Characterization of large energy solutions of the equivariant wave map problem: II

R. Côte, C. E. Kenig, A. Lawrie, W. Schlag

American Journal of Mathematics, Volume 137, Number 1, February 2015, pp. 209-250 (Article)

Published by Johns Hopkins University Press
DOI: *https://doi.org/10.1353/ajm.2015.0003*

CHARACTERIZATION OF LARGE ENERGY SOLUTIONS OF THE EQUIVARIANT WAVE MAP PROBLEM: II

By R. CÔTE, C. E. KENIG, A. LAWRIE, and W. SCHLAG

Abstract. We consider 1-equivariant wave maps from $\mathbb{R}^{1+2} \to \mathbb{S}^2$ of finite energy. We establish a classification of all degree one global solutions whose energies are less than three times the energy of the harmonic map Q. In particular, for each global energy solution of topological degree one, we show that the solution asymptotically decouples into a rescaled harmonic map plus a radiation term. Together with a companion article (Part I), where we consider the case of finite-time blow up, this gives a characterization of all 1-equivariant, degree 1 wave maps in the energy regime $[E(Q), 3E(Q))$.

1. Introduction. This paper is the companion article to [7]. Here we continue our study of the equivariant wave maps problem from $1+2$ dimensional Minkowski space to 2-dimensional surfaces of revolution. In local coordinates on the target manifold, (M,g), the Cauchy problem for wave maps is given by

$$\begin{aligned} \Box U^k &= -\eta^{\alpha\beta}\Gamma^k_{ij}(U)\partial_\alpha U^i \partial_\beta U^j \\ (U,\partial_t U)|_{t=0} &= (U_0,U_1), \end{aligned} \tag{1.1}$$

where Γ^k_{ij} are the Christoffel symbols on TM. As in [7] we will, for simplicity, restrict our attention to the case when the target $(M,g) = (\mathbb{S}^2,g)$ with g the round metric on the 2-sphere, $\mathbb{S}^2$. Our results here apply to more general compact surfaces of revolution as well, and we refer the reader to [7, Appendix A] for more details.

In spherical coordinates,

$$(\psi,\omega) \longmapsto (\sin\psi\cos\omega, \sin\psi\sin\omega, \cos\psi),$$

on $\mathbb{S}^2$, the metric, g, is given by the matrix $g = \mathrm{diag}(1,\sin^2(\psi))$. In the case of 1-equivariant wave maps, we require our wave map, U, to have the form

$$U(t,r,\omega) = (\psi(t,r),\omega) \longmapsto (\sin\psi(t,r)\cos\omega,\ \sin\psi(t,r)\sin\omega,\ \cos\psi(t,r)),$$

where (r,ω) are polar coordinates on $\mathbb{R}^2$. In this case, the Cauchy problem (1.1)

Manuscript received October 5, 2012.

Research of the first author supported by ERC project 291214 BLOWDISOL; research of the second author supported by NSF grant DMS-0968472; research of the fourth author supported by NSF grant DMS-1160817.

American Journal of Mathematics 137 (2015), 209–250.

reduces to

$$\begin{aligned} \psi_{tt} - \psi_{rr} - \frac{1}{r}\psi_r + \frac{\sin(2\psi)}{2r^2} &= 0 \\ (\psi, \psi_t)|_{t=0} &= (\psi_0, \psi_1). \end{aligned} \tag{1.2}$$

Wave maps exhibit a conserved energy, which in this equivariant setting is given by

$$\mathcal{E}(U, \partial_t U)(t) = \mathcal{E}(\psi, \psi_t)(t) = \int_0^\infty \left(\psi_t^2 + \psi_r^2 + \frac{\sin^2(\psi)}{r^2} \right) r\, dr = \text{const.},$$

and they are invariant under the scaling

$$\vec{\psi}(t,r) := (\psi(t,r), \psi_t(t,r)) \longmapsto (\psi(\lambda t, \lambda r), \lambda \psi_t(\lambda t \lambda r)).$$

The conserved energy is also invariant under this scaling which means that the Cauchy problem under consideration is energy critical.

We refer the reader to [7] for a more detailed introduction and history of the equivariant wave maps problem.

As in [7], we note that any wave map $\vec{\psi}(t,r)$ with finite energy and continuous dependence on $t \in I$ satisfies $\psi(t,0) = m\pi$ and $\psi(t,\infty) = n\pi$ for all $t \in I$ for fixed integers m, n. This determines a disjoint set of energy classes

$$\mathcal{H}_{m,n} := \{(\psi_0, \psi_1) \mid \mathcal{E}(\psi_0, \psi_1) < \infty \text{ and } \psi_0(0) = m\pi,\ \psi_0(\infty) = n\pi\}. \tag{1.3}$$

We will mainly consider the spaces $\mathcal{H}_{0,n}$ and we denote these by $\mathcal{H}_n := \mathcal{H}_{0,n}$. In this case we refer to n as the degree of the map. We also define $\mathcal{H} = \bigcup_{n \in \mathbb{Z}} \mathcal{H}_n$ to be the full energy space.

In our analysis, an important role is played by the unique (up to scaling) nontrivial harmonic map, $Q(r) = 2\arctan(r)$, given by stereographic projection. We note that Q solves

$$Q_{rr} + \frac{1}{r} Q_r = \frac{\sin(2Q)}{2r^2}. \tag{1.4}$$

Observe in addition that $(Q,0) \in \mathcal{H}_1$ and in fact $(Q,0)$ has minimal energy in $\mathcal{H}_1$ with $\mathcal{E}(Q) := \mathcal{E}(Q,0) = 4$. Note the slight abuse of notation above in that we will denote the energy of the element $(Q,0) \in \mathcal{H}_1$ by $\mathcal{E}(Q)$ rather than $\mathcal{E}(Q,0)$.

Recall that in [7] we showed that for any data $\vec{\psi}(0)$ in the zero topological class, $\mathcal{H}_0$, with energy $\mathcal{E}(\vec{\psi}) < 2\mathcal{E}(Q)$ there is a corresponding unique global wave map evolution $\vec{\psi}(t,r)$ that scatters to zero in the sense that the energy of $\vec{\psi}(t)$ on any arbitrary, but fixed compact region vanishes as $t \to \infty$, see [7, Theorem 1.1]. An equivalent way to view this scattering property is that there exists a decomposition

$$\vec{\psi}(t) = \vec{\varphi}_L(t) + o_{\mathcal{H}}(1) \quad \text{as } t \longrightarrow \infty \tag{1.5}$$

where $\vec{\varphi}_L(t) \in \mathcal{H}_0$ solves the linearized version of (1.2):

$$\varphi_{tt} - \varphi_{rr} - \frac{1}{r}\varphi_r + \frac{1}{r^2}\varphi = 0. \tag{1.6}$$

This result was proved via the concentration-compactness/rigidity method which was developed by the second author and Merle in [17, 18], and it provides a complete classification of all solutions in $\mathcal{H}_0$ with energy below $2\mathcal{E}(Q)$, namely, they all exist globally and scatter to zero. We note that this theorem is also a consequence of the work by Sterbenz and Tataru in [34] if one considers their results in the equivariant setting.

In [7] we also study *degree one* wave maps, $\vec{\psi}(t) \in \mathcal{H}_1$, with energy $\mathcal{E}(\vec{\psi}) = \mathcal{E}(Q) + \eta < 3\mathcal{E}(Q)$ that blow up in finite time. Because we are working in the equivariant, energy critical setting, blow-up can only occur at the origin in $\mathbb{R}^2$ and in an energy concentration scenario. We show that if blow-up does occur, say at $t = 1$, then there exists a scaling parameter $\lambda(t) = o(1-t)$, a degree zero map $\vec{\varphi} \in \mathcal{H}_0$ and a decomposition

$$\vec{\psi}(t,r) = \vec{\varphi}(r) + (Q(r/\lambda(t)),0) + o_{\mathcal{H}}(1) \quad \text{as } t \longrightarrow 1. \tag{1.7}$$

Here we complete our study of *degree one* solutions to (1.2), i.e., solutions that lie in $\mathcal{H}_1$, with energy below $3\mathcal{E}(Q)$, by providing a classification of such solutions with this energy constraint. Since the degree of the map is preserved for all time, scattering to zero is not possible for a degree one solution. However, we show that a decomposition of the form (1.7) holds in the global case. In particular we establish the following theorem:

THEOREM 1.1. (Classification of solutions in $\mathcal{H}_1$ with energies below $3\mathcal{E}(Q)$) *Let $\vec{\psi}(0) \in \mathcal{H}_1$ and denote by $\vec{\psi}(t) \in \mathcal{H}_1$ the corresponding wave map evolution. Suppose that $\vec{\psi}$ satisfies*

$$\mathcal{E}(\vec{\psi}) = \mathcal{E}(Q) + \eta < 3\mathcal{E}(Q).$$

Then, one of the following two scenarios occurs:

(1) Finite time blow-up: *The solution $\vec{\psi}(t)$ blows up in finite time, say at $t = 1$, and there exists a continuous function, $\lambda : [0,1) \to (0,\infty)$ with $\lambda(t) = o(1-t)$, a map $\vec{\varphi} = (\varphi_0, \varphi_1) \in \mathcal{H}_0$ with $\mathcal{E}(\vec{\varphi}) = \eta$, and a decomposition*

$$\vec{\psi}(t) = \vec{\varphi} + (Q(\cdot/\lambda(t)),0) + \vec{\epsilon}(t) \tag{1.8}$$

such that $\vec{\epsilon}(t) \in \mathcal{H}_0$ and $\vec{\epsilon}(t) \to 0$ in $\mathcal{H}_0$ as $t \to 1$.

(2) Global Solution: *The solution $\vec{\psi}(t) \in \mathcal{H}_1$ exists globally in time and there exists a continuous function, $\lambda : [0,\infty) \to (0,\infty)$ with $\lambda(t) = o(t)$ as $t \to \infty$, a*

solution $\vec{\varphi}_L(t) \in \mathcal{H}_0$ to the linear wave equation (1.6)*, and a decomposition*

$$\vec{\psi}(t) = \vec{\varphi}_L(t) + (Q(\cdot/\lambda(t)), 0) + \vec{\epsilon}(t) \tag{1.9}$$

such that $\vec{\epsilon}(t) \in \mathcal{H}_0$ and $\vec{\epsilon}(t) \to 0$ in $\mathcal{H}_0$ as $t \to \infty$.

Remark 1. One should note that the requirement $\lambda(t) = o(t)$ as $t \to \infty$ in part (2) above leaves open many possibilities for the asymptotic behavior of global degree one solutions to (1.2) with energy below $3\mathcal{E}(Q)$. If $\lambda(t) \to \lambda_0 \in (0,\infty)$ then our theorem says that the solution $\psi(t)$ asymptotically decouples into a soliton, Q_{λ_0}, plus a purely dispersive term, and one can call this *scattering to* Q_{λ_0}. If $\lambda(t) \to 0$ as $t \to \infty$ then this means that the solution is concentrating $\mathcal{E}(Q)$ worth of energy at the origin as $t \to \infty$ and we refer to this phenomenon as *infinite time blow-up*. Finally, if $\lambda(t) \to \infty$ as $t \to \infty$ then the solution can be thought of as concentrating $\mathcal{E}(Q)$ worth of energy at spacial infinity as $t \to \infty$ and we call this *infinite time flattening*.

We also would like to highlight the fact that global solutions of the type mentioned above, i.e., infinite time blow-up and flattening, have been constructed in the case of the $3d$ semi-linear focusing energy critical wave equation by Donninger and Krieger in [10]. No constructions of this type are known at this point for the energy critical wave maps studied here. In addition, a classification of all the possible dynamics for maps in $\mathcal{H}_1$ at energy levels $\geq 3\mathcal{E}(Q)$ remains open.

Remark 2. We emphasize that [7] goes hand-in-hand with this article and the two papers are intended to be read together. In fact, part (1) of Theorem 1.1 was established in [7, Theorem 1.3]. Therefore, in order to complete the proof of Theorem 1.1 we need to prove only part (2) and the rest of this paper will be devoted to that goal. The broad outline of the proof of Theorem 1.1 (2) is similar in nature to the proof of part (1). With this is mind we will often refer the reader to [7] where the details are nearly identical instead of repeating the same arguments here.

Remark 3. We remark that Theorem 1.1 is reminiscent of the recent works of Duyckaerts, the second author, and Merle in [11, 13, 12, 14] for the energy critical semi-linear focusing wave equation in 3 spacial dimensions and again we refer the reader to [7] for a more detailed description of the similarities and differences between these papers.

Remark 4. Finally, we would like to note that the same observations in [7, Appendix A] regarding 1-equivariant wave maps to more general targets, higher equivariance classes and the $4d$ equivariant Yang-Mills system hold in the context of the global statement in Theorem 1.1.

Acknowledgments. This first author wishes to thank the University of Chicago for its hospitality during the academic year 2011-12.

2. Preliminaries. For the reader's convenience, we recall a few facts and notations from [7] that are used frequently in what follows. We define the 1-equivariant energy space to be

$$\mathcal{H} = \{\vec{U} \in \dot{H}^1 \times L^2(\mathbb{R}^2;\mathbb{S}^2) \mid U \circ \rho = \rho \circ U,\ \forall \rho \in \mathrm{SO}(2)\}.$$

$\mathcal{H}$ is endowed with the norm

$$\mathcal{E}(\vec{U}(t)) = \|\vec{U}(t)\|^2_{\dot{H}^1 \times L^2(\mathbb{R}^2;\mathbb{S}^2)} = \int_{\mathbb{R}^2} \left(|\partial_t U|_g^2 + |\nabla U|_g^2\right) dx. \tag{2.1}$$

As noted in the introduction, by our equivariance condition we can write $U(t,r,\omega) = (\psi(t,r),\omega)$ and the energy of a wave map becomes

$$\mathcal{E}(U,\partial_t U)(t) = \mathcal{E}(\psi,\psi_t)(t) = \int_0^\infty \left(\psi_t^2 + \psi_r^2 + \frac{\sin^2(\psi)}{r^2}\right) r\, dr = \text{const.} \tag{2.2}$$

We also define the localized energy as follows: Let $r_1, r_2 \in [0,\infty)$. Then

$$\mathcal{E}_{r_1}^{r_2}(\vec{\psi}(t)) := \int_{r_1}^{r_2} \left(\psi_t^2 + \psi_r^2 + \frac{\sin^2(\psi)}{r^2}\right) r\, dr.$$

Following Shatah and Struwe [29], we set

$$G(\psi) := \int_0^\psi |\sin \rho|\, d\rho. \tag{2.3}$$

Observe that for any $(\psi,0) \in \mathcal{H}$ and for any $r_1, r_2 \in [0,\infty)$ we have

$$\begin{aligned} |G(\psi(r_2)) - G(\psi(r_1))| &= \left| \int_{\psi(r_1)}^{\psi(r_2)} |\sin \rho|\, d\rho \right| \\ &= \left| \int_{r_1}^{r_2} |\sin(\psi(r))|\, \psi_r(r)\, dr \right| \le \frac{1}{2} \mathcal{E}_{r_1}^{r_2}(\psi,0). \end{aligned} \tag{2.4}$$

We also recall from [7] the definition of the space $H \times L^2$.

$$\|(\psi_0,\psi_1)\|^2_{H \times L^2} := \int_0^\infty \left(\psi_1^2 + (\psi_0)_r^2 + \frac{\psi_0^2}{r^2}\right) r\, dr. \tag{2.5}$$

We note that for degree zero maps $(\psi_0,\psi_1) \in \mathcal{H}_0$ the energy is comparable to the $H \times L^2$ norm provided the L^∞ norm of ψ_0 is uniformly bounded below π. This equivalence of norms is detailed in [7, Lemma 2.1], see also [8, Lemma 2]. The space $H \times L^2$ is not defined for maps $(\psi_0,\psi_1) \in \mathcal{H}_1$, but one can instead consider the $H \times L^2$ norm of $(\psi_0 - Q_\lambda, 0)$ for $\lambda \in (0,\infty)$, and $Q_\lambda(r) = Q(r/\lambda)$. In fact, for maps $\vec{\psi} \in \mathcal{H}_1$ such that $\mathcal{E}(\vec{\psi}) - \mathcal{E}(Q)$ is small, one can choose $\lambda > 0$ so that

$$\|(\psi_0 - Q_\lambda, \psi_1)\|^2_{H \times L^2} \simeq \mathcal{E}(\vec{\psi}) - \mathcal{E}(Q).$$

This amounts to the coercivity of the energy near Q up to the scaling symmetry. For more details we refer the reader to [6, Proposition 4.3], [7, Lemma 2.5], and [2].

2.1. Properties of global wave maps. We will need a few facts about global solutions to (1.2). The results in this section constitute slight refinements and a few consequences of the work of Shatah and Tahvildar-Zadeh in [31, Section 3.1] on global equivariant wave maps and originate in the work of Christodoulou and Tahvildar-Zadeh on spherically symmetric wave maps, see [4].

