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On the spectrum of multi-frequency quasiperiodic Schrödinger operators with large coupling

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Abstract We study multi-frequency quasiperiodic Schrödinger operators on $\mathbb{Z}$. We prove that for a large real analytic potential satisfying certain restrictions the spectrum consists of a single interval. The result is a consequence of a criterion for the spectrum to contain an interval at a given location that we establish non-perturbatively in the regime of positive Lyapunov exponent.

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1 Introduction

In the last 40 years after the groundbreaking paper [11] the theory of quasiperiodic Schrödinger operators has been developed extensively, see the monograph [5] for an overview and [17] for a survey of the more recent results. For shifts on a one-dimensional torus $\mathbb{T}$ most of the results have been established nonperturbatively, i.e., either in the regime of almost reducibility or in the regime of positive Lyapunov exponent, and Avila's global theory, see [3], gives a qualitative spectral picture, covering both regimes, for generic potentials. One of the main results of the one-dimensional theory is the fact that the spectrum is a Cantor set. For the case of the almost Mathieu operator (corresponding to a cosine potential), this result has been proved for any non-zero coupling and any irrational shift, see [22] and [1,2]. For general analytic potentials in the regime of positive Lyapunov exponent with generic shift the Cantor structure of the spectrum has been obtained in [15].

On the other hand, shifts on a multidimensional torus $\mathbb{T}^d$ turned out to be harder to analyze and the theory is less developed, even in the perturbative setting. In particular, not much is known about the geometry of the spectrum for multidimensional shifts. In their pioneering paper [9], Chulaevsky and Sinai conjectured that in contrast to the shift on the one-dimensional torus, for the two-dimensional shift the spectrum can be an interval for generic large smooth potentials. In this paper we prove this conjecture for large analytic potentials.

Heuristically, gaps in the spectrum of the one-frequency operators are created by horizontal "forbidden zones" appearing at the points of intersection of the graphs of shifted finite scale eigenvalues parametrized by phase, see [15,23]. In contrast to this, the heuristic principle underlying [9] is that for multiple frequencies, the intersection curves of the graphs of shifted finite scale eigenvalues may not be too flat, thus preventing the appearance of the horizontal "forbidden zones" and stopping the formation of gaps. It is clear that some genericity assumption on the potential function is needed for this to be true, since potentials like $V(x, y) = v(x)$ lead to flat intersection curves and have Cantor spectrum. Furthermore, the largeness of the potential is also needed. Indeed, it is known that for small potentials with atypical frequency vector the spectrum has gaps, see [4].

Implementing such an argument, appears to be very challenging for a number of reasons. First, the analytical techniques available in finite volume are less favorable (mainly the large deviation theorems and everything that depends on them) as compared to the case of one frequency. In particular, it is difficult to implement an approach based on finite scale localization as in [15]. This is due to the fact that it is hard to handle long chains of resonances and to control the intersections of the resonant curves with the level sets of the eigenvalues. Second, it is inevitable that the intersection curves of the graphs of shifted finite scale eigenvalues flatten near the absolute extrema and handling this situation seems to be a delicate matter.

In [16] we addressed some of the issues regarding the analytical techniques, including establishing finite scale localization. We will use most of the basic tools from [16]. However, for the purpose of this paper one would need a refined version of finite scale localization, beyond what is achieved in that paper. We analyze the spectrum of the operator $H_N(x)$, $x \in \mathbb{T}^d$, on a finite interval $[1, N]$ subject to Dirichlet boundary conditions. To keep this spectrum under control requires resolving the following problem. Given E let $\mathcal{R}_N(E)$ be the set of all phases x such that E is in the spectrum of the operator $H_N(x)$. One has to identify phases $x \in \mathcal{R}_N(E)$ for which $x + n\omega$ is not too close to $\mathcal{R}_N(E)$ as n runs in the interval $N \ll n < N^A$, $A \gg 1$. This issue, commonly referred to as *double resonances*, is well-known. Similar strategies, leading to the formation of intervals in the spectrum, have been implemented for the skew-shift in [18] and for continuous two-dimensional Schrödinger operators in [19]. The main new device that we develop in this work, consists of an elimination of double resonances for *all* shifts $x + h$, and not just the "arithmetic ones" $x + n\omega$. Of course the shift h cannot be too small. Although this problem looks less accessible, it turns out to provide more control on the resonant set $\mathcal{R}_N(E)$ of the previous scale. The level sets $V(x) = E$ of the potential in question must satisfy the requirements of this more general elimination in order to launch the multi-scale analysis. This is exactly the origin of our main condition on the potential, see Definition 1.1 below.

Furthermore, in order to show that the spectrum is actually an interval, we develop a Cartan type estimate that controls the intersections of the level sets of an analytic function near a non-degenerate extremum with their shifts.

The core of our approach is *non-perturbative* and works in the regime of positive Lyapunov exponent. More precisely, we develop two non-perturbative inductive schemes, one leading to the formation of intervals in the bulk of the spectrum and the other leading to intervals at the edges of the spectrum. We will only use the largeness of the potential to check that the initial inductive conditions are satisfied.

We introduce some notation and definitions that we need to state our main result. We work with operators

$$[H_\lambda(x)\psi](n) = -\psi(n+1) - \psi(n-1) + \lambda V(x+n\omega)\psi(n), \tag{1.1}$$

with $\lambda > 0$ being a real parameter, and with the potential V a real analytic function on the torus $\mathbb{T}^d$, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, $d \geq 2$. We assume that the frequency vector $\omega \in \mathbb{T}^d$ obeys the standard Diophantine condition

$$\|k \cdot \omega\| \geq \frac{a}{|k|^b} \qquad \text{for all nonzero } k \in \mathbb{Z}^d, \tag{1.2}$$

where $a > 0$, $b > d$ are some constants, $\|\cdot\|$ denotes the usual norm on $\mathbb{T}$, and $|\cdot|$ denotes the sup-norm on $\mathbb{Z}^d$. Unless otherwise stated, throughout the paper a, b will refer to the constants from (1.2). In this paper we don't use elimination of frequencies and our results apply to any Diophantine frequency ω. To simplify notation, we omit dependence on ω from notation whenever possible. The dependence on frequency will still be reflected by having some of the constants depend on a, b.

Definition 1.1 We let $\mathfrak{G}$ be the class of real-analytic functions V on $\mathbb{T}^d$, $d \geq 2$, for which there exist constants $\mathfrak{c}_0 = \mathfrak{c}_0(V, d) \in (0, 1)$, $\mathfrak{c}_1 = \mathfrak{c}_1(d) \in (0, 1)$, $\mathfrak{C}_0 = \mathfrak{C}_0(V, d) > 1$, such that the following properties hold.

(i) V is a Morse function, i.e., all its critical points are non-degenerate.
(ii) V attains each global extremum at just one point.
(iii) Given $h \in \mathbb{T}^d$, let

$$g_{V,h,i,j}(x) = \det \begin{bmatrix} \partial_{x_i} V(x) & \partial_{x_j} V(x) \\ \partial_{x_i} V(x+h) & \partial_{x_j} V(x+h) \end{bmatrix}.$$

For any $i \neq j$, $K \geq \mathfrak{C}_0$, and any $\|h\| \geq \exp(-\mathfrak{c}_0 K)$ we have

$$\begin{aligned} &\mathrm{mes}\{x_{\hat{i}} \in \mathbb{T}^{d-1} : \min_{x_i} \big(|V(x+h) - V(x)| + |g_{V,h,i,j}(x)|\big) < \exp(-K)\} \\ &\quad \leq \exp(-K^{\mathfrak{c}_1}), \end{aligned}$$

where $x_{\hat{i}} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d)$.

(iv) For any i, $K \geq \mathfrak{C}_0$, $\eta \in \mathbb{R}$, and $h_0 \in \mathbb{R}^d$, $\|h_0\| = 1$, we have

$$\begin{aligned} &\mathrm{mes}\{x_{\hat{i}} \in \mathbb{T}^{d-1} : \min_{x_i} (|V(x) - \eta| + |\langle \nabla V(x), h_0 \rangle|) < \exp(-K)\} \\ &\quad \leq \exp(-K^{\mathfrak{c}_1}). \end{aligned}$$

Recall that spec $H_\lambda(x)$ is known not to depend on the phase. We will use the notation $\mathcal{S}_\lambda :=$ spec $H_\lambda(x)$. An essential feature of our inductive argument is the following one: we use the genericity of V only at the first step of the proof, and never change V at subsequent steps.

Theorem A *There exists $\lambda_0 = \lambda_0(V, a, b, d)$ such that the following statements hold for $\lambda \geq \lambda_0$.*

(a) Assume that V attains its global minimum at exactly one non-degenerate critical point $\underline{x}$. Then there exists $\underline{E} \in \mathbb{R}$, $|\lambda^{-1}\underline{E} - V(\underline{x})| < \lambda^{-1/4}$, such that

$$[\underline{E}, \underline{E} + \lambda \exp(-(\log \lambda)^{1/2})] \subset \mathcal{S}_\lambda \quad \textit{and} \quad (-\infty, \underline{E}) \cap \mathcal{S}_\lambda = \emptyset.$$

An analogous statement holds relative to the global maximum of V (using the notation $\overline{x}$, $\overline{E}$).

(b) Assume that $V \in \mathfrak{G}$ and let $\underline{E}$, $\overline{E}$ be as in (a). Then $\mathcal{S}_\lambda = [\underline{E}, \overline{E}]$.

Remark 1.2 (a) The constant $\lambda_0(V, a, b, d)$ can be expressed explicitly, see the proof of Theorem A.

(b) We conjecture the genericity of our assumptions on V. More precisely, we believe the following to be true: consider real trigonometric polynomials of the form

$$V(x) = \sum_{m \in \mathbb{Z}^d : |m| \leq n} c_m e^{2\pi i m \cdot x}, \quad x \in \mathbb{R}^d,$$

of a given cumulative degree $n \geq 1$, $|m| := \sum_{1 \leq j \leq d} |m_j|$. Then for almost all vectors $(c_m)_{|m| \leq n}$ one has $V \in \mathfrak{G}$. While genericity of admissible V remains a conjecture for general degrees, we do present specific examples of V of low degree in two variables, which obey our conditions.

(c) In fact, in Sect. 9, we show that

$$V(x, y) = \cos(2\pi x) + s \cos(2\pi y)$$

satisfies the assumptions of Definition 1.1 for all $s \in \mathbb{R} \backslash \{-1, 0, 1\}$. We note that as s approaches $\{-1, 0, 1\}$ our explicit value for λ_0 diverges to ∞ and the geometry of the spectrum cannot be decided by continuity. Of course, for $s = 0$ the spectrum is a Cantor set. However, for $s = \pm 1$, part (a) of Theorem A still applies and guarantees the existence of intervals at the edges of the spectrum.

(d) The measure estimates from conditions (iii) and (iv) of Definition 1.1 are Cartan type estimates (see Sect. 2.2). We note that one cannot apply Cartan's estimate directly to the functions from this conditions. Instead, the estimates can be obtained by applying Cartan's estimate to some resultants associated with these functions, see Sect. 9.

As mentioned above, the derivation of Theorem A is based on two non-perturbative statements in the regime of positive Lyapunov exponent, which appear later in Sect. 8. Namely, Theorem B produces an interval in the spectrum in the vicinity of a spectral value at which certain finite scale conditions hold, and Theorem C shows that the spectrum is an interval under certain additional finite scale conditions. Since they are rather technical, we do not state these theorems here. The inductive conditions and the theorems which provide the inductive step are discussed in Sect. 5 (see Theorem D) and Sect. 6 (see Theorem E). In Sect. 7 we show how these conditions hold at large coupling, given a potential as in Theorem A. Throughout the paper we will employ the basic tools discussed in Sect. 2 for the non-perturbative regime and in Sect. 3 for large coupling. The Cartan type estimate that we use to handle the edges of the spectrum is discussed in Sect. 4.

We conclude this introduction with more detailed comments on the aforementioned paper by Chulaevsky, Sinai [9], which is closely related to the one-frequency paper [23]. In [9] the authors propose an inductive perturbative scheme to establish localization, positive Lyapunov exponents, and the absence of gaps for the operators (1.1) for large λ and for ω outside a set of small measure. The potential V is assumed to be a generic (in a suitable sense) C^2 Morse function. The induction, of which [9] only provides a sketch with many details having been omitted, proceeds from the base case in which the eigenfunctions are taken to be δ-functions, to successively more accurate approximations of the true eigenfunctions. It is claimed that the corrections are obtained via first order eigenvalue perturbations only. It is well-understood by now that many delicate issues arise in the implementation of any inductive procedure aiming at Anderson localization. First and foremost, one needs to exclude the possibility of arbitrarily long chains of resonances between finite-volume Hamiltonians of successive scales.

The research literature devoted to Anderson localization with deterministic potentials has been almost entirely limited to the analytic category, i.e., V in (1.1) is either a trigonometric polynomial or an analytic function, see for example [5–8, 12–16]. In essence, resonances arise through intersections of level surfaces of the eigenvalue parametrizations of finite volume Hamiltonians. For algebraic curves Bezout's theorem gives a quantitative bound on the number of intersections. In the C^k category no analogous mechanism exists, and intersections can be extremely complicated. Bourgain [7] used semi-algebraic techniques such as the Gromov-Yomdin parametrization to limit the length of chains of resonances in any number of variables. For example, in the setting of (1.1) with $d = 2$ he needs to allow for chains of length 9. Bourgain's technique for eliminating variables via semi-algebraic methods (essentially, Bezout's theorem), played an important role in the implementation of an inductive argument for finite volume localization of (1.1) in the regime of positive Lyapunov exponent (i.e., without assuming large coupling as we do here), see [16]. In addition, we crucially relied on an effective separation between the eigenvalues in finite volume as in [15]. Complex variable tools such as the Weierstrass preparation theorem, and the resultant between polynomials are used to obtain these bounds.

Wang and Zhang [24] claim positive Lyapunov exponents for C^2-potentials *of one variable* with *two non-degenerate critical points* and large disorder λ. While they acknowledge Sinai's mechanism from [23] that resonances create gaps, their argument bears little resemblance with [23], and relies instead on techniques developed over the past 20 years such as the avalanche principle. Wang and Zhang's arguments are however entirely one-dimensional (for example, they use Rolle's theorem) and to our knowledge nothing comparable exists for C^k-potentials of several variables. We are therefore unable to reconcile the strategies which were proposed in [9,23] with the facts established over the past 20 years.

2 Basic tools

In this section we discuss some basic results that we will use throughout the paper. The results will apply to a family of discrete Schrödinger operators,

$$[H(x)\psi](n) = -\psi(n+1) - \psi(n-1) + V(x + n\omega)\psi(n) \tag{2.1}$$

with V real-analytic on $\mathbb{T}^d$ and ω as in (1.2). Note that we omit the coupling constant λ because the results of this section are non-perturbative. We also assume that V extends complex analytically to

$$\mathbb{T}^d_\rho := \{x + iy : x \in \mathbb{T}^d,\ \ y \in \mathbb{R}^d,\ \ |y| < \rho\},$$

with some $\rho > 0$. Note that we use $|\cdot|$ to denote the sup-norm on $\mathbb{R}^d$ and $\|\cdot\|$ to denote the Euclidean norm on $\mathbb{R}^d$. At the same time when we apply it to shifts on $\mathbb{T}^d$, $\|\cdot\|$ will stand for the usual norm on $\mathbb{T}^d$. It is well-known that for any real-analytic function on $\mathbb{T}^d$, such $\rho = \rho(V)$ exists. To simplify some later estimates we also assume $\rho \le 1$. Throughout the paper, with the exception of Sect. 4, we reserve ρ for this constant.

We recall some standard notation. Given an interval $[a, b] \subset \mathbb{Z}$, the transfer matrix is defined by

$$M_{[a,b]}(x, E) = \prod_{n=b}^{a} \begin{bmatrix} V(x + n\omega) - E & -1 \\ 1 & 0 \end{bmatrix}.$$

We let $H_{[a,b]}(x)$ be the restriction of $H(x)$ to the interval $[a, b]$ with Dirichlet boundary conditions and we denote the corresponding Dirichlet determinant by $f_{[a,b]}(x, E) := \det(H_{[a,b]}(x) - E)$. We use $E_j^{[a,b]}(x)$, $\psi_j^{[a,b]}(x, \cdot)$ to denote the eigenpairs of $H^{[a,b]}(x)$, with $\psi_j^{[a,b]}(x, \cdot)$ being ℓ^2-normalized. The transfer matrix is related to the Dirichlet determinants through the following formula

$$M_{[a,b]}(x, E) = \begin{bmatrix} f_{[a,b]}(x, E) & -f_{[a+1,b]}(x, E) \\ f_{[a,b-1]}(x, E) & -f_{[a+1,b-1]}(x, E) \end{bmatrix}. \tag{2.2}$$

We let $M_N := M_{[1,N]}$, $H_N := H_{[1,N]}$, $f_N := f_{[1,N]}$. The *Lyapunov exponent* is defined by

$$L(E) = \lim_{N\to\infty} L_N(E) = \inf_N L_N(E), \quad L_N(E) = \frac{1}{N} \int_{\mathbb{T}^d} \log \|M_N(x, E)\| \, dx.$$

Most of the results in this section *do not use* the fact that V assumes only real values on the torus $\mathbb{T}^d$ and therefore they also hold on $\mathbb{T}^d + iy$, $|y| < \rho/2$, by replacing V with $V(\cdot + iy)$. In particular, this applies to all the results up to and including Corollary 2.13. Of course, when we change the potential, we also need to adjust the Lyapunov exponents. To this end we define

$$\begin{aligned} L_N(y, E) &= \frac{1}{N} \int_{\mathbb{T}^d} \log \|M_N(x + iy, E)\| \, dx, \\ L(y, E) &= \lim_{N\to\infty} L_N(y, E). \end{aligned} \tag{2.3}$$

We will use some standard conventions. Unless stated otherwise, the constants denoted by c, C might have different values each time they are used. We let $a \lesssim b$ denote $a \le Cb$ with some positive C, $a \ll b$ denote $a \le Cb$ with a sufficiently large positive C, and $a \simeq b$ stand for $a \lesssim b$ and $b \lesssim a$. It will be clear from the context what the implicit constants are allowed to depend on. To emphasize the dependence on some parameter we may use it as a subscript for the above symbols (e.g., $a \simeq_d b$).

Our constants will depend on ω, V, E, d, and γ, where $\gamma > 0$ will stand for a lower bound on the Lyapunov exponent. The dependence on ω will be through the parameters a, b from (1.2). The dependence on V will be through ρ and

$$\|V\|_\infty := \sup\{|V(z)| : z \in \mathbb{T}^d_{3\rho/4}\}.$$

The dependence on E will be uniform on bounded sets. In most cases we leave the dependence on d implicit and, unless stated otherwise, all constants may depend on the dimension d.

When we work in the perturbative setting we will need to replace V by λV and we will need explicit knowledge of the dependence on λ. This means that we need to keep track explicitly of the dependence on $\|V\|_\infty$, E (because the range of energies we need to consider depends on V), and γ (note that ρ remains unchanged when we introduce the coupling constant). To this end we will use the quantity

$$S_{V,E} := \log(3 + \|V\|_\infty + |E|).$$

This definition is motivated by the fact that

$$\left\| \begin{bmatrix} V(x+n\omega) - E & -1 \\ 1 & 0 \end{bmatrix} \right\| \leq 1 + \|V\|_\infty + |E|$$

and therefore

$$0 \leq \log \|M_N(x, E)\| \leq N \log(1 + \|V\|_\infty + |E|), \tag{2.4}$$
$$0 \leq L_N(E) \leq \log(1 + \|V\|_\infty + |E|). \tag{2.5}$$

The choice of the absolute constant in the definition of $S_{V,E}$ is for the convenience of having $S_{V,E} > 1$. Since

$$\operatorname{spec} H_N(x) \subset [-2 - \|V\|_\infty, 2 + \|V\|_\infty],$$

it will actually be enough to work with $|E| \leq \|V\|_\infty + 4$ and when we want to suppress the dependence on E we will use

$$S_V := \log(3 + \|V\|_\infty). \tag{2.6}$$

Note that $S_{V,E} \simeq S_V$ for $|E| \leq \|V\|_\infty + 4$.

We will make repeated use of the observation that using the mean value theorem and Cauchy estimates, we have

$$|E_j^{[a,b]}(x) - E_j^{[a,b]}(x_0)| \leq \left\| H_{[a,b]}(x) - H_{[a,b]}(x_0) \right\| \leq C_\rho \|V\|_\infty |x - x_0|. \tag{2.7}$$

We will also use the following basic identity:

$$\operatorname{spec} H_{m+[a,b]}(x) = \operatorname{spec} H_{[a,b]}(x + m\omega). \tag{2.8}$$

2.1 Large deviations estimates

We recall the Large Deviations Theorem (LDT) for the transfer matrix. We refer to [5] and [13] for two different approaches to its proof. The particular formulation we give here is based on [13] (see Corollary 9.2 therein).

Theorem 2.1 *Assume* $E \in \mathbb{C}$*. There exist* $\sigma = \sigma(a, b)$*,* $\tau = \tau(a, b)$*,* $\sigma, \tau \in (0, 1)$*,* $C_0 = C_0(a, b, \rho)$*, such that for* $N \geq 1$ *one has*

$$\operatorname{mes}\left\{x \in \mathbb{T}^d : |\log \|M_N(x, E)\| - NL_N(E)| > C_0 S_{V,E} N^{1-\tau}\right\} < \exp(-N^\sigma).$$

In [14] it was shown (see Proposition 2.11 therein) that in the the regime of positive Lyapunov exponent, the large deviations estimate extends to the entries of the transfer matrix.

Theorem 2.2 *Assume* $E \in \mathbb{C}$*, and* $L(E) > \gamma > 0$*. There exist* $\sigma = \sigma(a, b)$*,* $\tau = \tau(a, b)$*,* $\sigma, \tau \in (0, 1)$*, such that for* $N \geq N_0(V, a, b, E, \gamma)$ *one has*

$$\operatorname{mes}\left\{x \in \mathbb{T}^d : |\log |f_N(x, E)| - NL_N(E)| > N^{1-\tau}\right\} < \exp(-N^\sigma).$$

Note that the large deviations estimates also hold with any other smaller choices of the actual exponents σ, τ. The sharpness of these exponents plays no role for us, so we will also assume without loss of generality that the exponents are the same in both statements and $\sigma \ll \tau \ll 1$.

We claim that by inspecting the proof from [14] it can be seen that the constant N_0 from Theorem 2.2 can be chosen to be $(S_{V,E} + \gamma^{-1})^C$, $C = C(a, b, \rho)$. In fact, all the large constants in our statements can be chosen of this form (though not optimally). Since the proof in [14] is quite lengthy and intricate, and we only need to be explicit about N_0 in the perturbative setting, we will give a simpler proof of the (LDT) for determinants at large coupling in Sect. 3.

The usefulness of the (LDT) is enhanced by the following result, known as the Avalanche Principle.

Proposition 2.3 ([13, Prop. 2.2]). *Let* $A_1, \dots, A_n$ *be a sequence of* 2×2*–matrices whose determinants satisfy*

$$\max_{1 \leq j \leq n} |\det A_j| \leq 1. \tag{2.9}$$

Suppose that

$$\min_{1 \leq j \leq n} \|A_j\| \geq \mu > n \quad \textit{and} \tag{2.10}$$

$$\max_{1\le j<n}[\log\|A_{j+1}\|+\log\|A_j\|-\log\|A_{j+1}A_j\|]<\frac{1}{2}\log\mu. \quad (2.11)$$

Then

$$\left|\log\|A_n\dots A_1\|+\sum_{j=2}^{n-1}\log\|A_j\|-\sum_{j=1}^{n-1}\log\|A_{j+1}A_j\|\right|<C\frac{n}{\mu} \quad (2.12)$$

with some absolute constant C.

To apply the Avalanche Principle one needs to be in the positive Lyapunov exponent regime and to be able to compare the Lyapunov exponents L_N at different scales. This can be achieved through the following result.

Proposition 2.4 ([13, Lem. 10.1]). *Assume* $E\in\mathbb{C}$, *and* $L(E)>\gamma>0$. *Then for any* $N\ge 2$,

$$0\le L_N(\omega,E)-L(\omega,E)<C_0\frac{(\log N)^{1/\sigma}}{N},$$

where $C_0=C_0(V,a,b,E,\gamma)$ *and* σ *is as in (LDT).*

The constant C_0 from the previous proposition can be evaluated explicitly by inspecting its proof in [13]. However, we will obtain an explicit perturbative version of this result in Sect. 3.

The remaining results that we state without proof in this section are proved in [16]. The specific constants from their statements are obtained by a simple inspection of the proofs in [16]. Note that in the choice of constants we favour simplicity over sharpness. Some of the constants will depend on the constants N_0 from Theorem 2.2 and C_0 from Proposition 2.4. To keep track of this we fix

$$B_0:=N_0+C_0. \quad (2.13)$$

As a consequence of the (LDT) and the submean value property for subharmonic functions one gets the following uniform upper estimate.

Proposition 2.5 ([16, Prop. 2.13]). *Let* $E\in\mathbb{C}$ *and* τ *as in (LDT). Then for all* $N\ge 1$,

$$\sup_{x\in\mathbb{T}^d}\log\|M_N(x,E)\|\le NL_N(E)+C_0S_{V,E}N^{1-\tau},$$

with $C_0=C_0(a,b,\rho)$.

To extend the uniform upper estimate to a complex neighborhood of $\mathbb{T}^d$ we need the following result.

Lemma 2.6 ([16, Cor. 2.12]). *Let $E \in \mathbb{C}$. For any $N \geq 1$ we have*

$$|L_N(y, E) - L_N(E)| \leq C_\rho S_{V,E} \sum_{i=1}^{d} |y_i|.$$

In particular, the same bound holds with L instead of L_N.

Corollary 2.7 *Let $E \in \mathbb{C}$ and τ as in (LDT). Then for all $N \geq 1$ and all $y \in \mathbb{R}^d$, $|y| < \min(\rho/2, 1/N)$,*

$$\sup_{x \in \mathbb{T}^d} \log \|M_N(x + iy, E)\| \leq N L_N(E) + C_0 S_{V,E} N^{1-\tau}, \tag{2.14}$$

with $C_0 = C_0(a, b, \rho)$. In particular we also have

$$\sup_{x \in \mathbb{T}^d} \log |f_N(x + iy, E)| \leq N L_N(E) + C_0 S_{V,E} N^{1-\tau}.$$

Proof The conclusion follows by applying Proposition 2.5 with $V(x + iy)$ instead of $V(x)$ and by using Corollary 2.6. □

Next we recall a way of obtaining off-diagonal decay for Green's function. We use the notation $\mathcal{G}_{[a,b]}(x, E) := (H_{[a,b]}(x) - E)^{-1}$.

Lemma 2.8 ([16, Lem. 2.24]). *Assume $x_0 \in \mathbb{T}^d$, $E_0 \in \mathbb{C}$, and $L(E_0) > \gamma > 0$. Let $K \in \mathbb{R}$ and τ as in (LDT). There exists $C_0 = C_0(a, b, \rho)$ such that if $N \geq (B_0 + S_{V,E_0} + \gamma^{-1})^C$, $C = C(a, b, \rho)$, and*

$$\log \left| f_N(x_0, E_0) \right| > N L_N(\omega_0, E_0) - K, \tag{2.15}$$

then for any $(x, E) \in \mathbb{T}^d \times \mathbb{C}$ with $|x - x_0|,\ |E - E_0| < \exp(-(K + C_0 N^{1-\tau}))$, we have

$$\left| \mathcal{G}_{[1,N]}(x, E; j, k) \right| \leq \exp\left(-\frac{\gamma}{2} |k - j| + K + 2C_0 N^{1-\tau} \right), \tag{2.16}$$

$$\left\| \mathcal{G}_{[1,N]}(x, E) \right\| \leq \exp(K + 3C_0 N^{1-\tau}). \tag{2.17}$$

2.2 Cartan's estimate

We recall the definition of Cartan sets from [14]. We use the notation $\mathcal{D}(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$.

Definition 2.9 Let $H \geq 1$. For an arbitrary set $\mathcal{B} \subset \mathcal{D}(z_0, 1) \subset \mathbb{C}$ we say that $\mathcal{B} \in \mathrm{Car}_1(H, K)$ if $\mathcal{B} \subset \bigcup_{j=1}^{j_0} \mathcal{D}(z_j, r_j)$ with $j_0 \leq K$, and

$$\sum_j r_j < e^{-H}. \tag{2.18}$$

If $d \geq 1$ is an integer and $\mathcal{B} \subset \prod_{j=1}^{d} \mathcal{D}(z_{j,0}, 1) \subset \mathbb{C}^d$, then we define inductively that $\mathcal{B} \in \mathrm{Car}_d(H, K)$ if for any $1 \leq j \leq d$ there exists $\mathcal{B}_j \subset \mathcal{D}(z_{j,0}, 1) \subset \mathbb{C}, \mathcal{B}_j \in \mathrm{Car}_1(H, K)$ so that $\mathcal{B}_z^{(j)} \in \mathrm{Car}_{d-1}(H, K)$ for any $z \in \mathbb{C} \backslash \mathcal{B}_j$, here $\mathcal{B}_z^{(j)} = \left\{(z_1, \ldots, z_d) \in \mathcal{B} : z_j = z\right\}$.

The definition is motivated by the following generalization of the usual Cartan estimate to several variables. Note that given a set S that has a centre of symmetry, we will let αS, $\alpha > 0$, stand for the set scaled with respect to its centre of symmetry.

Lemma 2.10 ([14, Lem. 2.15]). *Let $\varphi(z_1, \ldots, z_d)$ be an analytic function defined on a polydisk $\mathcal{P} = \prod_{j=1}^{d} \mathcal{D}(z_{j,0}, 1)$, $z_{j,0} \in \mathbb{C}$. Let $M \geq \sup_{z \in \mathcal{P}} \log |\varphi(z)|$, $m \leq \log |\varphi(z_0)|$, $z_0 = (z_{1,0}, \ldots, z_{d,0})$. Given $H \gg 1$ there exists a set $\mathcal{B} \subset \mathcal{P}$, $\mathcal{B} \in \mathrm{Car}_d\left(H^{1/d}, K\right)$, $K = C_d H(M - m)$, such that*

$$\log |\varphi(z)| > M - C_d H(M - m) \tag{2.19}$$

for any $z \in \frac{1}{6}\mathcal{P} \backslash \mathcal{B}$. Furthermore, when $d = 1$ we can take $K = C(M - m)$ and keep only the disks of $\mathcal{B}$ containing a zero of ϕ in them.

We note that the definition of the Cartan sets gives implicit information about their measure.

Lemma 2.11 *If $\mathcal{B} \in \mathrm{Car}_d(H, K)$ then*

$$\mathrm{mes}_{\mathbb{C}^d}(\mathcal{B}) \leq C(d) e^{-H} \quad \text{and} \quad \mathrm{mes}_{\mathbb{R}^d}(\mathcal{B} \cap \mathbb{R}^d) \leq C(d) e^{-H}.$$

Proof The case $d = 1$ follows immediately from the definition of Car_1. The case $d > 1$ follows by induction, using Fubini and the definition of Car_d. □

The following simple corollary of the Cartan estimate will allow us to upgrade estimates from $\mathbb{T}^d$, where we can take advantage of the fact that $H(x)$ is self-adjoint, to some complex neighborhood of $\mathbb{T}^d$.

Corollary 2.12 *Let $\varphi(z_1, \ldots, z_d)$ be an analytic function defined on a polydisk $\mathcal{P} = \prod_{j=1}^{d} \mathcal{D}(x_{j,0}, 1)$, $x_{j,0} \in \mathbb{R}$. Assume $\sup_{\mathcal{P}} \log |\varphi(z)| \leq 0$ and $\log |\varphi(x)| \leq m < 0$ for any $x \in \mathcal{P} \cap \mathbb{R}^d$. Then for any $z \in \frac{1}{24}\mathcal{P}$,*

$$\log |\varphi(z)| < c_0 m,$$

with some $c_0 \ll_d 1$.

Proof Assume, to the contrary, that there exists $z_0 = (z_{j,0})$, $|z_0 - x_0| < 1/24$, such that $\log\big|\varphi(z_0)\big| \geq c_0 m$, with c_0 to be specified later. Take $H \gg 1$ and find $\mathcal{B} \subset \prod_{j=1}^d \mathcal{D}(x_{j,0}, 1/2)$, $2(\mathcal{B} - z_0) \in \mathrm{Car}_d\left(H^{1/d}, K\right)$, $K = c_0 C_d H|m|$, such that

$$\log\big|\varphi(z)\big| > -c_0 C_d H|m| \tag{2.20}$$

for any $z \in \prod_{j=1}^d \mathcal{D}(z_{j,0}, 1/12)\backslash\mathcal{B}$. Note that since $|z_0 - x_0| < 1/24$,

$$\mathrm{mes}_{\mathbb{R}^d}\Big(\prod_{j=1}^d \mathcal{D}(z_{j,0}, 1/12) \cap \mathbb{R}^d\Big) \geq c_1(d),\ \ c_1 > 0.$$

On the other hand

$$\mathrm{mes}_{\mathbb{R}^d}\big(\mathcal{B} \cap \mathbb{R}^d\big) \leq C(d)\exp(-H^{\frac{1}{d}}) \ll c_1,$$

provided $H \gg 1$. So, there exists $x \in \left(\prod_{j=1}^d \mathcal{D}(z_{j,0}, 1/12)\backslash\mathcal{B}\right) \cap \mathbb{R}^d$. This implies $\log\big|\varphi(x)\big| > -c_0 C_d H|m| > \frac{m}{2}$, provided we choose $c_0 \ll 1$ appropriately. This contradicts our assumptions. □

Another simple consequence of Cartan's estimate is the following statement that we refer to as the *spectral form* of (LDT).

Corollary 2.13 ([16, Cor. 2.21]). *Assume* $x \in \mathbb{T}^d$, $E \in \mathbb{C}$, *and* $L(E) > \gamma > 0$. *Let* σ, τ *as in (LDT). If* $N \geq (B_0 + S_{V,E})^C$, $C = C(a, b, \rho)$, *and*

$$\big\|(H_N(x) - E)^{-1}\big\| \leq \exp(N^{\sigma/2}),$$

then

$$\log|f_N(x, E)| > NL_N(E) - N^{1-\tau/2}.$$

2.3 Poisson's formula

Recall that for any solution ψ of the difference equation $H(x)\psi = E\psi$, Poisson's formula reads

$$\psi(m) = \mathcal{G}_{[a,b]}(x,E;m,a)\psi(a-1)+\mathcal{G}_{[a,b]}(x,E;m,b)\psi(b+1), \quad m \in [a,b]. \tag{2.21}$$

With the help of Poisson's formula one gets the following *covering lemma*.

Lemma 2.14 ([16, Lem. 2.22]). *Let $x \in \mathbb{T}^d$, $E \in \mathbb{R}$, and $[a,b] \subset \mathbb{Z}$. If for any $m \in [a,b]$, there exists an interval $I_m = [a_m, b_m] \subset [a,b]$ containing m such that*

$$(1-\delta_{a,a_m})\left|\mathcal{G}_{I_m}(x,E;m,a_m)\right|+(1-\delta_{b,b_m})\left|\mathcal{G}_{I_m}(x,E;m,b_m)\right| < 1, \tag{2.22}$$

then $E \notin \operatorname{spec} H_{[a,b]}(x)$ (here $\delta_{\cdot,\cdot}$ stands for the Kronecker delta).

We refer to the next result as the *covering form* of (LDT).

Lemma 2.15 ([16, Lem. 2.25]). *Assume $N \geq 1$, $x_0 \in \mathbb{T}^d$, $E_0 \in \mathbb{R}$, and $L(E_0) > \gamma > 0$. Let σ, τ as in (LDT). Suppose that for each point $m \in [1,N]$ there exists an interval $I_m \subset [1,N]$ such that:*

(1) $dist(m,[1,N]\setminus I_m) \geq |I_m|/100$,
(2) $|I_m| \geq (B_0 + S_{V,E_0} + \gamma^{-1})^C$, $C = C(a,b,\rho)$,
(3) $\log|f_{I_m}(x_0,E_0)| > |I_m|L_{|I_m|}(E_0) - |I_m|^{1-\tau/4}$.

Then for any $(x,E) \in \mathbb{T}^d \times \mathbb{C}$ such that

$$|x-x_0|,\ |E-E_0| < \exp(-2\max_m |I_m|^{1-\tau/4}),$$

we have

$$dist(E, \operatorname{spec} H_N(x)) \geq \exp(-2\max_m |I_m|^{1-\tau/4}).$$

Remark 2.16 In some of the results to follow we will refer to intervals $[a,b] \subset \mathbb{Z}$, with $a, b \in \mathbb{Z}$. It should be clear that in this context the integers a, b are different from the real constants a, b from the Diophantine condition. We also note that in such results the dependence of the constants on a, b still refers to the dependence on the Diophantine condition.

We give another formulation of the covering form of (LDT) that is better suited for the setting of this paper.

Lemma 2.17 *Assume $x_0 \in \mathbb{T}^d$, $S \subset \mathbb{R}$, and $L(E) > \gamma > 0$ for $E \in S$. Let σ as in (LDT), and $a < b$ integers. Suppose that for each point $m \in [a,b]$ there exists an interval J_m such that $m \in J_m$ and:*

(1) $dist(m, \partial J_m) \geq |J_m|/100$,
(2) $dist(\operatorname{spec} H_{J_m}(x_0), S) \geq \exp(-K)$, with $K < \frac{1}{2}\min_m |J_m|^{\sigma/2}$,
(3) $K \geq (B_0 + S_V + \gamma^{-1})^C$, $C = C(a,b,\rho)$ (here a, b are as in (1.2)),

Let $J = \bigcup_{m\in[a,b]} J_m$. *Then for any* $|x - x_0| < \exp(-2K)$ *we have*

$$\mathit{dist}(\text{spec}\, H_J(x), S) \geq \frac{1}{2}\exp(-K).$$

Proof It is enough to consider the case $S = \{E_0\}$ because the full result follows by applying this particular case to each $E_0 \in S$. Furthermore, we can assume $|E_0| \leq \|V\|_\infty + 4$, because otherwise the conclusion holds trivially.

