2}\epsilon, \tag{2.24}$$

if $\widetilde{T}$ is sufficiently close to T. We can then use standard perturbation arguments to show that $h \in L_x^{6,2}L_t^\infty \cap L_x^\infty L_t^2(\mathbb{R}^3 \times [T,\widetilde{T}))$ with

$$\|h\|_{L_x^{6,2}L_t^\infty \cap L_x^\infty L_t^2(\mathbb{R}^3\times[T,\widetilde{T}))} \le K\epsilon,$$

as long as we choose ϵ to be sufficiently small. Here we take K large enough so that it dominates any constants appearing in the reverse Strichartz estimates. By a continuity argument, we shall prove that

$$\|h\|_{L_x^{6,2}L_t^\infty \cap L_x^\infty L_t^2(\mathbb{R}^3\times[T,\widetilde{T}))} \le K\epsilon, \tag{2.25}$$

for all $\widetilde{T}$, not just for $\widetilde{T}$ that are close to T. Suppose that (2.25) holds for $\widetilde{T}$, we shall show that for a small $\delta > 0$, (2.25) holds for $\widetilde{T} + \delta$.

Claim 2.2.2. Let h be a solution to the equation (2.16) with

$$\|h\|_{L_x^{6,2}L_t^\infty \cap L_x^\infty L_t^2(\mathbb{R}^3\times[T,\tilde{T}))} \le K\epsilon$$

and

$$\|h\|_{L_t^\infty L_x^6(\mathbb{R}^3\times[T,\tilde{T}))} \le \epsilon.$$

Suppose that $K > 10$. If ϵ is sufficiently small, then for $\delta > 0$ sufficiently small, we have

$$\|h\|_{L_x^{6,2}L_t^\infty \cap L_x^\infty L_t^2(\mathbb{R}^3\times[T,\tilde{T}+\delta))} \le C_1 K\epsilon,$$

where C_1 is a constant that only depends on V.

The claim will be proved at the end of the theorem. We note that due to the use of $L_x^{6,2}L_t^\infty$ type spaces, the continuity in time is not obvious. From the equation for $\lambda_i(t)$ in (2.18) and the uniform bound (2.22) on λ_i, we conclude that for $t \geq T$

$$\begin{aligned}
\lambda_i(t) &= \cosh\big(k_i(t-T)\big)\lambda_i(T) + \frac{1}{k_i}\sinh\big(k_i(t-T)\big)\dot{\lambda}_i(T)\\
&\quad + \frac{1}{k_i}\int_T^t \sinh\big(k_i(t-s)\big)N_{\rho_i}(s)\,ds\\
&= \frac{e^{k_i(t-T)}}{2}\left[\lambda_i(T) + \frac{1}{k_i}\dot{\lambda}_i(T) + \frac{1}{k_i}\int_T^t e^{k_i(T-s)}N_{\rho_i}(s)\,ds\right]\\
&\quad + e^{-k_i(t-T)}\left[\lambda_i(T) + \frac{1}{2k_i}\int_T^\infty e^{k_i(T-s)}N_{\rho_i}(s)ds\right]\\
&\quad - \frac{1}{2k_i}\int_T^\infty e^{-k_i|t-s|}N_{\rho_i}(s)ds,
\end{aligned}$$

where the last line remains bounded(in fact decays to 0 as $t \to +\infty$) for bounded $N_{\rho_i}(s)$. By (2.22) and the above formula, we obtain the following stability condition

$$\dot{\lambda}_i(T) = -k_i\lambda_i(T) - \int_T^\infty e^{k_i(T-s)}N_{\rho_i}(s)ds. \tag{2.26}$$

Under this condition we can rewrite equation (2.18) as the following integral equation

$$\begin{cases}
\lambda_i(t) = e^{-k_i(t-T)}\left[\lambda_i(T) + \dfrac{1}{2k_i}\displaystyle\int_T^\infty e^{k_i(T-s)}N_{\rho_i}(s)ds\right]\\
\qquad\qquad - \dfrac{1}{2k_i}\displaystyle\int_T^\infty e^{-k_i|t-s|}N_{\rho_i}(s)ds,\\
\qquad = e^{-k_i(t-T)}\left[\lambda_i(T) + \dfrac{1}{2k_i}\displaystyle\int_T^{\widetilde{T}+\delta} e^{k_i(T-s)}N_{\rho_i}(s)ds\right]\\
\qquad\qquad - \dfrac{1}{2k_i}\displaystyle\int_T^{\widetilde{T}+\delta} e^{-k_i|t-s|}N_{\rho_i}(s)ds\\
\qquad\qquad + e^{-k_i(t-T)}\dfrac{1}{2k_i}\displaystyle\int_{\widetilde{T}+\delta}^\infty e^{k_i(T-s)}N_{\rho_i}(s)ds\\
\qquad\qquad - \dfrac{1}{2k_i}\displaystyle\int_{\widetilde{T}+\delta}^\infty e^{-k_i|t-s|}N_{\rho_i}(s)ds,\\
\gamma(t) = \cos\big(\omega(t-T)\big)\gamma(T) + \dfrac{1}{\omega}\sin\big(\omega(t-T)\big)\dot{\gamma}(T)\\
\qquad\qquad + \displaystyle\int_T^t \frac{\sin\big(\omega(t-s)\big)}{\omega}N_c(s)ds.
\end{cases} \tag{2.27}$$

By (2.27) and the reversed Strichartz estimates in Lemma 2.1, we get that

(2.28)
$$\big\|\lambda_i(t)\big\|_{L^2([T,\widetilde{T}+\delta))} \le C\Big(\big|\lambda_i(T)\big| + \big\|N_{\rho_i}\big\|_{L^2_t([T,\widetilde{T}+\delta))} + \big\|N_{\rho_i}\big\|_{L^\infty_t([\widetilde{T}+\delta,\infty))}\Big),$$

and

$$\begin{aligned}
&\|\gamma\|_{L^{6,2}_x L^\infty_t \cap L^\infty_x L^2_t(\mathbb{R}^3\times[T,\widetilde{T}+\delta))}\\
&\le C\Big(\big\|\Gamma(t-T)\big(\gamma(T),\dot\gamma(T)\big)\big\|_{L^{6,2}_x L^\infty_t\cap L^\infty_x L^2_t(\mathbb{R}^3\times[T,\widetilde{T}+\delta))}\\
&\qquad + \|N_c\|_{L^{6/5,2}_x L^\infty_t \cap L^{3/2,1}_x L^2_t(\mathbb{R}^3\times[T,\widetilde{T}+\delta))}\Big).
\end{aligned} \tag{2.29}$$

Here the constant C depends on the L^1 and L^2 integrals of $e^{-k_i t}$ and on the constants in the reversed Strichartz estimates. Notice that instead of estimating the energy norm $\dot H^1\times L^2(\mathbb{R}^3)$ of $(\gamma(T),\dot\gamma(T))$ in (2.29), which may not be small, we estimate its free evolution in $L^{6,2}_x L^\infty_t \cap L^\infty_x L^2_t(\mathbb{R}^3\times[T,\widetilde{T}+\delta))$. Consequently, we can obtain smallness thanks to (2.23).

On the one hand, by the fact that

$$N_{\rho_i} = \langle \rho_i | N(\phi,h)\rangle, \quad N_\rho = \sum_i N_{\rho_i}\,\rho_i, \quad N_c = N - N_\rho$$

and the exponential decay of ρ_i, we have

$$\begin{aligned}
&\big\|N_{\rho_i}\big\|_{L^2_t([T,\widetilde{T}+\delta))} + \big\|N_c\big\|_{L^{6/5,2}_x L^\infty_t\cap L^{3/2,1}_x L^2_t(\mathbb{R}^3\times[T,\widetilde{T}+\delta))}\\
&\quad \le C\big\|N(\phi,h)\big\|_{L^{6/5,2}_x L^\infty_t\cap L^{3/2,1}_x L^2_t(\mathbb{R}^3\times[T,\widetilde{T}+\delta))}.
\end{aligned} \tag{2.30}$$

By the Hölder inequality in Lorentz spaces, noting that ϕ does not depend on time, we have

$$\begin{aligned}
&\big\|\phi^3 h^2\big\|_{L^{6/5,2}_x L^\infty_t\cap L^{3/2,1}_x L^2_t(\mathbb{R}^3\times[T,\widetilde{T}+\delta))}\\
&\lesssim \|\phi\|^3_{L^6_x}\Big(\|h\|^2_{L^{6,2}_x L^\infty_t(\mathbb{R}^3\times[T,\widetilde{T}+\delta))} + \|h\|_{L^{6,2}_x L^\infty_t(\mathbb{R}^3\times[T,\widetilde{T}+\delta))}\|h\|_{L^\infty_x L^2_t(\mathbb{R}^3\times[T,\widetilde{T}+\delta))}\Big),\\
&\big\|\phi^2 h^3\big\|_{L^{6/5,2}_x L^\infty_t\cap L^{3/2,1}_x L^2_t(\mathbb{R}^3\times[T,\widetilde{T}+\delta))}\\
&\lesssim \|\phi\|^2_{L^6_x}\Big(\|h\|^3_{L^{6,2}_x L^\infty_t(\mathbb{R}^3\times[T,\widetilde{T}+\delta))} + \|h\|^2_{L^{6,2}_x L^\infty_t(\mathbb{R}^3\times[T,\widetilde{T}+\delta))}\|h\|_{L^\infty_x L^2_t(\mathbb{R}^3\times[T,\widetilde{T}+\delta))}\Big),
\end{aligned}$$

as well as

$$\begin{aligned}
&\|\phi h^4\|_{L^{6/5,2}_x L^\infty_t\cap L^{3/2,1}_x L^2_t(\mathbb{R}^3\times[T,\widetilde{T}+\delta))}\\
&\lesssim \|\phi\|_{L^6_x}\Big(\|h\|^4_{L^{6,2}_x L^\infty_t(\mathbb{R}^3\times[T,\widetilde{T}+\delta))} + \|h\|^3_{L^{6,2}_x L^\infty_t(\mathbb{R}^3\times[T,\widetilde{T}+\delta))}\|h\|_{L^\infty_x L^2_t(\mathbb{R}^3\times[T,\widetilde{T}+\delta))}\Big),\\
&\|h^5\|_{L^{6/5,2}_x L^\infty_t\cap L^{3/2,1}_x L^2_t(\mathbb{R}^3\times[T,\widetilde{T}+\delta))}\\
&\lesssim \|h\|^5_{L^{6,2}_x L^\infty_t(\mathbb{R}^3\times[T,\widetilde{T}+\delta))} + \|h\|^4_{L^{6,2}_x L^\infty_t(\mathbb{R}^3\times[T,\widetilde{T}+\delta))}\|h\|_{L^\infty_x L^2_t(\mathbb{R}^3\times[T,\widetilde{T}+\delta))}.
\end{aligned}$$

Consequently

$$
\begin{aligned}
&\|N_{\rho_i}\|_{L_t^2([T,\widetilde{T}+\delta))} + \|N_c\|_{L_x^{6/5,2}L_t^\infty \cap L_x^{3/2,1}L_t^2(\mathbb{R}^3\times[T,\widetilde{T}+\delta))} \\
&\leq C\sum_{j=2}^{5} \|h\|^j_{L_x^{6,2}L_t^\infty\cap L_x^\infty L_t^2(\mathbb{R}^3\times[T,\widetilde{T}+\delta))}.
\end{aligned} \tag{2.31}
$$

On the other hand, by (2.21) and the exponential decay of ρ_i, we have

$$
\begin{aligned}
\big\|N_{\rho_i}\big\|_{L_t^\infty([\widetilde{T}+\delta,\infty))} &\leq C\big\|\rho_i\big\|_{L_x^6(\mathbb{R}^3)}\big\|N(\phi,h)\big\|_{L_t^\infty L_x^{\frac{6}{5}}([\widetilde{T}+\delta,\infty)\times\mathbb{R}^3)} \\
&\leq C\sum_{i=2}^{5}\|\phi\|^{5-i}_{L_x^6}\|h\|^i_{L_t^\infty L_x^6([\widetilde{T}+\delta,\infty)\times\mathbb{R}^3)} \leq C\epsilon^2.
\end{aligned} \tag{2.32}
$$

The bounds on λ_i and γ imply an estimate on h via

$$
h = \sum_i \lambda_i\rho_i + \gamma.
$$

In fact, combining estimates (2.28), (2.29), (2.30)–(2.32), with (2.22), (2.23) and Claim 2.2.2, one concludes that

$$
\|h\|_{L_x^{6,2}L^\infty\cap L_x^\infty L_t^2(\mathbb{R}^3\times[T,\widetilde{T}+\delta))} \leq \frac{K}{2}\epsilon + C\left\{\sum_{j=2}^{5}\big(4C_1K\epsilon+\epsilon\big)^j + \epsilon^2\right\},
$$

here C only depends on the constants in the reversed Strichartz inequalities and $\|\phi\|_{L_x^6}$ and $\|\rho_i\|_{L_x^\infty\cap L_x^{6,2}}$. If we choose $\epsilon \ll 1$, which can be achieved by taking T sufficiently large, such that

$$
\epsilon + \sum_{j=2}^{5}\big(4C_1K+1\big)^j\epsilon^{j-1} < 1,
$$

say, then it follows that

$$
\|h\|_{L_x^{6,2}L_t^\infty\cap L_x^\infty L_t^2(\mathbb{R}^3\times[T,\widetilde{T}+\delta))} \leq K\epsilon. \tag{2.33}
$$

Hence, by a standard continuity argument, we conclude that (2.25) holds for all $\widetilde{T} > T$ and

$$
h \in L_x^{6,2}L_t^\infty \cap L_x^\infty L_t^2\big(\mathbb{R}^3\times[T,\infty)\big).
$$

Step 3: Construction of the center-stable manifold near a solution U. Given a finite energy solution U to (1.1) satisfying (2.14), we consider another finite energy solution u, with $\|\vec{U}(T)-\vec{u}(T)\|_{\dot{H}^1\times L^2(\mathbb{R}^3)}$ small for a fixed large time T, taken from Step 2. We write $u = U + \eta$, then η satisfies

$$\eta_{tt} - \Delta\eta - V(x)\eta + (U+\eta)^5 - U^5 = 0, \quad (t,x) \in (T,\infty).$$

With $U = \phi + h$, we can rewrite the equation as

$$\eta_{tt} + \mathcal{L}_\phi \eta + \tilde{N}(\phi,h,\eta) = 0, \quad (t,x) \in (T,\infty), \tag{2.34}$$

with

$$\tilde{N}(\phi,h,\eta) = (\phi+h+\eta)^5 - (\phi+h)^5 - 5\phi^4\eta.$$

Note that $\tilde{N}$ contains terms which are linear in η. However, a further inspection shows that the coefficients of the linear terms in η contains the factor h and hence decay in both space and time, and can be made small if we choose T sufficiently large. First decompose η as

$$\eta = \tilde{\lambda}_1(t)\rho_1 + \cdots + \tilde{\lambda}_n(t)\rho_n + \tilde{\gamma}, \quad \tilde{\gamma} \perp \rho_i \tag{2.35}$$

for $i = 1,\ldots,n$. We shall use similar arguments as in step 2 to obtain a solution η which stays small for all large, positive times, with given $(\tilde{\lambda}_1(T),\ldots,\tilde{\lambda}_n(T))$ and $(\tilde{\gamma},\dot{\tilde{\gamma}})(T)$. Note that in order to determine the solution η, we still have to determine $\dot{\tilde{\lambda}}(T)$. We can obtain equations for $\tilde{\lambda}_i, \tilde{\gamma}$ similar to (2.18). Since we seek a forward solution which grows at most polynomially, we obtain a similar necessary and sufficient stability condition as (2.26)

$$\dot{\tilde{\lambda}}_i(T) = -k_i\tilde{\lambda}_i(T) - \int_T^\infty e^{k_i(T-s)}\tilde{N}_{\rho_i}(s)ds. \tag{2.36}$$

Using equations (2.34) and (2.36) we arrive at the system of equations for $\tilde{\lambda}_i$ and $\tilde{\gamma}$,

$$\begin{cases} \tilde{\lambda}_i(t) = e^{-k_i(t-T)}\left[\tilde{\lambda}_i(T) + \dfrac{1}{2k_i}\displaystyle\int_T^\infty e^{k_i(T-s)}\tilde{N}_{\rho_i}(s)\,ds\right] \\ \qquad\qquad - \dfrac{1}{2k_i}\displaystyle\int_T^\infty e^{-k_i|t-s|}\tilde{N}_{\rho_i}(s)\,ds \\ \tilde{\gamma}(t) = \cos(\omega(t-T))\tilde{\gamma}(T) + \dfrac{1}{\omega}\sin\big(\omega(t-T)\big)\dot{\tilde{\gamma}}(T) \\ \qquad\qquad + \dfrac{1}{\omega}\displaystyle\int_T^t \sin\big(\omega(t-s)\big)\tilde{N}_c(s)\,ds. \end{cases} \tag{2.37}$$

Define

$$\big\|\big(\tilde{\lambda}_1,\ldots,\tilde{\lambda}_n,\tilde{\gamma}\big)\big\|_X := \sum_{i=1}^n \big\|\tilde{\lambda}_i(t)\big\|_{L_t^\infty \cap L_t^2([T,\infty))} + \|\tilde{\gamma}\|_{L_x^{6,2}L_t^\infty \cap L_x^\infty L_t^2(\mathbb{R}^3\times[T,\infty))}. \tag{2.38}$$

Estimating system (2.37), we obtain that

$$\begin{aligned}\|\tilde{\lambda}_i(t)\|_{L^\infty\cap L^2([T,\infty))} &\lesssim |\tilde{\lambda}_i(T)| + \|\tilde{N}_{\rho_i}\|_{L^\infty_t\cap L^2_t([T,\infty))} \\ &\lesssim |\tilde{\lambda}_i(T)| + \|\tilde{N}\|_{L^{6/5,2}_x L^\infty_t \cap L^{3/2,1}_x L^2_t(\mathbb{R}^3\times[T,\infty))},\end{aligned} \tag{2.39}$$

and

$$\begin{aligned}\|\tilde{\gamma}\|_{L^{6,2}_x L^\infty_t\cap L^\infty_x L^2_t(\mathbb{R}^3\times[T,\infty))} &\lesssim \|(\tilde{\gamma}(T),\dot{\tilde{\gamma}}(T))\|_{\dot{H}^1\times L^2} \\ &\quad + \|\tilde{N}\|_{L^{6/5,2}_x L^\infty_t \cap L^{3/2,1}_x L^2_t(\mathbb{R}^3\times[T,\infty))}.\end{aligned} \tag{2.40}$$

Note that

$$|\tilde{N}| \lesssim \sum_{j=1}^{4} |\phi^{4-j}h^j\eta| + \sum_{k\geq 2, i+j+k=5} |\phi^i h^j \eta^k|. \tag{2.41}$$

For the linear term in η, by the Hölder inequalities in Lorentz spaces (2.1), we get that

$$\begin{aligned}&\|\phi^3 h\eta\|_{L^{6/5,2}_x L^\infty_t \cap L^{3/2,1}_x L^2_t(\mathbb{R}^3\times[T,\infty))} \\ &\quad \leq \|\phi\|^3_{L^6_x}\|h\|_{L^{6,2}_x L^\infty_t(\mathbb{R}^3\times[T,\infty))}\|\eta\|_{L^{6,2}_x L^\infty_t\cap L^\infty_x L^2_t(\mathbb{R}^3\times[T,\infty))}, \\ &\|\phi^2 h^2\eta\|_{L^{6/5,2}_x L^\infty_t \cap L^{3/2,1}_x L^2_t(\mathbb{R}^3\times[T,\infty))} \\ &\quad \leq \|\phi\|^2_{L^6_x}\|h\|^2_{L^{6,2}_x L^\infty_t(\mathbb{R}^3\times[T,\infty))}\|\eta\|_{L^{6,2}_x L^\infty_t\cap L^\infty_x L^2_t(\mathbb{R}^3\times[T,\infty))}, \\ &\|\phi h^3\eta\|_{L^{6/5,2}_x L^\infty_t \cap L^{3/2,1}_x L^2_t(\mathbb{R}^3\times[T,\infty))} \\ &\quad \leq \|\phi\|_{L^6_x}\|h\|^3_{L^{6,2}_x L^\infty_t(\mathbb{R}^3\times[T,\infty))}\|\eta\|_{L^{6,2}_x L^\infty_t\cap L^\infty_x L^2_t(\mathbb{R}^3\times[T,\infty))}, \\ &\|h^4\eta\|_{L^{6/5,2}_x L^\infty_t \cap L^{3/2,1}_x L^2_t(\mathbb{R}^3\times[T,\infty))} \\ &\quad \leq \|h\|^4_{L^{6,2}_x L^\infty_t(\mathbb{R}^3\times[T,\infty))}\|\eta\|_{L^{6,2}_x L^\infty_t\cap L^\infty_x L^2_t(\mathbb{R}^3\times[T,\infty))}.\end{aligned}$$

By (2.33), we have

$$\left\|\sum_{j=1}^{4}\phi^{4-j}h^j\eta\right\|_{L^{6/5,2}_x L^\infty_t \cap L^{3/2,1}_x L^2_t(\mathbb{R}^3\times[T,\infty))} \leq C\epsilon\|\eta\|_{L^{6,2}_x L^\infty_t\cap L^\infty_x L^2_t(\mathbb{R}^3\times[T,\infty))}. \tag{2.42}$$