PROPOSITION 2.1. *Let $\vec{\psi}(t) \in \mathcal{H}$ be a global wave map. Let $0 < \lambda < 1$. Then we have*

$$\limsup_{t\to\infty} \mathcal{E}_{\lambda t}^{t-A}(\vec{\psi}(t)) \longrightarrow 0 \quad \text{as } A \longrightarrow \infty. \tag{2.6}$$

In fact, we have

$$\mathcal{E}_{\lambda t}^{t-A}(\vec{\psi}(t)) \longrightarrow 0 \quad \text{as } t,\, A \longrightarrow \infty \text{ for } A \le (1-\lambda)t. \tag{2.7}$$

We note that Proposition 2.1 is a refinement of [31, (3.4)], see also [4, Corollary 1] where the case of spherically symmetric wave maps is considered. To prove this result, we follow [4, 31, 29] and introduce the following quantities:

$$\begin{aligned} e(t,r) &:= \psi_t^2(t,r) + \psi_r^2(t,r) + \frac{\sin^2(\psi(t,r))}{r^2} \\ m(t,r) &:= 2\psi_t(t,r)\psi_r(t,r). \end{aligned}$$

Observe that with this notation the energy identity becomes:

$$\partial_t e(t,r) = \frac{1}{r}\partial_r\left(r\, m(t,r)\right), \tag{2.8}$$

which we can conveniently rewrite as

$$\partial_t(re(t,r)) - \partial_r(r\, m(t,r)) = 0. \tag{2.9}$$

Using the notation in [4], we set

$$\begin{aligned} \alpha^2(t,r) &:= r(e(t,r) + m(t,r)) \\ \beta^2(t,r) &:= r(e(t,r) - m(t,r)) \end{aligned}$$

and we define null coordinates

$$u = t - r, \quad v = t + r.$$

Next, for $0 \le \lambda < 1$ set

$$\mathscr{E}_\lambda(u) := \int_{\frac{1+\lambda}{1-\lambda}u}^{\infty} \alpha^2(u,v)\,dv \tag{2.10}$$

$$\mathscr{F}(u_0,u_1) := \lim_{v\to\infty} \int_{u_0}^{u_1} \beta^2(u,v)\,du. \tag{2.11}$$

Also, let $\mathscr{C}_\rho^\pm$ denote the interior of the forward (resp. backward) light-cone with vertex at $(t,r) = (\rho,0)$ for $\rho > 0$ in (t,r) coordinates.

As in [31, Section 3.1], one can show that the integral in (2.10) and the limit in (2.11) exist for a wave map of finite energy, see also [4, Section 2] for the details of the argument for the spherically symmetric case.

By integrating the energy identity (2.9) over the region $(\mathscr{C}_{u_0}^+ \backslash \mathscr{C}_{u_1}^+) \cap \mathscr{C}_{v_0}^-$, where $0 < u_0 < u_1 < v_0$, we obtain the identity

$$\int_{u_0}^{u_1} \beta^2(u,v)\,du = \int_{u_0}^{v_0} \alpha^2(u_0,v)\,dv - \int_{u_1}^{v_0} \alpha^2(u_1,v)\,dv.$$

Letting $v_0 \to \infty$ we see that

$$0 \le \mathscr{F}(u_0,u_1) = \mathscr{E}_0(u_0) - \mathscr{E}_0(u_1), \tag{2.12}$$

which shows that $\mathscr{E}_0$ is decreasing. Next, note that

$$\mathscr{F}(u,u_2) = \mathscr{F}(u,u_1) + \mathscr{F}(u_1,u_2) \ge \mathscr{F}(u,u_1)$$

for $u_2 > u_1$, and thus $\mathscr{F}(u,u_1)$ is increasing in u_1. $\mathscr{F}(u,u_1)$ is also bounded above by $\mathscr{E}(u)$ so

$$\mathscr{F}(u) := \lim_{u_1\to\infty} \mathscr{F}(u,u_1)$$

exists and, as in [31, 4], we have

$$\mathscr{F}(u) \longrightarrow 0 \quad \text{as } u \longrightarrow \infty. \tag{2.13}$$

Finally note that the argument in [4, Lemma 1] shows that for all $0 < \lambda < 1$ we have

$$\mathscr{E}_\lambda(u) \longrightarrow 0 \quad \text{as } u \longrightarrow \infty, \tag{2.14}$$

which is stated in [31, (3.3)]. To deduce (2.14), follow the exact argument in [4, proof of Lemma 1] using the relevant multiplier inequalities for equivariant wave maps established in [29, proof of Lemma 8.2] in place of [4, equation (6)]. We can now prove Proposition 2.1.

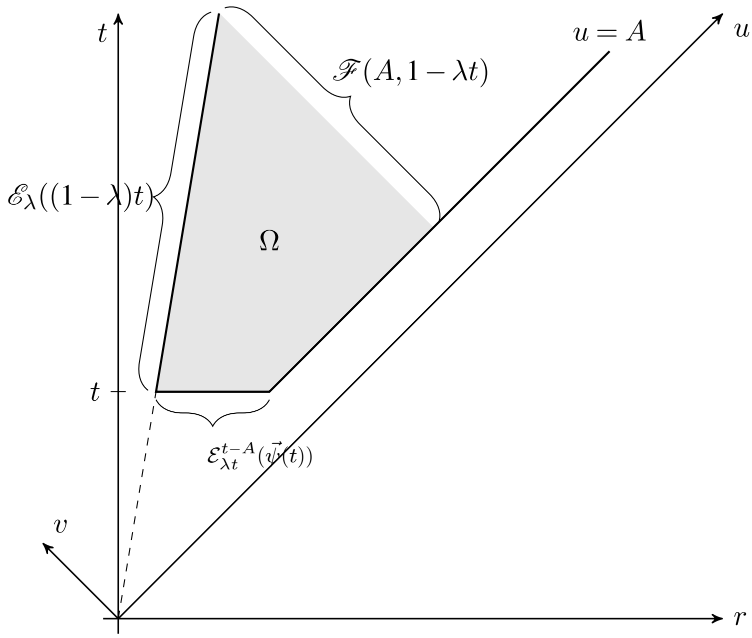

Figure 1. The quadrangle Ω over which the energy identity is integrated is the gray region above.

Proof of Proposition 2.1. Fix $\lambda \in (0,1)$ and $\delta > 0$. Find A_0 and T_0 large enough so that

$$0 \le \mathscr{F}(A) \le \delta, \quad 0 \le \mathscr{E}_\lambda((1-\lambda)t) \le \delta$$

for all $A \ge A_0$ and $t \ge T_0$. In (u,v)-coordinates consider the points

$$X_1 = ((1-\lambda)t, (1+\lambda)t), \quad X_2 = (A, 2t - A)$$
$$X_3 = (A, \bar{v}), \quad X_4 = ((1-\lambda)t, \bar{v})$$

where $\bar{v}$ is very large. Integrating the energy identity (2.9) over the region Ω bounded by the line segments X_1X_2, X_2X_3, X_3X_4, X_4X_1 we obtain,

$$\mathcal{E}_{\lambda t}^{t-A}(\vec{\psi}(t)) = -\int_{2t-A}^{\bar{v}} \alpha^2(A,v)\,dv + \int_A^{(1-\lambda)t} \beta^2(u,\bar{v})\,du + \int_{(1+\lambda)t}^{\bar{v}} \alpha^2((1-\lambda)t, v)\,dv.$$

Letting $\bar{v} \to \infty$ above and recalling that $\mathscr{F}(u,u_1)$ is increasing in u_1 we have

$$\mathcal{E}_{\lambda t}^{t-A}(\vec{\psi}(t)) \le \mathscr{E}_\lambda((1-\lambda)t) + \mathscr{F}(A,(1-\lambda)t) \le \mathscr{E}_\lambda((1-\lambda)t) + \mathscr{F}(A).$$

The proposition now follows from (2.14) and (2.13). $\square$

We will also need the following corollaries of Proposition 2.1:

COROLLARY 2.2. *Let $\vec{\psi}(t) \in \mathcal{H}$ be a global wave map. Then*

$$\limsup_{T\to\infty} \frac{1}{T}\int_A^T \int_0^{t-A} \psi_t^2(t,r) r\,dr\,dt \longrightarrow 0 \quad \textit{as } A \longrightarrow \infty. \tag{2.15}$$

Proof. We will use the following virial identity for solutions to (1.2):

$$\partial_t(r^2 m) - \partial_r(r^2\psi_t^2 + r^2\psi_r^2 - \sin^2\psi) + 2r\psi_t^2 = 0. \tag{2.16}$$

Now, fix $\delta > 0$ so that $\delta < 1/3$ and find A_0, T_0 so that for all $A \geq A_0$ and $t \geq T_0$ we have

$$\mathcal{E}_{\delta t}^{t-A}(\vec{\psi}(t)) \leq \delta.$$

Then,

$$\int_0^{\delta t} e(t,r) r^2\,dr \leq \mathcal{E}(\vec{\psi}(t))\delta t$$

and as long as we ensure that $A \leq 1/3t$, we obtain

$$\int_{\delta t}^{2t/3} e(t,r)\, r^2\,dr \leq \delta t.$$

This implies that

$$\int_0^{2t/3} e(t,r) r^2\,dr \leq C\delta t \quad \text{and} \quad \int_0^{2t/3} e(t,r) r^3\,dr \leq C\delta t^2.$$

Let $\chi : \mathbb{R} \to [0,1]$ be a smooth cut-off function such that $\chi(x) = 1$ for $|x| \leq 1/3$, $\chi(x) = 0$ for $|x| \geq 2/3$ and $\chi'(x) \leq 0$. Then, using the virial identity (2.16) we have

$$\begin{aligned}
\frac{d}{dt}\int_0^\infty m(t,r)\chi(r/t) r^2\,dr &= \int_0^\infty \partial_t(r^2 m(t,r))\chi(r/t)\,dr - \frac{2}{t^2}\int_0^\infty \psi_t\psi_r\, r^3\chi'(r/t)\,dr \\
&= \int_0^\infty \partial_r(r^2(\psi_t^2+\psi_r^2) - \sin^2(\psi))\chi(r/t)\,dr \\
&\quad - 2\int_0^\infty \psi_t^2(t,r)\chi(r/t)\, r\,dr + O(\delta) \\
&= \frac{1}{t^2}\int_0^\infty (r^2(\psi_t^2+\psi_r^2) - \sin^2(\psi))\chi'(r/t)\, r\,dr \\
&\quad - 2\int_0^\infty \psi_t^2(t,r)\chi(r/t)\, r\,dr + O(\delta) \\
&= -2\int_0^\infty \psi_t^2(t,r)\chi(r/t)\, r\,dr + O(\delta).
\end{aligned}$$

Integrating in t between 0 and T yields

$$\int_0^T \int_0^\infty \psi_t^2(t,r)\chi(r/t)\, r\, dr\, dt \le C\delta T$$

with an absolute constant $C > 0$. By the definition of $\chi(x)$ this implies

$$\int_0^T \int_0^{t/3} \psi_t^2(t,r)\, r\, dr\, dt \le C\delta T.$$

Next, note that we have

$$\begin{aligned}\int_A^T \int_{t/3}^{t-A} \psi_t^2(t,r)\, r\, dr\, dt &\le \int_A^{T_0} \mathcal{E}(\vec{\psi})\, dt + \int_{T_0}^T \int_{t/3}^{t-A} e(t,r)\, r\, dr\, dt \\ &\le (T_0 - A)\mathcal{E}(\vec{\psi}) + (T - T_0)\delta.\end{aligned}$$

Therefore,

$$\frac{1}{T}\int_A^T \int_0^{t-A} \psi_t^2(t,r)\, r\, dr\, dt \le C\delta + \frac{T_0}{T}\mathcal{E}(\vec{\psi}).$$

Hence,

$$\limsup_{T\to\infty} \frac{1}{T}\int_A^T \int_0^{t-A} \psi_t^2(t,r)\, r\, dr\, dt \le C\delta$$

for all $A \ge A_0$, which proves (2.15). □

COROLLARY 2.3. *Let $\vec{\psi}(t) \in \mathcal{H}$ be a smooth global wave map. Recall that $\vec{\psi}(t) \in \mathcal{H}$ implies that there exists $k \in \mathbb{Z}$ such that for all t we have $\psi(t,\infty) = k\pi$. Then for any $\lambda > 0$ we have*

$$\|\psi(t) - \psi(t,\infty)\|_{L^\infty(r\ge\lambda t)} \longrightarrow 0 \quad \text{as } t \longrightarrow \infty. \tag{2.17}$$

Before proving Corollary 2.3, we can combine Proposition 2.1 and Corollary 2.3 to immediately deduce the following result.

COROLLARY 2.4. *Let $\vec{\psi}(t) \in \mathcal{H}$ be a global wave map. Let $0 < \lambda < 1$. Then we have*

$$\limsup_{t\to\infty} \|\vec{\psi}(t) - (\psi(t,\infty), 0)\|^2_{H\times L^2(\lambda t \le r \le t-A)} \longrightarrow 0 \quad \text{as } A \longrightarrow \infty. \tag{2.18}$$

Proof. Say $\vec{\psi}(t) \in \mathcal{H}_k$. Observe that Corollary 2.3 shows that for t_0 large enough we have, say,

$$|\psi(t,r) - k\pi| \le \frac{\pi}{100}$$

for all $t \geq t_0$ and $r \geq \lambda t$. This in turn implies that for $t \geq t_0$ we can find a $C > 0$ such that

$$|\psi(t,r) - k\pi|^2 \leq C \sin^2(\psi(t,r)) \quad \forall t \geq t_0,\ r \geq \lambda t.$$

Now (2.18) follows directly from (2.6). □

The first step in the proof of Corollary 2.3 is the following lemma:

LEMMA 2.5. *Let $\vec{\psi}(t) \in \mathcal{H}$ be a smooth global wave map. Let $R > 0$ and suppose that the initial data $\vec{\psi}(0) = (\psi_0, \psi_1) \in \mathcal{H}_1$ satisfies* $\mathrm{supp}(\partial_r \psi_0), \mathrm{supp}(\psi_1) \subset B(0,R)$. *Then for any $t \geq 0$ and for any $A < t$ we have*

$$\|\psi(t) - \psi(t,\infty)\|_{L^\infty(r \geq t-A)} \leq \sqrt{\mathcal{E}(\vec{\psi})}\sqrt{\frac{A+R}{t-A}}. \tag{2.19}$$

Proof. By the finite speed of propagation we note that for each $t \geq 0$ we have $\mathrm{supp}(\psi_r(t)) \subset B(0, R+t)$. Hence, for all $t \geq 0$ we have

$$\begin{aligned}
|\psi(t,r) - \psi(t,\infty)| &\leq \int_r^\infty \left|\psi_r(t,r')\right| dr' \\
&\leq \left(\int_r^{R+t} \psi_r^2(t,r')\, r'\, dr'\right)^{\frac{1}{2}} \left(\int_r^{R+t} \frac{1}{r'}\, dr'\right)^{\frac{1}{2}} \\
&\leq \sqrt{\mathcal{E}(\vec{\psi})}\sqrt{\log\left(\frac{t+R}{r}\right)}.
\end{aligned}$$

Next observe that if $r \geq t - A$ then

$$\log\left(\frac{t+R}{r}\right) \leq \log\left(1 + \frac{A+R}{r}\right) \leq \log\left(1 + \frac{A+R}{t-A}\right) \leq \frac{A+R}{t-A}.$$

This proves (2.19). □

Proof of Corollary 2.3. Say $\psi(t) \in \mathcal{H}_k$, that is $\psi(t,\infty) = k\pi$ for all t. First observe that by an approximation argument, it suffices to consider wave maps $\vec{\psi}(t) \in \mathcal{H}_k$ with initial data $\vec{\psi}(0) = (\psi_0, \psi_1) \in \mathcal{H}_k$ with

$$\mathrm{supp}(\partial_r \psi_0), \mathrm{supp}(\psi_1) \subset B(0,R)$$

for $R > 0$ arbitrary, but fixed. Now, let $t_n \to \infty$ be any sequence and set

$$A_n := \sqrt{t_n}.$$

Then, for each $r \geq \lambda t_n$ we have

$$|\psi(t_n, r) - k\pi| \leq \|\psi(t_n) - k\pi\|_{L^\infty(\lambda t_n \leq r \leq t_n - A_n)} + \|\psi(t_n) - k\pi\|_{L^\infty(r \geq t_n - A_n)}.$$

By Lemma 2.5 we know that

$$\|\psi(t_n)-k\pi\|_{L^\infty(r\geq t_n-A_n)}\leq\sqrt{\mathcal{E}(\psi)}\sqrt{\frac{\sqrt{t_n}+R}{t_n-\sqrt{t_n}}}\longrightarrow 0\quad\text{as } n\longrightarrow\infty. \tag{2.20}$$

Hence it suffices to show that

$$\|\psi(t_n)-k\pi\|_{L^\infty(\lambda t_n\leq r\leq t_n-A_n)}\longrightarrow 0\quad\text{as } n\longrightarrow\infty.$$

To see this, first observe that (2.20) implies that

$$\psi(t_n,t_n-A_n)\longrightarrow k\pi$$

as $n\to\infty$. Therefore it is enough to show that

$$\|\psi(t_n)-\psi(t_n,t_n-A_n)\|_{L^\infty(\lambda t_n\leq r\leq t_n-A_n)}\longrightarrow 0\quad\text{as } n\longrightarrow\infty. \tag{2.21}$$

With G defined as in (2.3) we can combine (2.4) and Proposition 2.1 to deduce that for all $r\geq\lambda t_n$ we have

$$|G(\psi(t_n,r))-G(\psi(t_n,t_n-A_n))|\leq\frac{1}{2}\mathcal{E}_{\lambda t_n}^{t_n-A_n}(\vec{\psi}(t_n))\longrightarrow 0$$

as $n\to\infty$. This immediately implies (2.21) since G is a continuous, increasing function. □

3. Profiles for global degree one solutions with energy below $3\mathcal{E}(Q)$. In this section we carry out the proof of Theorem 1.1(2). We start by first deducing the conclusions along a sequence of times. To be specific, we establish the following proposition:

PROPOSITION 3.1. *Let $\psi(t)\in\mathcal{H}_1$ be a global solution to* (1.2) *with*

$$\mathcal{E}(\vec{\psi})=\mathcal{E}(Q)+\eta<3\mathcal{E}(Q).$$

Then there exist a sequence of times $\tau_n\to\infty$, a sequence of scales $\lambda_n\ll\tau_n$, a solution $\vec{\varphi}_L(t)\in\mathcal{H}_0$ to the linear wave equation (1.6)*, and a decomposition*

$$\vec{\psi}(\tau_n)=\vec{\varphi}_L(\tau_n)+(Q(\cdot/\lambda_n),0)+\vec{\epsilon}(\tau_n) \tag{3.1}$$

such that $\vec{\epsilon}(\tau_n)\in\mathcal{H}_0$ and $\vec{\epsilon}(\tau_n)\to 0$ in $H\times L^2$ as $n\to\infty$.