First we need to set up some intervals for which we will be able to apply the covering lemma. Let $J_m = [c_m, d_m]$. Then

$$J = [c, d], \quad c = \inf_m c_m, \quad d = \sup_m d_m.$$

Let

$$m_- = \sup\{m \in [a, b] : c_m = c\}, \quad m_+ = \inf\{m \in [a, b] : d_m = d\},$$
$$I_m = \begin{cases} J_{m_-}, & m \in [c, m_-] \\ J_m, & m \in [m_-, m_+] \\ J_{m+}, & m \in [m_+, d] \end{cases}.$$

Then $\text{dist}(m, J\backslash I_m) \geq |I_m|/100$.

Take $m \in [c, d]$. Using (2) and (3) (also recall (2.7)), for any

$$|x - x_0| < \exp(-2K), \quad |E' - E_0| \leq \frac{1}{2}\exp(-K)$$

we have

$$\text{dist}(\text{spec}\, H_{I_m}(x), E') \geq \frac{1}{4}\exp(-K) > \exp(-|I_m|^{\sigma/2}).$$

Combining the spectral form of (LDT) from Corollary 2.13 with Lemma 2.8 we get

$$\left|\mathcal{G}_{I_m}(x, E'; m, k)\right| \leq \exp\left(-\frac{\gamma}{2}|m - k| + \frac{3}{2}|I_m|^{1-\tau/2}\right).$$

Using (1) and (3) (which implies $|I_m| \gg 1$), the assumptions of Lemma 2.14 are satisfied, and therefore $E' \notin \text{spec}\, H_N(x)$ for any $|E' - E_0| \leq \frac{1}{2}\exp(-K)$. This yields the conclusion. □

Remark 2.18 Obviously, for the covering forms of (LDT) it is enough to have a collection of intervals that overlap near their edges for a fraction of their size. We will use this observation tacitly when we invoke the above results.

In connection with the estimates given by the covering form of (LDT) we recall the following elementary criterion for an energy not to be in the spectrum.

Lemma 2.19 ([16, Lem. 2.39]). *If for some $x \in \mathbb{T}^d$, $E \in \mathbb{R}$, $\rho > 0$, there exist sequences $N_s' \to \infty$, $N_s'' \to +\infty$ such that*

$$dist(E, \text{spec } H_{[-N_s', N_s'']}(x)) \geq \rho,$$

then

$$dist(E, \text{spec } H(x)) \geq \rho.$$

2.4 Finite scale localization

The covering and spectral forms of (LDT) can be used to obtain localization of the eigenfunctions on a finite interval. The following result is a version of [16, Prop. 3.1] that is better suited to the setting of Sects. 5 and 6.

Proposition 2.20 *Let $x_0 \in \mathbb{T}^d$, $E_0 \in \mathbb{R}$, and assume $L(E_0) > \gamma > 0$. Let σ as in (LDT) and $0 < \beta < \sigma/2$. Let $N \geq N_0$ be integers. Assume that for any $3N_0/2 < |m| \leq N$ there exists an interval J_m such that $m \in J_m$, $dist(m, \partial J_m) \geq N_0 - N_0^{1/2}$, $|J_m| \leq 10N_0$, and*

$$dist(\text{spec } H_{J_m}(x_0), E_0) \geq \exp(-N_0^{\beta}).$$

Let

$$[-N', N''] = [-3N_0/2, 3N_0/2] \cup \bigcup_{3N_0/2 < |m| \leq N} J_m.$$

Then the following holds provided $N_0 \geq (B_0 + S_V + \gamma^{-1})^C$, $C = C(a, b, \rho, \beta)$. If

$$|x - x_0| < \exp(-2N_0^{\beta}), \quad |E_k^{[-N', N'']}(x) - E_0| < \frac{1}{4}\exp(-N_0^{\beta}),$$

then

$$|\psi_k^{[-N', N'']}(x, n)| < \exp\left(-\gamma|n|/10\right), \quad |n| \geq 3N_0/4.$$

Proof Take x, $E = E_k^{[-N', N'']}(x)$, satisfying the assumptions, and without loss of generality assume $n \geq 3N_0/4$. Let $d = \lceil n - N_0/2 \rceil$. Note that $d > n/3$. Let

$$J = \bigcup \{J_m : m \in [n - d, n + d + N_0] \cap (3N_0/2, N]\}$$

(we add N_0 to make sure $3N_0/2 < n + d + N_0$, so that the intersection is not empty). Note that by the assumptions on J_m we have $m + [-(N_0 - N_0^{1/2}), N_0 - N_0^{1/2}] \subseteq J_m$, $N_0 < |J| \lesssim d$, $n \in J$, and $\operatorname{dist}(n, [-N', N'']\setminus J) \geq d$. Using the covering form of (LDT),

$$\operatorname{dist}(H_J(x), E) \geq \frac{1}{4}\exp(-N_0^{\beta}) > \exp(-|J|^{\sigma/2}),$$

and by the spectral form of (LDT),

$$\log|f_J(x, E)| \geq |J|L(E) - |J|^{1-\tau/2}. \tag{2.23}$$

Using Lemma 2.8 and Poisson's formula we get

$$\begin{aligned} \left|\psi_k^{[-N',N'']}(x, n)\right| &\leq 2\exp\left(-\frac{\gamma}{2}d + C|J|^{1-\tau/2}\right) \\ &< \exp\left(-\frac{\gamma}{3}d\right) < \exp\left(-\frac{\gamma}{10}n\right) \end{aligned}$$

(recall that ψ is normalized). □

Next we discuss the stability of localized eigenpairs when we increase the scale. Again, the particular set-up is motivated by the setting of Sects. 5 and 6. We will use the following elementary lemma from basic perturbation theory.

Lemma 2.21 ([16, Lem. 2.40]). *Let A be an $N \times N$ Hermitian matrix. Let $E, \varepsilon \in \mathbb{R}$, $\varepsilon > 0$, and suppose there exists $\phi \in \mathbb{R}^N$, $\|\phi\| = 1$, such that*

$$\|(A - E)\phi\| < \varepsilon. \tag{2.24}$$

Then the following statements hold.

(a) *There exists a normalized eigenvector ψ of A with an eigenvalue E_0 such that*

$$\begin{aligned} E_0 &\in (E - \varepsilon\sqrt{2}, E + \varepsilon\sqrt{2}), \\ |\langle\phi, \psi\rangle| &\geq (2N)^{-1/2}. \end{aligned} \tag{2.25}$$

(b) *If in addition there exists $\eta > \varepsilon$ such that the subspace of the eigenvectors of A with eigenvalues falling into the interval $(E - \eta, E + \eta)$ is at most of dimension one, then there exists a normalized eigenvector ψ of A with an eigenvalue $E_0 \in (E - \varepsilon, E + \varepsilon)$, such that*

$$\|\phi - \psi\| < \sqrt{2}\eta^{-1}\varepsilon. \tag{2.26}$$

Proposition 2.22 *We use the notation and assumptions of Proposition 2.20. We further assume that there exist integers $|N_0' - N_0| < N_0^{1/2}$, $|N_0'' - N_0| < N_0^{1/2}$, and k_0, such that the following conditions hold:*

(i) $|E_{k_0}^{[-N_0',N_0'']}(x_0) - E_0| < \exp(-2N_0^\beta)$,
(ii) $|E_j^{[-N_0',N_0'']}(x_0) - E_{k_0}^{[-N_0',N_0'']}(x_0)| > \exp(-N_0^\beta), \quad j \neq k_0$,
(iii) $|\psi_{k_0}^{[-N_0',N_0'']}(x_0, -N_0')|, |\psi_{k_0}^{[-N_0',N_0'']}(x_0, N_0'')| < \exp(-2N_0^\beta)$.

Then there exist $E_k^{[-N',N'']}$, $\psi_k^{[-N',N'']}$, such that the following estimates hold for any $|x - x_0| < \exp(-2N_0^\beta)$, provided $N_0 \geq (B_0 + S_V + \gamma^{-1})^C$, $C = (a, b, \rho, \beta)$:

(1) $|E_k^{[-N',N'']}(x) - E_{k_0}^{[-N_0',N_0'']}(x)| < \exp(-\gamma N_0/20)$,
(2) $|E_j^{[-N',N'']}(x) - E_k^{[-N',N'']}(x)| > \frac{1}{8}\exp(-N_0^\beta), \quad j \neq k$,
(3) $|\psi_k^{[-N',N'']}(x, n)| < \exp(-\gamma|n|/10), \quad |n| \geq 3N_0/4$,
(4) $\|\psi_k^{[-N',N'']}(x, \cdot) - \psi_{k_0}^{[-N_0',N_0'']}(x, \cdot)\| < \exp(-\gamma N_0/20)$.

Furthermore, if we additionally have

$$dist(\text{spec } H_{J_m}(x_0), (-\infty, E_0]) \geq \exp(-N_0^\beta), \quad 3N_0/2 < |m| \leq N \quad (2.27)$$

(J_m as in Proposition 2.20) and

(ii′) $E_j^{[-N_0',N_0'']}(x_0) - E_{k_0}^{[-N_0',N_0'']}(x_0) > \exp(-N_0^\beta), \quad j \neq k_0$,

then

(2′) $E_j^{[-N',N'']}(x) - E_k^{[-N',N'']}(x) > \frac{1}{8}\exp(-N_0^\beta), \quad j \neq k$.

Proof Due to condition (iii),

$$\|(H_{[-N',N'']}(x_0) - E_{k_0}^{[-N_0',N_0'']}(x_0))\psi_{k_0}^{[-N_0',N_0'']}(x_0, \cdot)\| \lesssim \exp(-2N_0^\beta),$$

where we naturally extend $\psi_{k_0}^{[-N_0',N_0'']}$ to $[-N', N'']$ by adding zero entries. Part (a) in Lemma 2.21 applies and we get that there exists $k = k(x_0)$ such that

$$|E_k^{[-N',N'']}(x_0) - E_{k_0}^{[-N_0',N_0'']}(x_0)| \lesssim \exp(-2N_0^\beta).$$

Then for $|x - x_0| < \exp(-2N_0^\beta)$ (recall (2.7)) we have

$$|E_k^{[-N',N'']}(x) - E_{k_0}^{[-N_0',N_0'']}(x)| \ll \exp(-N_0^\beta),$$
$$|E_k^{[-N',N'']}(x) - E_0| \ll \exp(-N_0^\beta).$$

Due to the last estimate, Proposition 2.20 applies and (3) follows. This implies

$$\begin{aligned}&\|(H_{[-N_0',N_0'']}(x) - E_k^{[-N',N'']}(x))\psi_k^{[-N',N'']}(x,\cdot)\|\\&\lesssim \exp(-\gamma(N_0 - N_0^{1/2})/10).\end{aligned}$$

Due to condition (ii), part (b) in Lemma 2.21 applies with $H_{[-N_0',N_0'']}(x)$ in the role of A and $\eta = c\exp(-N_0^{\beta})$, $c \ll 1$. This yields (1) and (4). To prove (2) assume to the contrary that there exist $j \neq k$ and x such that

$$|E_j^{[-N',N'']}(x) - E_k^{[-N',N'']}(x)| \le \frac{1}{8}\exp(-N_0^{\beta}).$$

It follows that

$$\begin{aligned}|E_j^{[-N',N'']}(x) - E_{k_0}^{[-N_0',N_0'']}(x)| &< \frac{1}{4}\exp(-N_0^{\beta}),\\ |E_j^{[-N',N'']}(x) - E_0| &< \frac{1}{4}\exp(-N_0^{\beta}).\end{aligned}$$

Proposition 2.20 applies and we get

$$|\psi_j^{[-N',N'']}(x,n)| < \exp(-\gamma|n|/10)\,, \quad |n| \ge 3N_0/4.$$

Now just as above we have

$$\|\psi_j^{[-N',N'']}(x,\cdot) - \psi_{k_0}^{[-N_0',N_0'']}(x,\cdot)\| < \exp(-\gamma N_0/20)$$

and hence

$$\|\psi_j^{[-N',N'']}(x,\cdot) - \psi_k^{[-N',N'']}(x,\cdot)\| \lesssim \exp(-\gamma N_0/20) < 1.$$

Since, $\psi_k^{[-N',N'']}(x,\cdot)$, $\psi_j^{[-N',N'']}(x,\cdot)$ are normalized eigenvectors with different eigenvalues

$$\|\psi_j^{[-N',N'']}(x,\cdot) - \psi_k^{[-N',N'']}(x,\cdot)\|^2 = 2.$$

This contradiction verifies (2).

Finally, we check (2′). Clearly all the estimates obtained so far hold with the extra assumptions. Assume to the contrary that there exist $j \neq k$ and x such that (4′) fails. By (4) we must have

$$E_j^{[-N',N'']}(x) < E_k^{[-N',N'']}(x) - \frac{1}{8}\exp(-N_0^{\beta}).$$

It follows that

$$E_j^{[-N',N'']}(x) < E_{k_0}^{[-N_0',N_0'']}(x) - \frac{1}{4}\exp(-N_0^\beta),$$
$$E_j^{[-N',N'']}(x) < E_0 - \frac{1}{4}\exp(-N_0^\beta).$$

By (ii′) and (2.27) (recall (2.7)) we get

$$\operatorname{dist}(\operatorname{spec} H_{[-N_0',N_0'']}(x), E_j^{[-N',N'']}(x)) > \frac{1}{4}\exp(-N_0^\beta),$$
$$\operatorname{dist}(\operatorname{spec} H_{J_m}(x), E_j^{[-N',N'']}(x)) > \frac{1}{2}\exp(-N_0^\beta).$$

It follows from Lemma 2.17 that $E_j^{[-N',N'']}(x) \notin \operatorname{spec} H_{[-N',N'']}(x)$. This contradiction concludes the proof. □

2.5 Semialgebraic sets

Recall that a set $\mathcal{S} \subset \mathbb{R}^n$ is called semialgebraic if it is a finite union of sets defined by a finite number of polynomial equalities and inequalities. More precisely, a semialgebraic set $\mathcal{S} \subset \mathbb{R}^n$ is given by an expression

$$\mathcal{S} = \cup_j \cap_{\ell \in L_j} \{P_\ell s_{j\ell} 0\},$$

where $\{P_1, \ldots, P_s\}$ is a collection of polynomials of n variables,

$$L_j \subset \{1, \ldots, s\} \text{ and } s_{j\ell} \in \{>, <, =\}.$$

If the degrees of the polynomials are bounded by d, then we say that the degree of $\mathcal{S}$ is bounded by sd. See [5, Ch. 9] for more information on semialgebraic sets.

In our context, semialgebraic sets can be introduced by approximating the analytic potential V with a polynomial $\tilde{V}$. More precisely, given $N \geq 1$, by truncating V's Fourier series and the Taylor series of the trigonometric functions, one can obtain a polynomial $\tilde{V}$ of degree less than

$$C(d, \rho)(1 + \log \|V\|_\infty) N^4$$

such that

$$\sup_{x \in \mathbb{T}^d} |V(x) - \tilde{V}(x)| \leq \exp(-N^2). \tag{2.28}$$

If we let $\tilde{H}$ be the operator with the truncated potential $\tilde{V}$, we have

$$\sup_{x\in\mathbb{T}^d} \left\| H_{[a,b]}(x) - \tilde{H}_{[a,b]}(x) \right\| \le \exp(-N^2) \tag{2.29}$$

for any $[a, b] \subseteq \mathbb{Z}$.

Our use of semialgebraic sets will be limited to applying the following result.

Lemma 2.23 ([5, Cor. 9.6]). *Let $\mathcal{S} \subset [0, 1]^n$ be semialgebraic of degree B. Let $\varepsilon > 0$ be a small number and $\operatorname{mes}_n(\mathcal{S}) < \varepsilon^n$. Then $\mathcal{S}$ may be covered by at most $B^C \left(\frac{1}{\varepsilon}\right)^{n-1}$ balls of radius ε*

2.6 Resultants

We briefly recall the definition of the resultant of two univariate polynomials and some of the basic properties that we will use in Sect. 9. Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \qquad Q(z) = b_m z^m + b_{m-1} z^{m-1} + \cdots + b_0$$

be polynomials, $a_i, b_j \in \mathbb{C}$, $a_n \neq 0$, $b_m \neq 0$. Let ζ_i, $1 \le i \le n$ and η_j, $1 \le j \le m$ be the zeros of P and Q respectively. The resultant of P and Q is the quantity

$$\operatorname{Res}(P, Q) = a_n^m b_m^n \prod_{i,j} (\zeta_i - \eta_j). \tag{2.30}$$

The resultant can be expressed explicitly in terms of the coefficients (see [20]):

$$\operatorname{Res}(P, Q) = \begin{vmatrix} a_n & & & b_m & & \\ a_{n-1} & \ddots & & b_{m-1} & \ddots & \\ \vdots & \ddots & a_n & \vdots & \ddots & b_m \\ a_0 & \ddots & a_{n-1} & b_0 & \ddots & b_{m-1} \\ & \ddots & \vdots & & \ddots & \vdots \\ & & a_0 & & & b_0 \end{vmatrix} \tag{2.31}$$

$$\underbrace{\qquad\qquad\qquad}_{m}\ \underbrace{\qquad\qquad\qquad}_{n}$$

Lemma 2.24 *Let P, Q, ζ_i, η_j as above and $r_P = \max_i |\zeta_i|$, $r_Q = \max_j |\eta_j|$. If there exists z such that*

$$\max(|P(z)|, |Q(z)|) < \min(|a_n|, |b_m|)\delta^{\max(m,n)}, \tag{2.32}$$

for some $\delta \in (0,1)$*, then*

$$\left|\text{Res}(P,Q)\right| < 2|a_n|^m|b_m|^n(2r)^{mn-1}\delta,$$

with $r = \max(r_P, r_Q)$.

Proof For (2.32) to hold there must exist ζ_{i_0}, η_{j_0} such that $|z-\zeta_{i_0}| < \delta$, $|z-\eta_{j_0}| < \delta$ and therefore, using (2.30),

$$|\text{Res}(P,Q)| \le |a_n|^m|b_m|^n(2r)^{mn-1}|\zeta_{i_0}-\eta_{j_0}| < |a_n|^m|b_m|^n(2r)^{mn-1}2\delta.$$

□

For the application of the previous lemma in Sect. 9 we will also need a couple of auxiliary results. First, recall the following elementary bound for the location of zeros of a polynomial due to Cauchy (see [21, Thm. (27,2)]).

Lemma 2.25 *All the zeros of a polynomial* $P(z) = a_nz^n + a_{n-1}z^{n-1}\cdots + a_0$, $a_n \neq 0$, $n \geq 1$, *are located in the disk* $|z| < 1 + \max_{k<n}|a_k/a_n|$.

Second, we will need the following consequence of Cartan's estimate.

Lemma 2.26 *Let* $P(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots + a_0$, $n \geq 1$, $a_n \neq 0$, $M = \max_i |a_i|$. *There exists an absolute constant* C_0 *such that for any* $H \gg 1$, *we have*

$$\text{mes}\{x \in [0,2\pi] : \log|P(\exp(ix))| < \log M - C_0 nH\} < \exp(-H/2).$$

Proof Using Cauchy estimates,

$$M \le \max_{|z|=1}|P(z)|.$$

In particular, there exists z_0, $|z_0| = 1$, such that $\log|P(z_0)| \geq \log M$. At the same time

$$\sup_{|z|\le 100}|P(z)| \le 2M100^n.$$

Given $H \gg 1$, by Cartan's estimate, there exists $\mathcal{B} = \bigcup_{k=1}^{k_0}\mathcal{D}(\zeta_k, r_k)$, $\sum_k r_k \lesssim \exp(-H)$, such that

$$\log|P(z)| \geq \log(2M100^n) - CH(\log(2M100^n) - \log M) \geq \log M - C'nH,$$

for any $z \in \mathcal{D}(0,2)\backslash\mathcal{B}$. The conclusion follows. □

3 Basic tools at large coupling

In this section we discuss some results that rely on having a large coupling constant. So, we work with operators of the form (1.1). As in the previous section we assume that V extends complex analytically to $\mathbb{T}^d_\rho$. Furthermore, we assume that V is not constant.

Our first goal is to give an explicit expression for the constant B_0 from the previous section (recall (2.13)). To this end we will obtain, in Proposition 3.4, a version of Theorem 2.2 and Proposition 2.4 at large coupling.

Let

$$\underline{\iota} = \underline{\iota}(V) := \inf_{x\in\mathbb{T}^d} \sup\{|V(x') - V(x)| : x' \in \mathbb{T}^d,\ |x' - x| \le \rho/100\}. \quad (3.1)$$

Since V is continuous and non-constant we have $\underline{\iota} > 0$.

Lemma 3.1 *Let $\eta \in \mathbb{C}$. For any $H \gg 1$ we have*

$$\operatorname{mes}\{x \in \mathbb{T}^d : |\log|V(x) - \eta|| > H_{V,\eta} H\} \le C(d)\exp(-H^{1/d}),$$

with

$$H_{V,\eta} = C(d)(1 + \max(0, \log(\|V\|_\infty + |\eta|)) + \max(0, \log \underline{\iota}^{-1})).$$

Proof Given $x_0 \in \mathbb{T}^d$ there exists $x_0' \in \mathbb{T}^d$ such that $|x_0 - x_0'| \le \rho/100$ and either

$$|V(x_0) - \eta| \ge \underline{\iota}/2 \quad \text{or} \quad |V(x_0') - \eta| \ge \underline{\iota}/2.$$

The conclusion follows by Lemmas 2.10, 2.11, and a covering argument. □

To keep track of the dependence of the various constants on the potential we introduce

$$T_V = 2 + \max(0, \log\|V\|_\infty) + \max(0, \log \underline{\iota}^{-1}). \quad (3.2)$$

Note that $S_{\lambda V} \le 2\log\lambda$, when $\log\lambda \gg T_V$. In what follows we will restrict ourselves to “spectral” values of E, that is, we will assume $|E| \le \lambda\|V\|_\infty + 4$.

Lemma 3.2 *There exists $\lambda_0(V) = \exp((T_V)^C)$, $C = C(d)$, such that the following hold for $\lambda \ge \lambda_0$ and $|E| \le \lambda\|V\|_\infty + 4$. For any $N \le \exp((\log\lambda)^{\frac{1}{4d}})$ we have*

$$|L_N(E) - 2L_2(E) + L_1(E)| \lesssim \frac{(\log\lambda)^{\frac{1}{2}}}{N},$$

$$|L_N(E) - \log\lambda| \lesssim (\log\lambda)^{\frac{1}{2}},$$

and there exists a set $\mathcal{B}_N$, $\operatorname{mes}(\mathcal{B}_N) < \exp(-(\log\lambda)^{\frac{1}{3d}})$, *such that*

$$\left|\log|f_N(x,E)| - \log\|M_N(x,E)\|\right| \lesssim (\log\lambda)^{1/2}, \tag{3.3}$$

for any $x \notin \mathcal{B}_N$.

Proof Denote by $\mathcal{B}$ the set from Lemma 3.1 with $\eta = E/\lambda$ and $H = (\log\lambda)^{\frac{1}{3}+\varepsilon}$, $\varepsilon \ll 1$. Set $\mathcal{B}_N = \bigcup_{1\le j\le N}(\mathcal{B} - j\omega)$. Note that we have $(\log\lambda)^{1/2} \ge H_{V,\eta}H$ and

$$\operatorname{mes}(\mathcal{B}_N) \le NC(d)\exp(-(\log\lambda)^{(\frac{1}{3}+\varepsilon)\frac{1}{d}}) < \exp(-(\log\lambda)^{\frac{1}{3d}}).$$

For $x \notin \mathcal{B}_N$, $1 \le j \le N$,

$$|\log|\lambda V(x+j\omega) - E| - \log\lambda| \le (\log\lambda)^{\frac{1}{2}}$$

and therefore

$$\left|\log|f_\ell(x+(j-1)\omega,E)| - \ell\log\lambda\right| \lesssim (\log\lambda)^{\frac{1}{2}},\ \ell = 1,2, \tag{3.4}$$

$$\left|\log\|M_\ell(x+(j-1)\omega,E)\| - \ell\log\lambda\right| \lesssim (\log\lambda)^{\frac{1}{2}},\ \ell = 1,2. \tag{3.5}$$

Applying the avalanche principle we get that for any $x \notin \mathcal{B}_N$,

$$\log\|M_N(x,E)\| = \sum_{j=0}^{N-2}\log\|M_2(x+j\omega,E)\| - \sum_{j=1}^{N-2}\log\|M_1(x+j\omega,E)\| + O(\lambda^{-\frac{1}{2}}) \tag{3.6}$$

and

$$\begin{aligned}\log|f_N(x,E)| &= \log\left\|M_2(x,E)\begin{bmatrix}1&0\\0&0\end{bmatrix}\right\| + \sum_{j=1}^{N-3}\log\|M_2(x+j\omega,E)\| \\ &\quad + \log\left\|\begin{bmatrix}1&0\\0&0\end{bmatrix}M_2(x+(N-2)\omega,E)\right\| \\ &\quad - \sum_{j=1}^{N-2}\log\|M_1(x+j\omega,E)\| + O(\lambda^{-\frac{1}{2}}).\end{aligned} \tag{3.7}$$

We used the fact that

$$\log|f_N(x)| = \log\left\|\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} M_N(x)\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\right\| \tag{3.8}$$

(recall (2.2)). It follows that (3.3) holds. Integrating (3.6) yields

$$\begin{aligned}|NL_N(E) - (N-1)2L_2(E) + (N-1)L_1(E)| &\le C\lambda^{-\frac{1}{2}} + 4\mathrm{mes}(\mathcal{B}_N)S_{\lambda V}\\ &\le \exp(-(\log\lambda)^{\frac{1}{4d}}).\end{aligned}$$

By integrating (3.4) we get

$$\begin{aligned}&|L_1(E) - \log\lambda|, |L_2(E) - \log\lambda|\\ &\quad\lesssim (\log\lambda)^{\frac{1}{2}} + (S_{\lambda V} + \log\lambda)\exp(-(\log\lambda)^{\frac{1}{3d}})\\ &\quad\lesssim (\log\lambda)^{\frac{1}{2}}.\end{aligned}$$

Therefore

$$\begin{aligned}|L_N(E) - 2L_2(E) + L_1(E)| &\le \exp(-(\log\lambda)^{\frac{1}{4d}}) + \frac{2(L_1(E) - L_2(E))}{N}\\ &\lesssim \frac{(\log\lambda)^{\frac{1}{2}}}{N}\end{aligned}$$

and

$$|L_N(E) - \log\lambda| \lesssim \frac{(\log\lambda)^{\frac{1}{2}}}{N} + (\log\lambda)^{\frac{1}{2}} \lesssim (\log\lambda)^{\frac{1}{2}}.$$

□

We use the avalanche principle to extend by induction the estimates of the previous lemma for arbitrarily large N.

Lemma 3.3 *Let $E \in \mathbb{C}$, and σ, τ as in Theorem 2.1. There exist $\ell_0(a, b, \rho)$ and $\lambda_0(V) = \exp((T_V)^C)$, $C = C(d)$, such that the following hold for $\lambda \ge \lambda_0$, $\ell \ge \ell_0$, and $|E| \le \lambda\,\|V\|_\infty + 4$. Assume that for any $\ell \le \ell', \ell'' \le 4\ell$ we have*

$$|L_{\ell'}(E) - L_{\ell''}(E)| \le \frac{(\log\lambda)\log\ell}{\ell}, \qquad L_{\ell'}(E) \ge \frac{1}{2}\log\lambda, \tag{3.9}$$

$$\begin{aligned}&\mathrm{mes}\left\{x \in \mathbb{T}^d : \left|\log|f_{\ell'}(x, E)| - \ell' L_{\ell'}(E)\right| > S_{\lambda V}(\ell')^{1-\tau/2}\right\}\\ &\quad < \exp(-(\ell')^{\sigma/2}).\end{aligned} \tag{3.10}$$

Then for $\ell^{10} \le N \le \ell^{100}$, $N \le N', N'' \le 4N$, *we have*

$$|L_{N'}(E) - L_{N''}(E)| \le \frac{(\log\lambda)\log N}{N},$$
$$L_{N'}(E) \ge L_\ell(E) - \frac{2(\log\lambda)\log\ell}{\ell} - \frac{(\log\lambda)\log N'}{3N'},$$
$$\operatorname{mes}\left\{x \in \mathbb{T}^d : \left|\log|f_{N'}(x,E)| - N'L_{N'}(E)\right| > S_{\lambda V}(N')^{1-\tau/2}\right\}$$
$$< \exp(-(N')^{\sigma/2}).$$

Proof We first prove the statements pertaining to the Lyapunov exponents. The derivation follows the method in [13, Lemma 4.2]. We omit some details. We also suppress E from most of the notation. To shorten the presentation we consider the case $N = n\ell$, $n \in \mathbb{N}$, only. By Theorem 2.1 and (3.9) we have

$$\log\|M_\ell(x + j\ell\omega)\| \ge \ell L_\ell - C_0 S_{\lambda V}\ell^{1-\tau} \ge \frac{1}{4}\ell\log\lambda \tag{3.11}$$

and

$$\log\|M_\ell(x+j\ell\omega)\| + \log\|M_\ell(x+(j+1)\ell\omega)\| - \log\|M_{2\ell}(x+j\ell\omega)\|$$
$$\le 2\ell(L_\ell - L_{2\ell}) + 2C_0S_{\lambda V}\ell^{1-\tau} + C_0S_{\lambda V}(2\ell)^{1-\tau} < \frac{1}{8}\ell\log\lambda, \tag{3.12}$$

for any $0 \le j \le N$, $x \notin \mathcal{B}$, $\operatorname{mes}(\mathcal{B}) \le 2n\exp(-\ell^\sigma) \le \exp(-\ell^\sigma/2)$. With these estimates in hand the avalanche principle kicks in and yields

$$\log\|M_N(x)\| = \sum_{j=0}^{n-2}\log\|M_{2\ell}(x+j\ell\omega)\| - \sum_{j=1}^{n-2}\log\|M_\ell(x+j\ell\omega)\|$$
$$+O(\exp(-(\ell\log\lambda)/8)), \tag{3.13}$$

for any $x \notin \mathcal{B}$. Recalling (2.5) and integrating (3.13) over x yields

$$\left|L_N - \frac{n-1}{n}2L_{2\ell} + \frac{n-2}{n}L_\ell\right| \le \frac{1}{N}C\exp(-(\ell\log\lambda)/8) + 4\operatorname{mes}(\mathcal{B})S_{\lambda V}$$
$$\le \exp(-c_0\ell^\sigma/4)\log\lambda.$$

Therefore

$$|L_N - 2L_{2\ell} + L_\ell| \le \exp(-c_0\ell^\sigma/4)\log\lambda + \frac{2}{n}(L_\ell - L_{2\ell}) \le \frac{3(\log\lambda)\log\ell}{N}$$
$$\le \frac{(\log\lambda)\log N}{3N}.$$

The same estimate also holds for general N (not just $N = n\ell$) and $N \leq N', N'' \leq 4N$. This implies the estimates for the Lyapunov exponents.

Next, we consider the statement about the determinants. The main tool here is the application of the avalanche principle to expand $\log|f_N|$. The argument is very close to the one in [14, Corollary 3.10]. Again we omit some details and assume $N = n\ell$, $n \in \mathbb{N}$. On top of (3.11) and (3.12), using Theorem 2.1 and (3.10) we have

$$\log\left\|M_\ell(x)\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\right\| \geq \log|f_\ell(x)| \geq \ell L_\ell - S_{\lambda V}\ell^{1-\tau/2} \geq \frac{1}{4}\ell\log\lambda,$$
$$\log\left\|\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}M_\ell(x+(n-1)\ell\omega)\right\| \geq \log|f_\ell(x+(n-1)\ell\omega)| \geq \frac{1}{4}\ell\log\lambda,$$
$$\log\|M_\ell(x)\| + \log\|M_\ell(x+\omega)\| - \log\|M_{2\ell}(x)\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\| < \tfrac{1}{8}\ell\log\lambda,$$
$$\log\|M_\ell(x+(n-2)\ell\omega)\| + \log\|M_\ell(x+(n-1)\ell\omega)\|$$
$$-\log\|\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}M_{2\ell}(x+(n-2)\ell\omega)\| < \tfrac{1}{8}\ell\log\lambda$$

for any $x \notin \mathcal{B}'$, $\mathrm{mes}(\mathcal{B}') \leq 4\exp(-\ell^{\sigma/2})$. So we can apply the avalanche principle to expand $\log|f_N(x)|$ for $x \notin \mathcal{B}\cup\mathcal{B}'$ (similarly to (3.7)). Combining this with (3.13) we get

$$\begin{aligned}\log|f_N(x)| &= \log\|M_N(x)\| + \log\left\|M_{2\ell}(x)\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\right\| - \log\|M_{2\ell}(x)\|\\ &\quad+\log\left\|\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}M_{2\ell}(x+(n-2)\ell\omega)\right\| - \log\|M_{2\ell}(x+(n-2)\ell\omega)\|\\ &\quad+O(\exp(-(\ell\log\lambda)/8)\\ &\geq \log\|M_N(x)\| - 2S_{\lambda V}(2\ell)^{1-\tau/2} - 2C_0S_{\lambda V}(2\ell)^{1-\tau} \geq NL_N - S_{\lambda V}N^{1-\tau}\end{aligned} \tag{3.14}$$

for any $x \notin \mathcal{B}\cup\mathcal{B}'$ (recall that $\tau \ll 1$). In particular, for any $x_0 \in \mathbb{T}^d$ there exists $x_1 \in \mathbb{T}^d$, $|x_1 - x_0| \ll \rho N^{-1}$ such that $\log|f_N(x_1)| \geq NL_N - S_{\lambda V}N^{1-\tau}$. On the other hand due to Corollary 2.7

$$\sup_{x\in\mathbb{T}^d,|y|<\rho N^{-1}} \log|f_N(x+iy)| \leq NL_N + C(a,b,\rho)S_{\lambda V}N^{1-\tau}.$$

Applying Cartan's estimate (with $H = N^{\tau/3}$) and using a covering argument we get

$$\operatorname{mes}\left\{x : |\log|f_N(x)| - NL_N| > S_{\lambda V} N^{1-\tau/2}\right\} \leq C(d)\exp(-N^{\tau/(3d)}) < \exp(-N^{\sigma/2}),$$

(recall that $\sigma \ll \tau$). The same estimate also holds for general N and $N \leq N', N'' \leq 4N$. □

Proposition 3.4 *Let $E \in \mathbb{C}$, and σ, τ as in Theorem 2.1. There exists $\lambda_0(a, b, \rho, V) = \exp((T_V)^C)$, $C = C(a, b, \rho)$, such that the following statements hold for $\lambda \geq \lambda_0$ and $|E| \leq \lambda \|V\|_\infty + 4$.*

(a) We have

$$L_N(E) - L(E) \leq \frac{C_0(\log\lambda)\log N}{N}, \quad N \geq 2,$$
$$L(E) \geq \log\lambda - C_1(\log\lambda)^{\frac{1}{2}} > \frac{1}{2}\log\lambda,$$

with $C_0 = C_0(a, b, \rho)$ and C_1 an absolute constant.

(b) For any $N \geq \log\lambda$ we have

$$\operatorname{mes}\left\{x \in \mathbb{T}^d : |\log|f_N(x, E)| - L_N(E)| > S_{\lambda V} N^{1-\tau/2}\right\} < \exp(-N^{\sigma/2}).$$

Proof of Proposition 3.4 (a) By Lemma 3.2, for $1 \ll \ell \leq \exp((\log\lambda)^{\frac{1}{4d}})/4$, $\ell \leq \ell', \ell'' \leq 4\ell$, we have

$$|L_{\ell'}(E) - L_{\ell'}(E)| \leq \frac{C(\log\lambda)^{\frac{1}{2}}}{\ell} \leq \frac{(\log\lambda)\log\ell}{\ell},$$
$$L_{\ell'}(E) \geq \log\lambda - C(\log\lambda)^{\frac{1}{2}} \geq \frac{1}{2}\log\lambda.$$

Let ℓ_0 as in Lemma 3.3. We choose λ_0 such that $\ell_0 \leq \log\lambda_0$. Using the above, Lemma 3.3, and induction we get that for any $N \geq \ell_0$, $N \leq N', N'' \leq 4N$ we have

$$|L_{N'}(E) - L_{N''}(E)| \leq \frac{(\log\lambda)\log N}{N}.$$

In particular we have

$$L_N(E) - L_{2^k N}(E) \leq \sum_{j=0}^{k-1} \frac{(\log\lambda)\log(2^j N)}{2^j N} \leq \frac{C(\log\lambda)\log N}{N},$$

with C an absolute constant. The first statement of part (a) follows by letting $k \to \infty$ and by adjusting the constant C to also cover the case $N < \ell_0$. The second statement follows from the fact that for $\ell = \lfloor \exp((\log \lambda)^{\frac{1}{4d}}) \rfloor$, we have

$$L(E) \geq L_\ell(E) - \frac{C(\log \lambda) \log \ell}{\ell} \geq \log \lambda - C(\log \lambda)^{\frac{1}{2}} - \exp(-(\log \lambda)^{\frac{1}{5d}})$$
$$\geq \log \lambda - C'(\log \lambda)^{\frac{1}{2}}.$$

(b) Take $\log \lambda \leq \ell \leq (\log \lambda)^{100}$. Using Lemma 3.2 and Theorem 2.1 we get

$$\operatorname{mes}\left\{x : |\log|f_\ell(x, E)| - \ell L_\ell(E)| > C_0 S_{\lambda V} \ell^{1-\tau} + C(\log \lambda)^{\frac{1}{2}}\right\}$$
$$< \exp(-(\log \lambda)^{\frac{1}{3d}}).$$

Note that with this choice of ℓ we have

$$C_0 S_{\lambda V} \ell^{1-\tau} + C(\log \lambda)^{\frac{1}{2}} < S_{\lambda V} \ell^{1-\tau/2}, \qquad \exp(-(\log \lambda)^{\frac{1}{3d}}) < \exp(-\ell^{\sigma/2})$$

(recall that $\sigma \ll \tau \ll 1$). Recalling that $\ell_0 \leq \log \lambda_0$, the conclusion follows by Lemma 3.3 and induction. □

Remark 3.5 (a) The previous proposition shows that for $\lambda \geq \lambda_0 \gg 1$ and $|E| \leq \lambda \|V\|_\infty + 4$, Theorem 2.2 holds with $N_0 = (\log \lambda)^{C(a,b)}$, and Proposition 2.4 holds with $C_0 = C(a, b, \rho) \log \lambda$. Therefore, for such λ and E we can take $B_0 = (\log \lambda)^{C(a,b,\rho)}$. By inspection of the previous proofs one can see that for $|E| > \lambda \|V\|_\infty + 4$ we can take $B_0 = (\log \lambda + \log |E|)^{C(a,b,\rho)}$, but we will not use this fact.

