ENHANCED DISSIPATION, HYPOELLIPTICITY FOR PASSIVE SCALAR EQUATIONS WITH FRACTIONAL DISSIPATION

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Dedicated to Yijun He

ABSTRACT. We consider the passive scalar equations subject to shear flow advection and fractional dissipation. The enhanced dissipation estimates are derived. For the classical passive scalar equation ($\gamma = 1$), our result agrees with the sharp one obtained in [46].

1. INTRODUCTION

We consider the passive scalar equations subject to shear advection and fractional dissipation:

$$(1.1) \qquad \partial_t n + u(y)\partial_x n = -\nu(-\Delta_x)^\gamma n - \nu(-\Delta_y)^\gamma n,$$
$$n(t = 0, x, y) = n_0(x, y), \quad (x, y) \in \mathbb{T}^2 = [-\pi, \pi]^2.$$

Their hypoelliptic counterparts read as follows

$$(1.2) \qquad \partial_t \eta + u(y)\partial_x \eta = -\nu(-\Delta_y)^\gamma \eta,$$
$$\eta(t = 0, x, y) = \eta_0(x, y), \quad (x, y) \in \mathbb{T}^2.$$

Here n, η denote the densities transported by the flow. The fractional dissipation order γ takes value in $(0, 2]$. The viscosity ν is small, i.e., $\nu \in (0, 1)$. Since the dynamics (1.1), (1.2) preserve the average of solutions, one can subtract the average and assume without loss of generality that ([18])

$$(1.3) \qquad \int_\mathbb{T} n_0(x, y)dx = 0, \quad \int_\mathbb{T} \eta_0(x, y)dx = 0, \quad \forall y \in \mathbb{T}.$$

Assume the shear flow profiles $u(y)$ have finitely many critical points $\{y_i^\star\}_{i=1}^N$. The *vanishing order* j_i associated with each critical point $y_i^\star$ is defined as the smallest integer such that

$$(1.4) \qquad u^{(\ell)}(y_i^\star) = 0, \quad u^{(j_i+1)}(y_i^\star) \neq 0, \quad \forall 1 \le \ell \le j_i,\, \ell \in \mathbb{N}.$$

The *maximal vanishing order* j_m of the shear flow profile $u(y)$ is $j_m := \max_{i=1}^N\{j_i\}$. Since any smooth shear flow profiles on the torus $\mathbb{T}$ have at least one critical point, the maximal vanishing orders j_m are greater than 1. If the maximal vanishing order is 1, the shear flow is *nondegenerate*.

The enhanced dissipation effect of the classical passive scalar equations ($\gamma = 1$) subject to shear flow has attracted much attentions from the mathematical fluid mechanics community in the recent years. In the paper [6], J. Bedrossian and M. Coti-Zelati applied hypocoercivity functional ([44, 3]) to show that if ν is smaller than a universal threshold ν_0, the following enhanced dissipation estimate holds for some universal constants $C > 1$, $\delta_{ED} \in (0, 1)$,

$$(1.5) \qquad \|\eta(t)\|_{L^2} \le C\|\eta_0\|_{L^2}e^{-\delta_{ED}d(\nu)|\log\nu|^{-2}t}, \quad d(\nu) = \frac{j_m + 1}{j_m + 3}, \forall t \in [0, \infty).$$

Their result was later improved by D. Wei [46]. Combining the resolvent estimates and a Gearhart-Prüss type theorem, D. Wei removed the logarithmic correction in the dissipation rate. Later, M. Coti Zelati and T. Drivas showed that the enhanced dissipation rate $d(\nu)$ appeared in (1.5) is sharp ([20]). The underlying mechanism of the enhanced dissipation effect is that the shear flow advection triggers the phase mixing phenomenon ([39, 49, 12]), which amplifies the damping effect of the dissipation operators (see, e.g., [14, 18]). Similar phase mixing phenomena play a fundamental role in Landau damping, see, e.g., [39, 13, 5]. Enhanced dissipation effect of the rough shear flows, and its relation to mixing are explored in [46, 16].

The shear flows' enhanced dissipation effect has found applications in various problems in fluid mechanics, plasma physics, and biology. First of all, the shear flows' enhanced dissipation is one of the stabilizing mechanisms in hydrodynamic stability. We refer the interested readers to the study of stability of the Couette flows ([41, 14, 9, 7, 8, 15, 10]), the Poiseuille flows ([21, 22]), and the Kolmogorov flows ([47, 30, 37]). In plasma physics, the enhanced collision effect, equivalent to the enhanced dissipation effect, stabilizes the plasma and prevents the echo-chain instability ([4]). In biology, the enhanced dissipation effect of the ambient shear flows suppresses the chemotactic blow-ups ([11, 27]).

The enhanced dissipation effect of the shear flow is *heterogeneous*. If the initial data n_0 of the passive scalar equation depends only on y-variables, $n(t, y)$ solves the heat equation, and no enhanced dissipation is possible. Hence the zero-average constraint (1.3) is enforced. However, there exist fluid flows inducing the enhanced dissipation effect in all directions. These are the relaxation-enhancing flows. The concept is first introduced by P. Constantin et al., [17]. In the papers [18, 25], the authors prove that flows with mixing properties are relaxation enhancing. Explicit constructions of mixing flows have attracted much attention in the dynamical system and fluid mechanics community, see, e.g., [45, 35, 42, 31, 1, 2, 48, 23], and the references therein. The relaxation enhancing flows find applications in various problems, see, e.g., [33, 28, 32, 24].

Much less is known for the systems (1.1) and (1.2). The enhanced dissipation result for the passive scalar equation (1.1) is obtained in [18]. However, the enhanced dissipation rate obtained is not sharp in general. Recently, an enhanced dissipation estimate for the $\gamma = 2$ case is derived in [19].

By taking the Fourier transform in the x-variable, one obtains the equations for each Fourier mode:

$$(1.6) \qquad \partial_t \widehat{n}_k + iu(y)k\widehat{n}_k = -\nu|k|^{2\gamma}\widehat{n}_k - \nu(-\Delta_y)^\gamma \widehat{n}_k, \quad \widehat{n}_k(t = 0, y) = \widehat{n}_{0;k}(y);$$

$$(1.7) \qquad \partial_t \widehat{\eta}_k + iu(y)k\widehat{\eta}_k = -\nu(-\Delta_y)^\gamma \widehat{\eta}_k, \quad \widehat{\eta}_k(t = 0, y) = \widehat{\eta}_{0;k}(y).$$

The first main theorem of the paper is the following

Theorem 1. *Consider the equation* (1.2) *subject to initial condition* $\eta_0 \in L^2(\mathbb{T}^2)$. *Assume that the shear flow profile* $u(\cdot) \in C^\infty(\mathbb{T})$ *has finitely many critical points* $\{y_i^\star\}_{i=1}^N$ *and the maximal vanishing order* $j_m \geq 1$ *is finite. Further assume that there exist* $R_i \in (0, \frac{\pi}{10})$, $i \in \{1, 2, ...N\}$ *such that in the neighborhood* $B(y_i^\star; R_i) \subset \mathbb{T}$, *the following estimate holds for some universal constant* $C_1(u) > 1$,

$$(1.8) \qquad \frac{1}{C_1(u)}|y - y_i^\star|^{j_i} \leq |u'(y)| \leq C_1(u)|y - y_i^\star|^{j_i}, \quad \forall y \in B(y_i^\star; R_i).$$

Then there exists a viscosity threshold $\nu_0 = \nu_0(u)$ *such that if* $\nu \leq \nu_0$, *the following enhanced dissipation estimate holds,*

$$(1.9) \quad \|\eta(t)\|_2 \leq C\|\eta_0\|_2 e^{-\delta_{ED}d(\nu)|\log \nu|^{-\beta(\gamma)}t}, \quad \forall t \geq 0, \quad d(\nu) = \frac{j_m + 1}{j_m + 1 + 2\gamma}, \quad \gamma \in (1/2, 2].$$

Here the constants $C > 1$, $\delta_{ED} > 0$ *depend only on the shear profile* u. *The parameter* $\beta = \beta(\gamma)$ *depends on the fractional dissipation order* γ *and vanishes for* $\gamma \in [1, 2]$. *The explicit form of* β *is* $\beta(\gamma) = 8\gamma(1 - \gamma)\mathbf{1}_{\gamma \in (1/2, 1)}$.

For the k-by-k system (1.7)*, the following enhanced dissipation estimate holds*

$$(1.10) \quad \|\widehat{\eta}_k(t)\|_2 \le C\|\widehat{\eta}_{0;k}\|_2 e^{-\delta_{ED}\nu^{\frac{j_m+1}{j_m+1+2\gamma}}|k|^{\frac{2\gamma}{j_m+1+2\gamma}}|\log(\nu|k|^{-1})|^{-\beta(\gamma)}t}, \ \forall t \ge 0, \ k \ne 0, \ \gamma \in (1/2, 2].$$

Remark 1.1. *The k-by-k estimate* (1.10) *implies that the solution to the hypoelliptic equation* (1.2) *gain Gevrey regularity in x-direction instantly. This gain in regularity is related to Hörmander's hypoellipticity theorem, see, e.g.,* [34, 29, 40, 38]*.*

Remark 1.2. *If the shear flow u is analytic near the critical points $y_i^\star$, then the condition* (1.8) *holds.*

Remark 1.3. *Our argument does not provide the enhanced dissipation estimate in the regime $\gamma \in (0, 1/2]$. The main reason is that our proof requires an apriori L^∞-bound of the solutions to the resolvent equation. If $\gamma \in (0, 1/2]$, Sobolev embedding does not guarantee such L^∞-estimate. However, a recent manuscript* [36] *seems to suggest that the enhanced dissipation estimate with rate $\nu^{\frac{j_m+1}{j_m+1+2\gamma}}$ might still hold in the range $\gamma \in (0, 1/2]$. We will leave that as a conjecture to pursuit in the future.*

Remark 1.4. *If the shear flow profile is non-degenerate, i.e., $j_m = 1$, then the enhanced dissipation rate (modulo logarithmic correction) is $\nu^{\frac{1}{1+\gamma}}$. When $\gamma = 1$, we recover the classical rate $\nu^{1/2}$.*

Remark 1.5. *The logarithmic loss here for $\gamma \in (1/2, 1)$ comes from the estimation of the $\dot{H}^{1/2}$-semi-norm of specific functions. New ideas are needed to drop the logarithmic factor or extend the result to $\gamma \in (0, 1/2]$. If γ ranges from $[1, 2]$, the $\dot{H}^1$-norm will be applied instead and no loss of $|\log(\nu|k|^{-1})|^\beta$ will appear in the dissipation rate* (1.10)*.*

The second main theorem provides the enhanced dissipation for the equations (1.1) and (1.6).

Theorem 2. *Consider the equation* (1.1) *subject to initial condition $n_0 \in L^2(\mathbb{T}^2)$. Assume the conditions in Theorem 1. Then there exists a viscosity threshold $\nu_0 = \nu_0(u)$ such that if $\nu \le \nu_0$, the following enhanced dissipation estimate holds*

$$(1.11) \quad \|n(t)\|_2 \le C\|n_0\|_2 e^{-\nu t - \delta_{ED}d(\nu)|\log \nu|^{-\beta(\gamma)}t}, \quad \forall t \ge 0, \quad d(\nu) = \frac{j_m+1}{j_m+1+2\gamma}, \gamma \in (1/2, 2].$$

Here the constants $C > 1$, $\delta_{ED} > 0$ depend only on the shear profile u. The parameter $\beta(\gamma) = 8\gamma(1-\gamma)\mathbf{1}_{\gamma \in (1/2,1)}$ vanishes for $\gamma \in [1, 2]$.

