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Homogeneity of the spectrum for quasi-periodic Schrödinger operators

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Abstract. We consider the one-dimensional discrete Schrödinger operator

$$[H(x,\omega)\varphi](n) \equiv -\varphi(n-1) - \varphi(n+1) + V(x+n\omega)\varphi(n),$$

$n \in \mathbb{Z}$, $x, \omega \in [0,1]$, with real-analytic potential $V(x)$. Assume $L(E,\omega) > 0$ for all E. Let $\mathcal{S}_\omega$ be the spectrum of $H(x,\omega)$. For all ω obeying the Diophantine condition $\omega \in \mathbb{T}_{c,a}$, we show the following: if $\mathcal{S}_\omega \cap (E', E'') \neq \emptyset$, then $\mathcal{S}_\omega \cap (E', E'')$ is homogeneous in the sense of Carleson [Car83]. Furthermore, we prove that if G_i, $i = 1, 2$, are two gaps with $1 > |G_1| \geq |G_2|$, then $|G_2| \lesssim \exp(-(\log \operatorname{dist}(G_1, G_2))^A)$, $A \gg 1$. Moreover, the same estimates hold for the gaps in the spectrum on a finite interval, that is, for $\mathcal{S}_{N,\omega} := \bigcup_{x\in\mathbb{T}} \operatorname{spec} H_{[-N,N]}(x,\omega)$, $N \geq 1$, where $H_{[-N,N]}(x,\omega)$ is the Schrödinger operator restricted to the interval $[-N, N]$ with Dirichlet boundary conditions. In particular, all these results hold for the almost Mathieu operator with $|\lambda| \neq 1$. For the supercritical almost Mathieu operator, we combine the methods of [GS08] with Jitomirskaya's approach from [Jit99] to establish most of the results from [GS08] with ω obeying a strong Diophantine condition.

Keywords. Quasiperiodic Schrödinger operators, Anderson localization, homogeneous set

1. Introduction

We consider quasi-periodic Schrödinger equations

$$[H(x,\omega)\varphi](n) \equiv -\varphi(n-1) - \varphi(n+1) + V(x+n\omega)\varphi(n) = E\varphi(n) \tag{1.1}$$

in the regime of positive Lyapunov exponents. We assume that $V(x)$ is a 1-periodic, real-analytic function. Recall that for irrational ω, the spectrum of $H(x,\omega)$ does not depend on x. We denote it by $\mathcal{S}_\omega$. It was shown in [GS11] that $\mathcal{S}_\omega$ is a Cantor set for almost

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every irrational ω, in the regime of positive Lyapunov exponent. The main objective of this work is to show that the structure of the gaps is "regular". More specifically, a closed set $\mathcal{S} \subset \mathbb{R}$ is called *homogeneous* if there is $\tau > 0$ such that for any $E \in \mathcal{S}$ and any $0 < \sigma \leq \operatorname{diam}(\mathcal{S})$, the estimate

$$|\mathcal{S} \cap (E - \sigma, E + \sigma)| > \tau\sigma \tag{1.2}$$

holds (see [Car83]). We then also say that $\mathcal{S}$ is τ-*homogeneous*.

Theorem H. *Let*

$$\omega \in \mathbb{T}_{c,a} := \left\{ \omega \in [0,1] : \|n\omega\| \geq \frac{c}{n(\log n)^a}, n \geq 1 \right\}.$$

If $E_0 \in \mathcal{S}_\omega$ and $L(\omega, E_0) \geq \gamma > 0$ then there exists $\sigma_0 = \sigma_0(V, c, a, \gamma)$ such that

$$|\mathcal{S}_\omega \cap (E_0 - \sigma, E_0 + \sigma)| \geq \sigma/2 \tag{1.3}$$

for all $\sigma \in (0, \sigma_0]$. In particular:

(a) *If $L(\omega, E) \geq \gamma$ for all $E \in \mathbb{R}$, then $\mathcal{S}_\omega$ is τ-homogeneous with some $\tau = \tau(V, c, a, \gamma)$.*
(b) *If $L(\omega, E) \geq \gamma$ for all $E \in (E', E'')$ and there exists $\varepsilon > 0$ such that*

$$\mathcal{S}_\omega \cap (E' - \varepsilon, E'' + \varepsilon) = \mathcal{S}_\omega \cap (E', E''), \tag{1.4}$$

then $\mathcal{S}_\omega \cap (E', E'')$ is either empty or τ-homogeneous with some $\tau = \tau(V, c, a, \gamma, \varepsilon)$.

The previous statements also hold with $\mathcal{S}_{N,\omega} := \bigcup_{x \in \mathbb{T}} \operatorname{spec} H_{[-N,N]}(x, \omega)$, $N \geq 1$, instead of $\mathcal{S}_\omega$ (here $H_{[-N,N]}(x, \omega)$ is the Schrödinger operator restricted to the interval $[-N, N]$ with Dirichlet boundary conditions).

Remarks. (1) If we introduce a coupling constant, that is, if we replace V by λV, we know by Sorets–Spencer [SS91] that part (a) of our theorem applies for $\lambda \geq \lambda_0(V)$. For part (b) we note that for energies near the edges of the interval (E', E'') we do not know how much of the nearby spectrum afforded by (1.3) sits inside (E', E''). We deal with this issue by imposing condition (1.4) which forces all the spectrum near (E', E'') to be in (E', E''). In general, if we assume that the Lyapunov exponent does not vanish throughout the spectrum, the existence of the intervals (E', E'') to which part (b) of our theorem applies follows from the continuity of the Lyapunov exponent (see [GS01], [BJ02]) and the density of gaps given by [GS11]. Indeed, given $E_0 \in \mathcal{S}_\omega$ such that $L(\omega, E_0) \geq \gamma > 0$, we can find an interval (E', E'') such that $E_0 \in (E', E'')$, $L(\omega, E) \geq \gamma/2 > 0$ for $E \in (E', E'')$, and $E', E'' \notin \mathcal{S}_\omega$. The last condition ensures that we have (1.4) with $\varepsilon = \varepsilon(\operatorname{dist}(\{E', E''\}, \mathcal{S}_\omega))$.

(2) In general our theorem does not guarantee that $\mathcal{S}_\omega \cap \{E \in \mathbb{R} : L(\omega, E) > 0\}$ is homogeneous (unless we are in the setting of part (a)). However, this is indeed true for typical analytic potentials, in the sense of the Main Theorem of [Avi15]. Recall that in [Avi15] it is shown that for typical analytic potentials there exist finitely many disjoint closed intervals I_k such that $\mathcal{S}_\omega \subset \bigcup_k I_k$ and $\mathcal{S}_\omega \cap I_k$ is either absolutely continuous or

pure point. Furthermore, one has spectral uniformity in both subcritical and supercritical regimes. For the supercritical regime this means that there exists $\gamma > 0$ such that $\mathcal{S}_\omega \cap \{E \in \mathbb{R} : L(\omega, E) > 0\} = \mathcal{S}_\omega \cap \{E \in \mathbb{R} : L(\omega, E) \geq \gamma\}$. One can now apply part (b) of our theorem on each non-empty interval $I_k \cap \{E \in \mathbb{R} : L(\omega, E) \geq \gamma\}$ to deduce the homogeneity of the spectrum in the supercritical regime.

(3) The strong Diophantine condition on ω can be relaxed. This is one of the results in the ongoing work of Tao and Voda [TV16]. In the current work we use the existing results developed assuming the strong Diophantine condition in [GS08] and [GS11].

The homogeneity property of the spectrum of quasi-periodic Schrödinger operators in the regime of *small coupling* was recently established in [DGL16]. There the continuum quasi-periodic Schrödinger operator

$$-\psi''(x) + V(x)\psi(x) = E\psi(x), \qquad x \in \mathbb{R}, \tag{1.5}$$

is considered, where

$$V(x) = \sum_{n \in \mathbb{Z}^\nu} c(n) e(xn\omega), \tag{1.6}$$

$$|c(m)| \leq \varepsilon \exp(-\kappa_0 |m|) \tag{1.7}$$

with $\kappa_0 > 0$, ε being small and with a Diophantine vector ω,

$$|n\omega| \geq a_0 |n|^{-b_0}, \qquad n \in \mathbb{Z}^\nu \setminus \{0\}, \tag{1.8}$$

for some

$$0 < a_0 < 1, \qquad \nu < b_0 < \infty.$$

The following relation between the gaps and the bands of the operator is established using the estimates from [DG14]:

There exists $\varepsilon_0 = \varepsilon_0(\kappa_0, a_0, b_0) > 0$ such that for $0 < \varepsilon < \varepsilon_0$, the gaps in the spectrum of the operator H can be labeled as $G_m = (E_m^-, E_m^+)$, $m \in \mathbb{Z}^\nu \setminus \{0\}$, $G_0 = (-\infty, \underline{E})$ so that the following estimates hold:

(i) For every $m \in \mathbb{Z}^\nu \setminus \{0\}$, we have

$$E_m^+ - E_m^- \leq 2\varepsilon \exp(-\kappa_0 |m|/2).$$

(ii) For every $m, m' \in \mathbb{Z}^\nu \setminus \{0\}$ with $m' \neq m$ and $|m'| \geq |m|$, we have

$$\operatorname{dist}([E_m^-, E_m^+], [E_{m'}^-, E_{m'}^+]) \geq a|m'|^{-b},$$

where $a, b > 0$ are constants depending on a_0, b_0, κ_0, ν.

(iii) For every $m \in \mathbb{Z}^\nu \setminus \{0\}$,

$$E_m^- - \underline{E} \geq a|m|^{-b},$$

This feature was not known for the almost Mathieu operator even in the regime of small coupling. The homogeneity property can be derived from (i)–(iii). In the current paper we establish a slightly weaker version of (i)–(iii).

Theorem G. *Let* $\omega \in \mathbb{T}_{c,a}$ *and assume* $L(\omega, E) \geq \gamma > 0$ *for any* $E \in (E', E'')$. *There exists* $N_0(V, c, a, \gamma)$ *such that if* $N \geq N_0$ *and* G_1, G_2 *are two gaps in* $\mathcal{S}_\omega \cap (E', E'')$ *with* $|G_1|, |G_2| > \exp(-N^{1-})$ *then* $\operatorname{dist}(G_1, G_2) > \exp(-(\log N)^{C_0})$ *with* $C_0 = C_0(V, c, a, \gamma)$. *The same statement holds for gaps in* $\mathcal{S}_{\bar{N},\omega} \cap (E', E'')$ *with* $\bar{N} \geq N$.

It is natural to inquire about the precise calibration between the gaps and the bands. In particular, is it true that, in Theorem G, one has

$$\operatorname{dist}(G_1, G_2) \geq a|N|^{-b},$$

with $a, b > 0$ being constants depending on V, ω and the lower bound $L(E) \geq \gamma > 0$? Moreover, if so, are these estimates optimal?

Consider the *almost Mathieu operator*

$$[H(x, \omega)\phi](n) = -\phi(n-1) - \phi(n+1) + 2\lambda \cos(2\pi(x + n\omega))\phi(n), \quad n \in \mathbb{Z}. \tag{1.9}$$

It is a fundamental fact that the Lyapunov exponent here obeys

$$L(\omega, E) \geq \log|\lambda|$$

for all E. Thus, as a particular case of Theorem H and as a consequence of Aubry duality, we have the following.

Theorem H′. *Let* $|\lambda| \neq 1$ *and* $\omega \in \mathbb{T}_{c,a}$. *Then the set* $\mathcal{S}_\omega$ *is* τ*-homogeneous for some* $\tau = \tau(c, a, \lambda)$. *Furthermore, the estimates in Theorem G hold.*

The relevance of the homogeneity property to the inverse spectral theory of almost periodic potentials (or Jacobi matrices with almost periodic coefficients) was established in the remarkable work by Sodin and Yuditskii [SY95, SY97]. They studied the inverse spectral problem for reflectionless Jacobi matrices whose spectrum is a given homogeneous set. The reflectionless potentials were introduced, in the continuum setting, by Craig [Cra89]. Reflectionless potentials are very relevant to the spectral theory of ergodic potentials. Different classes of potentials, which are in fact reflectionless, were studied, prior to [Cra89], in the basic works on ergodic potentials by Deift and Simon [DS83], Johnson [Joh82], Johnson and Moser [JM82], and Kotani [Kot84], [Kot87]. It was shown in [Cra89] that being reflectionless is the key feature which allows for the development of a number of fundamental objects from the periodic theory like auxiliary spectrum, trace formula, product expansions; see also the work by Gesztezy and Simon [GS96]. Employing the version of the trace formula from [GS96], Gesztezy and Yuditskii [GY06] found another remarkable consequence of the homogeneity property combined with being reflectionless: the spectrum is purely absolutely continuous. See also the paper by Poltoratski and Remling [PR09], where an even stronger result was established.

In view of these results and the results of the current paper it seems very natural to investigate the connection between the homogeneity property and the spectral phase transition theory of quasi-periodic potentials (see the work by Avila [Avi15]). Namely, we would like to pose the following question:

Problem 1. Consider

$$[H(x,\omega)\varphi](n) \equiv -\varphi(n-1) - \varphi(n+1) + \lambda V(x+n\omega)\varphi(n) = E\varphi(n) \tag{1.10}$$

with real-analytic V and Diophantine ω. It is known that for small λ, this operator has a complete set of Bloch–Floquet eigenfunctions. We expect that, in analogy to the small-coupling result in the continuum case from [DGL16], one can also prove that the spectrum is homogeneous, and moreover the calibration estimates (i)–(iii) for gaps and bands hold. Assume that the Lyapunov exponent $L(E,\lambda)$ vanishes on the spectrum for all $0 < \lambda < \lambda_0$. Can one find a complete set of Bloch–Floquet eigenfunctions for $0 < \lambda < \lambda_0$? The main issue here is how to control the homogeneity property of the spectrum using the zero Lyapunov exponent on the spectrum. Indeed, while vanishing Lyapunov exponents on a set of positive Lebesgue measure imply the presence of absolutely continuous spectrum, the homogeneity of the spectrum is a sufficient condition for *purely* absolutely continuous spectrum. Once the latter property has been established, the existence of a complete set of Bloch–Floquet eigenfunctions follows from the work of Kotani [Kot84], Deift–Simon [DS83], and Avila–Krikorian [AK06].

Finally, we emphasize that the analysis of "fine properties" of the spectral set $\mathcal{S}_{N,\omega} := \bigcup_{x\in\mathbb{T}} \operatorname{spec} H_{[-N,N]}(x,\omega)$ on **a finite interval**, especially with general analytic V, seems to be a very interesting problem in its own right. The numerical plots of the eigenvalues $E_j^{(N)}(x)$ of $H_{[-N,N]}(x,\omega)$ (Rellich parametrization) look very complicated (see Figures 1–3).

One can see some "almost gaps" shadowed by "rare graphs fragments"—see for instance Figure 1 between the spectral value levels $E = 2$ and $E = 4$. Even for the almost

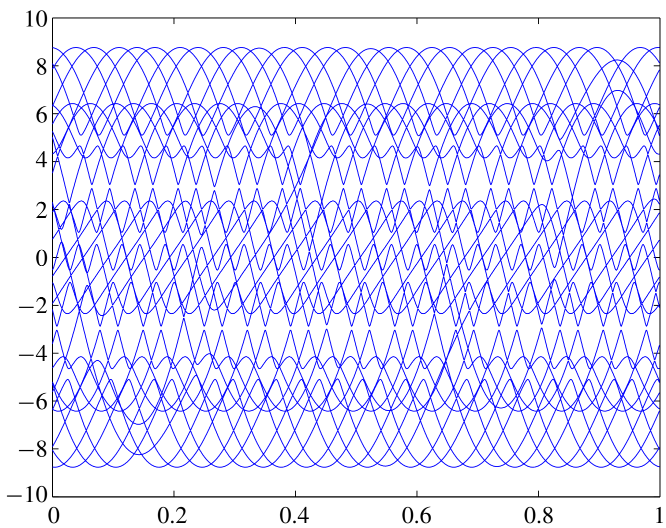

Fig. 1. Rellich functions, V 3rd degree, $N = 29$.

