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RELAXATION OF WAVE MAPS EXTERIOR TO A BALL TO HARMONIC MAPS FOR ALL DATA

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Abstract. In this paper we establish relaxation of an arbitrary 1-equivariant wave map from $\mathbb{R}^{1+3}_{t,x}\backslash(\mathbb{R}\times B(0,1)) \to S^3$ of finite energy and with a Dirichlet condition at $r = 1$, to the unique stationary harmonic map in its degree class. This settles a recent conjecture of Bizoń, Chmaj, Maliborski who observed this asymptotic behavior numerically.

1 Introduction

In this paper we describe all possible asymptotic dynamics for the 1-equivariant wave-map equation from

$$\mathbb{R}^{1+3}_{t,x}\backslash(\mathbb{R}\times B(0,1)) \to S^3$$

with a Dirichlet condition on the boundary of the ball $B(0,1)$, and data of finite energy. To be specific, consider the Lagrangian

$$\mathcal{L}(U,\partial_t U) = \int\limits_{\mathbb{R}^{1+3}\backslash(\mathbb{R}\times B(0,1))} \frac{1}{2}\left(-|\partial_t U|_g^2 + \sum_{j=1}^{3}|\partial_j U|_g^2\right) dtdx$$

where g is the round metric on S^3, and we only consider functions for which the boundary of the cylinder $\mathbb{R}\times B(0,1)$ gets mapped to a fixed point on S^3, say the north pole. Under the usual 1-equivariance assumption the Euler-Lagrange equation associated with this Lagrangian becomes

$$\psi_{tt} - \psi_{rr} - \frac{2}{r}\psi_r + \frac{\sin(2\psi)}{r^2} = 0 \tag{1.1}$$

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where $\psi(t,r)$ measures the angle from the north-pole on S^3. The imposed Dirichlet boundary condition is then $\psi(t,1) = 0$ for all $t \in \mathbb{R}$. In other words, we are considering the Cauchy problem

$$\begin{aligned} &\psi_{tt} - \psi_{rr} - \frac{2}{r}\psi_r + \frac{\sin(2\psi)}{r^2} = 0, \quad r \geq 1, \\ &\psi(t,1) = 0, \quad \forall\, t, \\ &\psi(0,r) = \psi_0(r), \quad \psi_t(0,r) = \psi_1(r). \end{aligned} \tag{1.2}$$

The conserved energy is

$$\mathcal{E}(\psi,\psi_t) = \int_1^\infty \frac{1}{2}\left(\psi_t^2 + \psi_r^2 + 2\frac{\sin^2(\psi)}{r^2}\right) r^2\, dr. \tag{1.3}$$

Any $\psi(t,r)$ of finite energy and continuous dependence on $t \in I := (t_0,t_1)$ must satisfy $\psi(t,\infty) = n\pi$ for all $t \in I$ where $n \in \mathbb{Z}$ is fixed. We can restrict to the case $n \geq 0$ since this covers the entire range $n \in \mathbb{Z}$ by the symmetry $\psi \mapsto -\psi$. We call n the *degree*, and denote by $\mathcal{E}_n$ the connected component of the metric space of all $\vec{\psi} = (\psi_0,\psi_1)$ with $\mathcal{E}(\vec{\psi}) < \infty$ and fixed degree n (of course obeying the boundary condition at $r = 1$), i.e.,

$$\mathcal{E}_n := \{(\psi_0,\psi_1) \mid \mathcal{E}(\psi_0,\psi_1) < \infty,\ \psi_0(1) = 0,\ \lim_{r\to\infty} \psi_0(r) = n\pi\}. \tag{1.4}$$

The advantage of this model lies with the fact that removing the unit ball eliminates the scaling symmetry and also renders the equation subcritical relative to the energy. This subcriticality immediately implies global wellposedness in the energy class. Both of these features are in stark contrast to the same equation on $1+3$-dimensional Minkowski space, which is known to be super-critical and to develop singularities in finite time, see Shatah [S88] and also Shatah and Struwe [SS98].

Another striking feature of this model, which fails for the $1+2$-dimensional analogue, lies with the fact that it admits infinitely many stationary solutions $(Q_n(r),0)$ which satisfy $Q_n(1) = 0$ and $\lim_{r\to\infty} Q_n(r) = n\pi$, for each $n \geq 1$. These solutions have minimal energy in the degree class $\mathcal{E}_n$, and they are the unique stationary solutions in that class.

The natural space to place the solution into for $n = 0$ is the *energy space* $\mathcal{H}_0 := (\dot{H}_0^1 \times L^2)(\mathbb{R}^3_*)$ with norm

$$\|\vec{\psi}\|_{\mathcal{H}_0}^2 := \int_1^\infty (\psi_r^2(r) + \psi_t^2(r))\, r^2\, dr, \qquad \vec{\psi} = (\psi,\psi_t). \tag{1.5}$$

Here, $\mathbb{R}^3_* := \mathbb{R}^3 \backslash B(0,1)$ and $\dot{H}_0^1(\mathbb{R}^3_*)$ is the completion under the first norm on the right-hand side of (1.5) of the smooth radial functions on $\{x \in \mathbb{R}^3 \mid |x| > 1\}$ with compact support. For $n \geq 1$, we denote $\mathcal{H}_n := \mathcal{E}_n - (Q_n,0)$ with "norm"

$$\|\vec{\psi}\|_{\mathcal{H}_n} := \|\vec{\psi} - (Q_n,0)\|_{\mathcal{H}_0}.$$

The point of this notation is that the boundary condition at $r=\infty$ is $\vec{\psi}-(Q_n,0)(r)\to 0$ as $r\to\infty$.

The exterior equation (1.2) was proposed by Bizon et al. [BCM12] as a model in which to study the problem of relaxation to the ground states given by the various equivariant harmonic maps. In the physics literature, this model was introduced in [BSS92] as an easier alternative to the Skyrmion equation. Moreover, [BSS92] stresses the analogy with the damped pendulum which plays an important role in our analysis. Both [BCM12, BSS92] obtain the existence and uniqueness of the ground state harmonic maps via the phase-plane of the damped pendulum, and they also observed stability of the linearized equation around the harmonic maps. Numerical simulations described in [BCM12] indicated that in each equivariance class and topological class given by the boundary value $n\pi$ at $r=\infty$ *every solution* scatters to the unique harmonic map Q_n that lies in this class. In this paper we verify this conjecture in the 1-equivariant setting, for all degrees and all data.

Our main result is as follows. It should be viewed as a verification of the *soliton resolution conjecture* for this particular case.

Theorem 1.1. *For any smooth energy data in $\mathcal{E}_n$ there exists a unique global and smooth solution to (1.2) which scatters to the harmonic map $(Q_n,0)$.*

Scattering here means that on compact regions in space one has $(\psi,\psi_t)(t)-(Q_n,0)\to(0,0)$ in the energy topology, or alternatively

$$(\psi,\psi_t)(t)=(Q_n,0)+(\varphi,\varphi_t)(t)+o_{\mathcal{H}_n}(1)\quad t\to\infty \tag{1.6}$$

where $(\varphi,\varphi_t)\in\mathcal{H}_0$ solves the linearized version of (1.2), i.e.,

$$\varphi_{tt}-\varphi_{rr}-\frac{2}{r}\varphi_r+\frac{2}{r^2}\varphi=0,\ r\geq 1,\ \varphi(t,1)=0. \tag{1.7}$$

We would like to emphasize that only the scattering part of Theorem 1.1 is difficult.

In [LS13] the second and third authors established this theorem for degree zero, and also proved asymptotic stability of the Q_n for $n\geq 1$. Here we are able to treat data of all sizes in the higher degree case. As in [LS13] we employ the method of concentration compactness from [KM06, KM08]. The main difference from [LS13] lies with the rigidity argument. While the virial identity was the key to rigidity in [LS13] for degree zero (which seems to be impossible for $n\geq 1$), here we follow an alternate route which was developed in a very different context in [DKM13, DKM12] for the three-dimensional energy critical nonlinear focusing wave equation. To be specific we rely on the *exterior asymptotic energy* arguments developed there. A novel feature of our work is that we elucidate the role of the Newton potential as an obstruction to linear energy estimates exterior to a cone in odd dimensions; in particular we do this for $\dim=5$, which is what is needed for equivariant wave maps in $\mathbb{R}^3$. It is precisely this feature which allows us to adapt the rigidity blueprint from [DKM13, DKM12] to the model under consideration.

Finally, let us mention that we expect the methods of this paper to carry over to higher equivariance classes as well.

2 Preliminaries

In this section we discuss the harmonic maps Q_n, as well as the reduction of the equivariant wave maps equation to a semi-linear equation in $\mathbb{R}^5_* := \mathbb{R}^5 \backslash B(0,1)$ with a Dirichlet condition at $r = 1$.

2.1 Exterior harmonic maps. In each energy class, $\mathcal{E}_n$ there is a unique finite energy exterior harmonic map, $(Q, 0) = (Q_n, 0)$. In fact $(Q_n, 0)$ can be seen to have minimal energy in $\mathcal{E}_n$. An exterior harmonic map is a stationary solution of (1.2), i.e.,

$$Q_{rr} + \frac{2}{r} Q_r = \frac{\sin(2Q)}{r^2} \tag{2.1}$$

$$Q(1) = 0, \quad \lim_{r\to\infty} Q(r) = n\pi. \tag{2.2}$$

LEMMA 2.1. *For all $\alpha \in \mathbb{R}$ there exists a unique solution $Q_\alpha \in \dot{H}^1(\mathbb{R}^3_*)$ to (2.1) with*

$$Q_\alpha(r) = n\pi - \alpha r^{-2} + O(r^{-6}).$$

The $O(\cdot)$ is determined by α, and vanishes for $\alpha = 0$. Moreover, there exists a unique α such that $Q_\alpha(1) = 0$, which we denote by α_0. One has $\alpha_0 > 0$.

The proof of Lemma 2.1 is standard so we just sketch an outline below. In order to study solutions to (2.1) it is convenient to introduce new variables $s = \log(r)$ and $\phi(s) = Q(r)$. With this change of variables we obtain an autonomous differential equation for ϕ, viz.,

$$\ddot{\phi} + \dot{\phi} = \sin(2\phi) \tag{2.3}$$

which is the equation for a damped pendulum. We can thus reduce matters to the phase portrait associated to (2.3). Setting $x(s) = \phi(s)$, $y(s) = \dot{\phi}(s)$ we rewrite (2.3) as the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -y + \sin(2x) \end{pmatrix} =: X(x, y) \tag{2.4}$$

and we denote by Φ_s the flow associated to X. The equilibria of (2.4) occur at points $v_{k/2} = (\frac{k\pi}{2}, 0)$ where $k \in \mathbb{Z}$. For each $\frac{k}{2} = n \in \mathbb{Z}$ the flow has a saddle with eigenvalues $\lambda_+ = 1$, $\lambda_- = -2$, and the corresponding unstable and stable invariant subspaces for the linearized flow are given by the spans of $(1, \lambda_+) = (1, 1)$, respectively $(1, \lambda_-) = (1, -2)$. In a neighborhood $V \ni v_n = (n\pi, 0)$ one can define the 1-dimensional invariant unstable manifold

$$W^u_n = \{(x, y) \in V \mid \Phi_s(x, y) \to v_n \text{ as } s \to -\infty\}$$

and the 1-dimensional invariant stable manifold

$$W^{st}_n = \{(x, y) \in V \mid \Phi_s(x, y) \to v_n \text{ as } s \to \infty\}$$

which are tangent at v_n to the invariant subspaces of the linearized flow. In particular, for each n one can parameterize the stable manifold W_n^{st} by

$$\phi_{n,\alpha}(s) = n\pi - \alpha e^{-2s} + O(e^{-6s})$$

with the parameter α determining all the coefficients of higher order. This proves the existence of the Q_α in Lemma 2.1. One can show that if the parameter α satisfies $\alpha > 0$ then $\phi_{n,\alpha}(s)$ lies on the branch of the stable manifold which stays below $n\pi$ for all $s \in \mathbb{R}$, i.e., $\phi_{n,\alpha}(s) < n\pi$ for all $s \in \mathbb{R}$. If $\alpha = 0$ then $\phi_{n,\alpha}(s) = n\pi$ for all s. Finally, if $\alpha < 0$ then $\phi_{n,\alpha}(s) > n\pi$ for all $s \in \mathbb{R}$. Different choices of α correspond to translations in s along the respective branches of the stable manifold, which is what we mean by uniqueness in the statement of Lemma 2.1.

To prove the existence and uniqueness of α_0, we note that an analysis of the phase portrait shows that any trajectory with $\alpha > 0$ must have crossed the y-axis at some finite time s_0, and once it has crossed can never do so again. Note that if the parameter α satisfies $\alpha < 0$ then the trajectory can never cross the y-axis since in this case $\phi_{n,\alpha}(s) > n\pi$ for all $s \in \mathbb{R}$.

Now, fix any $\alpha_+ > 0$ and $\alpha_- < 0$. Passing back to the original variables we have three trajectories

$$\begin{aligned} Q_{n,\alpha_\pm}(r) &= n\pi - \alpha_\pm r^{-2} + O(r^{-6}) \\ Q_{n,0}(r) &= n\pi \end{aligned} \tag{2.5}$$

where $Q_{n,\alpha_+}(r)$ is a trajectory on the branch of W_n^{st} that increases to $n\pi$ as $r \to \infty$, and $Q_{n,\alpha_-}(r)$ is a trajectory on the branch of W_n^{st} that decreases to $n\pi$ as $r \to \infty$. Since the trajectory Q_{n,α_+} satisfies $Q_{n,\alpha_+}(r_0) = 0$ for some $r_0 > 0$, we can obtain our solution $Q_n(r)$ to (2.1) which satisfies (2.2) by rescaling $Q_{n,\alpha_+}(r)$ by $\lambda_0 > 0$, i.e., we set

$$Q_n(r) = Q_n^+(r/\lambda_0) = n\pi - \lambda_0^2 \alpha_+ r^{-2} + O(r^{-6})$$

where we note that $\lambda_0 > 0$ is uniquely chosen to ensure that the boundary condition $Q_n(1) = 0$ is satisfied. Note that such rescalings amount to a translation in the s-variable above. Setting $\alpha_0 := \lambda_0^2 \alpha_+$, the unique harmonic map $(Q_n(r), 0) \in \mathcal{E}_n$ therefore satisfies

$$Q_n(r) = n\pi - \alpha_0 r^{-2} + O(r^{-6}) \tag{2.6}$$

as claimed above.