To prove Proposition 3.1 we proceed in several steps. We first construct the sequences τ_n and λ_n while identifying the large profile, $Q(\cdot/\lambda_n)$. Once we have done this, we extract the radiation term φ_L. In the last step, we prove strong convergence of the error

$$\vec{\epsilon}(\tau_n) := \vec{\psi}(\tau_n) - \vec{\varphi}_L(\tau_n) - (Q(\cdot/\lambda_n), 0) \longrightarrow 0$$

in the space $H \times L^2$.

3.1. The harmonic map at $t = +\infty$. Here we prove the analog of Struwe's result [35, Theorem 2.1] for global wave maps of degree different than zero, i.e., $\psi(t) \in \mathcal{H} \backslash \mathcal{H}_0$ for all $t \in [0, \infty)$. This will allow us to identify the sequences τ_n, λ_n and the harmonic maps $Q(\cdot/\lambda_n)$ in the decomposition (3.1).

THEOREM 3.2. *Let $\vec{\psi}(t) \in \mathcal{H} \backslash \mathcal{H}_0$ be a smooth, global solution to* (1.2). *Then, there exists a sequence of times $t_n \to \infty$ and a sequence of scales $\lambda_n \ll t_n$ so that the following results hold: Let*

$$\vec{\psi}_n(t,r) := \big(\psi(t_n + \lambda_n t, \lambda_n r), \lambda_n \dot{\psi}(t_n + \lambda_n t, \lambda_n r)\big) \tag{3.2}$$

be the global wave map evolutions associated to the initial data

$$\vec{\psi}_n(r) := (\psi(t_n, \lambda_n r), \lambda_n \dot{\psi}(t_n, \lambda_n, r)).$$

Then, there exists $\lambda_0 > 0$ so that

$$\vec{\psi}_n \longrightarrow (\pm Q(\cdot/\lambda_0), 0) \quad \textit{in } L^2_t([0,1); H^1 \times L^2)_{\mathrm{loc}}.$$

We begin with the following lemma, which follows from Corollary 2.2 and is the global-in-time version of [7, Corollary 2.9]. The statement and proof are also very similar to [12, Lemma 4.4] and [11, Corollary 5.3].

LEMMA 3.3. *Let $\vec{\psi}(t) \in \mathcal{H}$ be a smooth global wave map. Let $A : (0,\infty) \to (0,\infty)$ be any increasing function such that $A(t) \nearrow \infty$ as $t \to \infty$ and $A(t) \le t$ for all t. Then, there exists a sequence of times $t_n \to \infty$ such that*

$$\lim_{n\to\infty} \sup_{\sigma>0} \frac{1}{\sigma} \int_{t_n}^{t_n+\sigma} \int_0^{t-A(t_n)} \dot{\psi}^2(t,r)\, r\, dr\, dt = 0. \tag{3.3}$$

Proof. The proof is analogous to the argument given in [11, Corollary 5.3]. We argue by contradiction. The existence of a sequence of times t_n satisfying (3.3) is equivalent to the statement

$$\forall A(t) \nearrow \infty \text{ with } A(t) \le t \text{ as } t \longrightarrow \infty,\ \forall \delta > 0,\ \forall T_0 > 0,\ \exists \tau \ge T_0$$
$$\text{so that } \sup_{\sigma>0} \frac{1}{\sigma} \int_{\tau}^{\tau+\sigma} \int_0^{t-A(\tau)} \dot{\psi}^2(t,r)\, r\, dr\, dt \le \delta.$$

So we assume that (3.3) fails. Then,

$$\exists A(t) \nearrow \infty \text{ with } A(t) \le t \text{ as } t \longrightarrow \infty,\ \exists \delta > 0,\ \exists T_0 > 0,\ \forall \tau \ge T_0,\ \exists \sigma > 0 \\ \text{so that } \frac{1}{\sigma}\int_{\tau}^{\tau+\sigma}\int_0^{t-A(\tau)} \dot{\psi}^2(t,r)\, r\, dr\, dt > \delta. \tag{3.4}$$

Now, by Corollary 2.2 we can find a large A_1 and a $T_1 = T_1(A_1) > T_0$ so that for all $T \ge T_1$ we have

$$\frac{1}{T}\int_{A_1}^{T}\int_0^{t-A_1} \dot{\psi}^2(t,r)\, r\, dr\, dt \le \delta/100. \tag{3.5}$$

Since $A(t) \nearrow \infty$ we can fix $T > T_1$ large enough so that $A(t) \ge A_1$ for all $t \ge T$. Define the set X as follows:

$$X := \left\{ \sigma > 0 : \frac{1}{\sigma}\int_T^{T+\sigma}\int_0^{t-A(T)} \dot{\psi}^2(t,r)\, r\, dr\, dt \ge \delta \right\}.$$

Then X is nonempty by (3.4). Define $\rho := \sup X$. We claim that $\rho \le T$. To see this assume that there exists $\sigma \in X$ so that $\sigma \ge T$. Then we would have

$$T + \sigma \le 2\sigma.$$

This in turn implies, using (3.5), that

$$\frac{1}{2\sigma}\int_T^{T+\sigma}\int_0^{t-A(T)} \dot{\psi}^2(t,r)\, r\, dr\, dt \le \frac{1}{T+\sigma}\int_{A_1}^{T+\sigma}\int_0^{t-A_1} \dot{\psi}^2(t,r)\, r\, dr\, dt \le \delta/100$$

where we have also used the fact that $A(T) \ge A_1$. This would mean that

$$\frac{1}{\sigma}\int_T^{T+\sigma}\int_0^{t-A(T)} \dot{\psi}^2(t,r)\, r\, dr\, dt \le \delta/50,$$

which is impossible since we assumed that $\sigma \in X$. Therefore $\rho \le T$. Moreover, we know that

$$\int_T^{T+\rho}\int_0^{T-A(T)} \dot{\psi}^2(t,r)\, r\, dr\, dt \ge \delta\rho. \tag{3.6}$$

Now, since $T + \rho > T > T_1 > T_0$ we know that there exists $\sigma > 0$ so that

$$\int_{T+\rho}^{T+\rho+\sigma}\int_0^{t-A(T+\rho)} \dot{\psi}^2(t,r)\, r\, dr\, dt > \delta\sigma.$$

Since $A(t)$ is increasing, we have $A(T) \leq A(T+\rho)$ and hence the above implies that

$$\int_{T+\rho}^{T+\rho+\sigma} \int_0^{t-A(T)} \dot{\psi}^2(t,r)\, r\, dr\, dt > \delta\sigma. \tag{3.7}$$

Summing (3.6) and (3.7) we get

$$\int_{T}^{T+\rho+\sigma} \int_0^{t-A(T)} \dot{\psi}^2(t,r)\, r\, dr\, dt > \delta(\sigma+\rho),$$

which means that $\rho+\sigma \in X$. But this contradicts that fact that $\rho = \sup X$. $\square$

The rest of the proof of Theorem 3.2 will follow the same general outline of [35, proof of Theorem 2.1]. Let $\vec{\psi}(t) \in \mathcal{H}_1$ be a smooth global wave map.

We begin by choosing a scaling parameter. Let $\delta_0 > 0$ be a small number, for example $\delta_0 = 1$ would work. For each $t \in (0,\infty)$ choose $\lambda(t)$ so that

$$\delta_0 \leq \mathcal{E}_0^{2\lambda(t)}(\vec{\psi}(t)) \leq 2\delta_0. \tag{3.8}$$

Then using the monotonicity of the energy on interior cones we know that for each $|\tau| \leq \lambda(t)$ we have

$$\mathcal{E}_0^{\lambda(t)}(\vec{\psi}(t+\tau)) \leq \mathcal{E}_0^{2\lambda(t)-|\tau|}(\vec{\psi}(t+\tau)) \leq \mathcal{E}_0^{2\lambda(t)}(\vec{\psi}(t)) \leq 2\delta_0. \tag{3.9}$$

Similarly, we have

$$\delta_0 \leq \mathcal{E}_0^{2\lambda(t)+|\tau|}(\vec{\psi}(t+\tau)) \leq \mathcal{E}_0^{3\lambda(t)}(\vec{\psi}(t+\tau)). \tag{3.10}$$

LEMMA 3.4. *Let $\vec{\psi}(t) \in \mathcal{H}\backslash\mathcal{H}_0$ and $\lambda(t)$ be defined as above. Then we have $\lambda(t) \ll t$ as $t \to \infty$.*

Proof. Suppose $\vec{\psi} \in \mathcal{H}_k$ for $k \geq 1$. It suffices to show that for all $\lambda > 0$ we have $\lambda(t) \leq \lambda t$ for all t large enough. Fix $\lambda > 0$. By Corollary 2.3 we have

$$\|\psi(t) - k\pi\|_{L^\infty(r \geq \lambda t)} \longrightarrow 0 \tag{3.11}$$

as $t \to \infty$. For the sake of finding a contradiction, suppose that there exists a sequence $t_n \to \infty$ with $\lambda(t_n) \geq \lambda t_n$ for all $n \in \mathbb{N}$. By (2.4) and (3.11) we would then have that

$$\mathcal{E}_0^{2\lambda(t_n)}(\vec{\psi}(t_n)) \geq \mathcal{E}_0^{\lambda t_n}(\vec{\psi}(t_n)) \geq 2G(\psi(t_n,\lambda t_n)) \longrightarrow 2G(k\pi) \geq 4 > 2\delta_0,$$

which contradicts (3.8) as long as we ensure that $\delta_0 < 2$. $\square$

We can now complete the proof of Theorem 3.2.

Proof of Theorem 3.2. Let $\lambda(t)$ be defined as in (3.8). Choose another scaling parameter $A(t)$ so that $A(t) \to \infty$ and $\lambda(t) \le A(t) \ll t$ for $t \to \infty$, for example one could take $A(t) := \max\{\tilde{\lambda}(t), t^{1/2}\}$ where $\tilde{\lambda}(t) := \sup_{0 \le s \le t} \lambda(s)$. By Lemma 3.3 we can find a sequence $t_n \to \infty$ so that by setting $\lambda_n := \lambda(t_n)$ and $A_n := A(t_n)$ we have

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \int_{t_n}^{t_n+\lambda_n} \int_0^{t-A_n} \dot{\psi}^2(t,r)\, r\, dr\, dt = 0.$$

Now define a sequence of global wave maps $\vec{\psi}_n(t) \in \mathcal{H}\backslash\mathcal{H}_0$ by

$$\vec{\psi}_n(t,r) := \big(\psi(t_n+\lambda_n t, \lambda_n r), \lambda_n \dot{\psi}(t_n+\lambda_n t, \lambda_n r)\big)$$

and write the full wave maps in coordinates on $\mathbb{S}^2$ as $U_n(t,r,\omega) := (\psi_n(t,r),\omega)$. Observe that we have

$$\int_0^1 \int_0^{r_n} \dot{\psi}_n^2(t,r)\, r\, dr\, dt \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \tag{3.12}$$

where $r_n := (t_n - A_n)/\lambda_n \to \infty$ as $n \to \infty$ by our choice of A_n. Also note that

$$\mathcal{E}(\vec{\psi}_n(t)) = \mathcal{E}(\vec{\psi}(t_n+\lambda_n t)) = \mathcal{E}(\vec{\psi}) = C.$$

This implies that the sequence $\vec{\psi}_n$ is uniformly bounded in $L_t^\infty(\dot{H}^1 \times L^2)$. Note that (2.4) implies that ψ_n is uniformly bounded in $L_t^\infty L_x^\infty$. Hence we can extract a further subsequence so that

$$\vec{\psi}_n \rightharpoonup \vec{\psi}_\infty \quad \text{weakly in } L_t^2(H^1 \times L^2)_{\text{loc}}$$

and, in fact, locally uniformly on $[0,1) \times (0,\infty)$. By (3.12), the limit

$$\vec{\psi}_\infty(t,r) = (\psi_\infty(r), 0) \quad \forall (t,r) \in [0,1) \times (0,\infty)$$

and is thus a time-independent weak solution to (1.2) on $[0,1) \times (0,\infty)$. This means that the corresponding full, weak wave map $\tilde{U}_\infty(t,r,\omega) = U_\infty(r,\omega) := (\psi_\infty(r),\omega)$ is a time-independent weak solution to (1.1) on $[0,1) \times \mathbb{R}^2 \setminus \{0\}$. By Hélein's theorem [16, Theorem 2],

$$U_\infty : \mathbb{R}^2 \setminus \{0\} \longrightarrow \mathbb{S}^2$$

is a smooth finite energy, co-rotational harmonic map. By Sacks-Uhlenbeck [27], we can then extend U_∞ to a smooth finite energy, co-rotational harmonic map $U : \mathbb{R}^2 \to \mathbb{S}^2$. Writing $U(r,\omega) = (\psi_\infty(r),\omega)$, we have either $\psi_\infty \equiv 0$ or $\psi_\infty = \pm Q(\cdot/\lambda_0)$ for some $\lambda_0 > 0$.

Following Struwe, we can also establish strong local convergence

$$\vec{\psi}_n \longrightarrow (\psi_\infty, 0) \quad \text{in } L_t^2([0,1); H^1 \times L^2)_{\text{loc}} \tag{3.13}$$

using the equation (1.1) and the local energy constraints from (3.9):

$$\mathcal{E}_0^1(\vec{\psi}_n(t)) \le 2\delta_0, \quad \mathcal{E}_0^1(\psi_\infty) \le 2\delta_0,$$

which hold uniformly in n for $|t| \le 1$. For the details of this argument we refer the reader to [35, Proof of Theorem 2.1(ii)]. Finally we note that the strong local convergence in (3.13) shows that indeed $\psi_\infty \not\equiv 0$ since by (3.10) we have

$$\delta_0 \le \mathcal{E}_0^3(\vec{\psi}_n(t))$$

uniformly in n for each $|t| \le 1$. Therefore we can conclude that there exists $\lambda_0 > 0$ so that $\psi_\infty(r) = \pm Q(r/\lambda_0)$. □

As in [7], the following consequences of Theorem 3.2, which hold for global degree one wave maps with energy below $3\mathcal{E}(Q)$, will be essential in what follows.

COROLLARY 3.5. *Let $\psi(t) \in \mathcal{H}_1$ be a smooth global wave map such that $\mathcal{E}(\vec{\psi}) < 3\mathcal{E}(Q)$. Then we have*

$$\psi_n - Q(\cdot/\lambda_0) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \text{ in } L_t^2([0,1);H)_{\text{loc}}, \tag{3.14}$$

with $\psi_n(t,r)$, $\{t_n\}$, $\{\lambda_n\}$, and λ_0 as in Theorem 3.2.