For the k-by-k system (1.6) *($k \ne 0$), the following estimate holds for constant C, δ_{ED} which only depend on $u(\cdot)$,*

$$(1.12) \quad \|\widehat{n}_k(t)\|_2 \le C\|\widehat{n}_{0;k}\|_2 e^{-\nu|k|^{2\gamma}t - \delta_{ED}\nu^{\frac{j_m+1}{j_m+1+2\gamma}}|k|^{\frac{2\gamma}{j_m+1+2\gamma}}|\log(\nu|k|^{-1})|^{-\beta(\gamma)}t}, \ \forall t \ge 0, \ \gamma \in (1/2, 2].$$

Remark 1.6. *It is worth noting that our method can be adapted to provide the same enhanced dissipation estimate for passive scalar solutions subject to the classical fraction dissipation operator $-(-\Delta)^\gamma = -(|\partial_x|^2 + |\partial_y|^2)^\gamma$. Details of the adjustments are highlighted in Remark 2.1.*

Remark 1.7. *Similar argument yields enhanced dissipation for shear flows whose profile u are Lipschitz. Consider profile u with finitely many critical points. Furthermore, assume that the absolute value of the derivatives of the profile are strictly positive whenever they exist, i.e., $\min_{y \in \mathbb{T}}\{|u'(y)| \,|\, u'(y) \text{ exists.}\} > c > 0$. Then the enhanced dissipation estimate holds with rate $O(\nu^{\frac{1}{2\gamma+1}})$ (modulo logarithmic factors). The argument is similar to the proof of Theorem 4 in* [26]*.*

Our analysis combines a spectral gap estimate in the spirit of the work [6] and the Gearhart-Prüss type theorem proven in [46]. Furthermore, detailed resolvent estimates are carried out to prove the result. The resolvent estimate has found applications in various works in the hydrodynamics stability, see, e.g., [15, 37, 22].

The paper is organized as follows: in section 2, we present the proof of the main theorems; in section 3, we prove the main resolvent estimates (Proposition 2.1).

Notation: Throughout the paper, the constant B, C are constants independent of ν, k and are changing from line to line. In the section 3, the constant C can depend on a small constant δ and we will specify when it happens. The constants C_0, C_1, $C_{\text{spec}}, \ldots$ will be explicitly defined. The notations T_1, T_2, T_3, ... denote terms in long expressions and will be recycled after the proof of each lemma. Hence the meanings of $T_{(\cdot)}$ change from lemma to lemma. We use $|A|$ to denote the area of the set A.

We consider the Fourier transform *only in the x variable*, and denote it and its inverse as

$$\widehat{f}_k(y) := \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} f(x,y) dx, \quad \check{g}(x,y) := \sum_{k=-\infty}^{\infty} g_k(y) e^{ikx}.$$

If the function only depends on the y-variable, we use similar formulas to calculate the Fourier transform/inverse transform in y. The symbol $\langle f \rangle$ represents average on the torus $\mathbb{T}$, i.e., $\langle f \rangle = \frac{1}{2\pi} \int_{\mathbb{T}} f(y) dy$. The symbol $\overline{f}$ denotes the complex conjugate. For any measurable function $m(\xi)$, we define the Fourier multiplier $m(\nabla) f := (m(\xi) \widehat{f}(\xi))^{\vee}$. The L_y^p-norms are defined as

$$\|g\|_{L_y^p} = \left(\int_{\mathbb{T}} |g(y)|^p dy \right)^{\frac{1}{p}}, \quad p \in [1, \infty),$$

with natural extension to $p = \infty$. The $\dot{H}^{\gamma}$-seminorm and the H^{γ}-norm are defined as follows:

$$\|g\|_{\dot{H}_y^{\gamma}}^2 = \||\partial_y|^{\gamma} g\|_{L_y^2}^2 = \sum_{\ell \in \mathbb{Z}} |\ell|^{2\gamma} |\widehat{g}_{\ell}|^2, \quad \|g\|_{H_y^{\gamma}}^2 = \|g\|_{\dot{H}_y^{\gamma}}^2 + \|g\|_{L_y^2}^2.$$

2. Proof of Theorem 1 and Theorem 2

In this section, we prove Theorem 1 and Theorem 2. The main goal is to derive the k-by-k estimate (1.10), where k is horizontal wave number.

We make two preparations for the proof of the main theorem 1. First, we reduce the problem for general wave number $k \in \mathbb{Z} \backslash \{0\}$ to the case $k = 1$. Secondly, we present a semigroup estimate from [46].

If the estimate (1.10) is proven for $k = 1$, then by changing the sign of the shear $u(y)$, we obtain the estimate for $k = -1$. Now we consider the general case $k \in \mathbb{Z} \backslash \{0\}$ and rewrite the equation (1.7) as follows

$$\frac{1}{|k|} \partial_t \widehat{\eta}_k + iu(y) \frac{k}{|k|} \widehat{\eta}_k = -\frac{\nu}{|k|} (-\Delta_y)^{\gamma} \widehat{\eta}_k.$$

By rescaling in time $\tau = |k| t$ and setting $\widetilde{\nu} = \frac{\nu}{|k|}$, we obtain that

$$\partial_{\tau} \widehat{\eta}_k + iu(y) \frac{k}{|k|} \widehat{\eta}_k = -\widetilde{\nu} (-\Delta_y)^{\gamma} \widehat{\eta}_k.$$

Application of the enhanced dissipation estimate (1.10) for $k = \pm 1$ yields the following estimate

$$\|\widehat{\eta}_k(t)\|_2 = \|\widehat{\eta}_k(\tau)\|_2 \leq C \|\widehat{\eta}_{0;k}\|_2 e^{-\delta_{ED} d(\widetilde{\nu}) |\log \widetilde{\nu}|^{-\beta(\gamma)} \tau} = C \|\widehat{\eta}_k(0)\|_2 e^{-\delta d(\nu |k|^{-1}) |k| |\log(\nu |k|^{-1})|^{-\beta(\gamma)} t}.$$

Now recalling the definition of $d(\cdot)$ (1.9) yields the estimate (1.10) for general $k \neq 0$. Hence in the remaining part of the paper, we focus on the $k = 1$ case and drop the subscript k.

The second preparation involved in the proof is the Gearhart-Prüss type theorem proven by D. Wei, [46]. The theorem translates spectral estimate into quantitative semigroup estimate under suitable conditions. We recall the key concepts in the paper [46]. Let X be a complex Hilbert space. Let H be a linear operator in X with domain $D(H)$. Denote $\mathcal{B}(X)$ as the space of bounded

linear operators on X equipped with operator norm $\|\cdot\|$ and I as the identity operator. A closed operator H is m-accretive if the set $\{\lambda|\mathrm{Re}\lambda < 0\}$ is contained in the resolvent set of H, and

$$(H + \lambda I)^{-1} \in \mathcal{B}(X), \quad \|(H + \lambda I)^{-1}\| \leq (\mathrm{Re}\lambda)^{-1}, \quad \forall\,\mathrm{Re}\lambda > 0.$$

An m-accretive operator H is accretive and densely defined. The $-H$ is a generator of a semigroup e^{-tH}. The decay rate of the semigroup e^{-tH} is determined by the following quantity

$$\Psi(H) := \inf\{\|(H - i\lambda I)f\|_X ; f \in D(H), \lambda \in \mathbb{R}, \|f\|_X = 1\}.$$

This is the content of the main theorem of the paper [46].

Theorem 3. *Assume that H is an m-accretive operator in a Hilbert space X. Then the following estimate holds:*

$$\|e^{-tH}\| \leq e^{-t\Psi(H)+\pi/2}, \quad \forall t \geq 0.$$

We define the function space X to be $X = L^2$ and the differential operator to be

$$H := \nu(-\Delta_y)^\gamma + iu(y), \quad \gamma \in (1/2, 2].$$

The domain of the operator is $D(H) = H^{2\gamma}(\mathbb{T})$. By testing the equation $(H + \lambda I)w = f$ by w and taking the real part, we obtain that

$$\nu\||\partial_y|^\gamma w\|_2^2 + \mathrm{Re}\lambda\|w\|_2^2 = \mathrm{Re}\int f\overline{w}dy.$$

Hence if the real part of the spectral parameter $\mathrm{Re}\lambda > 0$, we have that $\mathrm{Re}\lambda\|w\|_2 \leq \|f\|_2$, which in turn yields that $\|(H + \lambda I)^{-1}\| \leq (\mathrm{Re}\lambda)^{-1}$, $\mathrm{Re}\lambda > 0$. Therefore, the operator H is m-accretive.

Now we are ready to prove the key estimate (1.10). To apply Theorem 3, we consider the following resolvent equation associated with the hypoelliptic passive scalar equation (1.7)

$$(2.1) \qquad\qquad (H - i\lambda)w = \nu(-\Delta_y)^\gamma w + i(u(y) - \lambda)w = F.$$

Recall that the shear flow profile $u(\cdot)$ has critical values $\{u_i^\star = u(y_i^\star)\}_{i=1}^N$, which locate at critical points $\{y_i^\star\}_{i=1}^N$ with vanishing order $\{j_i\}_{i=1}^N$ (1.4). We present the following proposition, whose proof is postponed to Section 3.

Proposition 2.1. *Consider the resolvent equation (2.1). Assume conditions in Theorem 1. Further assume that the spectral parameter λ in (2.1) ranges on the real line $\mathbb{R}$. Then the following resolvent estimate holds if $\nu > 0$ is smaller than a threshold $\nu_0 = \nu_0(u)$,*

$$\nu^{\frac{1}{2}(1+\frac{j_m+1}{j_m+1+2\gamma})}\||\partial_y|^\gamma w\|_2 + \nu^{\frac{j_m+1}{j_m+1+2\gamma}}\|w\|_2 \leq C_\star(u)\|F\|_2, \quad \gamma \in [1,2];$$

$$(2.2) \qquad \nu^{\frac{1}{2}(1+\frac{j_m+1}{j_m+1+2\gamma})}\||\partial_y|^\gamma w\|_2 + \nu^{\frac{j_m+1}{j_m+1+2\gamma}}\|w\|_2 \leq C_\star(u)|\log\nu|^{8\gamma(1-\gamma)}\|F\|_2, \quad \gamma \in (1/2, 1).$$

Here j_m is the maximal vanishing order of all the critical points (1.4), i.e., $j_m = \max_{i\in\{1,...,N\}} j_i$. The constant $C_\star = C_\star(u) \geq 1$ depends only on the shear profile $u(\cdot)$.

Combining Theorem 3 and Proposition 2.1 yields the k-by-k estimate (1.10). Summing up all k-modes yields the estimate (1.9). This concludes the proof of Theorem 1. The proof of the estimates in Theorem 2 follows from the observation that $\widehat{n}_k(t, y)e^{\nu|k|^{2\gamma}t}$ solves the equation (1.7). Therefore the L^2-norm is bounded as in (1.10), i.e.,

$$\|\widehat{n}_k(t)e^{\nu|k|^{2\gamma}t}\|_2 \leq C\|\widehat{n}_{0;k}\|_2 e^{-\delta_{ED}\nu^{\frac{j_m+1}{j_m+1+2\gamma}}|k|^{\frac{2\gamma}{j_m+1+2\gamma}}|\log(\nu|k|^{-1})|^{-\beta}t}, \quad \forall t \geq 0, \quad \gamma \in (1/2, 2].$$

By multiplying both side by $e^{-\nu|k|^{2\gamma}t}$, we obtain the estimate (1.12). Summing up all k-modes yields (1.11). Hence the proof of Theorem 2 is completed.

Remark 2.1. *The above arguments can be adapted to treat the passive scalar solutions subject to classical fractional dissipation operator* $(-\Delta)^\gamma = (|\partial_x|^2 + |\partial_y|^2)^\gamma$, $\gamma \in (1/2, 2]$. *One of the adjustments is that one will not re-scale the time variable to get rid of* $|k|$. *As a result, we consider the operator* $H_{k,\nu} = \nu(|k|^2 + |\partial_y|^2)^\gamma + iu(y)k$ *and its resolvent. The constructions of augmented functions in the next section are similar. We refer the interested readers to the appendix of* [19] *for the treatment in the bi-Laplacian case.*

3. Proof of Proposition 2.1

In this section, we prove Proposition 2.1.

