HIGHER RANK SEGRE INTEGRALS OVER THE HILBERT SCHEME OF POINTS

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ABSTRACT. Let S be a nonsingular projective surface. Each vector bundle V on S of rank s induces a tautological vector bundle over the Hilbert scheme of n points of S. When $s = 1$, the top Segre classes of the tautological bundles are given by a recently proven formula conjectured in 1999 by M. Lehn. We calculate here the Segre classes of the tautological bundles for all ranks s over all K-trivial surfaces. Furthermore, in rank $s = 2$, the Segre integrals are determined for all surfaces, thus establishing a full analogue of Lehn's formula. We also give conjectural formulas for certain series of Verlinde Euler characteristics over the Hilbert schemes of points.

1. INTRODUCTION

1.1. **The Hilbert scheme of points.** Let S be a nonsingular projective surface, and let $S^{[n]}$ denote the Hilbert scheme of n points on S. Each line bundle $L \to S$ gives rise to a tautological rank n vector bundle $L^{[n]} \to S^{[n]}$ via the assignment

$$\zeta \mapsto H^0(L \otimes \mathcal{O}_\zeta).$$

Tautological integrals over the Hilbert scheme of points or over their geometric subsets emerge often in enumerative geometry. We mention three situations studied previously:

(i) the count of n-nodal curves in fixed linear system $\mathbb{P}^n \subset |L|$ where $L \to S$ is sufficiently positive. These counts were recast by Göttsche [G1] in terms of Hilbert schemes. Specifically, for a suitable geometric subscheme

$$W_n \hookrightarrow S^{[3n]}$$

birational to $S^{[n]}$, the Severi degrees are encoded by the series

$$\mathsf{G}(z) = \sum_{n=0}^{\infty} z^n \int_{W_n} c(L^{[n]}).$$

(ii) the Verlinde series of holomorphic Euler characteristics [EGL]:

$$\mathsf{V}(z) = \sum_{n=0}^{\infty} z^n \cdot \chi(S^{[n]}, L_{(n)} \otimes E^r).$$

Here the line bundle $()_{(n)}$ is pulled back from the symmetric product, and E denotes $-\frac{1}{2}$ of the exceptional divisor on the Hilbert scheme.

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(iii) integrals of Segre classes of tautological bundles considered by Lehn [L]:

$$\mathsf{S}(z) = \sum_{n=0}^{\infty} z^n \int_{S^{[n]}} s(L^{[n]}).$$

There are strong reasons to study all three series above. The count of curves in (i) is a central enumerative question. The Verlinde series (ii) captures all holomorphic Euler characteristics of line bundles over the Hilbert scheme of points. Calculating the series has applications to the study of K-theoretic Donaldson invariants on moduli spaces of sheaves of arbitrary rank, as pursued in [GNY], [GY], [G3]. Indeed over elliptic surfaces, higher-rank invariants can in many cases be matched to Verlinde numbers on the Hilbert scheme via Fourier-Mukai techniques. Furthermore, knowledge of the series (ii) is crucial to the strange duality problem for sheaves on surfaces. Interesting variants of the Verlinde series, involving symmetric and exterior powers of tautological bundles, were recently studied in [A]; see also [Sc].

The main focus of this paper is the Segre series (iii) in its general form, when L is replaced by an arbitrary higher rank vector bundle. The Segre integrals appeared first in the study of rank 2 Donaldson invariants of rational surfaces [T, Ty]. More recently, the Segre series have turned up in the intersection theory of various parameter spaces of sheaves with sections (higher rank Quot schemes, higher rank stable pairs), where they naturally enter localization calculations via the obstruction bundles (cf [MOP1]). This is a setup which parallels the appearance of Hodge integrals in Gromov-Witten theory. The importance of systematically studying the Segre series (iii) and its higher rank generalizations was highlighted in [OP] in the case of curves.

In a significant development, a conjectural relationship between the Verlinde and Segre series was proposed in [J], aligned with the vaster conjectural framework of strange duality for moduli spaces of sheaves. This picture acquired further precision in [MOP2] where the series S and V were conjecturally matched through an explicit change of variables. As a consequence, the Segre series holds the key to understanding the Verlinde numbers (ii). This point will be addressed in Section 1.6.

The common feature of the expressions G, V, S is that all three factor (cf. [G1], [Tz], [KST], [EGL]) as products of four universal series

$$\mathsf{U}_1(z)^{L^2} \cdot \mathsf{U}_2(z)^{\chi(\mathcal{O}_S)} \cdot \mathsf{U}_3(z)^{L \cdot K_S} \cdot \mathsf{U}_4(z)^{K_S^2},$$

with

$$\mathsf{U}_1,\, \mathsf{U}_2, \mathsf{U}_3, \mathsf{U}_4 \in \mathbb{Q}[[z]].$$

The series U_1 and U_2 are uniquely determined by K-trivial geometries and are typically more accessible. They are known in examples (i)-(iii), see [G1], [EGL], [MOP1]. For

all types of tautological integrals considered over the past few decades, closed formulas for the remaining series $\mathsf{U}_3, \mathsf{U}_4$ have proven very difficult to calculate or even conjecture. Remarkably, the Segre geometry (iii) is the only nontrivial case known so far; the details are explained in Section 1.3.

1.2. **Higher rank.** We undertake the study of the Segre series in higher rank. To set the stage, let $V \to S$ be a vector bundle of rank s, inducing a tautological rank sn vector bundle

$$V^{[n]} \to S^{[n]}$$

over the Hilbert scheme via the assignment

$$\zeta \mapsto H^0(V \otimes \mathcal{O}_\zeta).$$

The construction extends to K-theory. For each K-theory class $\alpha \in K(S)$, there is an associated K-theory class

$$\alpha^{[n]} \to S^{[n]}$$

defined via locally free resolutions. We are interested in the Chern numbers of $\alpha^{[n]}$, but it will be more convenient to work with Segre classes. Fixing $\alpha \in K(S)$, we consider its associated Segre series

$$\begin{aligned} \mathsf{S}_\alpha(z) &= \sum_{n=0}^{\infty} z^n \int_{S^{[n]}} s(\alpha^{[n]}) \\ &= A_0(z)^{c_2(\alpha)} \cdot A_1(z)^{c_1(\alpha)^2} \cdot A_2(z)^{\chi(\mathcal{O}_S)} \cdot A_3(z)^{c_1(\alpha)\cdot K_S} \cdot A_4(z)^{K_S^2}. \end{aligned} \tag{1}$$

The above factorization follows by [EGL]. The five series

$$A_0,\, A_1,\, A_2,\, A_3,\, A_4 \in \mathbb{Q}[[z]]$$

are independent of the surface S, and depend on α only through the rank s.

1.3. **Lehn's conjecture.** Let us assume first that rank $\alpha = 1$ and α is represented by a line bundle

$$L \to S.$$

Then, the series A_0 is absent from (1). Lehn [L] conjectured closed formulas for *all* four series A_1, A_2, A_3, A_4. In the rephrasing of [MOP1], after the change of variables

$$z = t(1+2t)^2,$$

the universal Segre series are given by

$$\begin{aligned} A_1(z) &= (1+2t)^{\frac{1}{2}}\,, \\ A_2(z) &= (1+2t)^{\frac{3}{2}} \cdot (1+6t)^{-\frac{1}{2}}\,, \\ A_3(z) &= \frac{1}{2} \cdot (1+2t)^{-1} \cdot \left(\sqrt{1+2t}+\sqrt{1+6t}\right)\,, \\ A_4(z) &= 4 \cdot (1+2t)^{\frac{1}{2}} \cdot (1+6t)^{\frac{1}{2}} \cdot \left(\sqrt{1+2t}+\sqrt{1+6t}\right)^{-2}\,. \end{aligned}$$

The series A_1 and A_2 were confirmed via the study of the virtual geometry of the Quot scheme of K-trivial surfaces in [MOP1]. Using Reider type techniques, Voisin [V] proved the vanishing of certain Segre integrals in the case of the blowup of a $K3$ surface in one point, and showed these vanishings determined uniquely the series A_3, A_4. The series were shown to have Lehn's conjectured expressions via the residue calculations of [MOP2]. As a result, the Segre series are calculated for all line bundles over nonsingular projective surfaces S.

1.4. **The series A_0, A_1, A_2 in arbitrary rank.** Our first result gives a simple closed form expression for the Segre series of all K-trivial surfaces, in all ranks. We prove

Theorem 1. *Let S be a K-trivial surface, let α be a K-theory class of rank s, and set $r = s+1$. We have*

$$\mathsf{S}_\alpha(z) = \sum_{n=0}^{\infty} z^n \int_{S^{[n]}} s(\alpha^{[n]}) = A_0(z)^{c_2(\alpha)} \cdot A_1(z)^{c_1^2(\alpha)} \cdot A_2(z)^{\chi(\mathcal{O}_S)} \tag{2}$$

where the formulas for the series A_1, A_2, and A_3 are

$$\begin{aligned} A_0(z) &= (1+rt)^{-r} \cdot (1+(1+r)t)^{r-1}\,, \\ A_1(z) &= (1+rt)^{\frac{r-1}{2}} \cdot (1+(1+r)t)^{-\frac{r}{2}+1}\,, \\ A_2(z) &= (1+rt)^{\frac{r^2-1}{2}} \cdot (1+(1+r)t)^{-\frac{r^2}{2}+r} \cdot (1+r(1+r)t)^{-\frac{1}{2}}, \end{aligned}$$

after the change of variables

$$z = t\,(1+rt)^r\,. \tag{3}$$

Remark 1. In [J], Johnson computed the series A_0, A_1, A_2 up to order 6 and conjectured several connections between them. Building on these calculations, the above formulas for A_0, A_1, A_2 were proposed in [MOP2].