Corollary 3.5 is the global-in-time analog of [7, Corollary 2.13]. For the details, we refer the reader to [7, Proof of Lemma 2.11, Lemma 2.12, and Corollary 2.13]. At this point we note that we can, after a suitable rescaling, assume, without loss of generality, that λ_0 in Theorem 3.2, and Corollary 3.5, satisfies $\lambda_0 = 1$.

Arguing as in [7, Proof of Proposition 5.4] we can also deduce the following consequence of Theorem 3.2.

PROPOSITION 3.6. *Let $\psi(t) \in \mathcal{H}_1$ be a smooth global wave map such that $\mathcal{E}(\vec{\psi}) < 3\mathcal{E}(Q)$. Then, there exists a sequence $\alpha_n \to \infty$, a sequence of times $\tau_n \to \infty$, and a sequence of scales $\lambda_n \ll \tau_n$ with $\alpha_n\lambda_n \ll \tau_n$, so that*

(a) *As $n \to \infty$ we have*

$$\lim_{n\to\infty} \int_0^{\tau_n - A_n} \dot{\psi}^2(\tau_n, r)\, r\, dr \longrightarrow 0, \tag{3.15}$$

where $A_n \to \infty$ satisfies $\lambda_n \le A_n \ll \tau_n$.

(b) *As $n \to \infty$ we have*

(3.16)
$$\lim_{n\to\infty} \int_0^{\alpha_n\lambda_n} \left(\left| \psi_r(\tau_n, r) - \frac{Q_r(r/\lambda_n)}{\lambda_n} \right|^2 + \frac{|\psi(\tau_n, r) - Q(r/\lambda_n)|^2}{r^2} \right) r\, dr = 0.$$

Remark 5. Proposition 3.6 follows directly from Lemma 3.3, Corollary 3.5 and a diagonalization argument. As mentioned above, we refer the reader to [7, Proposition 5.4(a), (b)] for the details. Also note that $\tau_n \in [t_n, t_n + \lambda_n]$ where $t_n \to \infty$ is the sequence in Proposition 3.6. Finally $A_n := A(t_n)$ is the sequence that appears in the proof of Theorem 3.2.

As in [7] we will also need the following simple consequence of Proposition 3.6.

COROLLARY 3.7. *Let α_n, λ_n, and τ_n be defined as in Proposition 3.6. Let $\beta_n \to \infty$ be any sequence such that $\beta_n < c_0\alpha_n$ for some $c_0 < 1$. Then, for every $0 < c_1 < C_2$ such that $C_2 c_0 < 1$ there exists $\tilde{\beta}_n$ with $c_1\beta_n \le \tilde{\beta}_n \le C_2\beta_n$ such that*

$$\psi(\tau_n, \tilde{\beta}_n\lambda_n) \longrightarrow \pi \quad \text{as } n \longrightarrow \infty. \tag{3.17}$$

3.2. Extraction of the radiation term. In this section we construct what we will refer to as the radiation term, $\varphi_L(t) \in \mathcal{H}_0$ in the decomposition (3.1).

PROPOSITION 3.8. *Let $\psi(t) \in \mathcal{H}_1$ be a global wave map with $\mathcal{E}(\vec{\psi}) = \mathcal{E}(Q) + \eta < 3\mathcal{E}(Q)$. Then there exists a solution $\varphi_L(t) \in \mathcal{H}_0$ to the linear wave equation* (1.6) *so that for all $A \ge 0$ we have*

$$\|\vec{\psi}(t) - (\pi, 0) - \vec{\varphi}_L(t)\|_{H\times L^2(r\ge t-A)} \longrightarrow 0 \quad \text{as } t \longrightarrow \infty. \tag{3.18}$$

Moreover, for n large enough we have

$$\mathcal{E}(\vec{\varphi}_L(\tau_n)) \le C < 2\mathcal{E}(Q). \tag{3.19}$$

Proof. To begin we pick any $\alpha_n \to \infty$ and find τ_n, λ_n as in Proposition 3.6. Now let $\beta_n \to \infty$ be any other sequence such that $\beta_n \ll \alpha_n$. By Corollary 3.7 we can assume that

$$\psi(\tau_n, \beta_n\lambda_n) \longrightarrow \pi \tag{3.20}$$

as $n \to \infty$. We make the following definition:

$$\phi_n^0(r) = \begin{cases} \pi - \dfrac{\pi - \psi(\tau_n, \beta_n\lambda_n)}{\beta_n\lambda_n} r & \text{if } 0 \le r \le \beta_n\lambda_n \\ \psi(\tau_n, r) & \text{if } \beta_n\lambda_n \le r < \infty \end{cases} \tag{3.21}$$

$$\phi_n^1(r) = \begin{cases} 0 & \text{if } 0 \le r \le \beta_n\lambda_n \\ \dot{\psi}(\tau_n, r) & \text{if } \beta_n\lambda_n \le r < \infty. \end{cases} \tag{3.22}$$

We claim that $\vec{\phi}_n := (\phi_n^0, \phi_0^1) \in \mathcal{H}_{1,1}$ and $\mathcal{E}(\vec{\phi}_n) \le C < 2\mathcal{E}(Q)$. Clearly $\phi_n^0(0) = \pi$ and $\phi_n^0(\infty) = \pi$. We claim that

$$\mathcal{E}^{\infty}_{\beta_n\lambda_n}(\vec{\phi}_n) = \mathcal{E}^{\infty}_{\beta_n\lambda_n}(\vec{\psi}(\tau_n)) \le \eta + o_n(1). \tag{3.23}$$

Indeed, since $\psi(\tau_n, \beta_n\lambda_n) \to \pi$ we have $G(\psi(\tau_n, \beta_n\lambda_n)) \to 2 = \frac{1}{2}\mathcal{E}(Q)$ as $n \to \infty$. Therefore, by (2.4) we have

$$\mathcal{E}_0^{\beta_n\lambda_n}(\psi(\tau_n), 0) \ge 2G(\psi(\tau_n, \beta_n\lambda_n)) \ge \mathcal{E}(Q) - o_n(1)$$

for large n which proves (3.23) since $\mathcal{E}^{\infty}_{\beta_n\lambda_n}(\vec{\psi}(\tau_n)) = \mathcal{E}_0^{\infty}(\vec{\psi}(\tau_n)) - \mathcal{E}_0^{\beta_n\lambda_n}(\vec{\psi}(\tau_n))$.

We can also directly compute $\mathcal{E}_0^{\beta_n\lambda_n}(\phi_n^0, 0)$. Indeed,

$$\begin{aligned} \mathcal{E}_0^{\beta_n\lambda_n}(\phi_n^0, 0) &= \int_0^{\beta_n\lambda_n} \left(\frac{\pi - \psi(\tau_n, \beta_n\lambda_n)}{\beta_n\lambda_n}\right)^2 r\, dr \\ &\quad + \int_0^{\beta_n\lambda_n} \frac{\sin^2\left(\frac{\pi - \psi(\tau_n, \beta_n\lambda_n)}{\beta_n\lambda_n} r\right)}{r}\, dr \\ &\le C\,|\pi - \psi(\tau_n, \beta_n\lambda_n)|^2 \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned}$$

Hence $\mathcal{E}(\vec{\phi}_n) \le \eta + o_n(1)$. This means that for large enough n we have the uniform estimates $\mathcal{E}(\vec{\phi}_n) \le C < 2\mathcal{E}(Q)$. Therefore, by [7, Theorem 1.1], (which holds with exactly the same statement in $\mathcal{H}_{1,1}$ as in $\mathcal{H}_0 = \mathcal{H}_{0,0}$), we have that the wave map evolution $\vec{\phi}_n(t) \in \mathcal{H}_{1,1}$ with initial data $\vec{\phi}_n$ is global in time and scatters to π as $t \to \pm\infty$. The scattering statement means that for each n we can find initial data $\vec{\phi}_{n,L}$ so that the solution, $S(t)\vec{\phi}_{n,L}$, to the linear wave equation (1.6) satisfies

$$\|\vec{\phi}_n(t) - (\pi, 0) - S(t)\vec{\phi}_{n,L}\|_{H\times L^2} \longrightarrow 0 \quad \text{as } t \longrightarrow \infty.$$

Abusing notation, we set

$$\vec{\phi}_{n,L}(t) := S(t - \tau_n)\vec{\phi}_{n,L}.$$

By the definition of $\vec{\phi}_n$ and the finite speed of propagation observe that we have

$$\phi_n(t - \tau_n, r) = \psi(t, r) \quad \forall r \ge t - \tau_n + \beta_n\lambda_n.$$

Therefore, for all fixed m we have

$$\|\vec{\psi}(t) - (\pi, 0) - \vec{\phi}_{m,L}(t)\|_{H\times L^2(r \ge t - \tau_m + \beta_m\lambda_m)} \longrightarrow 0 \quad \text{as } t \longrightarrow \infty, \tag{3.24}$$

and, in particular

$$\|\vec{\phi}_n - (\pi, 0) - \vec{\phi}_{m,L}(\tau_n)\|_{H\times L^2(r \ge \tau_n - \tau_m + \beta_m\lambda_m)} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.25}$$

Now set $\vec{\varphi}_n = (\varphi_n^0, \varphi_n^1) := (\phi_n^0, \phi_n^1) - (\pi, 0) \in \mathcal{H}_0$. We have $\mathcal{E}(\vec{\varphi}_n) \le C < 2\mathcal{E}(Q)$ by construction. Therefore the sequence $S(-\tau_n)\vec{\varphi}_n$ is uniformly bounded in $H \times L^2$. Let $\vec{\varphi}_L = (\varphi_L^0, \varphi_L^1) \in \mathcal{H}_0$ be the weak limit of $S(-\tau_n)\vec{\varphi}_n$ in $H \times L^2$ as $n \to \infty$, i.e.,

$$S(-\tau_n)\vec{\varphi}_n \rightharpoonup \vec{\varphi} \quad \text{weakly in } H \times L^2$$

as $n \to \infty$. Denote by $\vec{\varphi}_L(t) := S(t)\vec{\varphi}_L$ the linear evolution of $\vec{\varphi}_L$ at time t. Following the construction in [1, Main Theorem] we have the following profile decomposition for $\vec{\varphi}_n$:

$$\begin{aligned} \vec{\varphi}_n(r) &= \vec{\varphi}_L(\tau_n, r) \\ &\quad + \sum_{j=2}^{k} \left(\varphi_L^j(t_n^j/\lambda_n^j, r/\lambda_n^j), \frac{1}{\lambda_n^j} \dot{\varphi}_L^j(t_n^j/\lambda_n^j, r/\lambda_n^j) \right) + \vec{\gamma}_n^k(r) \end{aligned} \tag{3.26}$$

where if we label $\varphi_L =: \varphi_L^1$, $\tau_n =: t_n^1$, and $\lambda_n^1 = 1$ this is exactly a profile decomposition as in [7, Corollary 2.15]. Now observe that for each fixed m we can write

$$\begin{aligned} &\vec{\varphi}_n(r) - \vec{\phi}_{m,L}(\tau_n, r) \\ &\quad = \vec{\varphi}_L(\tau_n, r) - \vec{\phi}_{m,L}(\tau_n, r) \\ &\quad\quad + \sum_{j=2}^{k} \left(\varphi_L^j(t_n^j/\lambda_n^j, r/\lambda_n^j), \frac{1}{\lambda_n^j} \dot{\varphi}_L^j(t_n^j/\lambda_n^j, r/\lambda_n^j) \right) + \vec{\gamma}_n^k(r) \end{aligned} \tag{3.27}$$

and (3.27) is still a profile decomposition in the sense of [7, Corollary 2.15] for the sequence $\vec{\varphi}_n(r) - \vec{\phi}_{m,L}(\tau_n, r)$. Since the pseudo-orthogonality of the $H \times L^2$ norm is preserved after sharp cut-offs, see [9, Corollary 8] or [7, Proposition 2.19], we then have

$$\begin{aligned} &\|\vec{\varphi}_n - \vec{\phi}_{m,L}(\tau_n)\|^2_{H \times L^2(r \ge \tau_n - \tau_m + \beta_m \lambda_m)} \\ &\quad = \|\vec{\varphi}_L(\tau_n) - \vec{\phi}_{m,L}(\tau_n)\|^2_{H \times L^2(r \ge \tau_n - \tau_m + \beta_m \lambda_m)} \\ &\quad\quad + \sum_{j=2}^{k} \|\vec{\varphi}_L^j(t_n^j/\lambda_n^j)\|^2_{H \times L^2(r \ge \tau_n - \tau_m + \beta_m \lambda_m)} + \|\vec{\gamma}_j^k\|^2_{H \times L^2(r \ge \tau_n - \tau_m + \beta_m \lambda_m)} + o_n(1). \end{aligned}$$

Note that (3.25) implies that the left-hand side above tends to zero as $n \to \infty$. Therefore, since all of the terms on right-hand side are nonnegative we can deduce that

$$\|\vec{\varphi}_L(\tau_n) - \vec{\phi}_{m,L}(\tau_n)\|^2_{H \times L^2(r \ge \tau_n - \tau_m + \beta_m \lambda_m)} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

Since,

$$\vec{\varphi}_L(\tau_n) - \vec{\phi}_{m,L}(\tau_n) = S(\tau_n)(\vec{\varphi} - S(-\tau_m)\vec{\phi}_{m,L})$$

is a solution to the linear wave equation, we can use the monotonicity of the energy on exterior cones to deduce that

$$\|\vec{\varphi}_L(t) - \vec{\phi}_{m,L}(t)\|^2_{H\times L^2(r\geq t-\tau_m+\beta_m\lambda_m)} \longrightarrow 0 \quad \text{as } t \longrightarrow \infty.$$

Combining the above with (3.24) we can conclude that

$$\|\vec{\psi}(t) - (\pi,0) - \vec{\varphi}_L(t)\|^2_{H\times L^2(r\geq t-\tau_m+\beta_m\lambda_m)} \longrightarrow 0 \quad \text{as } t \longrightarrow \infty.$$

The above holds for each $m \in \mathbb{N}$ and for *any* sequence $\beta_m \to \infty$ with $\beta_m < c_0\alpha_m$. Taking $\beta_m \ll \alpha_m$ and recalling that $\tau_m \to \infty$ and λ_m are such that $\alpha_m\lambda_m \ll \tau_m$ we have that $\tau_m - \beta_m\lambda_m \to \infty$ as $m \to \infty$. Therefore, for any $A > 0$ we can find m large enough so that $\tau_m - \beta_m\lambda_m > A$, which proves (3.18) in light of the above.

It remains to show (3.19). But this follows immediately from the decomposition (3.26) and the almost orthogonality of the nonlinear wave map energy for such a decomposition, see [7, Lemma 2.16], since we know that the left-hand side of (3.26) satisfies

$$\mathcal{E}(\vec{\varphi}_n) \leq C < 2\mathcal{E}(Q)$$

for large enough n. □

Now that we have constructed the radiation term $\vec{\varphi}_L(t)$ we denote by $\varphi(t) \in \mathcal{H}_0$ the global wave map that scatters to the linear wave $\vec{\varphi}_L(t)$, i.e., $\vec{\varphi}(t) \in \mathcal{H}_0$ is the global solution to (1.2) such that

$$\|\vec{\varphi}(t) - \vec{\varphi}_L(t)\|_{H\times L^2} \longrightarrow 0 \quad \text{as } t \longrightarrow \infty. \tag{3.28}$$

The existence of such a $\varphi(t) \in \mathcal{H}_0$ locally around $t = +\infty$ follows immediately from the existence of wave operators for the corresponding $4d$ semi-linear equation. The fact that $\varphi(t)$ is global-in-time follows from [7, Theorem 1] since (3.19) and (3.28) together imply that $\mathcal{E}(\vec{\varphi}) < 2\mathcal{E}(Q)$.

We will need a few facts about the degree zero wave map $\vec{\varphi}(t)$ which we collect in the following lemma.