The higher order terms in η are easier to estimate. Similar to the above, we can always estimate h in $L^{6,2}_x L^\infty_t$, hence

(2.43)

$$\left\|\sum_{k\geq 2, i+j+k=5}\phi^i h^j\eta^k\right\|_{L^{6/5,2}_x L^\infty_t \cap L^{3/2,1}_x L^2_t(\mathbb{R}^3\times[T,\infty))} \leq C\sum_{k=2}^{5}\|\eta\|^k_{L^{6,2}_x L^\infty_t\cap L^\infty_x L^2_t(\mathbb{R}^3\times[T,\infty))}.$$

By definition of X, $\|\eta\|_{L_x^{6,2}L_t^\infty \cap L_x^\infty L_t^2(\mathbb{R}^3\times[T,\infty))} \leq C\|(\tilde{\lambda}_1,\dots,\tilde{\lambda}_n,\tilde{\gamma})\|_X$. We can combine (2.39), (2.40) and (2.42), (2.43) to get

(2.44)
$$\big\|\big(\tilde{\lambda}_1,\dots,\tilde{\lambda}_n,\tilde{\gamma}\big)\big\|_X \leq L\left(\sum_{i=1}^n \big|\tilde{\lambda}_i(T)\big| + \big\|\big(\tilde{\gamma}(T),\dot{\tilde{\gamma}}(T)\big)\big\|_{\dot{H}^1\times L^2}\right) + L\epsilon\big\|\big(\tilde{\lambda}_1,\dots,\tilde{\lambda}_n,\tilde{\gamma}\big)\big\|_X + L\sum_{k=2}^5 \big\|\big(\tilde{\lambda}_1,\dots,\tilde{\lambda}_n,\tilde{\gamma}\big)\big\|_X^k,$$

where $L>1$ is a constant only depending on the constants in the reversed Strichartz estimates, $\|\phi\|_{L^6(\mathbb{R}^3)}$ and $\|\rho_i\|_{L_x^\infty\cap L_x^{6,2}}$ (for convenience of later use, we will also assume $L>n$). This inequality implies that if we take $\epsilon=\epsilon_0$ sufficiently small (which can be achieved by choosing T suitably large), with

$$\sum_{i=1}^n \big|\tilde{\lambda}_i(T)\big| + \big\|\big(\tilde{\gamma}(T),\dot{\tilde{\gamma}}(T)\big)\big\|_{\dot{H}^1\times L^2} \leq \epsilon_0, \tag{2.45}$$

such that $L^3\epsilon_0 < \frac{1}{32}$, then the map defined by the right-hand side of system (2.37) takes a ball $B_{2L\epsilon_0}(0)\subseteq X$ into itself. Moreover, we can check by the same argument that this map is in fact a contraction on $B_{2L\epsilon_0}(0)\subseteq X$. Thus for any given small $(\tilde{\lambda}_1(T),\dots,\tilde{\lambda}_n(T),\tilde{\gamma}(T))$ satisfying (2.45), we obtain a unique fixed point of (2.37). It follows that

$$u(t,x) := U(t,x) + \sum_{i=1}^k \tilde{\lambda}_i(t)\rho_i + \tilde{\gamma}(t,x) \tag{2.46}$$

solves (1.1) on $\mathbb{R}^3\times[T,\infty)$, satisfying

$$\|\vec{u}-\vec{U}\|_{L_t^\infty([T,\infty);\dot{H}^1\times L^2)} \lesssim \sum_{i=1}^n \big|\tilde{\lambda}_i(T)\big| + \big\|\big(\tilde{\gamma}(T),\dot{\tilde{\gamma}}(T)\big)\big\|_{\dot{H}^1\times L^2} \tag{2.47}$$

with Lipschitz dependence on the data $\tilde{\lambda}_i(T)$ and $(\tilde{\gamma}(T),\dot{\tilde{\gamma}}(T))$. By the smoothness of the nonlinearity $\tilde{N}$, the integral terms in (2.37) depend on $\tilde{\lambda}_i,\tilde{\gamma}$ smoothly. Hence $\tilde{\lambda}_i(t),\tilde{\gamma}(t,x)$ and the solution $u(t,x)$ actually have smooth dependence on the data.

Step 4: Proof of scattering. In this step, we prove that the solution $\vec{u}$ constructed in step 3 scatters to the same steady state $(\phi,0)$ as $\vec{U}$.

For each solution $\vec{u}$ with the decomposition (2.46) and any time $T'\geq T$, we denote

(2.48)
$$\big\|\big(\tilde{\lambda}_1,\dots,\tilde{\lambda}_n,\tilde{\gamma}\big)\big\|_{X[T',\infty)} := \sum_{i=1}^n \big\|\tilde{\lambda}_i(t)\big\|_{L_t^\infty\cap L_t^2([T',\infty))} + \|\tilde{\gamma}\|_{L_x^{6,2}L_t^\infty\cap L_x^\infty L_t^2(\mathbb{R}^3\times[T',\infty))}.$$

Here $X[T,\infty)$ is the space X from step 3, and from the construction we know that

$$\left\|\left(\tilde{\lambda}_1,\dots,\tilde{\lambda}_n,\tilde{\gamma}\right)\right\|_{X[T,\infty)} < 2L\epsilon_0 < \frac{1}{16}.$$

We will show that $\|(\tilde{\lambda}_1,\dots,\tilde{\lambda}_n,\tilde{\gamma})\|_{X[T',\infty)} \to 0$ as $T' \to \infty$.

We shall need the following property of the linear evolution, which will be proved towards the end of this section:

Claim 2.2.3. For $(f_0, f_1) \in P^{\perp}(\dot{H}^1 \times L^2)$, denote

$$f(t,x) = \cos(\omega t) f_0 + \frac{1}{\omega}\sin(\omega t) f_1,$$

then we have

$$\lim_{T_0\to\infty} \left\| f(t,x) \right\|_{L_x^{6,2} L_t^\infty \cap L_x^\infty L_t^2(\mathbb{R}^3\times[T_0,\infty))} = 0. \tag{2.49}$$

And there exists a free wave $f^L(t,x)$ with data $\vec{f}^L(0) \in \dot{H}^1 \times L^2$, such that

$$\lim_{t\to+\infty} \left\| \vec{f}(t,x) - \vec{f}^L(t,x) \right\|_{\dot{H}^1\times L^2} = 0. \tag{2.50}$$

Using (2.49) in Claim 2.2.3, for the ϵ_0 chosen in step 3, we can take $T_1 > T$ large enough such that

$$\left\| e^{-k_i(t-T)} \tilde{\lambda}_i(T) \right\|_{L_t^\infty \cap L_t^2([T_1,\infty))} < \epsilon_0^2, \tag{2.51}$$

$$\left\| \cos\left(\omega(t-T)\right)\tilde{\gamma}(T) + \frac{1}{\omega}\sin\left(\omega(t-T)\right)\dot{\tilde{\gamma}}(T) \right\|_{L_x^{6,2} L_t^\infty \cap L_x^\infty L_t^2(\mathbb{R}^3\times[T_1,\infty))} < \epsilon_0^2. \tag{2.52}$$

We control the system (2.37) on the interval $[T_1,\infty)$ in the following fashion: we estimate the linear part on the interval $[T_1,\infty)$ using (2.51)(2.52), and then estimate the nonlinear (integral) term over the larger interval $[T,\infty)$. This yields

$$\begin{aligned} &\left\|\tilde{\lambda}_i(t)\right\|_{L^\infty\cap L^2([T_1,\infty))} \\ &\quad\lesssim \left\| e^{-k_i(t-T)} \tilde{\lambda}_i(T) \right\|_{L_t^\infty \cap L_t^2([T_1,\infty))} + \left\|\tilde{N}\right\|_{L_x^{6/5,2} L_t^\infty \cap L_x^{3/2,1} L_t^2(\mathbb{R}^3\times[T,\infty))}, \end{aligned} \tag{2.53}$$

$$\begin{aligned} &\|\tilde{\gamma}\|_{L_x^{6,2} L_t^\infty \cap L_x^\infty L_t^2(\mathbb{R}^3\times[T,\infty))} \\ &\quad\lesssim \left\| \cos\left(\omega(t-T)\right)\tilde{\gamma}(T) + \frac{1}{\omega}\sin\left(\omega(t-T)\right)\dot{\tilde{\gamma}}(T) \right\|_{L_x^{6,2} L_t^\infty \cap L_x^\infty L_t^2(\mathbb{R}^3\times[T_1,\infty))} \\ &\qquad + \|\tilde{N}\|_{L_x^{6/5,2} L_t^\infty \cap L_x^{3/2,1} L_t^2(\mathbb{R}^3\times[T,\infty))}. \end{aligned} \tag{2.54}$$

Combing these estimates with (2.41), (2.42), (2.43), we infer that (notice we assumed $L > n$)

$$\begin{aligned}
&\big\|\big(\tilde\lambda_1,\dots,\tilde\lambda_n,\tilde\gamma\big)\big\|_{X[T_1,\infty)}\\
&\quad\le (n+1)\epsilon_0^2 + L\epsilon_0\big\|\big(\tilde\lambda_1,\dots,\tilde\lambda_n,\tilde\gamma\big)\big\|_{X[T,\infty)} + L\sum_{k=2}^{5}\big\|\big(\tilde\lambda_1,\dots,\tilde\lambda_n,\tilde\gamma\big)\big\|_{X[T,\infty)}^k\\
&\quad\le L\epsilon_0^2 + 2L^2\epsilon_0^2 + L\sum_{k=2}^{5}\big(2L\epsilon_0\big)^k < 2L\big(2L\epsilon_0\big)^2.
\end{aligned}$$

Next, fix our choice of T_1 and rewrite system (2.37) by breaking the integral into finer pieces,

$$\tag{2.55}\begin{cases}
\tilde\lambda_i(t) = e^{-k_i(t-T)}\left[\tilde\lambda_i(T) + \dfrac{1}{2k_i}\displaystyle\int_T^{T_1} e^{k_i(T-s)}\tilde N_{\rho_i}(s)\,ds\right]\\
\qquad\qquad - \dfrac{1}{2k_i}\displaystyle\int_T^{T_1} e^{-k_i|t-s|}\tilde N_{\rho_i}(s)\,ds\\
\qquad\qquad + e^{-k_i(t-T_1)}\dfrac{1}{2k_i}\displaystyle\int_{T_1}^{\infty} e^{k_i(T_1-s)}\tilde N_{\rho_i}(s)\,ds\\
\qquad\qquad - \dfrac{1}{2k_i}\displaystyle\int_{T_1}^{\infty} e^{-k_i|t-s|}\tilde N_{\rho_i}(s)\,ds,\\
\tilde\gamma(t) = \cos\big(\omega(t-T)\big)\tilde\gamma(T) + \dfrac{1}{\omega}\sin\big(\omega(t-T)\big)\dot{\tilde\gamma}(T)\\
\qquad\qquad + \dfrac{1}{\omega}\displaystyle\int_T^{T_1}\sin\big(\omega(t-s)\big)\tilde N_c(s)\,ds + \dfrac{1}{\omega}\displaystyle\int_{T_1}^{t}\sin\big(\omega(t-s)\big)\tilde N_c(s)\,ds.
\end{cases}$$

We can pick $T_2 > T_1$ large enough such that the first line in the expression of $\tilde\lambda_i$ is small in $L_t^\infty\cap L_t^2([T_2,\infty))$, also the first line in the expression of $\tilde\gamma$ is small in $L_x^{6,2}L_t^\infty\cap L_x^\infty L_t^2(\mathbb{R}^3\times[T_2,\infty))$. We can require that they are bounded by ϵ_0^3. Note that we used Claim 2.2.3 for the term $\frac{1}{\omega}\int_T^{T_1}\sin(\omega(t-s))\tilde N_c(s)\,ds$, which can be viewed as a superposition of linear evolutions.

Then estimating the second line of $\tilde\lambda_i$ and $\tilde\gamma$ over the larger interval $[T_1,\infty)$, we obtain

$$\begin{aligned}
&\big\|\big(\tilde\lambda_1,\dots,\tilde\lambda_n,\tilde\gamma\big)\big\|_{X[T_2,\infty)}\\
&\quad\le (n+1)\epsilon_0^3 + L\left(\epsilon_0\big\|\big(\tilde\lambda_1,\dots,\tilde\lambda_n,\tilde\gamma\big)\big\|_{X[T_1,\infty)} + \sum_{k=2}^{5}\big\|\big(\tilde\lambda_1,\dots,\tilde\lambda_n,\tilde\gamma\big)\big\|_{X[T_1,\infty)}^k\right),\\
&\quad\le 2L\big(2L\epsilon_0\big)^3.
\end{aligned}$$

It is clear that this process can be repeated indefinitely: once we fix T_j, we can rewrite the system (2.37) as in (2.55), and find $T_{j+1} > T_j$ such that the first line is bounded by ϵ_0^{j+1}, which implies the estimate

$$\left\|\left(\tilde{\lambda}_1,\ldots,\tilde{\lambda}_n,\tilde{\gamma}\right)\right\|_{X[T_{j+1},\infty)} \leq 2L\left(2L\epsilon_0\right)^{j+1}.$$

In view of (2.35), (2.41), (2.42), (2.43) we conclude that

$$\begin{aligned} \lim_{T'\to+\infty} \left\|\left(\tilde{\lambda}_1,\ldots,\tilde{\lambda}_n,\tilde{\gamma}\right)\right\|_{X[T',\infty)} &= 0, \\ \lim_{T'\to\infty} \|\eta\|_{L_x^{6,2}L_t^\infty \cap L_x^\infty L_t^2(\mathbb{R}^3\times[T',\infty))} &= 0, \\ \lim_{T'\to\infty} \|\tilde{N}\|_{L_x^{6/5,2}L_t^\infty \cap L_x^{3/2,1}L_t^2(\mathbb{R}^3\times[T',\infty))} &= 0. \end{aligned}$$

These asymptotics allow us to write the asymptotic profile of $\tilde{\gamma}$ in the form

$$\begin{aligned} \tilde{\gamma}_\infty(t) = \cos\left(\omega(t-T)\right)\tilde{\gamma}(T) + \frac{1}{\omega}\sin\left(\omega(t-T)\right)\dot{\tilde{\gamma}}(T) \\ + \frac{1}{\omega}\int_T^\infty \sin\left(\omega(t-s)\right)\tilde{N}_c(s)\,ds, \end{aligned} \tag{2.56}$$

with the property that

$$\left\|\vec{\tilde{\gamma}} - \vec{\tilde{\gamma}}_\infty\right\|_{L_t^\infty \dot{H}\times L^2[T',\infty)} \lesssim \|\tilde{N}\|_{L_x^{6/5,2}L_t^\infty\cap L_x^{3/2,1}L_t^2(\mathbb{R}^3\times[T',\infty))} \longrightarrow 0 \quad \text{as } T' \longrightarrow +\infty.$$

$\tilde{\gamma}_\infty(t)$ can be further replaced with a free wave by (2.50) in Claim 2.2.3. Combining the preceding with the fact $\|\tilde{\lambda}_i(t)\|_{L_t^\infty\cap L_t^2([T',\infty))} \to 0$ as $T' \to +\infty$, we conclude that $\vec{u}$ scatters to the same steady state $(\phi, 0)$ as $\vec{U}$. We can now define

$$\Psi : \vec{U}(T) + \left(B_{\epsilon_0}\left((0,0)\right)\cap X_{cs}\right) \longrightarrow \dot{H}^1 \times L^2, \tag{2.57}$$

as follows: for any $(\tilde{\gamma}_0,\tilde{\gamma}_1) \in P^\perp(\dot{H}^1\times L^2(\mathbb{R}^3))$ and $\tilde{\lambda}_i \in \mathbb{R}$ such that

$$\sim := \sum_{i=1}^n \tilde{\lambda}_i\left(\rho_i, -k_i\rho_i\right) + \left(\tilde{\gamma}_0,\tilde{\gamma}_1\right) + \vec{U}(T) \in \vec{U}(T) + \left(B_{\epsilon_0}\left((0,0)\right)\cap X_{cs}\right),$$

set

$$\tilde{\lambda}_i(T) = \tilde{\lambda}_i, \text{ for } i = 1,\ldots,n \text{ and } \left(\tilde{\gamma}(T),\dot{\tilde{\gamma}}(T)\right) = \left(\tilde{\gamma}_0,\tilde{\gamma}_1\right).$$

Then with $\dot{\tilde{\lambda}}_i(T)$ given by (2.36), we define

$$\Psi(\sim) := \left(\sum_{i=1}^n \tilde{\lambda}_i(T)\rho_i + \tilde{\gamma}_0, \sum_{i=1}^n \dot{\tilde{\lambda}}_i(T)\rho_i + \tilde{\gamma}_1\right) + \vec{U}(T).$$

If ϵ_0 is chosen sufficiently small, then $\dot{\tilde{\lambda}}_i$ is uniquely determined by contraction mapping in the above. We define $\widetilde{\mathcal{M}}$ as the graph of Ψ and let $\mathcal{M}$ be $\vec{S}(-T)(\widetilde{\mathcal{M}})$. We can then check that Ψ, $\mathcal{M}$, $\widetilde{\mathcal{M}}$ verify the requirements of the theorem. Since $\vec{S}(T)$ is a diffeomorphism, $\mathcal{M}$ is a C^1 manifold.

Step 5: Unconditional uniqueness. Now suppose that a solution u to equation (1.1) satisfies

$$\|\vec{u} - \vec{U}\|_{L^\infty([0,\infty);\dot{H}^1\times L^2)} \leq \epsilon_1 \ll \epsilon_0.$$

We need to show that $\vec{u}(T) \in \widetilde{\mathcal{M}}$. We denote

$$\eta(t,x) = u(t,x) - U(t,x) = \sum_{i=1}^n \tilde{\lambda}_i(t)\rho_i + \tilde{\gamma}(t,x),$$

then $\vec{\eta} \in L_t^\infty([0,\infty);\dot{H}^1 \times L^2)$ with norm smaller than ϵ_1. Using similar arguments as in Step 2, we can conclude that for sufficiently large T and $\widetilde{T}$ which is bigger than but close to T,

$$\begin{aligned} \big\|\tilde{\lambda}_i(t)\big\|_{L_t^\infty([T,\infty))} + \big\|\vec{\tilde{\gamma}}(t,x)\big\|_{L_t^\infty([T,\infty);\dot{H}^1\times L^2)} &\leq C\epsilon_1, \\ \big\|\tilde{\lambda}_i(t)\big\|_{L^2([T,\tilde{T}))} &\leq C\epsilon_1, \\ \|\tilde{\gamma}\|_{L_x^{6,2}L_t^\infty \cap L_x^\infty L_t^2(\mathbb{R}^3\times[T,\tilde{T}))} &\leq C\epsilon_1. \end{aligned} \tag{2.58}$$

Notice the L^∞ bound on $\tilde{\lambda}_i$ implies that the stability condition (2.36) must hold true, we are again reduced to (2.37). Now we wish to show that

$$\tilde{\lambda}_i(t) \in L^2\big([T,\infty)\big), \quad \tilde{\gamma}(t,x) \in L_x^{6,2}L_t^\infty \cap L_x^\infty L_t^2\big(\mathbb{R}^3 \times [T,\infty)\big), \tag{2.59}$$

with a small norm, which together with the fixed point theorem imply $\vec{u}(T) \in \widetilde{\mathcal{M}}$. Pulling back from T to 0, we can obtain the desired result. To show (2.59), we follow similar arguments as in step 2. Define the norm

$$\big\|(\tilde{\lambda}_1,\ldots,\tilde{\lambda}_n,\tilde{\gamma})\big\|_{X([T,\tilde{T}))} := \sum_{i=1}^n \big\|\tilde{\lambda}_i(t)\big\|_{L_t^2([T,\tilde{T}))} + \|\tilde{\gamma}\|_{L_x^{6,2}L_t^\infty \cap L_x^\infty L_t^2(\mathbb{R}^3\times[T,\tilde{T}))}.$$

Similar to (2.28), (2.29), (2.42) and (2.43), we get

$$
\begin{aligned}
&\sum_{i=1}^{n}\big\|\tilde{\lambda}_i(t)\big\|_{L^2([T,\tilde{T}))}+\|\tilde{\gamma}\|_{L_x^{6,2}L_t^\infty\cap L_x^\infty L_t^2(\mathbb{R}^3\times[T,\tilde{T}))}\\
&\quad\le C\Bigg(\sum_{i=1}^{n}\big|\tilde{\lambda}_i(T)\big|+\big\|\big(\tilde{\gamma}(T),\dot{\tilde{\gamma}}(T)\big)\big\|_{\dot{H}^1\times L^2}\\
&\qquad\qquad+\|\tilde{N}\|_{L_x^{6/5,2}L_t^\infty\cap L_x^{3/2,1}L_t^2(\mathbb{R}^3\times[T,\tilde{T}))}+\|\tilde{N}\|_{L_t^\infty L_x^{\frac{6}{5}}([\tilde{T},\infty))}\Bigg)\\
&\quad\le C\epsilon_1+C\epsilon_0\|\eta\|_{L_x^{6,2}L_t^\infty\cap L_x^\infty L_t^2(\mathbb{R}^3\times[T,\tilde{T}))}+C\sum_{k=2}^{5}\|\eta\|^k_{L_x^{6,2}L_t^\infty\cap L_x^\infty L_t^2(\mathbb{R}^3\times[T,\tilde{T}))}\\
&\qquad+C\sum_{i+j+k=5,\,k\ge1}\|\phi\|^i_{L_x^6}\|h\|^j_{L_t^\infty L_x^6([\tilde{T},\infty))}\|\eta\|^k_{L_t^\infty L_x^6([\tilde{T},\infty))}\\
&\quad\le C\epsilon_1+C\epsilon_0\|\eta\|_{L_x^{6,2}L_t^\infty\cap L_x^\infty L_t^2(\mathbb{R}^3\times[T,\tilde{T}))}+C\sum_{k=2}^{5}\|\eta\|^k_{L_x^{6,2}L_t^\infty\cap L_x^\infty L_t^2(\mathbb{R}^3\times[T,\tilde{T}))},
\end{aligned}
$$

where the constant C may change from line to line. Hence by (2.58), we have

$$
\begin{aligned}
\big\|\big(\tilde{\lambda}_1,\dots,\tilde{\lambda}_n,\tilde{\gamma}\big)\big\|_{X([T,\tilde{T}))}&\le C\epsilon_1+C\epsilon_0\|(\tilde{\lambda}_1,\dots,\tilde{\lambda}_n,\tilde{\gamma})\|_{X([T,\tilde{T}))}\\
&\quad+L\sum_{k=2}^{5}\big\|\big(\tilde{\lambda}_1,\dots,\tilde{\lambda}_n,\tilde{\gamma}\big)\big\|^k_{X[T,\tilde{T})}.
\end{aligned}
$$

By a continuity argument similar to the one used in Step 2, we can conclude that

$$
\big\|\big(\tilde{\lambda}_1,\dots,\tilde{\lambda}_n,\tilde{\gamma}\big)\big\|_{X([T,\infty))}\le\liminf_{\tilde{T}\to\infty}\big\|\big(\tilde{\lambda}_1,\dots,\tilde{\lambda}_n,\tilde{\gamma}\big)\big\|_{X([T,\tilde{T}))}\le C\epsilon_1<\epsilon_0,
$$

We omit the routine details. □

Now we give the proof for Claim 2.2.1. Claim 2.2.1 will be proved as a consequence of the following lemma.