Following the paper [6], we first introduce a partition of unity on the torus and localize the solution w to (2.1) around each critical point $y_i^\star$. To this end, we consider $2r_i$-neighborhood $B(y_i^\star; 2r_i)$ around each critical point $y_i^\star$ for $0 < r_i \leq \frac{1}{10}\pi$. Further assume that the dilated balls $\{B(y_i^\star; 4r_i)\}_{i=1}^N$ are pair-wise disjoint. Next we define $\{\xi_i^2\}_{i=0}^N$ to be a partition of unity on the torus such that $\xi_i \in C^\infty(\mathbb{T})$ for $i \in \{0, 1, ..., N\}$ and support$(\xi_i) = B(y_i^\star; 2r_i)$ for $i \in \{1, 2, ..., N\}$. Moreover, for the index i ranges from 1 to N, $\xi_i(y) \equiv 1$ in the neighborhood $B(y_i^\star; r_i)$ and decays to zero as y approaches $\partial B(y_i^\star; 2r_i)$. The function $\xi_0^2 = 1 - \sum_{i=1}^N \xi_i^2$ has support away from all the critical points and hence $\min_{y \in \text{support}\xi_0} |u'(y)| \geq \frac{1}{C(u)}$ for some positive constant $C(u) > 0$ which is independent of ν. At each critical point $(y_i^\star, u_i^\star)$ with vanishing order j_i, by Taylor's theorem, the shear flow profile has the following expansion

$$u(y) - u_i^\star = \frac{u^{(j+1)}(y_i^\star)}{(j+1)!}(y - y_i^\star)^{j+1} + O(|y - y_i^\star|^{j+2}).$$

We choose the radius r_i small enough such that on the neighborhood $B(y_i^\star; 2r_i)$, the first term on the right hand side dominates, i.e.,

$$(3.1) \quad \frac{1}{C_0(u)}|y - y_i^\star|^{j+1} \leq |u(y) - u_i^\star| \leq C_0(u)|y - y_i^\star|^{j+1}, \quad \forall y \in \text{support}(\xi_i), \ i \in \{1, 2, ..., N\}.$$

Here the constant $C_0 \geq 1$ only depends on the shear profile. Moreover, we can choose $r_i \leq \frac{1}{4}R_i$ in (1.8), such that on the support of ξ_i, $i \neq 0$, the following relation holds

$$(3.2) \quad \frac{1}{C_1(u)}|y - y_i^\star|^j \leq |u'(y)| \leq C_1(u)|y - y_i^\star|^j, \quad \forall y \in \text{support}(\xi_i), \ i \in \{1, 2, ..., N\}.$$

Since the above choice of r_i depends only on the shear profile u, there exists a constant $C(u)$, which is independent of the viscosity ν, such that the following estimate holds

$$(3.3) \quad \|\xi_i\|_{W^{4,\infty}(\mathbb{T})} \leq C(u), \quad \forall i \in \{0, 1, 2, ..., N\}.$$

Next we present some energy relations associated to the equation (2.1) and specify the interesting range of the spectral parameter λ. By testing the equation (2.1) against the conjugate $\overline{w}$ and taking the real and imaginary part, we obtain that

$$(3.4) \quad \nu \||\partial_y|^\gamma w\|_2^2 = \text{Re} \int_{\mathbb{T}} F\overline{w}dy \leq \|F\|_2\|w\|_2$$

$$(3.5) \quad \int_{\mathbb{T}} (u(y) - \lambda)|w|^2 dy = \text{Im} \int_{\mathbb{T}} F\overline{w}dy,$$

Testing the equation (2.1) with $g\overline{w}$, where g is any smooth real-valued function on $\mathbb{T}$, yields the following equation

$$(3.6) \quad \int_{\mathbb{T}} (u(y) - \lambda)g|w|^2 dy = \text{Im} \int_{\mathbb{T}} F\overline{w}gdy - \nu\text{Im} \int_{\mathbb{T}} |\partial_y|^\gamma w|\partial_y|^\gamma(g\overline{w})dy.$$

Direct application of these energy equalities ensures the estimate (2.2) given that the spectral parameter λ is away from the range of $u(\cdot)$. This is the content of the next lemma.

Lemma 3.1. *Assume that the spectral parameter $\lambda \in \mathbb{R}$ is away from the range of shear profile $u(\cdot)$ in the sense that*

$$(3.7) \qquad \min_{y \in \mathbb{T}} |\lambda - u(y)| \geq \delta \nu^{\frac{j_m+1}{j_m+1+2\gamma}}.$$

Here the small parameter $\delta \in (0,1)$ is independent of the viscosity ν and j_m is the maximal vanishing order of the shear u. Then following estimate holds

$$\nu^{1+\frac{j_m+1}{j_m+1+2\gamma}} \||\partial_y|^\gamma w\|_{L_y^2}^2 + \nu^{2\frac{j_m+1}{j_m+1+2\gamma}} \|w\|_{L_y^2}^2 \leq C(\delta^{-1}) \|F\|_{L_y^2}^2.$$

Remark 3.1. *The parameter $\delta = \delta(u) > 0$ will be chosen in (3.17). Hence the estimate we obtain is consistent with (2.2).*

Proof. The L^2-estimate is the key. Applying the relation (3.5), the fact that $u(y) - \lambda$ has fixed sign under the constraint (3.7), and the Hölder inequality yields the following estimate

$$(3.8) \qquad \delta \nu^{\frac{j_m+1}{j_m+1+2\gamma}} \|w\|_2 \leq C\|F\|_2.$$

Combining the relation (3.4) and the L^2-estimate (3.8), we have the higher regularity norm estimate

$$(3.9) \qquad \nu \||\partial_y|^\gamma w\|_2^2 \leq C\delta^{-1} \nu^{-\frac{j_m+1}{j_m+1+2\gamma}} \|F\|_2^2.$$

Combining the inequalities (3.8) and (3.9) yields the result. $\qquad \square$

Hence we focus on the case where the spectral parameter λ is close to the range of the shear profile u, i.e.,

$$\min_{y \in \mathbb{T}} |\lambda - u(y)| \leq \delta \nu^{\frac{j_m+1}{j_m+1+2\gamma}}.$$

Here $\delta(u)$ is chosen in (3.17). We decompose the L^2-norm $\|w\|_2^2$ into $N+1$ pieces with the partition of unity ξ_i^2,

$$\|w\|_2^2 = \sum_{i=0}^{N} \|w\xi_i\|_2^2.$$

Our primary goal is to derive L^2-estimate on each component $w\xi_i$, $\forall i \in \{0, 1, ..., N\}$,

$$(3.10) \qquad \begin{aligned} \|w\xi_i\|_2^2 \leq & C(B, u, \delta^{-1}, j_i)\nu^{-2\frac{j_i+1}{j_i+1+2\gamma}} |\log \nu|^{16\gamma\alpha(\gamma)} \|F\|_2^2 \\ & + \left(\frac{1}{B} + C(u, \delta^{-1}, j_i)\nu^{\frac{1}{j_m+1+2\gamma}} |\log\nu|^{4\alpha(\gamma)}\right) \|w\|_2^2, \end{aligned}$$

Here the universal constant $B \geq 1$ is arbitrary and will to be determined at the end of the proof (3.55). The j_i, $i \neq 0$ is the vanishing order of the critical points (1.4). If $i = 0$, $j_0 = 0$. The j_m is the maximal vanishing order. The parameter $\alpha = \alpha(\gamma)$ is

$$(3.11) \qquad \alpha(\gamma) = \mathbf{1}_{\gamma \in (1/2,1)}(1 - \gamma).$$

In the latter part of the proof, if the constant C depends on B or δ, we will explicitly spell out.

To prove the estimate (3.10), we first consider the components $\|w\xi_i\|_2^2$ with $i \neq 0$. We distinguish between three cases based on the relative position of λ and the value $u_i^\star = u(y_i^\star)$ at each critical point $y_i^\star$, $\forall i \in \{1, 2, ...N\}$:

$$(3.12) \qquad a) \quad i \in \mathcal{I}_{\text{near}}: \quad |\lambda - u_i^\star| \leq \delta \nu^{\frac{j_i+1}{j_i+1+2\gamma}};$$

$$(3.13) \qquad b) \quad i \in \mathcal{I}_{\text{interm}}: \quad |\lambda - u_i^\star| \geq \delta \nu^{\frac{j_i+1}{j_i+1+2\gamma}}, \quad \text{dist}(y_i^\star, \{z|u(z) = \lambda\}) \leq 3r_i;$$

$$(3.14) \qquad c) \quad i \in \mathcal{I}_{\text{far}}: \quad \text{dist}(y_i^\star, \{z|u(z) = \lambda\}) \geq 3r_i.$$

Here $2r_i$ is the radius of the support of ξ_i, $\forall i \in \{1, 2, ..., N\}$. We will prove the primary estimate (3.10) in case $a)$, $b)$, and $c)$ in Lemma 3.3, Lemma 3.4 and Lemma 3.5, respectively. Finally, we estimate the component $\|w\xi_0\|_2^2$ in Lemma 3.6. Once the estimate (3.10) is established, by summing all the contributions from different components, and taking B^{-1} and then ν_0 small enough, we will obtain the estimate (2.2).

Before proving the estimate (3.10) for $i \in \mathcal{I}_{\mathrm{near}}$, we introduce a crucial spectral gap estimate, which also plays a central role in [6].

Lemma 3.2. *Assume condition (1.8) and let $f \in H^\gamma(\mathbb{T})$. Consider a critical point $y_i^\star$ of the shear flow profile u with vanishing order $j = j_i \geq 1$ (1.4). The function $f_i = f\xi_i$ is supported in the $2r_i$-neighborhood of the critical point $y_i^\star$. Then the following estimate holds for some constant $C_{\mathrm{spec}}(u) \geq 1$,*

$$(3.15)\,\nu^{\frac{j+1}{j+1+2\gamma}}\|f_i\|_{L^2(\mathbb{T})}^2 \leq C_{\mathrm{spec}}(u)\nu\||\partial_y|^\gamma f_i\|_{L^2(\mathbb{T})}^2 + C_{\mathrm{spec}}(u)\nu^{\frac{1-j}{j+1+2\gamma}}\|u'f_i\|_{L^2(\mathbb{T})}^2, \quad \gamma \in \left(\frac{1}{2}, 2\right].$$

Remark 3.2. *As being discussed in the paper [6] (pages 12-13), the estimate (3.15) is related to the spectral gap of the differential operator $L := (-\Delta_z)^\gamma + |z|^{2j}$ on $\mathbb{R}$.*

Proof. First, we show that the following estimate on the torus implies (3.15),

$$(3.16)\qquad\qquad \sigma^{\frac{j}{j+\gamma}}\|f_i\|_{L^2(\mathbb{T})}^2 \leq C\sigma\||\partial_y|^\gamma f_i\|_{L^2(\mathbb{T})}^2 + C\||y-y_i^\star|^j f_i\|_{L^2(\mathbb{T})}^2.$$

Here the parameter $\sigma \in (0, 1)$ is any small enough number. Since $f_i = f\xi_i$ is localized near the i-th critical point $y_i^\star$, the last term makes sense. Combining the estimate (3.16) and the condition (1.8), we obtain

$$\sigma^{\frac{j}{j+\gamma}}\|f_i\|_{L^2(\mathbb{T})}^2 \leq C\sigma\||\partial_y|^\gamma f_i\|_{L^2(\mathbb{T})}^2 + C\|u'f_i\|_{L^2(\mathbb{T})}^2.$$

By setting $\sigma = \nu^{2\frac{j+\gamma}{j+1+2\gamma}}$ in the above inequality, we have (3.15).