LEMMA 3.9. *Let $\vec{\varphi}(t)$ be defined as above. Then we have*

$$\limsup_{t\to\infty} \|\vec{\varphi}(t)\|_{H\times L^2(|r-t|\geq A)} \longrightarrow 0 \quad \text{as } A \longrightarrow \infty, \tag{3.29}$$

$$\lim_{t\to\infty} \mathcal{E}^{\infty}_{t-A}(\vec{\varphi}(t)) \longrightarrow \mathcal{E}(\vec{\varphi}) \quad \text{as } A \longrightarrow \infty. \tag{3.30}$$

Proof. First we prove (3.29). We have

$$\|\vec{\varphi}(t)\|^2_{H\times L^2(|r-t|\geq A)} \leq \|\vec{\varphi}(t) - \vec{\varphi}_L(t)\|^2_{H\times L^2} + \|\varphi_L(t)\|^2_{H\times L^2(|r-t|\geq A)}.$$

By (3.28) the first term on the right-hand side above tends to 0 as $t \to \infty$ so it suffices to show that

$$\limsup_{t\to\infty} \|\varphi_L(t)\|^2_{H\times L^2(|r-t|\geq A)} \longrightarrow 0 \quad \text{as } A \longrightarrow \infty.$$

Since $\varphi_L(t)$ is a solution to (1.6) the above follows from [9, Theorem 4] by passing to the analogous statement for the corresponding $4d$ free wave $v_L(t)$ given by

$$rv_L(t,r) := \varphi_L(t,r).$$

To prove (3.30) we note that the limit as $t \to \infty$ exists for any fixed A due to the monotonicity of the energy on exterior cones. Next observe that we have

$$\lim_{t\to\infty} \mathcal{E}_0^{t-A}(\vec{\varphi}(t)) \leq \lim_{t\to\infty} \|\vec{\varphi}(t)\|^2_{H\times L^2(r\leq t-A)} \longrightarrow 0 \quad \text{as } A \longrightarrow \infty \tag{3.31}$$

by (3.29) and then (3.30) follows immediately from the conservation of energy. □

Now, observe that we can combine Proposition 3.8 and (3.28) to conclude that for all $A \geq 0$ we have

$$\|\vec{\psi}(t) - (\pi,0) - \vec{\varphi}(t)\|_{H\times L^2(r\geq t-A)} \longrightarrow 0 \quad \text{as } t \longrightarrow \infty. \tag{3.32}$$

With this in mind we define $a(t)$ as follows:

$$\vec{a}(t) := \vec{\psi}(t) - \vec{\varphi}(t) \tag{3.33}$$

and we aggregate some preliminary information about a in the following lemma:

LEMMA 3.10. *Let $\vec{a}(t)$ be defined as in* (3.33). *Then $\vec{a}(t) \in \mathcal{H}_1$ for all t. Moreover,*

- *for all $\lambda > 0$ we have*

$$\|\vec{a}(t) - (\pi,0)\|_{H\times L^2(r\geq\lambda t)} \longrightarrow 0 \quad \textit{as } t \longrightarrow \infty, \tag{3.34}$$

- *the quantity $\mathcal{E}(\vec{a}(t))$ has a limit as $t \to \infty$ and*

$$\lim_{t\to\infty} \mathcal{E}(\vec{a}(t)) = \mathcal{E}(\vec{\psi}) - \mathcal{E}(\vec{\varphi}). \tag{3.35}$$

Proof. By definition we have $a(t) \in \mathcal{H}_1$ for all t since

$$a(t,0) = 0, \quad a(t,\infty) = \pi.$$

To prove (3.34) observe that for every $A \leq (1-\lambda)t$ we have

$$\begin{aligned}\|\vec{a}(t) - (\pi,0)\|^2_{H\times L^2(r\geq\lambda t)} \leq{}& \|\vec{\psi}(t) - (\pi,0)\|^2_{H\times L^2(\lambda t\leq r\leq t-A)} \\ &+ \|\vec{\varphi}(t)\|^2_{H\times L^2(\lambda t\leq r\leq t-A)} \\ &+ \|\vec{a}(t) - (\pi,0)\|^2_{H\times L^2(r\geq t-A)}.\end{aligned}$$

Then (3.34) follows by combining (3.32), (3.29), and (2.18). To prove (3.35) we first claim that

$$\lim_{A\to\infty}\lim_{t\to\infty}\mathcal{E}_{t-A}^{\infty}(\vec{\psi}(t))=\mathcal{E}(\vec{\varphi}). \tag{3.36}$$

Indeed, we have

$$\begin{aligned}
\mathcal{E}_{t-A}^{\infty}(\vec{\psi}(t)) &= \int_{t-A}^{\infty}[(\psi_t(t)-\varphi_t(t)+\varphi_t(t))^2+(\psi_r(t)-\varphi_r(t)+\varphi_r(t))^2]\,r\,dr \\
&\quad+\int_{t-A}^{\infty}\frac{\sin^2(\psi(t)-\pi-\varphi(t)+\varphi(t))}{r}\,dr \\
&= \mathcal{E}_{t-A}^{\infty}(\vec{\varphi}(t))+\|\vec{\psi}(t)-(\pi,0)-\vec{\varphi}(t)\|^2_{\dot{H}^1\times L^2(r\geq t-A)} \\
&\quad+O\Big(\|\vec{\psi}(t)-(\pi,0)-\vec{\varphi}(t)\|_{\dot{H}^1\times L^2(r\geq t-A)}\|\vec{\varphi}(t)\|_{\dot{H}^1\times L^2(r\geq t-A)}\Big) \\
&\quad+\int_{t-A}^{\infty}\frac{\sin^2(\psi(t)-\pi-\varphi(t)+\varphi(t))-\sin^2(\varphi(t))}{r}\,dr \\
&= \mathcal{E}_{t-A}^{\infty}(\vec{\varphi}(t))+O\Big(\|\vec{\psi}(t)-(\pi,0)-\vec{\varphi}(t)\|^2_{H\times L^2(r\geq t-A)}\Big) \\
&\quad+O\Big(\sqrt{\mathcal{E}(\vec{\varphi})}\|\vec{\psi}(t)-(\pi,0)-\vec{\varphi}(t)\|_{H\times L^2(r\geq t-A)}\Big),
\end{aligned}$$

which proves (3.36) in light of (3.30) and (3.32). In the third equality above we have used the simple trigonometric inequality:

$$\left|\sin^2(x-y+y)-\sin^2(y)\right|\leq 2\,|\sin(y)|\,|x-y|+2\,|x-y|^2\,.$$

Now, fix $\delta>0$. By (3.29), (3.36), and (3.32) we can choose A, T_0 large enough so that for all $t\geq T_0$ we have

$$\begin{aligned}
\|\vec{\varphi}(t)\|_{H\times L^2(r\leq t-A)} &\leq \delta, \\
\left|\mathcal{E}_{t-A}^{\infty}(\vec{\psi}(t))-\mathcal{E}(\vec{\varphi})\right| &\leq \delta, \\
\|\vec{a}(t)-(\pi,0)\|^2_{H\times L^2(r\geq t-A)} &\leq \delta.
\end{aligned}$$

Then for all $t\geq T_0$ and A as above we can argue as before to obtain

$$\begin{aligned}
\mathcal{E}(\vec{a}(t)) &= \mathcal{E}_0^{t-A}(\vec{a}(t))+O\big(\|\vec{a}(t)-(\pi,0)\|^2_{H\times L^2(r\geq t-A)}\big) \\
&= \mathcal{E}_0^{t-A}(\vec{\psi}(t))+O\Big(\sqrt{\mathcal{E}(\vec{\psi})}\|\vec{\varphi}(t)\|_{H\times L^2(r\leq t-A)}\Big) \\
&\quad+O\big(\|\vec{\varphi}(t)\|^2_{H\times L^2(r\leq t-A)}\big)+O\big(\|\vec{a}(t)-(\pi,0)\|^2_{H\times L^2(r\geq t-A)}\big) \\
&= \mathcal{E}(\vec{\psi})-\mathcal{E}_{t-A}^{\infty}(\vec{\psi}(t))+O(\delta) \\
&= \mathcal{E}(\vec{\psi})-\mathcal{E}(\vec{\varphi})+O(\delta),
\end{aligned}$$

which proves (3.35). □

We will also need the following technical lemma in the next section.

LEMMA 3.11. *For any sequence $\sigma_n > 0$ with $\lambda_n \ll \sigma_n \ll \tau_n$ we have*

$$\lim_{n\to\infty} \frac{1}{\sigma_n} \int_{\tau_n}^{\tau_n+\sigma_n} \int_0^\infty \dot{a}^2(t,r)\, r\, dr\, dt = 0. \tag{3.37}$$

Proof. Fix $0 < \lambda < 1$. For each n we have

$$\frac{1}{\sigma_n} \int_{\tau_n}^{\tau_n+\sigma_n} \int_0^\infty \dot{a}^2(t,r)\, r\, dr\, dt \le \frac{1}{\sigma_n} \int_{\tau_n}^{\tau_n+\sigma_n} \int_0^{\lambda t} \dot{a}^2(t,r)\, r\, dr\, dt + \frac{1}{\sigma_n} \int_{\tau_n}^{\tau_n+\sigma_n} \int_{\lambda t}^\infty \dot{a}^2(t,r)\, r\, dr\, dt.$$

By (3.34) we can conclude that

$$\lim_{n\to\infty} \sup_{t\ge\tau_n} \int_{\lambda t}^\infty \dot{a}^2(t,r)\, r\, dr = 0.$$

Hence it suffices to show that

$$\lim_{n\to\infty} \frac{1}{\sigma_n} \int_{\tau_n}^{\tau_n+\sigma_n} \int_0^{\lambda t} \dot{a}^2(t,r)\, r\, dr\, dt = 0.$$

Observe that for every n we have

$$\frac{1}{\sigma_n} \int_{\tau_n}^{\tau_n+\sigma_n} \int_0^{\lambda t} \dot{a}^2(t,r)\, r\, dr\, dt \lesssim \frac{1}{\sigma_n} \int_{\tau_n}^{\tau_n+\sigma_n} \int_0^{\lambda t} \dot{\psi}^2(t,r)\, r\, dr\, dt + \frac{1}{\sigma_n} \int_{\tau_n}^{\tau_n+\sigma_n} \int_0^{\lambda t} \dot{\varphi}^2(t,r)\, r\, dr\, dt. \tag{3.38}$$

We first estimate the first integral on the right-hand side above. Let $A_n \to \infty$ be the sequence in Proposition 3.6, see also Remark 5, and let $t_n \to \infty$ be the sequence in Theorem 3.2. Recall that we have $\tau_n \in [t_n, t_n + \lambda_n]$ and $\lambda_n \le A_n \ll t_n$.

Observe that for n large enough we have that for each $t \in [\tau_n, \tau_n + \sigma_n]$ we have $\lambda t \le t - A_n$. Hence,

$$\frac{1}{\sigma_n} \int_{\tau_n}^{\tau_n+\sigma_n} \int_0^{\lambda t} \dot{\psi}^2(t,r)\, r\, dr\, dt \le \frac{1}{\sigma_n} \int_{\tau_n}^{\tau_n+\sigma_n} \int_0^{t-A_n} \dot{\psi}^2(t,r)\, r\, dr\, dt.$$

Next, note that since $\lambda_n \ll \sigma_n$ we can ensure that for n large enough we have $\lambda_n + \sigma_n \le 2\sigma_n$. Therefore,

$$\frac{1}{\sigma_n}\int_{\tau_n}^{\tau_n+\sigma_n}\int_0^{t-A_n} \dot{\psi}^2(t,r)\, r\, dr\, dt \le \frac{2}{\lambda_n+\sigma_n}\int_{t_n}^{t_n+\lambda_n+\sigma_n}\int_0^{t-A_n} \dot{\psi}^2(t,r)\, r\, dr\, dt \longrightarrow 0$$

as $n \to \infty$ by Lemma 3.3.

Lastly we estimate the second integral on the right-hand side of (3.38). For each $A > 0$ we can choose n large enough so that $\lambda t \le t - A$ for each $t \in [\tau_n, \tau_n + \sigma_n]$. So we have

$$\frac{1}{\sigma_n}\int_{\tau_n}^{\tau_n+\sigma_n}\int_0^{\lambda t} \dot{\varphi}^2(t,r)\, r\, dr\, dt \le \frac{1}{\sigma_n}\int_{\tau_n}^{\tau_n+\sigma_n}\int_0^{t-A} \dot{\varphi}^2(t,r)\, r\, dr\, dt.$$

Taking the limsup as $n \to \infty$ of both sides and then letting $A \to \infty$ on the right we have by (3.29) that the left-hand side above tends to 0 as $n \to \infty$. This concludes the proof. □

3.3. Compactness of the error. For the remainder of this section, we fix $\alpha_n \to \infty$ and find $\tau_n \to \infty$ and $\lambda_n \ll \tau_n$ as in Proposition 3.6. We define $\vec{b}_n = (b_{n,0}, b_{n,1}) \in \mathcal{H}_0$ as follows:

$$b_{n,0}(r) := a(\tau_n, r) - Q(r/\lambda_n), \tag{3.39}$$

$$b_{n,1}(r) := \dot{a}(\tau_n, r). \tag{3.40}$$

As in [7, Section 5.3], our goal in this subsection is to show that $\vec{b}_n \to 0$ in the energy space. Indeed we prove the following result:

PROPOSITION 3.12. *Define $\vec{b}_n \in \mathcal{H}_0$ as in* (3.39), (3.40). *Then,*

$$\|\vec{b}_n\|_{H\times L^2} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.41}$$

Remark 6. In light of (3.28), it is clear that Proposition 3.12 implies Proposition 3.1.

Remark 7. The proof of Proposition 3.12 will follow the same strategy as [7, Proposition 5.6] and we refer the reader to the outline given there for a general overview of the proof.

We begin with the following consequences of the previous sections.

LEMMA 3.13. *Let $\vec{b}_n \in \mathcal{H}_0$ be defined as above. Then we have*

(a) *As $n \to \infty$ we have*

$$\|b_{n,1}\|_{L^2} \longrightarrow 0. \tag{3.42}$$

(b) *As* $n \to \infty$ *we have*

$$\|b_{n,0}\|_{H(r \le \alpha_n \lambda_n)} \longrightarrow 0. \tag{3.43}$$

(c) *For any fixed* $\lambda > 0$ *we have*

$$\|b_{n,0}\|_{H(r \ge \lambda \tau_n)} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.44}$$

(d) *There exists a* $C > 0$ *so that*

$$\mathcal{E}(\vec{b}_n) \le C < 2\mathcal{E}(Q) \tag{3.45}$$

for n *large enough.*

Proof. To prove (3.42) fix $0 < \lambda < 1$ and observe that we have

$$\int_0^\infty b_{n,1}^2(r)\, r\, dr \le \int_0^{\lambda \tau_n} \dot{\psi}^2(\tau_n, r)\, r\, dr + \int_0^{\lambda \tau_n} \dot{\varphi}^2(\tau_n, r)\, r\, dr + \int_{\lambda \tau_n}^\infty \dot{a}(\tau_n, r)^2\, r\, dr.$$

Then (3.42) follows from (3.15), (3.29), and (3.34).

Next we prove (3.43). To see this, observe that for each n we have

$$\|b_{n,0}\|^2_{H(r \le \alpha_n \lambda_n)} \le \|\psi(\tau_n) - Q(\cdot/\lambda_n)\|^2_{H(r \le \alpha_n \lambda_n)} + \|\varphi(\tau_n)\|^2_{H(r \le \alpha_n \lambda_n)}.$$

The first term on the right-hand side tends to zero as $n \to \infty$ by (3.16). To estimate the second term on the right-hand side we note that for fixed $A > 0$ we can find n large enough so that $\alpha_n \lambda_n \le \tau_n - A$ and so we have

$$\|\varphi(\tau_n)\|^2_{H(r \le \alpha_n \lambda_n)} \le \|\varphi(\tau_n)\|^2_{H(r \le \tau_n - A)}.$$

Taking the limsup as $n \to \infty$ on both sides above and then taking $A \to \infty$ on the right and recalling (3.29) we see that the limit as $n \to \infty$ of the left-hand side above must be zero. This proves (3.43).

To deduce (3.44) note that

$$\|b_{n,0}\|^2_{H(r \ge \lambda \tau_n)} \le \|a(\tau_n) - \pi\|^2_{H(r \ge \lambda \tau_n)} + \|Q(\cdot/\lambda_n) - \pi\|^2_{H(r \ge \lambda \tau_n)}.$$

The first term on the right-hand side above tends to zero as $n \to \infty$ by (3.34). The second term tends to zero since $\lambda \tau_n / \lambda_n \to \infty$ as $n \to \infty$.