Remark 2. The simplicity of the expressions in Theorem 1 is deceiving. The individual Segre integrals are quite complicated. For instance, for a $K3$ surface S and $n = 2$, we

have

$$\int_{S^{[2]}} s_4(V^{[2]}) = \frac{1}{4}\left(\left(2(c_1^2)^2+2c_2^2-4c_1^2c_2-8c_1^2+6c_2\right)+s(-9c_1^2+6c_2+12)\right. \\ \left.+\, s^2(-3c_1^2+2c_2+22)+12s^3+2s^4\right), \tag{4}$$

where V is a vector bundle of rank s with Chern classes c_1 and c_2. For arbitrary surfaces, the answers are more involved.

Remark 3. Even in rank 1, Theorem 1 gives additional information not covered by Lehn's conjecture. While the series A_1, A_2 were predicted by Lehn's formula,

$$A_0(z) = (1+2t)^{-2}\cdot(1+3t) \text{ for } z = t(1+2t)^2$$

was absent from the line-bundle setup of the conjecture.

Remark 4. For nonsingular projective curves, the generating series of higher rank Segre integrals was determined in [MOP2]. A change of variables similar to (3) was needed to express the answer in closed form.

Remark 5. The expressions A_1, A_2 determined by Theorem 1 were subsequently connected by Mellit with certain generating series of Hurwitz numbers in genus zero [M].

1.5. **The series A_3, A_4.** For surfaces S which are not K-trivial, and an arbitrary class $\alpha \in K(S)$, the universal series A_3, A_4 appear in the Segre generating function $\mathsf{S}_\alpha(z)$ and are very difficult to evaluate.

We calculate the two remaining universal series for $\operatorname{rank}\alpha = 2$, thus proving a full rank-two analogue of Lehn's formulas. Thus, we provide a second nontrivial situation for which all series involved are determined. Using the change of variables

$$z = t(1+3t)^3,$$

Theorem 1 gives

$$\begin{aligned} A_0(z) &= \frac{(1+4t)^2}{(1+3t)^3}, \\ A_1(z) &= \frac{(1+3t)}{(1+4t)^{\frac{1}{2}}}, \\ A_2(z) &= \frac{(1+3t)^4}{(1+4t)^{\frac{3}{2}}\cdot(1+12t)^{\frac{1}{2}}}. \end{aligned}$$

To describe the remaining two series, let $\mathsf{y}(t)$ be the unique real solution of the quartic equation

$$\frac{y\cdot(1+y)^2}{(1-y)(1-y^3)} = \frac{t}{1+3t}, \;\; y(0)=0.$$

We have

$$\mathsf{y}(t) = t - 6t^2 + 41t^3 - 314t^4 + 2630t^5 + \ldots .$$

Theorem 2. *For rank $\alpha = 2$ the remaining universal Segre series are*

$$\begin{aligned} A_3(z) &= (1+3t)^{-1} \cdot \left(\frac{\mathsf{y}(t)}{t}\right)^{-1/2} \\ A_4(z) &= (1+3t) \cdot \left(\frac{\mathsf{y}(t)}{t}\right)^{3} \cdot \frac{(1+\mathsf{y})^2}{1-\mathsf{y}} \cdot \frac{1}{\mathsf{y}'}. \end{aligned}$$

Based on these expressions, we also formulate

Conjecture 1. *For all ranks, the functions A_3, A_4 are algebraic.*

Remark 6. Finding the series A_3 and A_4 for α of arbitrary rank remains an open question. In addition to Lehn's formula for rank one, and the substantial rank two analysis carried out here, a few cases are trivially known:

- the case rank $\alpha = -1$ is immediate. Set $\alpha = -L$ for some line bundle $L \to S$. For dimension reasons we have
$$\int_{S^{[n]}} s(\alpha^{[n]}) = \int_{S^{[n]}} c((-\alpha)^{[n]}) = 0 \text{ for } n \geq 1 \implies A_3(z) = A_4(z) = 1.$$
- for rank $\alpha = -2$, we can prove geometrically, see (18) for instance, that
$$\int_{S^{[n]}} s(\alpha^{[n]}) = \int_{S^{[n]}} c((-\alpha)^{[n]}) = \binom{c_2(-\alpha)}{n}.$$
We again obtain
$$A_3(z) = A_4(z) = 1.$$
- for rank $\alpha = 0$, we have $A_4(z) = 1$ just by setting $\alpha = 0$. Regarding the remaining series, we state the following

Conjecture 2. *In rank 0, we have*

$$A_3(z) = (1+t)^{-1} \cdot (1+2t)^{\frac{1}{2}} \text{ for } z = t(1+t).$$

1.6. **The Verlinde generating function.** We now turn to the generating series of holomorphic Euler characteristics of tautological line bundles over the Hilbert scheme. We set

$$\mathsf{V}_\alpha(w) = \sum_{n=0}^{\infty} \chi(S^{[n]}, (\det \alpha)_{(n)} \otimes E^r) \cdot w^n \,,$$

where the line bundle $()_{(n)}$ is pulled back from the symmetric product, and E denotes $-\frac{1}{2}$ of the exceptional divisor on the Hilbert scheme. Since V_α only depends on $\det \alpha$, only four power series are needed in the factorization

$$\mathsf{V}_\alpha(w) = B_1(w)^{\chi(c_1(\alpha))} \cdot B_2(w)^{\chi(\mathcal{O}_S)} \cdot B_3(w)^{c_1(\alpha)\cdot K_S - \frac{1}{2}K_S^2} \cdot B_4(w)^{K_S^2}.$$

The different form of the exponents used here relative to (1) is justified by the fact that with the current choices, we have the following symmetries

$$B_3 \mapsto B_3^{-1} \text{ as } r \to -r \tag{5}$$

$$B_4 \text{ is invariant under the symmetry } r \to -r.$$

This was noted in Theorem 5.3 in [EGL] as a consequence of Serre duality. Also by [EGL], we explicitly know B_1 and B_2. For

$$w = t(1+t)^{r^2-1},$$

we have

$$B_1(w) = 1 + t, \;\; B_2(w) = \frac{(1+t)^{\frac{r^2}{2}}}{(1+r^2t)^{\frac{1}{2}}}.$$

The series B_3, B_4 remain mysterious in general. The cases $r = 0$ and $r = \pm 1$ are immediate exceptions, obtained in [EGL], Lemma 5.1. Indeed for $r = 0$, we have

$$\mathsf{V}_\alpha(w) = (1-w)^{-\chi(c_1(\alpha))} \implies B_3(w) = B_4(w) = 1,$$

and for $r = \pm 1$,

$$\mathsf{V}_\alpha(w) = (1+w)^{\chi(c_1(\alpha))} \implies B_3(w) = B_4(w) = 1.$$

For all surfaces, motivated by strange duality, Johnson [J] predicted a connection between the Segre and Verlinde series

$$\mathsf{S}_{-\alpha}(z) \leftrightarrow \mathsf{V}_\alpha(w),$$

under an undetermined change of variables $z \leftrightarrow w$, and the shift

$$r = \text{rank } \alpha - 1. \tag{6}$$

These predictions were made precise in [MOP2] where the unknown change of variables was proposed:

$$z = t(1-rt)^{-r}, \;\; w = \frac{t(1-(r-1)t)^{r^2-1}}{(1-rt)^{r^2}}.$$

Using Theorem 1 we show under suitable numerics an equality between Verlinde numbers and tautological Chern integrals on Enriques surfaces, see Proposition 4. This is consistent with strange duality.

Since the Segre series is now calculated for rank $\alpha = 1$ (Lehn's formula) and rank $\alpha = 2$ (in Theorem 2), this precise conjectural Verlinde-Segre relationship gives further predictions for the unknown Verlinde universal series B_3 and B_4.

For $r = \pm 2$, the formulas for B_3 and B_4 are captured by Conjecture 2 of [MOP2]. Indeed for $r = 2$, setting [1]

$$w = t(1+t)^3,$$

we expect

$$\begin{aligned} B_3(w) &= \frac{1+\sqrt{1+4t}}{2(1+t)} \\ B_4(w) &= (1+t)^{\frac{1}{2}} \cdot (1+4t)^{\frac{1}{2}} \cdot \left(\frac{1+\sqrt{1+4t}}{2}\right)^{-\frac{5}{2}}. \end{aligned}$$

Thanks to Theorem 2, we obtain the following predictions about the Verlinde series for $r = \pm 3$. Let Y be the unique real solution of the quartic equation

$$\frac{y \cdot (1+y)^2}{(1-y)(1-y^3)} = t, \quad y(0) = 0.$$

We have

$$\mathsf{Y}(t) = \mathsf{y}\left(\frac{t}{1-3t}\right) = t - 3t^2 + 14t^3 - 80t^4 + 509t^5 - 3459t^6 + \ldots.$$

Conjecture 3. *For the Verlinde series V_α with $r = 3$, setting*

$$w = t(1+t)^8$$

we have

$$\begin{aligned} B_3(w) &= (1+t)^{-\frac{3}{2}} \cdot \left(\frac{\mathsf{Y}(t)}{t}\right)^{-\frac{1}{2}}, \\ B_4(w) &= (1+t)^{\frac{3}{4}} \cdot \left(\frac{\mathsf{Y}(t)}{t}\right)^{\frac{13}{4}} \cdot \frac{(1+\mathsf{Y})^2}{1-\mathsf{Y}} \cdot \frac{1}{\mathsf{Y}'}. \end{aligned}$$

The expressions for $r = -3$ are obtained via the symmetry (5).