(b) The positivity of the Lyapunov exponent for $\lambda \geq \lambda_0 \gg 1$ is well-known (see [6,10,13]). We only included the proof because it is an easy consequence of the lemmas we needed for the other statements.

Next we establish a version of the covering form of (LDT) and of the result on finite scale localization from Proposition 2.22, starting from the potential. We will need these results in Sect. 7 to connect the assumptions on the potential to the initial conditions required by our inductive schemes from Sects. 5 and 6.

Lemma 3.6 *Let $x_0 \in \mathbb{T}^d$, $[a, b] \subset \mathbb{Z}$, $a < b$. There exists $\lambda_0(V) = \exp((T_V)^C)$, $C = C(\rho)$, such that the following hold for $\lambda \geq \lambda_0$ and $|E_0| \leq \lambda \|V\|_\infty + 4$. Assume*

$$|V(x_0 + n\omega) - \lambda^{-1} E_0| \geq \exp(-K), \quad \textit{for any } n \in [a, b],$$

with some $K \geq (\log \lambda)^{1/3}$. Then for any $|x - x_0| < \exp(-2K)$, $\lambda^{-1}|E - E_0| < \frac{1}{2}\exp(-K)$,

(a) *dist*(spec $H_{[a,b]}(x), E_0) \geq \frac{1}{2}\lambda \exp(-K)$,
(b) $\left|\mathcal{G}_{[a,b]}(x, E; j, k)\right| \leq \exp\left(-(|j-k|+1)\log\lambda + C(b-a)K\right)$,

where C is an absolute constant.

Proof For any $|x - x_0| < \exp(-2K)$, $\lambda^{-1}|E - E_0| \leq \frac{1}{2}\exp(-K)$,

$$|V(x+n\omega) - \lambda^{-1}E| \geq \frac{1}{4}\exp(-K), \quad j \in [a,b]$$

(λ_0 depends on ρ because we used a Cauchy estimate). Then

$$|\log|\lambda V(x+n\omega) - E| - \log\lambda| \lesssim K, \quad n \in [a,b]$$

(note that $|V(x+n\omega) - \lambda^{-1}E| \leq \exp((\log\lambda)^{1/3}) \leq \exp(K)$, for large enough λ) and this implies

$$\left|\log|f_\ell(x+(n-1)\omega, E)| - \ell\log\lambda\right| \lesssim K, \quad n \in [a, b-\ell], \ell = 1, 2,$$
$$\left|\log\|M_\ell(x+(n-1)\omega, E)\| - \ell\log\lambda\right| \lesssim K, \quad n \in [a, b-\ell], \ell = 1, 2.$$

Applying the avalanche principle (as in the proof of Lemma 3.2) we have

$$\begin{aligned}\log|f_{[a,b]}(x,E)| = \log&\left\|M_2(x+(a-1)\omega, E)\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\right\| \\ &+ \sum_{n=a}^{b-a-2}\log\|M_2(x+n\omega, E)\| \\ &+ \log\left\|\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}M_2(x+(b-a-1)\omega, E)\right\| \\ &- \sum_{n=a}^{b-a-1}\log\|M_1(x+n\omega, E)\| + O(\lambda^{-\frac{1}{2}}).\end{aligned}$$

It then follows that

$$|\log|f_{[a,b]}(x,E)| - (b-a+1)\log\lambda| \lesssim (b-a+1)K,$$

In particular, $E \notin \operatorname{spec} H_{[a,b]}(x)$. This implies (a). Analogous estimates hold on any subinterval of $[a,b]$. Using these estimates and Cramer's rule for the resolvent we get (for $j \leq k$)

$$\begin{aligned}\log\left|\mathcal{G}_{[a,b]}(x,E;j,k)\right| = \log&\left|f_{[a,j-1]}(x,E)\right| + \log\left|f_{[k+1,b]}\big(x,E\big)\right| \\ &- \log\left|f_{[a,b]}(x,E)\right|\end{aligned}$$

$$\leq [(j-a)+(b-k)](\log\lambda + CK)$$
$$-(b-a+1)((j-a)(\log\lambda - CK))$$
$$\leq (j-k-1)\log\lambda + C'(b-a)K.$$

This implies (b). □

Corollary 3.7 *Let $x_0 \in \mathbb{T}^d$, $S \subset \mathbb{R}$, $[a,b] \subset \mathbb{Z}$, $a < b$. There exists $\lambda_0(V) = \exp((T_V)^C)$, $C = C(\rho)$, such that the following hold for $\lambda \geq \lambda_0$. If*

$$dist(V(x_0 + n\omega), \lambda^{-1}S) \geq \exp(-K), \quad \textit{for any } n \in [a,b],$$

with some $K \geq (\log\lambda)^{1/3}$, then for any $|x - x_0| < \exp(-2K)$,

$$dist(\text{spec } H_{[a,b]}(x), S) \geq \frac{1}{2}\lambda \exp(-K).$$

Proof This follows by applying Lemma 3.6 (a) for each $E_0 \in S$ with $|E_0| \leq \lambda\,\|V\|_\infty + 4$. Note that for $|E_0| > \lambda\,\|V\|_\infty + 4$, Lemma 3.6 (a) holds trivially. □

In the results of this section we could have used $(\log\lambda)^\varepsilon$, $\varepsilon \in (0,1)$, instead of $(\log\lambda)^{1/2}$. So far, working in such generality wasn't needed. However, we will need this setting for the applications of the next lemma. Recall Remark 2.16.

Lemma 3.8 *Let $x_0 \in \mathbb{T}^d$, $a < 0 < b$, $\varepsilon \in (0,1)$, and assume*

$$|V(x_0 + n\omega) - V(x_0)| \geq \exp(-(\log\lambda)^\varepsilon), \quad \textit{for any } n \in [a,b]\backslash\{0\}.$$

There exists $\lambda_0(V) = \exp((T_V)^C)$, $C = C(\rho, \varepsilon)$, such that the following hold for $\lambda \geq \lambda_0$. There exist $E_k^{[a,b]}$, $\psi_k^{[a,b]}$ such that for any $|x - x_0| < \exp(-3(\log\lambda)^\varepsilon)$ the following estimates hold:

(1) $|\lambda^{-1}E_k^{[a,b]}(x) - V(x)| \leq 2\lambda^{-1}$,
(2) $|\psi_k^{[a,b]}(x,n)| < \exp(-(\log\lambda)|n|/2), \quad |n| > 0$,
(3) $|\psi_k^{[a,b]}(x,0) - 1| < \exp(-(\log\lambda)/2)$,
(4) $\lambda^{-1}|E_j^{[a,b]}(x) - E_k^{[a,b]}(x)| \geq \frac{1}{8}\exp(-(\log\lambda)^\varepsilon), \quad j \neq k$.

Furthermore, if

$$V(x_0 + n\omega) - V(x_0) \geq \exp(-(\log\lambda)^\varepsilon), \quad \textit{for any } n \in [a,b]\backslash\{0\}, \quad (3.15)$$

then

(4') $\lambda^{-1}(E_j^{[a,b]}(x) - E_k^{[a,b]}(x)) \geq \frac{1}{8}\exp(-(\log\lambda)^\varepsilon), \quad j \neq k$.

Proof The proof is very similar to the one of Proposition 2.22. We have

$$\|(\lambda^{-1}H_{[a,b]}(x) - V(x))\delta_0\| \leq \sqrt{2}\lambda^{-1},$$

where δ_0 stands for the standard unit vector with mass concentrated at 0. By Lemma 2.21 there exists $k = k(x)$ such that (1) holds. At the end we will argue that $k(x) = k(x_0)$. Note that

$$\lambda^{-1}|E_k^{[a,b]}(x) - E_0| \ll \exp(-2(\log\lambda)^{\varepsilon}), \quad E_0 = \lambda V(x_0).$$

Estimate (2) now follows from Poisson's formula and Lemma 3.6 (b) (applied, for $n > 0$, on $[1, 2n] \cap [a, b]$). Since $\psi_k^{[a,b]}$ is normalized, estimate (3) follows from (2) (obviously, we choose $\psi_k^{[a,b]}$ such that $\psi_k^{[a,b]}(x, 0) \geq 0$). To prove (4) assume to the contrary that there exist $j \neq k$ and x such that

$$\lambda^{-1}|E_j^{[a,b]}(x) - E_k^{[a,b]}(x)| < \frac{1}{8}\exp(-2(\log\lambda)^{\varepsilon}).$$

Then

$$\lambda^{-1}|E_j^{[a,b]}(x) - E_0| < \exp(-2(\log\lambda)^{\varepsilon}), \quad E_0 = \lambda V(x_0).$$

and just as above we get

$$|\psi_j^{[a,b]}(x, n)| < \exp(-(\log\lambda)|n|/2), \quad |n| > 0,$$
$$|\psi_j^{[a,b]}(x, 0) - 1| < \exp(-(\log\lambda)/2).$$

Therefore $\left\|\psi_j^{[a,b]}(x, \cdot) - \psi_k^{[a,b]}(x, \cdot)\right\| \ll 1$, contradicting the fact that

$$\left\|\psi_j^{[a,b]}(x, \cdot) - \psi_k^{[a,b]}(x, \cdot)\right\|^2 = 2.$$

Now we argue that $k(x) = k(x_0)$. Since

$$|\lambda^{-1}E_{k(x_0)}^{[a,b]}(x_0) - V(x_0)| \leq 2\lambda^{-1},$$

we have

$$|\lambda^{-1}E_{k(x_0)}^{[a,b]}(x) - V(x)| \ll \exp(-2(\log\lambda)^{\varepsilon}),$$

and the conclusion follows using (1) and (4).

Finally, suppose that (3.15) holds. Clearly, estimates (1)–(4) still hold. Suppose to the contrary that there exist $j \neq k$ and x such that (4′) fails. By (4) we must have

$$\lambda^{-1}E_j^{[a,b]}(x) < \lambda^{-1}E_k^{[a,b]}(x) - \frac{1}{8}(\log\lambda)^{\varepsilon}.$$

By (1),

$$\lambda^{-1}E_j^{[a,b]}(x) < V(x) - \frac{1}{4}(\log\lambda)^{\varepsilon}.$$

Note that due to (3.15),

$$V(x+n\omega) - V(x) \geq \frac{1}{2}(\log\lambda)^{\varepsilon},$$

for $|x - x_0| < \exp(-3(\log\lambda)^{\varepsilon})$. It follows that

$$|V(x+n\omega) - \lambda^{-1}E_j^{[a,b]}(x)| \geq \frac{1}{4}(\log\lambda)^{\varepsilon}, \quad n \in [a,b],$$

and by Lemma 3.6, $E_j^{[a,b]}(x) \notin \operatorname{spec} H_{[a,b]}(x)$. This contradiction shows that (4′) holds. □

Corollary 3.9 *Using the assumptions and notation of Lemma 3.8 the following hold. For simplicity let $E^{[a,b]}$, $\psi^{[a,b]}$ be the eigenpair from Lemma 3.8. If $N \geq 1$, $[-N,N] \subset [a,b]$, then for any $|x - x_0| < \exp(-3(\log\lambda)^{\varepsilon})$,*

$$|E^{[a,b]}(x) - E^{[-N,N]}(x)| \lesssim \exp(-(\log\lambda)N/2).$$

Proof Using (2) from Lemma 3.8, we have

$$\left\|(H_{[-N,N]}(x) - E^{[a,b]}(x))\psi^{[a,b]}(x,\cdot)\right\| \lesssim \exp(-(\log\lambda)N/2).$$

The conclusion follows from Lemma 2.21, and (1) and (4) from the previous lemma. □

4 Cartan type estimates along level sets near a non-degenerate extremum point

The goal of this section is to prove the next proposition that we will use to handle the edges of the spectrum in Sect. 6. We let $\mathfrak{H}(f)$ stand for the Hessian of a function f. When the function is clear from the context, we will simply write $\mathfrak{H}$. Recall that $\|\cdot\|$ denotes the Euclidean norm, and $|\cdot|$ denotes the sup-norm.

Proposition 4.1 *Let $f(x)$ be a real-analytic function defined on $\{x \in \mathbb{R}^n : |x| < r_0\}$, $r_0 < 1$, which extends analytically to the polydisk $\mathcal{P} := \{z \in \mathbb{C}^n : |z| < r_0\}$. Assume that*

$$f(0) = 0, \quad \nabla f(0) = 0,$$

$$\mathfrak{H}(0) \geq \nu_0 I, \quad 0 < \nu_0 < 1.$$

Let $M(k) = \max_{|\alpha|=k} \sup_{\mathcal{P}} |\partial^\alpha f|$. *Set*

$$\nu_1 := c(n)\nu_0(1 + M(2) + M(3))^{-1}, \quad \rho = r_0 \nu_1^{10},$$

with $c(n)$ *a sufficiently small constant. Let* $0 < \|x_0\| < \rho$, $E_0 = f(x_0)$, $r = \nu_1 \|x_0\|$. *Then there exists a real-analytic map* $x(y, E)$, $(y, E) \in \mathbb{R}^{n-1} \times \mathbb{R}$, $|y| < r$, $|E - E_0| < r^2$, *such that*

$$f(x(y, E)) = E, \quad x(0, E_0) = x_0$$

and satisfying the following properties.

(I) The map $x(y, E)$ *extends analytically to* $\{(w, E) \in \mathbb{C}^{n-1} \times \mathbb{C} : |w| < r, |E - E_0| < r^2\}$ *and satisfies*

$$\|x(w, E) - x_0\| < \frac{\|x_0\|}{2}.$$

(II) For any $|E - E_0| < r^2$, *any vector* $h \in \mathbb{R}^n$ *with* $0 < \|h\| < \rho$, *and any* $H \gg 1$, *we have*

$$\operatorname{mes}\{y \in \mathbb{R}^{n-1}, |y| < r : \log|f(x(y, E) + h) - E| \leq H_0 H\}$$
$$\leq (\nu_1^{-2} r)^{n-1} \exp(-H^{\frac{1}{n-1}}),$$

with $H_0 = C(n)\log(\|h\|\,\|x_0\|)$.

(III) Let $h_0 \in \mathbb{R}^n$ *be an arbitrary unit vector. For any* $|E - E_0| < r^2$, *and any* $H \gg 1$, *we have*

$$\operatorname{mes}\{y \in \mathbb{R}^{n-1}, |y| < r : \log\left|\langle \nabla f(x(y, E)), h_0\rangle\right| \leq H_1 H\}$$
$$\leq (\nu_1^{-2} r)^{n-1} \exp(-H^{\frac{1}{n-1}}),$$

with $H_1 = C(n)\log(\nu_1\|x_0\|)$.

Part (I) of the proposition is a version of the implicit function theorem. For parts (II) and (III) we will apply Cartan's estimate to f along its level sets. To apply it we need a reference point with a "nice" lower bound estimate. So, it is important to accurately book-keep the size of the neighborhood where one can apply the implicit function theorem for it limits the search for the point in question. The same applies to all auxiliary estimates in the proof. For that matter we need to work out a version of the implicit function theorem, explicit enough for our purposes (see Lemma 4.4).

Lemma 4.2 *Let $f(z, w)$ be an analytic function defined on the polydisk*

$$\mathcal{P} = \{(z, w) \in \mathbb{C} \times \mathbb{C}^n : |z|, |w| < \rho_0\}.$$

Let $M_1 = \sup |\partial_z f|$, $M(2) = \max_{|\alpha|=2} \sup \left|\partial^\alpha f\right|$. Assume that $f(0, 0) = 0$, $\mu_0 := |\partial_z f(0, 0)| > 0$. Let

$$\rho_1 \leq \min(\rho_0/2, c(n)\mu_0 M(2)^{-1}), \quad r_i = c(n)\rho_1 \min(1, \mu_0/|\partial_{w_i} f(0, 0)|),$$

with $c(n)$ a sufficiently small constant. Then for any w, $|w_i| < r_i$, the equation

$$f(z, w) = 0$$

has a unique solution $|z(w)| < \rho_1$ which is an analytic function of w.

Proof Take arbitrary w, $|w_i| < r_i$, and z, $|z| = \rho_1$. Then by Taylor's formula and the definition of ρ_1, r_i,

$$\begin{aligned} \left|f(z, w)\right| &\geq |\partial_z f(0, 0)||z| - |\langle \nabla_w f(0, 0), w\rangle| - C(n)M(2) \left\|(z, w)\right\|^2 \\ &\geq \mu_0\rho_1/2. \end{aligned} \tag{4.1}$$

In particular we also have

$$|f(z, 0)| \geq |\partial_z f(0, 0)||z| - C(n)M(2)|z|^2 > 0,$$

for $0 < |z| \leq \rho_1$. So, $f(z, 0)$ has a simple root at $z = 0$ and no other roots in the disk $|z| < \rho_1$, hence

$$\frac{1}{2\pi i} \oint_{|z|=\rho_1} \frac{\partial_z f(z, 0)}{f(z, 0)} dz = 1.$$

By continuity,

$$\frac{1}{2\pi i} \oint_{|z|=\rho_1} \frac{\partial_z f(z, w)}{f(z, w)} dz = 1,$$

for $|w_i| < r_i$. This means $z \to f(z, w)$ has one simple root $z(w)$ in the disk $\{|z| < \rho_1\}$ and by the residue theorem

$$z(w) = \frac{1}{2\pi i} \oint_{|z|=\rho_1} z \frac{\partial_z f(z, w)}{f(z, w)} dz.$$

Clearly, the function on the right-hand side is analytic in w for $|w_i| < r_i$. □

For the proof of part (II) of Proposition 4.1 it will be crucial that the size in the direction of y of the polydisk where the implicit function is defined is of magnitude $\simeq \|\nabla f\|$ and in particular is much bigger than $\simeq \|\nabla f\|^2$ (assuming $\|\nabla f\| < 1$; see Lemma 4.9). This is one reason why in Lemma 4.4 we consider implicit functions in the direction of the gradient. The second reason is the fact that this way one gets some quadratic control over the implicit function (see (4.2)).

Definition 4.3 Given a function f differentiable at $x_0 \in \mathbb{R}^n$, with $\mu_{x_0} := \|\nabla f(x_0)\| > 0$, we let $\mathfrak{n}_{x_0} = \mu_{x_0}^{-1}\nabla f(x_0)$. Let $\mathfrak{e}_{x_0,j}$, $1 \le j \le n-1$ be an orthonormal basis in $\{\mathfrak{n}_{x_0}\}^{\perp}$. Given $(\xi, y) \in \mathbb{R} \times \mathbb{R}^{n-1}$ we denote

$$\varphi(\xi, y; x_0) := x_0 + \xi\mathfrak{n}_{x_0} + \sum_j y_j \mathfrak{e}_{x_0,j}.$$

The set-up of the lemmas to follow is tailored around that of Proposition 4.1.

Lemma 4.4 *Let $f(z)$ be an analytic function defined on $\mathcal{P} = \{z \in \mathbb{C}^n : |z - x_0| < \rho_0\}$, $x_0 \in \mathbb{R}^n$. Let $M(k) = \max_{|\alpha|=k} \sup|\partial^\alpha f|$. Assume $\mu_{x_0} := \|\nabla f(x_0)\| > 0$. Let $E_0 = f(x_0)$. Let*

$$\rho_1 \le c(n)\min(\rho_0, \mu_{x_0}M(2)^{-1}), \quad r = c(n)\rho_1, \quad r' = c(n)\rho_1 \min(1, \mu_{x_0}),$$

with $c(n)$ a sufficiently small constant. Then for any $(w, E) \in \mathbb{C}^{n-1} \times \mathbb{C}$, $|w| < r$, $|E - E_0| < r'$, the equation

$$f(\varphi(\xi, w; x_0)) = E$$

has a unique solution $\xi = g(w, E)$ in $|\xi| < \rho_1$ which is an analytic function of w, E. Furthermore, the following statements hold.

(a) For any $(w, E) \in \mathbb{C}^{n-1} \times \mathbb{C}$, $|w| < r$, $|E - E_0| < r'$ we have

$$|g(w, E)| \le 2\mu_{x_0}^{-1}(|E - E_0| + C(n)M(2)|w|^2). \tag{4.2}$$

(b) For any $x_0' \in \mathbb{R}^n$, $\|x_0' - x_0\| < r$, such that $f(x_0') = E$, $|E - E_0| < r'$, there exists $y \in \mathbb{R}^n$, $\|y\| \le \|x_0' - x_0\|$ such that $x_0' = \varphi(g(y, E), y; x_0)$.

Proof The existence and uniqueness of the solution $\xi = g(w, E)$ follows from Lemma 4.2 applied to $F(\xi, w, E) = f(\varphi(\xi, w; x_0)) - E$ on $\mathcal{P}' = \{(\xi, w, E) : |\xi|, |w|, |E - E_0| < c\rho_0\}$, with c small enough so that $|\varphi(\xi, w; x_0) - x_0| < \rho_0/2$, for $|\xi|, |w| < c\rho_0$. Note that

$$F(0, 0, E_0) = 0, \quad \partial_\xi F(0, 0, E_0) = \mu_{x_0},$$

$$\partial_{w_i} F(0, 0, E_0) = 0, \qquad \partial_E F(0, 0, E_0) = -1,$$

We just need to prove the claims (a),(b).

(a) Note that $\langle \nabla f(x_0), \varphi(\xi, w; x_0) - x_0 \rangle = \mu_{x_0}\xi$. Using Taylor's formula we have

$$f(\varphi(\xi, w; x_0)) - f(x_0) = \mu_{x_0}\xi + R(\xi, w),$$

with

$$|R(\xi, w)| \le C(n)M(2)(|\xi|^2 + |w|^2).$$

By setting $\xi = g(w, E)$ we get

$$\begin{aligned} |g(w, E)| &= \mu_{x_0}^{-1}|E - E_0 - R(g(w, E), w)| \\ &\le \mu_{x_0}^{-1}(|E - E_0| + C(n)M(2)(|g(w, E)|^2 + |w|^2)) \\ &\le \mu_{x_0}^{-1}(|E - E_0| + C(n)M(2)(\rho_1|g(w, E)| + |w|^2)) \\ &\le \frac{1}{2}|g(w, E)| + \mu_{x_0}^{-1}(|E - E_0| + C(n)M(2)|w|^2), \end{aligned}$$

provided ρ_1 is small enough, and (4.2) follows.

(b) Let $(\xi, y) \in \mathbb{R}^{n-1} \times \mathbb{R}$ be such that $x_0' = \varphi(\xi, y; x_0)$. We have $|\xi|, |y| \le \|x_0' - x_0\|$. Since $f(\varphi(\xi, y; x_0)) = E$, $|\xi| < r < \rho_1$, and $|y| < r$, uniqueness implies that $\xi = g(y, E)$. □

Remark 4.5 In Lemmas 4.2 and 4.4, if the function f is real-valued on $\mathbb{R}^n$, then the implicit functions are also real-valued on $\mathbb{R}^n$. Indeed, by the usual implicit function theorem, the implicit functions will be real valued on some small real polydisk, and by analyticity they will be real-valued on their whole real domain.

Part (I) of Proposition 4.1 will follow by letting $x(y, E) = \varphi(g(y, E), y; x_0)$, with g as in the previous lemma. For part (II) it will be enough to prove the result with $E = E_0$, so we focus on this case. To simplify notation we let $g(y) := g(y, E_0)$. Part (II) will follow from Cartan's estimate as soon as we find a point $|y| \ll r$ such that

$$|f(x(y, E_0) + h) - E_0| = |f(\varphi(g(y), y; x_0) + h) - f(x_0)| \ge \varepsilon,$$

with a certain $\varepsilon = \varepsilon(\|h\|, \|x_0\|)$. If $|f(x_0 + h) - f(x_0)| \ge \varepsilon$, then we can simply choose $y = 0$. We single out a simple case when this happens.

Lemma 4.6 *Let $f(x)$ be a smooth real function defined on $\{x \in \mathbb{R}^n : |x - x_0| < \rho_0\}$. Let $M(k) = \max_{|\alpha|=k} \sup |\partial^\alpha f|$. Assume $\mathfrak{H}(x_0) \ge \nu_{x_0} I > 0$ and set*

$$\nu_1 := c(n)\nu_{x_0}(1 + M(3))^{-1}.$$

with $c(n)$ a sufficiently small constant. If $\nu_1^{-1} \|\nabla f(x_0)\| \leq \|h\| < \min(\nu_1, \rho_0)$, then

$$|f(x_0 + h) - f(x_0)| \geq \frac{1}{4}\nu_{x_0} \|h\|^2 .$$

Proof Using Taylor's formula and the assumptions on h,

$$\begin{aligned} &|f(x_0 + h) - f(x_0)| \\ &\quad \geq \frac{1}{2} |\langle \mathfrak{H}(x_0)h, h\rangle| - |\langle \nabla f(x_0), h\rangle| - C(n)M(3) \|h\|^3 \\ &\quad \geq \frac{1}{2}\nu_{x_0} \|h\|^2 - \nu_1 \|h\|^2 - C(n)M(3)\nu_1 \|h\|^2 \geq \frac{1}{4}\nu_{x_0} \|h\|^2 . \end{aligned}$$

□

Suppose that $|f(x_0 + h) - f(x_0)| < \varepsilon$. Then we want to find $x_0' = \varphi(g(y), y; x_0)$, $f(x_0') = f(x_0)$, such that $|f(x_0' + h) - f(x_0')| \geq \varepsilon$. To this end it is enough to find x_0' such that $|f(x_0'+h) - f(x_0+h)| \geq 2\varepsilon$. By Taylor's formula

$$f(x_0' + h) - f(x_0 + h) = \langle \nabla f(x_0 + h), x_0' - x_0\rangle + O(|x_0' - x_0|^2).$$

The linear term will dominate the quadratic term if the projection of $x_0' - x_0$ onto $\nabla f(x_0 + h)$ is large relative to $x_0' - x_0$. By (4.2), the projection of $x_0' - x_0$ onto $\nabla f(x_0)$ is relatively small, so the projection onto $\{\nabla f(x_0)\}^{\perp}$ is relatively large. This means that if $\nabla f(x_0)$ and $\nabla f(x_0 + h)$ are not too close to being collinear, the projection of $x_0' - x_0$ onto $\nabla f(x_0 + h)$ will be relatively large (see Fig. 1),

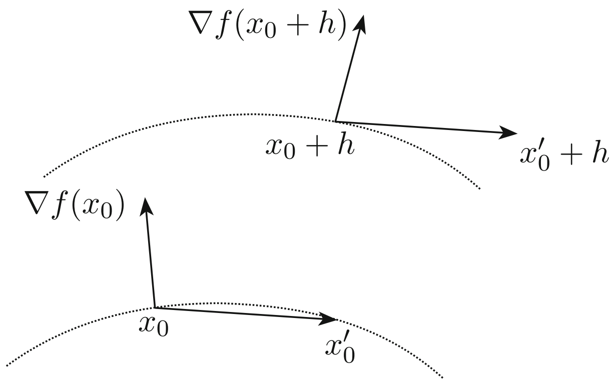

Fig. 1 If $\nabla f(x_0 + h)$ is not collinear with $\nabla f(x_0)$, then the projection of $x_0' - x_0$ onto $\nabla f(x_0 + h)$ is relatively large

and we should be able to find a lower bound on $|f(x_0'+h)-f(x_0+h)|$ via the linear term of the Taylor expansion. A quantitative version of this observation is given in the next lemma.

Lemma 4.7 *Using the notation and assumptions of Lemma 4.4 the following hold. Let $h \in \mathbb{R}^n$, $|h| < \rho_0/2$, $x_1 = x_0 + h$, $\mu_{x_1} := \|\nabla f(x_1)\|$. Assume*

$$\langle \nabla f(x_1), \nabla f(x_0)\rangle^2 \le (1-\delta_0^2)\|\nabla f(x_1)\|^2\|\nabla f(x_0)\|^2, \qquad 0 < \delta_0 \le 1.$$

Let

$$\rho \le c(n)\min(r, \mu M(2)^{-1}\delta_0^2) \ll r, \quad \mu = \min(\mu_{x_0}, \mu_{x_1}),$$

where $c(n)$ is a sufficiently small constant and r as in Lemma 4.4. Then there exists $\|x_0' - x_0\| \le 2\rho$, $x_0' = \varphi(g(y), y; x_0)$, $\|y\| \le \rho$, such that

$$|f(x_0'+h) - f(x_0+h)| \ge \frac{1}{2}\mu_{x_1}\delta_0^2\rho.$$

Proof The case $\mu_{x_1} = 0$ is trivial, so we assume $\mu_{x_1} > 0$. Given $\mathfrak{n} \in \mathbb{R}^n$, $y \in \mathbb{R}^{n-1}$ and using the notation of Definition 4.3 let

$$\mathfrak{p}(\mathfrak{n}; x_0) = \sum_j \langle \mathfrak{e}_{x_0,j}, \mathfrak{n}\rangle \mathfrak{e}_{x_0,j}, \qquad \mathfrak{q}(y; x_0) = \sum_j y_j \mathfrak{e}_{x_0,j}. \tag{4.3}$$

Let $\mathfrak{n}_{x_1} = \mu_{x_1}^{-1}\nabla f(x_1)$. We choose $y \in \mathbb{R}^{n-1}$ such that $\mathfrak{q}(y; x_0) = \rho\mathfrak{p}(\mathfrak{n}_{x_1}; x_0)$, with ρ as in the statement. Note that

$$1 \ge \left\|\mathfrak{p}(\mathfrak{n}_{x_1}; x_0)\right\|^2 = \left\|\mathfrak{n}_{x_1}\right\|^2 - \langle \mathfrak{n}_{x_1}, \mathfrak{n}_{x_0}\rangle^2 \ge 1 - (1-\delta_0^2) = \delta_0^2.$$

It follows that

$$\|y\| = \|\mathfrak{q}(y; x_0)\| = \rho\left\|\mathfrak{p}(\mathfrak{n}_{x_1}; x_0)\right\| \le \rho$$

and

$$\begin{aligned}\langle \nabla f(x_1), \mathfrak{q}(y; x_0)\rangle &= \mu_{x_1}\langle \mathfrak{n}_{x_1}, \mathfrak{q}(y; x_0)\rangle = \mu_{x_1}\langle \mathfrak{p}(\mathfrak{n}_{x_1}; x_0), \mathfrak{q}(y; x_0)\rangle \\ &= \mu_{x_1}\rho\left\|\mathfrak{p}(\mathfrak{n}_{x_1}; x_0)\right\|^2 \ge \mu_{x_1}\delta_0^2\rho.\end{aligned}$$

Let $x_0' = \varphi(g(y), y; x_0)$ with y as above. Then

$$\left\|x_0' - x_0\right\| \le |g(y)| + \|y\| \le \mu_{x_0}^{-1}C(n)M(2)\|y\|^2 + \|y\| \le 2\|y\| \le 2\rho,$$

provided ρ is small enough. Note that we used (4.2). By Taylor's formula

$$f(x_0'+h)-f(x_0+h)=\langle\nabla f(x_1),x_0'-x_0\rangle+R(x_0'-x_0)$$
$$=\langle\nabla f(x_1),g(y)\mathfrak{n}_{x_0}\rangle+\langle\nabla f(x_1),\mathfrak{q}(y;x_0)\rangle$$
$$+R(x_0'-x_0), \tag{4.4}$$

with

$$|R(x_0'-x_0)|\le C(n)M(2)\|x_0'-x_0\|^2\le 4C(n)M(2)\rho^2\le\frac{1}{4}\mu_{x_1}\delta_0^2\rho.$$

We also have

$$|\langle\nabla f(x_1),g(y)\mathfrak{n}_{x_0}\rangle|\le\mu_{x_1}\mu_{x_0}^{-1}C(n)M(2)\rho^2\le\frac{1}{4}\mu_{x_1}\delta_0^2\rho.$$

The conclusion follows by combining the estimates we obtained for the terms on the left hand side of (4.4). □

Now we have to deal with the situation when $|f(x_0+h)-f(x_0)|<\varepsilon$, and $\nabla f(x_0)$ and $\nabla f(x_0+h)$ are close to being collinear. We show that for small enough h this can only happen if h is very close to a particular "bad" direction.

Lemma 4.8 *Let $f(x)$ be a smooth real function defined on $\{x\in\mathbb{R}^n : |x-x_0|<\rho_0\}$. Let $M(k)=\max_{|\alpha|=k}\sup|\partial^\alpha f|$. Assume $\mathfrak{H}(x_0)\ge\nu_{x_0}I$, $0<\nu_{x_0}<1$, and set*

$$\nu_1:=c(n)\nu_{x_0}(1+M(2)+M(3))^{-1}$$

with $c(n)$ a sufficiently small constant. Let $0<\|h\|<\min(\rho_0,\nu_1^6)$. Assume that the following conditions hold:

$$|f(x_0+h)-f(x_0)|\le\|h\|^3, \tag{4.5}$$
$$\|\nabla f(x_0+h)-\lambda\nabla f(x_0)\|\le\|h\|^2, \tag{4.6}$$

with some $\lambda\in\mathbb{R}$. Then

$$\|h+2\mathfrak{H}(x_0)^{-1}\nabla f(x_0)\|\le\nu_1^{-8}\|\nabla f(x_0)\|^2. \tag{4.7}$$

Proof Note that (4.5) together with Lemma 4.6 imply $\|h\|\le\nu_1^{-1}\|\nabla f(x_0)\|$. In particular, this implies $\|\nabla f(x_0)\|>0$.

Combining (4.6) with Taylor's formula we get

$$\|(\lambda-1)\nabla f(x_0)-\mathfrak{H}(x_0)h\|\le C(n)(1+M(3))\|h\|^2.$$

Therefore

$$\|(\lambda-1)\mathfrak{H}(x_0)^{-1}\nabla f(x_0)-h\| \le \|\mathfrak{H}(x_0)^{-1}\|C(n)(1+M(3))\|h\|^2 \\ \le \nu_{x_0}^{-1}C(n)(1+M(3))\|h\|^2 \le \nu_1^{-1}\,\|h\|^2\,.$$

Combining (4.5) with Taylor's formula we get

$$\left|\langle\nabla f(x_0),h\rangle+\frac{1}{2}\langle\mathfrak{H}(x_0)h,h\rangle\right| \le \|h\|^3+C(n)M(3)\|h\|^3 \le \nu_1^{-1}\,\|h\|^3\,.$$

Let $v=(\lambda-1)\mathfrak{H}(x_0)^{-1}\nabla f(x_0)-h$. Combining the previous two estimates yields

$$\begin{aligned}&\left|(\lambda-1)\langle\nabla f(x_0),\mathfrak{H}(x_0)^{-1}\nabla f(x_0)\rangle+\frac{1}{2}(\lambda-1)^2\langle\nabla f(x_0),\mathfrak{H}(x_0)^{-1}\nabla f(x_0)\rangle\right|\\ &\le \nu_1^{-1}\|h\|^3+\|\nabla f(x_0)\|\,\|v\|+\frac{1}{2}\,\|\mathfrak{H}(x_0)\|\,(\|v\|^2+2\,\|v\|\,\|v+h\|)\\ &\le \nu_1^{-1}\,\|h\|^3+\|\nabla f(x_0)\|\,\nu^{-1}\,\|h\|^2+C(n)M(2)(\nu_1^{-2}\,\|h\|^4+\nu_1^{-1}\,\|h\|^3)\\ &\le \nu_1^{-1}\,\|\nabla f(x_0)\|\,\|h\|^2+\nu_1^{-2}\,\|h\|^3\,.\end{aligned}$$

Since $\langle\nabla f(x_0),\mathfrak{H}(x_0)^{-1}\nabla f(x_0)\rangle \ge \|\mathfrak{H}(x_0)\|^{-1}\,\|\nabla f(x_0)\|^2 \ge \nu_1\,\|\nabla f(x_0)\|^2$, it follows that

$$|(\lambda-1)(\lambda+1)| \le \varepsilon := \nu_1^{-2}\,\|\nabla f(x_0)\|^{-1}\,\|h\|^2+\nu_1^{-3}\,\|\nabla f(x_0)\|^{-2}\,\|h\|^3\,.$$

Since $\max(|\lambda-1|,|\lambda+1|)\ge 1$, we have

$$\min(|\lambda-1|,|\lambda+1|)\le\varepsilon.$$

If $|\lambda-1|\le\varepsilon$, then

$$\begin{aligned}\|h\| &\le \|h-(\lambda-1)\mathfrak{H}(x_0)^{-1}\nabla f(x_0)\|+\|(\lambda-1)\mathfrak{H}(x_0)^{-1}\nabla f(x_0)\|\\ &\le \nu_1^{-1}\,\|h\|^2+\nu_1^{-1}\varepsilon\,\|\nabla f(x_0)\| = \nu_1^{-1}\,\|h\|^2+\nu_1^{-3}\,\|h\|^2\\ &\quad +\nu_1^{-4}\,\|\nabla f(x_0)\|^{-1}\,\|h\|^3 \le \nu_1^{-6}\,\|h\|^2\end{aligned}$$

(recall that $\|h\|\le\nu_1^{-1}\,\|\nabla f(x_0)\|$). This is not compatible with the assumption that $0<\|h\|<\nu_1^6$. So, we must have $|\lambda+1|\le\varepsilon$ and therefore

$$\|h+2\mathfrak{H}(x_0)^{-1}\nabla f(x_0)\| \le \|h-(\lambda-1)\mathfrak{H}(x_0)^{-1}\nabla f(x_0)\| \\ +\|(\lambda+1)\mathfrak{H}(x_0)^{-1}\nabla f(x_0)\|$$

$$\leq \nu_1^{-1}\|h\|^2 + \nu_1^{-1}\varepsilon\|\nabla f(x_0)\| \leq \nu_1^{-6}\|h\|^2$$
$$\leq \nu_1^{-8}\|\nabla f(x_0)\|^2.$$

□

Finally, we show that (4.7) cannot hold over the entire piece of the $f(x_0)$-level set parametrized in Lemma 4.4.