LEMMA 2.3. *Let $\vec{U}^L$ be a finite energy free radiation and $(\phi,0)$ be a steady state to equation* (1.1). *Recall that*

$$
\omega=\sqrt{P^{\perp}(-\Delta-V+5\phi^4)}.
$$

Let γ be the solution to

$$
\begin{cases}\partial_{tt}\gamma+\omega^2\gamma=0, & \textit{in } [T,\infty)\times\mathbb{R}^3,\\ \vec{\gamma}(T)=P^{\perp}\big(\vec{U}^L(T)\big).\end{cases}\tag{2.60}
$$

For any $\epsilon > 0$*, if we take* $T = T(\epsilon, \vec{U}^L) > 0$ *sufficiently large, then*

$$\|\gamma\|_{L_x^{6,2}L_t^\infty \cap L_x^\infty L_t^2(\mathbb{R}^3\times[T,\infty))} < \epsilon. \tag{2.61}$$

Proof. For a given $\epsilon > 0$, fix $0 < \delta \ll \epsilon$ to be determined below. We can take a smooth compactly supported (in space) free radiation $\vec{\widetilde{U}}^L$ such that

$$\big\|\vec{U}^L(0) - \vec{\widetilde{U}}^L(0)\big\|_{\dot{H}^1\times L^2(\mathbb{R}^3)} \le \delta. \tag{2.62}$$

Let us assume that $\mathrm{supp}(\vec{\widetilde{U}}^L(0)) \Subset B_R(0)$ for some $R > 0$. Hence by the strong Huygens' principle, for large time t we have

$$\big|\widetilde{U}^L(t,x)\big| \le \frac{C}{t}\chi_{[t-R\le|x|\le t+R]}, \quad \text{for } t > R.$$

Now for $T \gg R$, by direct computation we get that

$$\begin{aligned}\|\widetilde{U}^L(t,x)\|_{L_x^{6,2}L_t^\infty(\mathbb{R}^3\times[T,\infty))} &\lesssim \left\|\frac{1}{t}\chi_{[t-R\le|x|\le t+R]}\right\|_{L_x^{6,2}L_t^\infty(\mathbb{R}^3\times[T,\infty))}\\ &\lesssim \left\|\frac{1}{|x|-R}\chi_{[|x|>T-R]}\right\|_{L_x^{6,2}}\\ &\lesssim \left\|\frac{1}{|x|}\chi_{[|x|>\frac{T}{2}]}\right\|_{L_x^{6,2}} \lesssim \frac{1}{\sqrt{T}}.\end{aligned}$$

Similarly,

$$\begin{aligned}\left\|\widetilde{U}^L(t,x)\right\|_{L_x^\infty L_t^2(\mathbb{R}^3\times[T,\infty))} &\lesssim \left\|\frac{1}{t}\chi_{[t-R\le|x|\le t+R]}\right\|_{L_x^\infty L_t^2(\mathbb{R}^3\times[T,\infty))}\\ &\lesssim \left\|\left(\frac{1}{|x|^2}\chi_{[|x|>T-R]}\cdot R\right)^{1/2}\right\|_{L_x^\infty} \lesssim_R \frac{1}{T}.\end{aligned}$$

Hence

$$\lim_{T\to\infty}\big\|\widetilde{U}^L\big\|_{L_x^{6,2}L_t^\infty\cap L_x^\infty L_t^2(\mathbb{R}^3\times[T,\infty))} = 0. \tag{2.63}$$

Since $\vec{\widetilde{U}}^L$ is a free radiation, we see that

$$\partial_{tt}\widetilde{U}^L - \Delta\widetilde{U}^L - V\widetilde{U}^L + 5\phi^4\widetilde{U}^L = -V\widetilde{U}^L + 5\phi^4\widetilde{U}^L, \quad \text{in } (0,\infty)\times\mathbb{R}^3. \tag{2.64}$$

By the decay property of V, $5\phi^4$ and (2.63), simple calculations show that

$$\lim_{T\to\infty}\|-V\widetilde{U}^L + 5\phi^4\widetilde{U}^L\|_{L_x^{6/5,2}L_t^\infty\cap L_x^{3/2,1}L_t^2(\mathbb{R}^3\times[T,\infty))} = 0.$$

Choose T sufficiently large, such that

$$\big\|\widetilde{U}^L\big\|_{L_x^{6,2}L_t^\infty\cap L_x^\infty L_t^2(\mathbb{R}^3\times[T,\infty))}\leq\delta, \tag{2.65}$$

$$\big\|-V\widetilde{U}^L+5\phi^4\widetilde{U}^L\big\|_{L_x^{6/5,2}L_t^\infty\cap L_x^{3/2,1}L_t^2(\mathbb{R}^3\times[T,\infty))}\leq\delta. \tag{2.66}$$

Note that $\vec{v}:=\vec{\gamma}-P^\perp\vec{\widetilde{U}}^L$ solves

$$\partial_{tt}v+\omega^2v=-P^\perp\big(-V\widetilde{U}^L+5\phi^4\widetilde{U}^L\big),\quad (t,x)\in[T,\infty)\times\mathbb{R}^3,$$

with initial data $\vec{v}(T)=P^\perp\big(\vec{U}^L(T)-P^\perp\vec{\widetilde{U}}^L(T)\big)$. We note that by definition, it is easy to see that $P^\perp$ is bounded in $L_x^{\frac{6}{5},2}L_t^\infty\cap L_x^{\frac{3}{2},1}L_t^2$. It is clear from the bounds (2.62) and energy conservation for the free radiation that

$$\big\|\vec{v}(T)\big\|_{\dot{H}^1\times L^2}\leq C\delta.$$

By (2.66) and reversed Strichartz estimates from Lemma 2.1, we can conclude that

$$\|v\|_{L_x^{6,2}L_t^\infty\cap L_x^\infty L_t^2(\mathbb{R}^3\times[T,\infty))}\leq C\delta. \tag{2.67}$$

Combining bounds (2.67) and (2.65), and fixing δ small, the lemma is proved. □

Now the proof of Claim 2.2.1 is easy. Note that due to the fact that

$$\lim_{T\to\infty}\big\|\vec{U}(T)-(\phi,0)-\vec{U}^L(T)\big\|_{\dot{H}^1\times L^2(\mathbb{R}^3)}=0,$$

we see that the initial data for γ satisfies

$$\lim_{T\to\infty}\big\|\vec{\gamma}(T)-P^\perp\vec{U}^L(T)\big\|_{\dot{H}^1\times L^2(\mathbb{R}^3)}=0.$$

Hence Claim 2.2.1 follows from the above lemma and reversed Strichartz estimates.

Proof of Claim 2.2.2. From the bound

$$\|h\|_{L_x^{6,2}L_t^\infty\cap L_x^\infty L_t^2(\mathbb{R}^3\times[T,\widetilde{T}))}\leq K\epsilon,$$

we check as in the proof of Theorem 2.2 that

$$\|f\|_{L_x^{\frac{6}{5},2}L_t^\infty\cap L_x^{\frac{3}{2},1}L_t^2(\mathbb{R}^3\times[T,\widetilde{T}))}\lesssim K^5\epsilon^2, \tag{2.68}$$

where $f=N(\phi,h)$. h satisfies

$$\partial_{tt}h-\Delta h-Vh+5\phi^4h+f=0,$$

and thus $\tilde{h} := P^{\perp}h$ satisfies

$$\partial_{tt}\tilde{h} - \Delta\tilde{h} - V\tilde{h} + 5\phi^4\tilde{h} + P^{\perp}f = 0.$$

By reverse Strichartz estimates and the estimates (2.68) on f, we conclude that the solution $\tilde{h}^L$ to

$$\partial_{tt}\tilde{h}^L - \Delta\tilde{h}^L - V\tilde{h}^L + 5\phi^4\tilde{h}^L = 0$$

with $\vec{\tilde{h}}^L(\widetilde{T}) = P^{\perp}(\vec{h}(\widetilde{T}))$ satisfies that

$$\left\|\tilde{h}^L - \tilde{h}\right\|_{L_x^{6,2}L_t^{\infty}\cap L_x^{\infty}L_t^2(\mathbb{R}^3\times[T,\widetilde{T}))} \leq CK^5\epsilon^2,$$

and hence

$$\left\|\tilde{h}^L\right\|_{L_x^{6,2}L_t^{\infty}\cap L_x^{\infty}L_t^2(\mathbb{R}^3\times[T,\widetilde{T}))} \leq C_0K\epsilon + CK^5\epsilon^2.$$

Using approximation by smooth and compactly supported data, it is easy to show that there exists sufficiently small $\delta > 0$ such that

$$\left\|\tilde{h}^L\right\|_{L_x^{6,2}L_t^{\infty}\cap L_x^{\infty}L_t^2(\mathbb{R}^3\times[T,\widetilde{T}+\delta))} \leq C_0K\epsilon + 2CK^5\epsilon^2.$$

Hence, by taking δ smaller if necessary so that the growth of the unstable modes can be controlled, we can conclude that the solution h^L to

$$\partial_{tt}h^L - \Delta h^L - Vh^L + 5\phi^4 h^L = 0$$

with $\vec{h}^L(\widetilde{T}) = \vec{h}(\widetilde{T})$ satisfies that

$$\left\|h^L\right\|_{L_x^{6,2}L_t^{\infty}\cap L_x^{\infty}L_t^2(\mathbb{R}^3\times[\widetilde{T},\widetilde{T}+\delta))} \leq C_0K\epsilon + \epsilon + 4CK^5\epsilon^2.$$

Then by a standard perturbation argument, we see that if ϵ is sufficiently small, then

$$\|h\|_{L_x^{6,2}L_t^{\infty}\cap L_x^{\infty}L_t^2(\mathbb{R}^3\times[\widetilde{T},\widetilde{T}+\delta))} \leq C_0K\epsilon + \epsilon + 8CK^5\epsilon^2.$$

Combining the above with estimates of h on the interval $[T,\widetilde{T})$ and choosing $C_1 \gg C_0$, the claim is proved. □

Proof of Claim 2.2.3. From the proof of Lemma 2.3, we know that for free wave U^L with smooth compactly supported data, we have

$$\lim_{T_0\to\infty}\left\|U^L\right\|_{L_x^{6,2}L_t^{\infty}\cap L_x^{\infty}L_t^2(\mathbb{R}^3\times[T_0,\infty))} = 0. \tag{2.69}$$

Then by approximation, (2.69) holds true for any free wave with finite energy.

Now let $f(t,x)$ be a solution to the equation (recall $\omega^2 = P^{\perp}(-\Delta - V + 5\phi^4)$)

$$\begin{cases} \partial_{tt} f + \omega^2 f = 0, \quad \text{in } [0,\infty) \times \mathbb{R}^3, \\ \qquad \vec{u}(0) = (f_0, f_1) \in P^{\perp}\big(\dot{H}^1 \times L^2\big). \end{cases} \tag{2.70}$$

For any given $\epsilon > 0$, we first take smooth and compactly supported data $(\tilde{f}_0, \tilde{f}_1)$ such that

$$\big\|(f_0, f_1) - (\tilde{f}_0, \tilde{f}_1)\big\|_{\dot{H}^1 \times L^2} \lesssim \epsilon,$$

which further implies

$$\big\|(f_0, f_1) - P^{\perp}(\tilde{f}_0, \tilde{f}_1)\big\|_{\dot{H}^1 \times L^2} \leq \big\|(f_0, f_1) - (\tilde{f}_0, \tilde{f}_1)\big\|_{\dot{H}^1 \times L^2} \lesssim \epsilon.$$

We take $g(t,x)$ to be the solution to the equation

$$\begin{cases} \partial_{tt} g + \omega^2 g = 0, \quad \text{in } [0,\infty) \times \mathbb{R}^3, \\ \qquad \vec{g}(0) = \big(g_0, g_1\big) := P^{\perp}\big(\tilde{f}_0, \tilde{f}_1\big). \end{cases} \tag{2.71}$$

From Strichartz estimates, we have $g \in L_x^{6,2} L_t^{\infty} \cap L_x^{\infty} L_t^2(\mathbb{R}^3 \times [0,\infty))$.

Let us recall an estimate from the proof of [3, Corollary 2] (page 27 in the journal version) which is slightly stronger than the estimate stated in the main result [3, Corollary 2]. Notice that it in fact follows from interpolation between the bounds in [3, Theorem 1]. For $0 \leq \theta_1, \theta_2 \leq 1$ and $\theta_1 + \theta_2 \leq 1$,

$$\left\| t^{1-\theta_1-\theta_2} \left(\cos(t\omega) g_0 + \frac{\sin t\omega}{\omega} g_1 \right) \right\|_{(\mathcal{K}^{\theta_2})_x^* L_t^{\infty}} \lesssim \big\|\Delta g_0\big\|_{\mathcal{K}^{\theta_1}} + \big\|\nabla g_1\big\|_{\mathcal{K}^{\theta_1}} \tag{2.72}$$

It is not necessary for us to give the detailed definition of $\mathcal{K}^{\theta}$ and $(\mathcal{K}^{\theta})^*$, as we only need the embedding property

$$L^{\frac{3}{3-\theta},1} \subset \mathcal{K}^{\theta}, \quad (\mathcal{K}^{\theta})^* \subset L^{3/\theta,\infty}.$$

Hence we can take $\theta_2 = \frac{1}{2}$ and $\theta_1 = 0$, and obtain the estimate we need, viz.

$$\left\| \cos(t\omega) g_0 + \frac{\sin t\omega}{\omega} g_1 \right\|_{L_x^{6,\infty} L_t^{\infty}[T_0,\infty)} \lesssim T_0^{-\frac{1}{2}} \left(\big\|\Delta g_0\big\|_{L^1} + \big\|\nabla g_1\big\|_{L^1} \right). \tag{2.73}$$

Notice that eigenfunctions ρ_i to $\mathcal{L}_{\phi} = -\Delta - V + 5\phi^4$ decay exponentially and $\rho_i \in W^{2,p}$, $1 \leq p \leq \infty$. Together with the fact $(\tilde{f}_0, \tilde{f}_1)$ is smooth and compactly supported and $(g_0, g_1) = P^{\perp}(\tilde{f}_0, \tilde{f}_1)$, we have $\Delta g_0, \nabla g_1 \in L^1$.

Define the matrix operator

$$J(t) = \begin{bmatrix} \cos\big(t|\nabla|\big) & |\nabla|^{-1} \sin\big(t|\nabla|\big) \\ -|\nabla| \sin\big(t|\nabla|\big) & \cos\big(t|\nabla|\big) \end{bmatrix},$$

and consider the free wave $g^L(t,x)$ with the initial data

$$\begin{bmatrix} g^L(0) \\ g_t^L(0) \end{bmatrix} = \begin{bmatrix} g_0 \\ g_1 \end{bmatrix} + \int_0^\infty J(-s) \begin{bmatrix} 0 \\ (V-5\phi^4)g(s) \end{bmatrix} ds. \tag{2.74}$$

We wish to compare g and g^L. By the decay property of V, $5\phi^4$ and the Strichartz estimate (2.7), we know the integral term in (2.74) converges in $\dot{H}^1 \times L^2$.

Then we have

$$\begin{bmatrix} g(t) \\ g_t(t) \end{bmatrix} - \begin{bmatrix} g^L(t) \\ g_t^L(t) \end{bmatrix} = -\int_t^\infty J(t-s) \begin{bmatrix} 0 \\ (V-5\phi^4)g(s) \end{bmatrix} ds.$$

In particular

$$g(t) = g^L(t) - \int_t^\infty \frac{\sin\big((t-s)|\nabla|\big)}{|\nabla|} \big((V-5\phi^4)g(s)\big) ds. \tag{2.75}$$

Since $g \in L_x^\infty L_t^2(\mathbb{R}^3 \times [0,\infty))$, by continuity of the norm in the time variable, we have $\|g\|_{L_x^\infty L_t^2(\mathbb{R}^3\times[T_0,\infty))} \to 0$ as $T_0 \to \infty$. Together with the fact $V - 5\phi^4 \in L^{\frac{3}{2},1}$ and from Hölder, we obtain

$$\big\|(V-5\phi^4)g\big\|_{L_x^{3/2,1} L_t^2(\mathbb{R}^3\times[T_0,\infty))} \longrightarrow 0 \quad \text{as } T_0 \longrightarrow \infty.$$

From (2.73), we also have

$$\begin{aligned} &\big\|(V-5\phi^4)g\big\|_{L_x^{6/5,2} L_t^\infty(\mathbb{R}^3\times[T_0,\infty))} \\ &\quad \lesssim \big\|V-5\phi^4\big\|_{L_x^{\frac{3}{2},2}} \|g\|_{L_x^{6,\infty} L_t^\infty[T_0,\infty)} \longrightarrow 0, \quad T_0 \longrightarrow +\infty. \end{aligned}$$

Now we can apply the Strichartz estimate (2.7) to (2.75) which implies

$$\begin{aligned} &\|g\|_{L_x^{6,2} L_t^\infty \cap L_x^\infty L_t^2(\mathbb{R}^3\times[T_0,\infty))} \\ &\quad \lesssim \|g^L\|_{L_x^{6,2} L_t^\infty \cap L_x^\infty L_t^2(\mathbb{R}^3\times[T_0,\infty))} \\ &\qquad + \big\|(V-5\phi^4)g\big\|_{L_x^{6/5,2} L_t^\infty \cap L_x^{3/2,1} L_t^2(\mathbb{R}^3\times[T_0,\infty))} \longrightarrow 0 \quad \text{as } T_0 \longrightarrow +\infty. \end{aligned}$$

Hence we can pick T_* large enough such that $\|g\|_{L_x^{6,2} L_t^\infty \cap L_x^\infty L_t^2(\mathbb{R}^3\times[T_0,\infty))} < \epsilon$ for $T_0 > T_*$. Combining this with the difference estimate

$$\big\|f(t,x) - g(t,x)\big\|_{L_x^{6,2} L_t^\infty \cap L_x^\infty L_t^2(\mathbb{R}^3\times[T_0,\infty))} \lesssim \big\|(f_0,f_1) - (g_0,g_1)\big\|_{\dot{H}^1\times L^2} \lesssim \epsilon,$$

we get

$$\|f\|_{L_x^{6,2} L_t^\infty \cap L_x^\infty L_t^2(\mathbb{R}^3\times[T_0,\infty))} < \epsilon, \quad \text{for } T_0 > T_*.$$

We have proved (2.49).