Finding the general expression for the unknown series B_3 and B_4 for arbitrary r, thus determining all rank 1 Verlinde numbers, is a central question in the enumerative theory of Hilbert schemes of points on surfaces.

Remark 7. Due to the rank shift (6), the Serre duality symmetry

$$s \to -s$$

on the Verlinde side translates into a *conjectural* transformation rule for the remaining unknown Segre universal series A_3, A_4, as

$$s \to -s - 2.$$

[1] To be in agreement with the formulas of [MOP2], one has to change $t \mapsto t/(2+2t)$.

In particular, we can also state predictions for the Segre series A_3, A_4 in rank $\alpha = -3$ and rank $\alpha = -4$ from the series in rank $\alpha = 1$ and rank $\alpha = 2$. The exact expressions are more cumbersome, but they support Conjecture 1.

1.7. **Strategy of proofs.** We first explain how Theorem 1 is derived. To find the three universal series A_0, A_1, A_2, we pick α to be the class of a suitable vector bundle

$$V \to S$$

over a $K3$ surface S. As witnessed by (4), the Segre integrals are generally very complicated. It is a key observation that the answers take a simpler form for sheaves with small deformation spaces. The most beautiful formulas are obtained for spherical vector bundles V and for vector bundles with isotropic Mukai vectors. Let

$$v = \operatorname{ch} V \cdot \sqrt{\operatorname{td}(S)} \in H^{2\star}(S, \mathbb{Z})$$

be the Mukai vector of V, and recall the Mukai pairing

$$\langle v, v \rangle = \int_S v_2^2 - 2v_0 v_4 \quad \text{where } v = (v_0, v_2, v_4) \in H^{2\star}(S).$$

Set $\chi = \chi(S, V)$. We show:

Theorem 3. *Let S be a K3 surface, and let $V \to S$ be a rank $s = r - 1$ vector bundle.*

(i) *If $\langle v, v \rangle = -2$, then*

$$\int_{S^{[n]}} s_{2n}(V^{[n]}) = r^n \binom{\chi - rn}{n}.$$

(ii) *If $\langle v, v \rangle = 0$, then*

$$\int_{S^{[n]}} s_{2n}(V^{[n]}) = r^n \left(-r + \frac{1}{r} + \frac{\chi}{n}\right) \binom{\chi - rn - 1}{n-1}.$$

Theorem 3 is proven by a direct geometric argument using Reider techniques [R], also employed in rank 1 in [V]. The crucial insight here is the identification of the optimal geometric setup for which the complicated Segre integrals become manageable. We will then show by a residue calculation that Theorems 1 and 3 are equivalent.

The analysis is quite intricate for Theorem 2, requiring in particular delicate excess calculations for Segre classes. The key statements are captured by the following

Theorem 4. *Let $\pi : S \to X$ be the blowup of a $K3$ surface X at a point, with exceptional divisor E. Let $V_0 \to X$ be a rank 2 bundle whose Mukai vector satisfies $\langle v_0, v_0 \rangle = -2$. Set*

$$V = \pi^\star V_0 \otimes E^{-k},$$

and assume that

$$\chi(V) = 4n - 1.$$

(i) *If* $k = n-2$, *then*

$$\int_{S^{[n]}} s_{2n}(V^{[n]}) = (-1)^n(2n+1).$$

(ii) *If* $k = n-1$, *we have*

$$\int_{S^{[n]}} s_{2n}(V^{[n]}) = 1 \ \textit{for} \ n \equiv 0 \mod 3.$$

$$\int_{S^{[n]}} s_{2n}(V^{[n]}) = 0 \ \textit{for} \ n \not\equiv 0 \mod 3.$$

We finally show by a residue calculation that Theorems 2 and 4 are equivalent.

1.8. **Moduli of surfaces.** The Segre integrals can be viewed as part of a richer theory which is important to explore further. For each flat family

$$\pi : S \to B$$

of nonsingular projective surfaces carrying line bundles $L_1, \ldots, L_\ell \to S$, we define the κ-classes

$$\kappa[a_1, \ldots, a_\ell, b] = \pi_\star \left(c_1(L_1)^{a_1} \cdots c_1(L_\ell)^{a_\ell} \cdot c_1(\omega_\pi)^b \right) \in A^\star(B)\,.$$

When π is the universal family of the moduli of polarized surfaces

$$\pi : \mathcal{S} \to \mathcal{M},$$

the κ-classes thus defined generate the tautological ring $\mathsf{R}^\star(\mathcal{M})$ analogous to the well-studied tautological ring $\mathsf{R}^\star(\mathcal{M}_g)$ of the moduli of curves. Finding relations between the κ-classes in $\mathsf{R}^\star(\mathcal{M})$ is a very interesting problem.

In the case of the moduli of $K3$ surfaces, a strategy for κ-relations was laid out in [MOP1] via the study of the virtual class of the Quot scheme; a different approach via Gromov-Witten theory was pursued in [PY]. The discussion however makes sense for arbitrary polarized surfaces as well. In this approach, the center stage is taken by the calculation of the push-forwards

$$\sum_{n=0}^{\infty} q^n \pi_\star^{[n]} \left(s_{i_1}(L_1^{[n]}) \cdots s_{i_\ell}(L_\ell^{[n]}) \right) \in A^\star(\mathcal{M})$$

in terms of the classes $\kappa[a_1, \ldots, a_\ell, b]$. In the K-trivial case, or more ambitiously for arbitrary surfaces, it becomes important to obtain explicit formulas, thus generalizing the results of this paper.

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2. $K3$ surfaces

2.1. **Residue calculations.** Our first goal is to prove Theorem 1. We begin by explaining how the special formulas of Theorem 3 are predicted by the series in Theorem 1. Conversely, we will prove in Section 2.4 that these predictions are equivalent to the statement of Theorem 1.

We keep the same notation as in the introduction. Let S be a $K3$ surface. For a vector bundle $V \to S$ with Mukai vector

$$v = \mathrm{ch}\ (V)\sqrt{\mathrm{td}(S)},$$

let

$$\chi = \chi(S,V)\,, \quad c_1 = c_1(V)\,, \quad c_2 = c_2(V)\,,$$

as in Section 1.7. Recall that $s = \mathrm{rk}\ V$ and $r = s+1$. Taking the K-theory class α to be V, the statement of Theorem 1 becomes

$$\mathsf{S}(z) = (1+(1+r)t)^{\left[(r-1)c_2+\left(-\frac{r}{2}+1\right)c_1^2-r^2+2r\right]} \cdot (1+tr)^{\left[-rc_2+\frac{r-1}{2}c_1^2+(r^2-1)\right]} \cdot \frac{1}{1+r(1+r)t}\,.$$

For convenience, we define

$$d = (r-1)c_2 + \left(-\frac{r}{2}+1\right)c_1^2 - r^2 + 2r\,. \tag{7}$$

We then need to prove

$$\mathsf{S}(z) = (1+(1+r)t)^{d} \cdot (1+tr)^{-d+\chi+1} \cdot \frac{1}{1+r(1+r)t}. \tag{8}$$

An important observation is that the quantity (7) is half the dimension of the moduli space $\mathfrak{M}_v$ of stable sheaves of type v on the K3 surface S,

$$\dim \mathfrak{M}_v = \langle v,\, v\rangle + 2 = 2d\,.$$

We can express the individual Segre integrals as the residues

$$\int_{S^{[n]}} s_{2n}(V^{[n]}) = \frac{1}{2\pi i} \oint \mathsf{S}(z) \cdot \frac{dz}{z^{n+1}}\,.$$

Substituting equation (8) in the above residue and using the change of variables (3)

$$z = t(1+rt)^r,$$

we are equivalently seeking to prove the following formula for the top Segre classes:

$$\begin{aligned}\int_{S^{[n]}} s_{2n}(V^{[n]}) &= \frac{1}{2\pi i} \oint (1+(1+r)t)^d \cdot (1+rt)^{-d+\chi-rn} \cdot \frac{dt}{t^{n+1}} \\ &= \mathrm{Coeff}_{t^n}\left[(1+(1+r)t)^d \cdot (1+rt)^{-d+\chi-rn}\right].\end{aligned}$$

The formula yields a remarkably simple answer in the following two cases, leading to the statement of Theorem 3.

(i) When $d = 0$, we expect

$$\int_{S^{[n]}} s_{2n}(V^{[n]}) = r^n \binom{\chi - rn}{n}.$$

In particular, when $r = 2$ we recover the known line bundle result of [MOP1],

$$\int_{S^{[n]}} s_{2n}(L^{[n]}) = 2^n \binom{\chi - 2n}{n}.$$

(ii) When $d = 1$, we expect

$$\int_{S^{[n]}} s_{2n}(V^{[n]}) = r^n \left(-r + \frac{1}{r} + \frac{\chi}{n}\right) \binom{\chi - rn - 1}{n-1}.$$

2.2. **Vanishing results.** We now prove the formulas above, thus establishing Theorem 3. The proof is guided by the simple form of the expected answers.