Lemma 4.9 *Let $f(x)$ be a smooth real function defined on $\{x \in \mathbb{R}^n : |x - x_0| < \rho_0\}$, $\rho_0 < 1$. Let $M(k) = \max_{|\alpha|=k} \sup|\partial^\alpha f|$. Assume $\mathfrak{H}(x_0) \geq \nu_{x_0} I$, $0 < \nu_{x_0} < 1$, and $0 < \|\nabla f(x_0)\| < \rho_0 \nu_1^9/20$ with*

$$\nu_1 := c(n)\nu_{x_0}(1 + M(2) + M(3))^{-1}$$

with $c(n)$ a sufficiently small constant. Then there exists $\|x_0' - x_0\| \ll r$, with r as in Lemma 4.4, $x_0' = \varphi(g(y), y; x_0)$, $\|y\| \ll r$, such that

$$\left\|\mathfrak{H}(x_0')^{-1}\nabla f(x_0') - \mathfrak{H}(x_0)^{-1}\nabla f(x_0)\right\| > \nu_1^{-8}(\|\nabla f(x_0)\|^2 + \left\|\nabla f(x_0')\right\|^2).$$

Proof Choose $y \in \mathbb{R}^{n-1}$ such that $\|y\| = \nu_1\|\nabla f(x_0)\|$ and let $x_0' = \varphi(g(y), y; x_0)$. Using (4.2) we have

$$\left\|x_0' - x_0\right\| \leq |g(y)| + \|y\| \leq \mu_{x_0}^{-1} C(n) M(2)\|y\|^2 + \|y\| \leq 2\nu_1\|\nabla f(x_0)\| \ll r,$$

provided ν_1 is small enough. Then

$$\left\|\mathfrak{H}(x_0') - \mathfrak{H}(x_0)\right\| \leq C(n)M(3)\left\|x_0' - x_0\right\| \leq \|\nabla f(x_0)\| \leq \frac{\nu_{x_0}}{2}$$

(recall that $\|\nabla f(x_0)\| < \rho_0\nu_1^{10}$, $\rho_0 < 1$) and therefore $\mathfrak{H}(x_0') \geq \frac{\nu_{x_0}}{2} I$ and $\left\|\mathfrak{H}(x_0')^{-1}\right\| \leq 2\nu_{x_0}^{-1}$. We have

$$\begin{aligned}
&\left\|\mathfrak{H}(x_0')^{-1}\nabla f(x_0') - \mathfrak{H}(x_0)^{-1}\nabla f(x_0)\right\| \\
&\quad \geq \left\|\mathfrak{H}(x_0')^{-1}(\nabla f(x_0) - \nabla f(x_0'))\right\| - \left\|(\mathfrak{H}(x_0')^{-1} - \mathfrak{H}(x_0)^{-1})\nabla f(x_0)\right\|.
\end{aligned}$$

On one hand using Taylor's formula applied to the gradient we get

$$\begin{aligned}
&\left\|\mathfrak{H}(x_0')^{-1}(\nabla f(x_0) - \nabla f(x_0'))\right\| \\
&\quad \geq \left\|x_0' - x_0\right\| - \left\|\mathfrak{H}(x_0')^{-1}\right\| C(n)M(3)\left\|x_0' - x_0\right\|^2 \\
&\quad \geq \left\|x_0' - x_0\right\| - \nu_1^{-1}\left\|x_0' - x_0\right\|^2 \geq \frac{\left\|x_0' - x_0\right\|}{2} \geq \frac{\nu_1\|\nabla f(x_0)\|}{2}.
\end{aligned}$$

On the other hand

$$\begin{aligned}&\left\|(\mathfrak{H}(x_0')^{-1} - \mathfrak{H}(x_0)^{-1})\nabla f(x_0)\right\| \\ &\quad \le \left\|\mathfrak{H}(x_0')^{-1}\right\| \left\|\mathfrak{H}(x_0)^{-1}\right\| \left\|(\mathfrak{H}(x_0') - \mathfrak{H}(x_0)\right\| \|\nabla f(x_0)\| \le \nu_1^{-1} \|\nabla f(x_0)\|^2 .\end{aligned}$$

Therefore

$$\begin{aligned}&\left\|\mathfrak{H}(x_0')^{-1}\nabla f(x_0') - \mathfrak{H}(x_0)^{-1}\nabla f(x_0)\right\| \\ &\quad \ge \frac{\nu_1 \|\nabla f(x_0)\|}{2} - \nu_1^{-1} \|\nabla f(x_0)\|^2 \ge \frac{\nu_1 \|\nabla f(x_0)\|}{4} > 5\nu_1^{-8} \|\nabla f(x_0)\|^2 .\end{aligned}$$

Since

$$\begin{aligned}\left\|\nabla f(x_0) - \nabla f(x_0')\right\| &\le C(n)M(2) \left\|x_0' - x_0\right\| \le 2C(n)M(2)\nu_1 \|\nabla f(x_0)\| \\ &\le \|\nabla f(x_0)\| ,\end{aligned}$$

we get that

$$\nu_1^{-8}(\left\|\nabla f(x_0')\right\|^2 + \|\nabla f(x_0)\|^2) \le 5\nu_1^{-8} \|\nabla f(x_0)\|^2 ,$$

and the conclusion follows. □

We will use the following simple consequence of Taylor's formula. We leave the proof as a simple exercise.

Lemma 4.10 *Let $f(x)$ be a smooth real function defined on $\{x \in \mathbb{R}^n : |x| < r_0\}$. Assume that*

$$\begin{aligned}&f(0) = 0, \quad \nabla f(0) = 0, \\ &\mathfrak{H}(0) \ge \nu_0 I, \quad \nu_0 > 0.\end{aligned}$$

Let $M(k) = \max_{|\alpha|=k} \sup_x |\partial^\alpha f|$. Then for $|x| < \min(r_0, c(n)\nu_0 M(3)^{-1})$, with $c(n)$ a sufficiently small constant, we have

$$\begin{aligned}\frac{\nu_0}{2}\|x\|^2 &\le f(x) \le (C(n)M(2) + 1)\|x\|^2, \\ \frac{\nu_0}{2}\|x\| &\le \|\nabla f(x)\| \le (C(n)M(2) + 1)\|x\|, \\ \mathfrak{H}(x) &\ge \frac{\nu_0}{2} I.\end{aligned}$$

Now we prove Proposition 4.1.

Proof of Proposition 4.1 Let $0 < \|x_0\| < \rho$, $E_0 = f(x_0)$. Using Lemma 4.10 we have

$$\frac{\nu_0}{2}\|x_0\| \le \|\nabla f(x_0)\| \le (C(n)M(2)+1)\|x_0\| \le \frac{r_0}{2}\nu_1^9 \ll 1. \tag{4.8}$$

Let $\mu_{x_0} = \|\nabla f(x_0)\|$,

$$\tilde{\rho}_1 = \tilde{c}(n)\min(r_0, \mu_{x_0}M(2)^{-1}), \quad \tilde{r} = \tilde{c}(n)\tilde{\rho}_1, \quad \tilde{r}' = \tilde{c}(n)\rho_1\min(1, \mu_{x_0}),$$

with $\tilde{c}(n)$ standing for the $c(n)$ constant from Lemma 4.4. By Lemma 4.4, for any $(w, E) \in \mathbb{C}^{n-1} \times \mathbb{C}$, $|w| < \tilde{r}$, $|E - E_0| < \tilde{r}'$, the equation

$$f(\varphi(\xi, w; x_0)) = E$$

has a unique solution $\xi = g(w, E)$ in $|\xi| < \tilde{\rho}_1$ which is an analytic function of w, E. Note that by the smallness of x_0 we have

$$\tilde{\rho}_1 = \tilde{c}(n)\mu_{x_0}M(2)^{-1}, \quad \tilde{r} = \tilde{c}(n)^2\mu_{x_0}M(2)^{-1}, \quad \tilde{r}' = \tilde{c}(n)^2\mu_{x_0}^2M(2)^{-1}, \tag{4.9}$$

$$r \ll \tilde{r}, \quad r^2 \ll \tilde{r}' \tag{4.10}$$

(we used the fact that $M(2) \ge c(n)\nu_0$). By (4.2),

$$\begin{aligned}\|\varphi(g(w,E), w; x_0) - x_0\| &\le |g(w,E)| + \|w\| \\ &\le 2\mu_{x_0}^{-1}(r^2 + C(n)M(2)r^2) + r\sqrt{n-1} \\ &< \frac{1}{2}\nu_1^{-1}r = \frac{1}{2}\|x_0\|,\end{aligned} \tag{4.11}$$

for any $|w| < r$, $|E - E_0| < r^2$. Now part (I) follows by setting $x(w, E) = \varphi(g(w, E), w; x_0)$.

We first prove (II) with $E = E_0$. Let $0 < \|h\| < \rho$. We claim that there exists y_0, $\|y_0\| \ll \tilde{r}$, such that

$$|f(x(y_0, E_0) + h) - E_0| = |f(\varphi(g(y_0), y_0; x_0) + h) - f(x_0)| \ge \|h\|^8\|x_0\|.$$

From the claim (also note that $|f(x(w, E) + h) - E| \ll 1$), Lemmas 2.10, and 2.11 it follows that for $H \gg 1$ we have

$$\begin{aligned}&\operatorname{mes}\{y \in \mathbb{R}^{n-1}, |y| < r : \log|f(x(y, E_0) + h) - E_0| \\ &\quad \ge C(n)H\log(\|h\|\|x_0\|)\} \le C(n)\tilde{r}^{n-1}\exp(-H^{\frac{1}{n-1}}) \\ &\quad \le (\nu_1^{-2}r)^{n-1}\exp(-H^{\frac{1}{n-1}})\end{aligned}$$

as stated in Proposition 4.1 (recall (4.8), (4.9), (4.10)). Now we check the claim. Let $x_1 = x_0 + h$, $\mu_{x_1} = \|\nabla f(x_1)\|$. If $|f(x_1) - f(x_0)| > \|h\|^8 \|x_0\|$, the claim holds with $y_0 = 0$. Suppose

$$|f(x_1) - f(x_0)| \le \|h\|^8 \|x_0\| \quad \text{and} \quad \|\nabla f(x_1) - \lambda \nabla f(x_0)\| > \|h\|^2,$$
$$\lambda = \frac{\langle \nabla f(x_0), \nabla f(x_1)\rangle}{\langle \nabla f(x_0), \nabla f(x_0)\rangle}.$$

Then a direct computation yields

$$\langle \nabla f(x_1), \nabla f(x_0)\rangle^2 \le (1 - \delta_0^2)\|\nabla f(x_1)\|^2 \|\nabla f(x_0)\|^2, \quad \delta_0 = \frac{\|h\|^4}{\mu_{x_1}^2}.$$

Note that

$$\|\nabla f(x_1)\| \ge \|\nabla f(x_1) - \lambda \nabla f(x_0)\| > \|h\|^2.$$

We choose a small enough constant $c(n)$ such that Lemma 4.7 applies with

$$\tilde{\rho} = c(n)\mu M(2)^{-1}\delta_0^2, \quad \mu = \min(\mu_{x_0}, \mu_{x_1})$$

instead of ρ, $\rho_0 = r_0/2$, $\tilde{r}$ instead of r, and δ_0 as above. Applying Lemma 4.7 we get that there exists y, $\|y\| \le \tilde{\rho} \ll \tilde{r}$, such that

$$|f(\varphi(g(y), y; x_0) + h) - f(x_1)| \ge \frac{1}{2}\mu_{x_1}\delta_0^2\tilde{\rho} \ge c(n)M(2)^{-1}\mu_{x_1}\mu\delta_0^4$$
$$\ge 2\|h\|^8 \|x_0\|.$$

We used (4.8) and the fact that

$$\|\nabla f(x_1)\| \le (C(n)M(2) + 1)\|x_0 + h\| \le r_0\nu_1^9 \ll 1.$$

Since $|f(x_1) - f(x_0)| \le \|h\|^8 \|x_0\|$, the claim follows with $y_0 = y$.

We are left with the case when

$$|f(x_1) - f(x_0)| \le \|h\|^8 \|x_0\| \quad \text{and} \quad \|\nabla f(x_1) - \lambda \nabla f(x_0)\| \le \|h\|^2.$$

Note that by Lemma 4.10, $\mathfrak{H}(x_0') \ge \frac{\nu_0}{2}I$ for any $\|x_0' - x_0\| < \tilde{r}$. Choosing sufficiently small constants $c(n)$ we can apply Lemmas 4.8 and 4.9 with the same ν_1 as in Proposition 4.1. Furthermore, we can apply Lemma 4.8 with any $\|x_0' - x_0\| < \tilde{r}$ instead of x_0. Lemmas 4.8 and 4.9 imply that there exists

$$\|x_0' - x_0\| \ll \tilde{r}, \quad f(x_0') = f(x_0), \quad x_0' = \varphi(g(y'), y'; x_0), \ \|y'\| \ll \tilde{r},$$

such that
$$\left\| h + 2\mathfrak{H}(x_0')^{-1}\nabla f(x_0') \right\| > \nu_1^{-8} \left\| \nabla f(x_0') \right\|^2 .$$
Lemma 4.8 (with x_0' instead of x_0) implies that

$$|f(x_1') - f(x_0')| > \|h\|^3 \quad \text{or} \quad \left\| \nabla f(x_1') - \lambda' \nabla f(x_0') \right\| > \|h\|^2 ,$$

with
$$x_1' = x_0' + h, \quad \lambda' = \frac{\langle \nabla f(x_0'), \nabla f(x_1') \rangle}{\langle \nabla f(x_0'), \nabla f(x_0') \rangle}.$$

If $|f(x_1') - f(x_0')| > \|h\|^3$, the claim holds with $y_0 = y'$. If $\left\| \nabla f(x_1') - \lambda' \nabla f(x_0') \right\| > \|h\|^2$, the reasoning above, based on Lemma 4.7, implies that there exists $\left\| x_0'' - x_0' \right\| \le 2\tilde{\rho}' \ll \tilde{r}$,

$$\tilde{\rho}' = c(n)\mu' M(2)^{-1}(\delta_0')^2, \quad \mu' = \min(\mu_{x_0'}, \mu_{x_1'}, \mu_{x_0}), \quad (\delta_0')^2 = \frac{\|h\|^4}{\mu_{x_1'}^2},$$

such that $f(x_0'') = f(x_0') = f(x_0)$ and

$$|f(x_0'' + h) - f(x_0' + h)| \ge \frac{1}{2}\mu_{x_1'}(\delta_0')^2 \tilde{\rho}' \ge 2 \|h\|^8 \|x_0\| .$$

Note that we added μ_{x_0} to the definition of μ' to ensure $\tilde{\rho}' \ll \tilde{r}$, and we used the fact that $\left\| x_0' \right\| \ge \|x_0\| /2$. We now have that either

$$|f(x_0' + h) - f(x_0')| > \|h\|^8 \|x_0\| \quad \text{or} \quad |f(x_0'' + h) - f(x_0'')| > \|h\|^8 \|x_0\| .$$

Since $\left\| x_0'' - x_0 \right\| \ll \tilde{r}$, Lemma 4.4 implies that there exists y'', $\left\| y'' \right\| \ll \tilde{r}$, such that $x_0'' = \varphi(g(y''), y''; x_0)$. Therefore the claim holds with either $y_0 = y'$ or $y_0 = y''$.

Next we consider part (II) with $|E - E_0| < r^2$. Let $x_0' = \varphi(g(0, E), 0; x_0)$. Repeating the above argument with x_0' instead of x_0 we get that there exists y_0',
$$\|y_0'\| \ll \tilde{c}(n)^2 \mu_{x_0'} M(2)^{-1}$$
(recall that $\tilde{r} = \tilde{c}(n)^2 \mu_{x_0} M(2)^{-1}$) such that

$$|f(\varphi(g(y_0'; x_0'), y_0'; x_0') + h) - E| \ge \|h\|^8 \left\| x_0' \right\| .$$

We used $g(y; x_0')$ to denote the analogue of $g(y)$ obtained by applying Lemma 4.4 with x_0' replacing x_0. By (4.11) we have $\left\| x_0' \right\| \ge \|x_0\| /2$. Let $x_0'' = \varphi(g(y_0'; x_0'), y_0'; x_0')$. Note that $f(x_0'') = f(x_0') = E$. We have

$$\left\|x_0'' - x_0\right\| \le \left\|x_0' - x_0\right\| + |g(y_0'; x_0')| + \left\|y_0'\right\| .$$

Using (4.2) we get

$$\left\|x_0' - x_0\right\| = |g(0, E)| \le 2\mu_{x_0}^{-1}|E - E_0| \le 2\left(\nu_0 \left\|x_0\right\|/2\right)^{-1} r^2 \le r \ll \tilde{r}$$

and

$$\begin{aligned}|g(y_0'; x_0')| &\le \mu_{x_0'}^{-1} C(n) M(2) \left\|y_0'\right\|^2 \le \mu_{x_0'}^{-1} C(n) M(2) \tilde{c}(n)^2 \mu_{x_0'} M(2)^{-1} \left\|y_0'\right\| \\ &\le \left\|y_0'\right\| ,\end{aligned}$$

provided $\tilde{c}(n)$ is made small enough. Since

$$\begin{aligned}|\mu_{x_0'} - \mu_{x_0}| &\le C(n) M(2) \left\|x_0' - x_0\right\| \le C(n) M(2) \left(\nu_0 \left\|x_0\right\|/2\right)^{-1} r^2 \\ &\le \nu_0 \left\|x_0\right\|/2 \le \mu_{x_0},\end{aligned}$$

we have

$$\left\|y_0'\right\| \ll \tilde{c}(n)^2 \mu_{x_0'} M(2)^{-1} \le 2\tilde{r}.$$

Therefore we have

$$\left\|x_0'' - x_0\right\| \ll \tilde{r}.$$

By Lemma 4.4 there exists y_0, $\|y_0\| \ll \tilde{r}$, such that $x_0'' = \varphi(g(y_0, E), y_0; x_0)$. Since

$$|f(\varphi(g(y_0, E), y_0; x_0) + h) - E| \ge \|h\|^8 \|x_0\|/2,$$

the conclusion follows as above from Cartan's estimate.

Next we prove (III) with $E = E_0$. We will argue that there exists y_0, $\|y_0\| \ll \tilde{r}$, such that

$$\log |\langle f(x(y_0, E_0)), h_0\rangle| \gtrsim \nu_1 \left\|x_0\right\| . \tag{4.12}$$

Recall that $x(y, E_0) = \varphi(g(y), y; x_0)$. If $|\langle \nabla f(x_0), h_0\rangle| \ge \|x_0\|^2$, we take $y_0 = 0$. We just need to deal with the case

$$|\langle \nabla f(x_0), h_0\rangle| < \left\|x_0\right\|^2 . \tag{4.13}$$

Let $x_0' = \varphi(g(y), y; x_0)$, with y to be specified later. By Taylor's formula

$$\begin{aligned}&|\langle \nabla f(x_0'), h_0\rangle - \langle \nabla f(x_0), h_0\rangle| \\ &\quad \ge |\langle \mathfrak{H}(x_0)(x_0' - x_0), h_0)\rangle| - C(n) M(3) \left\|x_0' - x_0\right\|^2 \\ &\quad = |\langle (x_0' - x_0), \mathfrak{H}(x_0) h_0)\rangle| - C(n) M(3) \left\|x_0' - x_0\right\|^2 .\end{aligned}$$

Using the notation from (4.3) we write

$$\mathfrak{H}(x_0)h_0 = \alpha_0 \mathfrak{n}_{x_0} + \mathfrak{p}(\mathfrak{H}(x_0)h_0; x_0)$$

and we choose y such that $\mathfrak{q}(y; x_0) = \rho\mathfrak{p}(\mathfrak{H}(x_0)h_0; x_0)$, $\rho = \nu_1^2 \|x_0\|$. Note that $\|y\| \leq r \ll \tilde{r}$ and

$$\begin{aligned}\langle (x_0' - x_0), \mathfrak{H}(x_0)h_0 \rangle &= \alpha_0 g(y) + \langle \mathfrak{q}(y; x_0), \mathfrak{p}(\mathfrak{H}(x_0)h_0; x_0)\rangle \\ &= \alpha_0 g(y) + \rho \|\mathfrak{p}(\mathfrak{H}(x_0)h_0; x_0)\|^2 .\end{aligned}$$

Using (4.2) it follows that

$$\begin{aligned}|\langle \nabla f(x_0'), h_0\rangle - \langle \nabla f(x_0), h_0\rangle| &\geq \rho \|\mathfrak{p}(\mathfrak{H}(x_0)h_0; x_0)\|^2 - |\alpha_0 g(y)| \\ &\quad - C(n)M(3)(|g(y)|^2 + \|y\|^2) \\ &\geq \frac{\rho}{2} \|\mathfrak{p}(\mathfrak{H}(x_0)h_0; x_0)\|^2\end{aligned}$$

(note that $|\alpha_0| \leq \|\mathfrak{H}(x_0)h_0\| \leq C(n)M(2)$). We claim that $\|\mathfrak{p}(\mathfrak{H}(x_0)h_0; x_0)\| \geq \|x_0\|$. We argue by contradiction. Assume that

$$\left\|\mathfrak{H}(x_0)h_0 - \alpha_0 \mathfrak{n}_{x_0}\right\| = \|\mathfrak{p}(\mathfrak{H}(x_0)h_0; x_0)\| < \|x_0\| .$$

By Taylor's formula (recall that $\nabla f(0) = 0$)

$$\|\nabla f(x_0) - \mathfrak{H}(x_0)x_0\| \leq C(n)M(3) \|x_0\|^2 .$$

So, using (4.8) we have

$$\left\|\mathfrak{n}_{x_0} - \mu_{x_0}^{-1}\mathfrak{H}(x_0)x_0\right\| \leq \mu_{x_0}^{-1} C(n)M(3) \|x_0\|^2 \leq \nu_1^{-1} \|x_0\| ,$$

and using (4.13) we have

$$|\langle \mathfrak{H}(x_0)h_0, x_0\rangle| = |\langle \mathfrak{H}(x_0)x_0, h_0\rangle| \leq (C(n)M(3) + 1) \|x_0\|^2 \leq \nu_1^{-1} \|x_0\|^2 .$$

Now we have

$$\left\|\mathfrak{H}(x_0)h_0 - \alpha_0\mu_{x_0}^{-1}\mathfrak{H}(x_0)x_0\right\| \leq (1 + \alpha_0\nu_1^{-1}) \|x_0\| \leq \nu_1^{-2} \|x_0\| .$$

and therefore

$$\begin{aligned}|\langle \alpha_0\mu_{x_0}^{-1}\mathfrak{H}(x_0)x_0, x_0\rangle| &\leq |\langle \mathfrak{H}(x_0)h_0 - \alpha_0\mu_{x_0}^{-1}\mathfrak{H}(x_0)x_0, x_0\rangle| + |\langle \mathfrak{H}(x_0)h_0, x_0\rangle| \\ &\leq \nu_1^{-2} \|x_0\|^2 + \nu_1^{-1} \|x_0\|^2 \leq 2\nu_1^{-2} \|x_0\|^2 . \qquad (4.14)\end{aligned}$$

On the other hand

$$|\langle \alpha_0 \mu_{x_0}^{-1} \mathfrak{H}(x_0) x_0, x_0 \rangle| \geq |\alpha_0| \mu_{x_0}^{-1} \frac{\nu_0}{2} \|x_0\|^2 \geq \nu_1^3 \|x_0\| . \tag{4.15}$$

We used Lemma 4.10, (4.8), and the fact that

$$|\alpha_0|^2 = \|\mathfrak{H}(x_0)h_0\|^2 - \|\mathfrak{p}(\mathfrak{H}(x_0)h_0; x_0)\|^2 \geq (\nu_0/2)^{-2} - \|x_0\|^2 \geq \nu_0^{-2}$$

(recall that $\|x_0\| < \nu_1^{11} \ll \nu_0$). The estimates (4.14) and (4.15) are incompatible due to the smallness of x_0. Therefore we have $\|\mathfrak{p}(\mathfrak{H}(x_0)h_0; x_0)\| \geq \|x_0\|$ and

$$|\langle \nabla f(x_0'), h_0 \rangle - \langle \nabla f(x_0), h_0 \rangle| \gtrsim \rho \|x_0\|^2 = \nu_1^2 \|x_0\|^3 .$$

This shows that (4.12) must hold either with $y_0 = 0$ or $y_0 = y$. From (4.12) (also note that $\|\nabla f(x(w, E))\| \ll 1$), Lemmas 2.10, and 2.11 it follows that for $H \gg 1$ we have

$$\begin{aligned}
&\operatorname{mes}\{y \in \mathbb{R}^{n-1}, |y| < r : \log |\langle \nabla f(x(y, E_0)), h_0 \rangle| \geq C(n) H \log(\nu_1 \|x_0\|)\} \\
&\quad \leq C(n) \tilde{r}^{n-1} \exp(-H^{\frac{1}{n-1}}) \leq (\nu_1^{-2} r)^{n-1} \exp(-H^{\frac{1}{n-1}})
\end{aligned}$$

as stated in Proposition 4.1. The case $|E - E_0| < r^2$ follows from the case $E = E_0$ analogously to the proof of (II). □

5 Inductive scheme for the bulk of the spectrum

In this section we assume the same non-perturbative setting as in Sect. 2. We introduce five conditions such that once they hold at a large enough initial scale they can be propagated to arbitrarily large scales (see Theorem D below) and lead to the formation of an interval in the spectrum, away from the edges (see Theorem B in Sect. 8).

For the statement of the conditions we need several exponents. Let $\sigma \ll \tau \ll 1$ be as in (LDT). Set $\delta = (\sigma')^{C_0}$, $\beta = (\sigma')^{C_1}$, $\mu = (\sigma')^{C_2}$ with $0 < \sigma' \leq \sigma$, and $C_0, C_1, C_2 > 1$, satisfying the following relations:

$$C_1 + 1 < C_2 < C_0 < 2C_1 .$$

Then we have

$$\beta^2 \ll \delta \ll \mu \ll \beta\sigma \ll \beta \ll \sigma, \tag{5.1}$$

with the constants implied by $\ll$ being as large as we wish, provided we take $\sigma' \leq c(C_0, C_1, C_2)\sigma$ small enough. The specific choice of the exponents δ, β, μ is not important. However, to carry out the induction with our set-up we will need that (5.1) holds.

Let $\gamma > 0$. Given an integer $s \geq 0$, let

$$E_s \in \mathbb{R}, \quad N_s \in \mathbb{N}, \quad r_s := \exp(-N_s^\delta).$$

The inductive conditions are as follows.
(A) There exist integers $|N_s' - N_s|, |N_s'' - N_s| < N_s^{1/2}$, a map $x_s : \Pi_s \to \mathbb{R}^d$,

$$\Pi_s = \mathcal{I}_s \times (E_s - r_s, E_s + r_s), \quad \mathcal{I}_s = \phi_s + (-r_s, r_s)^{d-1},$$

and k_s such that for any $(\phi, E) \in \Pi_s$ we have

$$E_{k_s}^{[-N_s', N_s'']}(x_s(\phi, E)) = E, \tag{5.2}$$

$$\left|E_j^{[-N_s', N_s'']}(x_s(\phi, E)) - E\right| > \exp(-N_s^\delta), \quad j \neq k_s. \tag{5.3}$$

To simplify notation we suppress k_s and use $E^{[-N_s', N_s'']}$, $\psi^{[-N_s', N_s'']}$ instead.
(B) The map $x_s(\phi, E)$ extends analytically on the domain

$$\mathcal{P}_s = \{(\phi, E) \in \mathbb{C}^d : \text{dist}((\phi, E), \Pi_s) < r_s\} \tag{5.4}$$

(the distance is with respect to the sup-norm) and

$$x_s(\mathcal{P}_s) \subset \mathbb{T}^d_{\rho/2}. \tag{5.5}$$

(C) For each $(\phi, E) \in \Pi_s$,

$$|\psi^{[-N_s', N_s'']}(x_s(\phi, E), n)| \leq \exp(-\gamma|n|/10), \quad |n| \geq N_s/4. \tag{5.6}$$

(D) Define

$$\mathfrak{T}_s = \{n\omega : 0 \leq |n| \leq 3N_s/2\}. \tag{5.7}$$

Take an arbitrary $h \in \mathbb{T}^d$ with $\text{dist}(h, \mathfrak{T}_s) \geq \exp(-N_s^\mu)$. Then for any $E \in (E_s - r_s, E_s + r_s)$,

$$\text{mes}\Big\{\phi \in \mathcal{I}_s : \max_{|n'|,|n''|<N_s^{1/2}} \text{dist}(\text{spec}\, H_{[-N_s+n', N_s+n'']}(x_s(\phi, E) + h), E) < \exp(-N_s^\beta/2)\Big\} < \exp(-N_s^{2\delta}).$$

(E) Take an arbitrary unit vector $h_0 \in \mathbb{R}^d$. Then for any $E \in (E_s - r_s, E_s + r_s)$,

$$\text{mes}\{\phi \in \mathcal{I}_s : \log|\langle \nabla E^{[-N_s', N_s'']}(x_s(\phi, E)), h_0\rangle| < -N_s^\mu/2\} < \exp(-N_s^{2\delta}).$$

Remark 5.1 (a) From the proof of Proposition 5.6 below it will become clear that in (A) it would be enough to have separation of eigenvalues by $\exp(-N_s^\beta)$. However, it will also be clear that even if we have separation by $\exp(-N_0^\beta)$, for $s = 0$, we will still get separation by $\exp(-N_s^\delta)$, for $s \geq 1$.

(b) The fact that condition (B) also increases the domain of x_s in $\mathbb{R}^d$ is not accidental. This buffer around the original domain is convenient for Cauchy estimates and for avoiding problems with "over-shooting" the domain of x_s in the E variable.

(c) The particular choices of the $\exp(-N_s^\beta/2)$ cutoff in (D) and of the $-N_s^\mu/2$ cutoff in (E) are made out of technical convenience. Specifically, the first choice allows us to have Lemma 5.3 with a $\exp(-N_s^\beta)$ cutoff, and the second choice spares us one application of Cartan's estimate in Lemma 5.10.

(d) For the measure estimate from (D) to be possible we need that the intervals $h + [-N_s + n', N_s + n'']$ do not overlap the localization centre from (C). This is the reason for the choice of $\mathfrak{T}_s$.

(e) The reason for working with non-symmetric intervals $[-N_s', N_s'']$, as well as for the set being used in (D) is explained in Remark 5.12 below.

To simplify notation, the dependence of the constants in this section on the choice of the exponents δ, β, μ will be kept implicit as part of the dependence on the parameters a, b of the Diophantine condition.

Theorem D *Assume the notation of the inductive conditions. Let $E_0 \in \mathbb{R}$, and assume $L(E) > \gamma > 0$ for $E \in (E_0 - 2r_0, E_0 + 2r_0)$. Let $N_0 \geq 1$, $N_s = \lfloor N_{s-1}^A \rfloor$, $A = \beta^{-1}$, $s \geq 1$. If $N_0 \geq (B_0 + S_V + \gamma^{-1})^C$, $C = C(a, b, \rho)$, and conditions (A)–(E) hold with $s = 0$, then for any $s \geq 1$ and $E_s \in (E_{s-1} - r_{s-1}, E_{s-1} + r_{s-1})$ the conditions (A)–(E) also hold with $\mathcal{I}_s \Subset \mathcal{I}_{s-1}$. Furthermore, for any $(\phi, E) \in \Pi_s$,*

$$|x_s(\phi, E) - x_{s-1}(\phi, E)| < \exp(-\gamma N_{s-1}/30), \tag{5.8}$$

$$\|\psi^{[-N_s', N_s'']}(x_s(\phi, E), \cdot) - \psi^{[-N_{s-1}', N_{s-1}'']}(x_{s-1}(\phi, E), \cdot)\| < \exp(-\gamma N_{s-1}/40). \tag{5.9}$$

Remark 5.2 Theorem D also holds with any $A \geq \beta^{-1}$, but the relations (5.1) would need to be adjusted. The reason for needing $A \geq \beta^{-1}$ will become clear at the end of the proof of Proposition 5.11 below (see Remark 5.12).

We split the proof into several auxiliary statements. Ultimately the theorem will follow by referring to these statements. We will check the theorem for the case $s = 1$. The inductive conditions and the auxiliary statements are designed so that the general inductive step follows from this particular one by simply changing indices. In what follows we fix E_0, N_0, such that the assumptions of

Theorem D are satisfied. We also fix $E_1 \in (E_0 - r_0, E_0 + r_0)$ and let N_1, A be as in the statement.

For simplicity, in all of the following statements we assume tacitly that N_0 is large enough. More precisely we assume $N_0 \geq (B_0 + S_V + \gamma^{-1})^C$, with $C = C(a, b, \rho)$ large enough. In particular this allows us to invoke any of the results from Sect. 2. It will be clear from the proofs that any further largeness constraints on N_0 can be accounted for by increasing C. Of course, it is then important that we only have finitely many additional constraints. To this end we note that the additional constraints are independent of s.

Our first goal is to identify $[-N_1', N_1'']$ and $E_{k_1}^{[-N_1', N_1'']}$. In what follows we let $\mathcal{B}_{0,E,h}$ be the set from the measure estimate in condition (D), with $s = 0$.

Lemma 5.3 *Let h as in (D), with $s = 0$. Set*

$$\mathcal{B}'_{0,E,h} = \Big\{\phi \in \mathcal{I}_0 : \max_{|n'|,|n''|<N_0^{1/2}} dist(\operatorname{spec} H_{[-N_0+n', N_0+n'']}(x_0(\phi, E) + h), E) < \exp(-N_0^{\beta})\Big\}.$$

Then for any $E \in (E_0 - r_0, E_0 + r_0)$, the set $\mathcal{B}'_{0,E,h}$ is contained in a semialgebraic set of degree less than N_0^{20} and with measure less than $\exp(-N_0^{2\delta})$.