In a similar fashion, we consider the free wave $f^L(t,x)$ with the initial data

$$\begin{bmatrix} f^L(0) \\ f_t^L(0) \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \end{bmatrix} + \int_0^\infty J(-s) \begin{bmatrix} 0 \\ (V-5\phi^4)f(s) \end{bmatrix} ds. \tag{2.76}$$

We know the integral term here converges in $\dot{H}^1 \times L^2$ and

$$f(t) = f^L(t) - \int_t^\infty \frac{\sin((t-s)|\nabla|)}{|\nabla|}((V-5\phi^4)f(s))\, ds. \tag{2.77}$$

Now that we have already proved $\|f\|_{L_x^{6,2}L_t^\infty \cap L_x^\infty L_t^2(\mathbb{R}^3\times[T_0,\infty))} \to 0$ as $T_0 \to +\infty$, we can apply Strichartz to obtain

$$\begin{aligned} &\big\|\vec{f}(t,x) - \vec{f}^L(t,x)\big\|_{\dot{H}^1\times L^2} \\ &\quad\lesssim \big\|(V-5\phi^4)f(s,x)\big\|_{L_x^{6/5,2}L_s^\infty \cap L_x^{3/2,1}L_s^2(\mathbb{R}^3\times[t,\infty))} \\ &\quad\lesssim \big\|V-5\phi^4\big\|_{L_x^{\frac{3}{2},1}}\big\|f(s,x)\big\|_{L_x^{6,2}L_s^\infty\cap L_x^\infty L_s^2(\mathbb{R}^3\times[t,\infty))} \longrightarrow 0 \quad \text{as } t \longrightarrow +\infty. \end{aligned}$$

This establishes (2.50). □

Remark 2.4. Due to the near optimal decay assumption on our potential V, we can not apply the structure formula from [4] to obtain scattering for solutions to the wave equation with potential. The proof above seems to provide a new perspective: scattering to a free wave occurs because the potential term becomes negligible for large times. This insight requires the use of reverse Strichartz estimates.

Before we end this section, let us prove the completeness of scattering operator.

PROPOSITION 2.5. *Let ϕ be a given unstable steady state as in Theorem 2.2. Then for any free wave u^L with finite energy, we can find a solution u to equation* (1.1) *such that*

$$\big\|\vec{u}(t) - (\phi,0) - \vec{u}^L(t)\big\|_{\dot{H}^1\times L^2} \longrightarrow 0, \quad \text{as } t \longrightarrow +\infty.$$

Proof. Notice the fact $V - 5\phi^4 \in L_x^{\frac{3}{2},1}$ and

$$\lim_{T\to+\infty} \|u^L\|_{L_x^{6,2}L_t^\infty \cap L_x^\infty L_t^2(\mathbb{R}^3\times[T,\infty))} = 0.$$

Hence for a given $\epsilon > 0$ to be chosen later, we find a large time $T > 0$ such that

(2.78)

$$\|u^L\|_{L_x^{6,2}L_t^\infty\cap L_x^\infty L_t^2(\mathbb{R}^3\times[T,\infty))} + \|(V-5\phi^4)u^L\|_{L_x^{6/5,2}L_t^\infty\cap L_x^{3/2,1}L_t^2(\mathbb{R}^3\times[T,\infty))} \le \epsilon.$$

Now we seek a solution of equation (1.1) with the form $u = \phi + u^L + \eta$. This means η satisfies the equation

$$\eta_{tt} + \big(-\Delta - V + 5\phi^4\big)\eta = \mathcal{N}\big(\phi, u^L, \eta\big)$$

with

$$\mathcal{N}\big(\phi, u^L, \eta\big) = \big(V - 5\phi^4\big)u^L + 10\phi^3\big(u^L + \eta\big)^2 + 10\phi^2\big(u^L + \eta\big)^3 + 5\phi\big(u^L + \eta\big)^4 + \big(u^L + \eta\big)^5,$$

hence we have

$$|\mathcal{N}| \lesssim \big|\big(V - 5\phi^4\big)u^L\big| + \sum_{k=2}^{5}\big|\phi^{5-k}\big(u^L + \eta\big)^k\big|.$$

As before, we write $\eta = \sum_{i=1}^n \lambda_i(t)\rho_i + \gamma(t,x)$ with $\gamma(t,x) \perp \rho_i$ for $1 \le i \le n$, and plug into the equation, we also apply stability condition as (2.26) and get the system

$$\begin{cases} \lambda_i(t) = e^{-k_i(t-T)}\left[\lambda_i(T) + \dfrac{1}{2k_i}\displaystyle\int_T^\infty e^{k_i(T-s)}\mathcal{N}_{\rho_i}(s)ds\right] \\ \qquad - \dfrac{1}{2k_i}\displaystyle\int_T^\infty e^{-k_i|t-s|}\mathcal{N}_{\rho_i}(s)ds \\ \begin{bmatrix}\gamma(t)\\ \gamma_t(t)\end{bmatrix} = \tilde{J}(t-T)\begin{bmatrix}\gamma(T)\\ \dot\gamma(T)\end{bmatrix} + \displaystyle\int_T^t \tilde{J}(t-s)\begin{bmatrix}0\\ \mathcal{N}_c(s)\end{bmatrix}ds. \end{cases} \tag{2.79}$$

with the notation $\omega = \sqrt{P^\perp(-\Delta - V + 5\phi^4)}$ and

$$\tilde{J}(t) = \begin{bmatrix}\cos(t\omega) & \omega^{-1}\sin(t\omega)\\ -\omega\sin(t\omega) & \cos(t\omega)\end{bmatrix}.$$

We seek a solution of the system (2.79) such that $\gamma(t,x)$ scatters to 0, i.e.,

$$\tilde{J}(-t)\begin{bmatrix}\gamma(t)\\ \gamma_t(t)\end{bmatrix} \longrightarrow_{\dot{H}^1\times L^2} \begin{bmatrix}0\\0\end{bmatrix}$$

which gives the relation between initial condition and solution

$$\begin{bmatrix}\gamma(T)\\ \dot\gamma(T)\end{bmatrix} = -\int_T^\infty \tilde{J}(T-s)\begin{bmatrix}0\\ \mathcal{N}_c(s)\end{bmatrix}ds. \tag{2.80}$$

Hence we use (2.80) and rewrite the system as

$$\begin{cases} \lambda_i(t) = e^{-k_i(t-T)}\left[\lambda_i(T) + \frac{1}{2k_i}\int_T^\infty e^{k_i(T-s)}\mathcal{N}_{\rho_i}(s)ds\right] \\ \qquad\qquad - \frac{1}{2k_i}\int_T^\infty e^{-k_i|t-s|}\mathcal{N}_{\rho_i}(s)ds \\ \begin{bmatrix}\gamma(t)\\ \gamma_t(t)\end{bmatrix} = -\int_t^\infty \tilde{J}(t-s)\begin{bmatrix}0\\ \mathcal{N}_c(s)\end{bmatrix}ds. \end{cases} \tag{2.81}$$

Once we solve to system (2.81) to get $\lambda_i(t), \gamma(t,x)$, we can use (2.80) to prescribe the initial data of γ at time T.

Again we define the norm

$$\left\|(\lambda_1,\dots,\lambda_n,\gamma)\right\|_X := \sum_{i=1}^n \left\|\lambda_i(t)\right\|_{L_t^\infty\cap L_t^2([T,\infty))} + \|\gamma\|_{L_x^{6,2}L_t^\infty\cap L_x^\infty L_t^2(\mathbb{R}^3\times[T,\infty))}. \tag{2.82}$$

Estimating system (2.81), we obtain that

$$\begin{aligned} \left\|\lambda_i(t)\right\|_{L^\infty\cap L^2([T,\infty))} &\lesssim \left|\lambda_i(T)\right| + \left\|\mathcal{N}_{\rho_i}\right\|_{L_t^\infty\cap L_t^2([T,\infty))} \\ &\lesssim \left|\lambda_i(T)\right| + \|\mathcal{N}\|_{L_x^{6/5,2}L_t^\infty\cap L_x^{3/2,1}L_t^2(\mathbb{R}^3\times[T,\infty))} \end{aligned} \tag{2.83}$$

and

$$\|\gamma\|_{L_x^{6,2}L_t^\infty\cap L_x^\infty L_t^2(\mathbb{R}^3\times[T,\infty))} \lesssim \|\mathcal{N}\|_{L_x^{6/5,2}L_t^\infty\cap L_x^{3/2,1}L_t^2(\mathbb{R}^3\times[T,\infty))}. \tag{2.84}$$

And the estimate for nonlinearity is almost identical to (2.42) and (2.43), just with extra forcing term controlled by (2.78)

$$\begin{aligned} &\|\mathcal{N}\|_{L_x^{6/5,2}L_t^\infty\cap L_x^{3/2,1}L_t^2(\mathbb{R}^3\times[T,\infty))} \\ &\quad \le \left\|(V-5\phi^4)u^L\right\|_{L_x^{6/5,2}L_t^\infty\cap L_x^{3/2,1}L_t^2(\mathbb{R}^3\times[T,\infty))} \\ &\qquad + \sum_{k=2}^5 \left\|\phi^{5-k}(u^L+\eta)^k\right\|_{L_x^{6/5,2}L_t^\infty\cap L_x^{3/2,1}L_t^2(\mathbb{R}^3\times[T,\infty))} \\ &\quad \lesssim \epsilon + \sum_{k=2}^5 \left(\epsilon + \|\eta\|_{L_x^{6,2}L_t^\infty\cap L_x^\infty L_t^2(\mathbb{R}^3\times[T,\infty))}\right)^k. \end{aligned}$$

By definition of X, $\|\eta\|_{L_x^{6,2}L_t^\infty\cap L_x^\infty L_t^2(\mathbb{R}^3\times[T,\infty))} \le C\|(\lambda_1,\dots,\lambda_n,\gamma)\|_X$. We get

$$\left\|(\lambda_1,\dots,\lambda_n,\gamma)\right\|_X \le L\left(\sum_{i=1}^n \left|\lambda_i(T)\right|\right) + L\epsilon + L\sum_{k=2}^5 \left(\epsilon + \left\|(\lambda_1,\dots,\lambda_n,\gamma)\right\|_X\right)^k,$$

with constant $L > 1$ is a constant only depending on the constants in the reversed Strichartz estimates, $\|\phi\|_{L^6(\mathbb{R}^3)}$, $\|V\|_{L^{\frac{3}{2},1}(\mathbb{R}^3)}$ and $\|\rho_i\|_{L_x^\infty\cap L^{6,2}(\mathbb{R}^3)}$. If we take ϵ small enough such that $(3L+1)^2\epsilon < \frac{1}{2}$ (this is achieved by taking T large enough), and

$$\sum_{i=1}^{n} |\lambda_i(T)| < \epsilon, \tag{2.85}$$

then the map defined by the right-hand side of (2.81) takes a ball $B_{3L\epsilon}(0) \subset X$ into itself. Moreover, by the same argument, we can check this is contraction mapping. This means that given small data $\lambda_i(T)$ satisfies (2.85) we have a unique solution.

Our estimate on nonlinearity guarantees the integral in (2.80) converges in $\dot{H}^1 \times L^2$, hence by taking initial data using (2.80), we also get a solution to system (2.79). Notice the size of initial data is $O(\epsilon)$.

Now we are left to check u scatters to ϕ with linear wave exactly u^L. The proof is identical to step 4 of the proof for Theorem 2.2. By showing

$$\lim_{T'\to+\infty} \|\eta\|_{L_x^{6,2} L_t^\infty \cap L_x^\infty L_t^2(\mathbb{R}^3\times[T',\infty))} + \|\mathcal{N}\|_{L_x^{6/5,2} L_t^\infty \cap L_x^{3/2,1} L_t^2(\mathbb{R}^3\times[T',\infty))} = 0,$$

we obtain the asymptotic profile of $\vec{\gamma}$

$$\begin{aligned}\gamma_\infty(t) &= \cos\big(\omega(t-T)\big)\gamma(T) + \frac{1}{\omega}\sin\big(\omega(t-T)\big)\dot{\gamma}(T)\\ &\quad + \frac{1}{\omega}\int_T^\infty \sin\big(\omega(t-s)\big) N_c(s)\,ds.\end{aligned}$$

Together with our initial condition (2.80), we proved that $\gamma_\infty(t) = 0$, which means γ scatters to 0. Combining with the fact $\|\lambda_i(t)\|_{L_t^2\cap L_t^\infty[T',\infty)} \to 0$ as $T' \to +\infty$, we conclude that u scatters to ϕ with a scattering profile u^L, i.e.,

$$\big\|\vec{u}(t) - (\phi,0) - \vec{u}^L(t)\big\|_{\dot{H}^1\times L^2} \longrightarrow 0, \quad \text{as } t \longrightarrow +\infty. \qquad \square$$

3. Channel of energy inequality. In this section, we first prove the channel of energy estimate for solutions to the linear wave equation with potential if the initial data has a dominating discrete mode. Then we show this estimate also holds for equation (1.1) as long as the initial data is small enough. Finally, for data which has a nontrivial but not dominant discrete mode, we prove a growth lemma which ensures that once we require the initial data to be sufficiently small, we can find a large time at which the solution is still small and the discrete mode becomes dominant.

For the following basic perturbation result, we refer the reader to [20, Lemma 2.1] for proof.

LEMMA 3.1. *Let $0 \in I \subset \mathbb{R}$ be an interval of time. Suppose $\tilde{u}(t,x) \in C_t(I, \dot{H}^1(\mathbb{R}^3))$ with $\|\tilde{u}\|_{L_t^5 L_x^{10}(I\times\mathbb{R}^3)} \le M < \infty$, $\|a\|_{L_t^{5/4} L_x^{5/2}(I\times\mathbb{R}^3)} \le \beta < \infty$ and $e(t,x), f(t,x) \in L_t^1 L_x^2(I\times\mathbb{R}^3)$, satisfy*

$$\partial_{tt}\tilde{u} - \Delta\tilde{u} + a(t,x)\tilde{u} + \tilde{u}^5 = e, \tag{3.1}$$

with initial data $\vec{\tilde{u}}(0) = (\tilde{u}_0, \tilde{u}_1) \in \dot{H}^1 \times L^2$. *Suppose for some sufficiently small positive* $\epsilon < \epsilon_0 = \epsilon_0(M, \beta)$,

$$\||e| + |f|\|_{L_t^1 L_x^2(I\times\mathbb{R}^3)} + \|(u_0, u_1) - (\tilde{u}_0, \tilde{u}_1)\|_{\dot{H}^1\times L^2} < \epsilon. \tag{3.2}$$

Then there is a unique solution $u \in C(I, \dot{H}^1)$ *with* $\|u\|_{L_t^5 L_x^{10}(I\times\mathbb{R}^3)} < \infty$, *satisfying the equation*

$$\partial_{tt} u - \Delta u + a(t,x)u + u^5 = f, \tag{3.3}$$

with initial data $\vec{u}(0) = (u_0, u_1)$. *Moreover, we have the following estimate*

$$\sup_{t\in I} \|\vec{u}(t) - \vec{\tilde{u}}(t)\|_{\dot{H}^1\times L^2} + \|u - \tilde{u}\|_{L_t^5 L_x^{10}(I\times\mathbb{R}^3)} < C(M,\beta)\epsilon. \tag{3.4}$$

We also need the following result on the precise asymptotics of eigenfunctions corresponding to negative eigenvalues of the Schrödinger operator $-\Delta - V$, which is a consequence of Theorem 4.2 in Meshkov [26].

LEMMA 3.2. *Let* V *satisfy* $\sup_{x\in\mathbb{R}^3}(1+|x|)^\beta |V(x)| < \infty$ *for some* $\beta > 2$, *and suppose that* $\rho \not\equiv 0$ *is an eigenfunction corresponding to the eigenvalue* $-k^2$ *of* $-\Delta - V$. *Then there exists* $f \in L^2(\mathbb{S}^2)$ *which does not vanish identically, such that*

$$\rho(x) = e^{-k|x|}|x|^{-1}\left(f\left(\frac{x}{|x|}\right) + \omega(x)\right), \tag{3.5}$$

where $\omega(x)$ *satisfies*

$$\int_{\mathbb{S}^2} \big|\omega(R\theta)\big|^2 \, d\sigma(\theta) = O\big(R^{-\frac{1}{2}}\big), \quad \text{as } R \longrightarrow +\infty. \tag{3.6}$$

An important observation in [15] is that the above precise asymptotics implies the following channel of energy inequality for the associated linear wave equation.

LEMMA 3.3. *Let* V *satisfy* $\sup_{x\in\mathbb{R}^3}(1+|x|)^\beta |V(x)| < \infty$ *for some* $\beta > 2$, *and suppose that* $\rho \not\equiv 0$ *is an eigenfunction corresponding to the eigenvalue* $-k^2$ *of the operator* $-\Delta - V$. *Suppose that* u *solves the equation*

$$u_{tt} - \Delta u - Vu = 0$$

with $\vec{u}(0) = \mu^+(\rho, k\rho)$, *then for any* $R > 0$ *the following channel of energy estimate holds for some constant* $c(\rho, V, R) > 0$

$$\int_{|x|\geq t+R} \big|\partial_t u\big|^2(t,x)\, dx \geq c(\rho, V, R)\big|\mu^+\big|^2, \quad \text{for } t \geq 0. \tag{3.7}$$

Similarly, if $\vec{u}(0) = \mu^-(\rho, -k\rho)$*, then*

$$\int_{|x|\geq|t|+R} \left|\partial_t u\right|^2(t,x)\,dx \geq c(\rho,V,R)\left|\mu^-\right|^2, \quad \text{for } t \leq 0. \tag{3.8}$$

Proof. We first prove the lemma for initial data $\vec{u}(0) = \mu^+(\rho, k\rho)$. In this case the solution u has the explicit form

$$u(t,x) = \mu^+ e^{kt}\rho.$$

From (3.6), we can take r_0 large enough such that when $r > r_0$, we have

$$\int_{\mathbb{S}^2} \left|\omega(r\theta)\right|^2 d\sigma(\theta) < \frac{1}{10}\int_{\mathbb{S}^2} \left|f(\theta)\right|^2 d\sigma(\theta). \tag{3.9}$$

By the asymptotics of ρ in (3.5), we get that

$$\begin{aligned}\int_{|x|\geq t+R} \left|\partial_t u\right|^2(t,x)\,dx &\geq \int_{r\geq t+R+r_0}\int_{\mathbb{S}^2} \left|\mu^+ k\right|^2 e^{-2k(r-t)}\big(f(\theta)+\omega(r\theta)\big)^2\,d\sigma(\theta)\,dr\\ &\gtrsim \int_{R+r_0}^{\infty}\int_{\mathbb{S}^2} \left|\mu^+ k\right|^2 e^{-2kr}\left|f(\theta)\right|^2 d\sigma(\theta)\,dr.\end{aligned}$$

Then (3.7) follows.

The case when $\vec{u}(0) = \mu^-(\rho, -k\rho)$ is similar, and we omit the detail. □

Lemma 3.3 can be generalized to the case when the initial data has finitely many discrete modes.

LEMMA 3.4. *Let* V *satisfy* $\sup_{x\in\mathbb{R}^3}(1+|x|)^\beta|V(x)| < \infty$ *for some* $\beta > 2$*, and suppose that* $-\Delta - V$ *has negative eigenvalues* $-k_1^2 \leq -k_2^2 \leq \cdots \leq -k_n^2 < 0$ *with corresponding orthonormal eigenmodes* $\rho_1, \rho_2, \ldots, \rho_n$*. Suppose that* u *solves the equation*

$$u_{tt} - \Delta u - Vu = 0 \tag{3.10}$$

with initial data $\vec{u}(0) = \sum_{i=1}^n \mu_i^+(\rho_i, k_i\rho_i)$*, then for any* $R > 0$*, there exists a constant* $c(R) > 0$ *such that we have the following channel of energy estimate forward in time*

$$\int_{|x|\geq t+R} \left|\partial_t u\right|^2(t,x)\,dx \geq c(R)\sum_{i=1}^n \left|\mu_i^+\right|^2, \quad \text{for } t > 0. \tag{3.11}$$

Similarly, if we consider data of the form $\vec{u}(0) = \sum_{i=1}^n \mu_i^-(\rho_i, -k_i\rho_i)$*, the channel of energy estimate holds backward in time.*

Proof. It suffices to prove the lemma for sufficiently large $R > 0$. By normalizing the coefficients, we will prove (3.11) when $\sum_{i=1}^n |\mu_i^+|^2 = 1$. We divide the proof into several steps.

Step 0: Computing the asymptotics. First notice that the solution has an explicit formula

$$u = \sum_{i=1}^{n} \mu_i^+ e^{k_i t} \rho_i.$$

From Lemma 3.2, we know that each ρ_i has the following asymptotic

$$\rho_i = e^{-k_i|x|} \frac{1}{|x|} \left(f_i\left(\frac{x}{|x|}\right) + \omega_i(x) \right)$$

with $f_i \in L^2(\mathbb{S}^2)$ which does not vanishing identically, and ω_i satisfies (3.6).