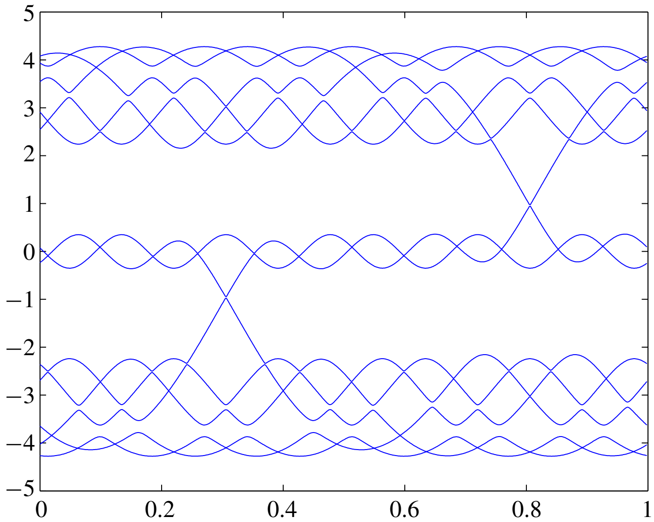

Fig. 2. Rellich functions, almost Mathieu, Dirichlet BC, $N = 12$.

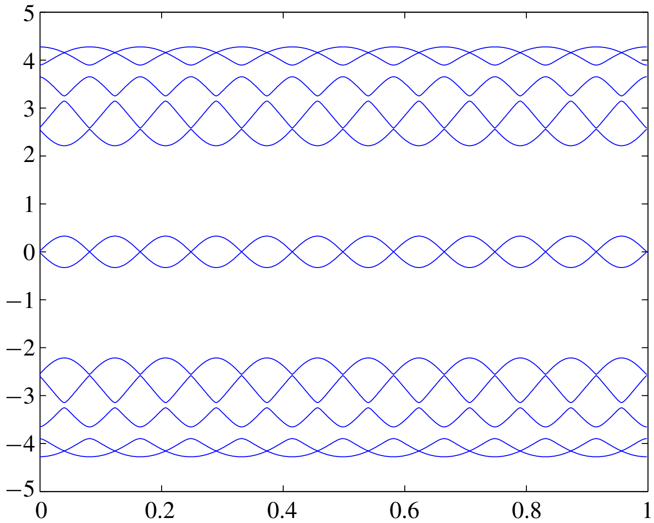

Fig. 3. Rellich functions, almost Mathieu, periodic case, $N = 12$.

Mathieu case, the picture still has some "gaps shadowing" (see Figure 2). The picture simplifies under periodic boundary conditions (see Figure 3).

Finally, we would like to pose the following problem:

Problem 2. (a) Describe as accurately as possible the "true" gaps in the spectral set $\mathcal{S}_{N,D} := \bigcup_{x \in \mathbb{T}} \operatorname{spec} H_{[-N,N],D}(x, \omega)$ on a finite interval with Dirichlet boundary conditions. In particular, determine the gaps of smallest size.

(b) Develop a description for the spectral set $\mathcal{S}_{N,P} := \bigcup_{x\in\mathbb{T}} \operatorname{spec} H_{[-N,N],P}(x,\omega)$ on a finite interval with periodic boundary conditions.

(c) Develop a description of the spectral sets on a finite interval for rational approximations of the frequency ω.

1.1. Description of the method

As mentioned before, the homogeneity of the spectrum for continuous Schrödinger operators and small coupling constant was recently established in [DGL16] via detailed quantitative results concerning the structure of the gaps in the spectrum. For results on homogeneity of the spectrum for limit-periodic Schrödinger operators see [Fil17, FL15]. We show in this paper that in the regime of positive Lyapunov exponent, homogeneity can be obtained with less machinery. In fact, one does not even need to use finite scale localization. Rather, we use finite scale approximate eigenvalues rather than eigenfunctions. This approach only relies on the availability of a large deviation estimate; compare [Bou02], where a similar idea was used.

This method has the advantage of avoiding the removal of a "non-arithmetic" set of frequencies, which would be needed to eliminate the double resonances, as required in order to establish localization. To capture the infinite volume spectrum, we establish a criterion for given E_0 to fall in the spectrum on the whole lattice (see Lemma 2.8).

The most basic mechanism behind the homogeneity of the spectrum is the *Wegner estimate*. It is a finite volume version of the fact that the *integrated density of states is Hölder continuous*. For any $x, \omega \in \mathbb{T}$, let

$$\{E_j^{(N)}(x,\omega)\}_{j=1}^N, \quad \{\psi_j^{(N)}(x,\omega,\cdot)\}_{j=1}^N \tag{1.11}$$

denote the eigenvalues and a choice of normalized eigenvectors of $H_{[1,N]}(x,\omega)$, respectively. The Wegner estimate amounts to the fact that the graphs of $E_j^{(N)}(x,\omega)$ cannot be "too flat". See the discussion of the quantitative version of this issue in Remark 2.2. The other main reason for the homogeneity of the spectrum is the fragmentary stabilization of the graphs of the Dirichlet eigenvalues plotted against the phases at different scales (see Figure 4). This allows for good control on the structure of the spectrum on the whole lattice $\mathbb{Z}$ via the spectrum on intervals $[-N, N]$ with large N. Thus, the Wegner estimate makes it possible to obtain finite scale spectral segments of considerable size that we can then screen, via fragmentary stabilization, to obtain relatively large sets in the infinite volume spectrum. Heuristically, this is how the proof of Theorem H proceeds.

As already mentioned, the resolution of the fragmentary stability picture is accurate enough to allow the proof of homogeneity to go through. However, it seems that for possible future refinements of the result, one needs the more detailed picture given by finite scale localization. This is why, after we prove the main result, we also develop the finite scale localization approach. The novelty here is that we focus on the almost Mathieu operator for which we can establish the results without removal of any frequencies $\omega \in \mathbb{T}_{c,a}$. This is due to the method of Jitomirskaya [Jit99] for eliminating resonances. The advantage of this method that it explicitly identifies the resonant phases as $x = m\omega/2 \bmod 1$, $m \in \mathbb{Z}$, and there is no need to eliminate any further ω's.

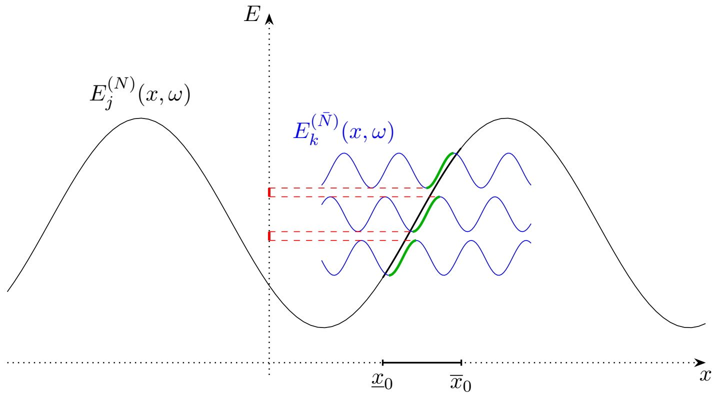

Fig. 4. *Fragmentary stability of spectral segments*: Fragments of graphs from scale $\bar{N} \gg N$ are very close to $E_j^{(N)}(x, \omega)$, $x \in [\underline{x}_0, \overline{x}_0]$. The small segments on the E axis represent the exceptional set.

We now give a rough outline of the main ingredients of both parts of the paper. Most of them were developed in [GS08, GS11]. In fact, we shall cite several results from these papers as part of our argument (see Propositions A–I below). The following three items describe the basic properties of the transfer matrix formalism. Throughout, it is essential that the Lyapunov exponents are positive.

- Large deviation estimate for the characteristic determinants of the Dirichlet problem on a finite interval.
- Hölder continuity of the Lyapunov exponent.
- Uniform upper bounds for the Dirichlet characteristic determinants.

The next three items build upon these foundations and describe essential features of the spectral theory, in particular the localization of the eigenfunctions.

- A version of the Wegner estimate.
- Elimination of double resonances on finite intervals.
- Exponential localization on finite intervals.

Finally, these tools feed into the following facts, which lie deeper and are of crucial importance to understanding the structure of the gaps in the spectrum.

- Quantitative separation of the Dirichlet eigenvalues on a finite interval.
- Formation of the spectrum on the whole lattice from the spectra on finite intervals.

This paper is not self-contained since we refer to reader to [GS08, GS11] for some of the rather involved proofs of the aforementioned technical ingredients. These results are used repeatedly in this work. On the other hand, some of the results derived from them, such as the Wegner estimate, are easy to obtain and we present their proofs.

Certain finer spectral properties, most notably the localization of the eigenfunctions and separation of the eigenvalues in the setting of Theorem H, require elimination of a Hausdorff dimension zero set of ω. So we cannot follow that route for the almost Mathieu operator. Therefore, approximately half of the work in this paper is devoted to establishing the needed ingredients for the Mathieu operator with $|\lambda| > 1$ and with arbitrary $\omega \in \mathbb{T}_{c,a}$.

On first reading, it is recommended to focus entirely on the proofs of Theorems G and H.

2. Transfer matrices and the Wegner estimate

We start by recalling the basics of the transfer matrix formalism. If ψ is a solution of the difference equation $H(x,\omega)\psi = E\psi$, then

$$\begin{bmatrix} \psi(b+1) \\ \psi(b) \end{bmatrix} = M_{[a,b]}(x,\omega,E) \begin{bmatrix} \psi(a) \\ \psi(a-1) \end{bmatrix},$$

where the transfer matrix is given by

$$M_{[a,b]}(x,\omega,E) = \prod_{k=b}^{a} \begin{bmatrix} V(x+k\omega) - E & -1 \\ 1 & 0 \end{bmatrix}.$$

We let $M_N = M_{[1,N]}$. The *Lyapunov exponent* is defined by

$$L(\omega,E) = \lim_{N\to\infty} \frac{1}{N} \int_0^1 \log \|M_N(x,\omega,E)\|\, dx \overset{\text{a.s.}}{=} \inf_N \frac{1}{N} \log \|M_N(x,\omega,E)\|.$$

The Hölder continuity of the Lyapunov exponent as a function of the energy was established in [GS01] under a strong Diophantine condition. The result was improved in [YZ14] to hold even for weak Liouville frequencies. The dependence of the Hölder exponent on γ (see the following proposition) was removed (for strongly Diophantine frequencies) in [Bou05, Prop. 8.3].

Proposition A. *Assume $\omega \in \mathbb{T}_{c,a}$ and $L(\omega,E_0) \geq \gamma > 0$. There exists $\varepsilon_0 = \varepsilon_0(V,c,a,\gamma)$ such that $L(\omega,E) \geq \gamma/2$ for any $E \in (E_0-\varepsilon_0, E_0+\varepsilon_0)$. Moreover, there exists $\alpha_0 = \alpha_0(V,c,a) > 0$ such that*

$$|L(\omega,E_1) - L(\omega,E_2)| \leq C(V,c,a,\gamma)|E_1 - E_2|^{\alpha_0}$$

for any $E_j \in (E_0-\varepsilon_0, E_0+\varepsilon_0)$, $j = 1,2$.

The above result is essentially [GS01, Thm. 6.1]. The first statement is implicit in [GS01], but it also follows explicitly from [BJ02].

Next we focus on results concerning the finite scale Dirichlet determinants. Let $H_{[a,b]}(x,\omega)$ be the Schrödinger operator defined via (1.1) on a finite interval $[a,b]$ with Dirichlet boundary conditions, $\psi(a-1) = 0$, $\psi(b+1) = 0$. Let $f_{[a,b]}(x,\omega,E) = \det(H_{[a,b]}(x,\omega) - E)$ be its characteristic polynomial. One has

$$f_{[a,b]}(x,\omega,E) = f_{b-a+1}(x+(a-1)\omega,\omega,E), \tag{2.1}$$

where

$$f_N(x,\omega,E) = \det(H_N(x,\omega)-E)$$
$$= \begin{vmatrix} V(x+\omega)-E & -1 & 0 & \cdots & \cdots & 0 \\ -1 & V(x+2\omega)-E & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & & & -1 \\ 0 & \cdots\cdots\cdots & 0 & -1 & & V(x+N\omega)-E \end{vmatrix}. \tag{2.2}$$

It is known that also

$$M_{[a,b]}(x,\omega,E) = \begin{bmatrix} f_{[a,b]}(x,\omega,E) & -f_{[a+1,b]}(x,\omega,E) \\ f_{[a,b-1]}(x,\omega,E) & -f_{[a+1,b-1]}(x,\omega,E) \end{bmatrix}. \tag{2.3}$$

It was shown in [GS08] that through this relation it is possible to pass from large deviation estimates for the transfer matrix to large deviation estimates for the determinants. The following large deviation estimate for the determinants is a basic tool in our approach (see [GS08, Cor. 3.6]).

Proposition B. *Let $\omega \in \mathbb{T}_{c,a}$ and $E \in \mathbb{R}$ be such that $L(\omega,E) \geq \gamma > 0$. There exists $C_0 = C_0(V,c,a,\gamma)$ such that*

$$\operatorname{mes}\left\{x \in \mathbb{T} : \left|\log|f_N(x,\omega,E)| - NL(\omega,E)\right| > H\right\} \leq C\exp(-H/(\log N)^{C_0}) \tag{2.4}$$

for all $H > (\log N)^{C_0}$ and $N \geq 2$. Moreover, the set on the left-hand side is contained in the union of $\lesssim N$ intervals, each of measure not exceeding the bound stated in (2.4).

Subharmonic functions can deviate only towards large negative values but not large positive ones. This explains the following result which is implied by [GS08, Prop. 4.3].

Proposition C. *Let $\omega \in \mathbb{T}_{c,a}$ and $E \in \mathbb{R}$ be such that $L(\omega,E) \geq \gamma > 0$. There exist $C_0 = C_0(V,c,a,\gamma)$ and $C = C(V)$ such that*

$$\sup_{x\in\mathbb{T}} \log|f_N(x,\omega,E)| \leq NL(\omega,E) + C(\log N)^{C_0} \quad \text{for any } N \geq 2. \tag{2.5}$$

Wegner's estimate now follows easily.

Proposition D. *Let $\omega \in \mathbb{T}_{c,a}$ and $E \in \mathbb{R}$ be such that $L(E,\omega) \geq \gamma > 0$. There exists $C_0 = C_0(V,c,a,\gamma)$ such that for $H > (\log N)^{2C_0}$ and $N \geq 2$, one has*

$$\operatorname{mes}\{x \in \mathbb{T} : \operatorname{dist}(E, \operatorname{spec} H_N(x,\omega)) < \exp(-H)\} \lesssim \exp(-H/(\log N)^{C_0}). \tag{2.6}$$

Moreover, the set on the left-hand side is contained in the union of $\lesssim N$ intervals, each of measure not exceeding the bound stated in (2.6).

Proof. By Cramer's rule,

$$|(H_N(x,\omega)-E)^{-1}(k,m)| = \frac{|f_{[1,k-1]}(x,\omega,E)|\,|f_{[m+1,N]}(x,\omega,E)|}{|f_N(x,\omega,E)|}.$$

By Proposition C,

$$\log|f_{[1,k-1]}(x),\omega,E)|+\log|f_{[m+1,N]}(x,\omega,E)|\le NL(\omega,E)+C(\log N)^{C_0}$$

for any $x\in\mathbb{T}$. Therefore,

$$\|(H_N(x,\omega)-E)^{-1}\|\le N^2\,\frac{\exp(NL(\omega,E)+C(\log N)^{C_0})}{|f_N(x,\omega,E)|}$$

for any $x\in\mathbb{T}$. Since

$$\operatorname{dist}(E,\operatorname{spec}H_N(x,\omega))=\|(H_N(x,\omega)-E)^{-1}\|^{-1},$$

the lemma follows from Proposition B. □

The following result is an immediate consequence of the Wegner estimate (2.6) and the continuity of the functions $E_j^{(N)}(x,\omega)$.