To prove the estimate (3.16), we first consider the $z \in \mathbb{R}$ variable. On $\mathbb{R}$, the following estimate holds

$$\|g\|_{L^2(\mathbb{R})}^2 \leq \int_\mathbb{R} \psi_{[-2\varepsilon,2\varepsilon]}(z)|g(z)|^2 dz + C(\varepsilon^{-1})\int_\mathbb{R} |g(z)|^2|z|^{2j}dz,$$

where $\varepsilon > 0$ is a universal small constant. Here, $\psi_{[-2\varepsilon,2\varepsilon]}$ is a smooth cut-off function which is 1 on $[-\varepsilon,\varepsilon]$ and has support in $[-2\varepsilon,2\varepsilon]$. Now we make the change of variables $g(z) = f_i(\kappa^{-1}z+y_i^\star) = (f\xi_i)(\kappa^{-1}z+y_i^\star)$, $y-y_i^\star = \kappa^{-1}z$, for $\kappa > 1$. In the y-coordinate, the estimate above can be rewritten as follows :

$$\kappa\int_\mathbb{R} |f_i(y)|^2 dy \leq \kappa\|f_i\|_{L^\infty(\mathbb{T})}^2 \int_\mathbb{R} \psi_{[-2\varepsilon,2\varepsilon]}(\kappa(y-y_i^\star))dy + C(\varepsilon^{-1})\kappa^{2j+1}\int_\mathbb{R} |f_i(y)|^2|y-y_i^\star|^{2j}dy$$

$$\leq C\varepsilon\|f_i\|_{L^\infty(\mathbb{T})}^2 + C(\varepsilon^{-1})\kappa^{2j+1}\int_\mathbb{R} |f_i(y)|^2|y-y_i^\star|^{2j}dy.$$

Now since f_i is localized, we have that the estimate also holds with the integral domain $\mathbb{R}$ replaced by the torus $\mathbb{T}$. Since $\gamma > 1/2$, the L^∞-norm can be controlled through H^γ-norm. Now applying the Hölder inequality, Young inequality and Gagliardo-Nirenberg interpolation inequality yields

that

$$\kappa \int_{\mathbb{T}} |f_i|^2 dy$$

$$\leq C\varepsilon(\|f_i - \langle f_i\rangle\|_{L^\infty(\mathbb{T})}^2 + \langle f_i\rangle^2) + C(\varepsilon^{-1})\kappa^{2j+1}\int_{\mathbb{T}} |f_i|^2 |y - y_i^\star|^{2j} dy$$

$$\leq C_{GN}\varepsilon\|f_i - \langle f_i\rangle\|_{L^2(\mathbb{T})}^{2-\frac{1}{\gamma}}\||\partial_y|^\gamma f_i\|_2^{\frac{1}{\gamma}} + C\varepsilon|\mathbb{T}|\|f_i\|_2^2 + C(\varepsilon^{-1})\kappa^{2j+1}\int |f_i|^2 |y - y_i^\star|^{2j} dy$$

$$\leq \frac{1}{2}\kappa\|f_i\|_{L^2(\mathbb{T})}^2 + C_{GN}\kappa^{1-2\gamma}\||\partial_y|^\gamma f_i\|_{L^2(\mathbb{T})}^2 + C\varepsilon|\mathbb{T}|\|f_i\|_2^2 + C(\varepsilon^{-1})\kappa^{2j+1}\|f_i|y - y_i^\star|^j\|_{L^2(\mathbb{T})}^2.$$

Recalling that $\kappa > 1$, choosing ε small enough and reorganizing the terms yield that

$$\kappa^{-2j}\|f_i\|_{L^2(\mathbb{T})}^2 \leq C\kappa^{-2j-2\gamma}\||\partial_y|^\gamma f_i\|_{L^2(\mathbb{T})}^2 + C\||y - y_i^\star|^j f_i\|_{L^2(\mathbb{T})}^2.$$

Now we set $\sigma = \kappa^{-2j-2\gamma} \leq 1$, then $\kappa^{-2j} = (\kappa^{-2j-2\gamma})^{\frac{2j}{2j+2\gamma}} = \sigma^{\frac{j}{j+\gamma}}$. As a result, we have derived the spectral gap estimate (3.16). This concludes the proof. $\qquad\square$

Next we consider critical points $\{y_i^\star\}_{i \in \mathcal{I}_{\text{near}}}$ in case a) (3.12).

Lemma 3.3. *Assume the conditions in Theorem 1. Assume that both of the following conditions hold:*
a) the parameter $\delta \in (0,1)$ is small, i.e.,

$$(3.17) \qquad\qquad 0 < \delta(u) \leq \frac{1}{9}C_1^2(u)C_{\text{spec}}(u),$$

where C_1 is defined in (1.8) and C_{spec} is defined in (3.15);
b) the threshold $\nu_0 > 0$ is smaller than a constant depending only on u and δ.
Then the estimate (3.10) holds for $i \in \mathcal{I}_{\text{near}}$ (3.12).

Proof. We focus on one critical point $y_i^\star$ and drop the subscript i in the vanishing order j_i. There are three main steps. In the first step, we introduce suitable cut-off functions and estimate their Sobolev norms. In the second and third step, we carry out the main estimates of the L^2-norm. Throughout the proof, we will choose the viscosity threshold $\nu_0(u, \delta)$ small in several occasions, and the final viscosity threshold will be chosen as the minimum of all.
Step #1: Cut-off functions and their Sobolev norms.
We define a smooth partition of unity on the domain $\mathbb{T}$, i.e., $1 = \psi_i^0 + \psi_i^+ + \psi_i^-$. First we choose the viscosity threshold $\nu_0(u)$ small so that $4\sqrt{C_0}\nu_0^{\frac{1}{j+1+2\gamma}} \leq \frac{1}{20}r_i$, where C_0 and r_i are defined in (3.1) and (3.2). The function ψ_i^0 has the following properties:

1) $\text{support}(\psi_i^0) = B(y_i^\star; 4\sqrt{C_0(u)}\delta^{\frac{1}{j+1}}\nu^{\frac{1}{j+1+2\gamma}})$;

2) $\psi_i^0(y) \equiv 1, \quad \forall y \in B(y_i^\star; 2\sqrt{C_0(u)}\delta^{\frac{1}{j+1}}\nu^{\frac{1}{j+1+2\gamma}})$;

3) $|u(y) - \lambda| \geq \delta\nu^{\frac{j+1}{j+1+2\gamma}}, \quad \forall y \in \text{support}(\xi_i)\backslash B(y_i^\star; 2\sqrt{C_0(u)}\delta^{\frac{1}{j+1}}\nu^{\frac{1}{j+1+2\gamma}})$.

Here $C_0(u) \geq 1$ is defined in (3.1). We check the last property as follows. By the condition (3.1), the assumption (3.12), and $j = j_i \geq 1$, we observe that for $y \in \text{support}(\xi_i)\backslash B(y_i^\star; 2\sqrt{C_0(u)}\delta^{\frac{1}{j+1}}\nu^{\frac{1}{j+1+2\gamma}})$,

$$|\lambda - u(y)| \geq \|u(y) - u_i^\star\| - |\lambda - u_i^\star\| \geq \frac{1}{C_0(u)}|y - y_i^\star|^{j+1} - \delta\nu^{\frac{j+1}{j+1+2\gamma}}$$

$$\geq \left(2^{j+1}C_0^{\frac{j+1}{2}}C_0^{-1} - 1\right)\delta\nu^{\frac{j+1}{j+1+2\gamma}} \geq \delta\nu^{\frac{j+1}{j+1+2\gamma}}.$$

The supports of the functions ψ_i^+ and ψ_i^- are adjacent to the support of ψ_i^0. Since the support of the functions $\psi_i^{\pm}\xi_i^2$ are included in $\text{support}(\xi_i)\backslash B(y_i^{\star};2\sqrt{C_0(u)}\delta^{\frac{1}{j+1}}\nu^{\frac{1}{j+1+2\gamma}})$, previous argument yields that

$$(3.18) \qquad |u(y)-\lambda| \geq \delta\nu^{\frac{j+1}{j+1+2\gamma}}, \quad \forall y \in \text{support}(\psi_i^{\pm}\xi_i^2).$$

Next we estimate the Sobolev norms. Since all three functions transition on interval of size $\mathcal{O}(\delta^{\frac{1}{j+1}}\nu^{\frac{1}{j+1+2\gamma}})$, their $\dot{H}^1, \dot{H}^2$-seminorms are bounded as follows

$$(3.19) \|\partial_y\psi_i^s\|_2 \leq C(u)\delta^{-\frac{1}{2(j+1)}}\nu^{-\frac{1}{2(j+1+2\gamma)}}, \quad \|\partial_y^2\psi_i^s\|_2 \leq C(u)\delta^{-\frac{3}{2(j+1)}}\nu^{-\frac{3}{2(j+1+2\gamma)}}, \quad \forall s \in \{0,+,-\}.$$

To estimate the $\dot{H}^{\gamma}$-seminorm for $\gamma \in (1/2,1)$, the $\dot{H}^{1/2}$-norm of the partition functions ψ_i^s are required:

$$\|\psi_i^s\|_{\dot{H}^{1/2}} \leq C(u,\delta^{-1},j)|\log(\nu)|, \quad s \in \{0,+,-\}.$$

The explicit estimate is as follows. Denote $\Lambda := 2\sqrt{C_0}\delta^{\frac{1}{j+1}}\nu^{\frac{1}{j+1+2\gamma}}$. Recall the Fourier characterization of the $\dot{H}^{\frac{1}{2}}$-seminorm: $\|\psi_i^s-\langle\psi_i^s\rangle\|_{\dot{H}^{1/2}}^2 = C\sum_{\ell\neq 0}|\widehat{\psi_i^s}(\ell)|^2|\ell|$, where $\widehat{\psi_i^s}(\ell) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\psi_i^s(y)e^{-i\ell y}dy$. Now we have the following two relations which are consequences of the integration by parts and the relation $e^{-i\ell y} = -\frac{1}{i\ell}\frac{d}{dy}e^{-i\ell y}$:

$$|\widehat{\psi_i^s}(\ell)| \leq \frac{1}{|\ell|}\int_{\mathbb{T}}|\partial_y\psi_i^s|dy \leq C\frac{1}{|\ell|}, \ell \neq 0;$$

$$|\widehat{\psi_i^s}(\ell)| \leq \frac{1}{|\ell|^2}\int_{\mathbb{T}}|\partial_{yy}\psi_i^s|dy \leq \frac{1}{|\ell|^2}\Lambda^{-1}, \ell \neq 0.$$

Now we estimate the $\dot{H}^{1/2}$-seminorm as follows

$$\|\psi_i^s\|_{\dot{H}^{1/2}}^2 = C\left(\sum_{0<|\ell|<\Lambda^{-1}} + \sum_{|\ell|\geq\Lambda^{-1}}\right)|\widehat{\psi_i^s}(\ell)|^2|\ell| \leq C\sum_{0<|\ell|<\Lambda^{-1}}\frac{1}{|\ell|} + C\sum_{|\ell|\geq\Lambda^{-1}}\frac{\Lambda^{-2}}{|\ell|^3} \leq C\log\Lambda^{-1} + C.$$

As a result, we have that $\|\psi_i^s\|_{\dot{H}^{1/2}} \leq C(u,\delta^{-1},j)|\log(\nu)|$.