Proof Fix $E \in (E_0 - r_0, E_0 + r_0)$. By truncating the Taylor series of $x_0(\cdot, E)$ we obtain a polynomial $\tilde{x}_0(\cdot, E)$ of degree less than $C(d)N_0^4$ such that

$$\sup_{\phi \in \mathcal{I}_0} |x_0(\phi, E) - \tilde{x}_0(\phi, E)| \leq \exp(-N_0^2)$$

To estimate the remainder of the Taylor series we used condition (B) and Cauchy estimates (also recall Remark 5.1 (a)). Note that for any $[a, b] \subset \mathbb{Z}$, $\phi \in \mathcal{I}_0$,

$$\begin{aligned}\left\| H_{[a,b]}(x_0(\phi, E)) - H_{[a,b]}(\tilde{x}_0(\phi, E))\right\| &\leq C_\rho \|V\|_\infty |x_0(\phi, E) - \tilde{x}_0(\phi, E)| \\ &\leq \exp(-N_0^2/2).\end{aligned}$$

Let $\tilde{V}$, $\tilde{H}$ be as in (2.28), (2.29) (with N_0 instead of N). We have

$$\left\| H_{[a,b]}(x_0(\phi, E)) - \tilde{H}_{[a,b]}(\tilde{x}_0(\phi, E))\right\| \leq \exp(-N_0^2/4) \tag{5.10}$$

for any $[a, b] \subset \mathbb{Z}$. Let

$$\tilde{\mathcal{B}}_{0,E,h} = \Big\{\phi \in \mathcal{I}_0 : \max_{|n'|,|n''|<N_0^{\frac{1}{2}}} \left\|(\tilde{H}_{[-N_0+n', N_0+n'']}(\tilde{x}_0(\phi, E) + h) - E)^{-1}\right\|_{\mathrm{HS}} > \exp(-3N_0^{\beta}/4)\Big\},$$

where $\|\cdot\|_{\rm HS}$ stands for the Hilbert-Schmidt norm. Then $\tilde{\mathcal{B}}_{0,E,h}$ is semialgebraic of degree less than N_0^{20} and using (5.10) we have

$$\mathcal{B}'_{0,E,h} \subset \tilde{\mathcal{B}}_{0,E,h} \subset \mathcal{B}_{0,E,h},$$

thus concluding the proof. □

Lemma 5.4 *For any $E \in (E_0 - r_0, E_0 + r_0)$ there exists a semialgebraic set $\mathcal{B}_{0,E,N_1}$,*

$$\deg(\mathcal{B}_{0,E,N_1}) \lesssim N_1 N_0^{20}, \quad \operatorname{mes}(\mathcal{B}_{0,E,N_1}) < \exp(-N_0^{2\delta}/2),$$

such that for any $\phi \in \mathcal{I}_0 \setminus \mathcal{B}_{0,E,N_1}$ and any $3N_0/2 < |m| \le N_1$, there exist $|n'(\phi, m)|, |n''(\phi, m)| < N_0^{1/2}$ such that with $J_m = m + [-N_0 + n'(\phi, m), N_0 + n''(\phi, m)]$

$$dist(\operatorname{spec} H_{J_m}(x_0(\phi, E)), E) \ge \exp(-N_0^{\beta}).$$

Proof Take arbitrary $3N_0/2 < |m| \le N_1$. Then $0 < |m - n| < 3N_1$ for any $n \in \mathfrak{T}_0$ (recall (5.7)) and due to the Diophantine condition we have

$$\operatorname{dist}(m\omega, \mathfrak{T}_0) > a(3N_1)^{-b} \ge a(CN_0^{A})^{-b} > \exp(-N_0^{\mu}).$$

Hence, for any $3N_0/2 < |m| \le N_1$ condition (D) applies with $h = m\omega$. We let $\mathcal{B}_{0,E,N_1} := \bigcup_m \tilde{\mathcal{B}}_{0,E,m\omega}$, where $\tilde{\mathcal{B}}_{0,E,m\omega}$ are the semialgebraic sets from the statement of Lemma 5.3. Then $\mathcal{B}_{0,E,N_1}$ is semialgebraic of degree $\lesssim N_1 N_0^{20}$ and we have

$$\operatorname{mes}(\mathcal{B}_{0,E,N_1}) \lesssim N_1 \exp(-N_0^{2\delta}) < \exp\left(-\frac{1}{2} N_0^{2\delta}\right).$$

Take $\phi \in \mathcal{I}_0 \setminus \mathcal{B}_{0,E,N_1}$. Since $\phi \in \mathcal{I}_0 \setminus \tilde{\mathcal{B}}_{0,E,m\omega}$, the conclusion follows from the definition of $\mathcal{B}_{0,E,m\omega}$ (recall (2.8)). □

The next lemma is not needed at the moment, but it motivates one of the choices we make in the statement of Proposition 5.6

Lemma 5.5 *(a) The function $E^{[-N'_0, N''_0]}$ is analytic on $\{z \in \mathbb{C}^d : |z - x_0(\phi, E)| < \exp(-2N_0^{\delta})\}$, for any $(\phi, E) \in \Pi_0$.*

(b) The function $E^{[-N'_0, N''_0]}(x_0(\phi, E))$ is analytic on

$$\mathcal{P}'_0 = \{(\phi, E) \in \mathbb{C}^d : dist((\phi, E), \Pi_0) < r_0^4\}.$$

Proof Statement (a) follows from the separation of eigenvalues in (A) and basic perturbation theory. Statement (b) follows from (a) by noticing that

$$|x_0(\phi+\zeta, E+\eta) - x_0(\phi, E)| \le C_\rho \exp(N_0^\delta)(|\zeta|+|\eta|) < \exp(-2N_0^\delta)$$

for any $(\zeta, \eta) \in \mathbb{C}^d$ with $|\zeta|, |\eta| < \exp(-4N_0^\delta)$ (we used (B) and Cauchy estimates). □

Proposition 5.6 *There exists $\phi_1 \in \mathbb{T}^d$, $|\phi_1 - \phi_0| \ll r_0^4$, and $|N_1' - N_1|, |N_1'' - N_1| \lesssim N_0$ such that the following hold.*

(i) $\mathcal{I}_1' \subset \mathcal{I}_0 \backslash \mathcal{B}_{0,E_1,N_1}$, $\mathcal{I}_1' = \phi_1 + (-r_1', r_1')^{d-1}$, $r_1' = \exp(-3N_0^\beta)$, with $\mathcal{B}_{0,E_1,N_1}$ as in Lemma 5.4.
(ii) There exists k_1 such that for any $\phi \in \mathcal{I}_1'$, $y \in \mathbb{R}^d$, $|y| < r_1'$, $E \in \mathbb{R}$, $|E - E_1| < r_1'$,

$$\left|E_{k_1}^{[-N_1',N_1'']}(x_0(\phi,E)+y) - E^{[-N_0',N_0'']}(x_0(\phi,E)+y)\right| < \exp(-\gamma N_0/20), \tag{5.11}$$

$$\left|E_{j}^{[-N_1',N_1'']}(x_0(\phi,E)+y) - E_{k_1}^{[-N_1',N_1'']}(x_0(\phi,E)+y)\right| > \frac{1}{8}\exp(-N_0^\beta), \quad j \neq k_1, \tag{5.12}$$

$$|\psi_{k_1}^{[-N_1',N_1'']}(x_0(\phi,E)+y,n)| < \exp(-\gamma|n|/10), \quad |n| \ge 3N_0/4, \tag{5.13}$$

$$\|\psi_{k_1}^{[-N_1',N_1'']}(x_0(\phi,E)+y,\cdot) - \psi^{[-N_0',N_0'']}(x_0(\phi,E)+y,\cdot)\| < \exp(-\gamma N_0/20). \tag{5.14}$$

Proof Using the information we have on $\mathcal{B}_{0,E_1,N_1}$ and Lemma 2.23, it follows that there exists ϕ_1, $|\phi_1 - \phi_0| \ll r_0^4$ (in fact, we could replace r_0^4 by r_0^C, with any fixed $C \ge 1$), such that $\mathcal{I}_1' \subset \mathcal{I}_0 \backslash \mathcal{B}_{0,E_1,N_1}$ (recall that $\beta \gg \delta$). Take the intervals $J_m = m + [-N_0 + n'(\phi_1, m), N_0 + n''(\phi_1, m)]$ from Lemma 5.4. Define

$$[-N_1', N_1''] = [-3N_0/2, 3N_0/2] \cup \bigcup_{3N_0/2 < |m| \le N_1} J_m. \tag{5.15}$$

Due to Lemma 5.4,

$$\mathrm{dist}(\mathrm{spec}\, H_{J_m}(x_0(\phi_1, E_1)), E_1) \ge \exp(-N_0^\beta).$$

Using condition (B) and Cauchy estimates we have that for $\phi \in \mathcal{I}_1'$, $|y| < \exp(-3N_0^\beta)$, $|E - E_1| < \exp(-3N_0^\beta)$,

$$|x_0(\phi, E) + y - x_0(\phi_1, E_1)| \le \exp(CN_0^{\delta})(|\phi - \phi_1| + |E - E_1|) + |y| < \exp(-2N_0^{\beta}).$$

The conclusion follows by invoking Proposition 2.22 (recall that $\beta \ll \sigma$) with $x_0 = x_0(\phi, E_1)$, $E_0 = E_1$. □

For the rest of this section we adopt the notation of Proposition 5.6. To simplify the notation, we suppress k_1 from the notation and use $E^{[-N_1', N_1'']}$, $\psi^{[-N_1', N_1'']}$ instead. Next we want to prove the existence of the parametrization x_1.

Lemma 5.7 *(a) The function $E^{[-N_1', N_1'']}$ is analytic on $\{z \in \mathbb{C}^d : |z - x_0(\phi, E)| < \exp(-2N_0^{\beta})\}$, for any $(\phi, E) \in \Pi_1'$.*

(b) The function $E^{[-N_1', N_1'']}(x_0(\phi, E))$ is analytic on

$$\mathcal{P}_1' = \{(\phi, E) \in \mathbb{C}^d : dist((\phi, E), \Pi_1') < r_1'\},$$

with $\Pi_1' = \mathcal{I}_1' \times (E_1 - r_1', E_1 + r_1')$. Furthermore, for any $(\phi, E) \in \frac{1}{50}\mathcal{P}_1'$,

$$\left|E^{[-N_1', N_1'']}(x_0(\phi, E)) - E\right| < \exp(-c_0 \gamma N_0), \tag{5.16}$$

$$\left|\partial_E E^{[-N_1', N_1'']}(x_0(\phi, E)) - 1\right| < \exp(-c_0 \gamma N_0/2). \tag{5.17}$$

with $c_0 = c_0(d)$.

Proof The analyticity statements follow as in Lemma 5.5. By Proposition 5.6, the estimate (5.16) holds for real $(\phi, E) \in \frac{1}{2}\mathcal{P}_1' \cap \mathbb{R}^d$ with $c_0 = 1/20$ (recall (5.2)). With the help of Corollary 2.12 one concludes that the estimate is also valid for complex ϕ, E, with some $c_0(d) < 1/20$. The estimate (5.17) follows from Cauchy estimates combined with (5.16). □

Proposition 5.8 *Let*

$$\mathcal{P}_1'' = \{(\phi, E) \in \mathbb{C}^d : |\phi - \phi_1|, |E - E_1| < \exp(-C_0 N_0^{\beta})\},$$

with $C_0 = C_0(d) \gg 1$. There exists a map $x_1 : \Pi_1'' \to \mathbb{R}^d$, $\Pi_1'' := \mathcal{P}_1'' \cap \mathbb{R}^d$, that extends analytically on $\mathcal{P}_1''$, such that

$$E^{[-N_1', N_1'']}(x_1(\phi, E)) = E, \quad (\phi, E) \in \mathcal{P}_1'', \tag{5.18}$$

$$x_1(\mathcal{P}_1'') \subset \mathbb{T}^d_{\rho/2}. \tag{5.19}$$

Furthermore, for any $(\phi, E) \in \Pi_1''$,

$$|x_1(\phi, E) - x_0(\phi, E)| < \exp(-\gamma N_0/30). \tag{5.20}$$

and for any $(\phi, E) \in \mathcal{P}_1''$,

$$|x_1(\phi, E) - x_0(\phi, E)| < \exp(-c_0\gamma N_0), \quad c_0 = c_0(d).$$

Proof By Proposition 5.6 one has

$$\left|E^{[-N_1', N_1'']}(x_0(\phi, E)) - E\right| < \exp(-\gamma N_0/20) \tag{5.21}$$

for any $\phi \in \mathcal{I}_1'$ and any real $|E - E_1| < \exp(-3N_0^{\beta})$. Given real $|E - E_1| < \frac{1}{2}\exp(-3N_0^{\beta})$, set $E_{\pm} = E \pm 2\exp(-\gamma N_0/20)$. Since $|E_{\pm} - E_1| < \exp(-3N_0^{\beta})$, using (5.21) we have

$$E^{[-N_1', N_1'']}(x_0(\phi, E_-)) < E < E^{[-N_1', N_1'']}(x_0(\phi, E_+)).$$

It follows that

$$E^{[-N_1', N_1'']}(x_0(\phi, \eta)) = E \tag{5.22}$$

has a solution $\eta \in (E_-, E_+)$. Let η_1 be the solution corresponding to $\phi = \phi_1$, $E = E_1$. Recall that due to (5.17) in Lemma 5.7 one has

$$\partial_\eta E^{[-N_1', N_1'']}(x_0(\phi, \eta)) \geq 1/2.$$

Therefore, due to the implicit function theorem for analytic functions, see Lemma 4.2, for

$$|\phi - \phi_1|, |E - E_1| < \exp(-2CN_0^{\beta}), \ C = C(d) > 3,$$

there exists a unique analytic solution $\eta(\phi, E)$, $|\eta(\phi, E) - \eta_1| < \exp(-CN_0^{\beta})$ of (5.22). Then (5.18) and (5.19) hold by setting $x_1(\phi, E) = x_0(\phi, \eta(\phi, E))$. By uniqueness, for real ϕ, E, $\eta(\phi, E) \in (E_-, E_+)$, and therefore

$$|\eta(\phi, E) - E| < 2\exp(-\gamma N_0/20)$$

and (5.20) follows. The last estimate is a consequence of Corollary 2.12 (note that we take $C_0 < 2C$). □

Corollary 5.9 *Using the notation of Proposition 5.8, for any* $(\phi, E) \in \Pi_1''$,

$$\left|E - E_j^{[-N_1', N_1'']}(x_1(\phi, E))\right| > \frac{1}{8}\exp(-N_0^{\beta}) > \exp(-N_1^{\delta}), \quad j \neq k_1, \tag{5.23}$$

$$|\psi^{[-N_1', N_1'']}(x_1(\phi, E), n)| < \exp(-\gamma|n|/10), \quad |n| \geq 3N_0/4, \tag{5.24}$$

$$\|\psi^{[-N_1',N_1'']}(x_1(\phi,E),\cdot)-\psi^{[-N_0',N_0'']}(x_1(\phi,E),\cdot)\|<\exp(-\gamma N_0/20), \quad (5.25)$$
$$\|\psi^{[-N_1',N_1'']}(x_1(\phi,E),\cdot)-\psi^{[-N_0',N_0'']}(x_0(\phi,E),\cdot)\|<\exp(-\gamma N_0/40). \quad (5.26)$$

Proof All statements, except the last one follow from (5.20) and Proposition 5.6 with $y = x_1(\phi, E)-x_0(\phi, E)$. In the first estimate we used $N_1 \simeq N_0^{\beta^{-1}}$ and $\beta^2 \ll \delta$. The last estimate follows from

$$\begin{aligned}
&\left\|(H_{[-N_0',N_0'']}(x_0(\phi,E)) - E^{[-N_1',N_1'']}(x_1(\phi,E)))\psi^{[-N_1',N_1'']}(x_1(\phi,E))\right\| \\
&\le \left\|(H_{[-N_0',N_0'']}(x_0(\phi,E)) - H_{[-N_0',N_0'']}(x_1(\phi,E)))\psi^{[-N_1',N_1'']}(x_1(\phi,E))\right\| \\
&\quad + \left\|(H_{[-N_0',N_0'']}(x_1(\phi,E)) - E^{[-N_1',N_1'']}(x_1(\phi,E)))\psi^{[-N_1',N_1'']}(x_1(\phi,E))\right\| \\
&\le C_\rho \|V\|_\infty |x_0(\phi,E) - x_1(\phi,E)| + 2\exp(-\gamma(N_0 - N_0^{1/2}/10)) \\
&< \exp(-\gamma N_0/35),
\end{aligned}$$

the separation of eigenvalues, and Lemma 2.21. □

Next we check condition (D) with $s = 1$. Let

$$\mathcal{I}_0' = \{\phi \in \mathbb{R}^{d-1} : |\phi - \phi_0| < r_0^4\}$$

(recall Lemma 5.5).

Lemma 5.10 *Let $h \in \mathbb{R}^d$, $\exp(-N_1^\mu) \le \|h\| < \exp(-N_0^\mu)$, and $E \in (E_1 - r_1, E_1 + r_1)$. Then for any $\nu > 0$,*

$$\begin{aligned}
&\mathrm{mes}\{\phi \in \mathcal{I}_0'/10 : \log\left|E^{[-N_0',N_0'']}(x_0(\phi,E)+h) - E\right| \le -N_1^{\mu+\nu}\} \\
&\quad < \exp(-c(d)(N_0^\delta + N_1^{\nu/(d-1)})).
\end{aligned}$$

Proof By Taylor's formula,

$$\begin{aligned}
&E^{[-N_0',N_0'']}(x_0(\phi,E)+h) - E^{[-N_0',N_0'']}(x_0(\phi,E)) = \langle \nabla E^{[-N_0',N_0'']}(x_0(\phi,E),h\rangle \\
&\quad + O(\exp(CN_0^\delta)\,\|h\|^2).
\end{aligned} \quad (5.27)$$

We used the fact that by Cauchy estimates (recall Lemma 5.5),

$$\left|\frac{d^2}{dh^2}E^{[-N_0',N_0'']}(x_0(\phi,E)+h)\right| \le \exp(CN_0^\delta).$$

Due to condition (E) we can find $|\hat\phi_0 - \phi_0| \ll r_0^4$ such that

$$|\langle \nabla E^{[-N_0',N_0'']}(x_0(\hat\phi_0,E_1)),h_0\rangle| \ge \exp(-N_0^\mu/2), \qquad h_0 := \frac{h}{\|h\|}.$$

Since

$$|\nabla E^{[-N_0',N_0'']}(x_0(\hat{\phi}_0, E_1)) - \nabla E^{[-N_0',N_0'']}(x_0(\hat{\phi}_0, E))| \\ \leq \exp(CN_0^\delta)|x_0(\hat{\phi}_0, E_1) - x_0(\hat{\phi}_0, E)| \leq \exp(C'N_0^\delta)|E - E_1|,$$

we have

$$|\langle \nabla E^{[-N_0',N_0'']}(x_0(\hat{\phi}_0, E)), h_0\rangle| \gtrsim \exp(-N_0^\mu/2),$$

for any $E \in (E_1 - r_1, E_1 + r_1)$ (note that $N_1^\delta \gg N_0^\mu$; recall that $\delta \gg \beta^2 \gg \beta\mu$).

Plugging the above in (5.27),

$$\left|E^{[-N_0',N_0'']}(x_0(\hat{\phi}_0, E) + h) - E\right| \gtrsim \|h\| \exp(-N_0^\mu/2) \geq \exp(-2N_1^\mu)$$

(we used $\exp(CN_0^\delta)\,\|h\| \leq \exp(CN_0^\delta - N_0^\mu) \ll \exp(-N_0^\mu/2)$). The conclusion follows by applying Cartan's estimate to $E^{[-N_0',N_0'']}(x_0(\phi, E) + h) - E$ on the polydisk $|\phi - \hat{\phi}_0| < r_0^4$, with $H = c\exp(N_1^\nu)$, $c \ll 1$. □

Proposition 5.11 *Let* $h \in \mathbb{T}^d$ *such that* $dist(h, \mathfrak{T}_1) \geq \exp(-N_1^\mu)$ *(recall (5.7)) and*

$$\mathcal{B}_{1,E,h}'' = \Big\{\phi \in \mathcal{I}_1'' : \max_{|n'|,|n''|<N_1^{1/2}} dist(\text{spec}\, H_{[-N_1+n',N_1+n'']}(x_1(\phi, E) + h), E) \\ < \exp(-N_1^\beta/2)\Big\}.$$

Then for any $E \in (E_1 - r_1, E_1 + r_1)$*,* $\text{mes}(\mathcal{B}_{1,E,h}'') < \exp(-N_1^{2\delta})$.

Proof Let $|m_1| \leq 3N_1/2$, $h_1 \in \mathbb{R}^d$ such that

$$\text{dist}(h, \mathfrak{T}_1) = \|h_1\|, \qquad h_1 = h - m_1\omega \,(\text{mod}\, \mathbb{Z}^d).$$

Note that for any $m_1 \in [-N_1, N_1]$ we have

$$\text{dist}(h + m\omega, \mathfrak{T}_0) = \text{dist}(h, -m\omega + \mathfrak{T}_0) \geq \text{dist}(h, \mathfrak{T}_1) = \|h_1\|\,, \tag{5.28}$$

since $-m + [-3N_0/2, 3N_0/2] \subset [-3N_1/2, 3N_1/2]$. At the same time, if $|m + m_1| > 3N_0/2$, using the Diophantine condition we get

$$\|h + m\omega - n\omega\| = \|h_1 + (m + m_1 - n)\omega\| \geq \|(m + m_1 - n)\omega\| - \|h_1\| \\ \geq a(CN_1)^{-b} - \|h_1\|\,, \tag{5.29}$$

for any $n \in \mathfrak{T}_0$.

We consider two cases: $\|h_1\| \geq \exp(-N_0^\mu)$ and $\|h_1\| < \exp(-N_0^\mu)$. In either case, by the above, we have $\text{dist}(h + m\omega, \mathfrak{T}_0) \geq \exp(-N_0^\mu)$ for all $m \in$

$[-N_1, N_1]$ with $|m+m_1| > 3N_0/2$. So, for such m, condition (D) implies that for each $\phi \in \mathcal{I}_0 \setminus \mathcal{B}_{0,E_1,h+m\omega}$ there exists $|n'(\phi, m, h)|, |n''(\phi, m, h)| < N_0^{1/2}$ such that with $J_m(\phi) = m + [-N_0 + n'(\phi, m, h), N_0 + n''(\phi, m, h)]$,

$$\operatorname{dist}(\operatorname{spec} H_{J_m(\phi)}(x_0(\phi, E_1) + h), E_1) \geq \exp(-N_0^\beta/2)$$

and therefore

$$\operatorname{dist}(\operatorname{spec} H_{J_m(\phi)}(x_0(\phi, E) + h), E) \geq \exp(-N_0^\beta/4) \geq \exp(-N_0^{\sigma/2})$$

for any $E \in (E_1 - r_1, E_1 + r_1)$ (note that $N_1^\delta \gg N_0^\beta$; recall that $\delta \gg \beta^2$). In particular, since $\operatorname{mes}(\mathcal{B}_{0,E_1,h+m\omega}) < \exp(-N_0^{2\delta})$, there exists $\phi_{0,m} \in \mathcal{I}_0 \setminus \mathcal{B}_{0,E_1,h+m\omega}$, $|\phi_{0,m} - \phi_0| \ll r_0^4$. Let $J_m := J_m(\phi_{0,m})$. Due to the spectral form of (LDT),

$$\log|f_{J_m}(x_0(\phi_{0,m}, E) + h), E)| > |J_m| L_{|J_m|}(E) - |J_m|^{1-\tau/2}.$$

Using the uniform upper estimate (see Corollary 2.7) we can apply Cartan's estimate to get

$$\begin{aligned} &\operatorname{mes}\{\phi \in \mathcal{I}_0'/10 : \log|f_{J_m}(x_0(\phi, E) + h, E)| < |J_m| L(E) - |J_m|^{1-\tau/4}\} \\ &\quad < \exp(-N_0^{\tau/8(d-1)}) \end{aligned} \tag{5.30}$$

(in fact, the estimate holds for $\phi \in \mathcal{I}_0/10$). Denote by $\mathcal{B}'_{0,E,m}$ the set in the above estimate and let

$$\mathcal{B}'_{0,E,N_1} = \bigcup_{-N_1 \leq m \leq N_1, |m+m_1| > 3N_0/2} \mathcal{B}'_{0,E,m}.$$

Since $\delta \ll \beta\sigma \ll \beta\tau$, we have

$$\operatorname{mes}(\mathcal{B}'_{0,E,N_1}) \lesssim N_1 \exp(-N_0^{\tau/8(d-1)}) < \exp(-N_0^{\tau/8(d-1)}/2) \ll \exp(-N_1^{2\delta}).$$

We now have to deal with $|m + m_1| \leq 3N_0/2$. It will be enough to focus on $m = -m_1$. We assume $m_1 \in [-N_1, N_1]$ so that (5.28) holds. If $\|h_1\| \geq \exp(-N_0^\mu)$, then by (5.28), $\operatorname{dist}(h + m_1\omega, \mathfrak{T}_0) \geq \exp(-N_0^\mu)$ and by the above reasoning there exists an interval J_{-m_1} such that (5.30) holds with $m = -m_1$. In this case we let $\mathcal{B}'_{0,E,-m_1}$ be the set from (5.30). Suppose that $\|h_1\| < \exp(-N_0^\mu)$. Let $J_{-m_1} := -m_1 + [-N_0', N_0'']$. We have

$$\operatorname{spec} H_{J_{-m_1}}(x + h) = \operatorname{spec} H_{[-N_0', N_0'']}(x + h_1).$$

Let

$$\mathcal{B}'_{0,E,-m_1} := \{\phi \in \mathcal{I}'_0/10 : \left|E^{[-N'_0,N''_0]}(x_0(\phi,E)+h_1)-E\right| \le \exp(-N_1^{\mu+\nu})\},$$

with $\nu = 3(d-1)\delta$. By Lemma 5.10,

$$\mathrm{mes}(\mathcal{B}'_{0,E,-m_1}) < \exp(-c(N_0^{\delta} + N_1^{\nu/(d-1)})) \ll \exp(-N_1^{2\delta}).$$

Since

$$\begin{aligned} E^{[-N'_0,N''_0]}(x_0(\phi,E)) &= E, \\ \left|E_j^{[-N'_0,N''_0]}(x_0(\phi,E)+h_1) - E_j^{[-N'_0,N''_0]}(x_0(\phi,E))\right| &\le C_\rho \|V\|_\infty \|h_1\| \\ &\ll \exp(-N_0^{\delta}), \end{aligned}$$

the separation of eigenvalues in condition (A) implies

$$\mathrm{dist}(\mathrm{spec}\, H_{J_{-m_1}}(x_0(\phi,E)+h), E) > \exp(-N_1^{\mu+\nu}),$$

for any $\phi \in \frac{1}{10}\mathcal{I}'_0 \backslash \mathcal{B}'_{0,E,-m_1}$. Note that $|J_{-m_1}|^{\sigma/2} \simeq N_0^{\sigma/2} \gg N_1^{\mu+\nu} \gg N_0^{\delta}$ since $\delta, \mu \ll \beta\sigma$. Therefore we can apply the spectral form of (LDT) to get

$$\log|f_{J_{-m_1}}(x_0(\phi,E)+h)| > |J_{-m_1}|L(E) - |J_{-m_1}|^{1-\tau/2},$$

for $\phi \in \frac{1}{10}\mathcal{I}'_0 \backslash \mathcal{B}'_{0,E,-m_1}$. So, in either case we identified an interval J_{-m_1} and got a similar conclusion.

Let

$$I := \begin{cases} J_{-m_1} \cup \left(\bigcup_{-N_1 \le m \le N_1, |m+m_1| > 3N_0/2} J_m\right), & m_1 \in [-N_1, N_1] \\ \bigcup_{-N_1+2N_0 \le m \le N_1-2N_0} J_m, & m_1 \notin [-N_1, N_1]. \end{cases} \tag{5.31}$$

Note that J_{-m_1} overlaps with the union of the other intervals and $|m+m_1| > 3N_0/2$ for all m's in the last union. By the above, we can use the covering form of (LDT) from Lemma 2.15 to get that

$$\begin{aligned} \mathrm{dist}(\mathrm{spec}\, H_I(x_0(\phi,E))+h), E) &\ge \exp(-2\max_m |J_m|^{1-\tau/4}) \\ &> \exp(-4N_0^{1-\tau/4}), \end{aligned}$$

for any $\phi \in \frac{1}{10}\mathcal{I}'_0 \backslash (\mathcal{B}'_{0,E,N_1} \cup \mathcal{B}'_{0,E,-m_1})$. Due to (5.20),

$$\mathrm{dist}(\mathrm{spec}\, H_I(x_1(\phi,E))+h), E) \gtrsim \exp(-4N_0^{1-\tau/4}) \gg \exp(-N_1^{\beta}/2), \tag{5.32}$$

for $\phi \in \mathcal{I}_1''\backslash(\mathcal{B}'_{0,E,N_1} \cup \mathcal{B}'_{0,E,-m_1})$. Therefore $\mathcal{B}''_{1,E,h} \subset \mathcal{B}'_{0,E,N_1} \cup \mathcal{B}'_{0,E,-m_1}$ and the conclusion holds. □

Remark 5.12 (a) Taking the maximum in the definition of the set $\mathcal{B}_{0,E,h}$ from condition (D) is a convenient way of capturing the fact that while we do not know precisely the interval I for which (5.32) holds, we do know that it is "close" to $[-N_1, N_1]$.

(b) If in the definition of $\mathcal{B}_{0,E,h}$ we would use symmetric intervals, then we could also choose I to be symmetric. However, even so, $[-N_1', N_1'']$ need not be symmetric because we don't have enough control over the sizes of the intervals J_m in (5.15) (for example we cannot say that J_m and J_{-m} have the same size).

(c) The reason for wanting $A \geq \beta^{-1}$, as noted in Remark 5.2, is the estimate (5.32).

Now we just need to check condition (E) with $s = 1$.

Lemma 5.13 *Let $h_0 \in \mathbb{R}^d$ be a unit vector. Then*

$$\left|\nabla E^{[-N_1',N_1'']}(x_1(\phi, E)) - \nabla E^{[-N_0',N_0'']}(x_0(\phi, E))\right| < \exp(-c_0\gamma N_0),$$
$$c_0 = c_0(d),$$

for any $(\phi, E) \in \Pi_1''$.

Proof Using (5.20), we have

$$\left|\nabla E^{[-N_0',N_0'']}(x_1(\phi, E)) - \nabla E^{[-N_0',N_0'']}(x_0(\phi, E))\right|$$
$$\leq \exp(CN_0^\delta)|x_1(\phi, E) - x_0(\phi, E)| < \exp(-\gamma N_0/35).$$

On the other hand, using (5.11), (5.20), Corollary 2.12, and Cauchy estimates, we have

$$\left|\nabla E^{[-N_1',N_1'']}(x_1(\phi, E)) - \nabla E^{[-N_0',N_0'']}(x_1(\phi, E))\right| \leq \exp(-c(d)\gamma N_0),$$

and the conclusion follows. □

Proposition 5.14 *Let $h_0 \in \mathbb{R}^d$ be a unit vector. Then for any $E \in (E_1 - r_1, E_1 + r_1)$,*

$$\mathrm{mes}\{\phi \in \mathcal{I}_1'' : \log|\langle\nabla E^{[-N_1',N_1'']}(x_1(\phi, E)), h_0\rangle| < -N_1^\mu/2\} < \exp(-N_1^{2\delta}).$$

Proof Due to condition (E) we can find $\hat{\phi}_0$, $|\hat{\phi}_0 - \phi_0| \ll r_0^4$, such that

$$\log|\langle\nabla E^{[-N_0',N_0'']}(x_0(\hat{\phi}_0, E)), h_0\rangle| \geq -N_0^\mu/2.$$

Applying Cartan's estimate we get

$$\begin{aligned}\operatorname{mes}\{\phi \in \mathcal{I}_0'/10 : \log|\langle \nabla E^{[-N_0',N_0'']}(x_0(\phi,E)), h_0\rangle| < -N_0^{\mu+\nu}\}\\ < \exp(-c(d)(N_0^{\delta} + N_0^{\nu/(d-1)})) < \exp(-N_1^{2\delta}),\end{aligned}$$

where $\nu = 3(d-1)\beta^{-1}\delta$. Let $\mathcal{B}$ be the set on the left-hand side. Note that $\mathcal{I}_1'' \subset \mathcal{I}_0'/10$, since $|\phi_1 - \phi_0| \ll r_0^4$. Since $\mu + \nu \ll 1$, Lemma 5.13 implies

$$\log|\langle \nabla E^{[-N_1',N_1'']}(x_1(\phi,E)), h_0\rangle| \geq -2N_0^{\mu+\nu} \geq -N_1^{\mu}/2,$$

for any $\phi \in \mathcal{I}_1''\backslash\mathcal{B}$ (recall that $\delta \ll \mu$). This concludes the proof. □

We briefly summarize how Theorem D follows from the previous statements.

Proof of Theorem D The existence of ϕ_1 was obtained in Proposition 5.6. Note that since $\delta \gg \beta^2$, we have $\mathcal{P}_1 \Subset \mathcal{P}_1''$. Conditions (A)–(C), and the estimates (5.8), (5.9), follow from Proposition 5.8 and Corollary 5.9. Condition (D) follows from Proposition 5.11. Condition (E) follows from Proposition 5.14. □

6 Inductive scheme for the edges of the spectrum

As in the previous section we assume the non-perturbative setting from Sect. 2. We introduce another set of conditions that will address the edges of the spectrum.

We assume the exponents δ, μ, β from the previous section and we introduce a new exponent $\mathfrak{d}$ such that $\mathfrak{d} \ll \delta$. Let $\gamma > 0$. Given an integer $s \geq 0$, let

$$\underline{x}_s \in \mathbb{T}^d, \quad N_s \in \mathbb{N}, \quad r_s := \exp(-N_s^{\mathfrak{d}}), \quad \underline{\Pi}_s = \{x \in \mathbb{R}^d : |x - \underline{x}_s| < r_s\}.$$

The inductive conditions for the lower edge are as follows.
(**A̲**) There exist integers $|N_s' - N_s|, |N_s'' - N_s| < N_s^{1/2}$, and $\underline{E}^{[-N_s',N_s'']} = E_{k_s}^{[-N_s',N_s'']}$, such that

$$E_j^{[-N_s',N_s'']}(x) - \underline{E}^{[-N_s',N_s'']}(x) \geq \exp(-N_s^{\mathfrak{d}}), \tag{6.1}$$

for any $x \in \underline{\Pi}_s$ and $j \neq k_s$.
(**B̲**) For any $x \in \underline{\Pi}_s$,

$$|\underline{\psi}^{[-N_s',N_s'']}(x,n)| \leq \exp(-\gamma|n|/10), \quad |n| \geq N_s/4.$$

($\underline{\mathrm{C}}$) The point $\underline{x}_s$, is a non-degenerate minimum of the function $\underline{E}^{[-N_s', N_s'']}$. Specifically, with $\nu_s = \exp(-N_s^{\mathfrak{d}})$,

$$\nabla \underline{E}^{[-N_s', N_s'']}(\underline{x}_s) = 0, \qquad \mathfrak{H}(\underline{E}^{[-N_s', N_s'']})(\underline{x}_s) \geq \nu_s I.$$

($\underline{\mathrm{D}}$) Let $\underline{E}_s = \underline{E}^{[-N_s', N_s'']}(\underline{x}_s)$. Let $\mathfrak{T}_s$ be as in (5.7). Take arbitrary $h \in \mathbb{T}^d$ with

$$\mathrm{dist}(h, \mathfrak{T}_s) \geq \exp(-N_s^{2\mathfrak{d}}).$$

There exist $|n'(h)|, |n''(h)| < N_s^{1/2}$ such that

$$\mathrm{dist}(\mathrm{spec}\, H_{[-N_s+n'(h), N_s+n''(h)]}(\underline{x}_s + h), (-\infty, \underline{E}_s]) \geq \exp(-N_s^{4\mathfrak{d}}).$$

The conditions ($\overline{\mathrm{A}}$), ($\overline{\mathrm{B}}$), ($\overline{\mathrm{C}}$), ($\overline{\mathrm{D}}$), for the upper edge are defined analogously, with obvious adjustments in notation.

Theorem E *Assume the notation of the inductive conditions. Let $\underline{x}_0 \in \mathbb{T}^d$, $N_0 \geq 1$, assume that the conditions ($\underline{\mathrm{A}}$)–($\underline{\mathrm{D}}$) hold with $s = 0$, and $L(E) > \gamma > 0$ for $E \in (\underline{E}_0 - 2r_0, \underline{E}_0 + 2r_0)$. Let $N_s = N_{s-1}^5$, $s \geq 1$. If $N_0 \geq (B_0 + S_V + \gamma^{-1})^C$, $C = C(a, b, \rho)$, then for any $s \geq 1$ there exists $\underline{x}_s \in \mathbb{T}^d$ such that the conditions ($\underline{\mathrm{A}}$)–($\underline{\mathrm{D}}$) hold and we have*

$$\begin{aligned} &|\underline{E}^{[-N_s', N_s'']}(x) - \underline{E}^{[-N_{s-1}', N_{s-1}'']}(x)| < \exp(-\gamma N_{s-1}/20), \quad x \in \underline{\Pi}_s, \\ &\left\| \underline{\psi}^{[-N_s', N_s'']}(x) - \underline{\psi}^{[-N_{s-1}', N_{s-1}'']}(x) \right\| < \exp(-\gamma N_{s-1}/20), \quad x \in \underline{\Pi}_s, \\ &|\underline{x}_s - \underline{x}_{s-1}| < \exp(-\gamma N_{s-1}/50), \quad |\underline{E}_s - \underline{E}_{s-1}| < \exp(-\gamma N_{s-1}/60). \end{aligned} \tag{6.2}$$

Furthermore, for any $E_s \in \mathbb{R}$, $\exp(-N_s^{100\mathfrak{d}}) \leq |E_s - \underline{E}_s| \leq \exp(-N_s^{2\mathfrak{d}})$, conditions (A)–(E) hold for $\underline{E}^{[-N_s', N_s'']}$. The analogous statements based on conditions ($\overline{\mathrm{A}}$)–($\overline{\mathrm{D}}$) also hold.

As for Theorem D, we only check Theorem E for $s = 1$, the general case following by simply replacing the indices. Furthermore, we only consider the statement with the conditions for the lower edge, the other case being completely analogous. Throughout the section we tacitly assume that $N_0 \geq (B_0 + S_V + \gamma^{-1})^C$, with $C = C(a, b, \rho)$ large enough. As in the previous section, the dependence on the exponents $\mathfrak{d}, \delta, \beta, \mu$ is left implicit. We split the proof of the first part of Theorem E into several auxiliary statements. In what follows we fix $\underline{x}_0$, N_0, such that the assumptions of Theorem E hold, and $N_1 = N_0^5$.