Corollary 2.1. *Let $\omega\in\mathbb{T}_{c,a}$ and assume $L(E,\omega)\ge\gamma>0$ for any $E\in(E',E'')$. There exists $C_0=C_0(V,c,a,\gamma)$ such that for $H>(\log N)^{2C_0}$ and $N\ge 2$, if I is an interval satisfying*

$$|I|\ge\exp(-H/(\log N)^{C_0})\quad\text{and}\quad E_j^{[-N,N]}(I,\omega)\subset(E',E''),$$

then

$$|E_j^{[-N,N]}(I,\omega)|\ge 2\exp(-H).$$

In the following remark, $a\sim b$ for $a,b>0$ means that these numbers are comparable up to fixed multiplicative constants (say, within a factor of 2). Moreover, $a\gg b$ means that $a/b\ge C$ for some large constant C.

Remark 2.2. The Wegner estimate is a fundamental tool which has been applied to the problem of localization of eigenfunctions in both the quasi-periodic and the random settings. For the problem under consideration here, namely the homogeneous nature of the spectrum, our reading of Wegner's estimate is as follows. Let $E\in\mathbb{R}$ be arbitrary and recall the eigenvalues as defined in (1.11). Assume that

$$|E-E_j^{(N)}(x_0,\omega)|\le\exp(-(\log N)^A)\tag{2.7}$$

for some N and x_0 and $A\gg 1$. Then, with σ calibrated against N such that

$$\sigma\sim\exp(-(\log N)^A),\tag{2.8}$$

the intersection

$$(E-\sigma,E+\sigma)\cap\big\{E_j^{(N)}(x,\omega): x\in\big(x_0-\exp(-(\log N)^B),x_0+\exp(-(\log N)^B)\big)\big\},\tag{2.9}$$

where $B:=A/2$, contains an interval $\mathcal{I}_E$ with

$$|\mathcal{I}_E|\ge\mathbf{w}_N:=\exp(-(\log N)^A)\sim\sigma.\tag{2.10}$$

Note that this is a special case of the previous corollary, using the largest possible values of H.

The next two results address the relation between the distance of an energy to the spectrum and the large deviation estimate from Proposition B. Recall that for any x_0, x, one has

$$|E_j^{(N)}(x,\omega) - E_j^{(N)}(x_0,\omega)| \le \|H_N(x,\omega) - H_N(x_0,\omega)\| \le C(V)|x - x_0|. \quad (2.11)$$

Lemma 2.3. *Let $\omega \in \mathbb{T}_{c,a}$ and $E \in \mathbb{R}$ be such that $L(E,\omega) \ge \gamma > 0$ and let $x \in \mathbb{T}$. There exist $N_0(V,c,a,\gamma,E)$ and $C_0(V,c,a,\gamma,E)$ such that for any $N \ge N_0$, if*

$$\mathrm{dist}(E, \mathrm{spec}\, H_N(x,\omega)) \ge \exp(-K)$$

for some $K \gg 1$, then

$$\log|f_N(x,\omega,E)| \ge NL(\omega,E) - K(\log N)^{C_0}.$$

Proof. Due to Proposition B there exists x' such that $|x' - x| < \exp(-K \log N)$ and

$$\log|f_N(x',\omega,E)| > NL(\omega,E) - K(\log N)^C.$$

From (2.11) and our assumption on E, we obtain

$$\begin{aligned} |E_j^{(N)}(x',\omega) - E_j^{(N)}(x,\omega)|\,|E_j^{(N)}(x,\omega) - E|^{-1} &\le C(V)|x' - x|\,|E_j^{(N)}(x,\omega) - E|^{-1} \\ &\le \exp(-(K \log N)/2) < 1/2, \end{aligned}$$

and therefore

$$\begin{aligned} &\big|\log|E - E_j^{(N)}(x',\omega)| - \log|E - E_j^{(N)}(x,\omega)|\big| \\ &\qquad \le 2|E_j^{(N)}(x',\omega) - E_j^{(N)}(x,\omega)|\,|E_j^{(N)}(x,\omega) - E|^{-1} \le 2\exp(-(K\log N)/2), \\ &\big|\log|f_N(x',\omega,E)| - \log|f_N(x,\omega,E)|\big| \le 2N\exp(-(K\log N)/2) < 1. \end{aligned}$$

This yields the desired conclusion. □

The usefulness of a lower bound on the determinant as in the previous lemma can be seen from the following result.

Lemma 2.4 ([GS11, Lem. 6.1]). *Let $\omega \in \mathbb{T}_{c,a}$, $E \in \mathbb{R}$, $L(\omega,E) > \gamma > 0$, and $N \ge N_0(V,a,c,\gamma,E)$. Furthermore, assume that*

$$\log|f_N(x,\omega,E)| > NL(\omega,E) - K/2$$

for some $x \in \mathbb{T}$ and $K > (\log N)^{C_0}$. Then

$$|(H_N(x,\omega) - E)^{-1}(j,k)| \le \exp(-\gamma|j-k| + K), \qquad \|(H_N(x,\omega) - E)^{-1}\| \le \exp(K).$$

In particular $\mathrm{dist}(E, \mathrm{spec}\, H_N(x,\omega)) \ge \exp(-K)$.

Proof. Apply Cramer's rule as in the proof of Wegner's estimate. □

We will use the following immediate consequence of Lemmas 2.3 and 2.4.

Lemma 2.5. *Let $\omega \in \mathbb{T}_{c,a}$ and $E \in \mathbb{R}$ be such that $L(\omega, E) \geq \gamma > 0$ and let $x \in \mathbb{T}$. There exist $N_0(V, c, a, \gamma, E)$ and $C_0(V, c, a, \gamma, E)$ such that for any $N \geq N_0$, if*

$$\operatorname{dist}(E, \operatorname{spec} H_N(x, \omega)) \geq \exp(-K)$$

for some $K \gg 1$, then

$$|(H_N(x, \omega) - E)^{-1}(j, k)| \leq \exp(-\gamma|j - k| + 2K(\log N)^{C_0}).$$

It is natural to link eigenfunctions of finite volume operators to (generalized) eigenfunctions in infinite volume. The standard tool for this is the *Poisson formula:* for any solution of the difference equation $H(x, \omega)\psi = E\psi$, we have

$$\psi(m) = (H_{[a,b]} - E)^{-1}(m, a)\psi(a-1) + (H_{[a,b]} - E)^{-1}(m, b+1)\psi(b+1), \quad m \in [a, b]. \tag{2.12}$$

This identity was introduced into the theory of localized eigenfunctions in the fundamental work on the Anderson model by Fröhlich and Spencer [FS83]. The Poisson formula tells us that the *decay of the Green function implies the decay of the eigenfunction wherever the Green function exists.* Lemmas 2.4 and 2.5 demonstrate how to effectively apply the Poisson formula in the regime of positive Lyapunov exponents, by being able to evaluate the decay of the Green function $(H_{[a,b]} - E)^{-1}(m, n)$ in terms of $|m - n|$. Lemma 2.4 explains how the large deviation estimate from Proposition B can be used to guarantee the conditions of Lemma 2.6. This leads to the following **localization principle**: *the eigenfunction $\psi_j^{[a,b]}$ defined by*

$$H_{[a,b]}(x, \omega)\psi_j^{[a,b]}(x, \omega) = E_j^{[a,b]}(x, \omega)\psi_j^{[a,b]}(x, \omega)$$

decays exponentially on any subinterval $[c, d] \subset [a, b]$ for which the large deviation estimate

$$\log|f_{[c,d]}(x, \omega, E_j^{[a,b]}(x, \omega))| > (c - d)L(\omega, E_j^{[a,b]}(x, \omega)) - (c - d)^{1-\delta} \tag{2.13}$$

is valid. This is of crucial importance to the theory of localization, and we shall make this precise later.

The following elementary observation links the spectra in finite volume to the decay of the Green function.

Lemma 2.6. *Let $x, \omega \in \mathbb{T}$, $E \in \mathbb{R}$, and $[a, b] \subset \mathbb{Z}$. If for any $m \in [a, b]$, there exists $\Lambda_m = [a_m, b_m] \subset [a, b]$ containing m such that*

$$(1 - \langle\delta_a, \delta_{a_m}\rangle)|(H_{\Lambda_m}(x, \omega) - E)^{-1}(a_m, m)| + (1 - \langle\delta_b, \delta_{b_m}\rangle)|(H_{\Lambda_m}(x, \omega) - E)^{-1}(b_m, m)| < 1,$$

then $E \notin \operatorname{spec} H_{[a,b]}(x, \omega)$.

Proof. Assume to the contrary that $E \in \operatorname{spec} H_{[a,b]}(x, \omega)$ and let ψ be a corresponding eigenvector. Let $m \in [a, b]$ be such that $|\psi(m)| = \max_n |\psi(n)|$. The hypothesis together with the Poisson formula (2.12) gives us that $|\psi(m)| < \max(|\psi(a_m)|, |\psi(b_m)|)$ if $a_m \neq a$ and $b_m \neq b$; $|\psi(m)| < |\psi(b_m)|$ if $a_m = a$; and $|\psi(m)| < |\psi(a_m)|$ if $b_m = b$. In each case we reach a contradiction, so we must have $E \notin \operatorname{spec} H_{[a,b]}(x, \omega)$. □

We use Lemma 2.6 to establish our criterion for an energy to be in the spectrum. For this we will also use the following well-known fact.

Lemma 2.7. *If for some $x, \omega \in \mathbb{T}$ and $E \in \mathbb{R}$ there exist $\delta > 0$ and sequences $a_k \to -\infty$ and $b_k \to \infty$ such that*

$$\operatorname{dist}(E, \operatorname{spec} H_{[a_k,b_k]}(x,\omega)) \geq \delta,$$

then

$$\operatorname{dist}(E, \operatorname{spec} H(x,\omega)) \geq \delta.$$

Proof. The hypothesis implies that for any $\phi \in \ell^2(\mathbb{Z})$ with finite support, there exists k such that

$$\|(H(x,\omega) - E)\phi\| = \|(H_{[a_k,b_k]}(x,\omega) - E)\phi\| \geq \delta\|\phi\|.$$

It follows by density that

$$\|(H(x,\omega) - E)\phi\| \geq \delta\|\phi\|$$

for any $\phi \in \ell^2(\mathbb{Z})$, and this yields the conclusion. □

We can now formulate the **spectrum criterion** lemma. In the following two results, the notation N^{1-} means $N^{1-\varepsilon}$ for some small absolute $\varepsilon > 0$. For example $\varepsilon = \frac{1}{100}$ will suffice (as in fact will large choices).

Lemma 2.8. *Let $\omega \in \mathbb{T}_{c,a}$ and $E \in \mathbb{R}$ be such that $L(\omega, E) \geq \gamma > 0$. There exists $N_0 = N_0(V, c, a, \gamma)$ such that the following statement holds for any $N \geq N_0$. If for any $x \in \mathbb{T}$, there exists $r(x) \in [-N/2, N/2]$ such that*

$$\operatorname{dist}(E, \operatorname{spec} H_{r(x)+[-N,N]}(x,\omega)) \geq \exp(-N^{1-}),$$

then

$$\operatorname{dist}(E, \mathcal{S}_\omega) \geq \tfrac{1}{2}\exp(-N^{1-}).$$

Proof. Fix $x \in \mathbb{T}$ and let $\bar{N} \geq N$ be arbitrary. Let

$$p = -\bar{N} - N + r(x - \bar{N}\omega), \qquad q = \bar{N} + N + r(x + \bar{N}\omega).$$

We will use Lemma 2.6 to show that $\tilde{E} \notin \operatorname{spec} H_{[p,q]}(x,\omega)$ for any $|\tilde{E} - E| \leq \exp(-N^{1-})/2$. Note that, by Proposition A, we have $L(\tilde{E}, \omega) \geq \gamma/2$. From the hypothesis we infer that

$$\operatorname{dist}(\tilde{E}, \operatorname{spec} H_{[p,p+2N]}) \geq \tfrac{1}{2}\exp(-N^{1-}).$$

It follows from Lemma 2.5 that

$$|(H_{[p,p+2N]}(x,\omega) - \tilde{E})^{-1}(p + 2N, m)| < 1$$

for any $m \in [p, p + N + [N/2]]$. Analogously

$$|(H_{[q-2N,q]}(x,\omega) - \tilde{E})^{-1}(q - 2N, m)| < 1$$

for any $m \in [q - N - [N/2], q]$. For $m \in [p + N + [N/2], q - N - [N/2]]$, let

$$a_m = m - N + r(x + m\omega), \ b_m = m + N + r(x + m\omega).$$

We clearly have $[a_m, b_m] \subset [p, q]$. Using the hypothesis and Lemma 2.5, we get

$$|(H_{[a_m,b_m]}(x, \omega) - \tilde{E})^{-1}(a_m, m)| + |(H_{[a_m,b_m]}(x, \omega) - \tilde{E})^{-1}(b_m, m)| < 1.$$

We can now apply Lemma 2.6 to find that $\tilde{E} \notin \operatorname{spec} H_{[p,q]}(x, \omega)$. Since this is true for any $|\tilde{E} - E| \le \exp(-N^{1-})/2$, it follows that

$$\operatorname{dist}(E, \operatorname{spec} H_{[p,q]}(x, \omega)) \ge \tfrac{1}{2} \exp(-N^{1-}).$$

Since $\bar{N}$ was arbitrary, we can choose sequences $a_k \to -\infty$ and $b_k \to \infty$ such that

$$\operatorname{dist}(E, \operatorname{spec} H_{[a_k,b_k]}(x, \omega)) \ge \tfrac{1}{2} \exp(-N^{1-}).$$

The conclusion follows from Lemma 2.7. □

The previous lemma relates the full spectrum $\mathcal{S}_\omega$ to the finite scale spectrum

$$\mathcal{S}_{N,\omega} := \bigcup_{x \in \mathbb{T}} \operatorname{spec} H_{[-N,N]}(x, \omega).$$

The proof of the lemma cannot be adjusted to give a relation between the finite scale spectra for different scales. Instead, we will use the following weaker result.

Lemma 2.9 ([GS11, Lem. 13.2]). *Let $\omega \in \mathbb{T}_{c,a}$ and $E \in \mathbb{R}$ be such that $L(\omega, E) \ge \gamma > 0$. There exists $N_0 = N_0(V, c, a, \gamma)$ such that the following statement holds for any $N \ge N_0$. If*

$$\operatorname{dist}(E, \mathcal{S}_{N,\omega}) \ge \exp(-N^{1-}),$$

then

$$\operatorname{dist}(E, \mathcal{S}_{\bar{N},\omega}) \ge \tfrac{1}{2} \exp(-N^{1-}) \quad \text{for any } \bar{N} \ge N.$$

Proof. The proof is analogous to that of Lemma 2.8. The only difference is that we now know that $r(x) = 0$. □

3. Stability of the spectrum

In this section we address the issue of how much of the finite scale spectrum $\mathcal{S}_{N,\omega}$ survives when we pass to a larger scale $\bar{N}$ or to the full scale.