2.2 5d reduction. In the higher topological classes, $\mathcal{E}_n$ for $n \geq 1$, we linearize about $Q = Q_n$ by writing

$$\psi = Q + \varphi$$

where $Q = Q_n$ is the unique harmonic map and energy minimizer in $\mathcal{E}_n$. If $\vec{\psi} \in \mathcal{E}_n$ is a wave map, then $\vec{\varphi} \in \mathcal{H}_n$ satisfies

$$\begin{aligned}
&\varphi_{tt} - \varphi_{rr} - \frac{2}{r}\varphi_r + \frac{2\cos(2Q)}{r^2}\varphi = Z(r,\varphi)\\
&Z(r,\varphi) := \frac{\cos(2Q)(2\varphi - \sin(2\varphi)) + 2\sin(2Q)\sin^2(\varphi)}{r^2}\\
&\varphi(t,1) = 0,\ \varphi(t,\infty) = 0 \quad \forall t, \quad \vec{\varphi}(0) = (\psi_0 - Q, \psi_1).
\end{aligned} \tag{2.7}$$

The standard $5d$ reduction is given by setting $ru := \varphi$ and then $\vec{u}$ solves

$$\begin{aligned}
&u_{tt} - u_{rr} - \frac{4}{r}u_r + V(r)u = F(r,u) + G(r,u), \quad r \geq 1\\
&u(t,1) = 0 \quad \forall t, \quad \vec{u}(0) = (u_0, u_1)\\
&V(r) := \frac{2(\cos(2Q) - 1)}{r^2}\\
&F(r,u) := 2\sin(2Q)\frac{\sin^2(ru)}{r^3}\\
&G(r,u) := \cos(2Q)\frac{(2ru - \sin(2ru))}{r^3}.
\end{aligned} \tag{2.8}$$

We will consider radial initial data $(u_0, u_1) \in \mathcal{H} := \dot{H}^1_0 \times L^2(\mathbb{R}^5_*)$ where $\mathbb{R}^5_* = \mathbb{R}^5 \backslash B(0,1)$,

$$\|(u_0,u_1)\|^2_{\mathcal{H}} := \int_1^\infty ((\partial_r u_0(r))^2 + u_1^2(r))\, r^4\, dr \tag{2.9}$$

where $\dot{H}^1_0(\mathbb{R}^5_*)$ is the completion under the first norm on the right-hand side above of all smooth radial compactly supported functions on $\{x \in \mathbb{R}^5 \mid |x| > 1\}$. We remark that the potential

$$V(r) := \frac{2(\cos(2Q) - 1)}{r^2} \tag{2.10}$$

is real-valued, radial, bounded, smooth and by (2.6) satisfies

$$V(r) = O(r^{-6}) \ \text{ as } \ r \to \infty. \tag{2.11}$$

Also, by (2.6) we can deduce that

$$\begin{aligned}
&|F(r,u)| \lesssim r^{-3}\,|u|^2\\
&|G(r,u)| \lesssim |u|^3.
\end{aligned} \tag{2.12}$$

For the remainder of the paper we deal exclusively with $u(t,r)$ in $\mathbb{R}^5_*$ rather than the equivariant wave map angle $\psi(t,r)$. In fact, one can check that the Cauchy problem (1.2) with data $(\psi_0, \psi_1) \in \mathcal{E}_n$ is equivalent to (2.8). To see this let $\vec{\psi} \in \mathcal{E}_n$ and set

$$r\vec{u}(r) := (\psi_0(r) - Q_n(r), \psi_1(r)). \tag{2.13}$$

We claim that

$$\|\vec{\psi}\|_{\mathcal{H}_n} \simeq \|\vec{u}\|_{\mathcal{H}}. \tag{2.14}$$

Indeed, setting $\varphi(r) := \psi_0(r) - Q_n(r)$ we see that

$$\int_1^\infty \varphi_r^2(r) r^2\, dr \simeq \int_1^\infty u_r^2(r) r^4\, dr \tag{2.15}$$

via Hardy's inequality and the relations

$$\varphi_r = r u_r + u = r u_r + \frac{\varphi}{r}.$$

Therefore for each topological class $\mathcal{E}_n$ the map

$$\vec{\psi} \mapsto \frac{1}{r}(\psi_0(r) - Q_n(r), \psi_1(r))$$

is an isomorphism between the spaces $\mathcal{E}_n$ and $\mathcal{H}$ respectively.

In particular, we will prove the analogous formulation of Theorem 1.1 in the u-setting rather than the original one. Scattering in this context will mean that we approach a solution of (2.8) but with $V = F = G = 0$.

3 Small Data Theory and Concentration Compactness

3.1 Global existence and scattering for data with small energy. Here we give a brief review of the small data well-posedness theory for (2.8) that was developed in [LS13]. As usual the small data theory rests on Strichartz estimates for the inhomogeneous linear, radial exterior wave equation with the potential V,

$$\begin{aligned} &u_{tt} - u_{rr} - \frac{4}{r} u_r + V(r) u = h \\ &u(t,1) = 0 \quad \forall t \\ &\vec{u}(0) = (u_0, u_1) \in \mathcal{H} \end{aligned} \tag{3.1}$$

where $V(r)$ is as in (2.10). We define $S_V(t)$ to be the exterior linear propagator associated to (3.1). The conserved energy associated to (3.1) with $h = 0$ is given by

$$\mathcal{E}_L(u, u_t) = \frac{1}{2} \int_1^\infty (u_t^2 + u_r^2 + V(r) u^2)\, r^4\, dr.$$

This energy has an important positive definiteness property: one has

$$\mathcal{E}_L(u, u_t) = \frac{1}{2}(\|u_t\|_2^2 + \langle Hu | u \rangle), \quad H = -\Delta + V. \tag{3.2}$$

It is shown in [BCM12, LS13] that H is a nonnegative self-adjoint operator in $L^2(\mathbb{R}^5_*)$ (with a Dirichlet condition at $r = 1$), and moreover, that the threshold energy zero is regular; in other words, if $Hf = 0$ where $f \in H^2 \cap \dot{H}^1_0$ then $f = 0$. It is now standard to conclude from this spectral information that for some constants $0 < c < C$,

$$c\|f\|^2_{\dot{H}^1_0} \le \langle Hf | f \rangle \le C\|f\|^2_{\dot{H}^1_0} \quad \forall\, f \in \dot{H}^1_0(\mathbb{R}^5_*). \tag{3.3}$$

We sometimes write $\|\vec{u}\|_{\mathcal{E}}^2 := \mathcal{E}_L(\vec{u})$, which satisfies

$$\|\vec{u}\|_{\mathcal{E}} \simeq \|\vec{u}\|_{\mathcal{H}} \quad \forall \vec{u} \in \mathcal{H}. \tag{3.4}$$

In what follows we say a triple (p, q, γ) is admissible if

$$\begin{aligned} &p > 2,\ q \geq 2 \\ &\frac{1}{p} + \frac{5}{q} = \frac{5}{2} - \gamma \\ &\frac{1}{p} + \frac{2}{q} \leq 1. \end{aligned}$$

For the free exterior $5d$ wave, i.e., the case $V = 0$ in (3.1), Strichartz estimates were established in [HMSSZ10]. Although the estimates in [HMSSZ10] hold in more general exterior settings, we state only the specific example of these estimates that we need here.

PROPOSITION 3.1 ([HMSSZ10]). *Let (p, q, γ) and (r, s, ρ) be admissible triples. Then any solution $\vec{v}(t)$ to*

$$\begin{aligned} &v_{tt} - v_{rr} - \frac{4}{r} v_r = h \\ &\vec{v}(0) = (f, g) \in \mathcal{H}(\mathbb{R}^5_*) \\ &v(t, 1) = 0 \quad \forall t \in \mathbb{R} \end{aligned} \tag{3.5}$$

with radial initial data satisfies

$$\| \, |\nabla|^{-\gamma} \nabla v\|_{L_t^p L_x^q} \lesssim \|(f, g)\|_{\mathcal{H}} + \| \, |\nabla|^{\rho} h\|_{L_t^{r'} L_x^{s'}}. \tag{3.6}$$

In [LS13], the second and third authors showed that in fact the same family of Strichartz estimates hold for (3.1).

PROPOSITION 3.2 ([LS13, Proposition 5.1]). *Let (p, q, γ) and (r, s, ρ) be admissible triples. Then any solution $\vec{u}(t)$ to (3.1) with radial initial data satisfies*

$$\| \, |\nabla|^{-\gamma} \nabla u\|_{L_t^p L_x^q} \lesssim \|\vec{u}(0)\|_{\mathcal{H}} + \| \, |\nabla|^{\rho} h\|_{L_t^{r'} L_x^{s'}}. \tag{3.7}$$

With these Strichartz estimates the following small data, global well-posedness theory for (2.8) follows from the standard contraction argument.

PROPOSITION 3.3 ([LS13, Theorem 1.2]). *The exterior Cauchy problem for (2.8) is globally well-posed in $\mathcal{H} := \dot{H}_0^1 \times L^2(\mathbb{R}^5_*)$. Moreover, a solution u scatters as $t \to \infty$ to a free wave, i.e., a solution $\vec{u}_L \in \mathcal{H}$ of*

$$\Box u_L = 0,\ r \geq 1,\ u_L(t, 1) = 0,\ \forall t \geq 0 \tag{3.8}$$

if and only if $\|u\|_{\mathcal{S}} < \infty$ where $\mathcal{S} = L_t^3([0, \infty); L_x^6(\mathbb{R}^5_*))$. In particular, there exists a constant $\delta > 0$ small so that if $\|\vec{u}(0)\|_{\mathcal{H}} < \delta$, then u scatters to free waves as $t \to \pm\infty$.

REMARK 1. We remark that in [LS13, Theorem 1.2], the conclusions of Proposition 3.3 were phrased in terms of the original wave map angle ψ where here the result is phased in terms of $u(t,r) := \frac{1}{r}(\psi(t,r) - Q_n(r))$. As we saw in Section 2 this passage to the $u-$formulation is allowed since the map $\vec{u} = \frac{1}{r}(\psi - Q_n, \psi_t)$ is an isomorphism between the energy class $\mathcal{E}_n$ and $\mathcal{H} := \dot{H}_0^{\ 1} \times L^2(\mathbb{R}^5_*)$, respectively.

We refer the reader to [LS13] for the details regarding Proposition 3.2 and Proposition 3.3. For convenience, we recall how the scattering norm $L_t^3 L_x^6$ is obtained. By Proposition 3.2, solutions to (3.1) satisfy

$$\|u\|_{L_t^3(\mathbb{R};\dot{W}_x^{\frac{1}{2},3}(\mathbb{R}^5_*))} \lesssim \|\vec{u}(0)\|_{\mathcal{H}} + \|h\|_{L_t^1 L_x^2 + L_t^{\frac{3}{2}} L_x^{\frac{30}{17}}}. \tag{3.9}$$

As in [LS13], we claim the embedding $\dot{W}_x^{\frac{1}{2},3} \hookrightarrow L_x^6$ for radial functions in $r \geq 1$ in $\mathbb{R}^5_*$. Indeed, one checks via the fundamental theorem of calculus that $\dot{W}_x^{1,3} \hookrightarrow L_x^\infty$. More precisely,

$$|f(r)| \leq r^{-\frac{2}{3}} \|f\|_{\dot{W}_x^{1,3}}. \tag{3.10}$$

Interpolating this with the embedding $L^3 \hookrightarrow L^3$ we obtain the claim. From (3.9) we infer the weaker Strichartz estimate

$$\|u\|_{L_t^3(\mathbb{R};L_x^6(\mathbb{R}^5_*))} \lesssim \|\vec{u}(0)\|_{\mathcal{H}} + \|h\|_{L_t^1(\mathbb{R};L_x^2(\mathbb{R}^5_*)) + L_t^{\frac{3}{2}}(\mathbb{R};L_x^{\frac{30}{17}}(\mathbb{R}^5_*))} \tag{3.11}$$

which suffices for our purposes. Indeed, using (3.11) on the nonlinear equation (2.8) gives

$$\begin{aligned}
\|u\|_{L_t^3(\mathbb{R};L_x^6(\mathbb{R}^5_*))} &\lesssim \|\vec{u}(0)\|_{\mathcal{H}} + \|F(r,u) + G(r,u)\|_{L_t^1 L_x^2 + L_t^{\frac{3}{2}} L_x^{\frac{30}{17}}} \\
&\lesssim \|\vec{u}(0)\|_{\mathcal{H}} + \|r^{-3}u^2\|_{L_t^{\frac{3}{2}} L_x^{\frac{30}{17}}} + \|u^3\|_{L_t^1 L_x^2} \\
&\lesssim \|\vec{u}(0)\|_{\mathcal{H}} + \|r^{-3}\|_{L_t^\infty L_x^{\frac{30}{7}}} \|u^2\|_{L_t^{\frac{3}{2}} L_x^3} + \|u\|_{L_t^3 L_x^6}^3 \\
&\lesssim \|\vec{u}(0)\|_{\mathcal{H}} + \|u\|_{L_t^3 L_x^6}^2 + \|u\|_{L_t^3 L_x^6}^3
\end{aligned}$$

where we have estimated the size of the nonlinearity $h = F(r,u) + G(r,u)$ using (2.12). Thus for small initial data, $\|\vec{u}(0)\|_{\mathcal{H}} < \delta$, we obtain the global a priori estimate

$$\|u\|_{L_t^3(\mathbb{R};L_x^6(\mathbb{R}^5_*))} \lesssim \|\vec{u}(0)\|_{\mathcal{H}} \lesssim \delta \tag{3.12}$$

from which the small data scattering statement in Proposition 3.3 follows.

3.2 Concentration compactness. We now formulate the concentration compactness principle relative to the linear wave equation with a potential, see (3.1) with $h = 0$. This is what we mean by "free" in Lemma 3.4. Note that this is a different meaning of "free" than the one used in Proposition 3.3. However, observe that any solution to (3.1) with $h = 0$, which is in $L_t^3 L_x^6$ must scatter to "free" waves, where "free" is in the sense of Proposition 3.3.

LEMMA 3.4. *Let $\{u_n\}$ be a sequence of free radial waves bounded in $\mathcal{H} = \dot{H}^1_0 \times L^2(\mathbb{R}^5_*)$. Then after replacing it by a subsequence, there exist a sequence of free solutions v^j bounded in $\mathcal{H}$, and sequences of times $t^j_n \in \mathbb{R}$ such that for γ^k_n defined by*

$$u_n(t) = \sum_{1\le j<k} v^j(t + t^j_n) + \gamma^k_n(t) \tag{3.13}$$

we have for any $j < k$,

$$\vec{\gamma}^k_n(-t^j_n) \rightharpoonup 0 \tag{3.14}$$

weakly in $\mathcal{H}$ as $n \to \infty$, as well as

$$\lim_{n\to\infty} |t^j_n - t^k_n| = \infty \tag{3.15}$$

and the errors γ^k_n vanish asymptotically in the sense that

$$\lim_{k\to\infty} \limsup_{n\to\infty} \|\gamma^k_n\|_{(L^\infty_t L^p_x \cap L^3_t L^6_x)(\mathbb{R}\times\mathbb{R}^5_*)} = 0 \quad \forall\ \tfrac{10}{3} < p < \infty. \tag{3.16}$$

Finally, one has orthogonality of the free energy with a potential, cf. (3.4),

$$\|\vec{u}_n\|^2_{\mathcal{E}} = \sum_{1\le j<k} \|\vec{v}^j\|^2_{\mathcal{E}} + \|\vec{\gamma}^k_n\|^2_{\mathcal{E}} + o(1) \tag{3.17}$$

as $n \to \infty$.

The proof is essentially identical with that of Lemma 3.2 in [LS13]. In fact, instead of the Strichartz estimates for $\Box$ in $\mathbb{R}^5_*$ we use those from Proposition 3.2 above.

Applying this decomposition to the nonlinear equation requires a perturbation lemma which we now formulate. All spatial norms are understood to be on $\mathbb{R}^5_*$. The exterior propagator $S_V(t)$ is as above.