Finally, we establish (3.45). First observe that for any fixed $\lambda > 0$, (3.44) implies that

$$\begin{aligned} \mathcal{E}(\vec{b}_n) &= \mathcal{E}_0^{\lambda \tau_n}(\vec{b}_n) + \mathcal{E}_{\lambda \tau_n}^\infty(\vec{b}_n) \\ &= \mathcal{E}_0^{\lambda \tau_n}(\vec{b}_n) + o_n(1) \end{aligned}$$

as $n \to \infty$. So it suffices to control $\mathcal{E}_0^{\lambda\tau_n}(\vec{b}_n)$. Next, observe that for n large enough, (3.31) gives that

$$\|\vec{\varphi}(\tau_n)\|_{H\times L^2(r\leq\lambda\tau_n)} \leq \|\vec{\varphi}(\tau_n)\|_{H\times L^2(r\leq\tau_n-A)}$$

and the right-hand side is small for n, A large. This means that the contribution of $\vec{\varphi}(\tau_n)$ is negligible on $r \leq \lambda\tau_n$, and thus

$$\mathcal{E}_0^{\lambda\tau_n}(\vec{b}_n) = \mathcal{E}_0^{\lambda\tau_n}(\vec{\psi}(\tau_n) - (Q(\cdot/\lambda_n), 0)) + o_n(1).$$

Next, recall that Proposition 3.6 implies that

$$\mathcal{E}_0^{\alpha_n\lambda_n}(\vec{\psi}(\tau_n) - Q(\cdot/\lambda_n), 0) = o_n(1), \tag{3.46}$$

which shows in particular that

$$\mathcal{E}_{\alpha_n\lambda_n}^{\infty}(\vec{\psi}(\tau_n)) \leq \eta + o_n(1) \tag{3.47}$$

where $\eta := \mathcal{E}(\vec{\psi}) - \mathcal{E}(Q) < 2\mathcal{E}(Q)$. Also, (3.46) means that it suffices to show that

$$\mathcal{E}_{\alpha_n\lambda_n}^{\lambda\tau_n}(\vec{\psi}(\tau_n) - (Q(\cdot/\lambda_n), 0)) \leq C < 2\mathcal{E}(Q).$$

Note that since $\alpha_n \to \infty$ we have

$$\mathcal{E}_{\alpha_n\lambda_n}^{\infty}(Q(\cdot/\lambda_n)) = \mathcal{E}_{\alpha_n}^{\infty}(Q) = o_n(1).$$

Hence,

$$\mathcal{E}_{\alpha_n\lambda_n}^{\lambda\tau_n}(\vec{\psi}(\tau_n) - (Q(\cdot/\lambda_n), 0)) = \mathcal{E}_{\alpha_n\lambda_n}^{\lambda\tau_n}(\vec{\psi}(\tau_n)) + o_n(1) \leq \eta + o_n(1),$$

which completes the proof. □

Next, we would like to show that the sequence $\vec{b}_n$ does not contain any nonzero profiles. This next result is the global-in-time analog of [7, Proposition 5.7] and the proof is again, reminiscent of the arguments given in [11, Section 5].

Denote by $\vec{b}_n(t) \in \mathcal{H}_0$ the wave map evolution with data $\vec{b}_n$. By (3.45) and [7, Theorem 1.1] we know that $\vec{b}_n(t) \in \mathcal{H}_0$ is global in time and scatters to zero as $t \to \pm\infty$.

The statements of the following proposition and its corollary are identical to the corresponding statements [7, Proposition 5.7 and Corollary 5.8] in the finite time blow-up case.

PROPOSITION 3.14. *Let $b_n \in \mathcal{H}_0$ and the corresponding global wave map evolution $\vec{b}_n(t) \in \mathcal{H}_0$ be defined as above. Then, there exists a decomposition*

$$\vec{b}_n(t, r) = b_{n,L}(t, r) + \vec{\theta}_n(t, r) \tag{3.48}$$

where $\vec{b}_{n,L}$ *satisfies the linear wave equation* (1.6) *with initial data* $\vec{b}_{n,L}(0,r) := (b_{n,0}, 0)$. *Moreover,* $b_{n,L}$ *and* $\vec{\theta}_n$ *satisfy*

$$\left\| \frac{1}{r} b_{n,L} \right\|_{L_t^3(\mathbb{R}; L_x^6(\mathbb{R}^4))} \longrightarrow 0 \tag{3.49}$$

$$\|\vec{\theta}_n\|_{L_t^\infty(\mathbb{R}; H\times L^2)} + \left\| \frac{1}{r} \theta_n \right\|_{L_t^3(\mathbb{R}; L_x^6(\mathbb{R}^4))} \longrightarrow 0 \tag{3.50}$$

as $n \to \infty$.

Before beginning the proof of Proposition 3.14 we use the conclusions of the proposition to deduce the following corollary which will be an essential ingredient in the proof of Proposition 3.12.

COROLLARY 3.15. *Let* $\vec{b}_n(t)$ *be defined as in Proposition 3.14. Suppose that there exists a constant* δ_0 *and a subsequence in* n *so that* $\|b_{n,0}\|_H \geq \delta_0$. *Then there exists* $\alpha_0 > 0$ *such that for all* $t > 0$ *and all* n *large enough along this subsequence we have*

$$\|\vec{b}_n(t)\|_{H\times L^2(r\geq t)} \geq \alpha_0 \delta_0. \tag{3.51}$$

Proof. First note that since $\vec{b}_{n,L}$ satisfies the linear wave equation (1.6) with initial data $\vec{b}_{n,L}(0) = (b_{n,0}, 0)$ we know by [9, Corollary 5] and [7, Corollary 2.3], that there exists a constant $\beta_0 > 0$ so that for each $t \geq 0$ we have

$$\|\vec{b}_{n,L}(t)\|_{H\times L^2(r\geq t)} \geq \beta_0 \|b_{n,0}\|_H.$$

On the other hand, by Proposition 3.14 we know that

$$\|\vec{b}_n(t) - \vec{b}_{n,L}(t)\|_{H\times L^2(r\geq t)} \leq \|\vec{\theta}_n(t)\|_{H\times L^2} = o_n(1).$$

Putting these two facts together gives

$$\begin{aligned} \|\vec{b}_n(t)\|_{H\times L^2(r\geq t)} &\geq \|b_{n,L}(t)\|_{H\times L^2(r\geq t)} - o_n(1) \\ &\geq \beta_0 \|b_{n,0}\|_H - o_n(1). \end{aligned}$$

This yields (3.51) by passing to a suitable subsequence and taking n large enough. □

The proof of Proposition 3.14 is very similar to the proof of [7, Proposition 5.7]. Instead of going through the entire argument again here, we establish the main ingredients of the proof and we refer the reader to [7] for the remainder of the argument.

Since $\vec{b}_n \in \mathcal{H}_0$ and $\mathcal{E}(\vec{b}_n) \leq C < 2\mathcal{E}(Q)$ we can, by [7, Corollary 2.15], consider the following profile decomposition for $\vec{b}_n$:

$$b_{n,0}(r) = \sum_{j\leq k} \varphi_L^j\left(\frac{-t_n^j}{\lambda_n^j}, \frac{r}{\lambda_n^j}\right) + \gamma_{n,0}^k(r), \tag{3.52}$$

$$b_{n,1}(r) = \sum_{j\leq k} \frac{1}{\lambda_n^j}\dot{\varphi}_L^j\left(\frac{-t_n^j}{\lambda_n^j}, \frac{r}{\lambda_n^j}\right) + \gamma_{n,1}^k(r), \tag{3.53}$$

where each $\vec{\varphi}_L^j$ is a solution to (1.6) and where we have for each $j \neq k$:

$$\frac{\lambda_n^j}{\lambda_n^k} + \frac{\lambda_n^k}{\lambda_n^j} + \frac{\left|t_n^j - t_n^k\right|}{\lambda_n^k} + \frac{\left|t_n^j - t_n^k\right|}{\lambda_n^j} \longrightarrow \infty \quad \text{as } n \longrightarrow \infty. \tag{3.54}$$

Moreover, if we denote by $\vec{\gamma}_{n,L}^k(t)$ the linear evolution of $\vec{\gamma}_n^k$, i.e., solution to (1.6), we have for $j \leq k$ that

$$\left(\gamma_{n,L}^k(\lambda_n^j t_n^j, \lambda_n^j \cdot), \lambda_n^j \dot{\gamma}_{n,L}^k(\lambda_n^j t_n^j, \lambda_n^j \cdot)\right) \rightharpoonup 0 \quad \text{in } H \times L^2 \quad \text{as } n \longrightarrow \infty \tag{3.55}$$

$$\limsup_{n\to\infty} \left\|\frac{1}{r}\gamma_{n,L}^k\right\|_{L_t^3 L_x^6(\mathbb{R}^4)} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{3.56}$$

Finally we have the following Pythagorean expansions:

$$\|b_{n,0}\|_H^2 = \sum_{j\leq k} \left\|\varphi_L^j\left(\frac{-t_n^j}{\lambda_n^j}\right)\right\|_H^2 + \|\gamma_{n,0}^k\|_H^2 + o_n(1) \tag{3.57}$$

$$\|b_{n,1}\|_{L^2}^2 = \sum_{j\leq k} \left\|\dot{\varphi}_L^j\left(\frac{-t_n^j}{\lambda_n^j}\right)\right\|_{L^2}^2 + \|\gamma_{n,1}^k\|_{L^2}^2 + o_n(1). \tag{3.58}$$

As in [7], the proof of Proposition 3.14 will consist of a sequence of steps designed to show that each of the profiles φ_L^j must be identically zero. Arguing exactly as in [7, Lemma 5.9] we can first deduce that the times t_n^j can be taken to be 0 for each n, j and that the initial velocities $\dot{\varphi}_L^j(0)$ must all be identically zero as well. We summarize this conclusion in the following lemma:

LEMMA 3.16. *In the decomposition* (3.52), (3.53) *we can assume, without loss of generality, that* $t_n^j = 0$ *for every* n *and for every* j. *In addition, we then have*

$$\dot{\varphi}_L^j(0, r) \equiv 0 \quad \textit{for every } j.$$

The proof of Lemma 3.16 is identical to the proof of [7, Lemma 5.9] and follows from the Pythagorean expansion (3.58), (3.42), and the asymptotic equipartition of energy for the corresponding $4d$ free waves. We refer the reader to [7] for the details.

By Lemma 3.16 we can rewrite our profile decomposition as follows:

$$b_{n,0}(r) = \sum_{j\le k} \varphi_L^j\left(0, r/\lambda_n^j\right) + \gamma_{n,0}^k(r) \tag{3.59}$$

$$b_{n,1}(r) = o_n(1) \text{ in } L^2 \quad \text{as } n \longrightarrow \infty. \tag{3.60}$$

Note that in addition to the Pythagorean expansions in (3.57) we also have the following almost-orthogonality of the nonlinear wave map energy, which was established in [7, Lemma 2.16]:

$$\mathcal{E}(\vec{b}_n) = \sum_{j\le k} \mathcal{E}(\varphi_L^j(0), 0) + \mathcal{E}(\gamma_{n,0}^k, 0) + o_n(1). \tag{3.61}$$

Note that $\varphi^j, \gamma_{n,0}^k \in \mathcal{H}_0$ for every j, for every n, and for every k. Using the fact that $\mathcal{E}(\vec{b}_n) \le C < 2\mathcal{E}(Q)$, (3.61) and [7, Theorem 1.1] imply that, for every j, the nonlinear wave map evolution of the data $(\varphi_L^j(0, r/\lambda_n^j), 0)$ given by

$$\vec{\varphi}_n^j(t,r) := \left(\varphi^j\left(\frac{t}{\lambda_n^j}, \frac{r}{\lambda_n^j}\right), \frac{1}{\lambda_n^j}\dot{\varphi}^j\left(\frac{t}{\lambda_n^j}, \frac{r}{\lambda_n^j}\right)\right) \tag{3.62}$$

is global in time and scatters as $t \to \pm\infty$. Moreover we have the following nonlinear profile decomposition guaranteed by [7, Proposition 2.17]:

$$\vec{b}_n(t,r) = \sum_{j\le k} \vec{\varphi}_n^j(t,r) + \vec{\gamma}_{n,L}^k(t,r) + \vec{\theta}_n^k(t,r) \tag{3.63}$$

where the $\vec{b}_n(t,r)$ are the global wave map evolutions of the data $\vec{b}_n$, $\vec{\gamma}_{n,L}^k(t,r)$ is the linear evolution of $(\gamma_n^k, 0)$, and the errors $\vec{\theta}_n^k$ satisfy

$$\limsup_{n\to\infty}\left(\|\vec{\theta}_n^k\|_{L_t^\infty(H\times L^2)} + \left\|\frac{1}{r}\theta_n^k\right\|_{L_t^3(\mathbb{R};L_x^6(\mathbb{R}^4))}\right) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{3.64}$$

Recall that we are trying to show that all of the profiles φ^j must be identically equal to zero. As in [7] we can make the following crucial observations about the scales λ_n^j. Since we have concluded that we can assume that all of the times $t_n^j = 0$ for all n, j we first note that the orthogonality condition (3.54) implies that for $j \ne k$:

$$\frac{\lambda_n^j}{\lambda_n^k} + \frac{\lambda_n^k}{\lambda_n^j} \longrightarrow \infty \quad \text{as } n \longrightarrow \infty.$$

Next, recall that by Lemma 3.13 we have

$$\|b_{n,0}\|_{H(r\le\alpha_n\lambda_n)} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{3.65}$$

$$\|b_{n,0}\|_{H(r\ge\lambda\tau_n)} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,\ \forall\lambda > 0 \text{ fixed.} \tag{3.66}$$

Combining the above two facts with [7, Proposition 2.19] we can conclude that for each λ_n^j corresponding to a nonzero profile φ^j we have

$$\lambda_n \ll \lambda_n^j \ll \tau_n \quad \text{as } n \longrightarrow \infty. \tag{3.67}$$

Now, let k_0 be the index corresponding to the first nonzero profile φ^{k_0}. We can assume, without loss of generality that $k_0 = 1$. By (3.65), (3.67) and [11, Appendix B] we can find a sequence $\tilde{\lambda}_n$ so that

$$\begin{aligned} &\tilde{\lambda}_n \ll \alpha_n \lambda_n \\ &\lambda_n \ll \tilde{\lambda}_n \ll \lambda_n^1 \\ &\tilde{\lambda}_n \ll \lambda_n^j \quad \text{or} \quad \lambda_n^j \ll \tilde{\lambda}_n \quad \forall j > 1. \end{aligned}$$

Define

$$\beta_n = \frac{\tilde{\lambda}_n}{\lambda_n} \longrightarrow \infty$$

and we note that $\beta_n \ll \alpha_n$ and $\tilde{\lambda}_n = \beta_n \lambda_n$. Therefore, up to replacing β_n by a sequence $\tilde{\beta}_n \simeq \beta_n$ and $\tilde{\lambda}_n$ by $\tilde{\tilde{\lambda}}_n := \tilde{\beta}_n \lambda_n$, we have by Corollary 3.7 and a slight abuse of notation that

$$\psi(\tau_n, \tilde{\lambda}_n) \longrightarrow \pi \quad \text{as } n \longrightarrow \infty. \tag{3.68}$$

We define the set

$$\mathcal{J}_{\text{ext}} := \{ j \geq 1 \mid \tilde{\lambda}_n \ll \lambda_n^j \}.$$

Note that by construction $1 \in \mathcal{J}_{\text{ext}}$.

The above technical ingredients are necessary for the proof of the following lemma and its corollary. The analog in the finite-time blow-up case is [7, Lemma 5.10].

LEMMA 3.17. *Let φ^1, λ_n^1 be defined as above. Then for all $\varepsilon > 0$ we have*

$$\frac{1}{\lambda_n^1} \int_0^{\lambda_n^1} \int_{\varepsilon\lambda_n^1 + t}^{\infty} \left| \sum_{j \in \mathcal{J}_{\text{ext}},\, j \leq k} \dot{\varphi}_n^j(t,r) + \dot{\gamma}_{n,L}^k(t,r) \right|^2 r\, dr\, dt = o_n^k \tag{3.69}$$

where $\lim_{k\to\infty} \limsup_{n\to\infty} o_n^k = 0$. *Also, for all $j > 1$ and for all $\varepsilon > 0$ we have*

$$\frac{1}{\lambda_n^1} \int_0^{\lambda_n^1} \int_{\varepsilon\lambda_n^1 + t}^{\infty} (\dot{\varphi}_n^j)^2(t,r)\, r\, dr\, dt \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.70}$$

Remark 8. We refer the reader to [7, Proof of Lemma 5.10] for the details of the proof of Lemma 3.17. The proof of (3.69) is nearly identical to [7, Proof of (5.57)] the one difference being that here we use Lemma 3.11 in place of the argument following [7, equation (5.66)]. The proof of (3.70) is identical to [7, Proof of (5.58)] and we omit it here.

Note that (3.69) and (3.70) together directly imply the following result:

COROLLARY 3.18. *Let* φ^1 *be as in Lemma 3.17. Then for all* $\varepsilon > 0$ *we have*

$$\frac{1}{\lambda_n^1}\int_0^{\lambda_n^1}\int_{\varepsilon\lambda_n^1+t}^{\infty}\left|\dot{\varphi}_n^1(t,r)+\dot{\gamma}_{n,L}^k(t,r)\right|^2 r\,dr\,dt = o_n^k \tag{3.71}$$

where $\lim_{k\to\infty}\limsup_{n\to\infty} o_n^k = 0$.

The proof of Proposition 3.14 now follows from the exact same argument as [7, Proof of Proposition 5.7]. We refer the reader to [7] for the details.

We can now complete the proof of Proposition 3.12.