Lemma 3.1. *Let $\omega \in \mathbb{T}_{c,a}$ and assume $L(\omega, E) \ge \gamma > 0$ for any $E \in (E', E'')$. There exist $c_0 = c_0(V, c, a, \gamma)$ and $N_0 = N_0(V, c, a, \gamma)$ such that*

$$\operatorname{mes}\big(\mathcal{S}_{N,\omega} \cap (E', E'') \setminus \mathcal{S}_{\bar{N},\omega}\big) \le \exp(-c_0 N) \quad \text{for any } \bar{N} \ge N \ge N_0.$$

Proof. Let $N^{(k)} = N^{2^k}$ and assume $\bar N \le N^{(1)}$. Let $E = E_j^{[-N,N]}(x,\omega) \in \mathcal{S}_{N,\omega} \cap (E', E'')$. Let $\ell = [c_1 N]$ with $c_1 \ll 1$. By Proposition B we can find an interval $n_0 + [-\ell, \ell]$, $|n_0| \le C\ell \ll N$, on which the large deviation estimate holds. By Lemma 2.4 and the Poisson formula it follows that

$$|\psi_j^{[-N,N]}(x,\omega;n)| \le \exp(-cN), \quad |n - n_0| \le \ell/2.$$

Let

$$\xi_l(n) = \begin{cases} \psi_j^{[-N,N]}(x,\omega;n), & n \in [-N, n_0], \\ 0, & \text{otherwise,} \end{cases}$$

$$\xi_r(n) = \begin{cases} \psi_j^{[-N,N]}(x,\omega;n), & n \in [n_0, N], \\ 0, & \text{otherwise.} \end{cases}$$

Then $\|\xi_l\| \ge 1/2$ or $\|\xi_r\| \ge 1/2$. If $\|\xi_l\| \ge 1/2$, then the fact that

$$\|(H_{[-N,-N+2\bar N]}(x,\omega) - E)\xi_l\| \le \exp(-cN)$$

implies $\operatorname{dist}(E, \mathcal{S}_{\bar N,\omega}) \le 2\exp(-cN)$. The same conclusion holds if $\|\xi_r\| \ge 1/2$. Since the finite spectra are unions of intervals, it follows that

$$\operatorname{mes}\big(\mathcal{S}_{N,\omega} \cap (E', E'') \setminus \mathcal{S}_{\bar N,\omega}\big) \lesssim \bar N \exp(-cN) \le \exp(-cN/2).$$

Recall that so far we are assuming $\bar N \le N^{(1)} = N^2$. In general, we can find k such that $N^{(k)} \le \bar N \le N^{(k+1)}$ and we have

$$\begin{aligned} &\operatorname{mes}\big(\mathcal{S}_{N,\omega} \cap (E', E'') \setminus \mathcal{S}_{\bar N,\omega}\big) \\ &\quad \le \operatorname{mes}\big(\mathcal{S}_{N,\omega} \cap (E', E'') \setminus \mathcal{S}_{N^{(1)},\omega}\big) + \operatorname{mes}\big(\mathcal{S}_{N^{(1)},\omega} \cap (E', E'') \setminus \mathcal{S}_{N^{(2)},\omega}\big) \\ &\qquad + \cdots + \operatorname{mes}\big(\mathcal{S}_{N^{(k)},\omega} \cap (E', E'') \setminus \mathcal{S}_{\bar N,\omega}\big) \\ &\quad \le \exp(-cN) + \exp(-cN^{(1)}) + \cdots + \exp(-cN^{(k)}) \le \exp(-cN/2). \end{aligned}$$ □

If the mass of an eigenvector $\psi_j^{[-N,N]}$ is concentrated near the edges of the interval, then we cannot guarantee that the corresponding eigenvalue is close to $\mathcal{S}_\omega$. We can only come close to the full scale spectrum provided that the mass of $\psi_j^{[-N,N]}$ is concentrated inside the interval. It is not clear whether each $E \in \mathcal{S}_{N,\omega}$ can be associated with such an eigenvector. However, we can produce spectral segments of considerable size for which this holds.

Lemma 3.2. *Let $\omega \in \mathbb{T}_{c,a}$ and $E \in \mathcal{S}_\omega$ and assume $L(E,\omega) \ge \gamma > 0$. There exist $c_0(V,c,a,\gamma)$, $C_0(V,c,a,\gamma)$, and $N_0(V,c,a,\gamma)$ such that the following statement holds for $N \ge N_0$. There exist $x_0 \in \mathbb{T}$ and $j_0 \in [-N,N]$ such that*

$$|E_{j_0}^{[-N,N]}(x_0,\omega) - E| \le \exp(-N^{1-}),$$

and for all $|x - x_0| < \exp(-(\log N)^{C_0})$, we have

$$|\psi_{j_0}^{[-N,N]}(x,\omega;n)| \le \exp(-c_0 N), \quad |n| \ge (1 - c_0)N.$$

Proof. Since $E \in \mathcal{S}_\omega$, Lemma 2.8 implies that there exists $x' \in \mathbb{T}$ such that

$$\max_{|n|\le N/2} \operatorname{dist}(E, \operatorname{spec} H_{n+[-N,N]}(x',\omega)) \le \exp(-N^{1-}). \tag{3.1}$$

Let $\ell = [cN]$, with $c < 1$ to be chosen later. We will argue that there exists $|n_0| \le N/2$ such that the Green function at scale ℓ has off-diagonal decay on the intervals

$$n_0 + [-N, -N+\ell-1] \quad \text{and} \quad n_0 + [N-\ell+1, N]$$

at the edges of $n_0 + [-N, N]$. Due to Proposition D we know that there exists $A(V, c, a, \gamma) \gg 1$ such that

$$\{x \in \mathbb{T} : \operatorname{dist}(E, \operatorname{spec} H_\ell(x_0,\omega)) < \exp(-(\log \ell)^A)\} \subset \bigcup_{k=1}^{k_0} I_k,$$

where I_k are intervals such that $|I_k| \le \exp(-(\log \ell)^{A/2})$, and $k_0 \le C\ell$. We now set $c = (4C)^{-1}$ so that we have $\ell \le N/4$. Due to the Diophantine condition, each I_k contains at most one point of the form

$$x' + (n - N)\omega \quad \text{or} \quad x' + (n + N - \ell + 1)\omega,$$

with $|n| \le N/2$. Since $k_0 \le N/4$, it follows that there exists $|n_0| \le N/2$ such that $x_0 + (n_0 - N)\omega$ and $x_0 + (n_0 + N - \ell + 1)\omega$ are not in any of the I_k, and therefore

$$\begin{aligned}\operatorname{dist}(E, \operatorname{spec} H_\ell(x_0 + (n_0 - N)\omega, \omega)), \operatorname{dist}(E, \operatorname{spec} H_\ell(x_0 + (n_0 + N - \ell + 1)\omega, \omega))\\ \ge \exp(-(\log \ell)^A) \gg \exp(-N^{1-}).\end{aligned} \tag{3.2}$$

Let $x_0 = x' + n_0\omega$. By (3.1) there exists j_0 such that

$$|E - E_{j_0}^{[-N,N]}(x_0,\omega)| \le \exp(-N^{1-}). \tag{3.3}$$

From this, (3.2), and (2.11) it follows that

$$\begin{aligned}&\operatorname{dist}(E_{j_0}^{[-N,N]}(x,\omega), \operatorname{spec} H_\ell(x - N\omega, \omega)),\\ &\qquad \operatorname{dist}(E_{j_0}^{[-N,N]}(x,\omega), \operatorname{spec} H_\ell(x + (N-\ell+1)\omega, \omega)) \ge \tfrac{1}{2}\exp(-(\log \ell)^A),\end{aligned}$$

for any $|x - x_0| \le c\exp(-(\log \ell)^A)$. Lemma 2.5 implies that

$$\begin{aligned}&\left|\left(H_\ell(x - N\omega, \omega) - E_{j_0}^{[-N,N]}(x,\omega)\right)^{-1}(j,k)\right| \le \exp(-\gamma|j-k| + (\log N)^C),\\ &\left|\left(H_\ell(x + (N-\ell+1)\omega, \omega) - E_{j_0}^{[-N,N]}(x,\omega)\right)^{-1}(j,k)\right| \le \exp(-\gamma|j-k| + (\log N)^C).\end{aligned}$$

The desired estimates on the eigenvector $\psi_{j_0}^{[-N,N]}(x,\omega)$ now follow by applying the Poisson formula on the intervals $[-N, -N+\ell-1]$ and $[N-\ell+1, N]$. □

Next we address the stability of the spectral segments produced via the previous lemma. As in the proof of Lemma 3.1 we need to argue by induction on scales. The inductive step that will be stated in Lemma 3.3 is essentially [Bou05, Lemma 12.22]. For the convenience of the reader we will sketch its proof. The original proof is for the case when the potential is a trigonometric polynomial. We will include the simple approximation argument needed to deal with analytic potentials. For this we recall some facts regarding the approximation of the potential by trigonometric polynomials.

Let

$$V(x) = \sum_{n=-\infty}^{\infty} v_n e(nx),$$

be the Fourier series expansion for V, where we use the notation $e(x) := \exp(2\pi i x)$. It is known that since V is real-analytic on $\mathbb{T}$, it can be extended to a strip of width $2\rho_0$ around the real axis, for some $\rho_0 > 0$, and that this implies the existence of C such that

$$|v_n| \le C \exp(-\pi\rho_0|n|).$$

In fact, we can take $C = \sup_{x\in\mathbb{T}} |V(x \pm i\rho_0/2)|$. Let

$$\tilde{V}(x) = \sum_{n=-K}^{K} v_n e(nx). \tag{3.4}$$

As a consequence of the bound on the Fourier coefficients, we have

$$\sup_{x\in\mathbb{T}} |V(x) - \tilde{V}(x)| \le C(\|V\|_\infty, \rho_0) \exp(-\pi\rho_0 K/3).$$

It follows that we always have

$$|E_j^{(N)}(x,\omega) - \tilde{E}_j^{(N)}(x,\omega)| \le \|H_N(x,\omega) - \tilde{H}_N(x,\omega)\| \le C\exp(-cK). \tag{3.5}$$

Lemma 3.3. *Let $\omega \in \mathbb{T}_{c,a}$ and assume $L(\omega, E) \ge \gamma > 0$ for any $E \in (E', E'')$. Let $I \subset [0,1]$ be an interval and let $j \in [-N, N]$. Assume that $E_j^{[-N,N]}(I,\omega) \subset (E', E'')$ and that for each $x \in I$, there exists ξ, $\|\xi\| = 1$, with support in $[-N+1, N-1]$, such that*

$$\|(H(x,\omega) - E_j^{[-N,N]}(x,\omega))\xi\| < e^{-c_0 N}, \tag{3.6}$$

where $c_0 > 0$ is some constant. Let

$$\log N_1 \gg \log N \gg \log\log N_1.$$

If $c_0 \le C(V, c, a, \gamma) \ll 1$ and $N \ge N_0(V, c, a, \gamma, c_0)$, then we can partition I into intervals I_m, $m \le N_1^C$, with C an absolute constant, so that for each I_m, there exists $j_1 \in [-N_1, N_1]$ such that

$$|E_j^{[-N,N]}(x,\omega) - E_{j_1}^{[-N_1,N_1]}(x,\omega)| \le \exp(-c_0 N/2), \quad x \in I_m,$$

and for each $x \in I_m$, there exists ξ, $\|\xi\| = 1$, with support in $[-N_1+1, N_1-1]$, satisfying

$$\|(H(x,\omega) - E_{j_1}^{[-N_1,N_1]}(x,\omega))\xi\| < e^{-c_0 N_1}.$$

Proof. Fix $x \in I$ and let ξ be as in (3.6). We have

$$\begin{aligned}H_{[-N,N]}(x,\omega) &= H_{[-N_1,N_1]}(x,\omega),\\ H_{[-N,N]}(x,\omega)\xi &= E_j^{[-N,N]}(x,\omega)\xi + O(e^{-c_0N})\\ &= E_j^{[-N,N]}(x)\sum\langle\xi,\psi_k^{[-N_1,N_1]}(x,\omega)\rangle\psi_k^{[-N_1,N_1]}(x,\omega) + O(e^{-c_0N}),\\ H_{[-N_1,N_1]}(x,\omega)\xi &= \sum E_k^{[-N_1,N_1]}(x,\omega)\langle\xi,\psi_k^{[-N_1,N_1]}(x,\omega)\rangle\psi_k^{[-N_1,N_1]}(x,\omega).\end{aligned}$$

Note that for the first two identities, we have used the fact that ξ is supported in $[-N+1, N-1]$. It follows that

$$\left(\sum|E_j^{[-N,N]}(x,\omega) - E_k^{[-N_1,N_1]}(x,\omega)|^2\langle\xi,\psi_k^{[-N_1,N_1]}(x,\omega)\rangle^2\right)^{1/2} < e^{-c_0N}. \tag{3.7}$$

Since $\|\xi\| = 1$, there exists $j_1(x) \in [-N_1, N_1]$ such that

$$|\langle\xi,\psi_{j_1}^{[-N_1,N_1]}(x,\omega)\rangle| \geq 1/\sqrt{N_1}. \tag{3.8}$$

The estimate (3.7) implies that

$$|E_j^{[-N,N]}(x,\omega) - E_{j_1}^{[-N_1,N_1]}(x,\omega)| < \sqrt{N_1}\,e^{-c_0N}. \tag{3.9}$$

As in the proof of Lemma 3.2 it can be seen that there exists $N_1/2 < k_1 < N_1$ such that

$$|\psi_{j_1}^{[-N_1,N_1]}(x,\omega;n)| \leq \exp(-3c_0N_1), \qquad k_1 \geq |n| \geq k_1 - 3c_0N_1. \tag{3.10}$$

For this we use the assumption that c_0 is small enough. Let η be the normalized projection of $\psi_{j_1}^{[-N_1,N_1]}(x,\omega)$ onto the subspace corresponding to the interval $[-k_1, k_1]$. By (3.8) and (3.10),

$$\|(H(x,\omega) - E_{j_1}^{[-N_1,N_1]})\eta\| \lesssim \sqrt{N_1}\exp(-3c_0N_1) < \exp(-2c_0N_1). \tag{3.11}$$

Now we just need to estimate the number of components of the set of phases x that satisfy (3.9) and (3.11). For this we need to approximate the potential V by a trigonometric polynomial. To do so, we note that the existence of η, $\|\eta\| = 1$, with support in $[-N_1+1, N_1-1]$, satisfying (3.11) is equivalent to

$$\|[P_{N_1}(H(x,\omega) - E_{j_1}^{[-N_1,N_1]})^*(H(x,\omega) - E_{j_1}^{[-N_1,N_1]})P_{N_1}]^{-1}\| > \exp(2c_0N_1), \tag{3.12}$$

where P_{N_1} is the projection onto the subspace corresponding to the interval $[-N_1+1, N_1-1]$. Choose $\tilde{V}$ as in (3.4) with $K = CN_1^2$. Then we have

$$|\tilde{E}_j^{[-N,N]}(x,\omega) - \tilde{E}_{j_1}^{[-N_1,N_1]}(x,\omega)| < 2\sqrt{N_1}\,e^{-c_0N},$$

and

$$\|[P_{N_1}(\tilde{H}(x,\omega) - \tilde{E}_{j_1}^{[-N_1,N_1]})^*(\tilde{H}(x,\omega) - \tilde{E}_{j_1}^{[-N_1,N_1]})P_{N_1}]^{-1}\| > \exp(2c_0N_1)/2.$$

The set of x's satisfying the above estimates can be given a semialgebraic description in terms of polynomials of degree at most N_1^C (see [Bou05, proof of Lemma 12.22]). It follows that I can be partitioned into intervals I_m, $m \le N_1^C$, such that $j_1(x)$ from the above estimates can be kept constant on each of the subintervals. If we now go back to the original potential, the estimates (3.9) and (3.11) hold up to a correction by a constant factor, and with the constant choice of j_1 on each I_m. This concludes the proof. □

The next lemma is our result on the stability of the spectral segments from Lemma 3.2.