LEMMA 3.5. *There are continuous functions $\varepsilon_0, C_0 : (0,\infty) \to (0,\infty)$ such that the following holds: Let $I \subset \mathbb{R}$ be an open interval (possibly unbounded), $u, v \in C(I; \dot{H}^1_0) \cap C^1(I; L^2)$ radial functions satisfying for some $A > 0$*

$$\|\vec{u}\|_{L^\infty(I;\mathcal{H})} + \|\vec{v}\|_{L^\infty(I;\mathcal{H})} + \|v\|_{L^3_t(I;L^6_x)} \le A$$
$$\|\mathrm{eq}(u)\|_{L^1_t(I;L^2_x)} + \|\mathrm{eq}(v)\|_{L^1_t(I;L^2_x)} + \|w_0\|_{L^3_t(I;L^6_x)} \le \varepsilon \le \varepsilon_0(A),$$

where $\mathrm{eq}(u) := (\Box + V)u - F(r,u) - G(r,u)$ in the sense of distributions, and $\vec{w}_0(t) := S_V(t - t_0)(\vec{u} - \vec{v})(t_0)$ with $t_0 \in I$ arbitrary but fixed. Then

$$\|\vec{u} - \vec{v} - \vec{w}_0\|_{L^\infty_t(I;\mathcal{H})} + \|u - v\|_{L^3_t(I;L^6_x)} \le C_0(A)\varepsilon.$$

In particular, $\|u\|_{L^3_t(I;L^6_x)} < \infty$.

The proof of this lemma is essentially identical with that of [LS13, Lemma 3.3]. The only difference is that we use the propagator S_V instead of S_0.

3.3 Critical element. We now turn to the proof of Theorem 1.1 following the concentration compactness methodology from [KM06, KM08]. We begin by noting that Theorem 1.1 was proved in the regime of all energies slightly above the ground state energy $\mathcal{E}(Q_n, 0)$ in [LS13, Theorem 1.2], see also Proposition 3.3 above. As usual, we assume that Theorem 1.1 fails and construct a *critical element* which is a non-scattering solution of minimal energy, E_*, which is necessarily strictly bigger than $\mathcal{E}(Q_n, 0)$. This is done in the following proposition on the level of the semi-linear formulation given by (2.8).

PROPOSITION 3.6. *Suppose that Theorem 1.1 fails. Then there exists a nonzero energy solution to (2.8) (referred to as a critical element) $\vec{u}_*(t)$ for $t \in \mathbb{R}$ with the property that the trajectory*

$$\mathcal{K} := \{\vec{u}_*(t) \mid t \in \mathbb{R}\} \tag{3.18}$$

is pre-compact in $\mathcal{H}(\mathbb{R}^5_*)$.

Proof. Suppose that the theorem fails. Then there exists a bounded sequence of $\vec{\psi}_j = (\psi_{0,j}, \psi_{1,j}) \in \mathcal{E}_n$ with

$$\mathcal{E}(\vec{\psi}_j) \to E_* > 0 \tag{3.19}$$

and a bounded sequence $\vec{u}_j := (u_{0,j}, u_{1,j}) \in \mathcal{H}$ where $\vec{u}_j(r) = \frac{1}{r}(\vec{\psi}_j(r) - (Q(r), 0))$ with

$$\|u_j\|_{\mathcal{S}} \to \infty$$

where u_n denotes the global evolution of $\vec{u}_n$ of (2.8). We may assume that E_* is minimal with this property. Applying Lemma 3.4 to the free evolutions S_V of $\vec{u}_j(0)$ yields free waves v^i and times t^i_j as in (3.13). Let U^i be the nonlinear profiles of (v^i, t^i_j), i.e., those energy solutions of (2.8) which satisfy

$$\lim_{t \to t^i_\infty} \|\vec{v}^i(t) - \vec{U}^i(t)\|_{\mathcal{H}} \to 0$$

where $\lim_{j\to\infty} t^i_j = t^i_\infty \in [-\infty, \infty]$. The U^i exist locally around $t = t^i_\infty$ by the local existence and scattering theory, see Proposition 3.3. Note that here and throughout we are using the equivalence of norms in (3.4). Locally around $t = 0$ one has the following *nonlinear profile decomposition*

$$u_j(t) = \sum_{i<k} U^i(t + t^i_j) + \gamma^k_j(t) + \eta^k_j(t) \tag{3.20}$$

where $\|\vec{\eta}^k_j(0)\|_{\mathcal{H}} \to 0$ as $j \to \infty$. Now suppose that either there are two non-vanishing v^j, say v^1, v^2, or that

$$\limsup_{k\to\infty} \limsup_{j\to\infty} \|\vec{\gamma}^k_j\|_{\mathcal{E}} > 0. \tag{3.21}$$

Note that the left-hand side does not depend on time since γ_j^k is a free wave. By the minimality of E_* and the orthogonality of the nonlinear energy–which follows from (3.15) and (3.14)–each U^i is a global solution and scatters with $\|U^i\|_{L_t^3 L_x^6} < \infty$.

We now apply Lemma 3.5 on $I = \mathbb{R}$ with $u = u_j$ and

$$v(t) = \sum_{i<k} U^i(t + t_j^i). \tag{3.22}$$

That $\|\mathrm{eq}(v)\|_{L_t^1 L_x^2}$ is small for large n follows from (3.15). To see this, note that with $N(v) := F(r, v) + G(r, v)$,

$$\begin{aligned} \mathrm{eq}(v) &= (\Box + V)v - F(r, v) - G(r, v) \\ &= \sum_{i<k} N(U^i(t + t_j^i)) - N\left(\sum_{i<k} U^i(t + t_j^i)\right). \end{aligned}$$

The difference on the right-hand side here only consists of terms which involve at least one pair of distinct i, i'. But then $\|\mathrm{eq}(v)\|_{L_t^1 L_x^2} \to 0$ as $j \to \infty$ by (3.15). In order to apply Lemma 3.5 it is essential that

$$\limsup_{j\to\infty} \left\| \sum_{i<k} U^i(t + t_j^i) \right\|_{L_t^3 L_x^6} \leq A < \infty \tag{3.23}$$

uniformly in k, which follows from (3.15), (3.17), and Proposition 3.3. The point here is that the sum can be split into one over $1 \leq i < i_0$ and another over $i_0 \leq i < k$. This splitting is performed in terms of the energy, with i_0 being chosen such that for all $k > i_0$

$$\limsup_{j\to\infty} \sum_{i_0 \leq i < k} \|\vec{U}^i(t_j^i)\|_{\mathcal{H}}^2 \leq \varepsilon_0^2 \tag{3.24}$$

where ε_0 is fixed such that the small data result of Proposition 3.3 applies. Clearly, (3.24) follows from (3.17). Using (3.15) as well as the small data scattering theory one now obtains

$$\begin{aligned} \limsup_{j\to\infty} \left\| \sum_{i_0 \leq i < k} U^i(\cdot + t_j^i) \right\|_{L_t^3 L_x^6}^3 &= \sum_{i_0 \leq i < k} \|U^i(\cdot)\|_{L_t^3 L_x^6}^3 \\ &\leq C \limsup_{j\to\infty} \left(\sum_{i_0 \leq i < k} \|\vec{U}^i(t_j^i)\|_{\mathcal{H}}^2 \right)^{\frac{3}{2}} \end{aligned} \tag{3.25}$$

with an absolute constant C. This implies (3.23), uniformly in k.

Hence one can take k and j so large that Lemma 3.5 applies to (3.20) whence

$$\limsup_{j\to\infty} \|u_j\|_{L_t^3 L_x^6} < \infty$$

which is a contradiction. Thus, there can be only one nonvanishing v^i, say v^1, and moreover

$$\limsup_{j\to\infty} \|\vec{\gamma}_j^2\|_{\mathcal{H}} = 0. \tag{3.26}$$

Thus, if we let $\vec{\psi}^1$ be the wave map angle associated to $\vec{U}^1$ then we have $\mathcal{E}(\vec{\psi}^1) = E_*$. By the preceding, necessarily

$$\|U^1\|_{L_t^3 L_x^6} = \infty. \tag{3.27}$$

Therefore, $U^1 =: u_*$ is the desired critical element. Suppose that

$$\|u_*\|_{L_t^3([0,\infty);L_x^6)} = \infty. \tag{3.28}$$

Then we claim that

$$\mathcal{K}_+ := \{\vec{u}_*(t) \mid t \geq 0\}$$

is precompact in $\mathcal{H}$. If not, then there exists $\delta > 0$ so that for some infinite sequence $t_n \to \infty$ one has

$$\|\vec{u}_*(t_n) - \vec{u}_*(t_m)\|_{\mathcal{H}} > \delta \quad \forall\, n > m. \tag{3.29}$$

Applying Lemma 3.4 to $U^1(t_n)$ one concludes via the same argument as before based on the minimality of E_* and (3.27) that

$$\vec{u}_*(t_n) = \vec{v}(\tau_n) + \vec{\gamma}_n(0) \tag{3.30}$$

where $\vec{v}$, $\vec{\gamma}_n$ are free waves in $\mathcal{H}$, and τ_n is some sequence in $\mathbb{R}$. Moreover, $\|\vec{\gamma}_n\|_{\mathcal{H}} \to 0$ as $n \to \infty$. If $\tau_n \to \tau_\infty \in \mathbb{R}$, then (3.30) and (3.29) lead to a contradiction. If $\tau_n \to \infty$, then

$$\|v(\cdot + \tau_n)\|_{L_t^3([0,\infty);L_x^6)} \to 0 \qquad \text{as } n \to \infty$$

implies via the local wellposedness theory that $\|u_*(\cdot + t_n)\|_{L_t^3([0,\infty);L_x^6)} < \infty$ for all large n, which is a contradiction to (3.28). If $\tau_n \to -\infty$, then

$$\|v(\cdot + \tau_n)\|_{L_t^3((-\infty,0];L_x^6)} \to 0 \qquad \text{as } n \to \infty$$

implies that $\|u_*(\cdot + t_n)\|_{L_t^3((-\infty,0];L_x^6)} < C < \infty$ for all large n where C is some fixed constant. Passing to the limit yields a contradiction to (3.27) and (3.29) is seen to be false, concluding the proof of compactness of $\mathcal{K}_+$.

Finally, we need to make sure that $u_*(t)$ is precompact with respect to both $t \to +\infty$ and $t \to -\infty$, see (3.18). To achieve the latter, we extract another critical element from the sequence

$$\{\vec{u}_*(n)\}_{n=1}^{\infty} \subset \mathcal{H}.$$

Indeed, by the compactness that we have already established we can pass to a strong limit $\vec{u}_n \to \vec{u}_\infty$ in $\mathcal{H}$, which has the same energy E_*. By construction, the nonlinear evolution (2.8) with data $\vec{u}_\infty$ has infinite $L_t^3 L_x^6$-norm in both time directions. Therefore, the same compactness argument as above concludes the proof. Indeed, the solution given by $\vec{u}_\infty$ is now our desired critical element. □

In Section 5 we will show that u_* cannot exist. In order to do so, we need to develop another tool for the linear evolution.

4 The Linear External Energy Estimates in $\mathbb{R}^5$

We now turn to our main new ingredient from the linear theory, which is Proposition 4.1. In order to motivate this result, we first review the analogous statements in dimensions $d = 1$ and $d = 3$.

Suppose $w_{tt} - w_{xx} = 0$ with smooth energy data $(w(0), \dot{w}(0)) = (f, g)$. Then by local energy conservation

$$\int\limits_{x>a} \frac{1}{2}(w_t^2 + w_x^2)(0,x)\,dx - \int\limits_{x>T+a} \frac{1}{2}(w_t^2+w_x^2)(T,x)\,dx = \frac{1}{2}\int\limits_0^T (w_t + w_x)^2(t, t+a)\,dt$$

for any $T > 0$ and $a \in \mathbb{R}$. Since $(\partial_t - \partial_x)(w_t + w_x) = 0$, we have that

$$\begin{aligned}\frac{1}{2}\int\limits_0^T (w_t + w_x)^2(t, t+a)\,dt &= \frac{1}{2}\int\limits_0^T (w_t + w_x)^2(0, a+2t)\,dt \\ &= \frac{1}{4}\int\limits_a^{a+2T} (w_t + w_x)^2(0,x)\,dx = \frac{1}{4}\int\limits_a^{a+2T} (f_x + g)^2(x)\,dx.\end{aligned}$$

Consequently,

$$\begin{aligned}&\int\limits_{x>a} \frac{1}{2}(w_t^2 + w_x^2)(0,x)\,dx - \lim_{T\to\infty} \int\limits_{x>T+a} \frac{1}{2}(w_t^2 + w_x^2)(T,x)\,dx \\ &\quad = \frac{1}{4}\int\limits_a^\infty (f_x + g)^2(x)\,dx\end{aligned}$$

and thus

$$\begin{aligned}&\min_{\pm}\left[\int\limits_{x>a} \frac{1}{2}(f_x^2 + g^2)(0,x)\,dx - \lim_{T\to\pm\infty} \int\limits_{x>|T|+a} \frac{1}{2}(w_t^2 + w_x^2)(T,x)\,dx\right] \\ &\quad \le \frac{1}{4}\int\limits_a^\infty (f_x^2 + g^2)(x)\,dx\end{aligned}$$

whence

$$\max_{\pm} \lim_{T\to\pm\infty} \int\limits_{x>|T|+a} \frac{1}{2}(w_t^2 + w_x^2)(T,x)\,dx \ge \frac{1}{4}\int\limits_a^\infty (f_x^2 + g^2)(x)\,dx. \tag{4.1}$$

Here we used that $t \mapsto -t$ leaves f unchanged, but turns g into $-g$.

Given $\Box u = 0$ radial in three dimensions, $w(t,r) = ru(t,r)$ solves $w_{tt} - w_{rr} = 0$. Consequently, (4.1) gives the following estimate from [DKM11, Lemma 4.2], see also [DKM12, DKM13, DKM12]: for any $a \geq 0$ one has

$$\max_{\pm} \lim_{T\to\pm\infty} \int\limits_{r>|T|+a} \frac{1}{2}((ru)_r^2 + (ru_t)^2)(T,r)\,dr \geq \frac{1}{4}\int\limits_{r>a} ((rf)_r^2 + (rg)^2)(r)\,dr \tag{4.2}$$

where $u(0) = f$, $\dot{u}(0) = g$. The left-hand side of (4.2) equals

$$\max_{\pm} \lim_{T\to\pm\infty} \int\limits_{r>|T|+a} \frac{1}{2}(u_r^2 + u_t^2)(T,r)\,r^2 dr \tag{4.3}$$

by the standard dispersive properties of the wave equation. The right-hand side, on the other hand, exhibits the following dichotomy: if $a = 0$, then it equals half of the full energy

$$\frac{1}{4}\int_0^\infty (f_r^2 + g^2)(r)\,r^2 dr.$$

However, if $a > 0$, then integration by parts shows that it equals (ignoring the constant from the spherical measure in $\mathbb{R}^3$)

$$\frac{1}{4}\int\limits_{r>a} (f_r^2 + g^2)(r) r^2\,dr - \frac{1}{4}af^2(a) = \frac{1}{4}\|\pi_a^\perp(f,g)\|^2_{\dot{H}^1\times L^2(r>a)}$$

where $\pi_a^\perp = \mathrm{Id} - \pi_a$ and π_a is the orthogonal projection onto the line

$$\{(cr^{-1}, 0) \mid c \in \mathbb{R}\} \subset \dot{H}^1 \times L^2(r > a).$$

The appearance of this projection is natural, in view of the fact that the Newton potential r^{-1} in $\mathbb{R}^3$ yields an explicit solution to $\Box u = 0$, $u(0,r) = r^{-1}, \dot{u}(0,r) = 0$: indeed, one has $u(r,t) = r^{-1}$ in $r > |t| + a$ for which (4.3) vanishes. Since $r^{-1} \notin L^2(r > 1)$ no projection appears in the time component. In contrast, the Newton potential in $\mathbb{R}^5$, viz. r^{-3}, does lie in $H^1(r > a)$ for any $a > 0$. This explains why in $\mathbb{R}^5$ we need to project away from a plane rather than a line, see (4.4) and the end of the proof of Proposition 4.1.