Throughout this Section, we let S be a $K3$ surface of Picard rank one,

$$\mathrm{Pic}\,(S) = \mathbb{Z}H.$$

Let $V \to S$ be an H-stable [2] vector bundle of rank $s > 1$ with $c_1(V) = H$, so that the Mukai vector equals

$$v = (s, H, \chi - s).$$

We assume that

$$\langle v,\, v\rangle = -2 \text{ or } \langle v,\, v\rangle = 0.$$

[2] H-stability is Gieseker stability with respect to the polarization H.

In both of these cases, we know the moduli space of H-stable sheaves with Mukai vector v is nonempty, see [Y3], Theorem 0.1 for a general result. The moduli space is either a point, or a $K3$ surface, see Theorems 1.4 and 3.6 in [Mu].

Since $s > 1$, the locus of locally free sheaves V in the moduli space with Mukai vector v is *nonempty*. Nonemptiness is obtained by invoking Yoshioka's classification of Mukai vectors yielding moduli consisting entirely of nonlocally free sheaves in Proposition 0.5 of [Y3]. His classification does not include vectors v as above.

Under these assumptions, we show the following vanishings.

Proposition 1. *We have*

(i) *If* $\langle v, v\rangle = -2$, *then* $s_{2n}(V^{[n]}) = 0$ *for* $rn \le \chi(V) < (r+1)n$.
(ii) *If* $\langle v, v\rangle = 0$, *then* $s_{2n}(V^{[n]}) = 0$ *for* $rn + 1 \le \chi(V) < (r+1)n$.

Proof. Recall that V is said $(n-1)$-*very ample* if the evaluation map

$$H^0(S, V) \to H^0(S, V \otimes \mathcal{O}_Z) \tag{9}$$

is surjective for all $Z \in S^{[n]}$. This is equivalent to the surjectivity of the natural vector bundle map

$$H^0(S, V) \otimes \mathcal{O}_{S^{[n]}} \longrightarrow V^{[n]} \text{ on } S^{[n]}.$$

We will show that this is the case under our numerics in Proposition 2 below.

By the definition of Segre classes, whenever the evaluation map is surjective, we have the vanishing

$$s_j(V^{[n]}) = 0 \text{ for } j > h^0(S, V) - \operatorname{rank} V^{[n]}\,.$$

In our situation

$$h^0(S, V) = \chi(V).$$

Indeed, by Serre duality and stability,

$$h^2(S, V) = h^0(S, V^\vee) = 0\,.$$

Furthermore, for $\chi > 0$ we have

$$h^1(S, V) = 0\,. \tag{10}$$

The vanishing (10) is established in the proof of Proposition 2 below, where it is argued that there do not exist nontrivial extensions

$$0 \to V^\vee \to E \to \mathcal{O}_S \to 0\,.$$

Therefore, $(n-1)$-very ampleness of V implies the vanishing

$$s_j(V^{[n]}) = 0 \text{ for } j > \chi(V) - sn\,.$$

In particular, we obtain

$$s_{2n}(V^{[n]}) = 0 \text{ for } (r+1)n > \chi(V).$$

□

Proposition 2. *We have*

(i) *If* $\langle v, v\rangle = -2$ *and* $\chi(V) \geq rn$, *then* V *is* $(n-1)$*-very ample.*
(ii) *If* $\langle v, v\rangle = 0$ *and* $\chi(V) \geq rn+1$, *then* V *is* $(n-1)$*-very ample.*

Proof. The evaluation map (9) is surjective if $H^1(V \otimes I_Z) = 0$ for all $Z \in S^{[n]}$. Assume, for contradiction, that

$$H^1(V \otimes I_Z) \neq 0\,. \tag{11}$$

The cohomology group (11) is Serre dual to $\mathrm{Ext}^1(I_Z, V^\vee)$ which is the space of extensions

$$0 \to V^\vee \to E \to I_Z \to 0\,. \tag{12}$$

By assumption, we have a non-split extension.

If V is H-stable with $c_1(V) = H$, then Lemma 2.1 of [Y2] shows that the middle term E of any non-split extension

$$0 \to V^\vee \to E \to I_Z \to 0$$

is H-stable.

For the benefit of the reader, let us recall the argument in [Y2] in our context. Since $c_1(V) = H$ is primitive, note first that the H-stability of V implies that V is in fact slope-stable, therefore $V^\vee$ is slope-stable, hence $V^\vee$ is H-stable as well.

Assuming now that E is not H-stable, let $G \hookrightarrow E$ be the maximal semistable destabilizing subsheaf. We have

$$\mathrm{rk}\, G < \mathrm{rk}\, E \text{ and } \mu_G \geq \mu_E > \mu_{V^\vee}\,.$$

We see that G cannot be a subsheaf of the kernel $V^\vee$ since this would contradict the H-stability of the latter. Therefore, we have a nonzero morphism

$$\phi : G \to I_Z\,.$$

The H-semistability of G now gives $\mu_G \leq 0$. Writing $c_1(G) = aH$ we deduce $a \leq 0$. Since $\mu_G \geq \mu_E$, we obtain

$$\frac{aH^2}{\mathrm{rk}\, G} \geq -\frac{H^2}{\mathrm{rk}\, E} \implies a > -1\,.$$

Therefore, we have

$$a = 0 \text{ and } c_1(G) = 0\,.$$

Furthermore, if nonzero, the kernel K of ϕ is a subsheaf of $V^\vee$ of slope greater than or equal to zero, contradicting the H-stability of $V^\vee$. Thus, we have

$$\phi : G \to I_Z \text{ is injective } \implies G = I_W$$

for a zero-dimensional subscheme $W \subset S$. We consider the exact sequence

$$0 \to G \to I_Z \to Q \to 0$$

and the associated sequence of extension groups

$$\mathrm{Ext}^1(Q, V^\vee) \to \mathrm{Ext}^1(I_Z, V^\vee) \xrightarrow{f} \mathrm{Ext}^1(G, V^\vee)\,.$$

The first group is zero by Serre duality since Q is supported at finitely many points. We conclude that f is injective. However, the image of the extension (12) in $\mathrm{Ext}^1(G, V^\vee)$ is trivial. The contradiction shows that E must be H-stable.

Now, we calculate

$$\begin{aligned}
\chi(E,E) &= \chi(V^\vee, V^\vee) + 2\,\chi\left(V^\vee,\, I_Z\right) + \chi(I_Z,\, I_Z) \\
&= -\langle v,\, v\rangle + 2(\chi(V) - sn) + 2 - 2n \\
&= 2\left(-\frac{\langle v,\, v\rangle}{2} + \chi(V) - rn + 1\right).
\end{aligned}$$

In both cases of Proposition 2, we obtain

$$\chi(E,E) \geq 4 \implies \mathrm{ext}^0(E,E) + \mathrm{ext}^2(E,E) \geq 4\,.$$

Therefore, by Serre duality,

$$\mathrm{ext}^0(E,E) \geq 2\,.$$

Since E is H-stable and therefore simple, we have a contradiction. Thus, (11) does not hold. The proofs of Proposition 2 and Proposition 1 are therefore complete. □

2.3. **Proof of Theorem 3.** We consider case (i) of the Theorem. Fix the rank $s > 1$ throughout. [3] By Theorem 4.1 in [EGL], over $K3$ surfaces, the Segre integral

$$\int_{S^{[n]}} s_{2n}(V^{[n]})$$

is given by a universal polynomial of degree n in $c_1(V)^2$ and $c_2(V)$. When

$$\langle v, v\rangle = -2$$

both $c_1(V)^2$ and $c_2(V)$ can be expressed in terms of $\chi = \chi(V)$, and therefore

$$\int_{S^{[n]}} s_{2n}(V^{[n]}) = \mathsf{P}_n(\chi),$$

for a degree n universal polynomial P_n.

[3] For $s = 1$, the functions A_1, A_2 are known by [MOP1, V]. All three functions A_0, A_1, A_2 for $s = 1$ follow from the case $s > 1$ via the polynomiality argument of Section 2.5.