Proposition 6.1 *There exist integers* $|N_1' - N_1|, |N_1'' - N_1| \lesssim N_0$, k_1, *such that the following hold with* $\underline{E}^{[-N_1',N_1'']} = E_{k_1}^{[-N_1',N_1'']}$ *and for any* $|x - \underline{x}_0| < \exp(-2N_0^{4\mathfrak{d}})$*:*

$$\left|\underline{E}^{[-N_1',N_1'']}(x) - \underline{E}^{[-N_0',N_0'']}(x)\right| < \exp(-\gamma N_0/20), \tag{6.3}$$

$$E_j^{[-N_1',N_1'']}(x) - \underline{E}^{[-N_1',N_1'']}(x) > \frac{1}{8}\exp(-N_0^{4\mathfrak{d}}), \quad j \neq k_1, \tag{6.4}$$

$$|\underline{\psi}^{[-N_1',N_1'']}(x,n)| < \exp(-\gamma|n|/10), \quad |n| > 3N_0/4, \tag{6.5}$$

$$\|\underline{\psi}^{[-N_1',N_1'']}(x,\cdot) - \underline{\psi}^{[-N_0',N_0'']}(x,\cdot)\| < \exp(-\gamma N_0/20). \tag{6.6}$$

Proof Take arbitrary $3N_0/2 < |m| \le N_1$. Using the Diophantine condition we have

$$\operatorname{dist}(m\omega, \mathfrak{T}_0) \ge a(CN_0)^{-b} \ge \exp(-N_0^{2\mathfrak{d}}).$$

Then by condition (D) with $h = m\omega$, there exist $|n'(m)|, |n''(m)| < N_0^{1/2}$ such that with $J_m = m + [-N_0 + n'(m), N_0 + n''(m)]$,

$$\operatorname{dist}(\operatorname{spec} H_{J_m}(\underline{x}_0), (-\infty, \underline{E}_0]) \ge \exp(-N_0^{4\mathfrak{d}})$$

(recall (2.8)). Define

$$[-N_1', N_1''] = [-3N_0/2, 3N_0/2] \cup \bigcup_{3N_0/2 < |m| \le N_1} J_m.$$

Using (6.1) and (B) we can apply Proposition 2.22 (with $x_0 = \underline{x}_0$, $E_0 = \underline{E}_0$, $\beta = 4\mathfrak{d}$) and all the estimates follow. □

For the rest of this section $\underline{E}^{[-N_1',N_1'']}$ will stand for the eigenvalue from the previous proposition. Let

$$\mathcal{P}_0' = \{z \in \mathbb{C}^d : |z - \underline{x}_0| < r_0'\}, \quad r_0' = \exp(-2N_0^{\mathfrak{d}}),$$
$$\mathcal{P}_1' = \{z \in \mathbb{C}^d : |z - \underline{x}_0| < r_1'\}, \quad r_1' = \exp(-3N_0^{4\mathfrak{d}}).$$

Lemma 6.2 *The functions* $\underline{E}^{[-N_0',N_0'']}$, $\underline{E}^{[-N_1',N_1'']}$ *are analytic on* $\mathcal{P}_0'$, $\mathcal{P}_1'$, *respectively, and*

$$\max_{|\alpha|=k} \sup_{\mathcal{P}_0'} |\partial^\alpha \underline{E}^{[-N_0',N_0'']}| \le \exp(C(k)N_0^{\mathfrak{d}}), \quad \max_{|\alpha|=k} \sup_{\mathcal{P}_1'} |\partial^\alpha \underline{E}^{[-N_1',N_1'']}| \le \exp(C(k)N_0^{4\mathfrak{d}}).$$

Furthermore,

$$\sup_{\mathcal{P}_1'} \left\| \nabla \underline{E}^{[-N_1',N_1'']} - \nabla \underline{E}^{[-N_0',N_0'']} \right\|, \sup_{\mathcal{P}_1'} \left\| \mathfrak{H}(\underline{E}^{[-N_1',N_1'']}) - \mathfrak{H}(\underline{E}^{[-N_0',N_0'']}) \right\| < \exp(-c_0 \gamma N_0),$$

with $c_0 = c_0(d)$.

Proof The analyticity of the functions follows from the separation of eigenvalues (see (6.1) and (6.4)) combined with basic perturbation theory. The derivative estimates are just Cauchy estimates. They hold on $\mathcal{P}_i'$ because the functions are in fact analytic on $100\mathcal{P}_i'$, $i = 0, 1$.

Using (6.3) and Corollary 2.12 we have

$$\sup_{2\mathcal{P}_1'} |\underline{E}^{[-N_1',N_1'']} - \underline{E}^{[-N_0',N_0'']}| < \exp(-c\gamma N_0), \quad c = c(d),$$

and the last estimates holds by Cauchy estimates (we chose $r_1' = \exp(-3N_0^{4\mathfrak{d}})$ instead of $\exp(-2N_0^{4\mathfrak{d}})$ to ensure we have the above estimate). □

Proposition 6.3 *There exists* $\underline{x}_1$, $|\underline{x}_1 - \underline{x}_0| < \exp(-\gamma N_0/50)$, *such that*

$$\underline{E}^{[-N_1',N_1'']}(\underline{x}_1) \le \underline{E}^{[-N_1',N_1'']}(x), \quad \textit{for any } |x - \underline{x}_0| < r_1',$$
$$\nabla \underline{E}^{[-N_1',N_1'']}(\underline{x}_1) = 0, \quad \mathfrak{H}(\underline{E}^{[-N_1',N_1'']})(\underline{x}_1) \ge \frac{\nu_0}{4} I.$$

Proof By Taylor's formula (recall Lemma 4.10 and (C))

$$\underline{E}^{[-N_0',N_0'']}(x) - \underline{E}_0 \ge \frac{\nu_0}{2} |x - \underline{x}_0|^2, \quad \text{for } |x - \underline{x}_0| < r_1'.$$

In particular,

$$\underline{E}^{[-N_0',N_0'']}(x) \ge \underline{E}_0 + 3\exp(-\gamma N_0/20),$$
$$\text{for} \quad \exp(-\gamma N_0/50) \le |x - \underline{x}_0| < r_1'.$$

Combining this with (6.3) we get

$$\underline{E}^{[-N_1',N_1'']}(x) \ge \underline{E}^{[-N_1',N_1'']}(\underline{x}_0) + \exp(-\gamma N_0/20),$$
$$\text{for } \exp(-\gamma N_0/50) \le |x - \underline{x}_0| < r_1'.$$

This implies the existence of a point $\underline{x}_1$, $|\underline{x}_1 - \underline{x}_0|$ where $\underline{E}^{[-N_1',N_1'']}$ attains its minimum on $|x - \underline{x}_0| < r_1'$. The estimate on the Hessian follows from

Lemma 6.2 and the fact that by Taylor's formula (again, recall Lemma 4.10 and (C)), we have $\mathfrak{H}(\underline{E}^{[-N_0',N_0'']})(\underline{x}_1) \geq (\nu_0/2)I$. □

We fix an $\underline{x}_1$ as in Proposition 6.3 (in fact, in can be argued that such $\underline{x}_1$ is unique).

Lemma 6.4 *We have* $|\underline{E}_1 - \underline{E}_0| < \exp(-\gamma N_0/60)$.

Proof By the mean value theorem, Lemma 6.2, and Proposition 6.3,

$$|\underline{E}^{[-N_0',N_0'']}(\underline{x}_1) - \underline{E}^{[-N_0',N_0'']}(\underline{x}_0)| \leq \exp(CN_0^{\mathfrak{d}})|\underline{x}_1 - \underline{x}_0| < \exp(-\gamma N_0/55).$$

Now the conclusion follows using (6.3). □

Proposition 6.5 *The condition* (D) *holds with* $s = 1$.

Proof The proof is similar to that of Proposition 5.11. Let $|m_1| \leq 3N_1/2$, $h_1 \in \mathbb{R}^d$ such that

$$\mathrm{dist}(h, \mathfrak{T}_1) = \|h_1\|, \qquad h_1 = h - m_1\omega \,(\mathrm{mod}\, \mathbb{Z}^d).$$

As in the proof of Proposition 5.11 (recall (5.28),(5.29)), we have

$$\begin{aligned} \mathrm{dist}(h + m\omega, \mathfrak{T}_0) &\geq \|h_1\|\,, \quad \text{provided } |m| \leq N_1, \\ \mathrm{dist}(h + m\omega, \mathfrak{T}_0) &\geq a(CN_1)^{-b} - \|h_1\|\,, \quad \text{provided } |m + m_1| > 3N_0/2. \end{aligned} \tag{6.7}$$

We consider two cases: $\|h_1\| \geq \exp(-N_0^{2\mathfrak{d}})$ and $\exp(N_1^{2\mathfrak{d}}) \leq \|h_1\| < \exp(-N_0^{\mathfrak{d}})$. In either case, by the above, we have $\mathrm{dist}(h + m\omega, \mathfrak{T}_0) \geq \exp(-N_0^{2\mathfrak{d}})$ for all $m \in [-N_1, N_1]$ with $|m + m_1| > 3N_0/2$. So, for such m, condition (D) (with $h = m\omega$) implies that there exists an interval $J_m = m + [-N_0 + n'(h), N_0 + n''(h)]$ such that

$$\mathrm{dist}(\mathrm{spec}\, H_{J_m}(\underline{x}_0 + h), (-\infty, \underline{E}_0]) \geq \exp(-N_0^{4\mathfrak{d}}). \tag{6.8}$$

Our goal is to apply Lemma 2.17 (with $S = (-\infty, \underline{E}_0]$). To this end we will deal with $|m + m_1| \leq 3N_0/2$ by focusing on $m = -m_1$. We assume $m_1 \in [-N_1, N_1]$. If $\|h_1\| \geq \exp(-N_0^{2\mathfrak{d}})$, then $\mathrm{dist}(h + m_1\omega, \mathfrak{T}_0) \geq \exp(-N_0^{2\mathfrak{d}})$ and by condition (D) there exists an interval J_{-m_1} such that (6.8) holds with $m = -m_1$. Suppose that $\exp(-N_1^{2\mathfrak{d}}) \leq \|h_1\| < \exp(-N_0^{2\mathfrak{d}})$. Let $J_{-m_1} := -m_1 + [-N_0', N_0'']$. We have

$$\mathrm{spec}\, H_{J_{-m_1}}(\underline{x}_0 + h) = \mathrm{spec}\, H_{[-N_0',N_0'']}(\underline{x}_0 + h_1). \tag{6.9}$$

By Taylor's formula (recall Lemma 4.10 and (C)),

$$\underline{E}^{[-N_0',N_0'']}(\underline{x}_0+h_1) \geq \underline{E}_0 + \frac{\nu_0}{2}\|h_1\|^2 \geq \underline{E}_0 + \exp(-3N_1^{2\mathfrak{d}}).$$

Using (A) it follows that

$$\operatorname{dist}(\operatorname{spec} H_{J_{-m_1}}(\underline{x}_0+h), (-\infty, \underline{E}_0]) \geq \exp(-3N_1^{2\mathfrak{d}}) > \exp(-N_0^{11\mathfrak{d}}).$$

We now have what we need to invoke the covering form of (LDT). Let I as in (5.31). By the above, we can use Lemma 2.17 (with $K = N_0^{11\mathfrak{d}}$; recall that $\mathfrak{d} \ll \delta \ll \sigma$) to get that

$$\operatorname{dist}(\operatorname{spec} H_I(\underline{x}_0+h), (-\infty, \underline{E}_0]) \geq \exp(-N_0^{12\mathfrak{d}}) \gg \exp(-N_1^{4\mathfrak{d}}).$$

Using Proposition 6.3 and Lemma 6.4 we have

$$\operatorname{dist}(\operatorname{spec} H_I(\underline{x}_1+h), (-\infty, \underline{E}_1]) \geq \exp(-N_1^{4\mathfrak{d}})$$

and the conclusion follows. □

We now proceed to the proof of Theorem E.

Proof of Theorem E The existence of $\underline{x}_1$ and $\underline{E}^{[-N_1',N_1'']}$ is given by Proposition 6.1 and Proposition 6.3. Note that due to Proposition 6.3, $\underline{\Pi}_1 \subset \{|x-\underline{x}_0| < r_1'\}$ (recall that $r_1' = \exp(-3N_0^{4\mathfrak{d}})$, $N_1 = N_0^5$). Now, for $s=1$, conditions (A) and (B) hold by Proposition 6.1, condition (C) holds by Proposition 6.3, and condition (D) holds by Proposition 6.5. The estimates (6.2) (with $s=1$) hold by Proposition 6.1, Proposition 6.3, and Lemma 6.4.

Fix x, $\exp(-N_1^{200\mathfrak{d}}) \leq |x-\underline{x}_0| \leq \exp(-N_1^{\mathfrak{d}})$. We will check that the conditions (A)–(E), with $s=1$, hold for $\underline{E}^{[-N_1',N_1'']}$ with $E_1 = \underline{E}^{[-N_0',N_0'']}(x)$. The conclusion then holds by noticing that

$$\begin{aligned}
&\{\underline{E}^{[-N_0',N_0'']}(x) : \exp(-N_1^{200\mathfrak{d}}) \leq |x-\underline{x}_0| \leq \exp(-N_1^{\mathfrak{d}})\}\\
&\quad \supset [\underline{E}_0 + \exp(-N_1^{100\mathfrak{d}})/2, \underline{E}_0 + 2\exp(-N_1^{2\mathfrak{d}})]\\
&\quad \supset [\underline{E}_1 + \exp(-N_1^{100\mathfrak{d}}), \underline{E}_1 + \exp(-N_1^{2\mathfrak{d}})]
\end{aligned}$$

(recall Lemma 4.10 and Lemma 6.4).

We apply Proposition 4.1 to $\underline{E}^{[-N_0',N_0'']}$ on $\mathcal{P}_0'$. Using the notation of Proposition 4.1, condition (C), and Lemma 6.2, we have

$$\nu_1 \simeq \exp(-CN_0^{\mathfrak{d}}), \quad \rho = r_0'\nu_1^{10} \simeq \exp(-C'N_0^{\mathfrak{d}}), \quad r = \nu_1\left\|x-\underline{x}_0\right\|.$$

Since $0 < \|x - \underline{x}_0\| < \rho$, Proposition 4.1 applies with x in the role of x_0 and we get the following:

(1) There exists a map $x_0 : \Pi_0 \to \mathbb{R}^d$,

$$\Pi_0 = \mathcal{I}_0 \times (E_1 - r^2, E_1 + r^2), \quad \mathcal{I}_0 = (-r, r)^{d-1}, \quad E_1 = \underline{E}^{[-N_0', N_0'']}(x),$$

such that

$$\underline{E}^{[-N_0', N_0'']}(x_0(\phi, E)) = E,$$

$x_0(\phi, E)$ extends analytically to

$$\mathcal{P}_0 = \{(\phi, E) \in \mathbb{C}^d : |\phi| < r, \ |E - E_1| < r^2\},$$

and

$$\|x_0(\phi, E) - \underline{x}_0\| < \|x - \underline{x}_0\| /2 \lesssim \exp(-N_0^{5\mathfrak{d}}) \tag{6.10}$$

In particular, from the last estimate it follows that $x_0(\mathcal{P}_0) \subset \mathbb{T}^d_{\rho/2}$.

(2) For any $|E - E_1| < r^2$, any vector $h \in \mathbb{R}^d$ with $0 < \|h\| < \rho$, and any $H \gg 1$, we have

$$\mathrm{mes}\{\phi \in \mathcal{I}_0 : \log |\underline{E}^{[-N_0', N_0'']}(x_0(\phi, E)) - E| \le H_0 H\} < \exp(-H^{1/(d-1)}), \tag{6.11}$$

with $H_0 = C(d) \log(\|h\| \|x - \underline{x}_0\|)$ (note that $\nu_1^{-2} r = \nu_1^{-1} \|x - \underline{x}_0\| < \nu_1^{-1} \rho = r_0' \nu_1^9 < 1$).

(3) Let h_0 be an arbitrary unit vector. For any $|E - E_1| < r^2$, and any $H \gg 1$, we have

$$\mathrm{mes}\{\phi \in \mathcal{I}_0 : \log |\langle \nabla \underline{E}^{[-N_0', N_0'']}(x_0(\phi, E)), h_0 \rangle| \le H_1 H\} < \exp(-H^{1/(d-1)}), \tag{6.12}$$

with $H_1 = C(d) \log(\nu_1 \|x - \underline{x}_0\|)$.

By (6.3) and (6.10) we have

$$\begin{aligned} &|\underline{E}^{[-N_1', N_1'']}(x_0(\phi, E)) - \underline{E}^{[-N_0', N_0'']}(x_0(\phi, E))| \\ &\quad = |\underline{E}^{[-N_1', N_1'']}(x_0(\phi, E)) - E| < \exp(-\gamma N_0/20), \end{aligned}$$

for $(\phi, E) \in \Pi_0$. Then, just as in Proposition 5.8, we can find a map $x_1 : \Pi_1'' \to \mathbb{R}^d$,

$$\Pi_1'' = \mathcal{P}_1'' \cap \mathbb{R}^d, \quad \mathcal{P}_1'' = \{(\phi, E) \in \mathbb{C}^d : |\phi| < r^{C_0}, |E - E_1| < r^{C_0}\},$$
$$C_0 = C_0(d) \gg 1,$$

that extends analytically to $\mathcal{P}_1''$, $x_1(\mathcal{P}_1'') \subset \mathbb{T}^d_{\rho/2}$, and such that

$$|x_1(\phi, E) - x_0(\phi, E)| < \exp(-\gamma N_0/30), \quad (\phi, E) \in \Pi_1''. \tag{6.13}$$

Since $r \gtrsim \exp(-N_1^{300\mathfrak{d}})$, we have that $\mathcal{P}_1$ as defined in condition (B) (with $\phi_1 = 0$), satisfies $\mathcal{P}_1 \subset \mathcal{P}_1''$ (recall that $\mathfrak{d} \ll \delta$). Note that $|x_1(\phi, E) - \underline{x}_0| \ll \exp(-2N_0^{4\mathfrak{d}})$. Now, by Proposition 6.1, conditions (A)–(C) hold with the above choice of parametrization x_1.

We proceed to check condition (D). The argument is based on applying the covering form of (LDT), similarly to Proposition 6.5. We assume everything from the proof of Proposition 6.5, up to and including (6.8), except that we take the lower bound for $\operatorname{dist}(h, \mathfrak{T}_1)$ to be $\exp(-N_1^{\mu})$. Fix $|E - E_1| < r^{C_0}$. By (6.8) and (6.10),

$$\begin{aligned}&\operatorname{dist}(\operatorname{spec} J_m(x_0(\phi, E) + h), (-\infty, \underline{E}^{[-N_0', N_0'']}(x_0(\phi, E))])\\&\quad = \operatorname{dist}(\operatorname{spec} J_m(x_0(\phi, E) + h), (-\infty, E]) \gtrsim \exp(-N_0^{4\mathfrak{d}}),\end{aligned} \tag{6.14}$$

provided $|m + m_1| > 3N_0/2$.

Now we focus on $m = -m_1$. We assume $m_1 \in [-N_1, N_1]$. If $\|h_1\| \geq \exp(-N_0^{2\mathfrak{d}})$, then $\operatorname{dist}(h + m_1\omega, \mathfrak{T}_0) \geq \exp(-N_0^{2\mathfrak{d}})$ and as above, there exists an interval J_{-m_1} such that (6.14) holds with $m = -m_1$. Suppose that $\exp(-N_1^{\mu}) \leq \|h_1\| < \exp(-N_0^{2\mathfrak{d}})$. Let $J_{-m_1} := -m_1 + [-N_0', N_0'']$ and recall (6.9). From (6.11) with $H = N_1^{2(d-1)\delta}$ (note that $\|h_1\| \leq \exp(-N_0^{2\mathfrak{d}}) < \rho$), it follows that

$$\operatorname{mes}\{\phi \in \mathcal{I}_0 : |\underline{E}^{[-N_0', N_0'']}(x_0(\phi, E) + h_1) - E| \leq \exp(-N_1^{2\mu})\} < \exp(-N_1^{2\delta}) \tag{6.15}$$

(we used $\mathfrak{d} \ll \delta \ll \mu$, $H_0 \gtrsim -(N_1^{200\mathfrak{d}} + N_1^{\mu}) \gtrsim -N_1^{\mu}$). Using ($\underline{\text{A}}$) it follows that

$$\operatorname{dist}(\operatorname{spec} H_{J_{-m_1}}(x_0(\phi, E) + h), E) > \exp(-N_1^{2\mu}),$$

for any $\phi \in \mathcal{I}_0 \backslash \mathcal{B}_1'$, where $\mathcal{B}_1'$ is the set from (6.15).

We now have what we need to invoke the covering form of (LDT). We let the interval I be as in the proof of Proposition 6.5. By the above, we can use Lemma 2.17 (with $K = N_1^{2\mu} = N_0^{10\mu} \ll N_0^{\sigma/2}$; recall that $\mu \ll \sigma$) to get that

$$\operatorname{dist}(\operatorname{spec} H_I(x_0(\phi, E) + h), E) \geq \exp(-2N_1^{2\mu}) = \exp(-2N_0^{10\mu}),$$

for any $\phi \in \mathcal{I}_0 \backslash \mathcal{B}_1'$. Let $\mathcal{I}_1'' = \operatorname{Proj}_{\phi} \Pi_1''$. Then, using (6.13), we get

$$\operatorname{dist}(\operatorname{spec} H_I(x_1(\phi, E) + h), E) \geq \exp(-3N_1^{2\mu}) > \exp(-N_1^{\beta}/2),$$

for any $\phi \in \mathcal{I}_1''\backslash\mathcal{B}_1'$ (recall that $\beta \gg \mu$). This implies that condition (D) holds.

Finally, we check condition (E). Fix $|E - E_1| < r^{C_0}$ and $h_0 \in \mathbb{R}^d$ a unit vector. By (6.12) with $H = N_1^{2(d-1)\delta}$,

$$\mathrm{mes}\{\phi \in \mathcal{I}_0 : \log|\langle \nabla \underline{E}^{[-N_0', N_0'']}(x_0(\phi, E)), h_0\rangle| < -N_1^{\mu}/4\} < \exp(-N_1^{2\delta})$$

(we used $HH_1 \gtrsim -N_1^{200\mathfrak{d}} N_1^{2(d-1)\delta} \gg N_1^{\mu}$; recall that $\mu \gg \delta \gg \mathfrak{d}$). Now condition (E) follows by using (6.13) and Lemma 6.2. □

7 From conditions on potential to inductive conditions

We start by assuming that V attains its absolute extrema at exactly one non-degenerate critical point and show that for large enough coupling we can satisfy the initial inductive conditions from Sect. 6. This means that we are working with operators of the form (1.1). Having the assumption be about both absolute extrema is just a matter of convenience, it will be clear that they can be handled separately.

Let $\underline{x}, \overline{x}$, be the points where the absolute minimum and maximum of V are attained. Since $\underline{x}, \overline{x}$ are assumed to be non-degenerate critical points they will be isolated from the other critical points. We give a quantitative version of this observation. We use $\mathfrak{C}$ to denote the set of critical points of V.

Lemma 7.1 *Given $x_0, x_1 \in \mathfrak{C}$, such that x_0 is non-degenerate, we have*

$$\|x_0 - x_1\| \geq c_\rho \left\|\mathfrak{H}(x_0)^{-1}\right\|^{-1} (1 + \|V\|_\infty)^{-1}.$$

Proof By Taylor's formula and Cauchy estimates,

$$\begin{aligned}\|\nabla V(x)\| &= \|\nabla V(x) - \nabla V(x_0)\| \\ &\geq \|\mathfrak{H}(x_0)(x - x_0)\| - C_\rho \|V\|_\infty \|x - x_0\|^2 \\ &\geq \frac{1}{2}\left\|\mathfrak{H}(x_0)^{-1}\right\|^{-1} \|x - x_0\|,\end{aligned}$$

provided $\|x - x_0\| \leq c_\rho \left\|\mathfrak{H}(x_0)^{-1}\right\|^{-1} (1 + \|V\|_\infty)^{-1}$. The conclusion follows. □

Note that $\mathfrak{C}$ is compact and since $\underline{x}, \overline{x}$ are isolated, $\mathfrak{C}\backslash\{\underline{x}, \overline{x}\}$ is also compact. Therefore there exists $\iota = \iota(V) > 0$, such that

$$V(\underline{x}) + \iota \leq V(x) \leq V(\overline{x}) - \iota, \quad x \in \mathfrak{C}\backslash\{\underline{x}, \overline{x}\}. \tag{7.1}$$

Let

$$\nu := \min(\left\|\mathfrak{H}(\underline{x})^{-1}\right\|^{-1}, \left\|\mathfrak{H}(\overline{x})^{-1}\right\|^{-1})$$

Note that since $\underline{x}$, $\overline{x}$ are non-degenerate extrema, we have

$$\mathfrak{H}(\underline{x}) \geq \nu I, \qquad \mathfrak{H}(\overline{x}) \leq -\nu I.$$

Lemma 7.2 *Let* $r = c\nu(1 + \|V\|_\infty)^{-1}$, *with* $c = c(\rho)$ *sufficiently small. Then*

$$\begin{gathered}\frac{\nu}{2}\left\|x-\underline{x}\right\|^2 \leq V(x) - V(\underline{x}) \leq C_\rho(1+\|V\|_\infty)\left\|x-\underline{x}\right\|^2, \quad \left\|x-\underline{x}\right\| \leq r,\\ \frac{\nu}{2}\left\|x-\underline{x}\right\| \leq \|\nabla V(x)\| \leq C_\rho(1+\|V\|_\infty)\left\|x-\underline{x}\right\|, \quad \left\|x-\underline{x}\right\| \leq r,\\ \min(\iota, \nu r^2/2) \leq V(x) - V(\underline{x}), \quad \left\|x-\underline{x}\right\| \geq r.\end{gathered} \tag{7.2}$$

Analogous estimates hold for $\overline{x}$.

Proof The estimates with $\left\|x-\underline{x}\right\| \leq r$ follow from Lemma 4.10 (we use Cauchy estimates to control $M(3)$). From Lemma 7.1 we have that, by choosing r small enough,

$$\mathfrak{E}\backslash\{\underline{x}\} \subset \mathbb{T}^d\backslash\{x : \left\|x-\underline{x}\right\| \leq r\}.$$

Then

$$\begin{aligned}&\min_{\|x-\underline{x}\|\geq r}(V(x) - V(\underline{x}))\\ &\quad= \min\left(\min_{x\in\mathfrak{E}\backslash\{\underline{x}\}}(V(x)-V(\underline{x})), \min_{\|x-\underline{x}\|=r}(V(x)-V(\underline{x}))\right)\end{aligned}$$

and the conclusion follows. □

For the purpose of the next result we update T_V (recall (3.2)), to be

$$\begin{aligned}T_V &= 2 + \max(0, \log\|V\|_\infty) + \max(0, \log\underline{\iota}^{-1}) + \max(0, \log\iota^{-1})\\ &\quad+ \max(0, \log\nu^{-1}).\end{aligned} \tag{7.3}$$

Clearly all the previous results using T_V also hold with this possibly larger T_V. The proofs of the next proposition and later of Proposition 7.5 are very similar to the proofs of Theorems E and D respectively, with some of the tools from Sect. 2 replaced by their analogues from Sect. 3. Due to the similarity we omit some details. However, for clarity, we do give complete proofs, as the key differences are spread out. Recall the exponent $\mathfrak{d}$ from the inductive conditions (A)–(D).

Proposition 7.3 *Assume the notation of conditions* (A)–(D) *from Sect.* 6. *Let* $\varepsilon > 0$. *There exists* $\lambda_0 = \exp((T_V)^C)$, $C = C(a, b, \rho, \varepsilon)$, *such that the following hold for* $\lambda \geq \lambda_0$. *For any* $(\log\lambda)^{C(a,b,\varepsilon)} \leq N_0 \leq \exp((\log\lambda)^{\varepsilon/2})$

there exists $\underline{x}_0 \in \mathbb{T}^d$, $|\underline{x}_0 - \underline{x}| \ll \lambda^{-1/3}$, *such that the conditions* (A)–(D) *hold with* $s = 0$, $\gamma = (\log\lambda)/2$, $[-N_0', N_0''] = [-N_0, N_0]$, *and* $|\lambda^{-1}\underline{E}_0 - V(\underline{x})| \ll \lambda^{-1/4}$. *Furthermore, for any* $E_0 \in \mathbb{R}$, $\exp(-N_0^{100\mathfrak{d}}) \le |E_0 - \underline{E}_0| \le \lambda\exp(-(\log\lambda)^{4\varepsilon})$, *conditions* (A)–(E), *with* $s = 0$, *hold for* $\underline{E}^{[-N_0,N_0]}$. *Analogous statements hold relative to conditions* ($\bar{\text{A}}$)–($\bar{\text{D}}$).

Proof To check (D) we will need to obtain conditions (A)–(C) not just for $[-N_0', N_0''] = [-N_0, N_0]$, but also for other intervals. By Lemma 7.2, for any $0 < |n| \le 2N_0$ we either have

$$V(\underline{x} + n\omega) - V(\underline{x}) \ge \frac{\nu}{2}\,\|n\omega\|^2 \ge \frac{\nu}{2}a(2N_0)^{-b},$$

or

$$V(\underline{x} + n\omega) - V(\underline{x}) \ge \min(\iota, \nu r^2/2).$$

Then for large enough λ (this is why we added $\max(0, \log\iota^{-1}) + \max(0, \log\nu^{-1})$ to T_V) and N_0 not too large, we have

$$V(\underline{x} + n\omega) - V(\underline{x}) \ge \exp(-(\log\lambda)^{\varepsilon}), \quad 0 < |n| \le N_0.$$

Let $a < 0 < b$, $[a, b] \subset [-2N_0, 2N_0]$. Then by Lemma 3.8, there exists $\underline{E}^{[a,b]} = E_k^{[a,b]}$ such that for any $|x - \underline{x}| < \exp(-3(\log\lambda)^{\varepsilon})$,

$$\begin{aligned} |\lambda^{-1}\underline{E}^{[a,b]}(x) - V(x)| &\le 2\lambda^{-1}, \\ |\underline{\psi}^{[a,b]}(x, n)| &< \exp(-(\log\lambda)|n|/2), \quad |n| > 0, \\ \lambda^{-1}(E_j^{[a,b]}(x) - \underline{E}^{[a,b]}(x)) &\ge \frac{1}{8}\exp(-(\log\lambda)^{\varepsilon}), \quad j \ne k. \end{aligned} \tag{7.4}$$

As in Lemma 6.2, $\underline{E}^{[a,b]}$ is analytic on

$$\mathcal{P}' = \{z \in \mathbb{C}^d : |z - \underline{x}| < r'\}, \quad r' = \exp(-4(\log\lambda)^{\varepsilon})$$

and

$$\sup_{\mathcal{P}'} \left\| \mathfrak{H}(\lambda^{-1}\underline{E}^{[a,b]}) - \mathfrak{H}(V) \right\| \le \lambda^{-c(d)}.$$

As in Proposition 6.3, we can find $\underline{\tilde{x}} = \underline{\tilde{x}}([a, b])$, $|\underline{\tilde{x}} - \underline{x}| \ll \lambda^{-1/3}$, such that

$$\begin{aligned} \underline{E}^{[a,b]}(\underline{\tilde{x}}) &\le \underline{E}^{[a,b]}(x), \quad \text{for any } |x - \underline{x}| < r', \\ \nabla\underline{E}^{[a,b]}(\underline{\tilde{x}}) = 0, &\quad \mathfrak{H}(\lambda^{-1}\underline{E}^{[a,b]})(\underline{\tilde{x}}) \ge \frac{\nu}{4}I. \end{aligned}$$

Also, as in Lemma 6.4, we have $|\lambda^{-1}\underline{\tilde{E}} - V(\underline{x})| \ll \lambda^{-1/4}$, where $\underline{\tilde{E}} = \underline{E}^{[a,b]}(\underline{\tilde{x}})$. We need to work around the weakness of the estimate $|\underline{\tilde{x}} - \underline{x}| \ll$

$\lambda^{-1/3}$. From now on assume $[a,b] \supset [-\hat{N}, \hat{N}]$, $\hat{N} = \lceil N_0^{1/4} \rceil$. By Corollary 3.9, we have

$$|\underline{E}^{[a,b]}(x) - \underline{E}^{[-\hat{N},\hat{N}]}(x)| \lesssim \exp(-(\log\lambda)\hat{N}/2),$$

for any $|x - \underline{x}| < \exp(-3(\log\lambda)^{\varepsilon})$. Let $\hat{\underline{x}} = \tilde{\underline{x}}([-\hat{N}, \hat{N}])$. As in Proposition 6.3, we can find, with a slight abuse of notation, $\tilde{\underline{x}} = \tilde{\underline{x}}([a,b])$,

$$|\tilde{\underline{x}} - \hat{\underline{x}}| < \exp(-(\log\lambda)\hat{N}/5), \tag{7.5}$$

such that

$$\begin{aligned} \underline{E}^{[a,b]}(\tilde{\underline{x}}) \le \underline{E}^{[a,b]}(x), \quad &\text{for any } |x - \hat{\underline{x}}| < \exp(-C(\log\lambda)^{\varepsilon}), \\ \nabla \underline{E}^{[a,b]}(\tilde{\underline{x}}) = 0, \quad &\mathfrak{H}(\lambda^{-1}\underline{E}^{[a,b]})(\tilde{\underline{x}}) \ge \frac{\nu}{8} I. \end{aligned} \tag{7.6}$$

Furthermore, as in Lemma 6.4,

$$|\tilde{\underline{E}} - \hat{\underline{E}}| < \exp(-(\log\lambda)\hat{N}/6), \tag{7.7}$$

with $\hat{\underline{E}} = \underline{E}^{[-\hat{N},\hat{N}]}(\hat{\underline{x}})$. Note that

$$|\tilde{\underline{x}} - \underline{x}| \ll \lambda^{-1/3}, \quad |\lambda^{-1}\tilde{\underline{E}} - V(\underline{x})| \ll \lambda^{-1/4}. \tag{7.8}$$

Let $\underline{x}_0 = \tilde{\underline{x}}([-N_0, N_0])$. Then the first statement, except for condition ($\underline{D}$), holds by all the above and by having $N_0^{\mathfrak{d}} \gg (\log\lambda)^{\varepsilon}$. As in Sect. 6 we incorporate the dependence on $\mathfrak{d}$ in the dependence on the Diophantine parameters.

Next we check condition ($\underline{D}$). First we consider the case $\operatorname{dist}(h, \mathfrak{T}_0) \ge \exp(-(\log\lambda)^{2\varepsilon})$. Since $\|h + n\omega\| \ge \exp(-(\log\lambda)^{2\varepsilon})$, we have, by Lemma 7.2,

$$V(\underline{x} + h + n\omega) - V(\underline{x}) \ge \exp(-3(\log\lambda)^{2\varepsilon}), \quad |n| \le N_0$$

(provided λ is large enough). By Corollary 3.7 we get

$$\operatorname{dist}(\operatorname{spec} H_{[-N_0,N_0]}(\underline{x} + h), (-\infty, \lambda V(\underline{x})]) \gtrsim \lambda \exp(-3(\log\lambda)^{2\varepsilon})$$

and by (7.8),

$$\operatorname{dist}(\operatorname{spec} H_{[-N_0,N_0]}(\underline{x}_0 + h), (-\infty, \underline{E}_0]) \gtrsim \lambda \exp(-3(\log\lambda)^{2\varepsilon}) \gg \exp(-N_0^{4\mathfrak{d}}). \tag{7.9}$$

Next we consider the case $\exp(-N_0^{2\mathfrak{d}}) \le \operatorname{dist}(h, \mathfrak{T}_0) < \exp(-(\log\lambda)^{2\varepsilon})$. Let n_1, $|n_1| \le 3N_0/2$, such that $\operatorname{dist}(h, \mathfrak{T}_0) = \|h - n_1\omega\|$. We consider two subcases depending on the position of n_1. If $n_1 \notin [-N_0 + N_0^{1/3}, N_0 - N_0^{1/3}]$, then for $n \in [-N_0 + N_0^{1/3}, N_0 - N_0^{1/3}]$

$$\|h + n\omega\| \geq \|(n - n_1)\omega\| - \|h - n_1\omega\| \geq aN_0^{-b} - \exp(-(\log\lambda)^{2\varepsilon}) \gg \exp(-(\log\lambda)^{2\varepsilon})$$

(recall that $N_0 \leq \exp((\log\lambda)^{\varepsilon/2})$) and as above we get

$$\mathrm{dist}(\mathrm{spec}\, H_{[-N_0+N_0^{1/3}, N_0-N_0^{1/3}]}(\underline{x}_0 + h), (-\infty, \underline{E}_0]) \gtrsim \lambda \exp(-3(\log\lambda)^{2\varepsilon}) \gg \exp(-N_0^{4\mathfrak{d}}).$$

Suppose $n_1 \in [-N_0 + N_0^{1/3}, N_0 - N_0^{1/3}]$. Let $h_1 = h - n_1\omega$ (so, $\|h_1\| = \mathrm{dist}(h, \mathfrak{T}_0)$), $[a_1, b_1] = n_1 + [-N_0, N_0]$, $\tilde{\underline{x}}_1 = \tilde{\underline{x}}([a_1, b_1])$, $\tilde{\underline{E}}_1 = \underline{E}^{[a_1,b_1]}(\tilde{\underline{x}}_1)$. Note that $[a_1, b_1] \supset [-\hat{N}, \hat{N}]$. By Taylor's formula (recall Lemma 4.10, (7.6)),

$$\underline{E}^{[a_1,b_1]}(\tilde{\underline{x}}_1 + h_1) - \underline{E}^{[a_1,b_1]}(\tilde{\underline{x}}_1) \geq \frac{\nu}{2}\|h_1\|^2 \geq \exp(-3N_0^{2\mathfrak{d}}).$$

Then, by (7.4) (recall (7.8)),

$$\mathrm{dist}(\mathrm{spec}\, H_{[a_1,b_1]}(\tilde{\underline{x}}_1 + h_1), (-\infty, \tilde{\underline{E}}_1]) \geq \exp(-3N_0^{2\mathfrak{d}}).$$

Since $\mathrm{spec}\, H_{[a_1,b_1]}(\tilde{\underline{x}}_1 + h_1) = \mathrm{spec}\, H_{[-N_0,N_0]}(\tilde{\underline{x}}_1 + h)$ and by (7.5), (7.7),

$$|\tilde{\underline{x}}_1 - \underline{x}_0| \lesssim \exp(-(\log\lambda)N_0^{1/4}/5), \quad |\tilde{\underline{E}}_1 - \underline{E}_0| \lesssim \exp(-(\log\lambda)N_0^{1/4}/6),$$

it follows that

$$\mathrm{dist}(\mathrm{spec}\, H_{[-N_0,N_0]}(\underline{x}_0 + h), (-\infty, \underline{E}_0]) \gtrsim \exp(-3N_0^{2\mathfrak{d}}) \gg \exp(-N_0^{4\mathfrak{d}})$$

Thus, condition (D) holds.