Proposition 4.1. *Let $\Box u = 0$ in $\mathbb{R}^{1+5}_{t,x}$ with radial data $(f,g) \in \dot{H}^1 \times L^2(\mathbb{R}^5)$. Then with some absolute constant $c > 0$ one has for every $a > 0$*

$$\max_{\pm} \limsup_{t\to\pm\infty} \int\limits_{r>a+|t|}^{\infty} (u_t^2 + u_r^2)(t,r) r^4\,dr \geq c\|\pi_a^\perp(f,g)\|^2_{\dot{H}^1\times L^2(r>a)} \tag{4.4}$$

where $\pi_a = \mathrm{Id} - \pi_a^{\perp}$ *is the orthogonal projection onto the plane*

$$\{(c_1 r^{-3}, c_2 r^{-3}) \mid c_1, c_2 \in \mathbb{R}\}$$

in the space $\dot{H}^1 \times L^2(r > a)$. *The left-hand side of* (4.4) *vanishes for all data in this plane.*

Remark 2. We note that by finite propagation speed Proposition 4.1 with $a > 1$ holds as well for solutions $v(t)$ to the free radial wave equation in $\mathbb{R} \times \mathbb{R}^5_*$ with a Dirichlet boundary condition at $r = 1$.

$$\begin{aligned} &v_{tt} - v_{rr} - \frac{4}{r} v_r = 0 \\ &\vec{v}(0) = (f, g) \\ &v(t, 1) = 0 \quad \forall t \in \mathbb{R}. \end{aligned} \tag{4.5}$$

Proof. By the basic energy estimate we may assume that f, g are compactly supported and smooth, say. We first note that it suffices to deal with data $(f, 0)$ and $(0, g)$ separately. Indeed, reversing the time direction keeps the former fixed, whereas the latter changes to $(0, -g)$. This implies that we may choose the time-direction so as to render the bilinear interaction term between the two respective solutions nonnegative on the left-hand side of (4.4).

We begin with data $(f, 0)$ and set $w(t, r) := r^{-1}(r^3 u(t, r))_r$, see [KM11]. Throughout this proof, the singularity at $r = 0$ plays no role due to the fact that $r \geq a + |t| \geq a > 0$. Then

$$w_{tt} - w_{rr} = r^2 \partial_r \big(u_{tt} - u_{rr} - \tfrac{4}{r} u_r\big) + 3r\big(u_{tt} - u_{rr} - \tfrac{4}{r} u_r\big) = 0.$$

From the d'Alembert formula,

$$\limsup_{t \to \infty} \int_{a+t}^{\infty} w^2(t, r)\, dr \geq \frac{1}{4} \int_a^{\infty} w^2(0, r)\, dr$$

which is the same as

$$\limsup_{t \to \infty} \int_{a+t}^{\infty} (r^2 u_r(t, r) + 3r u(t, r))^2\, dr \geq \frac{1}{4} \int_a^{\infty} (r^2 f'(r) + 3r f(r))^2\, dr. \tag{4.6}$$

By our assumption on the data, we have the point wise bound

$$|u(t, r)| \leq C t^{-2} \chi_{[R - t \leq r \leq R + t]}$$

for $t \geq 1$ and some large R. Hence, (4.6) equals

$$\limsup_{t \to \infty} \int_{a+t}^{\infty} u_r^2(t, r) r^4\, dr \geq \frac{1}{4} \left(\int_a^{\infty} r^4 f'(r)^2\, dr - 3a^3 f(a)^2 \right) \tag{4.7}$$

where we integrated by parts on the right-hand side. Finally, one checks that

$$\tilde{f}(r) := f(r) - \frac{a^3}{r^3} f(a)$$

is the orthogonal projection perpendicular to r^{-3} in $\dot{H}^1(r > a)$ in $\mathbb{R}^5$ and that it satisfies

$$\int_a^\infty r^4 \tilde{f}'(r)^2 \, dr = \int_a^\infty r^4 f'(r)^2 \, dr - 3a^3 f(a)^2$$

which agrees with the right-hand side of (4.7) and concludes the proof of (4.4) for data $(f, 0)$.

For data $(0, g)$ we use the new dependent variable

$$v(t, r) := \int_r^\infty s \partial_t u(t, s) \, ds. \tag{4.8}$$

By direct differentiation and integration by parts one verifies that v solves the 3-dimensional radial wave equation, viz.

$$v_{tt} - v_{rr} - \frac{2}{r} v_r = 0.$$

Moreover, $v_t(0, r) = 0$. From the exterior energy estimate in dim = 3, i.e., (4.2),

$$\limsup_{t \to \infty} \int_{a+t}^\infty ((rv)_t^2 + (rv)_r^2)(t, r) \, dr \geq \frac{1}{2} \int_a^\infty ((rv)_t^2 + (rv)_r^2)(0, r) \, dr \tag{4.9}$$

where we have used the fact that for data $(v_0, 0)$ or $(0, v_1)$ the estimate (4.2) holds in both time directions. By our assumption on the data and stationary phase

$$|v(t, r)| \leq C t^{-1} \chi_{[r \leq R+t]}, \quad |v_t(t, r)| \leq C t^{-2} \chi_{[r \leq R+t]}.$$

Hence (4.9) reduces to

$$\limsup_{t \to \infty} \int_{a+t}^\infty v_r^2(t, r) r^2 \, dr \geq \frac{1}{2} \int_a^\infty (r h'(r) + h(r))^2 \, dr \tag{4.10}$$

where $h(r) := \int_r^\infty s g(s) \, ds$. Inserting (4.8) on the left-hand side and integrating by parts on the right-hand side yields

$$\begin{aligned} \limsup_{t \to \infty} \int_{a+t}^\infty 2 u_t^2(t, r) r^4 \, dr \; &\geq \int_a^\infty h'(r)^2 r^2 \, dr - a h^2(a) \\ &= \int_a^\infty g(r)^2 r^4 \, dr - a \left(\int_a^\infty \rho g(\rho) \, d\rho \right)^2. \end{aligned} \tag{4.11}$$

Finally, the right-hand side here is $\|\tilde{g}\|^2_{L^2(r>a)}$ where

$$\tilde{g}(r) := g(r) - ar^{-3} \int_a^\infty sg(s)\, ds$$

is the orthogonal projection perpendicular to r^{-3} in $L^2(r > a)$ in $\mathbb{R}^5$.

For data $(r^{-3}, 0)$ the solution equals r^{-3} on $r > t + a \geq a > 0$ since r^{-3} is the Newton potential in $\mathbb{R}^5$. Similarly, data $(0, r^{-3})$ produce the solution tr^{-3} on the same region. In both cases, the left-hand side of (4.4) vanishes.

5 Rigidity Argument

In this section we will complete the proof of Thereom 1.1 by showing that a critical element as constructed in Section 3 does not exist. In particular, we prove the following proposition:

PROPOSITION 5.1 (Rigidity Property). *Let $\vec{u}(t) \in \mathcal{H} := \dot{H}^1_0 \times L^2(\mathbb{R}^5_*)$ be a global solution to (2.8) and suppose that the trajectory*

$$K := \{\vec{u}(t) \mid t \in \mathbb{R}\}$$

is pre-compact in $\mathcal{H}$. Then $\vec{u}(t) \equiv (0, 0)$.

First note that the pre-compactness of K immediately implies that the energy of $\vec{u}(t)$ on the exterior cone $\{r \geq R + |t|\}$ vanishes as $|t| \to \infty$.

COROLLARY 5.2. *Let $\vec{u}(t)$ be as in Proposition 5.1. Then for any $R \geq 1$ we have*

$$\|\vec{u}(t)\|_{\mathcal{H}(r\geq R+|t|)} \to 0 \quad \text{as} \quad |t| \to \infty. \tag{5.1}$$

The proof of Proposition 5.1 will proceed in several steps. The rough outline is to first use Corollary 5.2 together with Proposition 4.1 to determine the precise asymptotic behavior of $u_0(r) = u(0, r)$ and $u_1(r) = u_t(0, r)$ as $r \to \infty$. Namely, we show that

$$\begin{aligned} r^3 u_0(r) &= \ell_o + O(r^{-3}) \quad \text{as} \quad r \to \infty \\ r \int_r^\infty u_1(\rho)\rho\, d\rho &= O(r^{-1}) \quad \text{as} \quad r \to \infty. \end{aligned} \tag{5.2}$$

We will then argue by contradiction to show that $\vec{u}(t, r) = (0, 0)$ is the only possible solution that has both a pre-compact trajectory and initial data satisfying (5.2).

5.1 Step 1. We use the exterior estimates for the free radial wave equation in Proposition 4.1 together with Corollary 5.2 to deduce the following inequality for the pre-compact trajectory $\vec{u}(t)$.

LEMMA 5.3. *There exists $R_0 > 1$ such that for every $R \geq R_0$ and for all $t \in \mathbb{R}$ we have*

$$\begin{aligned}\|\pi_R^{\perp}\,\vec{u}(t)\|^2_{\mathcal{H}(r\geq R)} \lesssim\ & R^{-22/3}\|\pi_R\,\vec{u}(t)\|^2_{\mathcal{H}(r\geq R)} \\ & + R^{-11/3}\|\pi_R\,\vec{u}(t)\|^4_{\mathcal{H}(r\geq R)} + \|\pi_R\,\vec{u}(t)\|^6_{\mathcal{H}(r\geq R)} \end{aligned} \tag{5.3}$$

where again $P(R) := \{(k_1 r^{-3}, k_2 r^{-3}) \mid k_1, k_2 \in \mathbb{R},\ r > R\}$, π_R denotes the orthogonal projection onto $P(R)$ and ${\pi_R}^{\perp}$ denotes the orthogonal projection onto the orthogonal complement of the plane $P(R)$ in $\mathcal{H}(r > R; \mathbb{R}^5_*)$. We note that (5.3) holds with a constant that is uniform in $t \in \mathbb{R}$.

In order to prove Lemma 5.3 we need a preliminary result concerning the nonlinear evolution for a modified Cauchy problem which is adapted to capture the behavior of our solution $\vec{u}(t)$ only on the exterior cone $\{(t, r) \mid r \geq R + |t|\}$. Since we will only consider the evolution – and in particular the vanishing property (5.1) – on the exterior cone we can, by finite propagation speed, alter the nonlinearity and the potential term in (2.8) on the interior cone $\{1 \leq r \leq R + |t|\}$ without affecting the flow on the exterior cone. In particular, we can make the potential and the nonlinearity small on the interior of the cone so that for small initial data we can treat the potential and nonlinearity as small perturbations.

With this in mind, for every $R > 1$ we define $Q_R(t, r)$ by setting

$$Q_R(t,r) := \begin{cases} Q(R+|t|) & \text{for } \ 1 \leq r \leq R+|t| \\ Q(r) & \text{for } \ r \geq R+|t|\,. \end{cases} \tag{5.4}$$

Next, set

$$V_R(t,r) := \begin{cases} 2(R+|t|)^{-2}(\cos(2Q_R(t,r)) - 1) & \text{for } \ 1 \leq r \leq R+|t| \\ 2r^{-2}(\cos(2Q(r)) - 1) & \text{for } \ r \geq R+|t| \end{cases}$$

$$F_R(t,r,h) := \begin{cases} 2(R+|t|)^{-3}\sin(2Q_R(t,r))\sin^2((R+|t|)h) & \text{for } \ 1 \leq r \leq R+|t| \\ 2r^{-3}\sin(2Q(r))\sin^2(rh) & \text{for } \ r \geq R+|t| \end{cases}$$

$$G(r,h) := r^{-3}\cos(2Q(r))(2rh - \sin(2rh)) \quad \forall\, r \geq 1.$$

Note that for R large enough we have, using (2.6) and (2.11) that

$$|V_R(t,r)| \lesssim \begin{cases} (R+|t|)^{-6} & \text{for } \ 1 \leq r \leq R+|t| \\ r^{-6} & \text{for } \ r \geq R+|t| \end{cases} \tag{5.5}$$

$$|F_R(t,r,h)| \lesssim \begin{cases} (R+|t|)^{-3}\,|h(t,r)|^2 & \text{for } \ 1 \leq r \leq R+|t| \\ r^{-3}\,|h(t,r)|^2 & \text{for } \ r \geq R+|t| \end{cases} \tag{5.6}$$

$$|G(r,h)| \lesssim |h(t,r)|^3 \quad \text{for } \ r \geq 1, \quad \forall t \in \mathbb{R}. \tag{5.7}$$

We consider the modified Cauchy problem in $\mathbb{R} \times \mathbb{R}^5_*$:

$$\begin{aligned} &h_{tt} - h_{rr} - \frac{4}{r} h_r = \mathcal{N}_R(t, r, h) \\ &\mathcal{N}_R(t, r, h) := -V_R(t, r)h + F_R(t, r, h) + G(r, h) \\ &h(1, t) = 0 \quad \forall t \in \mathbb{R} \\ &\vec{h}(0) = (h_0, h_1) \in \mathcal{H}. \end{aligned} \tag{5.8}$$

LEMMA 5.4. *There exists $R_0 > 0$ and there exists $\delta_0 > 0$ small enough so that for all $R > R_0$ and all initial data $\vec{h}(0) = (h_0, h_1) \in \mathcal{H}$ with*

$$\|\vec{h}(0)\|^2_{\mathcal{H}} \leq \delta_0$$

there exists a unique global solution $\vec{h}(t) \in \mathcal{H}$ to (5.8). In addition $\vec{h}(t)$ satisfies

$$\|h\|_{L^3_t L^6_x(\mathbb{R}\times\mathbb{R}^5_*)} \lesssim \|\vec{h}(0)\|_{\mathcal{H}} \lesssim \delta_0. \tag{5.9}$$

Moreover, if we let $h_L(t) := S_0(t)\vec{h}(0) \in \mathcal{H}$ denote the free linear evolution, i.e., solution to (4.5), of the data $\vec{h}(0)$ we have

$$\sup_{t\in\mathbb{R}} \|\vec{h}(t) - \vec{h}_L(t)\|_{\mathcal{H}} \lesssim R^{-11/3}\|\vec{h}(0)\|_{\mathcal{H}} + R^{-11/6}\|\vec{h}(0)\|^2_{\mathcal{H}} + \|\vec{h}(0)\|^3_{\mathcal{H}}. \tag{5.10}$$

REMARK 3. Note that for each $t \in \mathbb{R}$,

$$\mathcal{N}_R(t, r, h) = -V(r)h + F(r, h) + G(r, h) \quad \forall r \geq R + |t| \tag{5.11}$$

where $V(r)$, $F(r, h)$, and $G(r, h)$ are as in (2.8). By finite propagation speed it is then immediate that solutions to (5.8) and (2.8) agree on the exterior cone $\{(t, r) \mid r \geq R + |t|\}$.