Proof of Proposition 3.12. We argue by contradiction. Assume that Proposition 3.12 fails. Then, up to extracting a subsequence, we can find a $\delta_0 > 0$ so that

$$\|b_{n,0}\|_H \geq \delta_0 \tag{3.72}$$

for every n. By Corollary 3.15 we know that there exists $\alpha_0 > 0$ so that for all t,

$$\|\vec{b}_n(t)\|_{H\times L^2(r\geq|t|)} \geq \alpha_0\delta_0.$$

We will show that the above is, in fact, impossible by constructing a sequence of times along which the left-hand side above tends to zero. It is convenient to carry out the argument in rescaled coordinates. Set

$$\mu_n := \frac{\lambda_n}{\tau_n}.$$

Since $\lambda_n \ll \tau_n$ as $n\to\infty$, our new scale $\mu_n \to 0$ as $n\to\infty$. We next define rescaled wave maps:

$$g_n(t,r) := \psi(\tau_n+\tau_n t, \tau_n r), \tag{3.73}$$

$$h_n(t,r) := \varphi(\tau_n+\tau_n t, \tau_n r). \tag{3.74}$$

Since $\vec{g}_n(t)$ and $\vec{h}_n(t)$ are defined by rescaling $\vec{\psi}$ and $\vec{\varphi}$ we have that $\vec{g}_n(t)\in\mathcal{H}_1$ is a global-in-time wave map and the wave map $\vec{\varphi}(t)\in\mathcal{H}_0$ is global-in-time and scatters to 0 as $t\to\pm\infty$. We then have

$$a(\tau_n+\tau_n t,\tau_n r) = g_n(t,r) - h_n(t,r).$$

Similarly, we define

$$\tilde{b}_{n,0}(r) := b_{n,0}(\tau_n r),$$
$$\tilde{b}_{n,1}(r) := \tau_n b_{n,1}(\tau_n r)$$

and the corresponding rescaled wave map evolutions

$$\tilde{b}_n(t,r) := b_n(\tau_n t, \tau_n r),$$
$$\partial_t \tilde{b}_n(t,r) := \tau_n \dot{b}_n(\tau_n t, \tau_n r).$$

After this rescaling, our decomposition becomes

$$g_n(0,r) = h_n(0,r) + Q\left(\frac{r}{\mu_n}\right) + \tilde{b}_{n,0}(r) \tag{3.75}$$

$$\dot{g}_n(0,r) = \dot{h}_n(0,r) + \tilde{b}_{n,1}(r). \tag{3.76}$$

We can rephrase (3.44) and (3.43) in terms of this rescaling and we obtain:

$$\forall \lambda > 0 \text{ fixed, } \|\tilde{b}_{n,0}\|_{H(r\geq\lambda)} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{3.77}$$

$$\|\tilde{b}_{n,0}\|_{H(r\leq\alpha_n\mu_n)} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.78}$$

Also, (3.29) implies that

$$\lim_{A\to\infty} \limsup_{n\to\infty} \|\vec{h}_n(0)\|_{H\times L^2(r\leq 1-A/\tau_n)} = 0, \tag{3.79}$$

$$\lim_{A\to\infty} \limsup_{n\to\infty} \|\vec{h}_n(0)\|_{H\times L^2(r\geq 1+A/\tau_n)} = 0. \tag{3.80}$$

Next, we claim that for every n a decomposition of the form (3.75) is preserved up to a small error if we replace the terms in (3.75) with their respective wave map evolutions at some future times to be defined precisely below.

By Corollary 3.7 we can choose a sequence $\gamma_n \to \infty$ with

$$\gamma_n \ll \alpha_n$$

so that

$$g_n(0, \gamma_n\mu_n) \longrightarrow \pi \quad \text{as } n \longrightarrow \infty.$$

Define $\delta_n \to 0$ by

$$|g_n(0,\gamma_n\mu_n) - \pi| =: \delta_n \longrightarrow 0.$$

Using (3.16) we define $\varepsilon_n \to 0$ by

$$\|\vec{g}_n(0) - (Q(\cdot/\mu_n), 0)\|_{H\times L^2(r\leq\alpha_n\mu_n)} =: \varepsilon_n \longrightarrow 0.$$

Finally, choose $\beta_n \to \infty$ so that

$$\begin{gathered} \beta_n \leq \min\left\{\sqrt{\gamma_n}, \delta_n^{-1/2}, \varepsilon_n^{-1/2}\right\} \\ g_n(0, \beta_n\mu_n/2) \longrightarrow \pi \quad \text{as } n \longrightarrow \infty. \end{gathered} \tag{3.81}$$

As in [7], we make the following claims:

(i) As $n \to \infty$ we have

$$\|\vec{g}_n(\beta_n\mu_n/2) - (Q(\cdot/\mu_n),0)\|_{H\times L^2(r\le\beta_n\mu_n)} \longrightarrow 0. \tag{3.82}$$

(ii) For each n, on the interval $r \in [\beta_n\mu_n, \infty)$ we have

$$\begin{aligned} \vec{g}_n\left(\frac{\beta_n\mu_n}{2}, r\right) - (\pi,0) = \vec{h}_n\left(\frac{\beta_n\mu_n}{2}, r\right) + \vec{\tilde{b}}_n\left(\frac{\beta_n\mu_n}{2}, r\right) + \vec{\tilde{\theta}}_n\left(\frac{\beta_n\mu_n}{2}, r\right), \\ \|\vec{\tilde{\theta}}_n\|_{L^\infty_t(H\times L^2)} \longrightarrow 0. \end{aligned} \tag{3.83}$$

We first prove (3.82). The proof is very similar to the corresponding argument in the finite-time blow-up case, see [7, Proof of (5.76)]. We repeat the argument here for completeness.

First note that we have

$$\|\vec{g}_n(0) - (Q(\cdot/\mu_n),0)\|_{H\times L^2(r\le\gamma_n\mu_n)} \le \varepsilon_n \longrightarrow 0.$$

Unscale the above by setting $\tilde{g}_n(t,r) = g_n(\mu_n t, \mu_n r)$, which gives

$$\|(\tilde{g}_n(0), \partial_t\tilde{g}_n(0)) - (Q(\cdot),0)\|_{H\times L^2(r\le\gamma_n)} \le \varepsilon_n \longrightarrow 0.$$

Now using [7, Corollary 2.6] and the finite speed of propagation we claim that we have

$$\|(\tilde{g}_n(\beta_n/2), \partial_t\tilde{g}_n(\beta_n/2)) - (Q(\cdot),0)\|_{H\times L^2(r\le\beta_n)} = o_n(1). \tag{3.84}$$

To see this, we need to show that [7, Corollary 2.6] applies. Indeed define

$$\hat{g}_{n,0}(r) := \begin{cases} \pi & \text{if } r \ge 2\gamma_n \\ \pi + \dfrac{\pi - \tilde{g}_n(0,\gamma_n)}{\gamma_n}(r - 2\gamma_n) & \text{if } \gamma_n \le r \le 2\gamma_n \\ \tilde{g}_n(0,r) & \text{if } r \le \gamma_n, \end{cases}$$

$$\hat{g}_{n,1}(r) = \begin{cases} \partial_t\tilde{g}_n(0,r) & \text{if } r \le \gamma_n \\ 0 & \text{if } r \ge \gamma_n. \end{cases}$$

Then, by construction we have $\vec{\hat{g}}_n \in \mathcal{H}_1$, and since

$$\|\vec{\hat{g}}_n - (\pi,0)\|_{H\times L^2(\gamma_n\le r\le 2\gamma_n)} \le C\delta_n$$

we then can conclude that

$$\begin{aligned} \|\vec{\hat{g}}_n - (Q,0)\|_{H\times L^2} &\le \|\vec{\hat{g}}_n - (Q,0)\|_{H\times L^2(r\le\gamma_n)} + \|\vec{\hat{g}}_n - (\pi,0)\|_{H\times L^2(\gamma_n\le r\le 2\gamma_n)} \\ &\quad + \|(\pi,0) - (Q,0)\|_{H\times L^2(r\ge\gamma_n)} \\ &\le C(\varepsilon_n + \delta_n + \gamma_n^{-1}). \end{aligned}$$

Now, given our choice of β_n, (3.84) follows from [7, Corollary 2.6] and the finite speed of propagation. Rescaling (3.84) we have

$$\|(g_n(\beta_n\mu_n/2), \partial_t g_n(\beta_n\mu_n/2)) - (Q(\cdot/\mu_n), 0)\|_{H\times L^2(r\le\beta_n\mu_n)} \longrightarrow 0.$$

This proves (3.82). Also note that by monotonicity of the energy on interior cones and the comparability of the energy and the $H\times L^2$ norm in $\mathcal{H}_0$, for small energies, we see that (3.42) and (3.78) imply that

$$\|(\tilde{b}_n(\beta_n\mu_n/2), \partial_t\tilde{b}_n(\beta_n\mu_n/2))\|_{H\times L^2(r\le\beta_n\mu_n)} \longrightarrow 0. \tag{3.85}$$

Next we prove (3.83). First we define

$$\tilde{g}_{n,0}(r) = \begin{cases} \pi - \dfrac{\pi - g_n(0,\mu_n\beta_n/2)}{\frac{1}{2}\mu_n\beta_n} r & \text{if } r \le \beta_n\mu_n/2 \\ g_n(0,r) & \text{if } r \ge \beta_n\mu_n/2 \end{cases}$$
$$\tilde{g}_{n,1}(r) = \dot{g}_n(0,r).$$

Then, let $\chi \in C^\infty([0,\infty))$ be defined so that $\chi(r) \equiv 1$ on the interval $[2,\infty)$ and $\operatorname{supp}\chi \subset [1,\infty)$. Define

$$\vec{\tilde{g}}_n(r) := \chi(4r/\beta_n\mu_n)(\vec{\tilde{g}}_n(r) - (\pi,0))$$
$$\vec{\tilde{b}}_n(r) := \chi(4r/\beta_n\mu_n)\vec{\tilde{b}}_n(r)$$

and observe that we have the following decomposition

$$\vec{\tilde{g}}_n(r) = \vec{h}_n(0,r) + \vec{\tilde{b}}_n(r) + o_n(1),$$

where the $o_n(1)$ is in the sense of $H\times L^2$—here we also have used (3.79). Moreover, the right-hand side above, without the $o_n(1)$ term, is a profile decomposition in the sense of [7, Corollary 2.15] because of Proposition 3.14 and [9, Lemma 11] or [7, Lemma 2.20]. We can then consider the nonlinear profiles. Note that by construction we have $\vec{\tilde{g}}_n \in \mathcal{H}_0$ and as in [7], we can use (3.81) to show that $\mathcal{E}(\vec{\tilde{g}}_n) \le C < 2\mathcal{E}(Q)$ for large n. The corresponding wave map evolution $\vec{\tilde{g}}_n(t) \in \mathcal{H}_0$ is thus global in time and scatters as $t\to\pm\infty$ by [7, Theorem 1.1]. We also need to check that $\mathcal{E}(\vec{\tilde{b}}_n) \le C < 2\mathcal{E}(Q)$. Note that by construction and the definition of $\tilde{b}_n$, we have

$$\begin{aligned}\mathcal{E}(\vec{\tilde{b}}_n) &\le \mathcal{E}(\vec{\tilde{b}}_n) + O\left(\int_0^\infty \frac{4r^2}{\beta_{n,0}^2\mu_n^2}(\chi')^2(4r/\beta_n\mu_n)\frac{b_n^2((1-\tau_n)r)}{r}\,dr\right) \\ &\quad + \int_{\beta_n\mu_n/2}^{\beta_n\mu_n} \frac{\sin^2(\chi(4r/\beta_n\mu_n)b_{n,0}((1-\tau_n)r))}{r}\,dr \\ &\le \mathcal{E}(\vec{\tilde{b}}_n) + O\left(\int_{\beta_n\lambda_n/2}^{\beta_n\lambda_n} \frac{b_{n,0}^2(r)}{r}\,dr\right) \\ &= \mathcal{E}(\vec{\tilde{b}}_n) + o_n(1) \le C < 2\mathcal{E}(Q),\end{aligned}$$

where the last line follows from (3.43) since $\beta_n \ll \alpha_n$.

Arguing as in [7], we can use Proposition 3.14, [7, Proposition 2.17] and [7, Lemma 2.18] to obtain the following nonlinear profile decomposition

$$\begin{aligned}\vec{\tilde{g}}_n(t,r) &= \vec{h}_n(t,r) + \vec{\tilde{b}}_n(t,r) + \vec{\tilde{\theta}}_n(t,r), \\ \|\vec{\tilde{\theta}}_n\|_{L_t^\infty(H\times L^2)} &\longrightarrow 0.\end{aligned}$$

Finally observe that by construction and the finite speed of propagation we have

$$\begin{aligned}\vec{\tilde{g}}_n(t,r) &= \vec{g}_n(t,r) - \pi, \\ \vec{\tilde{b}}_n(t,r) &= \vec{\tilde{b}}_n(t,r).\end{aligned}$$

for all $t \in \mathbb{R}$ and $r \in [\beta_n\mu_n/2 + |t|, \infty)$. Therefore, in particular we have

$$\vec{g}_n(\beta_n\mu_n/2, r) - (\pi, 0) = \vec{h}_n(\beta_n\mu_n/2, r) + \vec{\tilde{b}}_n(\beta_n\mu_n/2, r) + \vec{\tilde{\theta}}_n(\beta_n\mu_n/2, r)$$

for all $r \in [\beta_n\mu_n, \infty)$ which proves (3.83).

We can combine (3.82), (3.83), (3.85), and (3.79) together with the monotonicity of the energy on interior cones and the fact that $\|Q(\cdot/\mu_n) - \pi\|_{H(r\ge\beta_n\mu_n)} = o_n(1)$, to obtain the decomposition

$$\vec{g}_n(\beta_n\mu_n/2, r) = (Q(r/\mu_n), 0) + \vec{h}_n(\beta_n\mu_n/2, r) + \vec{\tilde{b}}_n(\beta_n\mu_n/2, r) + \vec{\tilde{\theta}}_n(r), \tag{3.86}$$

$$\|\vec{\tilde{\theta}}_n\|_{H\times L^2} \longrightarrow 0. \tag{3.87}$$

Now, let $s_n \to \infty$ be any sequence such that $s_n \ge \beta_n\mu_n/2$ for each n. The next step is to prove the following decomposition at time s_n:

$$\vec{g}_n(s_n, r) - (\pi, 0) = \vec{h}_n(s_n, r) + \vec{\tilde{b}}_n(s_n, r) + \vec{\zeta}_n(r) \quad \forall r \in [s_n, \infty), \tag{3.88}$$

$$\|\vec{\zeta}_n\|_{H\times L^2} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.89}$$

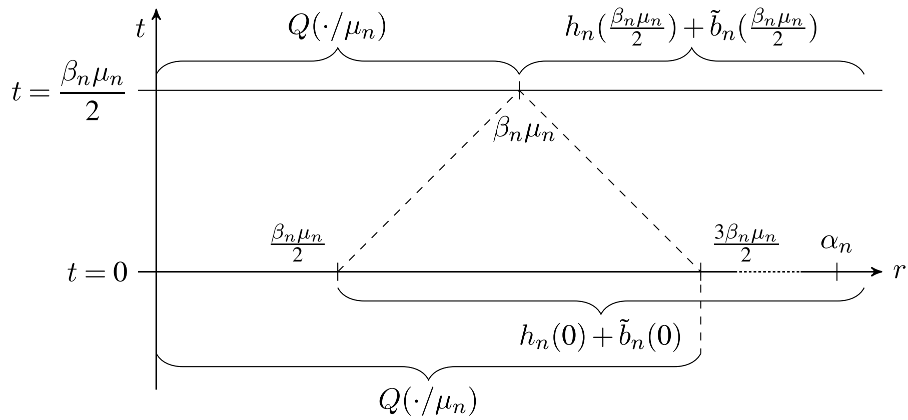

Figure 2. A schematic description of the evolution of the decomposition (3.75) from time $t = 0$ until time $t = \frac{\beta_n\mu_n}{2}$. At time $t = \frac{\beta_n\mu_n}{2}$ the decomposition (3.86) holds.