Finally, we apply Gagliardo-Nirenberg interpolation inequality to derive the $\dot{H}^{\gamma}$ semi-norm,

$$\|\psi_i^s\|_{\dot{H}^{\gamma}} \leq C_{GN}\|\psi_i^s\|_{\dot{H}^{1/2}}^{2-2\gamma}\|\psi_i^s\|_{\dot{H}^1}^{2\gamma-1} \leq C(u,\delta^{-1},j)|\log\nu|^{2-2\gamma}\nu^{-\frac{2\gamma-1}{2(j+1+2\gamma)}}, \quad \gamma \in (1/2,1);$$

$$\|\psi_i^s\|_{\dot{H}^{\gamma}} \leq C_{GN}\|\psi_i^s\|_{\dot{H}^1}^{2-\gamma}\|\psi_i^s\|_{\dot{H}^2}^{\gamma-1} \leq C(u,\delta^{-1},j)\nu^{-\frac{2\gamma-1}{2(j+1+2\gamma)}}, \quad \gamma \in [1,2].$$

We combine these two estimates with the parameter $\alpha(\gamma)$ (3.11):

$$(3.20) \qquad \|\psi_i^s\|_{\dot{H}^{\gamma}} \leq C(u,\delta^{-1},j_i)\nu^{-\frac{2\gamma-1}{2(j+1+2\gamma)}}|\log\nu|^{2\alpha(\gamma)}, \quad \gamma \in (1/2,2].$$

We apply the Minkowski inequality to decompose the L^2-norm as follows

$$(3.21) \qquad \|w\xi_i\|_2^2 \leq 3\|w\psi_i^0\xi_i\|_2^2 + 3\|w\psi_i^+\xi_i\|_2^2 + 3\|w\psi_i^-\xi_i\|_2^2.$$

This concludes step #1.

Step #2: Estimation of the L^2-norm $\|w\psi_i^0\xi_i\|_2^2$. We apply the following product rule for $f,g \in H^{\gamma}\cap L^{\infty}$:

$$(3.22) \qquad \||\partial_y|^{\gamma}(fg)\|_{L_y^2(\mathbb{T})} \leq C\|f\|_{H_y^{\gamma}(\mathbb{T})}\|g\|_{L_y^{\infty}(\mathbb{T})} + C\|f\|_{L_y^{\infty}(\mathbb{T})}\|g\|_{H_y^{\gamma}(\mathbb{T})}$$

The proof of the product rule on $\mathbb{R}$ can be found in various textbooks (see, e.g., appendix of [43].) and a small modification yields (3.22). Application of the spectral gap (3.15) and the product rule

yields that

$$\nu^{\frac{j+1}{j+1+2\gamma}}\|w\psi_i^0\xi_i\|_2^2 \le C_{\text{spec}}(u)\nu\||\partial_y|^\gamma(w\psi_i^0\xi_i)\|_2^2 + C_{\text{spec}}(u)\nu^{\frac{1-j}{j+1+2\gamma}}\|u'w\psi_i^0\xi_i\|_2^2$$

$$\le C(u)\nu(\||\partial_y|^\gamma w\|_2^2 + \|w\|_2^2)\|\psi_i^0\xi_i\|_\infty^2 + C(u)\nu\|w\|_\infty^2(\||\partial_y|^\gamma(\psi_i^0\xi_i)\|_2^2 + \|\psi_i^0\xi_i\|_2^2)$$

$$+ C_{\text{spec}}(u)\nu^{\frac{1-j}{j+1+2\gamma}}\|u'w\psi_i^0\xi_i\|_2^2$$

(3.23) $$=: T_1 + T_2 + T_3.$$

Combining the fact that $\|\xi_i\|_\infty$, $\|\psi_i^0\|_\infty$ are bounded by 1, and the estimate (3.4), we obtain that the first term is bounded, i.e.,

(3.24) $$T_1 \le C(u)\nu(\||\partial_y|^\gamma w\|_2^2 + \|w\|_2^2) \le C(u)\|F\|_2\|w\|_2 + C(u)\nu\|w\|_2^2.$$

To estimate the T_2 term, we first estimate the quantity $\||\partial_y|^\gamma(\psi_i^0\xi_i)\|_2$ with the product rule (3.22), the ξ_i-estimate (3.3), the L^∞-bounds $\|\xi_i\|_\infty + \|\psi_i^0\|_\infty \le 2$, and the $\dot{H}^\gamma$-estimate of ψ_i^s (3.20) as follows

(3.25) $$\||\partial_y|^\gamma(\psi_i^0\xi_i)\|_2 \le C(u, \|\xi_i\|_{H^2})(\|\psi_i^0\|_{\dot{H}^\gamma} + 1) \le C(u, \delta^{-1}, j)|\log(\nu)|^{2\alpha(\gamma)}\nu^{\frac{1-2\gamma}{2(j+1+2\gamma)}}.$$

Now we combine (3.25) with the L^∞-bounds $\|\xi_i\|_\infty + \|\psi_i^0\|_\infty \le 2$, and apply Hölder inequality, Young inequality and Gagliardo-Nirenberg interpolation inequality to derive the following

$$T_2 \le C(u, \delta^{-1}, j)\nu\left(\||\partial_y|^\gamma w\|_2^{\frac{1}{\gamma}}\|w\|_2^{2-\frac{1}{\gamma}} + \|w\|_2^2\right)\nu^{\frac{1-2\gamma}{j+1+2\gamma}}|\log(\nu)|^{4\alpha(\gamma)}$$

$$\le C(B, u, \delta^{-1}, j)\nu|\log\nu|^{8\gamma\alpha(\gamma)}\||\partial_y|^\gamma w\|_2^2 + \left(\frac{1}{B}\nu^{\frac{j+1}{j+1+2\gamma}} + C(u, \delta^{-1}, j)\nu^{\frac{j+2}{j+1+2\gamma}}|\log\nu|^{4\alpha(\gamma)}\right)\|w\|_2^2$$

Now we apply the relation (3.4) to obtain the following

(3.26)

$$T_2 \le C(B, u, \delta^{-1}, j)|\log\nu|^{8\gamma\alpha(\gamma)}\|F\|_2\|w\|_2 + \nu^{\frac{j+1}{j+1+2\gamma}}\left(\frac{1}{B} + C(u, \delta^{-1}, j)\nu^{\frac{1}{j+1+2\gamma}}|\log\nu|^{4\alpha(\gamma)}\right)\|w\|_2^2.$$

To estimate the last term T_3 in (3.23), we apply the condition (3.2) that $|u'(y)| \le C_1(u)|y - y_i^\star|^j \le C_1(u)\delta^{\frac{j}{j+1}}\nu^{\frac{j}{j+1+2\gamma}}$ on the support of $\psi_i^0\xi_i$ and choice of $\delta(u)$ (3.17) to obtain

$$T_3 \le C_{\text{spec}}(u)C_1^2(u)\delta^{\frac{2j}{j+1}}\nu^{\frac{1+j}{j+1+2\gamma}}\|w\psi_i^0\xi_i\|_2^2$$

(3.27) $$\le \left(C_{\text{spec}}(u)C_1^2(u)\delta\right)\nu^{\frac{1+j}{j+1+2\gamma}}\|w\psi_i^0\xi_i\|_2^2 \le \frac{1}{9}\nu^{\frac{1+j}{j+1+2\gamma}}\|w\psi_i^0\xi_i\|_2^2.$$

Here the crucial point is that the coefficients in the bound of T_3 only depends on the shear profile. By taking δ to be small compared to $C_{\text{spec}}(u)C_1^2(u)$, we have that the T_3 term can be absorbed by the left hand side of (3.23). Therefore, by combining the estimates (3.23), (3.24), (3.26), and (3.27), we obtain

$$\|w\psi_i^0\xi_i\|_2^2 \le C(B, u, \delta^{-1}, j)\nu^{-2\frac{j+1}{j+1+2\gamma}}|\log(\nu)|^{16\gamma\alpha(\gamma)}\|F\|_2^2$$

(3.28) $$+ \left(\frac{1}{B} + C(u, \delta^{-1}, j)\nu^{\frac{1}{j+1+2\gamma}}|\log\nu|^{4\alpha(\gamma)}\right)\|w\|_2^2.$$

This concludes the step #2.
Step #3: Estimation of the $\|w\psi_i^\pm\|_2^2$ terms in (3.21). Since $|u(y) - \lambda|$ has quantitative positive lower bound (3.18) on the domain of integration, we apply the relation (3.6), Hölder inequality and

the product rule (3.22) to obtain the following estimate,

$$\|w\psi_i^{\pm}\xi_i\|_2^2 = \left| \int_{\mathbb{T}} |w|^2 \frac{(u-\lambda)}{(u-\lambda)} |\psi_i^{\pm}|^2 \xi_i^2 dy \right| \leq \delta^{-1}\nu^{-\frac{j+1}{j+1+2\gamma}} \left| \int_{\mathbb{T}} |w|^2 (u-\lambda)|\psi_i^{\pm}|^2 \xi_i^2 dy \right|$$

$$\leq C\delta^{-1}\nu^{-\frac{j+1}{j+1+2\gamma}} \Bigg(\|F\|_2 \|w\|_2 \|\psi_i^{\pm}\xi_i\|_{\infty}^2 + \nu \||\partial_y|^{\gamma} w\|_2 (\||\partial_y|^{\gamma} w\|_2 + \|w\|_2) \|\psi_i^{\pm}\xi_i\|_{\infty}$$

$$+ \nu \||\partial_y|^{\gamma} w\|_2 \|w\|_{\infty} (\||\partial_y|^{\gamma} (\psi_i^{\pm}\xi_i)^2\|_2 + \|(\psi_i^{\pm}\xi_i)^2\|_2) \Bigg)$$

$$(3.29) \qquad =: T_4 + T_5 + T_6.$$

The T_4 can be estimated using the relation (3.4), the fact that $\|\xi_i\|_{\infty}, \|\psi_i^s\|_{\infty} \leq 1$, and the Young inequality as follows

$$(3.30) \qquad T_4 \leq C(B, \delta^{-1})\nu^{-2\frac{j+1}{j+1+2\gamma}} \|F\|_2^2 + \frac{1}{B} \|w\|_2^2.$$

Next we estimate T_5 term in (3.29). Application of the relation (3.4), Hölder inequality and Young inequality yields that

$$(3.31) \qquad T_5 \leq C(B, \delta^{-1}, j)\nu^{-2\frac{j+1}{j+1+2\gamma}} \|F\|_2^2 + \left(\frac{1}{B} + C(\delta^{-1}, j)\nu^{\frac{2\gamma}{j+1+2\gamma}} \right) \|w\|_2^2.$$

To estimate term T_6 in (3.29), we recall the quantitative estimates of ψ_i (3.19) and ξ_i-estimate (3.3), and apply a similar argument to (3.25) to obtain that

$$\||\partial_y|^{\gamma} (\psi_i^{\pm}\xi_i)^2\|_2 \leq C(u, \delta^{-1}, j)\nu^{\frac{1-2\gamma}{2(j+1+2\gamma)}} |\log(\nu)|^{2\alpha(\gamma)}.$$

Now we apply the relation (3.4), Young inequality, Gagliardo-Nirenberg interpolation inequality to obtain

$$T_6 \leq C\delta^{-1}\nu^{-\frac{j+1}{j+1+2\gamma}} \nu \left(\||\partial_y|^{\gamma} w\|_2^{1+\frac{1}{2\gamma}} \|w\|_2^{1-\frac{1}{2\gamma}} + \||\partial_y|^{\gamma} w\|_2 \|w\|_2 \right) (\||\partial_y|^{\gamma} (\psi_i^+\xi_i)^2\|_2 + 1)$$

$$\leq C(B, \delta^{-1}, j) \left(\nu^{-\frac{j+1}{j+1+2\gamma}} |\log(\nu)|^{\frac{8\gamma\alpha(\gamma)}{2\gamma+1}} + \nu^{-\frac{j}{j+1+2\gamma}} |\log \nu|^{4\alpha(\gamma)} \right) \nu \||\partial_y|^{\gamma} w\|_2^2 + \frac{1}{B}\|w\|_2^2$$

$$(3.32) \qquad \leq C(B, \delta^{-1}, j)\nu^{-2\frac{j+1}{j+1+2\gamma}} |\log(\nu)|^{16\gamma\alpha(\gamma)} \|F\|_2^2 + \frac{1}{B}\|w\|_2^2.$$