Proof of Lemma 5.4. The small data well-posedness theory, including estimate (5.9), follows from the usual contraction and continuity arguments based on the Strichartz estimates in Proposition 3.1. To prove (5.10) we note that by the Duhamel formula and Strichartz estimates we have

$$\begin{aligned} \|\vec{h}(t) - \vec{h}_L(t)\|_{\mathcal{H}} &\lesssim \|\mathcal{N}_R(\cdot, \cdot, h)\|_{L^1_t L^2_x(\mathbb{R}\times\mathbb{R}^5_*)} \\ &\lesssim \|V_R h\|_{L^1_t L^2_x(\mathbb{R}\times\mathbb{R}^5_*)} + \|F_R(\cdot, \cdot, h)\|_{L^1_t L^2_x(\mathbb{R}\times\mathbb{R}^5_*)} + \|G(\cdot, h)\|_{L^1_t L^2_x(\mathbb{R}\times\mathbb{R}^5_*)}. \end{aligned}$$

We can now estimate the three terms on the right-hand side above. First, we claim that

$$\|V_R h\|_{L^1_t L^2_x(\mathbb{R}\times\mathbb{R}^5_*)} \lesssim \|V_R\|_{L^{\frac{3}{2}}_t L^3_x} \|h\|_{L^3_t L^6_x} \lesssim R^{-11/3}\|h\|_{L^3_t L^6_x}.$$

To see this, we can use (5.5) to deduce that for each $t \in \mathbb{R}$

$$\begin{aligned} \|V_R\|^3_{L^3_x} &\lesssim \int_1^{R+|t|} (R + |t|)^{-18} r^4 \, dr + \int_{R+|t|}^{\infty} r^{-18} r^4 \, dr \\ &\lesssim (R + |t|)^{-13}. \end{aligned}$$

Therefore,

$$\|V_R\|_{L_t^{\frac{3}{2}}L_x^3} \lesssim \left(\int_{\mathbb{R}} (R+|t|)^{-13/2}\,dt\right)^{\frac{2}{3}} \lesssim R^{-11/3}$$

Similarly, we can show using (5.6) and (5.7) that

$$\|F_R(\cdot,\cdot,h)\|_{L_t^1L_x^2(\mathbb{R}\times\mathbb{R}^5_*)} \lesssim R^{-11/6}\|h\|^2_{L_t^3L_x^6}$$
$$\|G(\cdot,h)\|_{L_t^1L_x^2(\mathbb{R}\times\mathbb{R}^5_*)} \lesssim \|h\|^3_{L_t^3L_x^6}$$

which proves (5.10). □

We can now prove Lemma 5.3.

Proof of Lemma 5.3. We will first prove Lemma 5.3 for time $t = 0$. The fact that (5.3) holds at all times $t \in \mathbb{R}$ for $R > R_0$, with R_0 independent of t will follow from the pre-compactness of K.

For each $R \geq 1$, define truncated initial data $\vec{u}_R(0) = (u_{0,R}, u_{1,R})$ given by

$$\begin{aligned} u_{0,R}(r) &= \begin{cases} u_0(r) & \text{for } r \geq R \\ \frac{u_0(R)}{R-1}(r-1) & \text{for } r < R, \end{cases} \\ u_{1,R}(r) &= \begin{cases} u_1(r) & \text{for } r \geq R \\ 0 & \text{for } r < R. \end{cases} \end{aligned} \tag{5.12}$$

Observe that this truncated data has small energy for large R since

$$\|\vec{u}_R(0)\|_{\mathcal{H}} \lesssim \|\vec{u}(0)\|_{\mathcal{H}(r\geq R)}. \tag{5.13}$$

In particular, there exists $R_0 \geq 1$ so that for all $R \geq R_0$ we have

$$\|\vec{u}_R(0)\|_{\mathcal{H}} \leq \delta_0$$

where δ_0 is the small constant in Lemma 5.4. Let $\vec{u}_R(t)$ denote the solution to (5.8) given by Lemma 5.4 with data $\vec{u}_R(0)$ as in (5.12). Note that by finite propagation speed we have

$$\vec{u}_R(t,r) = \vec{u}(t,r) \quad \forall t \in \mathbb{R},\ \forall r \geq R + |t|.$$

Also let $\vec{u}_{R,L}(t) = S_0(t)\vec{u}_R(0)$ denote the solution to free wave equation (4.5) with initial data $\vec{u}_R(0)$. Now, by the triangle inequality we obtain for each $t \in \mathbb{R}$

$$\begin{aligned} \|\vec{u}(t)\|_{\mathcal{H}(r\geq R+|t|)} = \|\vec{u}_R(t)\|_{\mathcal{H}(r\geq R+|t|)} \ &\geq \|\vec{u}_{R,L}(t)\|_{\mathcal{H}(r\geq R+|t|)} \\ &\quad -\|\vec{u}_R(t) - \vec{u}_{R,L}(t)\|_{\mathcal{H}}. \end{aligned} \tag{5.14}$$

By (5.10) and (5.13) we can deduce that

$$\begin{aligned} \sup_{t\in\mathbb{R}} \|\vec{u}_R(t) - \vec{u}_{R,L}(t)\|_{\mathcal{H}} \ &\lesssim R^{-11/3}\|\vec{u}_R(0)\|_{\mathcal{H}} + R^{-11/6}\|\vec{u}_R(0)\|^2_{\mathcal{H}} + \|\vec{u}_R(0)\|^3_{\mathcal{H}} \\ &\lesssim R^{-11/3}\|\vec{u}(0)\|_{\mathcal{H}(r\geq R)} + R^{-11/6}\|\vec{u}(0)\|^2_{\mathcal{H}(r\geq R)} \\ &\quad +\|\vec{u}(0)\|^3_{\mathcal{H}(r\geq R)}. \end{aligned}$$

Therefore (5.14) gives

$$\begin{aligned}\|\vec{u}(t)\|_{\mathcal{H}(r\geq R+|t|)} \geq\ & \|\vec{u}_{R,L}(t)\|_{\mathcal{H}(r\geq R+|t|)} - C_0 R^{-11/3}\|\vec{u}(0)\|_{\mathcal{H}(r\geq R)} \\ & -C_0 R^{-11/6}\|\vec{u}(0)\|^2_{\mathcal{H}(r\geq R)} - C_0\|\vec{u}(0)\|^3_{\mathcal{H}(r\geq R)}.\end{aligned}$$

Letting t tend to either $\pm\infty$—the choice determined by Proposition 4.1—we can use Proposition 4.1 to estimate the right-hand side above and use Corollary 5.2 to see that the left-hand side above tends to zero, which gives

$$\|\pi_R^{\perp}\,\vec{u}_R(0)\|^2_{\mathcal{H}(r\geq R)} \lesssim R^{-22/3}\|\vec{u}(0)\|^2_{\mathcal{H}(r\geq R)} + R^{-11/3}\|\vec{u}(0)\|^4_{\mathcal{H}(r\geq R)} + \|\vec{u}(0)\|^6_{\mathcal{H}(r\geq R)}$$

after squaring both sides. Finally we note that by the definition of $\vec{u}_R(0)$,

$$\|\pi_R^{\perp}\,\vec{u}_R(0)\|^2_{\mathcal{H}(r\geq R)} = \|\pi_R^{\perp}\,\vec{u}(0)\|^2_{\mathcal{H}(r\geq R)}.$$

Therefore,

$$\begin{aligned}\|\pi_R^{\perp}\,\vec{u}(0)\|^2_{\mathcal{H}(r\geq R)} \lesssim\ & R^{-22/3}\left(\|\pi_R\,\vec{u}(0)\|^2_{\mathcal{H}(r\geq R)} + \|\pi_R^{\perp}\,\vec{u}(0)\|^2_{\mathcal{H}(r\geq R)}\right) \\ & +R^{-11/3}\left(\|\pi_R\,\vec{u}(0)\|^2_{\mathcal{H}(r\geq R)} + \|\pi_R^{\perp}\,\vec{u}(0)\|^2_{\mathcal{H}(r\geq R)}\right)^2 \\ & +\left(\|\pi_R\,\vec{u}(0)\|^2_{\mathcal{H}(r\geq R)} + \|\pi_R^{\perp}\,\vec{u}(0)\|^2_{\mathcal{H}(r\geq R)}\right)^3\end{aligned}$$

where we have used the orthogonality of the projection π_R to expand the right-hand side. To conclude the proof, simply choose R_0 large enough so that we can absorb all of the terms on the right-hand side involving $\pi^{\perp}$ into the left-hand side and deduce that

$$\begin{aligned}\|\pi_R^{\perp}\,\vec{u}(0)\|^2_{\mathcal{H}(r\geq R)} \lesssim\ & R^{-22/3}\|\pi_R\,\vec{u}(0)\|^2_{\mathcal{H}(r\geq R)} \\ & +R^{-11/3}\|\pi_R\,\vec{u}(0)\|^4_{\mathcal{H}(r\geq R)} + \|\pi_R\,\vec{u}(0)\|^6_{\mathcal{H}(r\geq R)}.\end{aligned}$$

This proves Lemma 5.3 for $t = 0$. To show that this inequality holds for all $t \in \mathbb{R}$ observe that by the pre-compactness of K we can choose $R_0 = R_0(\delta_0)$ so that

$$\|\vec{u}(t)\|_{\mathcal{H}(r\geq R)} \leq \delta_0 \tag{5.15}$$

uniformly in $t \in \mathbb{R}$. Now simply repeat the argument given above with the truncated initial data for time $t = t_0$ and $R \geq R_0$ defined by

$$u_{0,R,t_0}(r) = \begin{cases} u(t_0, r) & \text{for} \quad r \geq R \\ \frac{u(t_0,R)}{R-1}(r-1) & \text{for} \quad r < R, \end{cases}$$

$$u_{1,R,t_0}(r) = \begin{cases} u_t(t_0, r) & \text{for} \quad r \geq R \\ 0 & \text{for} \quad r < R. \end{cases}$$

This concludes the argument. □

5.2 Step 2. In this step we will deduce the asymptotic behavior of $\vec{u}(0, r)$ as $r \to \infty$ described in (5.2). In particular we will establish the following result.

LEMMA 5.5. *Let $\vec{u}(t)$ be as in Proposition 5.1 with $\vec{u}(0) = (u_0, u_1)$. Then there exists $\ell_0 \in \mathbb{R}$ such that*

$$r^3 u_0(r) \to \ell_0 \quad \text{as} \quad r \to \infty \tag{5.16}$$

$$r \int_r^\infty u_1(\rho)\rho \, d\rho \to 0 \quad \text{as} \quad r \to \infty. \tag{5.17}$$

Moreover, we have the following estimates for the rates of convergence,

$$\left| r^3 u_0(r) - \ell_0 \right| = O(r^{-3}) \quad \text{as} \quad r \to \infty \tag{5.18}$$

$$\left| r \int_r^\infty u_1(\rho)\rho \, d\rho \right| = O(r^{-1}) \quad \text{as} \quad r \to \infty. \tag{5.19}$$

To begin, we define

$$\begin{aligned} v_0(t,r) &:= r^3 u(t,r) \\ v_1(t,r) &:= r \int_r^\infty u_t(t,\rho)\rho \, d\rho \end{aligned} \tag{5.20}$$

and for simplicity we will write $v_0(r) := v_0(0,r)$ and $v_1(r) := v_1(0,r)$. By direct computation one can show that

$$\|\pi_R^\perp \vec{u}(t)\|^2_{\mathcal{H}(r\geq R)} = \int_R^\infty \left(\frac{1}{r} \partial_r v_0(t,r) \right)^2 dr + \int_R^\infty (\partial_r v_1(t,r))^2 \, dr \tag{5.21}$$

$$\|\pi_R \vec{u}(t)\|^2_{\mathcal{H}(r\geq R)} = 3R^{-3} v_0^2(t,R) + R^{-1} v_1^2(t,R). \tag{5.22}$$

For convenience, we can rewrite the conclusions of Lemma 5.3 in terms of (v_0, v_1):

LEMMA 5.6. *Let (v_0, v_1) be defined as in (5.20). There exists $R_0 > 1$ so that for all $R > R_0$ we have*

$$\begin{aligned} \int_R^\infty \left(\frac{1}{r} \partial_r v_0(t,r) \right)^2 dr + \int_R^\infty (\partial_r v_1(t,r))^2 \, dr &\lesssim R^{-\frac{22}{3}} \left(3R^{-3} v_0^2(t,R) + R^{-1} v_1^2(t,R) \right) \\ &\quad + R^{-\frac{11}{3}} \left(3R^{-3} v_0^2(t,R) + R^{-1} v_1^2(t,R) \right)^2 \\ &\quad + \left(3R^{-3} v_0^2(t,R) + R^{-1} v_1^2(t,R) \right)^3 \\ &\lesssim R^{-\frac{31}{3}} v_0^2(t,R) + R^{-\frac{29}{3}} v_0^4(t,R) + R^{-9} v_0^6(t,R) \\ &\quad + R^{-\frac{25}{3}} v_1^2(t,R) + R^{-\frac{17}{3}} v_1^4(t,R) + R^{-3} v_1^6(t,R) \end{aligned}$$

with the above estimates holding uniformly in $t \in \mathbb{R}$.

We will use Lemma 5.6 to prove a difference estimate. First, let $\delta_1 > 0$ be a small number to be determined below with $\delta_1 \leq \delta_0$ where δ_0 is as in Lemma 5.4. Let R_1 be large enough so that for all $R \geq R_1$ we have

$$\|\vec{u}(t)\|_{\mathcal{H}(r\geq R)} \leq \delta_1 \leq \delta_0 \quad \forall R \geq R_1, \quad \forall t \in \mathbb{R}$$
$$R_1^{-\frac{11}{3}} \leq \delta_1. \tag{5.23}$$

We note again that such an $R_1 = R_1(\delta_1)$ exists by the pre-compactness of K.

Corollary 5.7. *Let R_1 be as above. The for all r, r' with $R_1 \leq r \leq r' \leq 2r$ and for all $t \in \mathbb{R}$ we have*

$$\begin{aligned}\left|v_0(t,r) - v_0(t,r')\right| \lesssim\ & r^{-\frac{11}{3}} |v_0(t,r)| + r^{-\frac{10}{3}} |v_0(t,r)|^2 + r^{-3} |v_0(t,r)|^3 \\ & + r^{-\frac{8}{3}} |v_1(t,r)| + r^{-\frac{4}{3}} |v_1(t,r)|^2 + |v_1(t,r)|^3 \end{aligned} \tag{5.24}$$

and

$$\begin{aligned}\left|v_1(t,r) - v_1(t,r')\right| \lesssim\ & r^{-\frac{14}{3}} |v_0(t,r)| + r^{-\frac{13}{3}} |v_0(t,r)|^2 + r^{-4} |v_0(t,r)|^3 \\ & + r^{-\frac{11}{3}} |v_1(t,r)| + r^{-\frac{7}{3}} |v_1(t,r)|^2 + r^{-1} |v_1(t,r)|^3 \end{aligned} \tag{5.25}$$

with the above estimates holding uniformly in $t \in \mathbb{R}$.