We proceed as in the proof of (3.83). By (3.82) we can argue as in Corollary 3.7 and find $\rho_n \to \infty$ with $\rho_n \ll \beta_n$ so that

(3.90) $$g_n(\beta_n\mu_n/2,\, \rho_n\mu_n) \longrightarrow \pi \quad \text{as } n \longrightarrow \infty.$$

Define

$$\hat{f}_{n,0}(r) = \begin{cases} \pi - \dfrac{\pi - g_n(\beta_n\mu_n/2,\, \rho_n\mu_n)}{\rho_n\mu_n} r & \text{if } r \le \rho_n\mu_n \\ g_n(\beta_n\mu_n/2, r) & \text{if } r \ge \rho_n\mu_n \end{cases}$$

$$\hat{f}_{n,1}(r) = \dot{g}_n(\beta_n\mu_n/2, r).$$

Let $\chi \in C^\infty$ be as above and set

$$\vec{f}_n(r) := \chi(2r/\rho_n\mu_n)(\vec{\hat{f}}_n(r) - (\pi, 0)),$$
$$\vec{\tilde{b}}_n(r) := \chi(2r/\rho_n\mu_n)\vec{\tilde{b}}_n(\beta_n\mu_n/2, r).$$

Observe that we have the following decomposition:

$$\vec{f}_n(r) = \vec{h}_n(\beta_n\mu_n/2, r) + \vec{\tilde{b}}_n(r) + o_n(1),$$

where the $o_n(1)$ above is in the sense of $H \times L^2$. Moreover, the right-hand side above, without the $o_n(1)$ term, is a profile decomposition in the sense of [7, Corollary 2.15] because of Proposition 3.14 and [9, Lemma 11] or [7, Lemma 2.20]. We can then consider the nonlinear profiles. Note that by construction we have $\vec{f}_n \in \mathcal{H}_0$ and, as usual, we can use (3.90) to show that $\mathcal{E}(\vec{f}_n) \le C < 2\mathcal{E}(Q)$ for large n. The

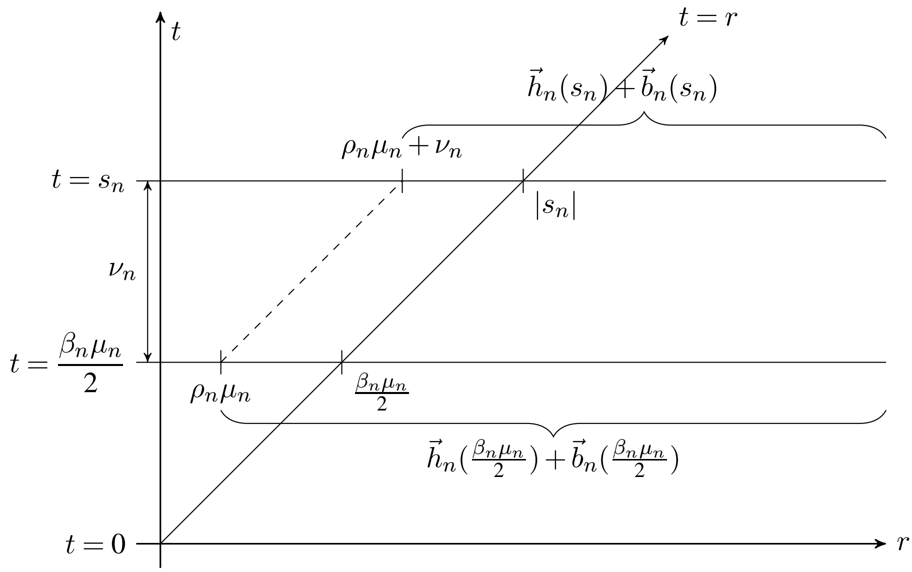

Figure 3. A schematic depiction of the evolution of the decomposition (3.86) up to time s_n. On the interval $[s_n, +\infty)$, the decomposition (3.88) holds.

corresponding wave map evolution $\vec{f}_n(t) \in \mathcal{H}_0$ is thus global in time and scatters as $t \to \pm\infty$ by [7, Theorem 1.1].

As in the proof of (3.83) it is also easy to show that $\mathcal{E}(\vec{\hat{b}}_n) \leq C < 2\mathcal{E}(Q)$ where here we use (3.85) instead of (3.43).

Again we can use Proposition 3.14, [7, Proposition 2.17] and [7, Lemma 2.18] to obtain the following nonlinear profile decomposition

$$\vec{f}_n(t,r) = \vec{h}_n(\beta_n\mu_n/2 + t, r) + \vec{\hat{b}}_n(t,r) + \vec{\tilde{\zeta}}_n(t,r),$$
$$\|\vec{\tilde{\zeta}}_n\|_{L^\infty_t(H\times L^2)} \longrightarrow 0.$$

In particular, for

$$\nu_n := s_n - \beta_n\mu_n/2$$

we have

$$\vec{f}_n(\nu_n, r) = \vec{h}_n(s_n, r) + \vec{\hat{b}}_n(\nu_n, r) + \vec{\tilde{\zeta}}_n(\nu_n, r).$$

By the finite speed of propagation we have that

$$\vec{f}_n(\nu_n, r) = \vec{g}_n(s_n, r) - (\pi, 0),$$
$$\vec{\hat{b}}_n(\nu_n, r) = \vec{\tilde{b}}_n(s_n, r)$$

as long as $r \geq \rho_n\mu_n + \nu_n$. Using the fact that $\rho_n \ll \beta_n$ we have that $s_n \geq \rho_n\mu_n + \nu_n$ and hence,

$$\vec{g}_n(s_n,r) - (\pi,0) = \vec{h}_n(s_n,r) + \vec{\tilde{b}}_n(s_n,r) + \vec{\tilde{\zeta}}_n(\nu_n,r) \quad \forall r \in [s_n,\infty).$$

Setting $\vec{\zeta}_n := \vec{\tilde{\zeta}}_n(\nu_n)$ we obtain (3.88) and (3.89). With this decomposition we can now complete the proof.

One the one hand observe that by rescaling, (3.34), and the fact that $2\tau_n s_n \geq \tau_n + \tau_n s_n$ for n large we have

$$\begin{aligned}
&\|\vec{g}_n(s_n) - \vec{h}_n(s_n) - (\pi,0)\|_{H\times L^2(r\geq s_n)} \\
&\quad = \|\vec{a}(\tau_n + \tau_n s_n, \tau_n\cdot) - (\pi,0)\|_{H\times L^2(r\geq s_n)} \\
&\quad = \|\vec{a}(\tau_n + \tau_n s_n) - (\pi,0)\|_{H\times L^2(r\geq \tau_n s_n)} \\
&\quad \leq \|\vec{a}(\tau_n + \tau_n s_n) - (\pi,0)\|_{H\times L^2(r\geq \frac{1}{2}(\tau_n+\tau_n s_n))} \\
&\quad \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned}$$

Combining the above with the decomposition (3.88) and (3.89) we obtain that

$$\|\vec{\tilde{b}}_n(s_n)\|_{H\times L^2(r\geq s_n)} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.91}$$

On the other hand, combining our assumption (3.72) and Corollary 3.15 we know that there exists $\alpha_0 > 0$ so that

$$\|\vec{\tilde{b}}_n(s_n)\|_{H\times L^2(r\geq s_n)} = \|\vec{b}_n(\tau_n s_n)\|_{H\times L^2(r\geq \tau_n s_n)} \geq \alpha_0\delta_0.$$

But this contradicts (3.91). □

We can now complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $\vec{a}(t)$ be defined as in (3.33). Recall that by (3.35) we have

$$\lim_{t\to\infty} \mathcal{E}(\vec{a}(t)) = \mathcal{E}(\vec{\psi}) - \mathcal{E}(\vec{\varphi}). \tag{3.92}$$

By Proposition 3.1 we have found a sequence of times $\tau_n \to \infty$ so that

$$\mathcal{E}(\vec{a}(\tau_n)) \longrightarrow \mathcal{E}(Q)$$

as $n \to \infty$. This then implies that

$$\lim_{t\to\infty} \mathcal{E}(\vec{a}(t)) = \mathcal{E}(Q).$$

We now use the variational characterization of Q to show that in fact $\|\dot{a}(t)\|_{L^2} \to 0$ as $t \to \infty$. To see this observe that since $a(t) \in \mathcal{H}_1$ we can deduce by [7, (2.18)] that

$$\mathcal{E}(Q) \longleftarrow \mathcal{E}(a(t), \dot{a}(t)) \geq \int_0^\infty \dot{a}^2(t,r)\, r\, dr + \mathcal{E}(Q).$$

Next observe that the decomposition in [7, Lemma 2.5] provides us with a function $\lambda : (0,\infty) \to (0,\infty)$ such that

$$\|a(t,\cdot) - Q(\cdot/\lambda(t))\|_H \leq \delta(\mathcal{E}(a(t),0) - \mathcal{E}(Q)) \longrightarrow 0.$$

This also implies that

$$\mathcal{E}(\vec{a}(t) - (Q(\cdot/\lambda(t)),0)) \longrightarrow 0 \tag{3.93}$$

as $t \to \infty$. Since $t \mapsto a(t)$ is continuous in H for $t \in [0,\infty)$ it follows from [7, Lemma 2.5] that $\lambda(t)$ is continuous on $[0,\infty)$. Therefore we have established that

$$\vec{\psi}(t) - \vec{\varphi}(t) - (Q(\cdot/\lambda(t)),0) \longrightarrow 0 \text{ in } H \times L^2 \quad \text{as } t \longrightarrow \infty.$$

It remains to show that $\lambda(t) = o(t)$. This follows immediately from the asymptotic vanishing of $\nabla_{t,r} a(t)$ outside the light cone and from (3.93). To see this observe that by (3.34) with $\lambda = 1$ we have that $a(t,r) - (\pi,0) = o(1)$ in $H \times L^2(r \geq t)$ as $t \to \infty$. Therefore we have

$$\mathcal{E}^\infty_{\frac{t}{\lambda(t)}}(Q) = \mathcal{E}^\infty_t(\pi - Q(\cdot/\lambda(t))) \leq \mathcal{E}(\vec{a}(t) - (Q(\cdot/\lambda(t)),0)) + o(1) \longrightarrow 0$$

as $t \to \infty$. But this then implies that $\frac{t}{\lambda(t)} \to \infty$ as $t \to \infty$. This completes the proof. □

CNRS AND ÉCOLE POLYTECHNIQUE, CENTRE DE MATHÉMATIQUES LAURENT SCHWARTZ UMR 7640, ROUTE DE PALAISEAU, 91128 PALAISEAU CEDEX, FRANCE
E-mail: cote@math.polytechnique.fr

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF CHICAGO, 5734 SOUTH UNIVERSITY AVENUE, CHICAGO, IL 60615
E-mail: cek@math.uchicago.edu, alawrie@math.uchicago.edu, schlag@math.uchicago.edu

---

REFERENCES

[1] H. Bahouri and P. Gérard, High frequency approximation of solutions to critical nonlinear wave equations, *Amer. J. Math.* **121** (1999), no. 1, 131–175.

[2] I. Bejenaru, J. Krieger, and D. Tataru, A codimension-two stable manifold of near soliton equivariant wave maps, *Anal. PDE* **6** (2013), no. 4, 829–857.

[3] A. Bulut, Maximizers for the Strichartz inequalities for the wave equation, *Differential Integral Equations* **23** (2010), no. 11-12, 1035–1072.

[4] D. Christodoulou and A. S. Tahvildar-Zadeh, On the asymptotic behavior of spherically symmetric wave maps, *Duke Math. J.* **71** (1993), no. 1, 31–69.

[5] ________, On the regularity of spherically symmetric wave maps, *Comm. Pure Appl. Math.* **46** (1993), no. 7, 1041–1091.

[6] R. Côte, Instability of nonconstant harmonic maps for the $(1+2)$-dimensional equivariant wave map system, *Internat. Math. Res. Notices IMRN* (2005), no. 57, 3525–3549.

[7] R. Côte, C. E. Kenig, A. Lawrie, and W. Schlag, Characterization of large energy solutions of the equivariant wave map problem: I, *Amer. J. Math.* **137** (2015), no. 1, 139–207.

[8] R. Côte, C. E. Kenig, and F. Merle, Scattering below critical energy for the radial 4D Yang-Mills equation and for the 2D corotational wave map system, *Comm. Math. Phys.* **284** (2008), no. 1, 203–225.

[9] R. Côte, C. E. Kenig, and W. Schlag, Energy partition for the linear radial wave equation, *Math. Ann.* **358** (2014), no. 3-4, 573–607.

[10] R. Donninger and J. Krieger, Nonscattering solutions and blowup at infinity for the critical wave equation, *Math. Ann.* **357** (2013), no. 1, 89–163.

[11] T. Duyckaerts, C. Kenig, and F. Merle, Universality of blow-up profile for small radial type II blow-up solutions of the energy-critical wave equation, *J. Eur. Math. Soc. (JEMS)* **13** (2011), no. 3, 533–599.

[12] ________, Profiles of bounded radial solutions of the focusing, energy-critical wave equation, *Geom. Funct. Anal.* **22** (2012), no. 3, 639–698.

[13] ________, Universality of the blow-up profile for small type II blow-up solutions of the energy-critical wave equation: the nonradial case, *J. Eur. Math. Soc. (JEMS)* **14** (2012), no. 5, 1389–1454.

[14] ________, Classification of the radial solutions of the focusing, energy-critical wave equation, *Cambridge Journal of Mathematics* **1** (2013), no. 1, 75–144.

[15] M. Grillakis, Classical solutions for the equivariant wave maps in $1+2$ dimensions, preprint, 1991.

[16] F. Hélein, *Harmonic Maps, Conservation Laws and Moving Frames. Translated from the 1996 French Original. With a Foreword by James Eells*, 2nd ed., *Cambridge Tracts in Math.*, vol. 150, Cambridge University Press, Cambridge, 2002.

[17] C. E. Kenig and F. Merle, Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case, *Invent. Math.* **166** (2006), no. 3, 645–675.

[18] ________, Global well-posedness, scattering and blow-up for the energy-critical focusing non-linear wave equation, *Acta Math.* **201** (2008), no. 2, 147–212.

[19] J. Krieger, Global regularity and singularity development for wave maps, *Surveys in Differential Geometry. Vol. XII. Geometric Flows*, *Surv. Differ. Geom.*, vol. 12, International Press, Somerville, MA, 2008, pp. 167–201.

[20] J. Krieger and W. Schlag, *Concentration Compactness for Critical Wave Maps*, *EMS Monogr. Math.*, European Mathematical Society (EMS), Zürich, 2012.

[21] J. Krieger, W. Schlag, and D. Tataru, Renormalization and blow up for charge one equivariant critical wave maps, *Invent. Math.* **171** (2008), no. 3, 543–615.

[22] ________, Renormalization and blow up for the critical Yang-Mills problem, *Adv. Math.* **221** (2009), no. 5, 1445–1521.

[23] A. Lawrie and W. Schlag, Scattering for wave maps exterior to a ball, *Adv. Math.* **232** (2013), 57–97.

[24] H. Lindblad and C. D. Sogge, On existence and scattering with minimal regularity for semilinear wave equations, *J. Funct. Anal.* **130** (1995), no. 2, 357–426.

[25] P. Raphaël and I. Rodnianski, Stable blow up dynamics for the critical co-rotational wave maps and equivariant Yang-Mills problems, *Publ. Math. Inst. Hautes Études Sci.* (2012), 1–122.

[26] I. Rodnianski and J. Sterbenz, On the formation of singularities in the critical $\mathrm{O}(3)$ σ-model, *Ann. of Math. (2)* **172** (2010), no. 1, 187–242.

[27] J. Sacks and K. Uhlenbeck, The existence of minimal immersions of 2-spheres, *Ann. of Math. (2)* **113** (1981), no. 1, 1–24.

[28] J. Shatah, Weak solutions and development of singularities of the $\mathrm{SU}(2)$ σ-model, *Comm. Pure Appl. Math.* **41** (1988), no. 4, 459–469.

[29] J. Shatah and M. Struwe, *Geometric Wave Equations*, *Courant Lect. Notes Math.*, vol. 2, American Mathematical Society, Providence, RI, 1998.

[30] ________, The Cauchy problem for wave maps, *Internat. Math. Res. Notices IMRN* (2002), no. 11, 555–571.

[31] J. Shatah and A. S. Tahvildar-Zadeh, On the Cauchy problem for equivariant wave maps, *Comm. Pure Appl. Math.* **47** (1994), no. 5, 719–754.

[32] C. D. Sogge, *Lectures on Non-Linear Wave Equations*, 2nd ed., International Press, Boston, MA, 2008.

[33] J. Sterbenz and D. Tataru, Energy dispersed large data wave maps in $2+1$ dimensions, *Comm. Math. Phys.* **298** (2010), no. 1, 139–230.

[34] ________, Regularity of wave-maps in dimension $2+1$, *Comm. Math. Phys.* **298** (2010), no. 1, 231–264.

[35] M. Struwe, Equivariant wave maps in two space dimensions, *Comm. Pure Appl. Math.* **56** (2003), no. 7, 815–823.

[36] T. Tao, Global regularity of wave maps. II. Small energy in two dimensions, *Comm. Math. Phys.* **224** (2001), no. 2, 443–544.

[37] ________, Global regularity of wave maps III–VII, preprints, 2008–2009.

[38] D. Tataru, On global existence and scattering for the wave maps equation, *Amer. J. Math.* **123** (2001), no. 1, 37–77.