For each $\chi > s$, let (S, H) be a $K3$ surface of Picard rank one $\mathrm{Pic}(S) = \mathbb{Z}H$ and of genus

$$H^2 = 2g - 2, \quad g = s(\chi - s).$$

Let V be the unique H-stable vector bundle with Mukai vector $v = (s, H, \chi - s)$ whose existence we noted in the beginning of Section 2.2. Since $\langle v, v \rangle = -2$, Proposition 1 (i) shows that the top Segre class of $V^{[n]}$ vanishes when

$$rn \leq \chi < (r+1)n.$$

These n values of χ are the n roots of the polynomial P_n. Therefore,

$$\mathsf{P}_n(\chi) = c \cdot \binom{\chi - rn}{n} \implies \int_{S^{[n]}} s_{2n}(V^{[n]}) = c \cdot \binom{\chi - rn}{n}$$

for some constant c. We will identify the constant $c = r^n$ by proving

$$\int_{S^{[n]}} s_{2n}(V^{[n]}) = \frac{r^n}{n!}\chi^n + \text{lower order terms (l.o.t.) in } \chi.$$

The argument is most naturally expressed by rewriting (2) in exponential form,

$$\sum_{n=0}^{\infty} z^n \int_{S^{[n]}} s_{2n}(V^{[n]}) = \exp\left(\bar{A}_0(z) \cdot c_1(V)^2 + \bar{A}_1(z) \cdot c_2(V) + \bar{A}_2(z)\right).$$

Restricting to the line $\langle v, v \rangle = -2$ in the (c_1^2, c_2)-plane, we obtain

$$\sum_{n=0}^{\infty} z^n \int_{S^{[n]}} s_{2n}(V^{[n]}) = \exp\Big(U(z) \cdot \chi + T(z)\Big),$$

for power series U and T. Let

$$U(z) = u_1 z + u_2 z^2 + \ldots.$$

Extracting the coefficient of z^n in the above expression yields

$$\int_{S^{[n]}} s_{2n}(V^{[n]}) = \frac{u_1^n}{n!}\chi^n + \text{ l.o.t. }.$$

In particular, for $n = 1$, we obtain

$$\int_S s_2(V) = u_1\chi + \text{ l.o.t. }.$$

Direct calculation shows $s_2(V) = r\chi - r^2$ so that $u_1 = r$. After substitution, we find the leading term to be

$$\frac{u_1^n}{n!} = \frac{r^n}{n!}.$$

For part (ii) of the Theorem, Proposition 1 (ii) gives only $n-1$ roots of the polynomial expressing the Segre integral. Combined with the leading term calculations, we conclude that if $\langle v, v \rangle = 0$ we have

$$\int_{S^{[n]}} s_{2n}(V^{[n]}) = \frac{r^n}{n}(\chi + c)\binom{\chi - rn - 1}{n-1},$$

for some constant c. We will prove

$$c = n\left(-r + \frac{1}{r}\right)$$

by computing the next term in the Segre polynomial in χ. Indeed, for the right hand side, the χ^{n-1} coefficient is easily seen to be

$$\frac{r^n}{n}\left(\frac{c}{(n-1)!} - \frac{n\left(r+\frac{1}{2}\right)}{(n-2)!}\right). \tag{13}$$

We compute the χ^{n-1} coefficient on the left. As before, we have

$$\sum_{n=0}^{\infty} z^n \int_{S^{[n]}} s_{2n}(V^{[n]}) = \exp\Big(U(z)\cdot\chi + T(z)\Big)$$

where

$$U(z) = u_1 z + u_2 z^2 + \dots\,, \quad T(z) = t_1 z + t_2 z^2 + \dots\,.$$

We obtain

$$\int_{S^{[n]}} s_{2n}(V^{[n]}) = \frac{u_1^n}{n!}\cdot\chi^n + \left(\frac{u_1^{n-1}t_1}{(n-1)!} + \frac{u_1^{n-2}u_2}{(n-2)!}\right)\cdot\chi^{n-1} + \dots\,.$$

When $n = 1$, we find

$$\chi r - r^2 + 1 = \int_S s_2(V) = u_1\chi + v_1 \implies u_1 = r,\ t_1 = -r^2 + 1\,.$$

When $n = 2$, for isotropic vectors $\langle v, v\rangle = 0$, equation (4) simplifies to

$$\int_{S^{[2]}} s_4(V^{[2]}) = r^2\left(-r + \frac{1}{r} + \frac{\chi}{2}\right)\cdot(\chi - 2r - 1)\,.$$

Using the above asymptotics, we have

$$\int_{S^{[2]}} s_4(V^{[2]}) = \frac{u_1^2}{2}\cdot\chi^2 + (u_1t_1 + u_2)\cdot\chi + \dots \implies u_2 = -r^3 - \frac{r^2}{2}.$$

Thus, for arbitrary n, the χ^{n-1}-coefficient equals

$$\frac{r^{n-1}(1-r^2)}{(n-1)!} + \frac{r^{n-2}(-r^3 - \frac{r^2}{2})}{(n-2)!}. \tag{14}$$

Comparison of (13) and (14) yields the requisite value for the constant c, completing the argument. □

2.4. **Proof of Theorem 1.** By [EGL], the generating series of Segre integrals takes the form

$$\mathsf{S}(z) = \sum_{n=0}^{\infty} z^n \int_{S^{[n]}} s_{2n}(V^{[n]}) = a_0(z)^{c_2(V)} \cdot a_1(z)^{c_1(V)^2} \cdot a_2(z)^{\chi(\mathcal{O}_S)}\,,$$

for three power series $a_0(z), a_1(z), a_2(z) \in \mathbb{Q}[[z]]$. We show that these power series are as claimed by Theorem 1:

$$a_0(z) = A_0(z), \;\; a_1(z) = A_1(z), \;\; a_2(z) = A_2(z).$$

Fix an integer $\chi > s$, and let $g = s(\chi - s)$. Consider (X, H) a $K3$ surface of genus g, and let V be a vector bundle with Mukai vector

$$v = (s, H, \chi - s) \implies \langle v, v \rangle = -2\,.$$

We calculate

$$c_2(V) = \chi(s-1) - s^2 + 2s - 1, \;\; c_1(V)^2 = 2(s\chi - 1 - s^2)\,.$$

The residue calculations of Section 2.1 and the first part of Theorem 3 together imply

$$\left(A_0^{s-1} A_1^{2s}\right)^{\chi} \left(A_0^{-s^2+2s-1} A_1^{-2-2s^2} A_2^2\right) = \left(a_0^{s-1} a_1^{2s}\right)^{\chi} \left(a_0^{-s^2+2s-1} a_1^{-2-2s^2} a_2^2\right).$$

Since χ is arbitrary, we obtain

$$A_0^{s-1} A_1^{2s} = a_0^{s-1} a_1^{2s}\,, \tag{15}$$

$$A_0^{-s^2+2s-1} A_1^{-2-2s^2} A_2^2 = a_0^{-s^2+2s-1} a_1^{-2-2s^2} a_2^2\,. \tag{16}$$

We now derive an additional equation using isotropic bundles corresponding to the second part of Theorem 3. To this end, let $g = s(\chi - s) + 1$. Let $V \to S$ be a vector bundle with Mukai vector

$$v = (s, H, \chi - s) \implies \langle v, v \rangle = 0\,.$$

We then calculate

$$c_2(V) = \chi(s-1) - s^2 + 2s\,, \quad c_1(V)^2 = 2(s\chi - s^2)\,.$$

Repeating the above argument for the new numerics, we replicate equation (15) and in addition we obtain

$$A_0^{-s^2+2s} A_1^{-2s^2} A_2^2 = a_0^{-s^2+2s} a_1^{-2s^2} a_2^2\,. \tag{17}$$

Constraints (15), (16), (17) show that $A_i = a_i$ for $1 \leq i \leq 3$, as claimed.

2.5. **Chern classes.** Let $\alpha \to S$ be a K-theory class of rank s on a $K3$ surface S, and let $s = r + 1$. As a corollary of Theorem 1, we also obtain expressions for the Chern classes of tautological bundles,

$$\sum_{n=0} z^n \int_{S^{[n]}} c_{2n}(\alpha^{[n]}) = \tilde{A}_0(z)^{c_2(\alpha)} \cdot \tilde{A}_1(z)^{c_1^2(\alpha)} \cdot \tilde{A}_2(z)^{\chi(\mathcal{O}_S)} .$$

After the change of variables

$$z = t(1-rt)^{-r} ,$$

the series are given by

$$\begin{aligned} \tilde{A}_0(z) &= (1-rt)^{-r} \cdot (1+(-r+1)t)^{r+1} , \\ \tilde{A}_1(z) &= (1-rt)^{\frac{r-1}{2}} \cdot (1+(-r+1)t)^{-\frac{r}{2}} , \\ \tilde{A}_2(z) &= (1-rt+r^2t)^{-\frac{1}{2}} \cdot (1-rt)^{\frac{r^2-1}{2}} \cdot (1+t(-r+1))^{-\frac{r^2}{2}-r}. \end{aligned}$$

Remark 8. The rank $s = 2$ case (corresponding to $r = 1$) is easily calculated from geomety by assuming the existence of a transverse section of V, see [J]. The derivation using our formulas is also simple,

$$\tilde{A}_0 = \frac{1}{1-t} , \quad \tilde{A}_1 = 1 , \quad \tilde{A}_2 = 1 , \quad z = \frac{t}{1-t} \implies \tilde{A}_0 = 1 + z.$$

We find

$$\sum_{n=0} z^n \int_{S^{[n]}} c_{2n}(V^{[n]}) = (1+z)^{c_2(V)} , \tag{18}$$

which is the correct answer.

The evaluation of $\tilde{A}_0$, $\tilde{A}_1$, and $\tilde{A}_2$ for Chern classes follows by regarding the Segre integrals of Theorem 1 as functions on the K-theory of the surface S which depend polynomially on

$$\text{rank } \alpha = s, \quad c_1(\alpha)^2 , \quad c_2(\alpha) ,$$

see [EGL]. Having established these polynomials for positive $s > 0$, we may then also allow s to be negative, and replace α by $-\alpha$.