Next we check the last statement. Let $N_1 = N_0^5$. Since all the statements of the proof hold for a range of N_0, they will also hold for N_1, by adjusting the range. In particular, let $\underline{x}_1 = \tilde{\underline{x}}([-N_1, N_1])$. Note that by (7.5), (7.7),

$$|\underline{x}_1 - \underline{x}_0| \lesssim \exp(-(\log\lambda)N_0^{1/4}/5), \quad |\underline{E}_1 - \underline{E}_0| \lesssim \exp(-(\log\lambda)N_0^{1/4}/6). \tag{7.10}$$

Fix x, $\exp(-N_1^{100\mathfrak{d}}) \leq |x - \underline{x}_0| \leq \exp(-(\log\lambda)^{2\varepsilon})$. We will check that conditions (A)–(E), with $s = 1$, hold for $\underline{E}^{[-N_1,N_1]}$ with $E_1 = \underline{E}^{[-N_0,N_0]}(x)$. Then the conclusion holds since

$$\{\underline{E}^{[-N_0,N_0]}(x) : \exp(-N_1^{200\mathfrak{d}}) \leq |x - \underline{x}_0| \leq \exp(-(\log\lambda)^{2\varepsilon})\} \supset [\underline{E}_0 + \lambda\exp(-N_1^{150\mathfrak{d}}), \underline{E}_0 + \lambda\exp(-(\log\lambda)^{3\varepsilon})]$$

$$\supset [\underline{E}_1 + \exp(-N_1^{100\eth}), \underline{E}_1 + \lambda \exp(-(\log \lambda)^{4\varepsilon})]$$

(we applied Lemma 4.10 to $\lambda^{-1}\underline{E}^{[-N_0,N_0]}$ and we used (7.10)). Note that since this statement will hold for a range of N_1, it will also hold for the stated range of N_0 by relabelling.

We apply Proposition 4.1 to $\lambda^{-1}\underline{E}^{[-N_0,N_0]}$ on

$$\mathcal{P}_0' = \{z \in \mathbb{C}^d : |z - \underline{x}_0| < \exp(-4(\log \lambda)^{\varepsilon})\}.$$

Using the notation of Proposition 4.1, we have

$$\nu_1 \simeq \exp(-C(\log \lambda)^{\varepsilon}), \quad \rho = \exp(-4(\log \lambda)^{\varepsilon})\nu_1^{10} \simeq \exp(-C'(\log \lambda)^{\varepsilon}),$$
$$r = \nu_1 \left\| x - \underline{x}_0 \right\| .$$

We chose to apply Proposition 4.1 to $\lambda^{-1}E^{[-N_0,N_0]}$ because of the $0 < \nu_0 < 1$ restriction in the statement of the proposition. Of course, we could artificially choose any $\nu_0 \in (0, 1)$ for $E^{[-N_0,N_0]}$, but this would result in a much smaller $\nu_1 \simeq \lambda^{-1}\exp(C(\log \lambda)^{\varepsilon})$, which is too small for our purposes. Since $0 < \left\| x - \underline{x}_0 \right\| < \rho$, Proposition 4.1 applies with x in the role of x_0 and we get the following:

(1) There exists a map $x_0 : \Pi_0 \to \mathbb{R}^d$,

$$\Pi_0 = \mathcal{I}_0 \times (E_1 - \lambda r^2, E_1 + \lambda r^2), \quad \mathcal{I}_0 = (-r, r)^{d-1}, \quad E_1 = \underline{E}^{[-N_0,N_0]}(x),$$

such that

$$\underline{E}^{[-N_0,N_0]}(x_0(\phi, E)) = E,$$

$x_0(\phi, E)$ extends analytically to

$$\mathcal{P}_0 = \{(\phi, \eta) \in \mathbb{C}^d : |\phi| < r, \ |E - E_1| < \lambda r^2\},$$

and

$$\left\| x_0(\phi, E) - \underline{x}_0 \right\| < \left\| x - \underline{x}_0 \right\| /2 \lesssim \exp(-(\log \lambda)^{2\varepsilon}). \tag{7.11}$$

From the last estimate it follows that $x_0(\mathcal{P}_0) \subset \mathbb{T}^d_{\rho/2}$. Of course, Proposition 4.1 actually gives a function $\tilde{x}_0(\phi, \eta)$, such that $\lambda^{-1}\underline{E}^{[-N_0,N_0]}(\tilde{x}_0(\phi, \eta)) = \eta$, and we get the above statement by setting $x_0(\phi, E) = \tilde{x}_0(\phi, \lambda^{-1}E)$.

(2) For any $|E - E_1| < \lambda r^2$, any vector $h \in \mathbb{R}^d$ with $0 < \|h\| < \rho$, and any $H \gg 1$, we have

$$\operatorname{mes}\{\phi \in \mathcal{I}_0 : \log |\underline{E}^{[-N_0,N_0]}(x_0(\phi, E)) - E| \le H_0 H\} < \exp(-H^{1/(d-1)}), \tag{7.12}$$

with $H_0 = C(d)\log(\|h\|\,\|x - \underline{x}_0\|)$.

(3) Let h_0 be an arbitrary unit vector. For any $|E - E_1| < \lambda r^2$, and any $H \gg 1$, we have

$$\begin{aligned}\operatorname{mes}\{\phi \in \mathcal{I}_0 : \log|\langle \nabla \underline{E}^{[-N_0,N_0]}(x_0(\phi, E)), h_0\rangle| \le H_1 H\}\\ < \exp(-H^{1/(d-1)}),\end{aligned} \tag{7.13}$$

with $H_1 = C(d)\log(\nu_1\,\|x - \underline{x}_0\|)$. By Corollary 3.9,

$$\begin{aligned}|\underline{E}^{[-N_1,N_1]}(x) - \underline{E}^{[-N_0,N_0]}(x)| \lesssim \exp(-(\log\lambda)N_0/2),\\ |x - \underline{x}| < \exp(-3(\log\lambda)^{\varepsilon}),\end{aligned} \tag{7.14}$$

and therefore

$$\begin{aligned}|\underline{E}^{[-N_1,N_1]}(x_0(\phi, E)) - \underline{E}^{[-N_0,N_0]}(x_0(\phi, E))| &= |\underline{E}^{[-N_1,N_1]}(x_0(\phi, E)) - E|\\ &\lesssim \exp(-(\log\lambda)N_0/2),\end{aligned}$$

for $(\phi, E) \in \Pi_0$. Then, just as in Proposition 5.8, we can find a map $x_1 : \Pi_1'' \to \mathbb{R}^d$,

$$\begin{aligned}&\Pi_1'' = \mathcal{P}_1'' \cap \mathbb{R}^d, \quad \mathcal{P}_1'' = \{(\phi, E) \in \mathbb{C}^d : |\phi| < r^{C_0}, |E - E_1| < r^{C_0}\},\\ &\quad C_0 = C_0(d) \gg 1,\end{aligned}$$

that extends analytically to $\mathcal{P}_1''$, $x_1(\mathcal{P}_1'') \subset \mathbb{T}^d_{\rho/2}$, and such that

$$|x_1(\phi, E) - x_0(\phi, E)| < \exp(-(\log\lambda)N_0/3), \quad (\phi, E) \in \Pi_1''. \tag{7.15}$$

In fact the domain in E is much larger, but we have no use for this improvement. Since $r \gtrsim \exp(-N_1^{200\mathfrak{d}})$, we have that $\mathcal{P}_1$ as defined in condition (B) (with $\phi_1 = 0$), satisfies $\mathcal{P}_1 \subset \mathcal{P}_1''$ (recall that $\mathfrak{d} \ll \delta$). Note that $|x_1(\phi, E) - \underline{x}_0| \ll \exp(-3(\log\lambda)^{\varepsilon})$. Now, conditions (A)–(C) hold with the above choice of parametrization x_1 (recall that we have (7.4) with $[a, b] = [-N_1, N_1]$).

We proceed to check condition (D). Let $|m_1| \le 3N_1/2$, $h_1 \in \mathbb{R}^d$ such that

$$\operatorname{dist}(h, \mathfrak{T}_1) = \|h_1\|, \qquad h_1 = h - m_1\omega \,(\operatorname{mod} \mathbb{Z}^d).$$

Recall that we have (6.7). We consider two cases: $\|h_1\| \ge \exp(-(\log\lambda)^{2\varepsilon})$ and $\exp(N_1^{\mu}) \le \|h_1\| < \exp(-(\log\lambda)^{2\varepsilon})$. In either case, by (6.7), we have $\operatorname{dist}(h + m\omega, \mathfrak{T}_0) \ge \exp(-(\log\lambda)^{2\varepsilon})$ for all $m \in [-N_1, N_1]$ with $|m + m_1| > 3N_0/2$. For such m, (7.9) implies

$$\operatorname{dist}(\operatorname{spec} H_{J_m}(\underline{x}_0 + h), (-\infty, \underline{E}_0]) \gtrsim \lambda\exp(-3(\log\lambda)^{2\varepsilon}), \tag{7.16}$$

with $J_m = m + [-N_0, N_0]$. Fix $|E - E_1| < r^{C_0}$. By (7.16) and (7.11),

$$\begin{aligned}&\mathrm{dist}(\mathrm{spec}\, H_{J_m}(x_0(\phi, E) + h), (-\infty, \underline{E}^{[-N_0', N_0'']}(x_0(\phi, E))])\\&\quad = \mathrm{dist}(\mathrm{spec}\, H_{J_m}(x_0(\phi, E) + h), (-\infty, E]) \gtrsim \lambda \exp(-3(\log \lambda)^{2\varepsilon}),\end{aligned} \tag{7.17}$$

provided $|m + m_1| > 3N_0/2$.

Now we focus on $m = -m_1$. We assume $m_1 \in [-N_1, N_1]$. Let $J_{-m_1} := -m_1 + [-N_0, N_0]$. If $\|h_1\| \geq \exp(-(\log \lambda)^{2\varepsilon})$, then $\mathrm{dist}(h + m_1\omega, \mathfrak{T}_0) \geq \exp(-(\log \lambda)^{2\varepsilon})$ and as above, (7.17) holds with $m = -m_1$. Suppose that $\exp(-N_1^{\mu}) \leq \|h_1\| < \exp(-(\log \lambda)^{2\varepsilon})$. From (7.12) with $H = N_1^{2(d-1)\delta}$, it follows that

$$\mathrm{mes}\{\phi \in \mathcal{I}_0 : |\underline{E}^{[-N_0, N_0]}(x_0(\phi, E) + h_1) - E| \leq \exp(-N_1^{2\mu})\} < \exp(-N_1^{2\delta}) \tag{7.18}$$

(we used $\mathfrak{d} \ll \delta \ll \mu$, $H_0 \gtrsim -(N_1^{200\mathfrak{d}} + N_1^{\mu}) \gtrsim -N_1^{\mu}$). Using (7.4) it follows that

$$\mathrm{dist}(\mathrm{spec}\, H_{J_{-m_1}}(x_0(\phi, E) + h), E) > \exp(-N_1^{2\mu}),$$

for any $\phi \in \mathcal{I}_0 \backslash \mathcal{B}_1'$, where $\mathcal{B}_1'$ is the set from (7.18).

Let I be an interval as in (5.31). By the above, we can use Lemma 2.17 (with $K = N_1^{2\mu} = N_0^{10\mu} \ll N_0^{\sigma/2}$; recall that $\mu \ll \sigma$) to get that

$$\mathrm{dist}(\mathrm{spec}\, H_I(x_0(\phi, E) + h), E) \geq \exp(-2N_1^{2\mu}) = \exp(-2N_0^{10\mu}),$$

for any $\phi \in \mathcal{I}_0 \backslash \mathcal{B}_1'$. Let $\mathcal{I}_1'' = \mathrm{Proj}_{\phi} \Pi_1''$. Then, using (7.15), we get

$$\mathrm{dist}(\mathrm{spec}\, H_I(x_1(\phi, E) + h), E) \geq \exp(-3N_1^{2\mu}) > \exp(-N_1^{\beta}/2),$$

for any $\phi \in \mathcal{I}_1'' \backslash \mathcal{B}_1'$ (recall that $\beta \gg \mu$). This implies that condition (D) holds.

Finally, we check condition (E). Fix $|E - E_1| < r^{C_0}$ and $h_0 \in \mathbb{R}^d$ a unit vector. By (7.13) with $H = N_1^{2(d-1)\delta}$,

$$\mathrm{mes}\{\phi \in \mathcal{I}_0 : \log |\langle \nabla \underline{E}^{[-N_0, N_0]}(x_0(\phi, E)), h_0 \rangle| < -N_1^{\mu}/4\} < \exp(-N_1^{2\delta})$$

(we used $HH_1 \gtrsim -N_1^{200\mathfrak{d}} N_1^{2(d-1)\delta} \gg N_1^{\mu}$; recall that $\mu \gg \delta \gg \mathfrak{d}$). Now condition (E) follows by using (7.15) and Cauchy estimates. □

For the rest of the section we assume that $V \in \mathfrak{G}$, recall Definition 1.1, and show that, for large enough coupling, we can we can satisfy the initial inductive conditions from Sect. 5. In fact, it will be clear that we only use properties (iii) and (iv) from the definition of $\mathfrak{G}$. The first two properties will only be needed in the proof of Theorem A (b). We fix the constants $\mathfrak{c}_0, \mathfrak{c}_1, \mathfrak{C}_0$ from Definition 1.1.

Proposition 7.4 *Let $x_0 \in \mathbb{T}^d$, $\eta_0 = V(x_0)$ and assume $\mu_0 := \|\nabla V(x_0)\| > 0$. Let*

$$r = \min(\rho/4, c\mu_0^2(1 + \|V\|_\infty)^{-2}),$$

with $c = c(\rho)$ small enough. There exists a map $x : \Pi \to \mathbb{R}^d$,

$$\Pi = \mathcal{I} \times (\eta_0 - r, \eta_0 + r), \quad \mathcal{I} = x_0 + (-r, r)^{d-1},$$

such that the following hold.

(a) The map extends analytically on the domain

$$\mathcal{P} = \{(\phi, \eta) \in \mathbb{C}^d : dist((\phi, \eta), \Pi) < r\},$$

and

$$x(\mathcal{P}) \subset \mathbb{T}^d_{\rho/2}, \qquad V(x(\phi, \eta)) = \eta, \quad (\phi, \eta) \in \mathcal{P}.$$

(b) For any $K \gg \mathfrak{C}_0 + C_\rho \max(0, \log \|V\|_\infty)$, $\|h\| \geq e^{-\mathfrak{c}_0 K}$, and $\eta \in (\eta_0 - r, \eta_0 + r)$,

$$\mathrm{mes}\{\phi \in \mathcal{I} : |V(x(\phi, \eta) + h) - \eta| < \exp(-K)\} < \exp(-K^{\mathfrak{c}_1}/10).$$

(c) Take an arbitrary unit vector $h_0 \in \mathbb{R}^d$. For any $K \geq \mathfrak{C}_0$, $\eta \in (\eta_0 - r, \eta_0 + r)$,

$$\mathrm{mes}\{\phi \in \mathcal{I} : \log |\langle \nabla V(x(\phi, \eta)), h_0\rangle| < -K\} < \exp(-K^{\mathfrak{c}_1}).$$

Proof There exists i such that $\partial_{x_i} V(x_0) \geq \mu_0/d$. To simplify the notation, we assume that $i = 1$. Let $\rho_1 \leq c_\rho \mu_0 (1 + \|V\|_\infty)^{-1}$ with c_ρ sufficiently small. Applying Lemma 4.2 (also recall Remark 4.5) to $V(x) - \eta$ near (x_0, η_0), we get that there exists an analytic function $x_1(x_2, \dots, x_d, \eta)$ on

$$|x_2 - x_{0,2}|, \dots, |x_d - x_{0,d}|, |\eta - \eta_0| < \rho_1^2$$

such that

$$|x_1(x_2, \dots, x_d, \eta) - x_{0,1}| < \rho_1,$$
$$V(x_1(x_2, \dots, x_d, \eta), x_2, \dots, x_d) = \eta.$$

The existence of the map and part (a) follow by setting

$$x(\phi, \eta) = (x_1(\phi, \eta), \phi), \quad \phi = (x_2, \dots, x_d).$$

Our choice of $r < \rho_1^2$ is made to ensure that $x(\mathcal{P}) \subset \mathbb{T}^d_{\rho/2}$.

Fix $\|h\| \geq \exp(-\mathfrak{c}_0 K)$, $\eta \in (\eta_0 - r, \eta_0 + r)$. Let

$$F(\phi) = V(x(\phi, \eta) + h) - \eta. \tag{7.19}$$

Let $g(x) := g_{V,h,1,2}(x)$ be as in Definition 1.1. We have

$$\begin{aligned}\partial_{x_2} F(\phi) &= \partial_{x_1} V(x(\phi,\eta)+h)\partial_{x_2} x_1(\phi,\eta) + \partial_{x_2} V(x(\phi,\eta)+h)\\ &= -\partial_{x_1} V(x(\phi,\eta)+h)\frac{\partial_{x_2} V(x(\phi,\eta))}{\partial_{x_1} V(x(\phi,\eta))} + \partial_{x_2} V(x(\phi,\eta)+h)\\ &= \frac{g(x(\phi,\eta))}{\partial_{x_1} V(x(\phi,\eta))}.\end{aligned} \tag{7.20}$$

Let $K \geq \mathfrak{C}_0$. By Definition 1.1 (iii) we have that

$$\operatorname{mes}\{x_{\hat{1}} : \min_{x_1}(|V(x+h) - V(x)| + |g(x)|) < \exp(-K)\} \leq \exp(-K^{\mathfrak{c}_1}).$$

In particular, it follows that

$$\operatorname{mes}\{\phi \in \mathcal{I} : |V(x(\phi,\eta)+h)-\eta| + |g(x(\phi,\eta))| < \exp(-K)\} \leq \exp(-K^{\mathfrak{c}_1}). \tag{7.21}$$

Let

$$\begin{aligned}\mathcal{B} &= \{\phi \in \mathcal{I} : |V(x(\phi,\eta)+h) - \eta| < \exp(-5K)\},\\ \mathcal{B}'' &= \{\phi \in \mathcal{I} : |V(x(\phi,\eta)+h) - \eta| < \exp(-5K),\\ &\qquad \|g(x(\phi,\eta))| \geq \exp(-K)/2\},\end{aligned}$$

and $\mathcal{B}'$ the set from (7.21). Then

$$\mathcal{B} \subset \mathcal{B}' \cup \mathcal{B}''.$$

We want to estimate $\operatorname{mes}(\mathcal{B}'')$. Let $z = (x_3, \ldots, x_d)$ and

$$\mathcal{B}''_z = \{x_2 : \phi = (x_2, z) \in \mathcal{B}''\}.$$

Fix $z = (x_3, \ldots, x_d)$ with $|x_i - x_{0,i}| < r$, $i = 3, \ldots, d$. By truncating the Taylor series (for both V and $x(\phi, \eta)$) we can find polynomials $P(x_2)$, $Q(x_2)$ (depending on z) of degree $\leq C \max(1, \log \|V\|_\infty) K^4$, such that for any $|x_2 - x_{0,2}| < r$,

$$\begin{aligned}&|F(x_2,z) - P(x_2)|,\ |\partial_{x_2} F(x_2,z) - P'(x_2)|,\ |g(x(x_2,z,\eta)) - Q(x_2)|\\ &\quad \leq \exp(-5K).\end{aligned}$$

Then

$$\mathcal{B}_z'' \subset \mathcal{B}_z''' := \{x_2 \in (x_{0,2} - r, x_{0,2} + r) : |P(x_2)| \le 2\exp(-5K),$$
$$|Q(x_2)| \ge \frac{1}{4}\exp(-K)\}.$$

Using (7.20) and Cauchy estimates, we have that for any $x_2 \in \mathcal{B}_z'''$,

$$|\partial_{x_2} F(x_2, z)| \gtrsim \rho \|V\|_\infty^{-1} |g(x(x_2, z, \eta))| \gtrsim \rho \|V\|_\infty^{-1} (|Q(x_2)| - e^{-5K})$$
$$\gtrsim \rho \|V\|_\infty^{-1} \exp(-K),$$
$$|P'(x_2)| \gtrsim (\rho \|V\|_\infty^{-1} \exp(-K) - \exp(-5K)) > \exp(-2K),$$

provided K is large enough. It follows that each connected component of $\mathcal{B}_z'''$ has length $\lesssim \exp(2K)\exp(-5K)$. Since $\mathcal{B}_z'''$ consists of the union of $\lesssim (\deg P + \deg Q)$ intervals, it follows that

$$\mathrm{mes}(\mathcal{B}_z''') \le C \max(1, \log \|V\|_\infty) K^4 \exp(-3K) < \exp(-2K).$$

Then we have $\mathrm{mes}(\mathcal{B}'') < \exp(-K)$ (recall that $\rho \le 1$, so $r \le 1/4$), $\mathrm{mes}(\mathcal{B}) < \exp(-K^{\mathfrak{c}_1}/2)$, and statement (b) follows.

Given $K \ge \mathfrak{C}_0$, by Definition 1.1 (iv) we have

$$\mathrm{mes}\{x_{\hat{1}} : \min_{x_1}(|V(x) - \eta| + |\langle \nabla V(x), h_0\rangle|) < \exp(-K)\} \le \exp(-K^{\mathfrak{c}_1}).$$

In particular, it follows that

$$\mathrm{mes}\{\phi \in \mathcal{I} : |V(x(\phi, \eta)) - \eta| + |\langle \nabla V(x(\phi, \eta), h_0\rangle|) < \exp(-K)\}$$
$$\le \exp(-K^{\mathfrak{c}_1}).$$

Since $V(x(\phi, \eta)) = \eta$, statement (c) follows. □

For the purpose of the next result we update T_V again to be to be

$$T_V = 2 + \max(0, \log \|V\|_\infty) + \max(0, \log \underline{\iota}^{-1}) + \max(0, \log \iota^{-1})$$
$$+ \max(0, \log \nu^{-1}) + \mathfrak{C}_0 + \mathfrak{c}_0^{-1}.$$

We don't include $\mathfrak{c}_1^{-1}$ because it doesn't depend on V.

Proposition 7.5 *There exists $\lambda_0 = \exp((T_V)^C)$, $C = C(a, b, \rho)$ such that the following hold for $\lambda \ge \lambda_0$. Let $x_0 \in \mathbb{T}^d$, $\eta_0 = V(x_0)$, and assume $\|\nabla V(x_0)\| \ge \exp(-(\log\lambda)^{\mathfrak{c}_1/3})$. Then for any $(\log\lambda)^{C(a,b)} \le N_0 \le \exp((\log\lambda)^{\mathfrak{c}_1/3})$, the conditions (A)–(E) hold with $s = 0$, $\gamma = (\log\lambda)/2$, $E_0 = \lambda\eta_0$, and some $\phi_0 \in \mathbb{R}^d$.*

Proof The proof is similar to that of Theorem D. As in Theorem D we leave the dependence on the exponents δ, β, μ implicit, as part of the dependence on the Diophantine condition parameters a, b.

Due to the lower bound on $\|\nabla V(x_0)\|$, we can apply Proposition 7.4 with $r = \exp(-3(\log \lambda)^{\mathfrak{c}_1/3})$. Furthermore, since λ is large enough, we can apply Proposition 7.4 (b),(c) with $K \gtrsim (\log \lambda)^{1/2}$ (this is why we added $\mathfrak{C}_0$ to T_V). In what follows we let $\mathcal{I}$, $x(\phi, \eta)$, be as in Proposition 7.4. Let

$$\mathcal{B}_{K,\eta,h} = \{\phi \in \mathcal{I} : |V(x(\phi, \eta)) - \eta| < \exp(-K)\}, \quad \mathcal{B}_{\eta,h} = \mathcal{B}_{(\log \lambda)^{1/2},\eta,h}.$$

By Proposition 7.4, for any $\eta \in (\eta_0 - r, \eta_0 + r)$, $\|h\| \geq \exp(-\mathfrak{c}_0(\log \lambda)^{1/2})$,

$$\operatorname{mes}(\mathcal{B}_{\eta,h}) < \exp(-(\log \lambda)^{\mathfrak{c}_1/2}).$$

As in Lemma 5.3 we can find a semialgebraic set $\tilde{\mathcal{B}}_{\eta,h}$ containing $\mathcal{B}_{\eta,h}$, of degree $\leq (\log \lambda)^3$, and with measure $\leq \exp(-(\log \lambda)^{\mathfrak{c}_1/2}/2)$. Let

$$\mathcal{B}_{\eta_0,N_0} = \bigcup_{0<|n|\leq 2N_0} \tilde{\mathcal{B}}_{\eta_0,n\omega}.$$

Since $N_0 \leq \exp((\log \lambda)^{\mathfrak{c}_1/3})$ we have $\|n\omega\| \geq \exp(-\mathfrak{c}_0(\log \lambda)^{1/2})$, $0 < |n| \leq 2N_0$ (provided λ is large enough; this why we added $\mathfrak{c}_0^{-1}$ to T_V), and $\operatorname{mes}(\mathcal{B}_{\eta_0,N_0}) < \exp(-(\log \lambda)^{\mathfrak{c}_1/2}/4)$. Since $\mathcal{B}_{\eta_0,N_0}$ is also semialgebraic of degree less than $\exp(2(\log \lambda)^{\mathfrak{c}_1/3})$, it follows, using Lemma 2.23, that there exists ϕ_0, $|\phi_0 - x_0| \ll r$, such that

$$\mathcal{I}_0' \Subset \mathcal{I} \backslash \mathcal{B}_{\eta_0,N_0}, \qquad \mathcal{I}_0' = \phi_0 + (-r_0', r_0')^{d-1}, \quad r_0' = \exp(-(\log \lambda)^{\mathfrak{c}_1/3}).$$

Let $a < 0 < b$, $[a, b] \subset [-2N_0, 2N_0]$. We consider such general intervals for reasons similar to the ones in Proposition 7.3. As in Proposition 5.6, but using Lemma 3.8 (with $x_0 = x(\phi, \eta_0)$, $\phi \in \mathcal{I}_0'$) instead of Proposition 2.22, we get that there exists k such that for any $\phi \in \mathcal{I}_0'$, $y \in \mathbb{R}^d$, $|y| < \exp(-4(\log \lambda)^{1/2})$, $|\eta - \eta_0| < \exp(-4(\log \lambda)^{1/2})$,

$$\begin{aligned}
&|\lambda^{-1} E_k^{[a,b]}(x(\phi, \eta) + y) - V(x(\phi, \eta) + y)| \leq 2\lambda^{-1},\\
&\lambda^{-1}|E_j^{[a,b]}(x(\phi, \eta) + y) - E_k^{[a,b]}(x(\phi, \eta) + y)|\\
&\quad > \frac{1}{8} \exp(-(\log \lambda)^{1/2}), \quad j \neq k,\\
&|\psi_k^{[a,b]}(x(\phi, \eta) + y, n)| < \exp(-(\log \lambda)|n|/2), \quad |n| > 0. \qquad (7.22)
\end{aligned}$$

To simplify notation we will drop the index k and write $E^{[a,b]}$, $\psi^{[a,b]}$. Let

$$\mathcal{P}_0'' = \{(\phi, E) \in \mathbb{C}^d : |\phi - \phi_0|, |E - E_0| < r_0''\}, \quad r_0'' = \exp(-C_0(\log \lambda)^{\mathfrak{c}_1/3}).$$

$C_0 = C_0(d) \gg 1$. Let $\Pi_0'' = \mathcal{P}_0'' \cap \mathbb{R}^d$, $\mathcal{I}_0'' = \mathrm{Proj}_\phi \Pi_0''$. As in Proposition 5.8, we can find an analytic map $\tilde{x}(\phi, \eta)$ such that

$$\lambda^{-1} E^{[a,b]}(\tilde{x}(\phi, \eta)) = \eta,$$

for any $(\phi, \lambda\eta) \in \mathcal{P}_0''$ and

$$|\tilde{x}(\phi, \eta) - x(\phi, \eta)| \le \lambda^{-1/2}, \tag{7.23}$$

for $(\phi, \lambda\eta) \in \Pi_0''$ (in fact, in the definition of $\mathcal{P}_0''$ we could take $|E - E_0| < \lambda \exp(-C_0(\log \lambda)^{1/2})$). We note that at this point, we have what we need for conditions (A)–(C) to hold. However, to check condition (D) we need to set things up more carefully. The problem we need to work around is the weakness of (7.23). From now on we assume that $[a, b] \supset [-\underline{N}, \underline{N}]$, $\underline{N} = \lceil N_0^{1/4} \rceil$. Let $\underline{x}$ be the parametrization obtained as above, so that

$$\lambda^{-1} E^{[-\underline{N}, \underline{N}]}(\underline{x}(\phi, \eta)) = \eta.$$

By Corollary 3.9 we have

$$|E^{[a,b]}(x(\phi, \eta) + y) - E^{[-\underline{N}, \underline{N}]}(x(\phi, \eta) + y)| \lesssim \exp(-(\log \lambda)\underline{N}/2).$$

for any $|y| < r_0'$, $(\phi, \lambda\eta) \in \Pi_0'$. Using (7.23) (with $\tilde{x} = \underline{x}$) it follows that

$$\begin{aligned} |E^{[a,b]}(\underline{x}(\phi, \eta)) - \lambda\eta| &= |E^{[a,b]}(\underline{x}(\phi, \eta)) - E^{[-\underline{N}, \underline{N}]}(\underline{x}(\phi, \eta))| \\ &\lesssim \exp(-(\log \lambda)\underline{N}/2). \end{aligned}$$

Again, as in Proposition 5.8, we get that there exists a map $\tilde{x}(\phi, \eta)$ such that

$$E^{[a,b]}(\tilde{x}(\phi, \eta)) = \lambda\eta, \qquad (\phi, \lambda\eta) \in \mathcal{P}_0'',$$

and for $(\phi, \lambda\eta) \in \Pi_0''$,

$$|\tilde{x}(\phi, \eta) - \underline{x}(\phi, \eta)| \le \exp(-(\log \lambda)\underline{N}/4). \tag{7.24}$$

To justify keeping the same domain $\mathcal{P}_0''$ as before we can increase the constant C_0 from its definition. Note that we still have

$$|\tilde{x}(\phi, \eta) - x(\phi, \eta)| \lesssim \lambda^{-1/2},$$

and therefore (using (7.22)) conditions (A)–(C) hold with $x_0(\phi, E) = \tilde{x}(\phi, \lambda^{-1}E)$, $[-N_0', N_0''] = [-N_0, N_0]$. Of course, we are assuming N_0 is large enough so that $r_0 = \exp(-N_0^\delta) \ll r_0''$.

Next we check condition (E), as in Proposition 5.14. Let $h_0 \in \mathbb{R}^d$ a unit vector, $\eta \in (\eta_0 - r_0'', \eta_0 + r_0'')$. By Proposition 7.4 (c),

$$\mathrm{mes}\{\phi \in \mathcal{I} : |\langle \nabla V(x(\phi, \eta)), h_0\rangle| < \exp(-(\log\lambda)^{1/2})\} < \exp(-(\log\lambda)^{\mathfrak{c}_1/2}).$$

Since $\exp(-(\log\lambda)^{\mathfrak{c}_1/2}) \ll \mathrm{mes}(\mathcal{I}_0'')$, it follows that there exists $\hat{\phi}$, $|\hat{\phi} - \phi_0| \ll r_0''$, such that

$$|\langle \nabla V(x(\hat{\phi}, \eta)), h_0\rangle| \geq \exp(-(\log\lambda)^{1/2})$$

and therefore

$$|\langle \nabla E^{[a,b]}(\tilde{x}(\hat{\phi}, \eta)), h_0\rangle| \gtrsim \lambda \exp(-(\log\lambda)^{1/2}) \tag{7.25}$$

(we used the first estimate in (7.22), (7.23), Corollary 2.12, and Cauchy estimates). Then Cartan's estimate yields that given $H \gg 1$,

$$\begin{aligned}&\mathrm{mes}\{\phi \in \mathcal{I}_0''/10 : |\langle \nabla E^{[a,b]}(\tilde{x}(\phi, \eta)), h_0\rangle| < \log\lambda - CH(\log\lambda)^{1/2}\}\\ &\quad < C(d)(r_0'')^{d-1}\exp(-H^{1/(d-1)}).\end{aligned}$$

In particular, condition (E) follows by setting $H = N_0^{2(d-1)\delta}$, with $[a, b] = [-N_0, N_0]$ (recall that $\mu \gg \delta$; we choose N_0 such that $N_0^\mu \gg \log\lambda$).

Finally, we check condition (D). Fix $\eta \in (\eta_0 - r_0'', \eta_0 + r_0'')$. For the rest of the proof $\tilde{x}$ stands for the parametrization associated with $[a, b] = [-N_0, N_0]$. Note that for condition (D) to hold it is enough that, given h, $\mathrm{dist}(h, \mathfrak{T}_0) \geq \exp(-N_0^\mu)$, we can find $|n'|, |n''| < N_0^{1/2}$ such that

$$\begin{aligned}&\mathrm{mes}\{\phi \in \mathcal{I}_0 : \mathrm{dist}(\mathrm{spec}\, H_{[-N_0+n', N_0+n'']}(\tilde{x}(\phi, \eta)), \lambda\eta) < \exp(-N_0^\beta/2)\}\\ &\quad < \exp(-N_0^{2\delta}).\end{aligned}$$

We first consider the case $\mathrm{dist}(h, \mathfrak{T}_0) \geq \exp(-\mathfrak{c}_0(\log\lambda)^{3/4})$. Let

$$\mathcal{B}_{\eta,h}' = \mathcal{B}_{(\log\lambda)^{3/4},\eta,h}, \qquad \mathcal{B}_{N_0,\eta,h}' = \bigcup_{|n|\leq N_0} \mathcal{B}_{\eta,h+n\omega}'.$$

Since $\|h + n\omega\| \geq \exp(-\mathfrak{c}_0(\log\lambda)^{3/4})$, using Proposition 7.4, we have

$$\mathrm{mes}(\mathcal{B}_{N_0,\eta,h}') < \exp(-(\log\lambda)^{3\mathfrak{c}_1/4}/2).$$

In particular, there exists $\hat{\phi} \in \mathcal{I}_0''\backslash\mathcal{B}'_{N_0,\eta,h}$, $|\hat{\phi}-\phi_0| \ll r_0''$, such that

$$|V(x(\hat{\phi},\eta)+h+n\omega)-\eta| \geq \exp(-(\log\lambda)^{3/4}), \quad |n| \leq N_0,$$

and therefore

$$|V(\tilde{x}(\hat{\phi},\eta)+h+n\omega)-\eta| \gtrsim \exp(-(\log\lambda)^{3/4}), \quad |n| \leq N_0.$$

Using Cartan's estimate

$$\begin{aligned}&\mathrm{mes}\{\phi \in \mathcal{I}_0''/10 : \log|V(\tilde{x}(\phi,\eta)+h+n\omega)-\eta| \\ &\quad < -C_d(\log\lambda)^{3/4}N_0^{3(d-1)\delta}\} < \exp(-2N_0^{2\delta}), \quad |n| \leq N_0.\end{aligned}$$

Using Lemma 3.6 we get

$$\begin{aligned}&\mathrm{mes}\{\phi \in \mathcal{I}_0''/10 : \mathrm{dist}(\mathrm{spec}\, H_{[-N_0,N_0]}(\tilde{x}(\phi,\eta)+h),\lambda\eta) \\ &\quad < \exp(-C(\log\lambda)^{3/4}N_0^{3(d-1)\delta})\} < \exp(-N_0^{2\delta}),\end{aligned}$$

and condition (D) holds, since $\beta \gg \delta$.