Combining the estimates (3.29), (3.30), (3.31) and (3.32) , we obtain that

$$\|w\psi_i^{\pm}\xi_i\|_2^2$$

$$(3.33) \leq C(B, u, \delta^{-1}, j)\nu^{-2\frac{j+1}{j+1+2\gamma}} |\log \nu|^{16\gamma\alpha(\gamma)} \|F\|_2^2 + \left(\frac{1}{B} + C(u, \delta^{-1}, j_i)\nu^{\frac{1}{j+1+2\gamma}} |\log \nu|^{4\alpha(\gamma)} \right) \|w\|_2^2,$$

for any $B > 1$. Now combining the decomposition (3.21), the estimates (3.28) and (3.33), we obtain the estimate (3.10) for $i \in \mathcal{I}_{\text{near}}$ (3.12). □
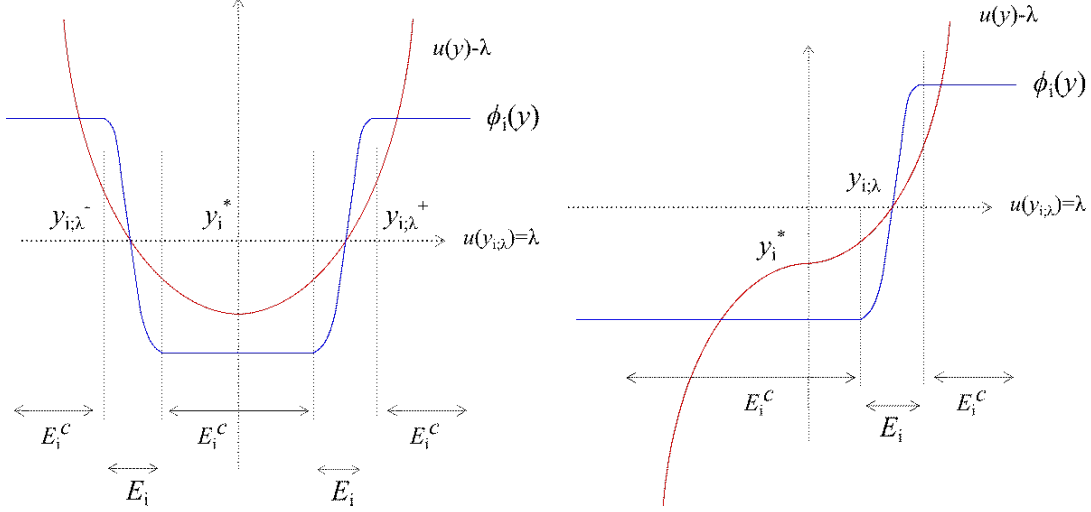
Next we treat the case where $i \in \mathcal{I}_{\text{interm}}$ (3.13).

Lemma 3.4. *Assume the condition in Theorem 1. For $i \in \mathcal{I}_{\text{interm}}$ (3.13) and for $\nu_0(u)$ small enough, the estimate (3.10) holds.*

Proof. Here we drop the subscript i in the vanishing order j_i. We organize the proof into three steps.
Step #1: Before estimating the L^2-norm $\|w\xi_i\|_{L^2(\mathbb{T})}$, several definitions are introduced. First, we consider the set E_i and its compliment E_i^c associated with each critical point:

$$(3.34) \qquad E_i := \left\{ y \,\Big|\, |u(y) - \lambda| \leq \frac{1}{9}\delta\nu^{\frac{j+1}{j+1+2\gamma}}, \, y \in \text{support}(\xi_i^2) \right\}, \qquad E_i^c = \mathbb{T}\backslash E_i.$$

FIGURE 1. Function $\phi_i(y)$ for j odd(left). Function $\phi_i(y)$ for j even (right).

Next, we define functions ϕ_i which change sign near the spectral parameter λ, and have transition layer adapted to the set E_i. Given the shear profile $u(y)$, we first define the $y_{i;\lambda}$ such that $u(y_{i;\lambda}) = \lambda$, $y_{i;\lambda} \in B(y_i^\star, 4r_i)$, support$(\xi_i^2) = B(y_i^\star, 2r_i)$. If the vanishing order j of the critical point is odd, then there are two points $y_{i;\lambda}$ such that $u(y_{i;\lambda}) = \lambda$. In this case, we use $y_{i;\lambda}^-$ and $y_{i;\lambda}^+$ to represent them and use $y_{i;\lambda}$ to denote the set $\{y_{i;\lambda}^+, y_{i;\lambda}^-\}$. Since the balls $\{B(y_i^\star, 4r_i)\}_{i=1}^N$ are mutually disjoint, the $y_{i;\lambda}$ associated with different critical points are distinct. We define the function $\{\phi_i\}_{i=1}^N \in (C^\infty(\mathbb{T}))^N$:

a) $\quad \phi_i = \begin{cases} \text{sign}(u(y) - \lambda), & |u(y) - \lambda| \geq \frac{1}{9}\delta\nu^{\frac{j+1}{j+1+2\gamma}}, \quad \text{dist}(y, y_i^\star) < 4r_i; \\ \text{monotone}, & |u(y) - \lambda| \leq \frac{1}{9}\delta\nu^{\frac{j+1}{j+1+2\gamma}}, \quad \text{dist}(y, y_i^\star) < 4r_i; \\ \text{smooth}, & \text{dist}\,(y, y_i^\star) \geq 4r_i = \text{diameter(support}(\xi_i)); \end{cases}$

b) $\quad \|\partial_y \phi_i\|_{L^2} \leq C\delta^{-\frac{1}{2(j+1)}}\nu^{-\frac{1}{2(j+1+2\gamma)}}, \quad \|\partial_y^2 \phi_i\|_{L^2} \leq C\delta^{-\frac{3}{2(j+1)}}\nu^{-\frac{3}{2(j+1+2\gamma)}};$

$\quad \||\partial_y|^{1/2}\phi_i\|_{L^2} \leq C(\delta^{-1}, j)|\log(\nu)|, \quad \||\partial_y|^\gamma \phi_i\|_{L^2} \leq C(\delta^{-1}, j)|\log(\nu)|^{2\alpha(\gamma)}\nu^{\frac{1-2\gamma}{2(j+1+2\gamma)}};$

c) $\quad \phi_i(y_{i;\lambda}) = \phi_i(y_{i;\lambda}^\pm) = 0, \quad \|\phi_i\|_{L^\infty} \leq 1;$

d) $\quad \phi_i(y) = \phi_i(y_{i;\lambda}) + \phi_i'(y_{i;\lambda})(y - y_{i;\lambda}), \quad |y - y_{i;\lambda}| \leq \frac{1}{L(u)}\delta^{\frac{1}{j+1}}\nu^{\frac{1}{j+1+2\gamma}}, \quad \text{if } j \text{ is even};$

$\quad \phi_i(y) = \phi_i(y_{i;\lambda}^\pm) + \phi_i'(y_{i;\lambda}^\pm)(y - y_{i;\lambda}^\pm), \quad |y - y_{i;\lambda}^\pm| \leq \frac{1}{L(u)}\delta^{\frac{1}{j+1}}\nu^{\frac{1}{j+1+2\gamma}}, \quad \text{if } j \text{ is odd}.$

Since the function ϕ_i is very similar to the function ψ_i^s in the proof of Lemma 3.3, the regularity estimates are similar to those of ψ_i^s. Here the universal constant $L(u)$ is chosen such that if $|y - y_{i;\lambda}^\pm| \leq \frac{1}{L(u)}\delta^{\frac{1}{j+1}}\nu^{\frac{1}{j+1+2\gamma}}$, then $|u(y) - \lambda| \leq \frac{1}{36}\delta\nu^{\frac{j+1}{j+1+2\gamma}}$. By the estimate of u on the support of ξ_i (3.1), the existence of this universal constant $L(u)$, which is independent of ν, is guaranteed.

Now we estimate the area of the set E_i. Based on the relative position between λ and $u_i^\star$, we distinguish between two cases:

(3.35) $\qquad\qquad (b_1) \quad |y_{i;\lambda} - y_i^\star| \leq r_i; \quad (b_2) \quad r_i < |y_{i;\lambda} - y_i^\star| \leq 3r_i.$

In case (b_1) (3.35), we first consider the case where the vanishing order j is even. As a result, the shear profile is strictly monotone near the critical point and there is a unique $y_{i;\lambda}$ such that $u(y_{i;\lambda}) = \lambda$. We consider arbitrary $y \in E_i$ (3.34). Note that in this case, the product $(y - y_i^\star)(y_{i;\lambda} - y_i^\star) > 0$.

Application of the mean value theorem, the u-estimates (3.1), (3.2) on the support of ξ_i and definition of E_i (3.34) yields that

$$|y - y_{i;\lambda}| \leq \frac{|u(y) - u(y_{i;\lambda})|}{\min_{z \in [y, y_{i;\lambda}] \cup [y_{i;\lambda}, y]} |u'(z)|} \leq C_1(u) \frac{|u(y) - \lambda|}{\min\{|y - y_i^\star|^j, |y_{i;\lambda} - y_i^\star|^j\}}$$

$$(3.36) \qquad \leq \frac{C(C_1(u), C_2(u))\delta \nu^{\frac{j+1}{j+1+2\gamma}}}{\min\{|u(y) - u_i^\star|^{\frac{j}{j+1}}, |\lambda - u_i^\star|^{\frac{j}{j+1}}\}} \leq C(u)\delta^{\frac{1}{j+1}} \nu^{\frac{1}{j+1+2\gamma}}.$$

Since the estimate holds for all $y \in E_i$, we have that

$$(3.37) \qquad |E_i| \leq C(u)\delta^{\frac{1}{j+1}} \nu^{\frac{1}{j+1+2\gamma}}.$$

If the vanishing order j is odd, then $y_{i;\lambda} = \{y_{i;\lambda}^-, y_{i;\lambda}^+\}$. Now we define $E_i^\pm = E \cap \{y | (y - y_i^\star)(y_{i;\lambda}^\pm - y_i^\star) > 0\}$. Now carrying out similar argument yields (3.37). In the case (b_2) (3.35), by picking ν_0 small enough in the definition of the set E_i (3.34), we have that for $y \in E_i$, the following the derivative lower bound of u holds,

$$\min_{z \in [y, y_{i;\lambda}] \cup [y_{i;\lambda}, y]} |u'(z)| \geq \frac{1}{C(u)} > 0.$$

Combining this derivative $|u'|$ lower bound and a similar argument to (3.36) yields that $|E_i| \leq C(u)\delta^{\frac{1}{j+1}} \nu^{\frac{1}{j+1+2\gamma}}$.