Now given any $R > 0$, using Lemma 3.2 we have

$$\begin{aligned}
&\lim_{t\to+\infty} \int_{|x|\geq t+R} \left|\partial_t u\right|^2(t,x)\,dx \\
&\quad = \lim_{t\to+\infty} \int_{r>t+R} \int_{\theta\in\mathbb{S}^2} \left[\sum_{i=1}^{n} \mu_i^+ k_i e^{-k_i(r-t)} \big(f_i(\theta) + \omega_i(r\theta)\big) \right]^2 d\sigma(\theta)\,dr \\
&\quad = \lim_{t\to+\infty} \int_{r>R} \int_{\theta\in\mathbb{S}^2} \left[\sum_{i=1}^{n} \mu_i^+ k_i e^{-k_i r} \big(f_i(\theta) + \omega_i\big((r+t)\theta\big)\big) \right]^2 d\theta\,dr \\
&\quad = \int_{r>R} \int_{\theta\in\mathbb{S}^2} \left[\sum_{i=1}^{n} \mu_i^+ k_i e^{-k_i r} f_i(\theta) \right]^2 d\sigma(\theta)\,dr.
\end{aligned}$$

Here we used the decay condition (3.6) for ω_i.

Step 1: Lower bound for the asymptotics. We claim that for any $R \geq 0$ fixed, there exists constant $c(R) > 0$ such that for any μ_i^+ satisfying $\sum_{i=1}^n |\mu_i^+|^2 = 1$, we have

$$\int_{r>R} \int_{\theta\in\mathbb{S}^2} \left[\sum_{i=1}^{n} \mu_i^+ k_i e^{-k_i r} f_i(\theta) \right]^2 d\sigma(\theta)\,dr \geq c(R). \tag{3.12}$$

Suppose (3.12) is not true, then for any $N > 0$, we find $\mu_i^+(N)$ satisfying $\sum_{i=1}^n |\mu_i^+(N)|^2 = 1$ such that

$$\int_{r>R} \int_{\theta\in\mathbb{S}^2} \left[\sum_{i=1}^{n} \mu_i^+(N) k_i e^{-k_i r} f_i(\theta) \right]^2 d\sigma(\theta)\,dr < \frac{1}{N}. \tag{3.13}$$

Using that $\mu_i^+(N)$ are bounded, we can extract a convergent subsequence. Hence we can assume that $\mu_i^+(N) \to a_i$ as $N \to \infty$, and $\sum_{i=1}^n a_i^2 = 1$. By the dominated convergence theorem, we pass to the limit in (3.13) and get

$$\int_{r>R}\int_{\theta\in\mathbb{S}^2}\left[\sum_{i=1}^{n}a_ik_ie^{-k_ir}f_i(\theta)\right]^2 d\sigma(\theta)\,dr=0$$

which implies that

$$\sum_{i=1}^{n}a_ik_ie^{-k_ir}f_i(\theta)=0 \quad \text{for } r>R,\ \theta\in\mathbb{S}^2. \tag{3.14}$$

Now we consider the problem in several cases:

Case 1. if k_i are different, then in (3.14) we first multiply with e^{-k_nr} and let $r\to\infty$, we conclude $a_nf_n=0$, and similarly we conclude

$$a_i\,f_i(\theta)=0, \quad \text{for } 1\le i\le n \text{ and } \theta\in\mathbb{S}^2.$$

Since $\|f_i\|_{L^2(\mathbb{S}^2)}\neq 0$, we conclude that $a_i=0$, which is a contradiction to $\sum a_i^2=1$.

Case 2. If one of the eigenvalues has multiplicity more than 1, say, k_{i_0} with multiplicity m, i.e., $k_{i_0}=k_{i_0+1}=k_{i_0+m-1}\neq k_j$ for any $j\in\{1,\dots,n\}\backslash\{i_0,i_0+1,\dots,i_0+m-1\}$. All other eigenvalue still have multiplicity 1. Then (3.14) now reads as

$$a_1k_1e^{-k_1r}f_1(\theta)+\cdots+e^{-k_{i_0}r}k_{i_0}\left[\sum_{i=i_0}^{i_0+m-1}a_if_i(\theta)\right]+\cdots+a_nk_ne^{-k_nr}f_n(\theta)=0$$
$$\text{for } r>R,\ \theta\in\mathbb{S}^2.$$

Applying the same method as in Case 1, we conclude that

$$a_1f_1(\theta)=0,\dots,\sum_{i=i_0}^{i_0+m-1}a_if_i(\theta)=0,\dots,a_nf_n(\theta)=0, \quad \text{for } \theta\in\mathbb{S}^2,$$

which implies $a_i=0$, for any $i\in\{1,\dots,n\}\backslash\{i_0,i_0+1,\dots,i_0+m-1\}$.

Now we consider the part $\sum_{i=i_0}^{i_0+m-1}a_if_i(\theta)=0$ and prove that all $a_i=0$. Denote $L=-\Delta-V$. By $L_\phi\rho_i=-k_{i_0}^2\rho_i$, $i_0\le i\le i_0+m-1$, we see that

$$L\left(\sum_{i=i_0}^{i_0+m-1}a_i\rho_i\right)=-k_{i_0}^2\left(\sum_{i=i_0}^{i_0+m-1}a_i\rho_i\right).$$

Assuming towards a contradiction that not all $a_i=0$, we conclude that $\sum_{i=i_0}^{i_0+m-1}a_i\rho_i$ is an eigenfunction for L with eigenvalue $-k_{i_0}^2$.

On the other hand,

$$\sum_{i=i_0}^{i_0+m-1} a_i\rho_i = e^{-k_{i_0}|x|}\frac{1}{|x|}\left[\sum_{i=i_0}^{i_0+m-1} a_i w_i(x)\right].$$

This contradicts Lemma 3.2, in particular (3.5). Hence we conclude that $a_i = 0$, $1 \le i \le n$, which is a contradiction to $\sum a_i^2 = 1$.

Case 3. In general, we could have several eigenvalues that have multiplicity more than 1. In that case we repeat the argument in Case 2 as needed.

Hence we conclude that our claim (3.12) is true.

Step 2: Refining the lower bound for asymptotics. Next we refine (3.12) by obtaining a better lower bound. Let $\alpha_i = k_i e^{-k_i r} f_i(\theta)$, $1 \le i \le n$ for $r > 0$, $\theta \in \mathbb{S}^2$, and

$$\langle \alpha_i, \alpha_j\rangle := \int_{r>0}\int_{\theta\in\mathbb{S}^2} \alpha_i\alpha_j \, d\sigma(\theta)dr, \quad \mathcal{A}_{n\times n} := [\langle \alpha_i, \alpha_j\rangle]_{1\le i,j\le n}.$$

Then (3.12) with $R = 0$ implies that $\mathcal{A}$ is a positive definite matrix. And for any $\vec{v} \in \mathbb{R}^n$, $\|\vec{v}\| = 1$, one has $\vec{v}^t A\vec{v} \ge c(0) > 0$.

Now for any $R > 0$, we change variables $r = s + R$ in (3.12) to wit

$$\begin{aligned}
&\int_{r>R}\int_{\theta\in\mathbb{S}^2}\left[\sum_{i=1}^{n}\mu_i^+ k_i e^{-k_i r} f_i(\theta)\right]^2 d\sigma(\theta)\,dr \\
&= \int_{s>0}\int_{\theta\in\mathbb{S}^2}\left[\sum_{i=1}^{n}\mu_i^+ k_i e^{-k_i s} e^{-k_i R} f_i(\theta)\right]^2 d\sigma(\theta)\,ds \\
&= \sum_{i,j}\mu_i^+ e^{-k_i R}\mu_j^+ e^{-k_j R}\langle \alpha_i, \alpha_j\rangle \\
&\ge c(0)\sum_{i=1}^{n}\left|\mu_i^+ e^{-k_i R}\right|^2.
\end{aligned} \tag{3.15}$$

Step 3: Channel of energy estimate. Now we prove (3.11). The computation from Step 0 implies that

$$\begin{aligned}
&\int_{|x|\ge t+R}\left|\partial_t u\right|^2(t,x)\,dx \\
&= \int_{r>R}\int_{\theta\in\mathbb{S}^2}\left[\sum_{i=1}^{n}\mu_i^+ k_i e^{-k_i r}\left(f_i(\theta)+\omega_i\big((r+t)\theta\big)\right)\right]^2 d\sigma(\theta)\,dr.
\end{aligned} \tag{3.16}$$

Expanding the square, this further equals

$$
\begin{aligned}
&= \sum_{i,j=1}^{n} \int_{r>R} \int_{\theta\in\mathbb{S}^2} \mu_i^+ \mu_j^+ k_i k_j e^{-(k_i+k_j)r} f_i(\theta) f_j(\theta)\, d\sigma(\theta)\, dr \\
&\quad + \sum_{i,j=1}^{n} \int_{r>R} \int_{\theta\in\mathbb{S}^2} \mu_i^+ \mu_j^+ k_i k_j e^{-(k_i+k_j)r} \\
&\quad \times \big[f_i(\theta)\omega_j\big((r+t)\theta\big) + f_j(\theta)\omega_i\big((r+t)\theta\big)\big]\, d\theta\, dr \\
&\quad + \sum_{i,j=1}^{n} \int_{r>R} \int_{\theta\in\mathbb{S}^2} \mu_i^+ \mu_j^+ k_i k_j e^{-(k_i+k_j)r} \omega_i\big((r+t)\theta\big)\omega_j\big((r+t)\theta\big) d\sigma(\theta) dr.
\end{aligned}
\tag{3.17}
$$

Using the decay estimate of ω_j in (3.6) and Cauchy-Schwarz inequality, we infer that

$$
\begin{aligned}
&\sum_{i,j=1}^{n} \left| \int_{r>R} \int_{\theta\in\mathbb{S}^2} \mu_i^+ \mu_j^+ k_i k_j e^{-(k_i+k_j)r} f_i(\theta) \omega_j\big((r+t)\theta\big)\, d\sigma(\theta)\, dr \right| \\
&\quad \le \sum_{i,j=1}^{n} \left(\int_{r>R} \int_{\theta\in\mathbb{S}^2} \big|\mu_i^+ k_i\big|^2 e^{-2k_i r} \big|f_i(\theta)\big|^2 d\sigma(\theta)\, dr \right)^{\frac12} \\
&\qquad \times \left(\int_{r>R} \int_{\theta\in\mathbb{S}^2} \big|\mu_j^+ k_j\big|^2 e^{-2k_j r} \big|\omega_j\big((r+t)\theta\big)\big|^2 d\sigma(\theta)\, dr \right)^{\frac12} \\
&\quad \lesssim R^{-\frac14} \sum_{i=1}^{n} \big|\mu_i^+\big|^2 e^{-2k_i R}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\sum_{i,j=1}^{n} \int_{r>R} \int_{\theta\in\mathbb{S}^2} \mu_i^+ \mu_j^+ k_i k_j e^{-(k_i+k_j)r} \omega_i\big((r+t)\theta\big) \omega_j\big((r+t)\theta\big)\, d\sigma(\theta)\, dr \\
&\quad \lesssim R^{-\frac12} \sum_{i=1}^{n} \big|\mu_i^+\big|^2 e^{-2k_i R}.
\end{aligned}
$$

Together with (3.15) we obtain

$$
\begin{aligned}
&\int_{|x|\ge t+R} \big|\partial_t u\big|^2 (t,x)\, dx \\
&\quad \ge c(0) \sum_{i=1}^{n} \big|\mu_i^+\big|^2 e^{-2k_i R} - C\big(R^{-\frac14} + R^{-\frac12}\big) \sum_{i=1}^{n} \big|\mu_i^+\big|^2 e^{-2k_i R} \\
&\quad \ge \frac{c(0)}{2} \sum_{i=1}^{n} \big|\mu_i^+ e^{-k_i R}\big|^2 \ge \frac{c(0)}{2} e^{-2k_1 R},
\end{aligned}
$$

where R is sufficiently large. The lemma is proved. □

Next we consider the case when there are several negative eigenvalues and prove that if one of the discrete modes is dominant, then we still have the channel of energy estimate.

COROLLARY 3.5. *Let V satisfy $\sup_{x\in\mathbb{R}^3}(1+|x|)^\beta|V(x)|<\infty$ for some $\beta>2$, and suppose that $-\Delta-V$ has no zero eigenvalue or zero resonance, and that it has negative eigenvalues $-k_1^2\le -k_2^2\le\cdots\le -k_n^2<0$ with corresponding orthonormal eigenmodes $\rho_1,\rho_2,\ldots,\rho_n$.*

Let $u(t)$ be a solution to (3.10) *with initial data*

$$\vec{u}(0)=\big(\gamma_0,\gamma_1\big)+\sum_{i=1}^{n}\big[\mu_i^+\big(\rho_i,k_i\rho_i\big)+\mu_i^-\big(\rho_i,-k_i\rho_i\big)\big]$$

satisfying the orthogonality conditions $\int\rho_i\gamma_0\,dx=\int\rho_i\gamma_1\,dx=0$, $1\le i\le n$.

(1) *For any $R\ge 0$, if we have*

$$|\mu_{i_0}^+|>K_0\left[\big\|\big(\gamma_0,\gamma_1\big)\big\|_{\dot{H}^1\times L^2}+\sum_{i=1}^{n}|\mu_i^-|\right] \tag{3.18}$$

for sufficiently large constant $K_0:=K_0(R)>0$, then there exists a constant $c(R)>0$ such that

$$\int_{|x|\ge t+R}\big|\partial_t u\big|^2(t,x)\,dx\ge c(R)\big|\mu_{i_0}^+\big|^2,\quad \textit{for all } t\ge 0. \tag{3.19}$$

(2) *For any $R\ge 0$, if we have*

$$|\mu_{i_0}^-|>K_0\left[\big\|\big(\gamma_0,\gamma_1\big)\big\|_{\dot{H}^1\times L^2}+\sum_{i=1}^{n}|\mu_i^+|\right]$$

for sufficiently large fixed constant $K_0:=K_0(R)>0$, then there exists a constant $c(R)>0$ such that

$$\int_{|x|\ge|t|+R}\big|\partial_t u\big|^2(t,x)\,dx\ge c(R)\big|\mu_{i_0}^-\big|^2,\quad \textit{for all } t\le 0. \tag{3.20}$$

Proof. To prove (1), first note that the solution is of the form

$$u=\sum_{i=1}^{n}\mu_i^+e^{k_it}\rho_i+\mu_i^-e^{-k_it}\rho_i+\gamma(t,x)$$

with the continuous part γ solving the equation

$$\gamma_{tt}+P^\perp(-\Delta-V)\gamma=0.$$

Hence from Lemma 3.4 and the Strichartz estimate for γ (2.7), we get for $t \geq 0$

$$\begin{aligned}
&\int_{|x|>t+R} \left|\partial_t u(t,x)\right|^2 dx \\
&\geq \frac{1}{2}\int_{|x|\geq t+R}\left|\sum_{i=1}^n \mu_i^+ k_i e^{k_i t}\rho_i\right|^2 dx - 2\sum_{i=1}^n \int_{|x|\geq t+R}\left|\mu_i^- k_i e^{-k_i t}\rho_i\right|^2 dx \\
&\quad - 2\int_{|x|\geq t+R}\left|\partial_t \gamma\right|^2 dx \\
&\geq c(R)\sum_{i=1}^n \left|\mu_i^+\right|^2 - C\sum_{i=1}^n \left|\mu_i^-\right|^2 - C\left\|(\gamma_0,\gamma_1)\right\|_{\dot H^1\times L^2}^2 \\
&\geq \frac{c(R)}{2}\left|\mu_{i_0}^+\right|^2,
\end{aligned}$$

if K_0 in (3.18) is sufficiently large.

Case (2) follows from (1) by time reversal. □

Next we shall see that the channel of energy estimate is stable with respect to nonlinear perturbations. In particular, the following lemma shows that if the initial data is very close to a steady state, and one discrete eigenmode of the initial data is dominant, then the solution will radiate energy outside the light cone either forward or backward in time.

LEMMA 3.6. *Fix any $R \geq 0$. Consider a finite energy solution u to the nonlinear equation* (1.1) *with initial data $(u_0,u_1)\in \dot H^1\times L^2$. Given a stationary solution ϕ and $\mathcal{L}_\phi = -\Delta - V + 5\phi^4$ with orthonormal eigenmodes $\rho_1,\rho_2,\ldots,\rho_n$ corresponding to negative eigenvalues $-k_1^2 \leq -k_2^2 \leq \cdots \leq -k_n^2 < 0$.*

(1) *Let (u_0,u_1) be of the form $\vec u(0) = (\phi,0)+(h_0,h_1)$ with*

$$(h_0,h_1) = (\gamma_0,\gamma_1) + \sum_{i=1}^n \left[\mu_i^+(\rho_i,k_i\rho_i) + \mu_i^-(\rho_i,-k_i\rho_i)\right]$$

and $\int \rho_i\gamma_0\,dx = \int \rho_i\gamma_1\,dx = 0$ for all $1\leq i\leq n$. Assume that

$$\left|\mu_{i_0}^+\right| := \max\{\left|\mu_i^+\right|,\ i=1,\ldots,n\}$$

and that

$$\left|\mu_{i_0}^+\right| > K\left[\left\|(\gamma_0,\gamma_1)\right\|_{\dot H^1\times L^2} + \sum_{i=1}^n \left|\mu_i^-\right|\right], \tag{3.21}$$

as well as

$$\big\|\big(h_0,h_1\big)\big\|_{\dot H^1\times L^2}<\epsilon_*,$$

for some sufficiently large constants $K\gg 1$ *and sufficiently small* $\epsilon_*>0$ *that only depend on the potential* V *and* R*. Then the solution satisfies the channel of energy estimate*

$$\int_{|x|\geq t+R}\big|\partial_t u\big|^2(t,x)\,dx\geq c(R)\big|\mu^+_{i_0}\big|^2,\quad \textit{for } t\geq 0 \tag{3.22}$$

for some constant $c(R)>0$.

(2) *Assume that* (u_0,u_1) *has the decomposition* $\vec u(0)=(\phi,0)+(h_0,h_1)$ *with*

$$\big(h_0,h_1\big)=\sum_{i=1}^n\mu_i^+\big(\rho_i,k_i\rho_i\big)+\big(\mathcal{R}_0,\mathcal{R}_1\big).$$

Furthermore, suppose that for $|\mu^+_{i_0}|:=\max\{|\mu_i^+|,\ i=1,\dots,n\}$*, we have*

$$\big|\mu^+_{i_0}\big|>K\big\|\big(\mathcal{R}_0,\mathcal{R}_1\big)\big\|_{\dot H^1\times L^2} \tag{3.23}$$

and $\|(h_0,h_1)\|_{\dot H^1\times L^2}<\epsilon_*$*, for sufficiently large* $K\gg 1$ *and sufficiently small* $\epsilon_*>0$ *that depend only on* V*,* R*. Then the solution* u *satisfies the channel of energy estimate*

$$\int_{|x|\geq t+R}\big|\partial_t u\big|^2(t,x)\,dx\geq c(R)\big|\mu^+_{i_0}\big|^2,\quad \textit{for } t\geq 0 \tag{3.24}$$

for some constant $c(R)>0$.

(3) *Similar results hold when we switch* μ_i^- *with* μ_i^+ *in (1), (2) and consider* $t\leq 0$.

Proof. (1) Write $u=\phi+h$. Then h solves the equation

$$h_{tt}+\big(-\Delta-V+5\phi^4\big)h=\mathcal{N}(h,\phi)$$

with $\mathcal{N}(h,\phi)=-(\phi+h)^5+\phi^5+5\phi^4h$.

Let h^L be the solution to the linear equation

$$h^L_{tt}+\big(-\Delta-V+5\phi^4\big)h^L=0. \tag{3.25}$$

Define

$$\widetilde V(t,x):=\begin{cases}V(x) & \text{if } |x|\geq|t|,\\ 0 & \text{if } |x|<|t|,\end{cases} \tag{3.26}$$

and

$$\widetilde{\phi}(t,x) := \begin{cases} \phi(x) & \text{if } |x| \geq |t|, \\ 0 & \text{if } |x| < |t|, \end{cases} \tag{3.27}$$

respectively. Let $\tilde{h}^L$ and $\tilde{h}$ be the solution to the linear and nonlinear wave equation with truncated potential, viz.

$$\tilde{h}^L_{tt} + \big(-\Delta - \widetilde{V} + 5\widetilde{\phi}^4\big)\tilde{h}^L = 0, \tag{3.28}$$

$$\tilde{h}_{tt} + \big(-\Delta - \widetilde{V} + 5\widetilde{\phi}^4\big)\tilde{h} = \mathcal{N}(\tilde{h}, \widetilde{\phi}). \tag{3.29}$$

It is easy to check that

$$\widetilde{V}, \widetilde{\phi} \in L_t^{\frac{5}{4}} L_x^{\frac{5}{2}}\big(\mathbb{R}^3 \times \mathbb{R}\big).$$

We take the initial data

$$\vec{h}(0) = \vec{h}^L(0) = \vec{\tilde{h}}^L(0) = \vec{\tilde{h}}(0) = \big(\gamma_0, \gamma_1\big) + \sum_{i=1}^{n} \big[\mu_i^+\big(\rho_i, k_i\rho_i\big) + \mu_i^-\big(\rho_i, -k_i\rho_i\big)\big]$$

which satisfy the condition (3.21) with a large constant K to be chosen later. By finite speed of propagation, $t \in \mathbb{R}$, $\vec{h} = \vec{\tilde{h}}$, $\vec{h}^L = \vec{\tilde{h}}^L$ for $|x| > |t|$. In view of Lemma 3.1,

$$\sup_{t\in[0,\infty)} \big\|\vec{\tilde{h}}(t)\big\|_{\dot{H}^1\times L^2} + \|\tilde{h}\|_{L_t^5 L_x^{10}([0,\infty)\times\mathbb{R}^3)} \lesssim \big\|\vec{\tilde{h}}(0)\big\|_{\dot{H}^1\times L^2} \lesssim \big|\mu_{i_0}^+\big| \tag{3.30}$$

and

$$\sup_{t\in[0,\infty)} \Big\|\vec{\tilde{h}}(t) - \vec{\tilde{h}}^L(t)\Big\|_{\dot{H}^1\times L^2} + \big\|\tilde{h} - \tilde{h}^L\big\|_{L_t^5 L_x^{10}([0,\infty)\times\mathbb{R}^3)} \lesssim \big|\mu_{i_0}^+\big|^2,$$

if ϵ_* is chosen sufficiently small depending on V.