Next we consider the case $\exp(-N_0^{\mu}) \leq \mathrm{dist}(h,\mathfrak{T}_0) < \exp(-\mathfrak{c}_0(\log\lambda)^{3/4})$. Let n_1, $|n_1| \leq 3N_0/2$, such that

$$\mathrm{dist}(h,\mathfrak{T}_0) = \|h-n_1\omega\|.$$

We consider two sub-cases. First, suppose $n_1 \notin [-N_0+N_0^{1/3}, N_0-N_0^{1/3}]$. Note that for $n \in [-N_0+N_0^{1/3}, N_0-N_0^{1/3}]$,

$$\begin{aligned}\|h+n\omega\| &\geq \|(n-n_1)\omega\| - \|h-n_1\omega\| \geq a(CN_0)^{-b} - \exp(-\mathfrak{c}_0(\log\lambda)^{3/4}) \\ &\geq \exp(-\mathfrak{c}_0(\log\lambda)^{3/4}).\end{aligned}$$

Then, as above, we get

$$\begin{aligned}&\mathrm{mes}\{\phi \in \mathcal{I}_0''/10 : \mathrm{dist}(\mathrm{spec}\, H_{[-N_0+N_0^{1/3},N_0-N_0^{1/3}]}(\tilde{x}(\phi,\eta)+h),\lambda\eta) \\ &\qquad < \exp(-C(\log\lambda)^{3/4}N_0^{3(d-1)\delta})\} \\ &\quad < \exp(-N_0^{2\delta}),\end{aligned}$$

and condition (D) holds. Next, we consider $n_1 \in [-N_0+N_0^{1/3}, N_0-N_0^{1/3}]$. Let

$$h_1 = h - n_1\omega, \quad [a_1,b_1] = n_1 + [-N_0,N_0]$$

and $\tilde{x}_1(\phi, \eta)$ the parametrization associated with $[a_1, b_1]$. Note that $[a_1, b_1] \supset [-\underline{N}, \underline{N}]$. Since

$$|\tilde{x}_1(\phi, \eta) + h_1 - x(\phi, \eta)| \lesssim \exp(-\mathfrak{c}_0(\log \lambda)^{3/4}),$$

using (7.22) we have

$$|\lambda^{-1} E^{[a_1,b_1]}(\tilde{x}_1(\phi, \eta) + h_1) - \eta| < \exp(-\mathfrak{c}_0(\log \lambda)^{3/4}/2),$$

for any $\phi \in \mathcal{I}_0''$. Due to the separation of eigenvalues in (7.22), we now have

$$\operatorname{dist}(\operatorname{spec} H_{[a_1,b_1]}(\tilde{x}_1(\phi, \eta) + h_1), \lambda\eta) = |E^{[a_1,b_1]}(\tilde{x}_1(\phi, \eta) + h_1) - \lambda\eta|.$$

Let $\hat{\phi}$ be as in (7.25), with $[a, b] = [a_1, b_1]$, $h_0 = \|h_1\|^{-1} h_1$. Then by Taylor's formula

$$\begin{aligned} |E^{[a_1,b_1]}(\tilde{x}_1(\hat{\phi}, \eta) + h_1) - \lambda\eta| &\geq |\langle \nabla E^{[a_1,b_1]}(\tilde{x}_1(\hat{\phi}, \eta)), h_1\rangle| \, \|h_1\| \\ &\quad - C_\rho \lambda \, \|V\|_\infty \, \|h_1\|^2 \\ &\gtrsim \lambda \exp(-(\log \lambda)^{1/2}) \, \|h_1\| \geq \exp(-2N_0^\mu). \end{aligned}$$

Using Cartan's estimate it follows that

$$\begin{aligned} \operatorname{mes}\{\phi \in \mathcal{I}_0''/10 : \operatorname{dist}(\operatorname{spec} H_{[a_1,b_1]}(\tilde{x}_1(\phi, \eta) + h_1, \lambda\eta)) \\ < \exp(-C(N_0^\mu + N_0^{3(d-1)\delta}))\} < \exp(-N_0^{2\delta}). \end{aligned}$$

Now the conclusion follows from the fact that $\operatorname{spec} H_{[-N_0,N_0]}(\tilde{x}(\phi, \eta) + h) = \operatorname{spec} H_{[a_1,b_1]}(\tilde{x}(\phi, \eta) + h_1)$, and that by (7.24),

$$|\tilde{x}_1(\phi, \eta) - \tilde{x}(\phi, \eta)| \lesssim \exp(-(\log \lambda) N_0^{1/4}) \ll \exp(-N_0^\beta/2)$$

(also recall that $\delta \ll \mu \ll \beta \ll 1$). □

8 Proofs of the main theorems

The first two results are non-perturbative and are stated for operators as in (2.1). For their statements recall the constants S_V and B_0 introduced in (2.6),(2.13), and the exponents δ, $\mathfrak{d}$ used for the inductive conditions in Sects. 5 and 6. We will use the notation $\mathcal{S} := \operatorname{spec} H(x)$.

Theorem B *Assume the notation of the inductive conditions* (A)–(E) *from Sect.* 5. *Let* $E_0 \in \mathbb{R}$, $N_0 \geq 1$, *and assume* $L(E) > \gamma > 0$ *for* $E \in (E_0 - 2r_0, E_0 + 2r_0)$, $r_0 = \exp(-N_0^\delta)$. *If* $N_0 \geq (B_0 + S_V + \gamma^{-1})^C$, $C = C(a, b, \rho)$, *and the conditions* (A)–(E) *hold with* $s = 0$ *for the given* E_0*, then* $[E_0 - r_0, E_0 + r_0] \subset \mathcal{S}$.

Proof Take an arbitrary $E \in (E_0 - r_0, E_0 + r_0)$ and apply Theorem D with $E_s = E$, $s \geq 1$. Since $\mathcal{I}_s \Subset \mathcal{I}_{s-1}$, there exists $\hat{\phi} \in \bigcap_s \mathcal{I}_s$. Due to (5.8) there exists $x(E)$ such that

$$|x(E) - x_s(\hat{\phi}, E)| < 2\exp(-\gamma N_s/30), \quad s \geq 0.$$

Due to (5.9) there exists $\psi(E, \cdot)$, $\|\psi(E, \cdot)\| = 1$, such that

$$\|\psi(E, \cdot) - \psi^{[-N_s', N_s'']}(x_s(\hat{\phi}, E), \cdot)\| < 2\exp(-\gamma N_s/40), \quad s \geq 0.$$

Note that

$$\|(H(x_s(\hat{\phi}, E)) - E)\psi^{[-N_s', N_s'']}(x_s(\hat{\phi}, E), \cdot)\| \lesssim \exp(-\gamma N_s/20)$$

(by condition (C)) and

$$\begin{aligned}\|H(x(E)) - H(x_s(\hat{\phi}, E))\| &\leq C_\rho \|V\|_\infty |x(E) - x_s(\hat{\phi}, E)| \\ &< \exp(-\gamma N_s/40).\end{aligned}$$

It follows that

$$\|(H(x(E)) - E)\psi(E, \cdot)\| \lesssim \exp(-\gamma N_s/40), \quad s \geq 0,$$

and therefore $H(x(E))\psi(E, \cdot) = E\psi(E, \cdot)$. In particular, $E \in \mathcal{S}$ and the conclusion holds (recall that $\mathcal{S}$ is closed). □

Theorem C *Assume the notation of the inductive conditions* ($\underline{\text{A}}$)–($\underline{\text{D}}$) *from Sect.* 6. *Let* $\underline{x}_0 \in \mathbb{T}^d$, $N_0 \geq 1$, *such that the conditions* ($\underline{\text{A}}$)–($\underline{\text{D}}$) *hold, and assume* $L(E) > \gamma > 0$ *for* $E \in (\underline{E}_0 - 2r_0, \underline{E}_0 + 2r_0)$, $r_0 = \exp(-N_0^{\mathfrak{d}})$. *If* $N_0 \geq (B_0 + S_V + \gamma^{-1})^C$, $C = C(a, b, \rho)$, *then there exists* $\underline{E} \in \mathbb{R}$, *such that* $|\underline{E} - \underline{E}_0| < \exp(-\gamma N_0/100)$, $\mathcal{S} \cap (-\infty, \underline{E}) = \emptyset$, *and* $[\underline{E}, \underline{E}_0 + \exp(-N_0^{20\mathfrak{d}})] \subset \mathcal{S}$. *Analogous statements hold relative to conditions* ($\overline{\text{A}}$)–($\overline{\text{D}}$).

Proof We choose N_0 large enough for Theorem E to hold. Using (6.2), we have that there exist

$$\underline{x} = \lim_{s\to\infty} \underline{x}_s, \quad \underline{E} = \lim_{s\to\infty} \underline{E}_s,$$

and we have

$$|\underline{x} - \underline{x}_s|, \; |\underline{E} - \underline{E}_s| < \exp(-\gamma N_s/100), \quad s \geq 1. \tag{8.1}$$

First we verify that $(-\infty, \underline{E}) \cap \mathcal{S} = \emptyset$. Take an arbitrary $E < \underline{E}$ and let $\rho = \underline{E} - E > 0$. By (8.1), for any $s \geq 1$ we have

$$\underline{E}_s - E > \rho - \exp(-\gamma N_s/100)$$

and therefore

$$\operatorname{dist}(\operatorname{spec} H_{[-N_s', N_s'']}(\underline{x}_s), E) > \rho - \exp(-\gamma N_s/100)$$

(recall condition ($\underline{\text{A}}$)). Using (8.1) again,

$$\operatorname{dist}(\operatorname{spec} H_{[-N_s', N_s'']}(\underline{x}), E) > \rho - \exp(-\gamma N_s/200) \geq \rho/2 > 0,$$

for $s \geq s_0$, with s_0 such that $\exp(-\gamma N_{s_0}/200) \leq \rho/2$. Then by Lemma 2.19 we have $\operatorname{dist}(E, \mathcal{S}) \geq \rho/2 > 0$, hence $E \notin \mathcal{S}$, as desired.

By Theorem E, the conditions (A)–(E) are satisfied for any E_s, $\exp(-N_s^{100\mathfrak{d}}) \leq |E_s - \underline{E}_s| \leq \exp(-N_s^{2\mathfrak{d}})$, $s \geq 1$. Then by Theorem B,

$$[\underline{E}_s + \exp(-N_s^{100\mathfrak{d}}), \underline{E}_s + \exp(-N_s^{2\mathfrak{d}})] \subset \mathcal{S}.$$

These intervals overlap for consecutive s (recall that $N_{s+1} = N_s^5$ and $|\underline{E}_{s+1} - \underline{E}_s| < \exp(-\gamma N_s/60)$) and we have

$$\begin{aligned}\mathcal{S} &\supset \bigcup_{s\geq 0} [\underline{E}_s + \exp(-N_s^{100\mathfrak{d}}), \underline{E}_s + \exp(-N_s^{2\mathfrak{d}})] \supset (\underline{E}, \underline{E}_1 + \exp(-N_1^{2\mathfrak{d}})] \\ &\supset (\underline{E}, \underline{E}_0 + \exp(-N_0^{20\mathfrak{d}})]\end{aligned}$$

The conclusion follows since $\mathcal{S}$ is closed. □

We are finally ready to prove Theorem A. We fix the constants $\mathfrak{c}_1, \mathfrak{c}_0, \mathfrak{C}_0$ from Definition 1.1.

Proof of Theorem A (a) Let T_V as in (7.3). Take $C_0 = C_0(a, b, \rho, d)$ large enough, such that for $\lambda \geq \exp((T_V)^{C_0})$, Proposition 7.3 with $\varepsilon = \mathfrak{c}_1/20$, Theorem B, and Theorem C hold for $N_0 = \lfloor \exp((\log\log\lambda)^2) \rfloor$ (recall Proposition 3.4 and Remark 3.5; of course, we take $\gamma = \log\lambda/2$). The choice of ε is made with part (b) in mind.

Let $\underline{E}_0$, $|\lambda^{-1}\underline{E}_0 - V(\underline{x})| \ll \lambda^{-1/4}$ be as in Proposition 7.3 and $\underline{E}$, $|\underline{E} - \underline{E}_0| < \exp(-(\log\lambda)N_0/2)$, be as in Theorem C. Combining Proposition 7.3 with Theorem B we have

$$[\underline{E}_0 + \exp(-N_0^{100\mathfrak{d}}), \underline{E}_0 + \lambda\exp(-(\log\lambda)^{\mathfrak{c}_1/5})] \subset \mathcal{S}_\lambda.$$

At the same time, combining Proposition 7.3 with Theorem C we have

$$[\underline{E}, \underline{E}_0 + \exp(-N_0^{20\mathfrak{d}})] \subset \mathcal{S}_\lambda, \qquad (-\infty, \underline{E}) \cap \mathcal{S}_\lambda = \emptyset.$$

Then

$$[\underline{E}, \underline{E}_0 + \lambda \exp(-(\log \lambda)^{\mathfrak{c}_1/5})] \subset \mathcal{S}_\lambda \tag{8.2}$$

This yields part (a). Of course, the proof the statement relative to the absolute maximum is completely analogous. Also, in the statement of part (a) we could replace $\exp(-(\log \lambda)^{1/2})$ by $\exp(-(\log \lambda)^{\varepsilon})$, for any $\varepsilon \in (0, 1)$, by adjusting the constant C_0 from above.

(b) Recall that $\mathfrak{C}$ denotes the set of critical points of V. Note that since all the critical points are assumed to be non-degenerate, by Lemma 7.1, $\mathfrak{C}$ is discrete and hence finite. Let

$$\nu = \min_{x \in \mathfrak{C}} \left\| \mathfrak{H}(x)^{-1} \right\|^{-1}$$

Using Lemmas 7.1 and 7.2 we choose $c = c(\rho)$ small enough so that with $r = c\nu(1 + \|V\|_\infty)^{-1}$ we have that $\mathbb{T}^d \setminus \bigcup_{x \in \mathfrak{C}} B(x, r)$ is connected and (7.2) holds. Let

$$\mathfrak{g} = \mathfrak{g}(V) := \min\{\|\nabla V(x)\| : x \in \mathbb{T}^d \setminus \bigcup_{x \in \mathfrak{C}} B(x, r)\} > 0,$$

and increase T_V to be

$$\begin{aligned} T_V = 2 &+ \max(0, \log \|V\|_\infty) + \max(0, \log \underline{\iota}^{-1}) \\ &+ \max(0, \log \iota^{-1}) + \max(0, \log \nu^{-1}) + \mathfrak{C}_0 + \mathfrak{c}_0^{-1} + \max(0, \log \mathfrak{g}^{-1}). \end{aligned} \tag{8.3}$$

Take $C_0 = C_0(a, b, \rho, d)$ large enough, such that for $\lambda \geq \exp((T_V)^{C_0})$ in addition to the assumptions for part (a) we also have

$$\exp(-(\log \lambda)^{\mathfrak{c}_1/3}) \leq \min(\nu r/2, \mathfrak{g}), \tag{8.4}$$

and Proposition 7.5 holds with $N_0 = \lfloor \exp((\log \log \lambda)^2) \rfloor$.

Let r_λ such that $\nu r_\lambda/2 = \exp(-(\log \lambda)^{\mathfrak{c}_1/3})$. By (8.4), $r_\lambda \leq r$ and therefore $\mathcal{G}_\lambda := \mathbb{T}^d \setminus \bigcup_{x \in \mathfrak{C}} B(x, r_\lambda)$ is connected. By (8.4) and (7.2),

$$\|\nabla V(x)\| \geq \exp(-(\log \lambda)^{\mathfrak{c}_1/3}), \quad x \in \mathcal{G}_\lambda.$$

Combining Proposition 7.5 and Theorem B we have

$$\{\lambda V(x) : x \in \mathcal{G}_\lambda\} \subset \mathcal{S}_\lambda.$$

Take $\underline{x}', \overline{x}' \in \mathcal{G}_\lambda$, $\|\underline{x}' - \underline{x}\| = \|\overline{x}' - \overline{x}\| = r_\lambda$. Since $\mathcal{G}_\lambda$ is connected we have

$$[\lambda V(\underline{x}'), \lambda V(\overline{x}')] \subset \{\lambda V(x) : x \in \mathcal{G}_\lambda\} \subset \mathcal{S}_\lambda.$$

Let $\underline{E}_0$, $\underline{E}$ as in part (a). By (7.2) and by increasing C_0 if needed,

$$\exp(-3(\log\lambda)^{\mathfrak{c}_1/3}) \le V(\underline{x}') - V(\underline{x}) \le \exp(-(\log\lambda)^{\mathfrak{c}_1/3})$$

and therefore

$$\lambda\exp(-3(\log\lambda)^{\mathfrak{c}_1/3}) \lesssim |\lambda V(\underline{x}') - \underline{E}_0| \lesssim \lambda\exp(-(\log\lambda)^{\mathfrak{c}_1/3}).$$

From the above and (8.2) it follows that $[\underline{E}, \lambda V(\overline{x}')] \subset \mathcal{S}_\lambda$. Let $\overline{E}$ be as in Theorem C with respect to the conditions ($\overline{\text{A}}$)–($\overline{\text{D}}$). Analogously, we get $[\lambda V(\underline{x}'), \overline{E}] \subset \mathcal{S}_\lambda$ and therefore $[\underline{E}, \overline{E}] \subset \mathcal{S}_\lambda$. Since

$$(-\infty, \underline{E}) \cap \mathcal{S}_\lambda = (\overline{E}, \infty) \cap \mathcal{S}_\lambda = \emptyset,$$

we conclude that $\mathcal{S}_\lambda = [\underline{E}, \overline{E}]$. □

Remark 8.1 The constant $\underline{\iota}$ in the definition of T_V from the proof of Theorem A (b) is redundant and can be dropped at the cost of slightly increasing C_0 in the lower bound for λ. More precisely, it can be seen, by using Taylor's formula, that $\underline{\iota}$ can be bound below in terms of ν, $\mathfrak{g}$, $\|V\|_\infty$, and ρ.

9 An example

For the purpose of this section it is convenient to redefine $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$. Let

$$V(x, y) = \cos(x) + s\cos(y).$$

We will check that V satisfies the conditions of Definition 1.1 for $s \notin \{-1, 0, 1\}$.

First, a direct computation shows that conditions (i),(ii) of Definition 1.1 are satisfied for $s \neq 0$ and they fail for $s = 0$.

Next we show that condition (iii) holds for $s \notin \{-1, 0, 1\}$. Take

$$H \gg 1 + \max(\log|s|, \log|s|^{-1}, \log|1 - s^2|^{-1}), \tag{9.1}$$

$h \in \mathbb{T}^2$, $h = (\alpha, \beta)$, $\|h\| \ge \exp(-H)$. The largeness of H will be used tacitly in most of the estimates to follow. Let $g(x, y, \alpha, \beta) := g_{V,h,1,2}(x, y)$, with

$g_{V,h,1,2}$ as in Definition 1.1. Note that $|g_{V,h,1,2}| = |g_{V,h,2,1}|$. In what follows it is useful to "complexify" the functions involved in condition (iii). Let

$$z = \exp(ix), \quad w = \exp(iy), \quad A = \exp(i\alpha), \quad B = \exp(i\beta).$$

Then

$$V(x+\alpha, y+\beta) - V(x, y) = \frac{1}{2zw} P_1(z, w), \quad g(x, y, \alpha, \beta) = -\frac{s}{4zw} Q_1(z, w), \tag{9.2}$$

with

$$\begin{aligned} P_1(z, w) &= (A-1)z^2 w + s(B-1)zw^2 + (A^{-1}-1)w + s(B^{-1}-1)z, \\ Q_1(z, w) &= (B-A)z^2 w^2 + (A-B^{-1})z^2 + (A^{-1}-B)w^2 + B^{-1} - A^{-1}. \end{aligned}$$

Recall that when $\|\cdot\|$ is applied to the shifts h, α, β, it stands for the usual norm on the torus.

Lemma 9.1 *If* $\|\alpha\| < \exp(-3H)$ *or* $\|\beta\| < \exp(-3H)$, *then*

$$\begin{aligned} &\operatorname{mes}\{y \in \mathbb{T} : \min_x(|V(x+\alpha, y+\beta) - V(x, y)| + |g(x, y, \alpha, \beta)|) \\ &\qquad < \exp(-3H)\} \lesssim \exp(-H/2), \\ &\operatorname{mes}\{x \in \mathbb{T} : \min_y(|V(x+\alpha, y+\beta) - V(x, y)| + |g(x, y, \alpha, \beta)|) \\ &\qquad < \exp(-3H)\} \lesssim \exp(-H/2). \end{aligned}$$

Proof We only check the first estimate, the other one following analogously. First we assume $\|\alpha\| < \exp(-3H)$. Since $\|h\| \geq \exp(-H)$, we must have $\|\beta\| \gtrsim \exp(-H)$. Note that

$$\begin{aligned} P_1(z, w) &= (A-1)z^2 w + s(B-1)zw^2 + (A^{-1}-1)w + s(B^{-1}-1)z \\ &= w(A-1)(z^2 - A^{-1}) + sz(B-1)(w^2 - B^{-1}). \end{aligned}$$

Then

$$\left| \frac{1}{2zw} P_1(z, w) \right| \geq cs \exp(-H)|w^2 - B^{-1}| - C \exp(-3H) > \exp(-3H),$$

provided $|w^2 - B^{-1}| \geq \exp(-H/2)$. Therefore the first estimate holds (recall (9.2)).

Next we assume $\|\beta\| < \exp(-3H)$. As before, we must have $\|\alpha\| \gtrsim \exp(-H)$. Note that

$$Q_1(z,w) = (B-1)z^2w^2 + (1-B^{-1})z^2 + (1-B)w^2 + B^{-1} - 1$$
$$+(w^2-1)[(1-A)z^2 + A^{-1} - 1].$$

We have

$$\left|\frac{1}{2zw}P_1(z,w)\right| \geq c|w|\exp(-H)|z^2 - A^{-1}| - Cs\exp(-3H)$$

and

$$\left|\frac{s}{4zw}Q_1(z,w)\right| \geq cs|w^2-1||(1-A)z^2 + A^{-1} - 1| - Cs\exp(-3H)$$
$$\geq cs|w^2-1|\exp(-H)|z^2$$
$$+(1-A)^{-1}(A^{-1}-1)| - Cs\exp(-3H).$$

Note that

$$|z^2 - A^{-1}| + |z^2 + (1-A)^{-1}(A^{-1}-1)| \geq |A^{-1} + (1-A)^{-1}(A^{-1}-1)| = |2/A| = 2.$$

Then

$$\left|\frac{1}{2zw}P_1(z,w)\right| + \left|\frac{s}{4zw}Q_1(z,w)\right|$$
$$\geq c\exp(-2H)(|z^2 - A^{-1}| + |z^2 + (1-A)^{-1}(A^{-1}-1)|)$$
$$-Cs\exp(-3H) > \exp(-3H),$$

provided $|w|, |w^2-1| \geq \exp(-H/2)$. The conclusion follows. □

Lemma 9.2 *There exists an absolute constant* $C_0 \gg 1$ *such that if* $\|\alpha - \beta\| < \exp(-2C_0H)$ *or* $\|\alpha + \beta\| < \exp(-2C_0H)$*, then*

$$\text{mes}\{y \in \mathbb{T} : \min_x(|V(x+\alpha, y+\beta) - V(x,y)| + |g(x,y,\alpha,\beta)|)$$
$$< \exp(-C_0H)\} \lesssim \exp(-H/2),$$
$$\text{mes}\{x \in \mathbb{T} : \min_y(|V(x+\alpha, y+\beta) - V(x,y)| + |g(x,y,\alpha,\beta)|)$$
$$< \exp(-C_0H)\} \lesssim \exp(-H/2).$$

Proof We only prove the first estimate under the assumption that $\|\alpha - \beta\|$ is small. The other cases are completely analogous. We have

$$P_1(z,w) = \tilde{P}_1(z,w) + s(B-A)zw^2 + s(B^{-1} - A^{-1})z, \quad (9.3)$$
$$\tilde{P}_1(z,w) = (A-1)z^2w + s(A-1)zw^2 + (A^{-1}-1)w + s(A^{-1}-1)z, \quad (9.4)$$

$$Q_1(z,w) = \tilde{Q}_1(z,w) + (B-A)z^2w^2 + B^{-1} - A^{-1}, \tag{9.5}$$
$$\tilde{Q}_1(z,w) = (A-B^{-1})z^2 + (A^{-1}-B)w^2. \tag{9.6}$$

Let a_i, b_i be the polynomials in w such that

$$\tilde{P}_1(z,w) = a_2z^2 + a_1z + a_0, \quad \tilde{Q}_1(z,w) = b_2z^2 + b_1z + b_0.$$

Let

$$\tilde{R}_1(w) = \mathrm{Res}_z(\tilde{P}_1, \tilde{Q}_1) = \det \begin{bmatrix} a_2 & 0 & b_2 & 0 \\ a_1 & a_2 & b_1 & b_2 \\ a_0 & a_1 & b_0 & b_1 \\ 0 & a_0 & 0 & b_0 \end{bmatrix}.$$

Analyzing the degrees of the terms from the Leibniz formula for the above determinant, one sees that $\tilde{R}_1(w)$ is a polynomial of degree 6 and the only terms containing a monomial of degree 6 are

$$a_2^2b_0^2 = [(A-1)w]^2[(A^{-1}-B)w^2]^2,$$
$$a_1^2b_2b_0 = [s(A-1)w^2 + s(A^{-1}-1)]^2(A-B^{-1})(A^{-1}-B)w^2$$

corresponding to the even permutations

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}.$$

It follows that the leading coefficient is

$$\begin{aligned} c_6 &:= (A-1)^2(A^{-1}-B)[(A^{-1}-B) + s^2(A-B^{-1})] \\ &= (A-1)^2(A^{-1}-B)^2(1-s^2B^{-1}A). \end{aligned}$$

Since $\|\alpha-\beta\| \ll \exp(-H)$ and $\|h\| \geq \exp(-H)$, we have

$$\|\alpha\|, \|\beta\|, \|\alpha+\beta\| \gtrsim \exp(-H)$$

and therefore

$$|c_6| \gtrsim \exp(-2H)\exp(-2H)|1-|s|^2| > \exp(-5H).$$

Then, using Lemma 2.26,

$$|\tilde{R}_1(\exp(iy))| \geq \exp(-CH),$$

for $y \in \mathbb{T}\setminus\mathcal{B}$, $\operatorname{mes}(\mathcal{B}) < \exp(-H/2)$, with C an absolute constant. Note that

$$|a_2(\exp(iy))| = |(A-1)\exp(iy)| \gtrsim \exp(-H),$$
$$|b_2(\exp(iy))| = |A - B^{-1}| \gtrsim \exp(-H)$$

for any $y \in \mathbb{T}$. Let $r(\exp(iy))$ be the maximum of the absolute values of the roots of $\tilde{P}_1(\cdot, \exp(iy))$ and $\tilde{Q}_1(\cdot, \exp(iy))$. Using Lemma 2.25 we have $r(\exp(iy)) \le \exp(CH)$, for $y \in \mathbb{T}$. It follows that

$$|\tilde{R}_1(\exp(iy))| \ge 2|a_2(\exp(iy))|^2 |b_2(\exp(iy))|^2 r(\exp(iy))^3 \delta, \quad y \in \mathbb{T}\setminus\mathcal{B},$$

where $\delta = \exp(-CH)$, with C a sufficiently large absolute constant. By Lemma 2.24,

$$\max(\tilde{P}_1(z, \exp(iy)), \tilde{Q}_1(z, \exp(iy))) \ge \min(|a_2(\exp(iy))|, |b_2(\exp(iy))|)\delta^2$$
$$> \exp(-CH)$$

for any z and $y \in \mathbb{T}\setminus\mathcal{B}$. The conclusion follows by recalling (9.2) and (9.3). □

Lemma 9.3 *If* $\|\alpha\|, \|\beta\| \ge \exp(-3H)$*, then there exists an absolute constant* $C_0 \gg 1$ *such that*

$$\operatorname{mes}\{y \in \mathbb{T} : \min_x (|V(x+\alpha, y+\beta) - V(x,y)| + |g(x,y,\alpha,\beta)|)$$
$$< \exp(-C_0 H)\} \lesssim \exp(-H/2),$$
$$\operatorname{mes}\{x \in \mathbb{T} : \min_y (|V(x+\alpha, y+\beta) - V(x,y)| + |g(x,y,\alpha,\beta)|)$$
$$< \exp(-C_0 H)\} \lesssim \exp(-H/2).$$

Proof We only check the first estimate, the second one being completely analogous. The proof is similar to that of the previous lemma. Let a_i, b_i be the polynomials in w such that

$$P_1(z,w) = a_2 z^2 + a_1 z + a_0, \quad Q_1(z,w) = b_2 z^2 + b_1 z + b_0.$$

Let

$$R_1(w) = \operatorname{Res}_z(P_1, Q_1) = \det \begin{bmatrix} a_2 & 0 & b_2 & 0 \\ a_1 & a_2 & b_1 & b_2 \\ a_0 & a_1 & b_0 & b_1 \\ 0 & a_0 & 0 & b_0 \end{bmatrix}.$$

Analyzing the degrees of the terms from the Leibniz formula for the above determinant, one sees that $R_1(w)$ is a polynomial of degree 8 and the only term containing a monomial of degree 8 is

$$a_1^2 b_2 b_0 = [s(B-1)w^2 + s(B^{-1}-1)]^2[(B-A)w^2 + A - B^{-1}]$$
$$[(A^{-1}-B)w^2 + B^{-1} - A^{-1}],$$

corresponding to the even permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}.$$

It follows that the leading coefficient is

$$c_8 := s^2(B-1)^2(B-A)(A^{-1}-B).$$

If $|B-A| < \exp(-CH)$ or $|A^{-1}-B| < \exp(-CH)$, with $C \gg 1$ a sufficiently large absolute constant, the conclusion follows by Lemma 9.2. So, we just need to consider the case when $|B-A|, |A^{-1}-B| \geq \exp(-CH)$, $C \gg 1$. Note that we have $|c_8| \geq \exp(-CH)$. Using Lemma 2.26, we have

$$|R_1(\exp(iy))| \geq \exp(-CH),$$

for $y \in \mathbb{T}\setminus\mathcal{B}_1$, $\mathrm{mes}(\mathcal{B}_1) < \exp(-H/2)$, with C an absolute constant. Applying Lemma 2.26 again to $b_2(w) = (B-A)w^2 + A - B^{-1}$, we get that

$$|b_2(\exp(iy))| \geq \exp(-CH),$$

for $y \in \mathbb{T}\setminus\mathcal{B}_2$, $\mathrm{mes}(\mathcal{B}_2) < \exp(-H/2)$. At the same time,

$$|a_2(\exp(iy))| = |(A-1)\exp(iy)| \gtrsim \exp(-3H),$$

for any $y \in \mathbb{T}$. Let $r(\exp(iy))$ be the maximum of the absolute values of the roots of $P_1(\cdot, \exp(iy))$ and $Q_1(\cdot, \exp(iy))$. Using Lemma 2.25 we have that the $r(\exp(iy)) \leq \exp(CH)$, for $y \in \mathbb{T}\setminus\mathcal{B}_2$.

Fix $y \in \mathbb{T}\setminus\mathcal{B}$, $\mathcal{B} := \mathcal{B}_1 \cup \mathcal{B}_2$. It follows that

$$|R_1(\exp(iy))| \geq 2|a_2(\exp(iy))|^2|b_2(\exp(iy))|^2 r(\exp(iy))^3\delta,$$

where $\delta = \exp(-CH)$, with C a sufficiently large absolute constant. By Lemma 2.24,

$$\max(P_1(z, \exp(iy)), Q_1(z, \exp(iy))) \geq \min(|a_2(\exp(iy))|, |b_2(\exp(iy))|)\delta^2$$
$$> \exp(-CH)$$

for any z and any $y \in \mathbb{T}\setminus\mathcal{B}$, and the conclusion follows (recall (9.2)). ◻

Now condition (iii) follows from Lemmas 9.1 and 9.3, by setting $K = C_0 H$, with C_0 as in Lemma 9.3, and by taking $\mathfrak{c}_0 = 1/C_0$, $\mathfrak{c}_1 = 1/2$,

$$\mathfrak{C}_0 = C(C_0^2 + C_0 \max(\log|s|, \log|s|^{-1}, \log|1-s^2|^{-1})), \quad C \gg 1. \tag{9.7}$$

Finally, we check that condition (iv) holds for $s \notin \{-1, 0, 1\}$. Take H as in (9.1), $\eta \in \mathbb{R}$, and $h_0 \in \mathbb{R}^2$ a unit vector. With some abuse of notation we let $h_0 = (\alpha, \beta)$, $\alpha^2 + \beta^2 = 1$.

Lemma 9.4 *(a) If* $|\alpha| < \exp(-2H)$*, then*

$$\operatorname{mes}\{y \in \mathbb{T} : \min_x |\langle \nabla V(x, y), h_0 \rangle| < \exp(-2H)\} < \exp(-H).$$

(b) If $|\beta| < \exp(-2H)$*, then*

$$\operatorname{mes}\{x \in \mathbb{T} : \min_y |\langle \nabla V(x, y), h_0 \rangle| < \exp(-2H)\} < \exp(-H).$$

Proof (a) Since $|\alpha| < \exp(-2H)$, we have $|\beta| \geq (1 - \exp(-4H))^{1/2} > 1/2$, and therefore

$$\begin{aligned} |\langle \nabla V(x, y), h_0 \rangle| = |\alpha \sin x + s\beta \sin y| &\geq \frac{1}{2}|s \sin y| - \exp(-2H) \\ &\geq \exp(-2H), \end{aligned}$$

for all $x \in \mathbb{T}$, and y such that $|\sin y| > \exp(-3H/2)$. The conclusion follows. The proof for (b) is analogous. □

Lemma 9.5 *(a) If* $|\alpha| \geq \exp(-2H)$*, then there exists an absolute constant* $C_0 \gg 1$ *such that*

$$\begin{aligned} &\operatorname{mes}\{y \in \mathbb{T} : \min_x (|V(x, y) - \eta| + |\langle \nabla V(x, y), h_0 \rangle|) < \exp(-C_0 H)\} \\ &\quad < \exp(-H/2). \end{aligned}$$

(b) If $|\beta| \geq \exp(-2H)$*, then there exists an absolute constant* $C_0 \gg 1$ *such that*

$$\begin{aligned} &\operatorname{mes}\{x \in \mathbb{T} : \min_y (|V(x, y) - \eta| + |\langle \nabla V(x, y), h_0 \rangle|) < \exp(-C_0 H)\} \\ &\quad < \exp(-H/2). \end{aligned}$$

Proof We only prove (a), the proof of the second statement being analogous. By letting $z = \exp(ix)$, $w = \exp(iy)$, we have

$$V(x, y) - \eta = \frac{1}{2zw} P_2(z, w), \quad \langle \nabla V(x, y), h_0 \rangle = -\frac{1}{2izw} Q_2(z, w), \tag{9.8}$$

with

$$P_2(z,w) = z^2 w + s z w^2 - 2\eta z w + w + s z,$$
$$Q_2(z,w) = \alpha z^2 w + \beta z w^2 - \alpha w - \beta z.$$

Let a_i, b_i be the polynomials in w such that

$$P_2(z,w) = a_2 z^2 + a_1 z + a_0, \quad Q_2(z,w) = b_2 z^2 + b_1 z + b_0.$$

In particular, $a_2(w) = w$ and $b_2(w) = \alpha w$. A direct computation yields

$$\begin{aligned} R_2(w) &= \mathrm{Res}_z(P_2, Q_2) = \sum_{k=0}^{6} c_k w^k \\ &= w^6\left(-\alpha^2 s^2 + \beta^2\right) + w^5(4\alpha^2 \eta s) \\ &\quad + w^4\left(-4\alpha^2\eta^2 - 2\alpha^2 s^2 + 4\alpha^2 - 2\beta^2\right) + w^3(4\alpha^2\eta s) \\ &\quad + w^2\left(-\alpha^2 s^2 + \beta^2\right) \\ &= w^6\left(1 - \alpha^2(1+s^2)\right) + w^5(4\alpha^2\eta s) \\ &\quad + w^4\left(\alpha^2(6 - 4\eta^2 - 2s^2) - 2\right) + w^3(4\alpha^2\eta s) + w^2\left(1 - \alpha^2(1+s^2)\right). \end{aligned}$$

We will argue that not all of the coefficients of R_2 are too small. To this end, note that

$$2\alpha^{-2} c_6 + \alpha^{-2} c_4 = 4 - 4s^2 - 4\eta^2.$$

If $|\eta| < \exp(-H)$, then

$$|2\alpha^{-2} c_6 + \alpha^{-2} c_4| > 4|1 - s^2| - 4\exp(-2H) > 2|1 - s^2| > \exp(-H),$$

and therefore, either

$$|c_6| \gtrsim \exp(-H)\alpha^2 \geq \exp(-5H) \qquad \text{or} \qquad |c_4| \gtrsim \exp(-H)\alpha^2 \geq \exp(-5H).$$

On the other hand, if $|\eta| \geq \exp(-H)$, then

$$|c_5| \gtrsim \alpha^2 \exp(-H) s > \exp(-6H).$$

Thus, $\max_k |c_k| \gtrsim \exp(-6H)$. Then, using Lemma 2.26,

$$|R_2(\exp(iy))| \geq \exp(-CH),$$

for $y \in \mathbb{T} \setminus \mathcal{B}$, $\mathrm{mes}(\mathcal{B}) < \exp(-H/2)$. Let $r(\exp(iy))$ be the maximum of the absolute values of the roots of $P_2(\cdot, \exp(iy))$ and $Q_2(\cdot, \exp(iy))$. By

Lemma 2.25, $r(\exp(iy)) < \exp(3H)$. Then

$$|R_2(\exp(iy))| \geq 2|a_2(\exp(iy))|^2 |b_2(\exp(iy))|^2 r(\exp(iy))^3 \delta,$$

for $y \in \mathbb{T} \backslash \mathcal{B}$, with $\delta = \exp(-CH)$. By Lemma 2.24,

$$\max(P_2(z, \exp(iy)), Q_2(z, \exp(iy))) \geq \min(|a_2(\exp(iy))|, |b_2(\exp(iy))|)\delta^2 > \exp(-CH)$$

for any z, and $y \in \mathbb{T} \backslash \mathcal{B}$. The conclusion follows by recalling (9.8). □

Now condition (iv) follows from Lemmas 9.4 and 9.5, by setting $K = C_0 H$, with C_0 as in Lemma 9.5, and by taking $\mathfrak{c}_1 = 1/2$ and $\mathfrak{C}_0$ as in (9.7), with the new C_0. Obviously, we can arrange for both condition (iii) and (iv) to hold with the same $\mathfrak{C}_0$.

Remark 9.6 (a) It should be clear that for $s \in \{-1, 0, 1\}$ not all of the conditions are satisfied. Indeed, we noted that conditions (i) and (ii) fail for $s = 0$, and for $s = \pm 1$, for example, condition (iv) fails for $\eta = 0$ and h_0 proportional to $(\pm 1, 1)$.

(b) Due to the choices of $\mathfrak{C}_0$ in (9.7) and λ_0 implied by the proof of Theorem A (recall (8.3)), we have that as s approaches $\{-1, 0, 1\}$, λ_0 approaches ∞, as claimed in Remark 1.2 (c).

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