To conclude step #1, we estimate a specific function which will be applied later. We first consider the case where j is even. We apply the properties of the functions ϕ_i, ξ_i and the mean value theorem to estimate the following function for $\forall y \in \mathbb{T}$,

$$\xi_i^2(y) \frac{|\phi_i(y)|}{|u(y) - \lambda|} \leq \xi_i^2(y) \mathbf{1}_{E_i}(y) \frac{\|\partial_y \phi_i\|_\infty |y - y_{i;\lambda}|}{|u(y) - u(y_{i;\lambda})|} + \xi_i^2(y) \mathbf{1}_{E_i^c}(y) 9\delta^{-1} \nu^{-\frac{j+1}{j+1+2\gamma}}$$

$$(3.38) \qquad \leq \xi_i^2(y) \mathbf{1}_{E_i}(y) \|\partial_y \phi_i\|_\infty \frac{1}{\min_{z \in E_i} |u'(z)|} + \xi_i^2(y) \mathbf{1}_{E_i^c}(y) 9\delta^{-1} \nu^{-\frac{j+1}{j+1+2\gamma}}.$$

Now by the behavior of the shear profile u on the support of ξ_i (3.1), (3.2), the λ constraint (3.13), the definition of E_i (3.34), we observe that

$$\min_{y \in E_i} |u'(y)| \geq C_1^{-1}(u) \min_{y \in E_i} |y - y_i^\star|^j \geq C_1^{-1} C_0^{-1} \min_{y \in E_i} |u(y) - u(y_i^\star)|^{\frac{j}{j+1}}$$

$$\geq C_1^{-1} C_0^{-1} \min_{y \in E_i} \||u_i^\star - \lambda| - |u(y) - \lambda|\|^{\frac{j}{j+1}} \geq C_1^{-1} C_0^{-1} \left| \delta\nu^{\frac{j+1}{j+1+2\gamma}} - \frac{1}{9}\delta\nu^{\frac{j+1}{j+1+2\gamma}} \right|^{\frac{j}{j+1}}$$

$$\geq C(u)^{-1} \delta^{\frac{j}{j+1}} \nu^{\frac{j}{j+1+2\gamma}}.$$

Combining the lower bound, the estimate (3.38) and the property b) of the function ϕ_i, we have obtained the following

$$(3.39) \qquad \xi_i^2(y) \frac{|\phi_i(y)|}{|u(y) - \lambda|} \leq C\delta^{-1} \nu^{-\frac{j+1}{j+1+2\gamma}}, \quad \forall y \in \mathbb{T}.$$

Here if j is odd, then we can replace the $y_{i;\lambda}$ above by either $y_{i;\lambda}^+$ or $y_{i;\lambda}^-$. Moreover, we will decompose the domain $\mathbb{T}$ into the three subdomains E_i^c, $E_i^\pm$, where $E_i^\pm$ denotes the connect component of E_i which contains $y_{i;\lambda}^\pm$. Application of similar estimate above yields the estimate (3.39). This concludes step #1.

Step #2: Estimation of $\|w\xi_i\|_{L^2(E_i^c)}$. Direct application of the constraint (3.34) and property a) of the function ϕ_i yields that

$$(3.40) \qquad \|w\xi_i\|^2_{L^2(E_i^c)} = \int_{E_i^c} \xi_i^2 \frac{|u-\lambda||\phi_i|}{|u-\lambda||\phi_i|}|w|^2 dy \leq C\delta^{-1}\nu^{-\frac{j+1}{j+1+2\gamma}} \int_{\mathbb{T}} |u-\lambda|\xi_i^2|\phi_i||w|^2 dy.$$

Now we test the equation (2.1) by $\overline{w}\phi_i\xi_i^2$,

$$\int |u-\lambda||\phi_i|\xi_i^2|w|^2 dy = \int (u-\lambda)\phi_i\xi_i^2|w|^2 dy = \mathrm{Im}\int_{\mathbb{T}} F\overline{w}\phi_i\xi_i^2 dy - \nu\mathrm{Im}\int_{\mathbb{T}} |\partial_y|^\gamma w|\partial_y|^\gamma(\phi_i\xi_i^2\overline{w})dy.$$

Now we apply the bound (3.39), property a) and c) of the functions ϕ_i and the product estimate (3.22), Hölder inequality, Young's inequality and Gagliardo-Nirenberg interpolation inequality to get the following

$$\int |u-\lambda||\phi_i|\xi_i^2|w|^2 dy$$

$$\leq C\left(\left\|\frac{F\sqrt{|\phi_i|}\xi_i}{\sqrt{|u-\lambda|}}\right\|_2 \|\sqrt{|u-\lambda|}\sqrt{|\phi_i|}\xi_i w\|_2 + \nu\||\partial_y|^\gamma w\|_2(\||\partial_y|^\gamma w\|_2 + \|w\|_2)\|\phi_i\xi_i^2\|_\infty\right.$$

$$\left. + \nu\||\partial_y|^\gamma w\|_2\|w\|_\infty(\|\phi_i\xi_i^2\|_{\dot{H}^\gamma} + \|\phi_i\xi_i^2\|_{L^2})\right)$$

$$\leq C(\|\xi_i\|_{W^{1,\infty}})\left(\left\|\frac{F\sqrt{|\phi_i|}\xi_i}{\sqrt{|u-\lambda|}}\right\|_2 \|\sqrt{|u-\lambda|}\sqrt{|\phi_i|}\xi_i w\|_2 + \nu\||\partial_y|^\gamma w\|_2(\||\partial_y|^\gamma w\|_2 + \|w\|_2)\right.$$

$$\left. + \nu(\||\partial_y|^\gamma w\|_2^{1+\frac{1}{2\gamma}}\|w\|_2^{1-\frac{1}{2\gamma}} + \||\partial_y|^\gamma w\|_2\|w\|_2)(\||\partial_y|^\gamma \phi_i\|_2 + 1)\right)$$

$$\leq C(u)\left\|\frac{F\sqrt{|\phi_i|}\xi_i}{\sqrt{|u-\lambda|}}\right\|_2^2 + \frac{1}{2}\|\sqrt{|u-\lambda|}\sqrt{|\phi_i|}\xi_i w\|_2^2 + C(u)\nu\||\partial_y|^\gamma w\|_2(\||\partial_y|^\gamma w\|_2 + \|w\|_2)$$

$$+ C(u)\nu(\||\partial_y|^\gamma w\|_2^{1+\frac{1}{2\gamma}}\|w\|_2^{1-\frac{1}{2\gamma}} + \||\partial_y|^\gamma w\|_2\|w\|_2)(\|\phi_i\|_{\dot{H}^\gamma} + 1).$$

Note that the second term on the right hand side gets absorbed by the left hand side, which implies the following estimate

$$\int |u-\lambda||\phi_i|\xi_i^2|w|^2 dy \leq C(u)\left\|\frac{F\sqrt{|\phi_i|}\xi_i}{\sqrt{|u-\lambda|}}\right\|_2^2 + C(u)\nu\left(\||\partial_y|^\gamma w\|_2^2 + \||\partial_y|^\gamma w\|_2\|w\|_2\right)$$

$$+ C(u)\nu\left(\||\partial_y|^\gamma w\|_2^{1+\frac{1}{2\gamma}}\|w\|_2^{1-\frac{1}{2\gamma}} + \||\partial_y|^\gamma w\|_2\|w\|_2\right)(\|\phi_i\|_{\dot{H}^\gamma} + 1).$$

Combining this relation with (3.40) and the properties of the function ϕ_i yields the following

$$\|w\xi_i\|^2_{L^2(E_i^c)} \leq C(u,\delta^{-1})\nu^{-\frac{j+1}{j+1+2\gamma}}\left(\left\|\frac{F\sqrt{|\phi_i|}\xi_i}{\sqrt{|u-\lambda|}}\right\|_{L^2}^2 + \nu(\||\partial_y|^\gamma w\|_2^2 + \||\partial_y|^\gamma w\|_2\|w\|_2)\right.$$

$$\left. + \nu(\||\partial_y|^\gamma w\|_2^{1+\frac{1}{2\gamma}}\|w\|_2^{1-\frac{1}{2\gamma}} + \||\partial_y|^\gamma w\|_2\|w\|_2)(\|\phi_i\|_{\dot{H}^\gamma} + 1)\right)$$

$$(3.41) \qquad =: T_1 + T_2 + T_3.$$

Now we estimate each term in (3.41). Combining the estimate (3.39) and the relation (3.4), we obtain

$$(3.42) \qquad\qquad\qquad T_1 \leq C(u,\delta^{-1})\nu^{-2\frac{j+1}{j+1+2\gamma}}\|F\|_2^2.$$

Now we estimate T_2 term with the relation (3.4) as follows

$$(3.43) \qquad T_2 \leq C(B, \delta^{-1})\nu^{-2\frac{j+1}{j+1+2\gamma}}\|F\|_2^2 + \frac{1}{B}\|w\|_2^2.$$

Next we estimate T_3. The estimate of T_3 is similar to the estimate of T_6 (3.32) in the proof of Lemma 3.3. Combining the property b) of the function ϕ_i, the relation (3.4), Gagliardo-Nirenberg inequality, and Young inequality, we obtain

$$(3.44) \qquad T_3 \leq C(B, u, \delta^{-1}, j)\nu^{-2\frac{j+1}{j+1+2\gamma}}|\log(\nu)|^{\frac{16\gamma\alpha(\gamma)}{2\gamma+1}}\|F\|_2^2 + \frac{1}{B}\|w\|_2^2.$$

Combining the estimates (3.42), (3.43), (3.44) and the relation (3.41), we obtain the estimate

$$(3.45) \quad \|w\xi_i\|_{L^2(E_i^c)}^2 \leq C(B, \delta^{-1}, u, j)\nu^{-2\frac{j+1}{j+1+2\gamma}}|\log(\nu)|^{\frac{16\gamma\alpha(\gamma)}{2\gamma+1}}\|F\|_2^2 + \frac{1}{B}\|w\|_2^2, \quad \forall i \in \mathcal{I}_{\text{interm}}.$$

This concludes step #2.

Step #3: Estimation of $\|\xi_i w\|_{L^2(E_i)}^2$ (3.34). Applying the area estimate (3.37), the relation (3.4), Gagliardo-Nirenberg interpolation inequality and Young inequality, we estimate the L^2 contribution as follows:

$$\|w\xi_i\|_{L^2(E_i)}^2 \leq |E_i|\|w\|_{L^\infty(\mathbb{T})}^2$$

$$\leq C\delta^{\frac{1}{j+1}}\nu^{\frac{1}{j+1+2\gamma}}\||\partial_y|^\gamma w\|_{L^2(\mathbb{T})}^{\frac{1}{\gamma}}\|w\|_{L^2(\mathbb{T})}^{2-\frac{1}{\gamma}} + C\delta^{\frac{1}{j+1}}\nu^{\frac{1}{j+1+2\gamma}}\|w\|_{L^2(\mathbb{T})}^2$$

$$\leq C(B)\nu^{-\frac{j+1}{j+1+2\gamma}}\nu\||\partial_y|^\gamma w\|_{L^2(\mathbb{T})}^2 + \left(\frac{1}{B} + C\delta^{\frac{1}{j+1}}\nu^{\frac{1}{j+1+2\gamma}}\right)\|w\|_{L^2(\mathbb{T})}^2$$

$$\leq C(B)\nu^{-\frac{j+1}{j+1+2\gamma}}\|F\|_{L^2(\mathbb{T})}\|w\|_{L^2(\mathbb{T})} + \left(\frac{1}{B} + C\nu^{\frac{1}{j+1+2\gamma}}\right)\|w\|_{L^2(\mathbb{T})}^2$$

$$(3.46) \qquad \leq C(B)\nu^{-2\frac{j+1}{j+1+2\gamma}}\|F\|_{L^2(\mathbb{T})}^2 + \left(\frac{1}{B} + C\nu^{\frac{1}{j+1+2\gamma}}\right)\|w\|_{L^2(\mathbb{T})}^2.$$

Combining (3.46) and estimate (3.45) implies (3.10) for $i \in \mathcal{I}_{\text{interm}}$. $\qquad\square$

Now we consider $i \in \mathcal{I}_{\text{far}}$ (3.14).