We will also need a trivial consequence of the preceding result which we state as another corollary for convenience.

Corollary 5.8. *Let R_1 be as above. The for all r, r' with $R_1 \leq r \leq r' \leq 2r$ and for all $t \in \mathbb{R}$ we have*

$$\left|v_0(t,r) - v_0(t,r')\right| \lesssim \delta_1 |v_0(t,r)| + r\delta_1 |v_1(t,r)| \tag{5.26}$$

and

$$\left|v_1(t,r) - v_1(t,r')\right| \lesssim r^{-1}\delta_1 |v_0(t,r)| + \delta_1 |v_1(t,r)| \tag{5.27}$$

with the above estimates holding uniformly in $t \in \mathbb{R}$.

We remark that Corollary 5.8 follows immediately from Corollary 5.7 in light of (5.22) and (5.23).

Proof of Corollary 5.7. This is a simple consequence of Lemma 5.6. Indeed, for $r \geq R_1$ and $r' \in [r, 2r]$ we use Lemma 5.6 to see that

$$\begin{aligned}\left|v_0(t,r) - v_0(t,r')\right|^2 &\leq \left(\int_r^{r'} |\partial_r v_0(t,\rho)|\, d\rho\right)^2 \\ &\leq \left(\int_r^{r'} \rho^2\, d\rho\right)\left(\int_r^{r'} \left|\frac{1}{\rho}\partial_r v_0(t,\rho)\right|^2 d\rho\right) \\ &\lesssim r^3\left(r^{-\frac{31}{3}} v_0^2(t,r) + r^{-\frac{29}{3}} v_0^4(t,r) + r^{-9} v_0^6(t,r)\right) \\ &\quad + r^3\left(r^{-\frac{25}{3}} v_1^2(t,r) + r^{-\frac{17}{3}} v_1^4(t,r) + r^{-3} v_1^6(t,r)\right).\end{aligned}$$

Similarly,

$$\begin{aligned}\left|v_1(t,r)-v_1(t,r')\right|^2 &\leq \left(\int_r^{r'} |\partial_r v_1(t,\rho)|\, d\rho\right)^2\\ &\leq \left(\int_r^{r'} d\rho\right)\left(\int_r^{r'} |\partial_r v_1(t,\rho)|^2\, d\rho\right)\\ &\lesssim r\left(r^{-\frac{31}{3}}v_0^2(t,r)+r^{-\frac{29}{3}}v_0^4(t,r)+r^{-9}v_0^6(t,r)\right)\\ &\quad +r\left(r^{-\frac{25}{3}}v_1^2(t,r)+r^{-\frac{17}{3}}v_1^4(t,r)+r^{-3}v_1^6(t,r)\right)\end{aligned}$$

as claimed. □

The next step towards establishing Lemma 5.5 is to provide an upper bound on the growth rates of $v_0(t,r)$ and $v_1(t,r)$.

Claim 5.9. *Let $v_0(t,r)$, $v_1(t,r)$ be as in (5.20). Then,*

$$|v_0(t,r)| \lesssim r^{\frac{1}{6}} \tag{5.28}$$

$$|v_1(t,r)| \lesssim r^{\frac{1}{18}} \tag{5.29}$$

uniformly in $t \in \mathbb{R}$.

Proof. First, note that it suffices to prove Claim 5.9 only for $t=0$ since the ensuing argument relies exclusively on results in this section that hold uniformly in $t \in \mathbb{R}$. Fix $r_0 \geq R_1$ and observe that by (5.26), (5.27)

$$\left|v_0(2^{n+1}r_0)\right| \leq (1+C_1\delta_1)\left|v_0(2^n r_0)\right| + (2^n r_0)C_1\delta_1\left|v_1(2^n r_0)\right| \tag{5.30}$$

$$\left|v_1(2^{n+1}r_0)\right| \leq (1+C_1\delta_1)\left|v_1(2^n r_0)\right| + (2^n r_0)^{-1}C_1\delta_1\left|v_0(2^n r_0)\right|. \tag{5.31}$$

To simply the exposition, we introduce the notation

$$a_n := |v_1(2^n r_0)| \tag{5.32}$$

$$b_n := (2^n r_0)^{-1}\left|v_0(2^n r_0)\right|. \tag{5.33}$$

Then, combining (5.30) and (5.31) gives

$$\begin{aligned}a_{n+1}+b_{n+1} &\leq \left(1+\frac{3}{2}C_1\delta_1\right)a_n + \left(\frac{1}{2}+\frac{3}{2}C_1\delta_1\right)b_n\\ &\leq \left(1+\frac{3}{2}C_1\delta_1\right)(a_n+b_n).\end{aligned}$$

Arguing inductively we then see that for each n we have

$$(a_n+b_n) \leq \left(1+\frac{3}{2}C_1\delta_1\right)^n (a_0+b_0).$$

Choosing δ_1 small enough so that $(1+\frac{3}{2}C_1\delta_1) \leq 2^{\frac{1}{18}}$ allows us to conclude that

$$a_n \leq C(2^n r_0)^{\frac{1}{18}} \tag{5.34}$$

where the constant $C > 0$ above depends on r_0 which is fixed. In light of (5.32) we have thus proved (5.29) for all $r = 2^n r_0$. Now define

$$c_n := |v_0(2^n r_0)|. \tag{5.35}$$

By (5.22), (5.23), (5.24), and (5.34) we have

$$c_{n+1} \leq (1 + C_1\delta_1)c_n + C(2^n r_0)^{\frac{1}{6}}$$

Inductively, we can deduce that

$$\begin{aligned} c_n &\leq (1+C_1\delta_1)^n c_0 + C r_0^{\frac{1}{6}} \sum_{k=1}^{n} (1+C_1\delta_1)^{n-k} 2^{\frac{k-1}{6}} \\ &\leq C(2^n r_0)^{\frac{1}{6}} \end{aligned}$$

where we have used that $(1 + C_1\delta_1) \leq 2^{\frac{1}{18}}$, and again the constant $C > 0$ depends on r_0, which is fixed. This proves (5.28) for $r = 2^n r_0$. The general estimates (5.28) and (5.29) follow from the difference estimates (5.24) and (5.25). □

Claim 5.10. *For each $t \in \mathbb{R}$ there exists a number $\ell_1(t) \in \mathbb{R}$ such that*

$$|v_1(t,r) - \ell_1(t)| = O(r^{-1}) \quad \text{as} \quad r \to \infty \tag{5.36}$$

where the $O(\cdot)$ is uniform in t.

Proof. Again, it suffices to show this for $t = 0$. Let $r_0 \geq R_1$ where $R_1 > 1$ is as in (5.23). By (5.25) and Claim 5.9 we have

$$\begin{aligned} \left|v_1(2^{n+1}r_0) - v_1(2^n r_0)\right| &\lesssim (2^n r_0)^{-\frac{9}{2}} + (2^n r_0)^{-4} + (2^n r_0)^{-\frac{7}{2}} \\ &\quad + (2^n r_0)^{-\frac{65}{18}} + (2^n r_0)^{-\frac{20}{9}} + (2^n r_0)^{-\frac{5}{6}} \\ &\lesssim (2^n r_0)^{-\frac{5}{6}}. \end{aligned}$$

This implies that the series

$$\sum_n \left|v_1(2^{n+1}r_0) - v_1(2^n r_0)\right| < \infty$$

which in turn implies that there exists $\ell_1 \in \mathbb{R}$ such that

$$\lim_{n\to\infty} v_1(2^n r_0) = \ell_1.$$

The fact that $\lim_{r\to\infty} v_1(r) = \ell_1$ follows from the difference estimates (5.24), (5.25), and the growth estimates (5.28), (5.29). To establish the estimates on the rate of convergence in (5.36) we note that by the difference estimate (5.25) and the fact that we now know that $|v_1(r)|$ is bounded, for large enough r we have

$$\left|v_1(2^{n+1}r) - v_1(2^n r)\right| \lesssim (2^n r)^{-1}.$$

Hence,

$$|v_1(r) - \ell_1| = \left|\sum_{n\geq 0}(v_1(2^{n+1}r) - v_1(2^n r))\right| \lesssim r^{-1}\sum_{n\geq 0} 2^{-n} \lesssim r^{-1}$$

as desired. □

Next we show that the limit $\ell_1(t)$ is actually independent of t.

CLAIM 5.11. *The function $\ell_1(t)$ in Claim 5.10 is independent of t, i.e., $\ell_1(t) = \ell_1$ for all $t \in \mathbb{R}$.*

Proof. By the definition of $v_1(t,r)$ we have shown that

$$\ell_1(t) = r\int_r^\infty u_t(t,\rho)\rho\, d\rho + O(r^{-1}).$$

Fix $t_1, t_2 \in \mathbb{R}$ with $t_1 \neq t_2$. We will show that

$$\ell_1(t_2) - \ell_1(t_1) = 0.$$

To see this observe that for each $R \geq R_1$ we have

$$\begin{aligned}
\ell_1(t_2) - \ell_1(t_1) &= \frac{1}{R}\int_R^{2R}(\ell_1(t_2) - \ell_1(t_1))\, ds \\
&= \frac{1}{R}\int_R^{2R}\left(s\int_s^\infty (u_t(t_2,r) - u_t(t_1,r))r\, dr\right) ds + O(R^{-1}) \\
&= \frac{1}{R}\int_R^{2R}\left(s\int_s^\infty\int_{t_1}^{t_2} u_{tt}(t,r)\, dt\, r\, dr\right) ds + O(R^{-1}).
\end{aligned}$$

Using the fact that u is a solution to (2.8), we can rewrite the above integral as

$$\begin{aligned}
&= \int_{t_1}^{t_2}\frac{1}{R}\int_R^{2R}\left(s\int_s^\infty (ru_{rr}(t,r) + 4u_r(t,r))\, dr\right) ds\, dt \\
&\quad + \int_{t_1}^{t_2}\frac{1}{R}\int_R^{2R}\left(s\int_s^\infty (-rV(r)u(t,r) + rN(r,u(t,r)))\, dr\right) ds\, dt \\
&\quad + O(R^{-1}) \\
&= I + II + O(R^{-1}). \qquad (5.37)
\end{aligned}$$

To estimate I we integrate by parts:

$$\begin{aligned} I &= \int_{t_1}^{t_2} \frac{1}{R} \int_R^{2R} \left(s \int_s^\infty \frac{1}{r^3} \partial_r (r^4 u_r(t,r))\, dr \right) ds\, dt \\ &= 3 \int_{t_1}^{t_2} \frac{1}{R} \int_R^{2R} \left(s \int_s^\infty u_r(t,r)\, dr \right) ds\, dt - \int_{t_1}^{t_2} \frac{1}{R} \int_R^{2R} s^2 u_r(t,s)\, ds\, dt \\ &= -3 \int_{t_1}^{t_2} \frac{1}{R} \int_R^{2R} r\, u(t,r)\, dr\, dt - \int_{t_1}^{t_2} \frac{1}{R} \int_R^{2R} r^2\, u_r(t,r)\, dr\, dt \\ &= -\int_{t_1}^{t_2} \frac{1}{R} \int_R^{2R} r\, u(t,r)\, dr\, dt + \int_{t_1}^{t_2} (Ru(t,R) - 2Ru(t,2R))\, dt. \end{aligned} \tag{5.38}$$

Finally, we note that (5.28) and the definition of $v_0(t,r)$ give us

$$r^3 |u(t,r)| = |v_0(t,r)| \lesssim r^{\frac{1}{6}}. \tag{5.39}$$

Using this estimate for $|u(t,r)|$ in the last line in (5.38) shows that

$$I = |t_2 - t_1|\, O(R^{-\frac{11}{6}}).$$

To estimate II we can use (5.39) to see that for $r > R$ large enough

$$\begin{aligned} |-V(r)u(t,r) + N(r,u(t,r))| &\lesssim r^{-6} |u(t,r)| + r^{-3} |u(t,r)|^2 + |u(t,r)|^3 \\ &\lesssim r^{-6-\frac{17}{6}} + r^{-3-\frac{17}{3}} + r^{-\frac{17}{2}} \\ &\lesssim r^{-8}. \end{aligned}$$

Hence,

$$II \lesssim \int_{t_1}^{t_2} \frac{1}{R} \int_R^{2R} s \int_s^\infty r^{-8}\, dr\, ds\, dt = |t_2 - t_1|\, O(R^{-6}).$$

Putting this together we get

$$|\ell_1(t_2) - \ell_1(t_1)| = O(R^{-1})$$

which implies that $\ell_1(t_2) = \ell_1(t_1)$. □

We next show that ℓ_1 is necessarily equal to 0.

Claim 5.12. $\ell_1 = 0$.

Proof. Suppose $\ell_1 \neq 0$. We know that for all $R \geq R_1$ and for all $t \in \mathbb{R}$ we have

$$R \int_R^\infty u_t(t,r)\, r\, dr = \ell_1 + O(R^{-1})$$

where $O(\cdot)$ is uniform in t. Hence, for R large, the left-hand side above has the same sign as ℓ_1, for all t. Thus we can choose $R \geq R_1$ large enough so that for all $t \in \mathbb{R}$,

$$\left| R \int_R^\infty u_t(t,r)\, r\, dr \right| \geq \frac{|\ell_1|}{2}.$$

Integrating from $t = 0$ to $t = T$ gives

$$\left| \int_0^T R \int_R^\infty u_t(t,r)\, r\, dr\, dt \right| \geq T \frac{|\ell_1|}{2}.$$

However, we integrate in t on the left-hand side and use (5.39) to obtain

$$\begin{aligned} \left| R \int_R^\infty \int_0^T u_t(t,r)\, r\, dt\, dr \right| &= \left| R \int_R^\infty [u(T,r) - u(0,r)]\, r\, dr \right| \\ &\lesssim R \int_R^\infty r^{-\frac{11}{6}}\, dr \lesssim R^{\frac{1}{6}}. \end{aligned}$$

Therefore for fixed large R we have

$$T \frac{|\ell_1|}{2} \lesssim R^{\frac{1}{6}}$$

which gives a contradiction by taking T large. □

Now that we have shown that $v_1(r) \to 0$ as $r \to \infty$, we can prove that $v_0(r)$ also converges and complete the proof of Lemma 5.5.