Lemma 3.4. *Let $\omega \in \mathbb{T}_{c,a}$ and assume $L(\omega, E) \ge \gamma > 0$ for any $E \in (E', E'')$. Let $I \subset [0, 1]$ be an interval and let $j \in [-N, N]$. Assume that $E_j^{[-N,N]}(I, \omega) \subset (E', E'')$ and for each $x \in I$, there exists ξ, $\|\xi\| = 1$, with support in $[-N + 1, N - 1]$, such that*

$$\|(H(x, \omega) - E_j^{[-N,N]}(x, \omega))\xi\| < e^{-c_0 N}, \tag{3.13}$$

where $c_0 > 0$ is some constant. If $c_0 \le C(V, c, a, \gamma) \ll 1$ and $N \ge N_0(V, c, a, \gamma, c_0)$, then

$$\mathrm{mes}(E_j^{[-N,N]}(I, \omega) \setminus \mathcal{S}_\omega) < \exp(-c_0 N/4).$$

Proof. Let $N_1 = N^2$. Using Lemma 3.3, we partition I into intervals I_m, $m \le N_1^C$, so that for each I_m, there exists $j_1 \in [-N_1, N_1]$ such that

$$|E_j^{[-N,N]}(x, \omega) - E_{j_1}^{[-N_1,N_1]}(x, \omega)| \le \exp(-c_0 N/2), \qquad x \in I_m, \tag{3.14}$$

and for each $x \in I_m$, there exists ξ, $\|\xi\| = 1$, with support in $[-N_1+1, N_1-1]$, satisfying

$$\|(H(x, \omega) - E_{j_1}^{[-N_1,N_1]}(x, \omega))\xi\| < e^{-c_0 N_1}. \tag{3.15}$$

Let

$$\mathcal{E}_{N,1,\omega} = \bigcup_m \big(E_j^{[-N,N]}(I_m, \omega) \ominus E_{j_1}^{[-N_1,N_1]}(I_m, \omega)\big),$$

where $\ominus$ denotes symmetric difference. By the continuity of the parametrization of the eigenvalues and (3.14), it follows that $\mathrm{mes}(\mathcal{E}_{N,1,\omega}) \le \exp(-c_0 N/3)$.

Note that (3.13) implies that $\mathrm{dist}(E, \mathcal{S}_\omega) < \exp(-c_0 N)$ for all $E \in E_j^{[-N,N]}(I, \omega)$. At the same time, if $E \in E_j^{[-N,N]}(I, \omega) \setminus \mathcal{E}_{N,1,\omega}$, then $E \in E_{j_1}^{[-N_1,N_1]}(I_m, \omega)$ for some m, and (3.15) implies that $\mathrm{dist}(E, \mathcal{S}_\omega) < \exp(-c_0 N_1)$.

Let $N_k = N^{2^k}$. Through iteration we obtain sets $\mathcal{E}_{N,k,\omega}$ such that $\mathrm{mes}(\mathcal{E}_{N,k,\omega}) < \exp(-c_0 N_{k-1}/3)$ and if $E \in E_j^{[-N,N]}(I, \omega) \setminus \bigcup_{l \le k} \mathcal{E}_{N,l,\omega}$, then $\mathrm{dist}(E, \mathcal{S}_\omega) < \exp(-c_0 N_k)$. Finally, we note that

$$E_{j_0}^{[-N,N]}(I, \omega) \setminus \mathcal{S}_\omega \subset \bigcup_k \mathcal{E}_{N,k,\omega},$$

and we are done. □

4. Proofs of Theorems G and H

Proof of Theorem G. First we prove the full scale statement. Let J be the interval between G_1 and G_2. Then there exists $E_0 \in \mathcal{S}_\omega \cap J$. We will argue that the size of J is bounded below because $\mathcal{S}_\omega \cap J$ must be relatively large. Lemma 3.2 implies the existence of $j \in [-N, N]$ and of a segment I, $|I| > \exp(-(\log N)^C)$, centered at a point x_0, such that

$$|E_j^{[-N,N]}(x_0, \omega) - E_0| \leq \exp(-N^{1-}), \tag{4.1}$$

and for any $x \in I$ there exists ξ, $\|\xi\| = 1$, with support in $[-N+1, N-1]$, such that

$$\|(H(x,\omega) - E_j^{[-N,N]}(x,\omega))\xi\| < \exp(-cN)$$

(the vector ξ can be chosen to be the normalized projection of $\psi_j^{[-N,N]}(x,\omega)$ onto the subspace corresponding to $[-N+1, N-1]$). Using Proposition A we can guarantee that $L(\omega, E) \geq \gamma/2$ for all $E \in E_j^{[-N,N]}(I, \omega)$. Thus, we can apply Lemma 3.4 to get

$$\operatorname{mes}(E_j^{[-N,N]}(I,\omega) \setminus \mathcal{S}_\omega) \leq \exp(-cN).$$

By the continuity of $E_j^{[-N,N]}(\cdot, \omega)$ and (4.1), it follows that

$$E_j^{[-N,N]}(I,\omega) \cap \mathcal{S}_\omega \subset J \cap \mathcal{S}_\omega$$

(otherwise, we would have $\operatorname{mes}(E_j^{[-N,N]}(I,\omega) \setminus \mathcal{S}_\omega) \geq \exp(-N^{1-})/2$). At the same time, Corollary 2.1 implies

$$\operatorname{mes}(E_j^{[-N,N]}(I,\omega)) \geq \exp(-(\log N)^C).$$

Putting all these together we have

$$|J| \geq |J \cap \mathcal{S}_\omega| \geq |E_j^{[-N,N]}(I,\omega) \cap \mathcal{S}_\omega| \geq \exp(-(\log N)^C) - \exp(-cN).$$

The conclusion follows immediately with an appropriate choice of C_0.

The proof of the finite scale statement is analogous. One just needs to use Lemmas 2.9 and 3.1 instead of Lemmas 2.8 and 3.4. As before, there exists $E_0 \in \mathcal{S}_{\bar{N},\omega} \cap J$. By Lemma 2.9, there exist $j \in [-N, N]$ and $x_0 \in \mathbb{T}$ such that

$$|E_j^{[-N,N]}(x_0, \omega) - E_0| \leq \exp(-N^{1-}). \tag{4.2}$$

Let $I = (x_0 - \exp(-(\log N)^C), x_0 + \exp(-(\log N)^C))$. Using Proposition A we can guarantee that $L(\omega, E) \geq \gamma/2$ for all $E \in E_j^{[-N,N]}(I, \omega)$. Thus, we can apply Lemma 3.1 to get

$$\operatorname{mes}(E_j^{[-N,N]}(I,\omega) \setminus \mathcal{S}_{\bar{N},\omega}) \leq \exp(-cN).$$

By the continuity of $E_j^{[-N,N]}(\cdot, \omega)$ and (4.2), it follows that

$$E_j^{[-N,N]}(I,\omega) \cap \mathcal{S}_{\bar{N},\omega} \subset J \cap \mathcal{S}_{\bar{N},\omega}$$

(otherwise, we would have $\mathrm{mes}(E_j^{[-N,N]}(I,\omega) \setminus \mathcal{S}_{\bar{N},\omega}) \geq \exp(-N^{1-})/2)$. At the same time, Corollary 2.1 implies

$$\mathrm{mes}(E_j^{[-N,N]}(I,\omega)) \geq \exp(-(\log N)^C).$$

Putting all these together we have

$$|J| \geq |J \cap \mathcal{S}_{\bar{N},\omega}| \geq |E_j^{[-N,N]}(I,\omega) \cap \mathcal{S}_{\bar{N},\omega}| \geq \exp(-(\log N)^C) - \exp(-cN).$$

The conclusion follows immediately with an appropriate choice of C_0. □

Proof of Theorem H. We only prove the full scale version of the result. The finite scale statement is proved analogously to the proof of Theorem G. We only need to prove part (b). The other statements follow from its proof. Assume that $\mathcal{S}_\omega \cap (E', E'') \neq \emptyset$. Let $E_0 \in \mathcal{S}_\omega \cap (E', E'')$. Let $N \geq N_0$ be large enough. From the proof of Theorem G we know that there exist $j \in [-N, N]$ and an interval I, $|I| > \exp(-(\log N)^C)$, centered at a point x_0, such that

$$|E_j^{[-N,N]}(x_0,\omega) - E_0| \leq \exp(-N^{1-}),$$
$$\mathrm{mes}(E_j^{[-N,N]}(I,\omega) \setminus \mathcal{S}_\omega) \leq \exp(-cN),$$
$$\mathrm{mes}(E_j^{[-N,N]}(I,\omega)) \geq \exp(-(\log N)^C).$$

Putting all this information together and assuming $\exp(-(\log N)^C) < \varepsilon$ (with ε from (1.4)), we have

$$\begin{aligned}&\big|\big(E_0 - \exp(-(\log N)^C), E_0 + \exp(-(\log N)^C)\big) \cap (\mathcal{S}_\omega \cap (E', E''))\big| \\ &\qquad \geq \exp(-(\log N)^C) - \exp(-N^{1-}) - \exp(-cN) \geq \tfrac{1}{2}\exp(-(\log N)^C).\end{aligned}$$

Since this is true for arbitrary large enough N, it follows that (1.2) holds for $\sigma \leq \sigma_0$, with $\tau = 1/4$. The conclusion follows because (1.2) holds trivially with $\tau \simeq \sigma_0$ for $\sigma > \sigma_0$. □

5. Double resonances

Of key importance in the theory of localization is the notion of *double resonance*. This refers to the situation where inside of a large window $[-N, N]$, there are two smaller ones, say $I := [k_1, k_2]$, $J := [k_3, k_4]$, which are not too close to each other and $H_I(x,\omega)$ and $H_J(x,\omega)$ have two eigenvalues E_1, E_2, respectively, with $|E_1 - E_2|$ very small. If these eigenvalues correspond to eigenfunctions $H_I(x,\omega)\psi = E_1\psi$, $H_J(x,\omega)\phi = E_1\phi$, respectively, which are well localized within these respective windows, then $H_{[-N,N]}(x,\omega)$ exhibits an eigenvalue close to E_1 with two eigenfunctions $\psi \pm \phi$ (with the understanding that we set $\psi = 0$ outside of I and $\phi = 0$ outside of J). It is a delicate matter to turn this idea into a quantitative, rigorous machinery. Localization happens precisely if such long chains of resonances cannot occur.

For the almost Mathieu operator, double resonances can be handled explicitly via Lagrange interpolation for the trigonometric polynomial given by the finite-volume determinant. This is the method of Jitomirskaya [Jit99], which we will use in this section. Since the method does not apply to general potentials V, the problem is treated via elimination of “exceptional frequencies” ω in [GS11]. Alternatively, one can use semialgebraic techniques to eliminate “bad” ω’s. This technique is quite robust and applies for example to higher-dimensional tori. One can find a very effective and beautiful development of this method in the monograph [Bou05].

For the purposes of this paper (as well as for the analysis of the gaps in [GS11]), it is necessary to achieve a level of resolution in the double resonance problem that is considerably higher than the one required for localization. The reason for this lies in the distances between the eigenvalues on a finite interval. We use this separation to control the process of formation of the spectrum $\mathcal{S}_\omega$ on the whole lattice from the spectra on finite intervals (see Section 8). More specifically, to obtain points in $\mathcal{S}_\omega$ via $\mathcal{S}_N$ we need to keep the essential supports of the eigenfunctions in question bounded in order to obtain a spectral value of H_ω (see Remark 6.1). To keep the essential support compact, we make sure that the eigenfunction at scale N gives rise to an eigenfunction at scale $\bar{N} \gg N$ that is very close to the initial function and with an eigenvalue that is very close to the initial eigenvalue (see Propositions I and I′).

To derive the quantitative separation for two eigenvalues, say $E_j^{(N)}(x,\omega)$ and $E_k^{(N)}(x,\omega)$, we need to verify that the sizes of the essential supports of $\psi_j^{(N)}(x,\omega)$ and $\psi_k^{(N)}(x,\omega)$ are bounded by

$$|\text{essential support}| \le \underline{N} := Q_N := \exp((\log\log N)^C), \tag{5.1}$$

for some $C \gg 1$. This is the estimate that allows one to evaluate

$$|f_N(x,\omega,E_1) - f_N(x,\omega,E_2)| \tag{5.2}$$

from Proposition B for two close but distinct values E_1, E_2. Heuristically, Proposition C states that the exceptional set in the large deviation estimate is close to an algebraic curve of degree $\le (\log N)^A$. This level of resolution is fine enough to see the scale (5.1) in the setting of general potentials V for which we use frequency modulation to eliminate double resonances.

The next result, which follows from [GS11, Prop. 5.5] and Proposition D, is a tool designed to obtain the desired resolution in the elimination of double resonances. This result employs the notions of *measure* and *complexity*. To be specific,

$$\operatorname{mes}(\mathcal{S}) \le \varepsilon, \qquad \operatorname{compl}(\mathcal{S}) \le K$$

means that for some intervals I_k,

$$\mathcal{S} \subset \bigcup_{k=1}^{K} I_k, \qquad \sum_{k=1}^{K} |I_k| \le \varepsilon.$$

Therefore, for the purposes of this paper we can assume that the sets from the next result are just unions of intervals.

Proposition E. *Consider operators* (1.1) *with real-analytic* V. *Assume that* $L(\omega, E) \geq \gamma > 0$ *for any* ω *and any* $E \in (E', E'')$. *There exists* $\ell_0 = \ell_0(V, c, a, \gamma)$ *such that for any* $\ell_1 \geq \ell_2 \geq \ell_0$, *the following holds: Given* $t > \exp((\log \ell_1)^{C_0})$, $H \geq 1$, *there exists a set* $\Omega_{\ell_1,\ell_2,t,H} \subset \mathbb{T}$ *with*

$$\mathrm{mes}(\Omega_{\ell_1,\ell_2,t,H}) < \exp((\log \ell_1)^{C_1})e^{-\sqrt{H}}, \quad \mathrm{compl}(\Omega_{\ell_1,\ell_2,t,H}) < t\exp((\log \ell_1)^{C_1})H$$

such that for any $\omega \in \mathbb{T}_{c,a} \setminus \Omega_{\ell_1,\ell_2,t,H}$ *there exists a set* $\mathcal{B}_{\ell_1,\ell_2,t,H,\omega} \subset \mathbb{T}$ *with*

$$\mathrm{mes}(\mathcal{B}_{\ell_1,\ell_2,t,H,\omega}) < t\exp((\log \ell_1)^{C_1})\, e^{-\sqrt{H}},$$
$$\mathrm{compl}(\mathcal{B}_{\ell_1,\ell_2,t,H,\omega}) < t\exp((\log \ell_1)^{C_1})H$$

such that for any $x \in \mathbb{T} \setminus \mathcal{B}_{\ell_1,\ell_2,t,H,\omega}$ *one has*

$$\mathrm{dist}((E', E'') \cap \mathrm{spec}\, H_{\ell_1}(x, \omega), \mathrm{spec}\, H_{\ell_2}(x + t\omega, \omega)) \geq e^{-H(\log \ell_1)^{3C_2}}.$$

Even though no upper bound on the translation t is stated here, note that the estimates are only meaningful if $t < \bar{t}(H)$, where the latter makes the right-hand side in the measure estimate on the order of 1. In view of Lemmas 2.3 and 2.4 it is natural to recast the problem of double or multiple resonances as the following question: *for how many subintervals* $[c, d]$ *can the large deviation estimate*

$$\log |f_{[c,d]}(x, \omega, E))| > (c - d)L(\omega, E) - (c - d)^{1-\delta} \tag{5.3}$$

fail with the ***same*** x, E*?* The large deviation estimate in Proposition B tells us that the set where it fails is "almost a curve" in the plane of two variables x, ω. Therefore a Bézout type argument should tell us that two intervals occur only for special values of x, which is borne out by the previous proposition.