As before, if V is a bundle with $\langle v, v \rangle = -2$, we obtain

$$\int_{S^{[n]}} c_{2n}(V^{[n]}) = (-r)^n \binom{-\chi + rn}{n} .$$

In particular, paralleling Proposition 1, when $\langle v, v \rangle = -2$ we obtain the vanishing

$$c_{2n}(V^{[n]}) = 0 \text{ for } (s-2)n < \chi(V) \leq (s-1)n.$$

3. Abelian and Enriques surfaces

3.1. **Abelian surfaces.** Having established the $K3$ case, we can also determine the Segre integrals over abelian or bielliptic surfaces S. Indeed, by the universality results of [EGL], we have

$$\sum_{n=0}^{\infty} z^n \int_{S^{[n]}} s_{2n}(V^{[n]}) = A_0(z)^{c_2(V)} \cdot A_1(z)^{c_1^2(V)}.$$

Residue calculations give the coefficients of the right hand side

$$\int_{S^{[n]}} s_{2n}(V^{[n]}) = \mathrm{Coeff}_{t^n}\left[(1+(1+r)t)^d \cdot (1+rt)^{-d+\chi-rn-1} \cdot (1+r(r+1)t)\right],$$

where $\langle v, v\rangle = 2d$. In particular, we obtain the following generalization of the $s = 1$ result of [MOP1].

Proposition 3. *Let $V \to S$ be a vector bundle of rank $s = r-1$ on an abelian or bielliptic surface S. If $\langle v, v\rangle = 0$, then*

$$\int_{S^{[n]}} s_{2n}(V^{[n]}) = r^n \cdot \frac{\chi}{n} \cdot \binom{\chi - rn - 1}{n-1}.$$

3.2. **Enriques surfaces and strange duality.** When S is an Enriques surface, the link between the Segre and Verlinde series yields individual equalities of intersection numbers, as the numerical data can be suitably matched. The Proposition below parallels Conjecture 2.2 of [J] formulated for del Pezzo surfaces. The corresponding result does not hold for other K-trivial surfaces.

Proposition 4. *Let $V \to S$ be a vector bundle of rank $s = r+1$, determinant L, and $\chi(V) = (r-1)n + 1$. We have*

$$\int_{S^{[n]}} c_{2n}(V^{[n]}) = \chi(S^{[n]}, L_{(n)} \otimes E^r).$$

The left hand side interprets enumeratively the Verlinde number on the right as an integral over the Hilbert scheme – counting the expected (finite) number of quotients $V^\vee \to I_Z$ with Z a subscheme of length n. Implications of Proposition 4 for strange duality over $K3$ and Enriques surfaces will be taken up elsewhere.

Proof. Write $\chi = \chi(L)$ and note that the assumption of the Proposition translates into

$$c_2(V) = \chi - (r-1)(n-1).$$

By Subsection 2.5, the Chern integral on the left hand side is the coefficient of z^n in the series

$$\tilde{A}_0(z)^{c_2(V)} \cdot \tilde{A}_1(z)^{c_1^2(V)} \cdot \tilde{A}_2(z),$$

or equivalently, the residue of the differential form

$$\tilde{A}_0(z)^{\chi-(r-1)(n-1)} \cdot \tilde{A}_1(z)^{2\chi-2} \cdot \tilde{A}_2(z) \cdot \frac{dz}{z^{n+1}}\,.$$

Using the change of variables $z = t(1-rt)^{-r}$, the differential form becomes

$$(1-tr(1-r))^{\frac{1}{2}} \cdot (1-tr)^{-\chi+r^2\left(n-\frac{1}{2}\right)-\frac{1}{2}} \cdot (1+(1-r)\,t)^{\chi-r^2\left(n-\frac{1}{2}\right)+(n-1)} \cdot \frac{dt}{t^{n+1}} \tag{19}$$

For the right hand side, we use the calculations of Lemma 5.2 of [EGL]:

$$\sum w^n \chi(S^{[n]}, L_{(n)} \otimes E^r) = F(w)^{\frac{1}{2}} \cdot G(w)^{\chi}$$

where

$$F(w) = \frac{(1+u)^{r^2}}{1+r^2u}\,, \quad G(w) = 1+u$$

for the change of variables

$$w = u(1+u)^{r^2-1}.$$

Therefore, $\chi(S^{[n]}, L_{(n)} \otimes E^r)$ is the residue of the differential form

$$F(w)^{\frac{1}{2}} \cdot G(w)^{\chi} \cdot \frac{dw}{w^{n+1}} = (1+r^2u)^{\frac{1}{2}} \cdot (1+u)^{\chi-r^2\left(n-\frac{1}{2}\right)+(n-1)} \cdot \frac{du}{u^{n+1}} \tag{20}$$

The change of variables

$$u = \frac{t}{1-tr}$$

matches the two differential forms (19) and (20) and completes the proof. □

4. $K3$ BLOWUPS

In order to determine the functions A_3, A_4, we need to consider surfaces which are not K-trivial. We will look at two different families of examples over the blowup of a $K3$ at a point. These examples are stated in Theorem 4. We will prove this theorem first. As before, Reider-type arguments play a key role in the calculation. In addition, several excess intersection calculations are needed. Afterwards we show Theorem 2: the integrals calculated in Theorem 4 give the rank 2 series for all surfaces.

4.1. **Stability in extensions.** Let

$$\pi : S \to X$$

be the blowup of a $K3$ surface X at one point. We assume that $\operatorname{Pic} X = \mathbb{Z}H$. Then

$$\operatorname{Pic} S = \mathbb{Z}H + \mathbb{Z}E,$$

where E is the exceptional divisor on S and H denotes the pullback to S of the ample Picard generator on X. Note that H is a nef line bundle on S.

The notion of H-stability and H-semistability have the usual meaning:

Definition 1. We say that a torsion-free sheaf V on S is H-stable if for any nonzero subsheaf G of strictly smaller rank, we have

$$\frac{c_1(G)\cdot H}{\mathrm{rk}\, G} < \frac{c_1(V)\cdot H}{\mathrm{rk}\, V}.$$

For the notion of H-semistability the inequality is not required to be strict.

We first prove several statements regarding vanishing of cohomology which will be useful in the proof of Theorem 4. We consider a vector bundle $V_0 \to X$ with Mukai vector v_0 such that

$$\text{rank } V_0 = 2,\ \ c_1(V_0) = H,\ \ \langle v_0, v_0\rangle = -2,\ \ V_0 \text{ is Gieseker } H\text{-stable.} \tag{21}$$

The existence of the bundle V_0 was noted in the beginning of Section 2.2: V_0 is the unique point in the moduli space of Gieseker H-stable bundles with Mukai vector v_0. Such vector bundles are necessarily *rigid*, that is

$$\mathrm{Ext}^1(V_0, V_0) = 0.$$

Let

$$V = \pi^\star V_0 \otimes E^{-k} \text{ on } S, \text{ for } k \geq 0. \tag{22}$$

In order to compute Segre classes of $V^{[n]} \to S^{[n]}$, we analyze the surjectivity of the evaluation map:

$$H^0(V)\otimes \mathcal{O}_{S^{[n]}} \to V^{[n]} \text{ on } S^{[n]}. \tag{23}$$

Thus, for a zero dimensional subscheme Z of length n, we investigate the vanishing of

$$H^1(S, V\otimes I_Z) = \mathrm{Ext}^1(V^\vee, I_Z) = \mathrm{Ext}^1(I_Z, V^\vee \otimes E)^\vee.$$

Assume that a non-zero element exists in the extension group. We prove

Lemma 1. (i) *Assume $k \geq n-1$. For any nonsplit extension*

$$0 \to V^\vee \otimes E \to W \to I_Z \to 0 \tag{24}$$

with $Z \in S^{[n]}$, the middle sheaf W is H-stable.

(ii) *Assume $k = n-2$. The same conclusion holds, unless the scheme Z is supported on the exceptional divisor E.*

Proof. Assume $G \hookrightarrow W$ is an H-semistable destabilizing subsheaf. We then have

$$\mu_G \geq \mu_W > \mu_{V^\vee \otimes E}.$$

Since V_0 is H-slope stable on X, $V_0^\vee$ is also H-slope stable. The pullback $\pi^\star V_0^\vee$ is H-stable on S. Thus $V^\vee \otimes E$ is also H-stable. As a consequence, we see that G cannot

be a subsheaf of the kernel $V^\vee \otimes E$ since this would contradict the H-stability of the latter.

Thus we have a nonzero morphism $\phi : G \to I_Z$. The H-semistability of G now gives $\mu_G \leq 0$. If $c_1(G) = aH + bE$, we deduce

$$0 \geq \mu_G \geq \mu_W \implies 0 \geq \frac{aH^2}{\operatorname{rk} G} \geq -\frac{H^2}{\operatorname{rk} W},$$

from which we conclude that $a = 0$ and $\mu_G = 0$. Furthermore, if nonzero, the kernel K of ϕ is a subsheaf of $V^\vee \otimes I_Z$ of slope greater than or equal to zero, contradicting the H-semistability of $V^\vee \otimes E$. Thus we have in fact that $\phi : G \to I_Z$ is injective, so

$$G = I_U \otimes E^{-m}$$

for a zero-dimensional subscheme $U \subset S$ and $m \geq 0$. In particular Z is supported on the exceptional curve E and on U. We turn to the exact sequence

$$0 \to G \to I_Z \to Q \to 0$$

and the associated sequence of extension groups

$$\operatorname{Ext}^1(Q, V^\vee \otimes E) \to \operatorname{Ext}^1(I_Z, V^\vee \otimes E) \xrightarrow{\alpha} \operatorname{Ext}^1(G, V^\vee \otimes E).$$

The image of the extension (24)

$$0 \neq e \in \operatorname{Ext}^1(I_Z, V^\vee \otimes E)$$

under α is trivial since the resulting extension is seen to be split. Turning to the first extension group, we have

$$\operatorname{Ext}^1(Q, V^\vee \otimes E)^\vee = H^1(V \otimes Q).$$

Recall the notation $V = \pi^\star V_0 \otimes E^{-k}$. We claim that

$$H^1(V \otimes Q) = 0, \text{ for } k \geq n - 1.$$

This would imply α is injective, and in turn that the original extension e splits – a contradiction.