Take $K > K_0(R)$ where $K_0(R)$ is the constant from part (1) of Corollary 3.5, then we get that the linear solution h^L satisfies the channel estimate,

$$\int_{|x|\geq t+R} \big|\partial_t h^L(t,x)\big|^2\, dx \geq c(R)\big|\mu_{i_0}^+\big|^2, \quad \text{for } t \geq 0.$$

Hence, for all $t \geq 0$

$$\begin{aligned}\int_{|x|\geq t+R} \left|\partial_t h\right|^2(t,x)\,dx &= \int_{|x|\geq t+R} \left|\partial_t \tilde{h}\right|^2(t,x)\,dx \\ &\geq \int_{|x|\geq t+R} \left|\partial_t \tilde{h}^L\right|^2(t,x)\,dx - C\left|\mu_{i_0}^+\right|^4 \\ &= \int_{|x|\geq t+R} \left|\partial_t h^L\right|^2(t,x)\,dx - C\left|\mu_{i_0}^+\right|^4 \\ &\geq c(R)\left|\mu_{i_0}^+\right|^2 - C\left|\mu_{i_0}^+\right|^4 \geq \frac{c(R)}{2}\left|\mu_{i_0}^+\right|^2.\end{aligned}$$

The last line holds provided $\epsilon_* = \epsilon_*(R) \gtrsim |\mu_{i_0}^+|$ is small enough.

(2) Consider two solutions to equation (1.1) u and v, with data

$$\begin{aligned}\vec{u}(0) &= (\phi,0) + \sum_{i=1}^{n} \mu_i^+\left(\rho_i, k_i\rho_i\right) + \left(\mathcal{R}_0,\mathcal{R}_1\right), \\ \vec{v}(0) &= (\phi,0) + \sum_{i=1}^{n} \mu_i^+\left(\rho_i, k_i\rho_i\right),\end{aligned}$$

respectively. If we set $u = \phi + h$ and $v = \phi + \ell$, then h, ℓ satisfy

$$\begin{aligned}h_{tt} + \left(-\Delta - V + 5\phi^4\right)h &= \mathcal{N}(h,\phi) \\ \ell_{tt} + \left(-\Delta - V + 5\phi^4\right)\ell &= \mathcal{N}(\ell,\phi)\end{aligned}$$

with initial data

$$\vec{h}(0) = \sum_{i=1}^{n} \mu_i^+\left(\rho_i, k_i\rho_i\right) + \left(\mathcal{R}_0,\mathcal{R}_1\right), \quad \vec{\ell}(0) = \sum_{i=1}^{n} \mu_i^+\left(\rho_i, k_i\rho_i\right).$$

As in the proof for (1), we define $\widetilde{V}, \widetilde{\phi}$ and consider truncated versions $\tilde{h}, \tilde{\ell}$ that satisfy the equation (3.29), with data $\vec{\tilde{h}}(0) = \vec{h}(0)$, $\vec{\tilde{\ell}}(0) = \vec{\ell}(0)$. Then from finite speed of propagation we infer $\vec{h} = \vec{\tilde{h}}, \vec{\ell} = \vec{\tilde{\ell}}$ for $|x| \geq |t|$. The perturbation Lemma 3.1 and (3.23) yield the bound

$$\sup_{t\in[0,\infty)} \left\|\vec{\tilde{h}}(t) - \vec{\tilde{\ell}}(t)\right\|_{\dot{H}^1\times L^2} \leq C\left\|\vec{\tilde{h}}(0) - \vec{\tilde{\ell}}(0)\right\|_{\dot{H}^1\times L^2} \leq \frac{C}{K}\left|\mu_{i_0}^+\right|.$$

Note that

$$\left\|\vec{\ell}(0)\right\|_{\dot{H}^1\times L^2} \lesssim \left|\mu_{i_0}^+\right|.$$

From part (1) we know that there exists $\epsilon_*(R) > 0$ small enough, such that if $\|\vec{\ell}(0)\| < \epsilon_*$, then $\ell(t,x)$ satisfy the channel of energy inequality

$$\int_{|x|\geq t+R} \big|\partial_t \ell\big|^2(t,x)\,dx \geq c(R)\big|\mu_{i_0}^+\big|^2 \quad \text{for } t\geq 0.$$

Hence we get for $t \geq 0$,

$$\begin{aligned}\int_{|x|\geq t+R} \big|\partial_t h\big|^2(t,x)\,dx &= \int_{|x|\geq t+R} \big|\partial_t \tilde{h}\big|^2(t,x)\,dx \\ &\geq \int_{|x|>t+R} \big|\partial_t \tilde{\ell}\big|^2(t,x)\,dx - \frac{C^2}{K^2}\big|\mu_{i_0}^+\big|^2 \\ &= \int_{|x|\geq t+R} \big|\partial_t \ell\big|^2(t,x)\,dx - \frac{C^2}{K^2}\big|\mu_{i_0}^+\big|^2 \\ &\geq c(R)\big|\mu_{i_0}^+\big|^2 - \frac{C^2}{K^2}\big|\mu_{i_0}^+\big|^2 \geq \frac{c(R)}{2}\big|\mu_{i_0}^+\big|^2.\end{aligned}$$

The last line holds if we pick $K := K(R)$ large enough.

(3) The proof is similar to (1) and (2) and we omit the details here. □

Initially, the discrete spectral component may not be large enough as required by (3.23). But since any eigenmode grows exponentially either forward or backward in time, we might expect that it will take over the dispersive term for large times as long as it is not too small initially. The following lemma makes this logic precise.

LEMMA 3.7. *Given a steady state solution ϕ to the nonlinear equation* (1.1)*, suppose that $\mathcal{L}_\phi = -\Delta - V + 5\phi^4$ has orthonormal eigenmodes $\rho_1, \rho_2, \dots, \rho_n$ corresponding to eigenvalues $-k_1^2 \leq -k_2^2 \leq \cdots \leq -k_n^2 < 0$. Suppose that $\vec{u}$ is a solution to equation* (1.1) *with initial data*

$$\vec{u}(0) = (\phi,0) + \sum_{i=1}^{n}\left[\mu_i^+\big(\rho_i, k_i\rho_i\big) + \mu_i^-\big(\rho_i, -k_i\rho_i\big)\right] + \big(\gamma_0,\gamma_1\big)$$

obeying the orthogonality conditions $\int \rho_i\gamma_0\,dx = \int \rho_i\gamma_1\,dx = 0$, for all $1\leq i\leq n$. Write the solution as

$$\vec{u}(t) = (\phi,0) + \vec{h}(t).$$

(1) *Suppose*

$$\big|\mu_{i_0}^+\big| := \max\left\{\big|\mu_i^+\big|,\ i=1,\dots,n\right\} \geq \kappa\big\|\vec{h}(0)\big\|_{\dot{H}^1\times L^2}$$

for some constant $\kappa > 0$*. Then for any* $\epsilon_* > 0$*,* $K > 1$*, there exist* $T(\kappa, K) > 0$ *sufficiently large and* $\varepsilon(\kappa, \epsilon_*, K, T) > 0$ *sufficiently small, such that if* $\|\vec{h}(0)\|_{\dot{H}^1 \times L^2} < \varepsilon$ *then*

$$\vec{h}(T) = \sum_{i=1}^{n} e^{k_i T} \mu_i^+ \big(\rho_i, k_i \rho_i\big) + \big(\mathcal{R}_0, \mathcal{R}_1\big),$$

with

$$\big\|\vec{h}(T)\big\|_{\dot{H}^1 \times L^2} < \epsilon_*$$

and

$$\big\|\big(\mathcal{R}_0, \mathcal{R}_1\big)\big\|_{\dot{H}^1 \times L^2} \le \frac{1}{K} e^{k_{i_0} T} \big|\mu_{i_0}^+\big| \big\|\big(\rho_{i_0}, k_{i_0} \rho_{i_0}\big)\big\|_{\dot{H}^1 \times L^2}.$$

(2) *Suppose*

$$\big|\mu_{i_0}^-\big| := \max\big\{\mu_i^- : i = 1, \dots, n\big\} \ge \kappa \big\|\vec{h}(0)\big\|_{\dot{H}^1 \times L^2}$$

for some constant $\kappa > 0$*. Then for any* $\epsilon_* > 0$*,* $K > 1$*, there exist* $T(\kappa, K) > 0$ *sufficiently large and* $\varepsilon(\kappa, \epsilon_*, K, T) > 0$ *sufficiently small, such that if* $\|\vec{h}(0)\|_{\dot{H}^1 \times L^2} < \varepsilon$ *then*

$$\vec{h}(-T) = \sum_{i=1}^{n} e^{k_i T} \mu_i^- \big(\rho_i, -k_i \rho_i\big) + \big(\mathcal{R}_0, \mathcal{R}_1\big),$$

with

$$\big\|\vec{h}(-T)\big\|_{\dot{H}^1 \times L^2} < \epsilon_*$$

and

$$\big\|\big(\mathcal{R}_0, \mathcal{R}_1\big)\big\|_{\dot{H}^1 \times L^2} \le \frac{1}{K} e^{k_{i_0} T} \big|\mu_{i_0}^-\big| \big\|\big(\rho_{i_0}, -k_{i_0} \rho_{i_0}\big)\big\|_{\dot{H}^1 \times L^2}.$$

Proof. The proof of (2) is again the time reversal of (1), so it suffices to consider the latter.

Step 1: Bound on h*.* Writing $u = \phi + h$, we see that h solves the equation (with $\mathcal{N}$ as above)

$$h_{tt} + \big(-\Delta - V + 5\phi^4\big) h = \mathcal{N}(h, \phi).$$

Let h^L be the solution to the linear equation

$$h_{tt}^L + \big(-\Delta - V + 5\phi^4\big) h^L = 0 \tag{3.31}$$

with data $\vec{h}^L(0)=\vec{h}(0)$. We denote by $S(t)g$ the solution to the linear equation (3.31) with data $(0,g)$ for any $g\in L^2$. By decomposing the data into continuous and discrete modes, the Strichartz estimates (2.7) for the continuous modes, and the explicit formula for the evolution of discrete modes, we can find absolute constants $C,A\geq 1$ such that

$$\sup_{\tau\in[0,t)}\big\|\vec{S}(\tau)g\big\|_{\dot{H}^1\times L^2}+\big\|S(\tau)g\big\|_{L_t^5L_x^{10}([0,t)\times\mathbb{R}^3)}\leq Ce^{k_1t}\|g\|_{L^2} \tag{3.32}$$

$$\sup_{\tau\in[0,t)}\big\|\vec{h}^L(\tau)\big\|_{\dot{H}^1\times L^2}+\big\|h^L(\tau)\big\|_{L_t^5L_x^{10}([0,t)\times\mathbb{R}^3)}\leq \frac{A}{8}e^{k_1t}\big\|\vec{h^L}(0)\big\|_{\dot{H}^1\times L^2}. \tag{3.33}$$

Denote $\epsilon:=\|\vec{h}(0)\|_{\dot{H}^1\times L^2}<\varepsilon$. Now on an interval $[0,T)$ with $e^{3k_1T}\varepsilon$ sufficiently small, we will use a continuity argument to show that for $t\in[0,T)$

$$\sup_{\tau\in[0,t)}\big\|\vec{h}(\tau)\big\|_{\dot{H}^1\times L^2}+\big\|h(\tau)\big\|_{L_t^5L_x^{10}([0,t)\times\mathbb{R}^3)}\leq Ae^{k_1t}\big\|\vec{h}(0)\big\|_{\dot{H}^1\times L^2}. \tag{3.34}$$

In fact, assuming that the bound (3.34) holds for $0\leq t\leq t_0$ with some $0<t_0<T$, we will show that we actually have

$$\begin{aligned}&\sup_{\tau\in[0,t)}\big\|\vec{h}(\tau)\big\|_{\dot{H}^1\times L^2}+\big\|h(\tau)\big\|_{L_t^5L_x^{10}([0,t)\times\mathbb{R}^3)}\\ &\qquad\leq \frac{A}{2}e^{k_1t}\big\|\vec{h}(0)\big\|_{\dot{H}^1\times L^2},\quad \text{for all } 0\leq t\leq t_0.\end{aligned} \tag{3.35}$$

Then a simple continuity argument finishes the proof of proof of (3.34). From Duhamel's formula

$$h(\tau)=h^L(\tau)+\int_0^\tau S(\tau-s)\mathcal{N}(h,\phi)(s)\,ds$$

Denote $F(\tau,x)=\int_0^\tau S(\tau-s)\mathcal{N}(h,\phi)(s)\,ds$, then from (3.32) we get

$$\begin{aligned}&\sup_{\tau\in[0,t)}\Big(\big\|F(\tau)\big\|_{\dot{H}^1}+\big\|\partial_\tau F(\tau)\big\|_{L^2}\Big)\\ &\qquad\leq \sup_{\tau\in[0,t)}C\int_0^\tau e^{k_1(\tau-s)}\big\|\mathcal{N}(h,\phi)(s)\big\|_{L^2}\,ds\\ &\qquad\leq \sup_{\tau\in[0,t)}Ce^{k_1\tau}\big\|\mathcal{N}(h,\phi)(s)\big\|_{L_s^1L_x^2([0,\tau)\times\mathbb{R}^3)}\\ &\qquad\leq Ce^{k_1t}\big\|\mathcal{N}(h,\phi)(s)\big\|_{L_s^1L_x^2([0,t)\times\mathbb{R}^3)}\end{aligned} \tag{3.36}$$

and

$$\begin{aligned}
&\|F\|_{L^5_\tau L^{10}_x([0,t)\times\mathbb{R}^3)} \\
&\le \left\| \int_0^t \chi_{\{\tau-s\ge 0\}} \big\| S(\tau-s)\mathcal{N}(h,\phi)(s)\big\|_{L^{10}_x} ds \right\|_{L^5_\tau[0,t)} \\
&\le \int_0^t \big\| S(\tau-s)\mathcal{N}(h,\phi)(s)\big\|_{L^5_\tau L^{10}_x([s,t)\times\mathbb{R}^3)} ds \\
&\le C\int_0^t e^{k_1(t-s)} \big\|\mathcal{N}(h,\phi)(s)\big\|_{L^2} ds \le Ce^{k_1 t}\big\|\mathcal{N}(h,\phi)(s)\big\|_{L^1_s L^2_x([0,t)\times\mathbb{R}^3)}.
\end{aligned} \tag{3.37}$$

Note that $|\mathcal{N}(h,\phi)(s)| \lesssim \sum_{j=2}^5 |\phi|^{5-j}|h|^j$. Assuming the bound (3.34) on $[0,t_0)$, for any $t\in[0,t_0)$ we pick an integer $J_0\ge 0$ such that $J_0 < t \le J_0+1$. This leads to

$$\begin{aligned}
\|\phi^3 h^2\|_{L^1_s L^2_x([0,t)\times\mathbb{R}^3)} &= \sum_{q=0}^{J_0} \|\phi^3 h^2\|_{L^1_s L^2_x([q,q+1)\times\mathbb{R}^3)} + \big\|\phi^3 h^2\big\|_{L^1_s L^2_x([J_0,t)\times\mathbb{R}^3)} \\
&\lesssim \sum_{q=0}^{J_0} \|h\|^2_{L^5_s L^{10}_x([q,q+1)\times\mathbb{R}^3)} + \|h\|^2_{L^5_s L^{10}_x([J_0,t)\times\mathbb{R}^3)} \\
&\lesssim \sum_{q=0}^{J_0} \big(Ae^{k_1(q+1)}\epsilon\big)^2 + \big(Ae^{k_1 t}\epsilon\big)^2 \lesssim A^2\epsilon^2 e^{2k_1 t}.
\end{aligned}$$

We can control the other terms in $\mathcal{N}(h,\phi)$ in an analogous fashion, whence

$$\big\|\mathcal{N}(h,\phi)(s)\big\|_{L^1_s L^2_x([0,t)\times\mathbb{R}^3)} \lesssim \sum_{j=2}^5 \big(A\epsilon e^{k_1 t}\big)^j \quad \text{for } 0\le t<t_0.$$

Using (3.33), we therefore obtain

$$\begin{aligned}
&\sup_{\tau\in[0,t)} \big\|\vec{h}(\tau)\big\|_{\dot H^1\times L^2} + \|h\|_{L^5_t L^{10}_x([0,t)\times\mathbb{R}^3)} \\
&\quad\le \sup_{\tau\in[0,t)} \big\|\vec{h}^L(\tau)\big\|_{\dot H^1\times L^2} + \big\|h^L\big\|_{L^5_t L^{10}_x([0,t)\times\mathbb{R}^3)} \\
&\quad\quad + Ce^{k_1 t}\big\|\mathcal{N}(h,\phi)(s)\big\|_{L^1_s L^2_x([0,t)\times\mathbb{R}^3)} \\
&\quad\le \frac{A}{8}e^{k_1 t}\epsilon + Ce^{k_1 t}\left[\sum_{j=2}^5 \big(Ae^{k_1 t}\epsilon\big)^j\right] \le \frac{A}{2}e^{k_1 t}\epsilon,
\end{aligned}$$

provided $e^{3k_1 T}\varepsilon \ll 1$ is sufficiently small. Hence (3.34) holds on $[0,T)$ as long as T,ε satisfy the relation $e^{3k_1 T}\varepsilon < \epsilon_1$ with a small fixed constant ϵ_1.

Step 2: Decide the constants. Now we consider the linear solution h^L with data

$$\vec{h}^L(0) = \sum_{i=1}^{n} \left[\mu_i^+ \big(\rho_i, k_i\rho_i\big) + \mu_i^- \big(\rho_i, -k_i\rho_i\big)\right] + \big(\gamma_0, \gamma_1\big),$$

then we have the explicit formula for the linear solution

$$\vec{h}^L(t) = \sum_{i=1}^{n} \mu_i^+ e^{k_i t} \big(\rho_i, k_i\rho_i\big) + \sum_{i=1}^{n} \mu_i^- e^{-k_i t} \big(\rho_i, -k_i\rho_i\big) + \vec{\gamma}(t).$$

For any given κ, K, we can choose a large constant $T(\kappa, K)$ such that

$$\begin{aligned} &\left\| \sum_{i=1}^{n} \mu_i^- e^{-k_i T} \big(\rho_i, -k_i\rho_i\big) + \vec{\gamma}(T) \right\|_{\dot{H}^1 \times L^2} \\ &\quad \lesssim \frac{T}{\kappa} |\mu_{i_0}^+| \leq \frac{1}{2K} |\mu_{i_0}^+| e^{k_{i_0} T} \big\| \big(\rho_{i_0}, k_{i_0}\rho_{i_0}\big) \big\|_{\dot{H}^1 \times L^2}. \end{aligned} \tag{3.38}$$

Next from Duhamel's formula and the estimate of $\mathcal{N}$ in Step 1, we have

$$\begin{aligned} \left\| \vec{h}(T) - \vec{h}^L(T) \right\|_{\dot{H}^1 \times L^2} &= \left\| \int_0^T S(T-s) \mathcal{N}(h, \phi)(s)\, ds \right\|_{\dot{H}^1 \times L^2} \\ &\leq C e^{k_1 T} \left[\sum_{j=2}^{5} \big(A\epsilon e^{k_1 T}\big)^j \right] < \frac{1}{2K} |\mu_{i_0}^+|, \end{aligned}$$

if $e^{3k_1 T}\varepsilon$ is sufficiently small.