Proof of Lemma 5.5. It remains to show that there exists $\ell_0 \in \mathbb{R}$ such that

$$|v_0(r) - \ell_0| = O(r^{-3}) \quad \text{as} \quad r \to \infty. \tag{5.40}$$

Using the difference estimate (5.24) as well as (5.28) and the fact that $|v_1(r)| \lesssim r^{-1}$ for $r \geq R_1$ we have for $r_0 \geq R_1$

$$\begin{aligned} \left|v_0(2^{n+1}r_0) - v_0(2^n r_0)\right| &\lesssim (2^n r_0)^{-\frac{11}{3}} (2^n r_0)^{\frac{1}{6}} + (2^n r_0)^{-\frac{10}{3}} (2^n r_0)^{\frac{1}{3}} + (2^n r_0)^{-3} (2^n r_0)^{\frac{1}{2}} \\ &\quad + (2^n r_0)^{-\frac{8}{3}} (2^n r_0)^{-1} + (2^n r_0)^{-\frac{4}{3}} (2^n r_0)^{-2} + (2^n r_0)^{-3} \\ &\lesssim (2^n r_0)^{-\frac{5}{2}}. \end{aligned}$$

Hence,

$$\sum_{n \geq 0} \left|v_0(2^{n+1}r_0) - v_0(2^n r_0)\right| < \infty$$

and therefore there exists $\ell_0 \in \mathbb{R}$ so that

$$\lim_{n \to \infty} v_0(2^n r_0) = \ell_0.$$

By the difference estimate (5.24) and the fact that $v_1(r) \to 0$ we can conclude that in fact $\lim_{r\to\infty} v_0(r) = \ell_0$. To establish the convergence rate, we note that since we now know that $|v_0(r)|$ is bounded we have the improved difference estimate

$$\left|v_0(2^{n+1}r) - v_0(2^n r)\right| \lesssim (2^n r)^{-3} \tag{5.41}$$

which holds for all $r \geq R$. Therefore,

$$|v_0(r) - \ell_0| = \left|\sum_{n\geq 0}(v_0(2^{n+1}r) - v_0(2^n r))\right| \lesssim r^{-3}\sum_{n\geq 0} 2^{-3n} \tag{5.42}$$

as claimed. □

5.3 Step 3. Finally, we complete the proof of Proposition 5.1 by showing that $\vec{u}(t) = (0,0)$. We divide this argument into two separate cases depending on whether the number ℓ_0 found in the previous step is zero or nonzero.

Case 1: $\ell_0 = 0$ implies $\vec{u}(0) = (0,0)$:

In this case we show that if $\ell_0 = 0$, then $\vec{u}(t) = (0,0)$.

LEMMA 5.13. *Let $\vec{u}(t)$ be as in Proposition 5.1 and let ℓ_0 be as in Lemma 5.5. Suppose that $\ell_0 = 0$. Then $\vec{u}(t) = (0,0)$.*

We begin by showing that if $\ell_0 = 0$ then (u_0, u_1) must be compactly supported.

CLAIM 5.14. *Let ℓ_0 be as in Lemma 5.5. If $\ell_0 = 0$ then (u_0, u_1) must be compactly supported.*

Proof. The assumption $\ell_0 = 0$ means that

$$\begin{aligned} |v_0(r)| &= O(r^{-3}) \quad \text{as} \quad r \to \infty \\ |v_1(r)| &= O(r^{-1}) \quad \text{as} \quad r \to \infty. \end{aligned} \tag{5.43}$$

Therefore, for $r_0 \geq R_1$ we have

$$|v_0(2^n r_0)| + |v_1(2^n r_0)| \lesssim (2^n r_0)^{-3} + (2^n r_0)^{-1} \lesssim (2^n r_0)^{-1}. \tag{5.44}$$

On the other hand, using the difference estimates (5.24)–(5.27) as well as our assumption (5.43) we obtain

$$\begin{aligned} \left|v_0(2^{n+1}r_0)\right| &\geq (1 - C_1\delta_1)\,|v_0(2^n r_0)| - C_1(2^n r_0)^{-2}\,|v_1(2^n r_0)| \\ \left|v_1(2^{n+1}r_0)\right| &\geq (1 - C_1\delta_1)\,|v_1(2^n r_0)| - C_1(2^n r_0)^{-4}\,|v_0(2^n r_0)|. \end{aligned}$$

This means that

$$\left|v_0(2^{n+1}r_0)\right| + \left|v_1(2^{n+1}r_0)\right| \geq (1 - C_1\delta_1 - C_1 r_0^{-2})\,(|v_0(2^n r_0)| + |v_1(2^n r_0)|).$$

Choose r_0 large enough and δ_1 small enough so that $C_1(\delta_1 + r_0^{-2}) < \frac{1}{4}$. Arguing inductively we can conclude that

$$|v_0(2^n r_0)| + |v_1(2^n r_0)| \geq \left(\frac{3}{4}\right)^n (|v_0(r_0)| + |v_1(r_0)|).$$

Estimating the left hand side above using (5.44) gives

$$\left(\frac{3}{4}\right)^n (|v_0(r_0)| + |v_1(r_0)|) \lesssim 2^{-n} r_0^{-1}$$

which means that

$$\left(\frac{3}{2}\right)^n (|v_0(r_0)| + |v_1(r_0)|) \lesssim 1.$$

Hence $\vec{v}(r_0) := (v_0(r_0), v_1(r_0)) = (0,0)$. But then (5.22) implies that

$$\|\pi_{r_0} \vec{u}(0)\|_{\mathcal{H}(r \geq r_0)} = 0.$$

Using Lemma 5.3 we can also deduce that

$$\|\pi_{r_0}^{\perp} \vec{u}(0)\|_{\mathcal{H}(r \geq r_0)} = 0$$

and hence

$$\|\vec{u}(0)\|_{\mathcal{H}(r \geq r_0)} = 0$$

which concludes the proof since $\lim_{r \to \infty} u_0(r) = 0$. □

Proof of Lemma 5.13. Assume that $\ell_0 = 0$. Then by Claim 5.14, (u_0, u_1) is compactly supported. We assume that $(u_0, u_1) \neq (0,0)$ and argue by contradiction. In this case we can find $\rho_0 > 1$ so that

$$\rho_0 := \inf\{\rho \,:\, \|\vec{u}(0)\|_{\mathcal{H}(r \geq \rho)} = 0\}.$$

Let $\varepsilon > 0$ small to be determined below and find $1 < \rho_1 < \rho_0$, $\rho_1 = \rho_1(\varepsilon)$ so that

$$0 < \|\vec{u}(0)\|^2_{\mathcal{H}(r \geq \rho_1)} \leq \varepsilon \leq \delta_1^2$$

where $\delta_1 > 0$ is as in (5.23). With (v_0, v_1) as in (5.20) we have

$$\begin{aligned} &\int_{\rho_1}^{\infty} \left(\frac{1}{r} \partial_r v_0(r)\right)^2 dr + \int_{\rho_1}^{\infty} (\partial_r v_1(r))^2 \, dr + 3\rho_1^{-3} v_0^2(\rho_1) + \rho_1^{-1} v_1^2(\rho_1) \\ &\quad = \|\pi_{\rho_1}^{\perp} \vec{u}(0)\|^2_{\mathcal{H}(r \geq \rho_1)} + \|\pi_{\rho_1} \vec{u}(0)\|^2_{\mathcal{H}(r \geq \rho_1)} = \|\vec{u}(0)\|^2_{\mathcal{H}(r \geq \rho_1)} < \varepsilon. \end{aligned} \tag{5.45}$$

By Lemma 5.6 we also have

$$\int_{\rho_1}^{\infty}\left(\frac{1}{r}\partial_r v_0(r)\right)^2 dr+\int_{\rho_1}^{\infty}(\partial_r v_1(r))^2\,dr\lesssim \rho_1^{-\frac{31}{3}}v_0^2(\rho_1)+\rho_1^{-\frac{29}{3}}v_0^4(\rho_1)+\rho_1^{-9}v_0^6(\rho_1)$$
$$+\rho_1^{-\frac{25}{3}}v_1^2(\rho_1)+\rho_1^{-\frac{17}{3}}v_1^4(\rho_1)+\rho_1^{-3}v_1^6(\rho_1). \tag{5.46}$$

Arguing as in Corollary 5.8 and using the fact that $v_0(\rho_0)=v_1(\rho_0)=0$ gives

$$|v_0(\rho_1)|=|v_0(\rho_1)-v_0(\rho_0)|\lesssim \varepsilon\,|v_0(\rho_1)|+\rho_1\varepsilon\,|v_1(\rho_1)| \tag{5.47}$$

and

$$|v_1(\rho_1)|=|v_1(\rho_1)-v_1(\rho_0)|\lesssim \rho_1^{-1}\varepsilon\,|v_0(\rho_1)|+\varepsilon\,|v_1(\rho_1)|. \tag{5.48}$$

Plugging (5.47) into (5.48) gives

$$|v_1(\rho_1)|\lesssim \rho_1^{-1}\varepsilon^2\,|v_0(\rho_1)|+\varepsilon(1+\varepsilon)\,|v_1(\rho_1)|$$

which means that for ε small enough we have

$$|v_1(\rho_1)|\lesssim \rho_1^{-1}\varepsilon^2\,|v_0(\rho_1)|. \tag{5.49}$$

Putting this estimate back into (5.47) we obtain

$$|v_0(\rho_1)|\lesssim \varepsilon\,|v_0(\rho_1)|+\varepsilon^3\,|v_0(\rho_1)|\lesssim \varepsilon(1+\varepsilon^2)\,|v_0(\rho_1)|$$

which implies that $v_0(\rho_1)=0$ as long as ε is chosen small enough. By (5.49) we can conclude that $v_1(\rho_1)=0$ as well. By (5.46) and (5.45) we then have that

$$\|\vec{u}(0)\|_{\mathcal{H}(r\geq\rho_1)}=0$$

which is a contradiction since $\rho_1<\rho_0$. □

We next consider the case $\ell_0\neq 0$.

Case 2: $\ell_0\neq 0$ is impossible.

In this final step we show that the case $\ell_0\neq 0$ is impossible. Indeed we prove that if $\ell_0\neq 0$ then our original wave map $\vec{\psi}(t)$ is equal to a rescaled solution Q_{ℓ_0} to (2.1) that does not satisfy the Dirichlet boundary condition, $Q_{\ell_0}(1)\neq 0$, which is a contradiction since $\psi(t,1)=0$ for all $t\in\mathbb{R}$.

We have shown that

$$r^3u_0(r)=\ell_0+O(r^{-3}).$$

Recall that $ru_0(r)=\varphi_0(r)=\psi_0(r)-Q(r)$ and that

$$Q(r)=n\pi-\frac{\alpha_0}{r^2}+O(r^{-6})$$

where $\alpha_0 > 0$ is *uniquely* determined by the boundary condition $Q(1) = 0$. Hence,

$$\psi_0(r) = n\pi - \frac{\alpha_0 - \ell_0}{r^2} + O(r^{-5}). \tag{5.50}$$

By Lemma 2.1 there is a solution $Q_{\alpha_0-\ell} \in \dot{H}^1(\mathbb{R}^3_*)$ to (2.1) satisfying

$$Q_{\alpha_0-\ell_0}(r) = n\pi - \frac{\alpha_0 - \ell_0}{r^2} + O(r^{-6}) \tag{5.51}$$

and from here out we write $Q_{\ell_0} := Q_{\alpha_0-\ell_0}$. Note, by Lemma 2.1, $\ell_0 \neq 0$ implies that

$$Q_{\ell_0}(1) \neq 0.$$

Indeed, recall from the discussion following Lemma 2.1 that if $\alpha_0 - \ell_0 > 0$ then Q_{ℓ_0} is a nontrivial rescaling of the harmonic map Q and hence no longer satisfies the boundary condition. If $\alpha_0 - \ell_0 = 0$ then $Q_{\ell_0}(r) = n\pi$ for all r. Finally, we recall that $\alpha_0 - \ell_0 < 0$ implies that $Q_{\ell_0}(r) > n\pi$ for all r. Now set

$$\begin{aligned} u_{\ell_0,0}(r) &:= \frac{1}{r}(\psi_0(r) - Q_{\ell_0}(r)) \\ u_{\ell_0,1}(r) &:= \frac{1}{r}\psi_1(r). \end{aligned} \tag{5.52}$$

For each $t \in \mathbb{R}$ define $u_{\ell_0}(t,r) := \frac{1}{r}(\psi(t,r) - Q_{\ell_0}(r))$. We record a few properties of $\vec{u}_{\ell_0} := (u_{\ell_0}, \partial_t u_{\ell_0})$. Note that by construction we have

$$\begin{aligned} v_{\ell_0,0}(r) &:= r^3 u_{\ell_0}(r) = O(r^{-3}) \quad \text{as} \quad r \to \infty \\ v_{\ell_0,1}(r) &:= r\int_r^\infty \rho\, u_{\ell_0,1}(\rho)\, d\rho = O(r^{-1}) \quad \text{as} \quad r \to \infty. \end{aligned} \tag{5.53}$$

Also, $\vec{u}_{\ell_0}(t)$ satisfies

$$\partial_{tt} u_{\ell_0} - \partial_{rr} u_{\ell_0} - \frac{4}{r}\partial_r u_{\ell_0} = -V_{\ell_0}(r)u + N_{\ell_0}(r, u_{\ell_0}) \tag{5.54}$$

where

$$\begin{aligned} V_{\ell_0}(r) &:= \frac{2(\cos(2Q_{\ell_0}) - 1)}{r^2} \\ N_{\ell_0}(r, u_{\ell_0}) &:= \cos(2Q_{\ell_0})\frac{(2ru_{\ell_0} - \sin(2ru_{\ell_0}))}{r^3} + 2\sin(2Q_{\ell_0})\frac{\sin^2(ru_{\ell_0})}{r^3}. \end{aligned} \tag{5.55}$$

Crucially, we remark that $\vec{u}_{\ell_0}(t)$ inherits the compactness property from $\vec{\psi}(t)$. Indeed, the trajectory

$$\widetilde{K} := \{\vec{u}_{\ell_0}(t) \mid t \in \mathbb{R}\}$$

is pre-compact in $\dot{H}^1 \times L^2(\mathbb{R}^5_*)$. However, since we have assumed that $\ell_0 \neq 0$ we see that

$$u_{\ell_0}(t,1) = \psi_0(t,1) - Q_{\ell_0}(1) = -Q_{\ell_0}(1) \neq 0. \tag{5.56}$$

On the other hand, below we will show that $\vec{u}_{\ell_0} = (u_{\ell_0}, \partial_t u_{\ell_0}) = (0,0)$ which contradicts (5.56).

LEMMA 5.15. *Suppose $\ell_0 \neq 0$. Let $\vec{u}(t)$ be as in Proposition 5.1 and define $\vec{u}_{\ell_0}$ as in (5.52). Then $\vec{u}_{\ell_0} = (0,0)$.*

The argument that we will use to prove Lemma 5.15 is nearly identical to the one presented in the previous steps to reach the desired conclusion for $\ell_0 = 0$ and we omit many details here.

We start by showing that $(\partial_r u_{\ell_0,0}, u_{\ell_0,1})$ must be compactly supported. As before we can argue as in the proof of Lemma 5.3, by modifying (5.54) inside the interior cone $\{(t,r) \mid 1 \leq r \leq R + |t|\}$, and using the linear exterior estimates in Proposition 4.1 to produce the same type of inequality as (5.3).