We shall now establish the following version of Proposition E for the almost Mathieu operator. A key improvement over the general version is that no further elimination of frequencies is required. However, this comes at the cost of the count of intervals where the large deviation estimate might fail.

Proposition E′. *Consider the almost Mathieu operator* (1.9) *with* $|\lambda| > 1$ *and* $\omega \in \mathbb{T}_{c,a}$. *Let* $\sigma > 0$. *There exist* $N_0 = N_0(c, a, |\lambda|, \sigma)$ *such that for* $N \geq N_0$, $4N \leq |t| \leq \exp(N^{\sigma-})$, *and* $x \in \mathbb{T}$ *satisfying*

$$\min_{|n| \leq |t|+4N} \|x - n\omega/2\| \geq \exp(-N^{\sigma}),$$

we have

$$\max\Big(\max_{|n|\leq[N/2]} \log |f_{n+[-N,N]}(x, \omega, E)|, \max_{|n|\leq[N/2]+1} \log |f_{n+t+[-N,N]}(x, \omega, E)|\Big) \geq (2N + 1)L(\omega, E) - N^{\sigma+}.$$

The key idea is to invoke Lagrange interpolation for the determinants, because one can estimate the size of the Lagrange basis polynomials (see Lemma 5.2). It is clear from (2.2) that $f_{[-N,N]}$ is an even trigonometric polynomial of degree $2N+1$:

$$f_{[-N,N]}(x,\omega,E) = \sum_{k=0}^{2N+1} a_k(\lambda,\omega,E)\cos^k 2\pi x =: Q(\cos 2\pi x).$$

Given $\theta_k \in \mathbb{T}$, $k = 1,\dots,2N+1$, the Lagrange interpolation formula reads

$$Q(\cos 2\pi y) = \sum_{k=1}^{2N+1} Q(\cos 2\pi\theta_k)\frac{\prod_{n\neq k}(\cos 2\pi y - \cos 2\pi\theta_n)}{\prod_{n\neq k}(\cos 2\pi\theta_k - \cos 2\pi\theta_n)}. \tag{5.4}$$

For the purposes of Proposition E′ we will take $\{\theta_k\}$ to be made of two pieces of the orbit of the irrational shift. We will control the size of the Lagrange basis polynomials by invoking the following elementary estimate (cf. [GS01, Lem. 3.1], [Jit99, Lem. 11]).

Lemma 5.1. *Let $\omega \in \mathbb{T}_{c,a}$, $x, y \in \mathbb{T}$ and $f(\theta) = \log|\cos 2\pi y - \cos 2\pi\theta|$ for $\theta \in \mathbb{T}$. There exists $C_0(a,c)$ such that*

$$\left|\sum_{k=1}^{n} f(x+k\omega) - n\int_0^1 f(\theta)\,d\theta\right| \lesssim n\delta\log\frac{1}{\delta} + C_0(\log n)^{a+2}\log\frac{1}{\delta}$$

for any $n > 1$ and any $0 < \delta \ll 1$ such that $\delta \le \min_k \|(x+k\omega \pm y)/2\|$.

Proof. Let

$$g_\delta^{\pm}(\theta) = \begin{cases} \log|\sin 2\pi\,\|(y\pm\theta)/2\|\,|, & \|(y\pm\theta)/2\| \ge \delta,\\ \log|\sin 2\pi\delta|, & \|(y\pm\theta)/2\| < \delta. \end{cases}$$

If $\delta \le \min_k \|(x+k\omega \pm y)/2\|$, then

$$\sum_{k=0}^{n} f(x+k\omega) = n\log 2 + \sum_{k=1}^{n} g_\delta^{+}(x+k\omega) + \sum_{k=1}^{n} g_\delta^{-}(x+k\omega).$$

By Koksma's inequality (see [KN74, Thm. 2.5.1]) we have

$$\left|\sum_{k=1}^{n} g_\delta^{\pm}(x+k\omega) - n\int_0^1 g_\delta^{\pm}(\theta)\,d\theta\right| \le nD_n\mathrm{Var}(g_\delta^{\pm}) \lesssim C(a,c)(\log n)^{a+2}\log\frac{1}{\delta}.$$

The discrepancy D_n is evaluated via the Erdős–Turán theorem (see [KN74, Lem. 2.3.2–3]). Finally, one has

$$\left|\int_0^1 f(\theta)\,d\theta - \log 2 - \int_0^1 g_\delta^{+}(\theta)\,d\theta - \int_0^1 g_\delta^{-}(\theta)\,d\theta\right| \lesssim \delta\log\frac{1}{\delta},$$

and the lemma follows. ◻

Lemma 5.2. *Let* $\omega \in \mathbb{T}_{c,a}$, $x, y \in \mathbb{T}$, $N \geq 1$, $t \in \mathbb{Z}$, *and*

$$\theta_n = \begin{cases} x + (-N + n - [N/2] - 1)\omega, & n = 1, \dots, 2[N/2] + 1, \\ x + (-N + t + n - [N/2] - N - 1)\omega, & n = 2[N/2] + 2, \dots, 2N + 1. \end{cases}$$

Given $\sigma > 0$, *there exists* $N_0(c, a, \sigma)$ *such that if*

$$\min_{|n| \leq |t| + 4N} \left\| x - \frac{n\omega}{2} \right\| \geq \exp(-N^\sigma), \qquad \min_{|n| \leq |t| + 4N} \left\| \frac{x + n\omega \pm y}{2} \right\| \geq \exp(-N^\sigma)$$

for $N \geq N_0$ *and* $4N \leq |t| \leq \exp(-N^{\sigma-})$, *then*

$$\left| \frac{\prod_{n \neq k} (\cos 2\pi y - \cos 2\pi \theta_n)}{\prod_{n \neq k} (\cos 2\pi \theta_k - \cos 2\pi \theta_n)} \right| \leq \exp(N^{\sigma+}) \quad \textit{for } k = 1, \dots, 2N + 1.$$

Proof. Recall that

$$\int_0^1 \log |\xi - \cos 2\pi\theta| \, d\theta = -\log 2, \qquad |\xi| \leq 1.$$

Our assumptions on x, y, and t, together with the Diophantine condition on ω, guarantee that we can apply Lemma 5.1 with $\delta = \exp(-N^\sigma)$ to get

$$\left| \log \left| \prod_{n \neq k} (\cos 2\pi y - \cos 2\pi \theta_n) \right| + 2N \log 2 \right| \leq N^{\sigma+},$$
$$\left| \log \left| \prod_{n \neq k} (\cos 2\pi \theta_k - \cos 2\pi \theta_n) \right| + 2N \log 2 \right| \leq N^{\sigma+}.$$

Note that the Diophantine condition and the assumption that $|t| \leq \exp(N^{\sigma-})$ are needed to ensure that $\|(\theta_k - \theta_n)/2\| \geq \exp(-N^\sigma)$. The conclusion follows immediately. □

Proof of Proposition E′. Fix $x \in \mathbb{T}$ satisfying the assumptions of the proposition. Due to the large deviation estimate of Proposition B, we know there exists $y \in \mathbb{T}$ satisfying the assumptions of Lemma 5.2 and such that

$$\log |f_{[-N,N]}(y, \omega, E)| > (2N + 1)L(\omega, E) - (\log N)^C.$$

At the same time, from Lemma 5.2 and (5.4), we have

$$|f_{[-N,N]}(y, \omega, E)| = |Q(\cos 2\pi y)| \leq (2N + 1) \exp(N^{\sigma+}) \max_n |Q(\cos 2\pi \theta_n)|.$$

Therefore,

$$\max_n \log |f_{[-N,N]}(\theta_n, \omega, E)| \geq (2N + 1)L(\omega, E) - (N^{\sigma+})^{1-}.$$

This yields the desired conclusion. □

6. Localized eigenfunctions on finite intervals

We now continue by proving results on finite scale localization of eigenfunctions. We give a detailed proof for the almost Mathieu case and only briefly discuss the proof for a general analytic potential. For the latter case, a slightly different statement with detailed proof can be found in [GS08].

The result which we present here is adjusted to our criterion for identifying finite scale energies that are close to the full spectrum, as stated in Lemma 2.8. In turn, this criterion is adapted to the elimination of resonances afforded by Proposition E′. Due to the weaker elimination of resonances, in the case of the almost Mathieu operator we cannot immediately exclude the possibility of the eigenvector having some mass concentrated at the edges of a given interval. Instead we will see that we can work around this issue by shifting the edges of the interval. The shift is phase and energy dependent, and it will be crucial for Proposition I′ that our result addresses the stability of the shift.

Proposition F′. *Consider the almost Mathieu operator (1.9) with $|\lambda| > 1$ and $\omega \in \mathbb{T}_{c,a}$. Let*

$$\mathcal{B}_{\ell,M,\omega} := \Big\{x \in \mathbb{T} : \min_{|n|\le M+C_0\ell} \|x - n\omega/2\| < \exp(-\ell^{1/2})\Big\}.$$

There exist $\ell_0(|\lambda|, c, a)$, $c_0(|\lambda|, c, a)$, and $C_0(|\lambda|, c, a)$ such that the following statement holds for any $\ell \ge \ell_0$, $4\ell \le M \le \exp(\ell^{c_0})$. Given $E_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}$, there exists $N = N(x_0, E_0, \ell)$ such that $0 \le N - M \le C_0\ell$ and if

$$\begin{aligned}
&x \in \big(x_0 - \exp(-(\log\ell)^{C_0}), x_0 + \exp(-(\log\ell)^{C_0})\big) \setminus \mathcal{B}_{\ell,M,\omega},\\
&|E_j^{[-N,N]}(x,\omega) - E_0| \le \exp(-(\log\ell)^{C_0}),\\
&\max_{|s|\le[\ell/2]} \operatorname{dist}(E_j^{[-N,N]}(x,\omega), \operatorname{spec} H_{s+[-\ell,\ell]}(x,\omega)) \lesssim \exp(-\ell^{1/2+}),
\end{aligned} \tag{6.1}$$

then

$$|\psi_j^{[-N,N]}(x_0,\omega;n)| \le \exp(-|n|\log|\lambda| + C_0\ell), \qquad |n| > 4\ell.$$

Proof. From Proposition D we know that there exists $A(V, c, a, \gamma)$ such that

$$\{x \in \mathbb{T} : \operatorname{dist}(E_0, \operatorname{spec} H_{[-\ell,\ell]}(x,\omega)) < \exp(-(\log\ell)^A)\} \subset \bigcup_{k=1}^{k_0} I_k,$$

where I_k are intervals such that $|I_k| \le \exp(-(\log\ell)^{A/2})$, and $k_0 \le C\ell$. Due to the Diophantine condition, each interval I_k contains at most one point of the form

$$x_0 + (-M - n + \ell)\omega \quad \text{or} \quad x + (M + n - \ell)\omega,$$

respectively, with $|n| \le \ell^C$. It follows that there exists $|n_0| \le C\ell$ such that

$$x_0 + (-M - n_0 + \ell)\omega, x_0 + (M + n_0 - \ell)\omega \notin \bigcup_{k=1}^{k_0} I_k. \tag{6.2}$$

We let $N = M + n_0$. Suppose

$$x \in \big(x_0 - \exp(-(\log \ell)^{2A}), x_0 + \exp(-(\log \ell)^{2A})\big)$$

and $E = E_j^{[-N,N]}(x,\omega)$ are such that $|E - E_0| \le \exp(-(\log \ell)^{2A})$. From (6.2) and (2.11) it follows that

$$\operatorname{dist}(E, \operatorname{spec} H_{[-N,-N+2\ell]}(x,\omega)), \operatorname{dist}(E, \operatorname{spec} H_{[N-2\ell,N]}(x,\omega)) \gtrsim \exp(-(\log \ell)^{A}).$$

We will use Lemma 2.6 to show that any

$$E' \in (E - \exp(-\ell^{1/2+}), E + \exp(-\ell^{1/2+}))$$

is not in the spectrum of H on certain large intervals. Note that

$$\operatorname{dist}(E', \operatorname{spec} H_{[-N,-N+2\ell]}(x,\omega)), \operatorname{dist}(E', \operatorname{spec} H_{[N-2\ell,N]}(x,\omega)) \gtrsim \exp(-(\log \ell)^{A}).$$

Lemmas 2.3 and 2.4 imply that

$$\begin{aligned} |(H_{[-N,-N+2\ell]}(x,\omega) - E')^{-1}(m, -N+2\ell)| &< 1, \quad m \in [-N, -N+2\ell-\ell/4],\\ |(H_{[N-2\ell,N]}(x,\omega) - E')^{-1}(m, N-2\ell)| &< 1, \quad m \in [N-2\ell+\ell/4, N]. \end{aligned}$$

Since we also have

$$\max_{|s| \le [\ell/2]} \operatorname{dist}(E', \operatorname{spec} H_{s+[-\ell,\ell]}(x,\omega)) \lesssim \exp(-\ell^{1/2+}),$$

Lemma 2.4 implies that

$$\max_{|s| \le [\ell/2]} \log |f_{s+[-\ell,\ell]}(x,\omega,E')| \le (2\ell+1)L(E',\omega) - \ell^{1/2+}.$$

Proposition E′ and Lemma 2.4 imply that for any

$$m \in [-N+2\ell-\ell/4, -4\ell] \cup [4\ell, N-2\ell+\ell/4]$$

there exists $\Lambda_m = [a_m, b_m] \subset [-N, N]$ containing m such that $m - a_m, b_m - m > \ell/4$ and

$$\begin{aligned} |(H_{\Lambda_m}(x,\omega) - E')^{-1}(m', a_m)| &\ll 1, \quad m' - a_m \ge \ell/4,\\ |(H_{\Lambda_m}(x,\omega) - E')^{-1}(m', b_m)| &\ll 1, \quad b_m - m' \ge \ell/4. \end{aligned}$$

Lemma 2.6 now implies that E' is not in the spectrum of H restricted to $[-N, b_{-4\ell}]$ and $[a_{4\ell}, N]$. Since this is true for any $E' \in (E - \exp(-\ell^{1/2+}), E + \exp(-\ell^{1/2+}))$, it follows that

$$\operatorname{dist}(E, \operatorname{spec} H_{[-N,b_{-4\ell}]}(x,\omega)), \operatorname{dist}(E, \operatorname{spec} H_{[a_{4\ell},N]}(x,\omega)) \gtrsim \exp(-\ell^{1/2+}).$$

Lemma 2.3 implies that

$$\begin{aligned}\log|f_{[-N,b_{-4\ell}]}(x,\omega,E)|,\ &\log|f_{[a_{4\ell},N]}(x,\omega,E)| \\ &\geq NL(\omega,E) - C\ell - \ell^{1/2+}(\log N)^C \geq NL(\omega,E) - C'\ell, \qquad (6.3)\end{aligned}$$

provided that $N \leq \exp(\ell^{c_0})$ with c_0 small enough. The conclusion follows by using Lemma 2.4 and the Poisson formula. □

We will now state the analogous result for general potentials.