Note that Q is supported on the exceptional divisor and a finite number of points in S. To prove the claimed vanishing, let us first assume that U is empty. The argument is best illustrated by the case $m = 1$. In this case, the defining sequence

$$0 \to E^{-1} \to I_Z \to Q \to 0$$

shows that $Q = I_{Z/E}$ is the ideal sheaf of n points on the exceptional divisor. Since

$$V|_E = \mathbb{C}^2 \otimes \mathcal{O}_E(k),$$

we see that

$$H^1(V \otimes Q) = H^1(\mathcal{O}_E(k - n)) \otimes \mathbb{C}^2 = 0 \text{ for } k \geq n - 1.$$

We now consider the case of arbitrary m, continuing to assume that $U = \emptyset$. We let $W_\ell = \ell E$ be the scheme defined by the ideal sheaf $E^{-\ell}$. Set $Z_m = Z$ and $Q_m = Q$. The defining exact sequence

$$0 \to E^{-m} \to I_Z \to Q \to 0$$

shows that

$$0 \to Q_m \to \mathcal{O}_{W_m} \to \mathcal{O}_{Z_m} \to 0.$$

Inductively define the scheme-theoretic intersection

$$Z_\ell = Z_{\ell+1} \cap W_\ell \hookrightarrow W_{\ell+1}$$

for $1 \leq \ell \leq m-1$, and let Q_ℓ be given by the exact sequence

$$0 \to Q_\ell \to \mathcal{O}_{W_\ell} \to \mathcal{O}_{Z_\ell} \to 0.$$

Clearly,

$$\text{length } (Z_\ell) \leq \text{length } (Z_m) = n.$$

We will show inductively that

$$H^1(V \otimes Q_\ell) = 0 \tag{25}$$

for all $\ell \leq m$. The base case $\ell = 1$ was verified above. For the inductive step, form the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_\ell & \longrightarrow & \mathcal{O}_E(\ell-1) & \longrightarrow & M_\ell & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q_\ell & \longrightarrow & \mathcal{O}_{W_\ell} & \longrightarrow & \mathcal{O}_{Z_\ell} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q_{\ell-1} & \longrightarrow & \mathcal{O}_{W_{\ell-1}} & \longrightarrow & \mathcal{O}_{Z_{\ell-1}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Note that the support of M_ℓ has length at most n. As we already noted

$$V|_E = \mathbb{C}^2 \otimes \mathcal{O}_E(k).$$

If $\ell \geq 1$ and $k+1 \geq n$, we have just enough positivity to ensure that the morphism

$$H^0(V \otimes \mathcal{O}_E(\ell-1)) = \mathbb{C}^2 \otimes H^0(E, \mathcal{O}_E(k+(\ell-1)) \to H^0(V \otimes M_\ell) \tag{26}$$

is surjective. Using the first row we conclude

$$H^1(V \otimes K_\ell) = 0.$$

Using the first column and invoking the inductive hypothesis, we conclude

$$H^1(V \otimes Q_\ell) = 0.$$

This completes the argument when U is empty. For the general case, let T denote the scheme with ideal $E^{-m} \otimes I_U$, and recall W_m had ideal E^{-m}. The defining exact sequence gives

$$0 \to I_T \to I_Z \to Q \to 0 \implies 0 \to Q \to \mathcal{O}_T \to \mathcal{O}_Z \to 0.$$

By composing the first map with the canonical restriction $\mathcal{O}_T \to \mathcal{O}_{W_m}$ we obtain an exact sequence

$$0 \to Q \to \mathcal{O}_{W_m} \to A \to 0. \tag{27}$$

Furthermore

$$\mathcal{O}_Z \to A \to 0,$$

so $A = \mathcal{O}_{\widetilde{Z}}$ is supported on at most n points. Then the previous argument applied to the exact sequence (27)

$$0 \to Q \to \mathcal{O}_{W_m} \to \mathcal{O}_{\widetilde{Z}} \to 0$$

gives the vanishing $H^1(V \otimes Q) = 0$. This finishes the proof when $k = n - 1$.

When $k = n-2$, the same argument carries through, unless $m = 1$ and Z is contained in E, as one can easily check going through the details, in particular by examining (26). □

Lemma 2. *If W is H-stable, then for any $k \geq 0$, the dimension of $Hom(W, W \otimes E^k)$ equals* 1.

Proof. The H-stability of W implies that any nonzero homomorphism $\phi : W \to W \otimes E^k$ is injective. This can be seen as usual by examining the kernel and image of ϕ. Let λ be an eigenvalue of ϕ at a point p in S where W is locally free and $p \notin E$. Let I be the canonical homomorphism $W \to W \otimes E^k$ obtained by W-twisting the unique section $\mathcal{O} \to E^k$. We claim

$$\phi = \lambda I.$$

Indeed, assuming otherwise, set

$$\psi := \phi - \lambda I \neq 0.$$

By the first line of the proof, the morphism $\psi : W \to W \otimes E^k$ must be injective. Consider the induced morphism

$$\det \psi : \det W \to \det W \otimes E^{kr}.$$

Writing as before I for the $\det W$-twisting of the unique section of E^{kr}, we conclude

$$\det \psi = \mu I$$

for some constant μ. By construction, $\det \psi$ vanishes at p. However I only vanishes along E and $p \notin E$. This shows that $\mu = 0$, so $\det \psi = 0$. This contradicts the fact that ψ is injective. □

Lemma 3. *Let $V = \pi^\star V_0 \otimes E^{-k}$ with V_0 a rank 2 bundle satisfying (21). Assume* [4] $\chi(V) = 4n - 1$.

(i) *If $k = n - 2$ then*

$$H^1(V \otimes I_Z) = 0$$

for all $Z \in S^{[n]}$ unless $Z \subset E$.

(ii) *If $k = n - 1$ then any nontrivial extension W in (24) is a rigid sheaf.*

(iii) *In both cases, $H^1(V) = H^2(V) = 0$.*

Proof. We continue working with the exact sequence (24). If $Z \not\subset E$, W must be H-stable by Lemma 1. Using Lemma 2, we compute

$$\chi(W, W) \leq \operatorname{ext}^0(W, W) + \operatorname{ext}^2(W, W) = \operatorname{ext}^0(W, W) + \operatorname{ext}^0(W, W \otimes E) = 2.$$

On the other hand, from the defining exact sequence (24), we calculate:

$$\begin{aligned}
\chi(W, W) &= \chi(V^\vee, V^\vee) + \chi\left(V^\vee \otimes E,\, I_Z\right) + \chi\left(V^\vee,\, I_Z\right) + \chi(I_Z,\, I_Z) \\
&= -\langle v_0,\, v_0\rangle + (\chi(V_0) - 2n - (k+1)(k+2)) + (\chi(V_0) - 2n - k(k+1)) + 2 - 2n \\
&= 2\left(-\frac{\langle v_0,\, v_0\rangle}{2} + \chi(V_0) - 3n + 1 - (k+1)^2\right) \\
&= 2\left(-\frac{\langle v_0,\, v_0\rangle}{2} + \chi(V) - 3n - k\right) = 2\,(n-k)\,.
\end{aligned}$$

Note now that if $k = n - 2$, then

$$\chi(W, W) = 2(n - k) = 4$$

which is a contradiction. This establishes (i).

If $k = n - 1$, then $\chi(W, W) = 2$. Since $\operatorname{Ext}^0(W, W) = \operatorname{Ext}^2(W, W) = \mathbb{C}$ we find that $\operatorname{Ext}^1(W, W) = 0$ so W is rigid. This establishes (ii).

Note that the same argument for $Z = \emptyset$ shows that $H^1(V) = 0$, while $H^2(V) = 0$ follows by stability. □

We will analyze the two situations (i) and (ii) in Propositions 5, 6, and 7 below.

[4] This assumption uniquely specifies the genus $H^2 = 2g - 2$ of the $K3$ surface X in terms of n and k, as well as the Mukai vector v_0 for each such $K3$ surface.

4.2. **Excess calculations for $k = n-2$.** The goal of this section is to prove the following result which corresponds to Theorem 4 (i).

Proposition 5. *Let S be the blowup of a $K3$ surface at one point. Let*

$$V = \pi^\star V_0 \otimes E^{-(n-2)}$$

with V_0 a rank 2 bundle satisfying (21). Assume furthermore that $\chi(V) = 4n-1$. Then

$$\int_{S^{[n]}} s(V^{[n]}) = (-1)^n(2n+1).$$

Proof. We compute $s_{2n}(V^{[n]})$ as an excess intersection over the nonsurjectivity locus of the evaluation map (23). By Lemma 3, this nonsurjectivity locus consists of those Z with $Z \subset E$, or in other words

$$Z \in E^{[n]} \simeq \mathbb{P}^n.$$

For the excess calculation, we need to interpret the Segre class s_{2n} in connection with the top Chern class of a vector bundle, where the excess formula is easier to understand. This connection is as follows, cf. Fulton [F].