Lemma 3.5. *Assume the conditions in Theorem 1. For $i \in \mathcal{I}_{\text{far}}$ (3.14), the estimate (3.10) holds for $\nu_0(u) > 0$ small enough.*

Proof. Combining the assumption $\text{dist}(y_i^\star, \{z|u(z) = \lambda\}) \geq 3r_i$ and the fact that the derivative u' is away from zero in the region $B(y_i^\star, 3r_i)\backslash\text{support}(\xi_i^2)$, we have that $|u(y) - \lambda| \geq \frac{1}{C(u)}$ for $y \in \text{support}(\xi_i^2) = B(y_i^\star, 2r_i)$. Now we apply the relation (3.6) with $g = \xi_i^2$, the product estimate (3.22), the Gagliardo-Nirenberg interpolation inequality and Young inequality to obtain that

$$\|w\xi_i\|_2^2 \leq C\bigg(\|F\|_2\|w\xi_i\|_2 + \nu\||\partial_y|^\gamma w\|_2(\||\partial_y|^\gamma w\|_2 + \|w\|_2)\|\xi_i\|_\infty$$

$$+ \nu\||\partial_y|^\gamma w\|_2\|w\|_\infty(\||\partial_y|^\gamma(\xi_i^2)\|_2 + \|\xi_i^2\|_2)\bigg)\frac{1}{\min_{y\in\text{support}(\xi_i^2)}|u(y) - \lambda|}$$

$$\leq C(u, \delta^{-1}, B)\|F\|_2^2 + \frac{1}{B}\|w\|_2^2 + C(u, \|\xi_i\|_{H^2})\nu(\||\partial_y|^\gamma w\|_2^2 + \|w\|_2^2)$$

$$\leq C(u, \delta^{-1}, B)\|F\|_2^2 + \left(\frac{1}{B} + C(u)\nu\right)\|w\|_2^2$$

for any positive constant B. This implies the estimate (3.10). $\qquad\square$

Finally, we estimate $\|\xi_0 w\|_2^2$.

Lemma 3.6. *Assume the condition in Theorem 1. Then the following estimate holds for any positive constant $B > 1$ and $\nu_0(u) > 0$ small enough,*

$$(3.47) \quad \|\xi_0 w\|_2^2 \leq C(B, \delta^{-1}, u) \nu^{-\frac{2}{2\gamma+1}} |\log \nu|^{16\gamma\alpha(\gamma)} \|F\|_2^2 + \left(\frac{1}{B} + C(u, \delta^{-1}) \nu^{\frac{1}{2\gamma+1}} \right) \|w\|_{L^2(\mathbb{T})}^2.$$

Proof. Note that on the support of ξ_0, the function $u(y)$ intersects the value λ at different points $\{y_h^\dagger\}_{h=1}^M$, $M \leq N+1$, and near each intersection, the derivative of u is away from zero, i.e., $\|u'(y_h^\dagger)\|_\infty \geq \frac{1}{C(u)} > 0$. We use the transition function trick again. To this end, we consider the following transition function ψ_0:

$$a) \quad \psi_0(y) = \text{sign}(u(y) - \lambda), \quad |y - y_h^\dagger| \geq \delta \nu^{\frac{1}{2\gamma+1}};$$

$$(3.48) b) \quad \|\psi_0\|_\infty \leq 1, \quad \|\partial_y \psi_0\|_\infty \leq C(u) \delta^{-1} \nu^{-\frac{1}{2\gamma+1}}, \quad \|\psi_0\|_{\dot{H}^1} \leq C(u) \delta^{-1/2} \nu^{-\frac{1}{2(2\gamma+1)}},$$

$$\||\partial_y|^{1/2} \psi_0\|_2 \leq C(u, \delta^{-1}) |\log \nu|, \quad \|\psi_0\|_{\dot{H}^2} \leq C(u) \delta^{-3/2} \nu^{-\frac{3}{2(2\gamma+1)}};$$

$$c) \quad \text{Near the zero points } y_h^\dagger, \quad C^{-1} |\partial_y \psi_0(y_h^\dagger)(y - y_h^\dagger)| \leq |\psi_0(y)| \leq C |\partial_y \psi_0(y_h^\dagger)(y - y_h^\dagger)|.$$

Now we focus on one intersection and comment that other intersections are similar. We test the equation (2.1) with $\psi_0 \xi_0^2 \overline{w}$ and obtain the following relation

$$(3.49)$$
$$\int_\mathbb{T} |(u - \lambda) \psi_0| \xi_0^2 |w|^2 dy = \int_\mathbb{T} (u - \lambda) \psi_0 \xi_0^2 |w|^2 dy = \text{Im} \int_\mathbb{T} F \psi_0 \xi_0^2 \overline{w} dy - \nu \text{Im} \int_\mathbb{T} |\partial_y|^\gamma w |\partial_y|^\gamma (\psi_0 \xi_0^2 \overline{w}) dy.$$

We further decompose the domain into the following two component:
a) $E_0 := \{y | \exists h \in \{1, ..., M\} \text{ such that } |y - y_h^\dagger| \leq \delta \nu^{\frac{1}{2\gamma+1}}\}$; b) E_0^c. To estimate the solution on E_0, we apply the Gagliardo-Nirenberg interpolation inequality inequality and $\gamma > 1/2$ to obtain:

$$\|\xi_0 w\|_{L^2(E_0)}^2 \leq |E_0| \|w\|_{L^\infty(\mathbb{T})}^2$$

$$\leq C \nu^{\frac{1}{2\gamma+1}} \||\partial_y|^\gamma w\|_{L^2(\mathbb{T})}^{\frac{2}{2\gamma}} \|w\|_{L^2(\mathbb{T})}^{2\frac{2\gamma-1}{2\gamma}} + C \nu^{\frac{1}{2\gamma+1}} \|w\|_{L^2(\mathbb{T})}^2$$

$$\leq C(B) \nu^{\frac{2\gamma}{2\gamma+1}} \||\partial_y|^\gamma w\|_{L^2(\mathbb{T})}^2 + \left(\frac{1}{B} + C \nu^{\frac{1}{2\gamma+1}} \right) \|w\|_{L^2(\mathbb{T})}^2$$

$$(3.50) \qquad \leq C(B) \nu^{-\frac{2}{2\gamma+1}} \|F\|_{L^2(\mathbb{T})}^2 + \left(\frac{1}{B} + C \nu^{\frac{1}{2\gamma+1}} \right) \|w\|_{L^2(\mathbb{T})}^2.$$

Now we estimate the E_0^c component of the solution. To this end, we collect the relation (3.49), the fact that $|u'|(y)$ is bounded below on the set E_0, and the fact that $|y - y_h^\dagger| \geq \delta \nu^{\frac{1}{2\gamma+1}}$ on E_0^c. Then we apply the product estimate (3.22), the Gagliardo-Nirenberg interpolation inequality to estimate

the E_0^c part as follows,

$$\|\xi_0 w\|_{L^2(E_0^c)}^2 \le C(u)\delta^{-1}\left(\min_{y \in E_0}|u'(y)|\right)^{-1}\nu^{-\frac{1}{2\gamma+1}}\int_{E_0^c}|u-\lambda|\|\psi_0|\xi_0|^2|w|^2 dy$$

$$\le C(u)\delta^{-1}\left(\min_{y \in E_0}|u'(y)|\right)^{-1}\nu^{-\frac{1}{2\gamma+1}}\int_{\mathbb{T}}(u-\lambda)\psi_0\xi_0^2|w|^2 dy$$

$$\le C(\delta^{-1},u,\|\xi_0\|_{H^2})\nu^{-\frac{1}{2\gamma+1}}\left(\left\|\frac{F\sqrt{|\psi_0|}\xi_0}{\sqrt{|u-\lambda|}}\right\|_{L^2}^2 + \nu\||\partial_y|^\gamma w\|_{L^2}(\||\partial_y|^\gamma w\|_{L^2}+\|w\|_{L^2})\|\psi_0\|_{L^\infty}\right.$$

$$\left. + \nu\left(\||\partial_y|^\gamma w\|_2^{\frac{2\gamma+1}{2\gamma}}\|w\|_2^{\frac{2\gamma-1}{2\gamma}} + \||\partial_y|^\gamma w\|_2\|w\|_2\right)(\|\psi_0\|_{\dot{H}^\gamma}+1)\right)$$

$$(3.51) =: \sum_{i=1}^3 T_i.$$

We estimate each term in the expression (3.51). Before estimating the T_1 term, we apply property c) of the function ψ_0, the fact that the derivative of u does not vanish in the set $E_0 \cap \text{support}(\xi_0)$ to derive the following relation

$$\xi_0^2(y)\frac{|\psi_0(y)|}{|u(y)-\lambda|} \le \xi_0^2(y)\mathbf{1}_{E_0}(y)\frac{\|\partial_y\psi_0\|_\infty|y-y_h^\dagger|}{|y-y_h^\dagger|\min_{z \in E_0}|u'(z)|} + \xi_0^2(y)\mathbf{1}_{E_0^c}(y)\frac{1}{\min_{z \in E_0}|u'(z)|}\delta^{-1}\nu^{-\frac{1}{2\gamma+1}}$$

$$\le C(u)\delta^{-1}\nu^{-\frac{1}{2\gamma+1}}, \quad \forall y \in \mathbb{T}.$$

The estimate yields that

$$(3.52) \qquad\qquad\qquad T_1 \le C(\delta^{-1},u)\nu^{-\frac{2}{2\gamma+1}}\|F\|_2^2.$$

Now we estimate T_2 in the decomposition using Hölder inequality, Young's inequality and the estimate (3.4), as follows

$$(3.53) \qquad\qquad\qquad T_2 \le C(B,\delta^{-1},u)\nu^{-\frac{2}{2\gamma+1}}\|F\|_2^2 + \frac{1}{B}\|w\|_2^2.$$

Finally, we estimate T_3 in (3.51). We estimate the H^γ-norm of ψ_0. By the Gagliardo-Nirenberg interpolation inequality and the regularity of ψ_0 (3.48), we have the following estimate

$$\|\psi_0\|_{\dot{H}^\gamma} \le C(\delta^{-1})|\log\nu|^{2\alpha(\gamma)}\nu^{-\frac{2\gamma-1}{2(2\gamma+1)}}, \quad \gamma \in (1/2,2].$$

Combining these two estimate together with Hölder inequality, Young's inequality and the estimate (3.4), we obtain that for $\gamma \in (1/2,2]$,

$$T_3 \le C(B,\delta^{-1},u)|\log\nu|^{\frac{8\gamma\alpha(\gamma)}{2\gamma+1}}\nu^{\frac{2\gamma}{2\gamma+1}}\||\partial_y|^\gamma w\|_2^2 + C(B,\delta^{-1},u)|\log\nu|^{4\alpha(\gamma)}\nu\||\partial_y|^\gamma w\|_2^2 + \frac{1}{B}\|w\|_2^2$$

$$(3.54) \le C(B,\delta^{-1},u)\nu^{-\frac{2}{2\gamma+1}}|\log\nu|^{16\gamma\alpha(\gamma)}\|F\|_2^2 + \frac{1}{B}\|w\|_2^2.$$

Combining the estimates (3.51), (3.52), (3.53), (3.54) above and (3.50) yields the estimate (3.47).
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Proof of Proposition 2.1. We combine Lemma 3.3, Lemma 3.4, Lemma 3.5, and Lemma 3.6, to obtain that if $\min_{y \in \mathbb{T}}|\lambda - u(y)| \le \delta(u)\nu^{\frac{jm+1}{jm+1+2\gamma}}$ for $\delta(u)$ chosen in (3.17), the following estimate

holds

$$\|w\|_2^2 = \sum_{i=0}^{N} \|w\xi_i\|_2^2 \leq C(B,\delta^{-1},u,N,\{j_i\}_{i=1}^N)\nu^{-2\frac{jm+1}{jm+1+2\gamma}}|\log(\nu)|^{16\gamma\alpha(\gamma)}\|F\|_2^2$$

$$+ CN\left(\frac{1}{B} + C(u,\delta^{-1},\{j_i\}_{i=1}^N)\nu^{\frac{1}{jm+1+2\gamma}}|\log\nu|^{4\alpha(\gamma)}\right)\|w\|_2^2.$$

If we choose B^{-1}, and then ν_0 small enough, then for $0 < \nu \leq \nu_0$, the following estimate holds

$$(3.55) \qquad \|w\|_2^2 \leq C(u)\nu^{-2\frac{jm+1}{jm+1+2\gamma}}|\log(\nu)|^{16\gamma\alpha(\gamma)}\|F\|_2^2.$$

Combining the L^2-estimate with the relation (3.4) yields the conclusion (2.2). Recalling Lemma 3.1, this completes the proof of Proposition 2.1. $\qquad\square$

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