Hence we have $\vec{h}(T) = \sum_{i=1}^{n} \mu_i^+ e^{k_i T} (\rho_i, k_i\rho_i) + (\mathcal{R}_0, \mathcal{R}_1)$ with

$$\big(\mathcal{R}_0, \mathcal{R}_1\big) = \sum_{i=1}^{n} \mu_i^- e^{-k_i t} \big(\rho_i, -k_i\rho_i\big) + \vec{\gamma}(t) + \vec{h}(T) - \vec{h}^L(T)$$

and

$$\big\| \big(\mathcal{R}_0, \mathcal{R}_1\big) \big\|_{\dot{H}^1 \times L^2} \leq \frac{1}{K} |\mu_{i_0}^+| e^{k_{i_0} T} \big\| \big(\rho_{i_0}, k_{i_0}\rho_{i_0}\big) \big\|_{\dot{H}^1 \times L^2}.$$

We also have

$$\big\| \vec{h}(T) \big\|_{\dot{H}^1 \times L^2} \leq \sum_{i=1}^{n} e^{k_i T} |\mu_i^+| + \big\| \big(\mathcal{R}_0, \mathcal{R}_1\big) \big\|_{\dot{H}^1 \times L^2} \lesssim e^{k_1 T}\varepsilon < \epsilon_*,$$

by choosing ε sufficiently small. □

Remark 3.8. While part (1) of Lemma 3.7 guarantees that at time T the unstable mode

$$e^{k_{i_0}T}\mu_{i_0}^{+}(\rho_{i_0}, k_{i_0}\rho_{i_0})$$

dominates the continuous part and the stable mode, we cannot be sure of its size compared to the other unstable modes, which might grow faster. However, we can easily conclude that the largest mode at time T, say $e^{k_jT}\mu_j^{+}(\rho_j, k_j\rho_j)$, satisfies

$$\left\| e^{k_jT}\mu_j^{+}(\rho_j, k_j\rho_j) \right\|_{\dot{H}^1\times L^2} \geq \frac{1}{n+1}\|\vec{h}(T)\|_{\dot{H}^1\times L^2}.$$

4. Global center stable manifold of unstable excited states. In this section we prove our main result. Before giving the detailed proof, let us briefly summarize the main ideas in physical terms. The crucial fact that we establish can be explained roughly as follows. Take any solution $U(t)$ which scatters to an unstable steady state ϕ. We have shown in Section 2 that in a small neighborhood of $\vec{U}(0)$ in the energy space $\dot{H}^1 \times L^2$, there exists a local, finite co-dimensional manifold $\mathcal{M}$ such that if $\vec{u}(t)$ starts on the manifold, i.e., if $\vec{u}(0) \in \mathcal{M}$, then $\vec{u}(t)$ stays close to $\vec{U}(t)$ for all positive times and scatters to $(\phi, 0)$. On the other hand, if $\vec{u}(t)$ starts in a small neighborhood of $\vec{U}(0)$ but *off* the manifold, then

$$\sup_{t\geq 0}\left\|\vec{u}(t) - \vec{U}(t)\right\|_{\dot{H}^1\times L^2} \geq \epsilon_1 > 0,$$

no matter how small $\left\|\vec{u}(0) - \vec{U}(0)\right\|_{\dot{H}^1\times L^2}$ is. Suppose that $\left\|\vec{u}(0) - \vec{U}(0)\right\|_{\dot{H}^1\times L^2}$ is sufficiently small, then dynamically $\vec{u}(t)$ will stay close to $\vec{U}(t)$ for a long time, say for $0 \leq t \leq T_0$. Since $\vec{U}(t)$ scatters to $(\phi, 0)$, we can write (in the energy space)

$$\vec{U}(t) \approx (\phi, 0) + \vec{U}^L(t)$$

for large times. Hence for large $t \leq T_0$,

$$\vec{u}(t) \approx (\phi, 0) + \vec{U}^L(t)$$

in the energy space. After time T_0, $\vec{u}(t)$ starts to deviate from $\vec{U}(t)$ as $\vec{u}(0) \notin \mathcal{M}$. By an expansion of the energy functional near the steady state, we shall show that the deviation is due to growth in the unstable mode. Then it is not hard to conclude that at a large time $T_1 > T_0$, $\vec{u}(t) - (\phi, 0) - \vec{U}^L(t)$ concentrates most of its energy in the discrete mode and has energy $\gtrsim \epsilon_1$. These arguments finally set the stage for us to apply the channel of energy inequalities proved in the previous section. We will show that besides the radiated energy that $\vec{U}^L$ carries to spatial infinity, $\vec{u}(t)$ emits a *second radiation*. The total radiated energy for $\vec{u}(t)$ will therefore exceed the radiated energy for $\vec{U}(t)$ by a fixed amount. Now note that $\vec{u}(t)$ has almost the same amount of energy as $\vec{U}(t)$, a comparison argument of the energy in the local region then implies that $\vec{u}(t)$, having strictly less energy than $(\phi,0)$ in the local region, can no longer scatter to $(\phi,0)$. Hence, locally the set $\mathcal{M}_\phi$ of all initial data for which the solution scatters to $(\phi,0)$ coincide with $\mathcal{M}$. Thus the set $\mathcal{M}_\phi$ has a manifold structure. This is the key property showing that scattering to unstable steady states is non-generic.

Now we turn to the main argument. Let us first compute the expansion of energy around any steady state $(\phi,0)$.

LEMMA 4.1. *Let* $(u_0,u_1)=(\phi,0)+(\Lambda_0,\Lambda_1)$*, where* $(\Lambda_0,\Lambda_1)\in\dot{H}^1\times L^2$*. Assume that*

$$\|\Lambda_0\|_{L^6(\mathbb{R}^3)}<\beta\ll 1,$$

then we have

$$\mathcal{E}\big((u_0,u_1)\big)=\mathcal{E}(\phi,0)+\frac{1}{2}\big(\mathcal{L}_\phi\Lambda_0,\Lambda_0\big)+\frac{1}{2}\big(\Lambda_1,\Lambda_1\big)+O\big(\beta^3\big), \tag{4.1}$$

where $\mathcal{L}_\phi=-\Delta-V+5\phi^4$.

Suppose $\mathcal{L}_\phi$ *has orthonormal eigenmodes* $\rho_1,\rho_2,\dots,\rho_n$ *corresponding to eigenvalues* $-k_1^2\le -k_2^2\le\cdots\le -k_n^2<0$*. If we further decompose*

$$\big(\Lambda_0,\Lambda_1\big)=\big(X_0,X_1\big)+\big(w_0,w_1\big), \tag{4.2}$$

$$\big(w_0,w_1\big)=\sum_{i=1}^{n}\big[\mu_i^+\big(\rho_i,k_i\rho_i\big)+\mu_i^-\big(\rho_i,-k_i\rho_i\big)\big]+\big(\gamma_0,\gamma_1\big), \tag{4.3}$$

with $(X_0,X_1)\in\dot{H}^1\times L^2$ *and the orthogonality condition* $\int\rho_j\gamma_0\,dx=\int\rho_j\gamma_1\,dx=0$*, for all* $1\le j\le n$*. Then we have*

$$\begin{aligned}\mathcal{E}\big((u_0,u_1)\big)&=\mathcal{E}(\phi,0)+\frac{1}{2}\big[\big(\mathcal{L}_\phi X_0,X_0\big)+\big(X_1,X_1\big)\big]\\&\quad+\frac{1}{2}\big[\big(\mathcal{L}_\phi\gamma_0,\gamma_0\big)+\big(\gamma_1,\gamma_1\big)\big]-\sum_{i=1}^{n}2\mu_i^+\mu_i^-k_i^2\\&\quad+\big(\mathcal{L}_\phi X_0,w_0\big)+\big(X_1,w_1\big)+O\big(\beta^3\big).\end{aligned} \tag{4.4}$$

Proof. The proof is by direct computation

$$\begin{aligned}\mathcal{E}\big((u_0,u_1)\big)&=\mathcal{E}\big((\phi,0)+\big(\Lambda_0,\Lambda_1\big)\big)\\&=\int\frac{\big|\nabla\phi+\nabla\Lambda_0\big|^2}{2}+\frac{\big|\Lambda_1\big|^2}{2}-\frac{V(\phi+\Lambda_0)^2}{2}+\frac{\big(\phi+\Lambda_0\big)^6}{6}\,dx\end{aligned}$$

$$
\begin{aligned}
&= \int \frac{1}{2}\left(|\nabla\phi|^2 - \frac{1}{2}V\phi^2 + \frac{1}{6}\phi^6\right)dx + \int\left(-\Delta\phi - V\phi + \phi^5\right)\Lambda_0\,dx \\
&\quad + \int \frac{1}{2}\left[\left|\nabla\Lambda_0\right|^2 - \frac{1}{2}V\Lambda_0^2 + \frac{5}{2}\phi^4\Lambda_0^2 + \frac{1}{2}\left|\Lambda_1\right|^2 + \frac{1}{6}\sum_{j\geq 3} C_6^j\phi^{6-j}\Lambda_0^j\right]dx \\
&= \mathcal{E}(\phi,0) + \frac{1}{2}\left(\mathcal{L}_\phi\Lambda_0,\Lambda_0\right) + \frac{1}{2}\left(\Lambda_1,\Lambda_1\right) + O\left(\beta^3\right).
\end{aligned}
$$

This finishes the proof of (4.1).

Next we further expand the energy functional using (4.2)

$$
\begin{aligned}
&\mathcal{E}\left((u_0,u_1)\right) \\
&= \mathcal{E}(\phi,0) + \frac{1}{2}\left(\mathcal{L}_\phi\left(X_0+w_0\right), X_0+w_0\right) + \frac{1}{2}\left(X_1+w_1, X_1+w_1\right) + O\left(\beta^3\right) \\
&= \mathcal{E}(\phi,0) + \frac{1}{2}\left[\left(\mathcal{L}_\phi w_0, w_0\right) + \left(w_1,w_1\right)\right] + \frac{1}{2}\left[\left(\mathcal{L}_\phi X_0, X_0\right) + \left(X_1,X_1\right)\right] \\
&\quad + \left(\mathcal{L}_\phi X_0, w_0\right) + \left(X_1, w_1\right) + O\left(\beta^3\right).
\end{aligned}
$$

Since $\mathcal{L}_\phi\rho_i = -k_i^2\rho_i$, we get

$$
\begin{aligned}
\left(\mathcal{L}_\phi w_0, w_0\right) &= -\sum_{i=1}^{n}\left(\mu_i^+ + \mu_i^-\right)^2 k_i^2 + \left(\mathcal{L}_\phi\gamma_0,\gamma_0\right), \\
\left(w_1,w_1\right) &= \sum_{i=1}^{n}\left(\mu_i^+ - \mu_i^-\right)^2 k_i^2 + \left(\gamma_1,\gamma_1\right).
\end{aligned}
$$

Combining the calculations above, we get (4.4). □

Now we are ready to present the main idea of our paper, which is crucial to conclude that the set of initial data for which the solution scatters to an unstable steady state $(\phi,0)$ has a manifold structure, and hence is a "thin set".

THEOREM 4.2. *Let $V\in Y$ be a potential such that equation* (1.1) *has only finitely many steady states, all of which are hyperbolic. Suppose that the finite energy solution $\vec{U}(t)$ to equation* (1.1) *scatters to an unstable excited state $(\phi,0)$. Let $\mathcal{M}$ be the local center-stable manifold around $\vec{U}(0)$ and let ϵ_0, ϵ_1 be as defined in Theorem 2.2. Then there exist ϵ with $0<\epsilon<\epsilon_1<\epsilon_0$ and $\delta(\epsilon_1)\gg\epsilon$, such that for any solution u with finite energy initial data $(u_0,u_1)\notin\mathcal{M}$ with*

$$
\left\|\left(u_0,u_1\right) - \vec{U}(0)\right\|_{\dot{H}^1\times L^2} < \epsilon,
$$

we can find $A > 0$ such that for all $t \geq A$

$$\int_{|x|\geq t-A} \left[\frac{|\nabla u|^2}{2} + \frac{(\partial_t u)^2}{2}\right](t,x)\,dx \geq \mathcal{E}\big(\vec{U}(t)\big) - \mathcal{E}\big((\phi,0)\big) + \delta. \tag{4.5}$$

As a consequence, $\vec{u}(t)$ will not scatter to $(\phi,0)$.

We remark that by a simple adaptation of the result in [19], we know that the collection of potential V which satisfies the condition in Theorem 4.2 are dense in Y.

Proof. We divide our proof into several steps.

Step 1: Set up the parameters. By the local center-stable manifold theorem of Section 3, the locally defined finite co-dimensional manifold $\mathcal{M}$ satisfies the property that any solution to equation (1.1) with initial data on $\mathcal{M}$ scatters to $(\phi,0)$. Moreover, if a solution $\vec{u}(t)$ with initial data $(u_0,u_1) \in B_{\epsilon_1}(\vec{U}_0)$ satisfies

$$\big\|\vec{u}(t) - \vec{U}(t)\big\|_{\dot{H}^1\times L^2} < \epsilon_1 \quad \text{for all } t \geq 0, \tag{4.6}$$

then $(u_0,u_1) \in \mathcal{M}$. Take $\epsilon < \epsilon_1$ sufficiently small to be chosen below. Since the solution $\vec{U}(t)$ scatters to $(\phi,0)$ as $t \to \infty$, denoting by $\vec{U}^L$ the scattered linear wave, we have the property that

$$\lim_{t\to\infty} \big\|\vec{U}(t) - \vec{U}^L(t) - (\phi,0)\big\|_{\dot{H}^1\times L^2} = 0. \tag{4.7}$$

This implies that

$$\mathcal{E}(\vec{U}) = \mathcal{E}(\phi,0) + \frac{1}{2}\big\|\vec{U}^L\big\|^2_{\dot{H}^1\times L^2}. \tag{4.8}$$

By (4.7), the fact that $\phi \in \dot{H}^1(\mathbb{R}^3)$ and $U^L \in L^5_t L^{10}_x([0,\infty)\times\mathbb{R}^3)$, for any small $\delta_1 > 0$, we can first fix some large L and then choose $T_1 > L$ sufficiently large, such that for all $t \geq T_1$,

- (Free wave small in L^6 norm)

$$\big\|U^L(t)\big\|_{L^6(\mathbb{R}^3)} \leq \delta_1 \tag{4.9}$$

- (Closeness of $\vec{U}$ to $\vec{U}^L + (\phi,0)$ and choice of the bounded region)

$$\big\|\vec{U}(t) - \vec{U}^L(t) - (\phi,0)\big\|_{\dot{H}^1\times L^2} + \big\|\vec{U}^L(0)\big\|_{\dot{H}^1\times L^2(|x|\geq L)} + \|\phi\|_{\dot{H}^1(|x|\geq L)} \leq \delta_1; \tag{4.10}$$

- (Most energy of the free radiation is exterior)

$$\int_{|x|\geq t-T_1+L} \big|\nabla_{x,t}U^L\big|^2(t,x)\,dx \geq \int_{\mathbb{R}^3} \big|\nabla_{t,x}U^L\big|^2(t,x)\,dx - \delta_1^2; \tag{4.11}$$

- (Control on the Strichartz norm of the radiation) Let

$$D := \{(t,x): |x| \le T_1 + L - t,\ 0 \le t \le T_1\}.$$

Then we have

$$\|U^L\|_{L_t^5 L_x^{10}((0,\infty)\times\mathbb{R}^3\setminus D)} < \delta_1. \tag{4.12}$$

We remark that (4.11) is a consequence of the strong Huygens principle and approximation by free waves with compactly supported initial data. (4.11) ensures that U^L can essentially be taken as zero for our purposes inside the region $|x| \le t - T_1 + L$ for $t \ge T_1$, which will be important to keep in mind later, in order to distinguish the second piece of radiation. By the continuous dependence of the solution to equation (1.1) on the initial data in $\dot H^1 \times L^2(\mathbb{R}^3)$ and by finite speed of propagation, if we take ϵ sufficiently small and initial data $(u_0, u_1) \in \dot H^1 \times L^2 \setminus \mathcal{M}$ with

$$\|(u_0, u_1) - \vec U(0)\|_{\dot H^1 \times L^2} < \epsilon, \tag{4.13}$$

then

$$\|\vec u(T_1) - \vec U(T_1)\|_{\dot H^1 \times L^2} \tag{4.14}$$

can be made sufficiently small. Hence, noting that $\|V\|_{L_t^{5/4} L_x^{5/2}(|x|\ge|t|)}$ is finite, we can apply Lemma 3.1 to conclude that

$$\|\vec u(t) - \vec U(t)\|_{\dot H^1 \times L^2(|x|\ge t - T_1)} \le \delta_1, \quad \text{for all } t \ge T_1. \tag{4.15}$$

(4.15) means that we can effectively identify $\vec u$ with $\vec U$ in the exterior region

$$|x| \ge t - T_1, \quad t \ge T_1.$$

Hence by (4.10), we see that

$$\|\vec u(t) - \vec U^L(t)\|_{\dot H^1 \times L^2(|x|\ge t - T_1 + L)} \le 3\delta_1, \tag{4.16}$$

that is, we can also identify $\vec u$ with $\vec U^L$ in the exterior region $|x| \ge t - T_1 + L$, $t \ge T_1$.

In order to avoid any possibility of confusion due to the many parameters, we remark that δ_1 and ϵ can be made as small as we wish, and will be chosen later. T_1, L depend on δ_1 and $\vec U$ only. ϵ is a small free parameter below some threshold determined by δ_1. The key point for us is that $\epsilon_1 > 0$ is fixed no matter how small ϵ is chosen, see (4.6).

Since $(u_0, u_1) \notin \mathcal{M}$, there exists an exit time $T_2 > 0$ from the ϵ_1 ball, i.e., such that

$$\big\|\vec{u}(T_2) - \vec{U}(T_2)\big\|_{\dot{H}^1\times L^2(\mathbb{R}^3)} = \epsilon_1. \tag{4.17}$$

Note that the choices of T_1 and L do not depend on ϵ. Therefore, by the continuous dependence of the solution on its initial data in $\dot{H}^1 \times L^2$, if we choose ϵ sufficiently small, we can assume $T_2 > 2(L + T_1 + 1)$.

Step 2: Analyze the size of discrete mode at time T_2. Let us analyze $\vec{u}(T_2)$ in more detail. By the estimates (4.10) and (4.17) we can write

$$\vec{u}(T_2) = (\phi, 0) + \vec{U}^L(T_2) + \big(w_0, w_1\big), \tag{4.18}$$

where $\vec{w} = (w_0, w_1) \in \dot{H}^1 \times L^2$ satisfies

$$2\epsilon_1 \geq \epsilon_1 + \delta_1 \geq \big\|\vec{w}\big\|_{\dot{H}^1\times L^2(\mathbb{R}^3)} \geq \epsilon_1 - \delta_1 \geq \epsilon_1/2, \tag{4.19}$$

if δ_1 is chosen smaller than $\frac{\epsilon_1}{2}$. We now list several facts:

(i) From (4.13), we infer that

$$\big|\mathcal{E}(\vec{u}) - \mathcal{E}(\vec{U})\big| \lesssim \epsilon. \tag{4.20}$$

(ii) Rewrite the decompostion (4.18) in the form

$$\vec{u}(T_2) = (\phi, 0) + \big(\Lambda_0, \Lambda_1\big), \tag{4.21}$$

$$\big(\Lambda_0, \Lambda_1\big) = \vec{U}^L(T_2) + \big(w_0, w_1\big), \tag{4.22}$$

$$\big(w_0, w_1\big) = \sum_{i=1}^{n} \big[\mu_i^+(\rho_i, k_i\rho_i) + \mu_i^-(\rho_i, -k_i\rho_i)\big] + \big(\gamma_0, \gamma_1\big), \tag{4.23}$$

with orthogonality conditions $\int \rho_0\gamma_j\,dx = \int \rho_1\gamma_j\,dx = 0$, for all $1 \leq j \leq n$. (4.19) implies that

$$\epsilon_1^2 \lesssim \big\|\big(\gamma_0, \gamma_1\big)\big\|^2_{\dot{H}^1\times L^2} + \sum_{i=1}^{n}\Big[k_i^2\big(\mu_i^+ - \mu_i^-\big)^2 + \big(\mu_i^+ + \mu_i^-\big)^2\Big]. \tag{4.24}$$

(iii) Expand the energy functional at T_2. Since $\Lambda_0 = U^L(T_2) + w_0$, from (4.9), (4.19) and our a priori choice $\delta_1 < \frac{1}{2}\epsilon_1$, we have $\|\Lambda_0\|_{L^6} \lesssim \epsilon_1$. We now apply

Lemma 4.1 and obtain

$$\begin{aligned}\mathcal{E}\big(\vec{u}(T_2)\big) = \mathcal{E}(\phi,0) &+ \frac{1}{2}\big[\big(\mathcal{L}_\phi U^L(T_2),U^L(T_2)\big)+\big(U_t^L(T_2),U_t^L(T_2)\big)\big]\\ &-\sum_{i=1}^n 2\mu_i^+\mu_i^- k_i^2+\frac{1}{2}\big[\big(\mathcal{L}_\phi\gamma_0,\gamma_0\big)+\big(\gamma_1,\gamma_1\big)\big]\\ &+\big(\mathcal{L}_\phi U^L(T_2),w_0\big)+\big(U_t^L(T_2),w_1\big)+O(\epsilon_1^3).\end{aligned} \tag{4.25}$$

Note that using the L^6 estimate of U^L in (4.9), we further have

$$\begin{aligned}&\frac{1}{2}\Big[\big(\mathcal{L}_\phi U^L(T_2),U^L(T_2)\big)+\big(U_t^L(T_2),U_t^L(T_2)\big)\Big]\\ &\quad=\frac{1}{2}\big\|\vec{U}^L(T_2)\big\|^2_{\dot{H}^1\times L^2}+\big((-V+5\phi^4)U^L(T_2),U^L(T_2)\big)\\ &\quad=\frac{1}{2}\big\|\vec{U}^L(T_2)\big\|^2_{\dot{H}^1\times L^2}+O\big(\delta_1^2\big).\end{aligned} \tag{4.26}$$

In view of (4.18), (4.16) together with (4.10) implies that

$$\big\|w_0\big\|_{\dot{H}^1(|x|\geq T_2-T_1+L)}+\big\|w_1\big\|_{L^2(|x|\geq T_2-T_1+L)}\leq 4\delta_1.$$