Proposition F. *Consider the Schrödinger operator* (1.1) *with real-analytic* V. *Assume that* $L(\omega,E) \geq \gamma > 0$ *for any* ω *and any* $E \in (E',E'')$. *There exist* $\ell_0(V,c,a,\gamma)$, $C_0(V,c,a,\gamma)$, *and* $C_1(V,c,a,\gamma)$ *such that the following statement holds for any* $\ell \geq \ell_0$, $N \geq \exp((\log\ell)^{2C_0})$, $\omega \in \mathbb{T}_{c,a} \setminus \Omega_{\ell,N}$, *and* $x \in \mathbb{T} \setminus \mathcal{B}_{\ell,N,\omega}$, *where, using the notation of Proposition* E, *we have*

$$\Omega_{\ell,N} = \bigcup_{\exp((\log\ell)^{C_0}) \leq |t| \leq N} \Omega_{2\ell+1,2\ell+1,t,\ell^{1/2}},$$

$$\mathcal{B}_{\ell,N,\omega} = \bigcup_{\exp((\log\ell)^{C_0}) \leq |t| \leq N, |s| \leq \ell} (s\omega + \mathcal{B}_{2\ell+1,2\ell+1,t,\ell^{1/2},\omega}).$$

If $E_j^{[-N,N]}(x,\omega) \in (E',E'')$ *is such that*

$$\max_{|s| \leq [\ell/2]} \operatorname{dist}(E_j^{[-N,N]}(x,\omega), H_{s+[-\ell,\ell]}(x,\omega)) \lesssim \exp(-\ell^{1/2+}), \qquad (6.4)$$

then

$$|\psi_j^{[-N,N]}(x,\omega;n)| \leq \exp\big(-|n|\gamma + C_1\exp((\log\ell)^{C_0})\big), \qquad |n| > \exp((\log\ell)^{C_0}).$$

Proof. The main idea is to use Lemma 2.6 to show that any

$$E \in \big(E_j^{[-N,N]}(x,\omega) - \exp(-\ell^{1/2+}), E_j^{[-N,N]}(x,\omega) + \exp(-\ell^{1/2+})\big)$$

is not in the spectrum of H on certain large intervals. Since

$$\max_{|s| \leq [\ell/2]} \operatorname{dist}(E, \operatorname{spec} H_{s+[-\ell,\ell]}(x,\omega)) \lesssim \exp(-\ell^{1/2+}),$$

Proposition E implies that

$$\operatorname{dist}(E, \operatorname{spec} H_{t+[-\ell,\ell]}(x,\omega)) \gtrsim \exp(-\ell^{1/2}(\log\ell)^{C_0}), \qquad \exp((\log\ell)^C) < |t| \leq N.$$

By the same reasoning as in the proof of Proposition F′, we obtain

$$\begin{aligned}\operatorname{dist}(E_j^{[-N,N]}&(x,\omega), \operatorname{spec} H_{[-N,-a]}(x,\omega)), \\ &\operatorname{dist}(E_j^{[-N,N]}(x,\omega), \operatorname{spec} H_{[a,N]}(x,\omega)) \gtrsim \exp(-\ell^{1/2}(\log\ell)^{C_0}),\end{aligned}$$

with $a = [\exp((\log\ell)^{C_0})] + 1$. The conclusion follows by using Lemma 2.3, Lemma 2.4, and the Poisson formula. □

Note that, due to Proposition E, in the above proposition we have

$$\begin{aligned} &\mathrm{mes}(\Omega_{\ell,N}) \le N \exp(-\ell^{1/4}/2), &&\mathrm{compl}(\Omega_{\ell,N}) \le N^2 \exp((\log \ell)^C), \\ &\mathrm{mes}(\mathcal{B}_{\ell,N,\omega}) \le N^2 \exp(-\ell^{1/4}/2), &&\mathrm{compl}(\mathcal{B}_{\ell,N,\omega}) \le N^2 \exp(\exp(\log \ell)^C), \end{aligned} \tag{6.5}$$

provided ℓ is large enough. This shows that the above result is meaningful as long as $N \le \exp(l^\varepsilon)$.

Remark 6.1. Condition (6.4) in Propositions F and F′ implies that the essential support of the eigenfunction remains close to the origin as N grows. This condition serves as a criterion for a given value E_0 to fall into the spectrum in the regime of positive Lyapunov exponents. This is the meaning of Lemma 2.8. Let us note that the elimination of resonances in Propositions E and E′ combined with the Poisson formula ensures only that the essential support of the eigenfunction cannot be too spread out. However, this obviously does not specify where the essential support is located.

7. Separation of finite scale eigenvalues

Next we discuss the separation of finite scale eigenvalues. The basic idea is that if two distinct eigenvalues are too close, then we can show that their corresponding eigenfunctions are also close, contradicting their orthogonality. It follows from (2.3) that the eigenvector ψ for the Dirichlet problem on $[a, b]$, normalized by $\psi(a) = 1$, is given by

$$\psi(n) = f_{[a,n-1]}(x, \omega, E), \qquad n \in [a, b], \tag{7.1}$$

with the convention that $f_{[a,a-1]} = 1$. Thus, we can estimate the distance between the eigenvectors corresponding to different energies by using the following consequence of the uniform upper bound estimate.

Corollary 7.1 ([GS11, Cor. 2.14]). *Fix $\omega_0 \in \mathbb{T}_{c,a}$ and $E_0 \in \mathbb{C}$. Assume that $L(\omega_0, E_0) \ge \gamma > 0$. Let ∂ denote one of the partial derivatives ∂_x, ∂_E, ∂_ω. Then*

$$\sup\{\log \|\partial M_N(x, \omega, E)\| : |E - E_0| + |\omega - \omega_0| < N^{-C},\ x \in \mathbb{T}\} \le N L(\omega_0, E_0) + C(\log N)^{C_0}$$

for all $N \ge 2$. Here $C_0 = C_0(a)$ and $C = C(V, a, c, \gamma, E_0)$.

We are ready to prove separation of eigenvalues for the almost Mathieu operator.

Proposition G′. *Using the notation and the assumptions of Proposition F′, we have*

$$|E_k^{[-N,N]}(x, \omega) - E_j^{[-N,N]}(x, \omega)| > \exp(-C_2 \ell), \qquad k \ne j,$$

with $C_2 = C_2(|\lambda|, c, a) \gg C_1$.

Proof. To reach a contradiction, let

$$E_1 = E_j^{[-N,N]}(x,\omega), \qquad E_2 = E_k^{[-N,N]}(x,\omega),$$

and assume that $|E_1 - E_2| < \exp(-C_2\ell)$. We have $|E_2 - E_0| \lesssim \exp(-\ell^{1/2})$, so Proposition F′ applies to E_2 also. We know from (7.1) that

$$\psi_i(n) = f_{[-N,n-1]}(x,\omega,E_i), \qquad i = 1,2,$$

are eigenvectors corresponding to E_1 and E_2. Proposition F′ implies that

$$\sum_{|n|>C\ell} |\psi_i(n)|^2 \le \exp(-C\ell\log|\lambda|)\sum|\psi_i(n)|^2$$

provided $C \gg C_1$. From Corollary 7.1 it follows that

$$\begin{aligned}\sum_{|n|\le C\ell} |\psi_1(n) - \psi_2(n)|^2 &\le |E_1 - E_2|^2 \exp(2NL(E_1,\omega) + C\ell)\\ &\le \exp(-2C_2\ell + 2NL(E_1,\omega) + C\ell).\end{aligned}$$

From (6.3) we know that

$$\sum|\psi_1(n)|^2 \ge \exp(2NL(E_1,\omega) - C\ell).$$

Therefore

$$\sum_{|n|\le C\ell} |\psi_1(n) - \psi_2(n)|^2 \le \exp(-C_2\ell)\sum|\psi_1(n)|^2$$

provided C_2 is large enough. We arrive at the estimate

$$\sum|\psi_1(n)|^2 + \sum|\psi_2(n)|^2 = \|\psi_1 - \psi_2\|^2 \le \exp(-C\ell)\Big(\sum|\psi_1(n)|^2 + \sum|\psi_2(n)|^2\Big).$$

This is impossible and concludes the proof. □

We only state the analogous result for general analytic potentials. Its proof is completely analogous to Proposition G′. The difference in the results comes from the difference in the sizes of the localization windows. Note that, for this reason, the separation is much better for the almost Mathieu operator.

Proposition G. *Using the notation and the assumptions of Proposition F, we have*

$$|E_k^{[-N,N]}(x,\omega) - E_j^{[-N,N]}(x,\omega)| > \exp(-C_2\exp((\log\ell)^{C_0})), \qquad k \ne j,$$

with $C_2 = C_2(V,c,a,\gamma) \gg C_1$.

8. Stabilization of finite scale spectral segments

Propositions G and G′ allow us to obtain a *stability property* of the finite volume spectra as we pass from one scale to the next bigger one. This paves the way for a multi-scale control of the spectrum in infinite volume.

We first recall some well-known estimates on the stabilization of finite scale eigenvalues and eigenfunctions as the scale increases.

Lemma 8.1. *Let $x, \omega \in \mathbb{T}$. For any intervals $\Lambda_0 = [a_0, b_0] \subset \Lambda \subset \mathbb{Z}$ and any j_0, we have*

$$\operatorname{dist}(E_{j_0}^{\Lambda_0}(x,\omega), \operatorname{spec} H_\Lambda(x,\omega)) \le |\psi_{j_0}^{\Lambda_0}(x,\omega; a_0)| + |\psi_{j_0}^{\Lambda_0}(x,\omega; b_0)|.$$

Proof. Let ψ_0 be the extension, with zero entries, of $\psi_{j_0}^{\Lambda_0}(x,\omega)$ to Λ. Since $\|\psi_0\| = 1$, the conclusion follows from the fact that

$$\|(H_\Lambda(x,\omega) - E_{j_0}^{\Lambda_0}(x,\omega))\psi_0\| \le |\psi_{j_0}^{\Lambda_0}(x,\omega; a_0)| + |\psi_{j_0}^{\Lambda_0}(x,\omega; b_0)|.$$

Indeed, this implies that $\|(H_\Lambda(x,\omega) - E_{j_0}^{\Lambda_0}(x,\omega))^{-1}\|^{-1}$ is also bounded by the right-hand side and the lemma follows by self-adjointness of $H_\Lambda(x,\omega)$. □

Lemma 8.2. *Let A be a finite-dimensional Hermitian operator. Let $E, \eta \in \mathbb{R}$, $\eta > 0$. Assume that the subspace of the eigenvectors of A with eigenvalues in the interval $(E-\eta, E+\eta)$ is at most of dimension one. If there exists ϕ such that $\|\phi\| = 1$ and*

$$\|(A - E)\phi\| < \varepsilon < \eta,$$

then there exists an eigenvector ψ_0 with an eigenvalue $E_0 \in (E - \varepsilon, E + \varepsilon)$ such that

$$\|\phi - \psi_0\| \lesssim \varepsilon\eta^{-1}.$$

Proof. Let $\{\psi_j\}$ be an orthonormal basis of eigenvectors of A, $A\psi_j = E_j\psi_j$. Then

$$\varepsilon^2 > \|(A-E)\phi\|^2 = \sum_j |\langle \phi, \psi_j\rangle|^2 (E_j - E)^2 \ge \min_j (E_j - E)^2.$$

This implies that $E_k \in (E - \varepsilon, E + \varepsilon)$ for some k, and $E_j \notin (E - \eta, E + \eta)$ for any $j \neq k$. We have

$$\varepsilon^2 > \|(A-E)\phi\|^2 \ge \sum_{j \neq k} |\langle \phi, \psi_j\rangle|^2 (E_j - E)^2 \ge \eta^2 \sum_{j\neq k} |\langle \phi, \psi_j\rangle|^2,$$

and therefore

$$1 - |\langle \phi, \psi_k\rangle|^2 = \|\phi - \langle \phi, \psi_k\rangle\psi_k\|^2 = \sum_{j\neq k} |\langle \phi, \psi_j\rangle|^2 \le \varepsilon^2\eta^{-2}.$$

The conclusion now follows from the fact that $\|\phi - \psi_k\|^2 = 2(1 - \operatorname{Re}\langle \phi, \psi_k\rangle)$. □

We will also use the following well-known result (which could be replaced by considerations about semialgebraic sets).

Lemma 8.3. *Let $\omega \in \mathbb{T}$, $N_1, N_2 \geq 1$, $\delta > 0$, and assume that the potential V in (1.1) is a trigonometric polynomial of degree d_0. Then the number of connected components of*

$$\{x \in \mathbb{T} : |E_{j_1}^{(N_1)}(x,\omega) - E_{j_2}^{(N_2)}(x,\omega)| \leq \delta\}$$

is $\lesssim N_1 N_2 d_0^2$.

Proof. It can be seen from (2.2) that

$$\exp(2\pi i d_0 N x) f_N(x,\omega,E) = P_N(\exp(2\pi i x), E),$$

with P_N being a polynomial of degree $2d_0 N$. Since the eigenvalues are continuous in x, the number of components of the set we are interested in is bounded by the number of solutions of the system

$$\begin{cases} 0 = P_{N_1}(z,E), \\ 0 = P_{N_2}(z, E \pm \delta). \end{cases}$$

The conclusion follows by using Bézout's Theorem. □

We are now ready to prove a detailed result on the stability of the finite scale spectra for the almost Mathieu operator. To be more precise, the result only applies to certain spectral segments. However, by Lemmas 2.8 and 3.2 we know that these are precisely the spectral segments that we need to get control of the full scale spectrum.

Proposition I′. *Consider the almost Mathieu operator (1.9) with $|\lambda| > 1$ and $\omega \in \mathbb{T}_{c,a}$. Let*

$$\mathcal{B}_{N,k,\omega} = \left\{x \in \mathbb{T} : \min_{|n| \leq 2N^{(k)}} \|x - n\omega/2\| < \exp(-(N^{(k-2)})^{1/2})\right\},$$

where $N^{(k)} = N^{2^k}$, $k \geq 0$, $N^{(-1)} = N$. Let C_0 be as in Proposition F′. There exists $N_0(|\lambda|, c, a)$ such that the following statement holds for any $N \geq N_0$, $k_0 \geq 1$. If there exists j_0 such that

$$|\psi_{j_0}^{[-N,N]}(x,\omega;\pm N)| \leq \exp(-c_0 N)$$

for some constant $c_0 < 1$ and for any x in an interval I, $|I| \leq \exp(-(\log 2N^{(k_0-2)})^{C_0})$, then $I \setminus \bigcup_{k=1}^{k_0} \mathcal{B}_{N,k,\omega}$ can be partitioned into subintervals I_m, $m \leq (N^{(k_0)})^C$, with C an absolute constant, so that for each I_m, there exist $N_k = N_k(I_m) \simeq N^{(k)}$ and $j_k = j_k(I_m)$, $k = 1, \dots, k_0$, such that for any $x \in I_m$ and $k \leq k_0 - 1$, we have

$$\begin{aligned}
|E_{j_0}^{[-N,N]}(x,\omega) - E_{j_1}^{[-N_1,N_1]}(x,\omega)| &\lesssim \exp(-c_0 N), \\
|E_{j_{k+1}}^{[-N_{k+1},N_{k+1}]}(x,\omega) - E_{j_k}^{[-N_k,N_k]}(x,\omega)| &\leq \exp(-(N_k \log|\lambda|)/2), \\
|\psi_{j_1}^{[-N_1,N_1]}(x,\omega;n)| &\leq \exp(-(|n| \log|\lambda|)/2), \\
&\qquad |n| > C(|\lambda|, a, c)N, \\
|\psi_{j_{k+1}}^{[-N_{k+1},N_{k+1}]}(x,\omega;n)| &\leq \exp(-(|n| \log|\lambda|)/2), \quad |n| > N_k, \\
|\psi_{j_{k+1}}^{[-N_{k+1},N_{k+1}]}(x,\omega;n) - \psi_{j_k}^{[-N_k,N_k]}(x,\omega;n)| &\leq \exp(-(N_k \log|\lambda|)/2), \quad |n| \leq N_k.
\end{aligned} \tag{8.1}$$

Proof. Let $\mathcal{B} = \bigcup_{k=1}^{k_0} \mathcal{B}_{N,k,\omega}$. Note that $I \setminus \mathcal{B}$ has $\lesssim (N^{(k_0)})^2$ components. Let x_0 be the midpoint of I and let $E_0 = E_{j_0}^{[-N,N]}(x_0, \omega)$. Let $N' = 3N$. We choose $N_i(x_0, E_0, N') \simeq N^{2i}$, $i = 1, 2$, as in Proposition F′. Since

$$s + [-N', N'] \supset [-N, N] \quad \text{for any } |s| \leq [N'/2],$$

it follows from Lemma 8.1 that

$$\max_{|s| \leq [N'/2]} \operatorname{dist}(E_{j_0}^{[-N,N]}(x, \omega), \operatorname{spec} H_{s+[-N',N']}(x, \omega)) \leq 2 \exp(-c_0 N). \tag{8.2}$$

Lemma 8.1 also implies that there exists $j_1(x)$ such that

$$|E_{j_0}^{[-N,N]}(x, \omega) - E_{j_1}^{[-N_1,N_1]}(x, \omega)| \leq 2 \exp(-c_0 N). \tag{8.3}$$

It follows from Lemma 8.3 that we can partition I into fewer than N_1^C subintervals, with C an absolute constant, such that we can choose $j_1(x)$ to be constant on each of the subintervals. Let $I_m^{(1)}$, $m \lesssim (N^{(k_0)})^C$, be the intervals of the partition induced on $I \setminus \mathcal{B}$. Because of (8.2), (8.3) we have

$$\max_{|s| \leq [N'/2]} \operatorname{dist}(E_{j_1}^{[-N_1,N_1]}(x, \omega), \operatorname{spec} H_{s+[-N',N']}(x, \omega)) \leq \exp(-(N')^{1/2+}),$$

so we can apply Proposition F′ to get

$$|\psi_{j_1}^{[-N_1,N_1]}(x, \omega; n)| \leq \exp(-|n| \log |\lambda| + C'N') \leq \exp(-|n| \log |\lambda|/2), \quad |n| > CN,$$

for all $x \in I \setminus \mathcal{B}$. Note that

$$|E_{j_1}^{[-N_1,N_1]}(x, \omega) - E_0| \leq \exp(-(\log N')^{C_0}),$$

because of (8.3), (2.11), and our assumption on the length of I.