LEMMA 5.16. *There exists $R_0 > 1$ so that for all $R \geq R_0$ we have*

$$\begin{aligned} \|\pi_R^{\perp} \vec{u}_{\ell_0}\|^2_{\mathcal{H}(r\geq R)} &\lesssim R^{-22/3} \|\pi_R \vec{u}_{\ell_0}\|^2_{\mathcal{H}(r\geq R)} \\ &\quad + R^{-11/3} \|\pi_R \vec{u}_{\ell_0}\|^4_{\mathcal{H}(r\geq R)} + \|\pi_R \vec{u}_{\ell_0}\|^6_{\mathcal{H}(r\geq R)} \end{aligned} \tag{5.57}$$

where again $P(R) := \{(k_1 r^{-3}, k_2 r^{-3}) \mid k_1, k_2 \in \mathbb{R}, r > R\}$, π_R denotes the orthogonal projection onto $P(R)$ and ${\pi_R}^{\perp}$ denotes the orthogonal projection onto the orthogonal complement of the plane $P(R)$ in $\mathcal{H}(r > R; \mathbb{R}^5_*)$.

We remark that the proof of Lemma 5.16 follows exactly as the proof of Lemma 5.3 where we simply replace Q with Q_{ℓ_0} and $\vec{u}$ with $\vec{u}_{\ell_0}$ in the arguments given for the proof of Lemma 5.3. We note that since the trajectory $\widetilde{K}$ is pre-compact in $\dot{H}^1 \times L^2(\mathbb{R}^5_*)$, $\vec{u}_{\ell_0}$ satisfies the conclusions of Corollary 5.2, namely for each $R > 1$ we have

$$\|\vec{u}_{\ell_0}(t)\|_{\mathcal{H}(r\geq R+|t|)} \to 0 \quad \text{as} \quad |t| \to \infty$$

where the condition $R > 1$ allows the interchange of the norms $\mathcal{H} = \dot{H}^1_0 \times L^2(\mathbb{R}^5_*)$ and $\dot{H}^1 \times L^2(\mathbb{R}^5_*)$. With $(v_{\ell_0,0}, v_{\ell_0,1})$ defined as in (5.53) we can then conclude that for all $R > R_0$ large enough we have

$$\begin{aligned} &\int_R^{\infty} \left(\frac{1}{r}\partial_r v_{\ell_0,0}(r)\right)^2 dr + \int_R^{\infty} (\partial_r v_{\ell_0,1}(r))^2 \, dr \lesssim R^{-\frac{31}{3}} v^2_{\ell_0,0}(R) + R^{-\frac{29}{3}} v^4_{\ell_0,0}(R) \\ &\quad + R^{-9} v^6_{\ell_0,0}(R) + R^{-\frac{25}{3}} v^2_{\ell_0,1}(R) + R^{-\frac{17}{3}} v^4_{\ell_0,1}(R) + R^{-3} v^6_{\ell_0,1}(R) \\ &\lesssim R^{-7}(v^2_{\ell_0,0}(R) + v^2_{\ell_0,1}(R)) \end{aligned} \tag{5.58}$$

where the first inequality follows by rewriting (5.57) in terms of $\vec{v}_{\ell_0} = (v_{\ell_0,0}, v_{\ell_0,1})$ and the last line following from the known decay estimates in (5.53). Next, mimicking the proof of Corollary 5.7 we can again establish difference estimates using (5.58). Indeed, for all $R_0 \leq r \leq r' \leq 2r$ we have

$$\begin{aligned} \left|v_{\ell_0,0}(r) - v_{\ell_0,0}(r')\right|^2 &\lesssim r^{-4}(v^2_{\ell_0,0}(r) + v^2_{\ell_0,1}(r)) \\ \left|v_{\ell_0,1}(r) - v_{\ell_0,1}(r')\right|^2 &\lesssim r^{-6}(v^2_{\ell_0,0}(r) + v^2_{\ell_0,1}(r)). \end{aligned} \tag{5.59}$$

In terms of the vector $\vec{v}_{\ell_0} = (v_{\ell_0,0}, v_{\ell_0,1})$ we then have

$$|\vec{v}_{\ell_0}(r) - \vec{v}_{\ell_0}(r')| \lesssim r^{-2}\,|\vec{v}_{\ell_0}(r)|. \tag{5.60}$$

Hence for fixed $r_0 \geq R_0$ large enough we can deduce that

$$\left|\vec{v}_{\ell_0}(2^{n+1}r_0)\right| \geq \frac{3}{4}\,|\vec{v}_{\ell_0}(2^n r_0)|.$$

Therefore for each n,

$$|\vec{v}_{\ell_0}(2^n r_0)| \geq \left(\frac{3}{4}\right)^n |\vec{v}_{\ell_0}(r_0)|.$$

On the other hand, by (5.53) we have

$$|\vec{v}_{\ell_0}(2^n r_0)| \lesssim (2^n r_0)^{-1}.$$

Combining the last two lines we see that

$$\left(\frac{3}{2}\right)^n |\vec{v}_{\ell_0}(r_0)| \lesssim 1,$$

which implies that $\vec{v}_{\ell_0}(r_0) = (0,0)$. By (5.58) we can deduce that

$$\int_{r_0}^{\infty} \left(\frac{1}{r}\partial_r v_{\ell_0,0}(r)\right)^2 dr + \int_{r_0}^{\infty} (\partial_r v_{\ell_0,1}(r))^2\, dr = 0.$$

Therefore,

$$\|\vec{u}_{\ell_0}\|^2_{\mathcal{H}(r\geq r_0)} = \int_{r_0}^{\infty} \left(\frac{1}{r}\partial_r v_{\ell_0,0}(r)\right)^2 dr + \int_{r_0}^{\infty} (\partial_r v_{\ell_0,1}(r))^2\, dr + 3r_0^{-3} v^2_{\ell_0,0}(r_0) + r_0^{-1} v^2_{\ell_0,1}(r_0) = 0$$

which means that $(\partial_r u_{\ell_0,0}, u_{\ell_0,1})$ is compactly supported. We conclude by showing that $\vec{u}_{\ell_0} = (0,0)$.

Proof of Lemma 5.15. The proof is nearly identical to the proof of Lemma 5.13. Suppose

$$(\partial_r u_{\ell_0,0}, u_{\ell_0,1}) \neq (0,0)$$

and we argue by contradiction. By the preceding arguments $(\partial_r u_{\ell_0,0}, u_{\ell_0,1})$ is compactly supported. Then we can define $\rho_0 > 1$ by

$$\rho_0 := \inf\{\rho \,:\, \|\vec{u}_{\ell_0}\|_{\mathcal{H}(r\geq\rho)} = 0\}.$$

Let $\varepsilon > 0$ small to be determined below and find $1 < \rho_1 < \rho_0$, $\rho_1 = \rho_1(\varepsilon)$ so that

$$0 < \|\vec{u}_{\ell_0}\|_{\mathcal{H}(r\geq\rho_1)} \leq \varepsilon.$$

We then have

$$\int_{\rho_1}^{\infty}\left(\frac{1}{r}\partial_r v_{\ell_0,0}(r)\right)^2 dr + \int_{\rho_1}^{\infty}(\partial_r v_{\ell_0,1}(r))^2\,dr + 3\rho_1^{-3}v_{\ell_0,0}^2(\rho_1) + \rho_1^{-1}v_{\ell_0,1}^2(\rho_1)$$
$$= \|\pi_{\rho_1}^{\perp}\vec{u}_{\ell_0}\|^2_{\mathcal{H}(r\geq\rho_1)} + \|\pi_{\rho_1}\vec{u}_{\ell_0}\|^2_{\mathcal{H}(r\geq\rho_1)} = \|\vec{u}_{\ell_0}\|^2_{\mathcal{H}(r\geq\rho_1)} < \varepsilon \tag{5.61}$$

By (5.58) we also have

$$\int_{\rho_1}^{\infty}\left(\frac{1}{r}\partial_r v_{\ell_0,0}(r)\right)^2 dr + \int_{\rho_1}^{\infty}(\partial_r v_{\ell_0,1}(r))^2\,dr \lesssim \rho_1^{-\frac{31}{3}}v_{\ell_0,0}^2(\rho_1) + \rho_1^{-\frac{29}{3}}v_{\ell_0,0}^4(\rho_1)$$
$$+\rho_1^{-9}v_{\ell_0,0}^6(\rho_1) + \rho_1^{-\frac{25}{3}}v_{\ell_0,1}^2(\rho_1) + \rho_1^{-\frac{17}{3}}v_{\ell_0,1}^4(\rho_1) + \rho_1^{-3}v_{\ell_0,1}^6(\rho_1). \tag{5.62}$$

Arguing as in Corollary 5.8 and using the fact that $v_0(\rho_0) = v_1(\rho_0) = 0$ gives

$$|v_{\ell_0,0}(\rho_1)| = |v_{\ell_0,0}(\rho_1) - v_{\ell_0,0}(\rho_0)| \lesssim \varepsilon\,|v_{\ell_0,0}(\rho_1)| + \rho_1\varepsilon\,|v_{\ell_0,1}(\rho_1)| \tag{5.63}$$

and

$$|v_{\ell_0,1}(\rho_1)| = |v_{\ell_0,1}(\rho_1) - v_{\ell_0,1}(\rho_0)| \lesssim \rho_1^{-1}\varepsilon\,|v_{\ell_0,0}(\rho_1)| + \varepsilon\,|v_{\ell_0,1}(\rho_1)|. \tag{5.64}$$

Plugging (5.63) into (5.64) gives

$$|v_{\ell_0,1}(\rho_1)| \lesssim \rho_1^{-1}\varepsilon^2\,|v_{\ell_0,0}(\rho_1)| + \varepsilon(1+\varepsilon)\,|v_{\ell_0,1}(\rho_1)|$$

which means that for ε small enough we have

$$|v_{\ell_0,1}(\rho_1)| \lesssim \rho_1^{-1}\varepsilon^2\,|v_{\ell_0,0}(\rho_1)|. \tag{5.65}$$

Putting this estimate back into (5.63) we obtain

$$|v_{\ell_0,0}(\rho_1)| \lesssim \varepsilon\,|v_{\ell_0,0}(\rho_1)| + \varepsilon^3\,|v_{\ell_0,0}(\rho_1)| \lesssim \varepsilon(1+\varepsilon^2)\,|v_{\ell_0,0}(\rho_1)|$$

which implies that $v_{\ell_0,0}(\rho_1) = 0$ as long as ε is chosen small enough. By (5.65) we can conclude that $v_{\ell_0,1}(\rho_1) = 0$ as well. By (5.62) and (5.61) we then have that

$$\|\vec{u}_{\ell_0}\|_{\mathcal{H}(r\geq\rho_1)} = 0$$

which is a contradiction since $\rho_1 < \rho_0$. Therefore, $(\partial_r u_{\ell_0,0}, u_{\ell_0,1}) = (0,0)$ Since $u_{\ell_0}(r) \to 0$ as $r \to \infty$ we can also conclude that $(u_{\ell_0,0}, u_{\ell_0,1}) = (0,0)$. □

5.4 Proof of Proposition 5.1 and Proof of Theorem 1.1. For clarity, we summarize what we have done in the proof of Proposition 5.1.

Proof of Proposition 5.1. Let $\vec{u}(t)$ be a solution to (2.8) and suppose that the trajectory

$$K = \{\vec{u}(t) \mid t \in \mathbb{R}\}$$

is pre-compact in $\mathcal{H}$. We recall that

$$r\vec{u}(t,r) = \vec{\psi}(t,r) - (Q_n(r), 0)$$

where $\vec{\psi}(t) \in \mathcal{H}_n$ is a degree n wave map, i.e., a solution to (1.2). By Lemma 5.5 there exists $\ell_0 \in \mathbb{R}$ so that

$$\left|r^3 u_0(r) - \ell_0\right| = O(r^{-3}) \quad \text{as} \quad r \to \infty \tag{5.66}$$

$$\left| r \int_r^\infty u_1(\rho)\rho\, d\rho \right| = O(r^{-1}) \quad \text{as} \quad r \to \infty. \tag{5.67}$$

If $\ell_0 \neq 0$ then by Lemma 5.15, $\psi(0,r) = Q_{\ell_0}$ where Q_{ℓ_0} is defined in (5.51). However, this is impossible since $Q_{\ell_0}(1) \neq 0$, which contradicts the Dirichlet boundary condition $\psi(t,1) = 0$ for all $t \in \mathbb{R}$.

Hence, $\ell_0 = 0$. Then by Lemma 5.13 we can conclude that $\vec{u}(0) = (0,0)$, which proves Proposition 5.1. □

The proof of Theorem 1.1 is now complete. We conclude by summarizing the argument. *Proof of Theorem 1.1* Suppose that Theorem 1.1 fails. Then by Proposition 3.6 there exists a critical element, that is, a nonzero solution $\vec{u}_*(t) \in \mathcal{H}$ to (2.8) such that the trajectory $K = \{\vec{u}_*(t) \mid t \in \mathbb{R}\}$ is pre-compact in $\mathcal{H}$. However, Proposition 5.1 implies that any such solution is necessarily identically equal to $(0,0)$, which contradicts the fact that the critical element $\vec{u}_*(t)$ is nonzero. □

References

[BSS92] B. Balakrishna, V. Sanyuk, J. Schechter, and A. Subbaraman. Cutoff quantization and the skyrmion. *Physical Review D*, 45(1) (1992), 344–351

[BCM12] P. Bizoń, T. Chmaj, and M. Maliborski. Equivariant wave maps exterior to a ball. *Nonlinearity*, 25(5) (2012), 1299–1309

[DKM11] T. Duyckaerts, C. Kenig, and F. Merle. Universality of the blow-up profile for small radial type II blow-up solutions of the energy critical wave equation. *Journal of the European Mathematical Society* (JEMS), 13(3) (2011), 533–599

[DKM12] T. Duyckaerts, C. Kenig, and F. Merle. Profiles of bounded radial solutions of the focusing, energy-critical wave equation. *Geometric and Functional Analysis*, 22(3) (2012), 639–698

[DKM13] T. Duyckaerts, C. Kenig, and F. Merle. Classification of radial solutions of the focusing, energy critical wave equation. *Cambridge Journal of Mathematics*, 1(1) (2013), 74–144

[DKM12] T. Duyckaerts, C. Kenig, and F. Merle. Scattering for radial, bounded solutions of focusing supercritical wave equations. To appear in I.M.R.N, Preprint (2012)

[HMSSZ10] K. Hidano, J. Metcalfe, H. Smith, C. Sogge, and Y. Zhou. On abstract Strichartz estimates and the Strauss conjecture for nontrapping obstacles. *Transactions of the American Mathematical Society*, 362(5) (2010), 2789–2809

[KM06] C. Kenig and F. Merle. Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case. *Inventiones Mathematicae*, 166(3) (2006), 645–675

[KM08] C. Kenig and F. Merle. Global well-posedness, scattering and blow-up for the energy-critical focusing non-linear wave equation. *Acta Mathematica* 201(2) (2008), 147–212

[KM11] C. Kenig and F. Merle. Radial solutions to energy supercritical wave equations in odd dimensions. *Discrete and Continuous Dynamical Systems*, 31(4) (2011), 1365–1381

[LS13] A. Lawrie and W. Schlag. Scattering for wave maps exterior to a ball. *Advances in Mathematics*, 232(1) (2013), 57–97

[S88] J. Shatah. Weak solutions and development of singularities of the SU(2)σ-model. *Communications on Pure and Applied Mathematics*, 41(4) (1988), 459–469

[SS98] J. Shatah and M. Struwe. *Geometric wave equations*. Courant Lecture notes in Mathematics, New York University, Courant Institute of Mathematical Sciences, New York. American Mathematical Society, Providence (1998)

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