Thus (w_0,w_1) is small inside the region $\{|x|\geq T_2-T_1+L\}$, while (4.11) implies that $\vec{U}^L$ is small inside the region $\{|x|<T_2-T_1+L\}$:

$$\big\|\vec{U}^L(T_2)\big\|_{\dot{H}^1\times L^2(|x|<T_2-T_1+L)}\leq\delta_1.$$

Hence we get that

$$\big|\big(U_t^L(T_2),w_1\big)\big|=\left|\int_{\{|x|\geq T_2-T_1+L\}\cup\{|x|<T_2-T_1+L\}}U_t^L(x,T_2)\,w_1(x)\,dx\right|\lesssim\delta_1; \tag{4.27}$$

and that

$$\begin{aligned}&\big|\big(\mathcal{L}_\phi U^L(T_2),w_0\big)\big|\\ &\quad=\left|\int\big(-\Delta-V+5\phi^4\big)U^L(x,T_2)\,w_0(x)\,dx\right|\\ &\quad=\left|\int_{\{|x|\geq T_2-T_1+L\}\cup\{|x|<T_2-T_1+L\}}\nabla U^L(T_2)\cdot\nabla w_0\,dx\right.\\ &\qquad\qquad\left.+\int\big(-V+5\phi^4\big)U^L(T_2)\,w_0\,dx\right|\lesssim\delta_1.\end{aligned} \tag{4.28}$$

Now let us combine estimates (4.26), (4.27), (4.28) with (4.25), noting (4.8), we deduce

$$\mathcal{E}(\vec{u}) = \mathcal{E}(\vec{U}) + \frac{1}{2}\big[\big(\mathcal{L}_\phi \gamma_0, \gamma_0\big) + \big(\gamma_1, \gamma_1\big)\big] - \sum_{i=1}^{k} 2\mu_i^+ \mu_i^- k_i^2 + O\big(\delta_1 + \epsilon_1^3\big). \tag{4.29}$$

(iv) Since (γ_0, γ_1) is in the continuous spectrum and $\mathcal{L}_\phi$ has no zero eigenvalues or zero resonance, we have

$$\big(\mathcal{L}_\phi \gamma_0, \gamma_0\big) + \big(\gamma_1, \gamma_1\big) \gtrsim \big\|\big(\gamma_0, \gamma_1\big)\big\|^2_{\dot{H}^1 \times L^2}$$

In combination with (4.20), (4.24), and (4.29), this coercivity yields

$$\begin{aligned}
&\mathcal{E}(\vec{u}) - \mathcal{E}(\vec{U}) + \sum_{i=1}^{n} 2\mu_i^+ \mu_i^- k_i^2 \\
&\quad = \frac{1}{2}\big[\big(\mathcal{L}_\phi \gamma_0, \gamma_0\big) + \big(\gamma_1, \gamma_1\big)\big] + O\big(\delta_1 + \epsilon_1^3\big) \\
&\quad \gtrsim \big\|\big(\gamma_0, \gamma_1\big)\big\|^2_{\dot{H}^1 \times L^2} + O\big(\delta_1 + \epsilon_1^3\big) \\
&\quad \gtrsim c\epsilon_1^2 - \sum_{i=1}^{n} \Big[k_i^2 \big(\mu_i^+ - \mu_i^-\big)^2 + \big(\mu_i^+ + \mu_i^-\big)^2\Big] + O\big(\delta_1 + \epsilon_1^3\big).
\end{aligned}$$

This implies that

$$\begin{aligned}
&\sum_{i=1}^{n} 2\mu_i^+ \mu_i^- k_i^2 + \frac{C}{2}\sum_{i=1}^{n} k_i^2 \big(\mu_i^+ - \mu_i^-\big)^2 + \frac{C}{2}\sum_{i=1}^{n} \big(\mu_i^+ + \mu_i^-\big)^2 \\
&\quad \gtrsim \epsilon_1^2 - \big|\mathcal{E}(\vec{U}) - \mathcal{E}(\vec{u})\big| - O\big(\delta_1 + \epsilon_1^3\big) \\
&\quad \gtrsim \epsilon_1^2 - C\epsilon - O\big(\delta_1 + \epsilon_1^3\big).
\end{aligned}$$

Since all the constants depend only on U, we can choose $\delta_1, \epsilon \ll \epsilon_1^2$ and conclude that

$$\sum_{i=1}^{n} \big|\mu_i^+\big|^2 + \big|\mu_i^-\big|^2 \gtrsim \epsilon_1^2. \tag{4.30}$$

Now we denote $|\mu_{\max}| = \max\{|\mu_i^+|, |\mu_i^-|,\ 1 \le i \le n\}$. We can find $1 \le i_0 \le n$ such that either $|\mu_{i_0}^+| = |\mu_{\max}|$ or $|\mu_{i_0}^-| = |\mu_{\max}|$. From (4.19) and (4.30), we get

$$2n|\mu_{\max}|^2 \ge c\epsilon_1^2 \ge \frac{c}{4}\|\vec{w}\|^2_{\dot{H}^1 \times L^2},$$

hence $|\mu_{\max}| \ge \sqrt{\frac{c}{8n}}\|\vec{w}\|_{\dot{H}^1 \times L^2}$. The constant c only depends on V and ϕ.

Step 3: Show the second emission of energy and finish the proof.

Case 1: $|\mu_{max}| = |\mu^+_{i_0}|$. Consider the solution $\tilde{u}$ to equation (1.1) with

$$\vec{\tilde{u}}(T_2) = (\phi, 0) + (w_0, w_1).$$

Take $\kappa = \sqrt{\frac{C}{8n}}$, K and ϵ_* corresponding to $R = 0$ in Lemma 3.6. Note that both parameters depend only on V. With these choices of parameters, we get $T(\kappa, K, V)$ and $\varepsilon(\kappa, K, \epsilon_*, T)$ from Lemma 3.7. Shrinking ϵ_1 if necessary, we can assume that $\epsilon_1 < \varepsilon(\kappa, K, \epsilon_*, T)$. We emphasize that none of these parameters depend on δ_1 or ϵ, which are free parameters at this point. This is very important of course, in order not to run into a circular argument. We also note that $T = T(\kappa, K, V)$ from Lemma 3.7 does not depend on δ_1 or ϵ.

We can now apply part (1) of Lemma 3.7 and part (2) of Lemma 3.6 to conclude that

$$\int_{|x|>t-(T_2+T)} \left|\partial_t \tilde{u}\right|^2(t,x)\,dx \geq c(\epsilon_1) > 0, \quad \text{for } t \geq T_2 + T. \tag{4.31}$$

Denote $\Xi := R^3 \times [T_2, T_2 + T] \bigcup \{(t,x) \colon |x| > t - T_2 - T, t \geq T_2 + T\}$. Note that

$$\left(|V| + \phi^4\right)\chi_\Xi \in L_t^{\frac{5}{4}} L_x^{\frac{5}{2}}\left(\mathbb{R}^3 \times \mathbb{R}\right),$$

and that $U^L + \tilde{u}$ is an approximate solution to (1.1) with a right-hand side f with $\|f\|_{L^1_t L^2_x} \lesssim \delta_1$. By bound (4.12) and Lemma 3.1 (by treating u as perturbation of $U^L + \tilde{u}$), if we choose δ_1 sufficiently small, then for $(x,t) \in \Xi$,

$$\vec{u}(t,x) = \vec{U}^L(t,x) + \vec{\tilde{u}}(t,x) + \vec{r}(t,x), \tag{4.32}$$

where the remainder term $\vec{r}$ satisfies

$$\sup_{t \in \mathbb{R}} \left\|\vec{r}(t)\right\|_{\dot{H}^1 \times L^2(\mathbb{R}^3)} \leq C\delta_1. \tag{4.33}$$

The estimate (4.16), decomposition (4.32) and the estimate on the remainder term (4.33) imply for $t \geq T_2$ (in particular, $t \geq T_2 + T$)

$$\int_{|x| \geq t - T_1 + L} \left|\nabla_{t,x} \tilde{u}\right|^2(t,x)\,dx \leq C\delta_1, \tag{4.34}$$

this combined with (4.31) implies that for $t \geq T_2 + T$

$$\int_{t - T_1 + L \geq |x| \geq t - (T_2 + T)} \left|\nabla_{t,x} \tilde{u}\right|^2(t,x)\,dx \geq c(\epsilon_1) - C\delta_1. \tag{4.35}$$

Hence by estimating $\|\vec{u}(t)\|_{\dot{H}^1\times L^2}$ in different regions $\{|x|\geq t-T_1+L\}$ and $\{t-T_1+L\geq |x|\geq t-(T_2+T)\}$, we get that

$$
\begin{aligned}
&\|\vec{u}\|^2_{\dot{H}^1\times L^2(|x|\geq t-(T_2+T))}\\
&\quad\geq \big\|\vec{U}^L+\vec{\tilde{u}}\big\|^2_{\dot{H}^1\times L^2(|x|\geq t-(T_2+T))}-C_1\delta_1\\
&\quad\geq \big\|\vec{U}^L\big\|^2_{\dot{H}^1\times L^2(|x|\geq t-T_1+L)}+c\big(\epsilon_1\big)-C_2\delta_1\\
&\quad\geq \big\|\vec{U}^L\big\|^2_{\dot{H}^1\times L^2(\mathbb{R}^3)}+c(\epsilon_1)-C_3\delta_1\geq \big\|\vec{U}^L\big\|^2_{\dot{H}^1\times L^2(\mathbb{R}^3)}+\frac{1}{2}c\big(\epsilon_1\big).
\end{aligned}\tag{4.36}
$$

The last line holds when we choose δ_1 sufficiently small. (4.5) is then proved with $A=T_2+T$ and $\delta=\frac{1}{2}c(\epsilon_1)>0$.

Now we prove that u cannot scatter to $(\phi,0)$ $t\to+\infty$. Suppose it does so with free radiation $\vec{u}^L$, i.e.,

$$\big\|\vec{u}(t)-(\phi,0)-\vec{u}^L(t)\big\|_{\dot{H}^1\times L^2}\longrightarrow 0,\quad \text{as } t\longrightarrow+\infty.$$

Then (4.36) implies that

$$\big\|\vec{u}^L(t)\big\|^2_{\dot{H}^1\times L^2(\mathbb{R}^3)}\geq \lim_{t\to\infty}\big\|\vec{u}(t)\big\|^2_{\dot{H}^1\times L^2(|x|\geq t-A)}\geq \big\|\vec{U}^L(t)\big\|^2_{\dot{H}^1\times L^2(\mathbb{R}^3)}+\delta.\tag{4.37}$$

Note that

$$\big\|\vec{U}^L\big\|^2_{\dot{H}^1\times L^2}=\mathcal{E}(\vec{U})-\mathcal{E}(\phi,0),\quad \big\|\vec{u}^L\big\|^2_{\dot{H}^1\times L^2}=\mathcal{E}(\vec{u})-\mathcal{E}(\phi,0).$$

We have reached contradiction with (4.37) if $\|\vec{u}(0)-\vec{U}(0)\|_{\dot{H}^1\times L^2(\mathbb{R}^3)}$ is chosen small, and thus have proved the theorem in case 1.

Case 2: $|\mu_{max}|=|\mu^-_{i_0}|$. We will show this is impossible if we take ϵ small enough. In fact, again applying part (1) of Lemma 3.7 and part (2) of Lemma 3.6, consider the solution $\tilde{u}$ to equation (1.1) with data

$$\vec{\tilde{u}}\big(T_2\big)=(\phi,0)+\big(w_0,w_1\big).$$

We can find a time $T>0$ such that

$$\int_{|x|>|t-(T_2-T)|}\big|\partial_t\tilde{u}\big|^2(t,x)\,dx\geq c(\epsilon_1)>0,\quad \text{for } t\leq T_2-T.\tag{4.38}$$

By taking ϵ sufficiently small, we can assume $T_2>2T$. Now setting time $t=0$ in (4.38), we get $\|\vec{\tilde{u}}(0)\|_{\dot{H}^1\times L^2(|x|>\frac{1}{2}T_2)}>c(\epsilon_1)$.

Introduce the set

$$\Xi':=R^3\times[T_2-T,T_2]\bigcup\big\{(t,x)\colon |x|>\big|t-(T_2-T)\big|,\ 0\leq t\leq T_2-T\big\}.$$

In analogy to case 1, if we choose $\delta_1 > 0$ sufficiently small, then for $(x,t) \in \Xi'$ we have

$$\vec{u}(t,x) = \vec{U}^L(t,x) + \vec{\tilde{u}}(t,x) + \vec{r}(t,x), \tag{4.39}$$

with the remainder term $\vec{r}$ satisfying

$$\sup_{t\in\mathbb{R}} \big\|\vec{r}(t)\big\|_{\dot{H}^1\times L^2(\mathbb{R}^3)} \le C\delta_1. \tag{4.40}$$

From (4.10) and by our choice of T_2, i.e., $T_2 > 2(L+T_1+1)$ and $T_2 > 2T$, we have $\|\vec{U}^L(0)\|_{\dot{H}^1\times L^2(|x|>\frac{1}{2}T_2)} < \delta_1$ and

$$\begin{aligned}&\big\|(u_0,u_1)\big\|_{\dot{H}^1\times L^2(|x|>\frac{1}{2}T_2)}\\ &\quad \ge \big\|\nabla_{x,t}\tilde{u}(0)\big\|_{\dot{H}^1\times L^2(|x|>\frac{1}{2}T_2)} - C\delta_1 > c(\epsilon_1) - C\delta_1 > \frac{1}{2}c(\epsilon_1).\end{aligned}$$

The last inequality holds provided we take δ_1 small enough. This yields a contradiction to the finite energy of $\vec{U}(0)$ by choosing ϵ sufficiently small and T_2 sufficiently large. Hence case 2 does not arise and we are done. □

Next we prove the property of path connectedness.

THEOREM 4.3. *For any unstable excited state* $(\phi,0)$*, the corresponding center-stable manifold* $\mathcal{M}_\phi$ *is path connected.*

Proof. Given data $(u_0,u_1),(\tilde{u}_0,\tilde{u}_1) \in \mathcal{M}_\phi$, we denote the corresponding solutions by $u,\tilde{u}$. Write $h = u-\phi$, $\ell = \tilde{u}-\phi$. Repeat step 1 and step 2 in the proof of Theorem 2.2. Then given any $\epsilon \ll 1$, we can find $T = T(\epsilon,u,\tilde{u})$, such that

$$\|h\|_{L_x^{6,2}L_t^\infty\cap L_x^\infty L_t^2(\mathbb{R}^3\times[T,\infty))} < \epsilon, \quad \|\ell\|_{L_x^{6,2}L_t^\infty\cap L_x^\infty L_t^2(\mathbb{R}^3\times[T,\infty))} < \epsilon. \tag{4.41}$$

Now we seek a function $w(\theta,t,x)$ of the form

$$\begin{aligned} w(\theta,t,x) &= (1-\theta)u + \theta\tilde{u} + \eta\\ &= \phi + (1-\theta)h + \theta\ell + \sum_{i=1}^n \lambda_i(\theta,t)\rho_i + \gamma(\theta,t,x)\end{aligned} \tag{4.42}$$

with initial data $\lambda_i(\theta,T)$ and $\gamma(\theta,T,x)$ decided later, such that for all $\theta \in [0,1]$, $\gamma(\theta,t,x) \perp \rho_i$, $i = 1,\dots,n$ and $w(\theta,t,x)$ is a solution to equation (1.1) that scatters to ϕ.

For $\theta \in [0,1]$ fixed, the equation satisfied by $\eta = \sum_{i=1}^n \lambda_i(\theta,t)\rho_i + \gamma(\theta,t,x)$ is:

$$\eta_{tt} - \Delta\eta - V(x)\eta + 5\phi^4\eta + N(\theta,h,\ell,\phi,\eta) = 0,$$

where

$$N(\theta,h,\ell,\phi,\eta)=\big(\phi+(1-\theta)h+\theta\ell+\eta\big)^5-(1-\theta)(\phi+h)^5-\theta(\phi+\ell)^5-5\phi^4\eta.$$

Now we can repeat the stability condition (2.36) and obtain the reduced system of the form (2.37).

In $N(\theta,h,\ell,\phi,\eta)$, the terms independent of η are of the form

$$\big(\phi+(1-\theta)h+\theta\ell\big)^5-(1-\theta)(\phi+h)^5-\theta(\phi+\ell)^5=\sum_{i+j+k=5,\, i\le 3}C(\theta,i,j,k)\phi^i h^j\ell^k.$$

Notice that there are no terms ϕ^5 or $\phi^4 h, \phi^4\ell$.

Also, the linear term of η in $N(\theta,h,\ell,\phi,\eta)$ is

$$5\big(\phi+(1-\theta)h+\theta\ell\big)^4\eta-5\phi^4\eta$$

hence all linear terms involve a factor of h or ℓ.

Now we can repeat estimates (2.39)(2.40), then (2.42) for the linear term in η, (2.43) for higher order terms in η. We also have the following estimate on terms independent of η

$$\left\|\sum_{i+j+k=5,\, i\le 3}C(\theta,i,j,k)\phi^i h^j\ell^k\right\|_{L^1_tL^2_x([T,\infty)\times\mathbb{R}^3)}\lesssim\epsilon^2.$$

To sum up, using the X norm defined in (2.38), we conclude that

$$\begin{aligned}&\big\|\big(\lambda_1,\dots,\lambda_n,\gamma\big)\big\|_{X([T,\infty))}\\ &\quad\le L\epsilon^2+L\left(\sum_{i=1}^n\big|\lambda_i(\theta,T)\big|+\big\|\big(\gamma(\theta,T),\dot\gamma(\theta,T)\big)\big\|_{\dot H^1\times L^2}\right)\\ &\qquad+L\epsilon\big\|\big(\lambda_1,\dots,\lambda_n,\gamma\big)\big\|_{X([T,\infty))}+L\sum_{k=2}^5\big\|\big(\lambda_1,\dots,\lambda_n,\gamma\big)\big\|^k_{X([T,\infty))},\end{aligned}$$

where $L>1$ is a constant only depending on the constants in the reversed Strichartz estimates, $\|\phi\|_{L^6(\mathbb{R}^3)}$ and $\|\rho_i\|_{L^\infty_x\cap L^{6,2}_x}$.

Moreover, in a similar fashion one sees that the difference of two solutions satisfies a similar estimate in which the first two terms disappear. Following step 3 of the proof for Theorem 2.2, we can use the contraction mapping principle and conclude that for sufficiently small data

$$\sum_{i=1}^n\big|\lambda_i(\theta,T)\big|+\big\|\big(\gamma(\theta,T),\dot\gamma(\theta,T)\big)\big\|_{\dot H^1\times L^2}\le\delta$$

there is a solution w as in (4.42) which solves (1.1). We can also check that w scatters to ϕ as in step 4 of the proof for Theorem 2.2.

In particular, let us take $\lambda_i(\theta,T) = \frac{1}{n}\delta\theta(1-\theta)$ and $\vec{\gamma}(\theta,T,x) = \vec{0}$. We claim that the corresponding solution $w(\theta,t,x)$ satisfies the following relation

$$w(0,t,x) = u(t,x), \quad w(1,t,x) = \tilde{u}(t,x), \quad \text{for all } t \in \mathbb{R}. \tag{4.43}$$

In fact, notice that $\lambda_i(0,T) = 0$, $\vec{\gamma}(0,T,x) = \vec{0}$ implies $\lambda_i(0,t) = 0$, $\vec{\gamma}(0,t,x) = \vec{0}$ for $t \geq T$, which further implies $w(0,t,x) = u(t,x)$, $t \geq T$. Similarly we have $w(1,t,x) = \tilde{u}(t,x)$, $t \geq T$. Then (4.43) follows from the uniqueness of solutions to equation (1.1).

Hence $\{\vec{w}(\theta,0,x),\ \theta \in [0,1]\}$ is a path in $\mathcal{M}_\phi$ connecting the two data (u_0,u_1), $(\tilde{u}_0,\tilde{u}_1)$. □

Now we can finish the proof for our main theorem.

Proof of Theorem 1.1. We only consider the case in which $(\phi,0)$ is unstable; stable $(\phi,0)$ can be handled using standard perturbation arguments and the reversed Strichartz estimates. We only note that due to the lack of local wellposedness of equation (1.1) in the reverse Strichartz space $L^{6,2}_x L^\infty_t \cap L^\infty_x L^2_t$, we need to use the fact that

$$\lim_{T\to\infty} \|U-\phi\|_{L^{6,2}_x L^\infty_t \cap L^\infty_x L^2_t(\mathbb{R}^3\times[T,\infty))} = 0,$$

if $U(t)$ scatters to ϕ as $t \to \infty$. This fact can be easily deduced by using the same argument as in Claim 2.2.1. In some small neighborhood of any point $\vec{U}(0)$ on $\mathcal{M}_\phi$, $\mathcal{M}_\phi$ coincides with the local center-stable manifold $\mathcal{M}$ of codimension n which we constructed in Section 2. By Theorem 4.2, $\mathcal{M}_\phi$ is thus a global manifold of co-dimension n. The path-connectedness follows from Theorem 4.3. □

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