Now Lemma 8.1 yields the existence of $j_2(x)$ for each $x \in I \setminus \mathcal{B}$ such that

$$|E_{j_1}^{[-N_1,N_1]}(x, \omega) - E_{j_2}^{[-N_2,N_2]}(x, \omega)| \leq 2 \exp(-N_1 \log |\lambda| + CN') \leq \exp(-N_1 \log |\lambda|/2).$$

Using Lemma 8.3 we obtain a refined partition $I_m^{(2)}$, $m \lesssim (N^{(k_0)})^C$, of $I \setminus \mathcal{B}$, that contains at most $\lesssim N_2^C$ more intervals than the previous one, and such that the choice of j_2 is constant on each $I_m^{(2)}$. Again we have

$$\max_{|s| \leq [N'/2]} \operatorname{dist}(E_{j_2}^{[-N_2,N_2]}(x, \omega), \operatorname{spec} H_{s+[-N',N']}(x, \omega)) \leq \exp(-(N')^{1/2+}),$$

and

$$|E_{j_2}^{[-N_2,N_2]}(x, \omega) - E_0| \leq \exp(-(\log N')^{C_0}),$$

so we can apply Proposition F′ to get the localization estimate for $\psi_{j_2}^{[-N_2,N_2]}(x,\omega)$. Furthermore, we can also apply Proposition G′ together with Lemma 8.2 to get

$$\begin{aligned}&\|\psi_{j_2}^{[-N_2,N_2]}(x,\omega)-\tilde{\psi}_{j_1}^{[-N_1,N_1]}(x,\omega)\|\\&\qquad\lesssim \exp(-N_1\log|\lambda|+CN')\exp(C'N')\le\exp(-(N_1\log|\lambda|)/2),\end{aligned}$$

where $\tilde{\psi}_{j_1}^{[-N_1,N_1]}(x,\omega)$ is the extension, with zero entries, of $\psi_{j_1}^{[-N_1,N_1]}(x,\omega)$ to $[-N_2,N_2]$.

The conclusion follows through iteration. For the sake of clarity we set up the next step. Let x_1 be the midpoint of $I_m^{(2)}$ and let $E_1=E_{j_1}^{[-N_1,N_1]}(x_1,\omega)$. Let $N_1'=3N_1$. We have

$$\max_{|s|\le[N_1'/2]}\operatorname{dist}(E_{j_1}^{[-N_1,N_1]}(x,\omega),\operatorname{spec}H_{s+[-N_1'+N_1']}(x,\omega))\le\exp(-(N_1')^{1/2+}),$$

and we choose $N_3(x_1,E_1,N_1')\simeq N^{(3)}$. As before we obtain a refined partition $I_m^{(3)}$, $m\lesssim(N^{(k_0)})^C$, and for each interval, there exists j_3 such that

$$|E_{j_1}^{[-N_2,N_2]}(x,\omega)-E_{j_3}^{[-N_3,N_3]}(x,\omega)|\le\exp(-N_2\log|\lambda|/2)$$

for all x in the interval. As before we can apply Proposition F′, Proposition G′, and Lemma 8.2 to deduce the desired estimates on the eigenvectors. □

The result for general analytic potentials is analogous. Its proof is similar to that of Proposition I′, but we have to approximate the potential V by trigonometric polynomials in order to be able to use Lemma 8.3.

Proposition I. *Consider the Schrödinger operator (1.1) with real-analytic V. Assume that $L(\omega,E)\ge\gamma>0$ for any ω and any $E\in(E',E'')$. Let $\mathcal{B}_{\ell,N,\omega}$, $\Omega_{\ell,N}$ be as in Proposition F and*

$$\mathcal{B}_{N,k,\omega}=\mathcal{B}_{\ell_k,N_k,\omega},\quad \Omega_{N,k}=\Omega_{\ell_k,N_k}$$

with $N_k=[\exp(N^{1/10})]^{2^{k-1}}$, $\ell_k=3[\log N_k]^{10}$. There exists $N_0(V,c,a,\gamma)$ such that the following statement holds for any $N\ge N_0$, $k_0\ge 1$ and $\omega\in\mathbb{T}_{c,a}\setminus\bigcup_{k=1}^{k_0}\Omega_{N,k}$. If there exists j_0 such that

$$|\psi_{j_0}^{[-N,N]}(x,\omega;\pm N)|\le\exp(-c_0N)$$

for some constant $c_0<1$ and for any x in an interval I, and $E_{j_0}^{[-N,N]}(I,\omega)\subset(E',E'')$, then $I\setminus\bigcup_{k=1}^{k_0}\mathcal{B}_{N,k,\omega}$ can be partitioned into subintervals I_m, $m\le N_{k_0}^C$, with C an absolute constant, so that for each I_m, there exist $N_k=N_k(I_m)\simeq N^{(k)}$ and $j_k=j_k(I_m)$, $k=1,\dots,k_0$, such that for any $x\in I_m$ and $k\le k_0-1$, we have

$$
\begin{aligned}
|E_{j_0}^{[-N,N]}(x,\omega) - E_{j_1}^{[-N_1,N_1]}(x,\omega)| &\lesssim \exp(-c_0 N),\\
|E_{j_{k+1}}^{[-N_{k+1},N_{k+1}]}(x,\omega) - E_{j_k}^{[-N_k,N_k]}(x,\omega)| &\le \exp(-(N_k\gamma)/2),\\
|\psi_{j_1}^{[-N_1,N_1]}(x,\omega;n)| &\le \exp(-(|n|\gamma)/2),\\
&\qquad |n| > \exp((\log N)^{C(V,c,a,\gamma)}),\\
|\psi_{j_{k+1}}^{[-N_{k+1},N_{k+1}]}(x,\omega;n)| &\le \exp(-(|n|\gamma)/2), \quad |n| > N_k,\\
|\psi_{j_{k+1}}^{[-N_{k+1},N_{k+1}]}(x,\omega;n) - \psi_{j_k}^{[-N_k,N_k]}(x,\omega;n)| &\le \exp(-(N_k\gamma)/2), \quad |n| \le N_k.
\end{aligned}
\tag{8.4}
$$

Proof. Let $\mathcal{B} = \bigcup_{k=1}^{k_0} \mathcal{B}_{N,k,\omega}$. It follows from (6.5) that $I \setminus \mathcal{B}$ has $\lesssim N_{k_0}^C$ intervals. Note that $\ell_1 \simeq 3N$, so $s + [-\ell_1, \ell_1] \supset [-N, N]$ for any $|s| \le [\ell_1/2]$, and therefore by Lemma 8.1,

$$\max_{|s|\le[\ell_1/2]} \operatorname{dist}(E_{j_0}^{[-N,N]}(x,\omega), \operatorname{spec} H_{s+[-\ell_1,\ell_1]}(x,\omega)) \le 2\exp(-c_0 N).$$

Lemma 8.1 also implies that there exists $j_1(x)$ such that

$$|E_{j_0}^{[-N,N]}(x,\omega) - E_{j_1}^{[-N_1,N_1]}(x,\omega)| \le 2\exp(-c_0 N).$$

Choose $\tilde{V}$ as in (3.4) with $K = CN_1$ such that, by (3.5), we have

$$|\tilde{E}_{j_0}^{[-N,N]}(x,\omega) - \tilde{E}_{j_1}^{[-N_1,N_1]}(x,\omega)| \le 3\exp(-c_0 N).$$

By Lemma 8.3 we can partition I into fewer than N_1^C subintervals such that $j_1(x)$ can be kept constant on each of the subintervals. Using (3.5) again, on these intervals we have

$$|E_{j_0}^{[-N,N]}(x,\omega) - E_{j_1}^{[-N_1,N_1]}(x,\omega)| \le 4\exp(-c_0 N),$$

and one can proceed as in the proof of Proposition I′. We just note that the choice of ℓ_k and N_k is such that the separation obtained from Proposition G is by $\exp(-N_k^{\varepsilon}) > \exp(-N_{k_1}^{1/2})$. This is crucial for obtaining the desired estimates from Lemma 8.2. □

Finally, let us note that our main results also follow from the results on stabilization (though for general potentials the result is weaker because we have to remove a measure zero set of bad frequencies). This is simply because we can establish the following two analogues of Lemma 3.4. Their proofs mirror that of Lemma 3.4. For the convenience of the reader, we include the proof for the almost Mathieu case.

Proposition 8.4. *Consider the almost Mathieu operator (1.9) with $|\lambda| > 1$ and $\omega \in \mathbb{T}_{c,a}$. There exists $N_0(|\lambda|, c, a)$ such that the following statement holds for any $N \ge N_0$. If there exist j_0 and an interval $I \subset \mathbb{T}$ such that*

$$|\psi_{j_0}^{[-N,N]}(x,\omega;\pm N)| \le \exp(-c_0 N), \quad x \in I,$$

for some constant $c_0 < 1$, then

$$\operatorname{mes}(E_{j_0}^{[-N,N]}(I,\omega) \setminus \mathcal{S}_\omega) \le \exp(-c_1 N^{1/2}),$$

with c_1 an absolute constant.

Proof. Let $\mathcal{B}_{N,k,\omega}$ be as in Proposition I′. Let C_0 be as in Proposition F′. Partition I into intervals $I_m^{(0)}$, $m \lesssim \exp((\log 2N)^{C_0})$, such that $|I_m^{(0)}| \le \exp(-(\log 2N)^{C_0})$. Let $I_m^{(1)}$, with

$$m \lesssim N^C \exp((\log 2N)^{C_0}) \le \exp((\log N)^C),$$

be the partition of $I \setminus \mathcal{B}_{N,1,\omega}$ obtained by applying Proposition I′ with $k_0 = 1$ on each $I_m^{(0)}$. Since on each $I_m^{(1)}$ we have

$$|E_{j_0}^{[-N,N]}(x,\omega) - E_{j_1}^{[-N_1,N_1]}(x,\omega)| \le \exp(-cN)$$

with $N_1 = N_1(I_m^{(1)}) \simeq N^2$, $j_1 = j_1(I_m^{(1)})$, it follows by the continuity of the parametrization of the eigenvalues that

$$\operatorname{mes}\big(E_{j_0}^{[-N,N]}(I_m^{(1)},\omega) \ominus E_{j_1}^{[-N_1,N_1]}(I_m^{(1)},\omega)\big) \lesssim \exp(-cN),$$

where $\ominus$ denotes symmetric difference. From (2.11) it follows that

$$\operatorname{mes}(E_{j_0}^{[-N,N]}(\mathcal{B}_{N,1,\omega},\omega)) \le \exp(-cN^{1/2}).$$

Let

$$\mathcal{E}_{N,1,\omega} = E_{j_0}^{[-N,N]}(\mathcal{B}_{N,1,\omega},\omega) \cup \bigcup_m \big(E_{j_0}^{[-N,N]}(I_m^{(1)},\omega) \ominus E_{j_1}^{[-N_1,N_1]}(I_m^{(1)},\omega)\big).$$

We clearly have $\operatorname{mes}(\mathcal{E}_{N,1,\omega}) \le \exp(-cN^{1/2})$.

Note that Lemma 8.1 implies that

$$\operatorname{dist}(E,\mathcal{S}_\omega) \lesssim \exp(-c_0 N), \qquad E \in E_{j_0}^{[-N,N]}(I,\omega).$$

Since any $E \in E_{j_0}^{[-N,N]}(I,\omega) \setminus \mathcal{E}_{N,1,\omega}$ also belongs to some $E_{j_1}^{[-N_1,N_1]}(I_m^{(1)},\omega)$, it follows from Proposition I′ and Lemma 8.1 that

$$\operatorname{dist}(E,\mathcal{S}_\omega) \le \exp(-cN_1), \qquad E \in E_{j_0}^{[-N,N]}(I,\omega) \setminus \mathcal{E}_{N,1,\omega}.$$

By applying Proposition I′ repeatedly, we obtain sets

$$\mathcal{E}_{N,k,\omega} = E_{j_0}^{[-N,N]}(\mathcal{B}_{N,k,\omega},\omega) \cup \bigcup_m \big(E_{j_{k-1}}^{[-N_{k-1},N_{k-1}]}(I_m^{(k)},\omega) \ominus E_{j_k}^{[-N_k,N_k]}(I_m^{(k)},\omega)\big)$$

such that $\operatorname{mes}(\mathcal{E}_{N,k,\omega}) \le \exp(-c(N^{(k-2)})^{1/2})$ (recall that $N^{(k)} = N^{2^k}$) and

$$\operatorname{dist}(E,\mathcal{S}_\omega) \le \exp(-cN^{(k)}), \qquad E \in E_{j_0}^{[-N,N]}(I,\omega) \setminus \bigcup_{\ell=1}^{k} \mathcal{E}_{N,\ell,\omega}.$$

Finally, we note that

$$E_{j_0}^{[-N,N]}(I,\omega) \setminus \mathcal{S}_\omega \subset \bigcup_k \mathcal{E}_{N,k,\omega},$$

and we are done. □

Proposition 8.5. *Consider the Schrödinger operator* (1.1) *with real-analytic* V. *Assume that* $L(\omega, E) \geq \gamma > 0$ *for any* ω *and any* $E \in (E', E'')$. *Let* $\Omega_{N,k}$ *be as in Proposition I. There exists* $N_0(V, c, a, \gamma)$ *such that the following statement holds for any* $N \geq N_0$ *and any* $\omega \in \mathbb{T}_{c,a} \setminus \bigcup_{k \geq 1} \Omega_{N,k}$. *If there exist* j_0 *and an interval* $I \subset \mathbb{T}$ *such that*

$$|\psi_{j_0}^{[-N,N]}(x, \omega; \pm N)| \leq \exp(-c_0 N), \qquad x \in I,$$

for some constant $c_0 < 1$, *and* $E_{j_0}^{[-N,N]}(I, \omega) \subset (E', E'')$, *then*

$$\operatorname{mes}(E_{j_0}^{[-N,N]}(I, \omega) \setminus \mathcal{S}_\omega) \leq \exp(-c_1 N^{1/2}),$$

with c_1 *an absolute constant.*

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