Let $G(2n, 4n-1)$ be the Grassmannian of $2n$ planes $\Lambda \hookrightarrow H^0(V)$, with tautological bundle $\mathcal{E}$ of rank $2n$. Consider the product $S^{[n]} \times G(2n, 4n-1)$ with projection

$$\pi : S^{[n]} \times G(2n, 4n-1) \to S^{[n]}.$$

By [F], Proposition 14.2.2,

$$s_{2n}(V^{[n]}) = \pi_\star \left(c_{\mathrm{top}}(\mathcal{E}^\vee \otimes V^{[n]}) \right).$$

Here $\mathcal{E}$ and $V^{[n]}$ are pulled back to the product from $G(2n, 4n-1)$ and $S^{[n]}$ respectively. The vector bundle $\mathcal{E}^\vee \otimes V^{[n]}$ has a natural section s. This is induced by the morphism

$$\mathcal{E} \to V^{[n]} \text{ on } S^{[n]} \times G(2n, 4n-1)$$

obtained by composing the inclusion

$$\mathcal{E} \hookrightarrow H^0(V) \otimes \mathcal{O}$$

on $G(2n, 4n-1)$ with the evaluation map (23)

$$H^0(V) \otimes \mathcal{O} \to V^{[n]} \text{ on } S^{[n]}.$$

The zero locus $\mathbb{D}$ of the section s is expected to be zero-dimensional, but let us suppose it is in excess with dimension d, so that

$$\mathbb{D} \subset S^{[n]} \times G(2n, 4n-1), \text{ with normal bundle } N.$$

The excess intersection formula of Section 14.4 in [F] reads

$$s_{2n}(V^{[n]}) = \pi_\star \left(c_d \left(\mathcal{E}^\vee \otimes V^{[n]}|_{\mathbb{D}} - N \right) \right), \tag{28}$$

where $\pi : \mathbb{D} \to S^{[n]}$.

We now compute the right hand side of equation (28). By definition,

$$\mathbb{D} = \{(Z, \Lambda) \text{ so the map } \Lambda \hookrightarrow H^0(V) \to H^0(V \otimes \mathcal{O}_Z) \text{ is zero}\}.$$

We describe $\mathbb{D}$ in concrete terms. Observe that

$$V|_E = \mathbb{C}^2 \otimes \mathcal{O}_{\mathbb{P}^1}(n-2),$$

and set

$$\Lambda_0 = \ker H^0(V) \to H^0(V|_E).$$

The argument of Lemma 3 shows that $H^1(V \otimes \mathcal{O}(-E)) = 0$. Therefore

$$\dim \Lambda_0 = (4n-1) - (2n-2) = 2n+1.$$

Observe next that

$$\mathbb{D} = \{(Z, \Lambda) \text{ with } Z \in E^{[n]} \text{ and } \Lambda \subset \Lambda_0\} \subset S^{[n]} \times G(2n, 4n-1).$$

Indeed, if $Z \not\subset E$, by Lemma 3

$$H^0(V) \to H^0(V|_Z)$$

is surjective. The kernel is of dimension $(4n-1) - 2n = 2n-1$, hence it cannot contain a subspace W of dimension $2n$. Furthermore, for $Z \subset E$, the restriction

$$H^0(V) \to H^0(V|_Z)$$

factors through $H^0(V|_E)$. Since $V|_E = \mathcal{O}_E(n-2) \otimes \mathbb{C}^2$, the restriction

$$H^0(V|_E) \to H^0(V|_Z)$$

is injective. Therefore,

$$W \hookrightarrow H^0(V) \to H^0(V|_E) \text{ is zero } \implies \Lambda \subset \Lambda_0.$$

Consequently, we have

$$\mathbb{D} = E^{[n]} \times G(2n, \Lambda_0) \subset S^{[n]} \times G(2n, 4n-1),$$

so that

$$\mathbb{D} \simeq \mathbb{P}^n \times \mathbb{P}^{2n}.$$

This identification holds scheme-theoretically. Indeed, it is easy to check that the above pointwise arguments can also be carried out in families. The key observation is that cohomology and base change commute for all relative constructions involved.

In K-theory, the normal bundle to $\mathbb{D}$ is

$$N = N_{E^{[n]}/S^{[n]}} + N_{\mathbb{P}^{2n}/G(2n,4n-1)} = \mathcal{O}(E)^{[n]} + \mathcal{E}^\vee \otimes \mathbb{C}^{2n-2}.$$

Here $2n-2$ is the dimension of the quotient $H^0(V)/\Lambda_0$ and the two summands are restricted to $\mathbb{D}$. We now calculate

$$\begin{aligned} s_{2n}(V^{[n]}) &= \pi_\star \left(c_{3n} \left(\mathcal{E}^\vee \otimes V^{[n]} - \mathcal{O}(E)^{[n]} - \mathcal{E}^\vee \otimes \mathbb{C}^{2n-2}|_{\mathbb{D}} \right) \right) \\ &= \pi_\star \left(c_{3n} \left(\mathcal{E}^\vee \otimes \mathbb{C}^2 \otimes \mathcal{O}(n-2)^{[n]} - \mathcal{O}(-1)^{[n]} - \mathcal{E}^\vee \otimes \mathbb{C}^{2n-2} \right) \right) \end{aligned}$$

Let ζ denote the hyperplane class on $E^{[n]} \simeq \mathbb{P}^n$ and h denote the hyperplane in the Grassmannian $G(2n, \Lambda_0) \simeq \mathbb{P}^{2n}$. We have

$$\begin{aligned} c(\mathcal{E}^\vee) &= \frac{1}{1-h}, \\ c\left(\mathcal{O}(n-2)^{[n])} \right) &= 1-\zeta, \\ c\left(\mathcal{O}(-1)^{[n]} \right) &= (1-\zeta)^n. \end{aligned}$$

The last two formulas for the total Chern classes of tautological vector bundles over the Hilbert scheme of points on $\mathbb{P}^1$ are explained for instance in the proof of Theorem 2 of [MOP2]. Completing the calculation,

$$\begin{aligned} s_{2n}(V^{[n]}) &= \mathrm{Coeff}_{h^{2n}\zeta^n} \left[(1-\zeta)^{3n+2} \cdot (1-h)^{2n-2} \cdot \frac{1}{c(h^\vee \otimes \mathcal{O}(n-2)^{[n]})^2} \right] \\ &= \mathrm{Coeff}_{h^{2n}\zeta^n} \frac{(1-\zeta)^{3n+2}}{(1-h-\zeta)^2} \\ &= (-1)^n(2n+1). \quad \square \end{aligned}$$

4.3. **The case $k = n-1$.** Consider now the case $k = n-1$, so that

$$V = V_0 \otimes E^{-(n-1)}$$

with V_0 satisfying (21), and $\chi(V) = 4n-1$. By Lemma 3 (ii), all nontrivial extensions

$$0 \to V^\vee \otimes E \to W \to I_Z \to 0 \tag{29}$$

must have the middle term W a rigid H-stable sheaf of rank 3. Of course,

$$c_1(W) = -H + 2nE. \tag{30}$$

We record the following

Lemma 4. *There are no rigid H-stable sheaves of rank* 3 *satisfying* (30) *when $n \not\equiv 0$ mod 3.*

When $n \equiv 0 \mod 3$, the sheaf W must necessarily be of the form

$$W = \pi^\star W_0 \otimes E^{\frac{2n}{3}}$$

where $W_0 \to X$ is a bundle over the $K3$ surface with

$$\text{rank } W_0 = 3, \;\; c_1(W_0) = -H, \;\; W_0 \text{ rigid and Gieseker } H\text{-stable.} \tag{31}$$

Proof. As noted in [G2], [Y1], stability with respect to the non-ample divisor H on the blowup is equivalent to $H - \epsilon E$-stability for ϵ sufficiently small. Thus the moduli space of H-stable sheaves on the blowup admits a natural scheme structure. In general, the moduli space $M_H^S(3, -H + 2nE, c_2)$ of H-stable sheaves on the blowup has expected dimension $6d - 16$ where d denotes the discriminant. For rigid moduli spaces, we have $d = \frac{8}{3}$. The corresponding moduli of H-stable sheaves is zero dimensional and consists of isolated points. To count the stable rigid sheaves, we compute the Euler characteristic of the moduli space.

To this end, we use the blowup formulas of Proposition 3.4 in [Y1] or Proposition 3.1 (2) in [G2]. The generating series of Euler characteristics on the blowup is computed in terms of the same series on the underlying $K3$ surface X as

$$\sum_d q^{d-\frac{8}{3}} \mathsf{e}(M_H^S(3, -H + 2nE, d)) = \prod_{m=1}^{\infty} \frac{1}{(1-q^m)^3} \cdot \left(\sum_{(x,y)} q^{x^2+xy+y^2} \right) \cdot \left(\sum_d q^{d-\frac{8}{3}} \mathsf{e}(M_H^X(3, -H, d)) \right)$$

where $x, y \in \mathbb{Z} + \frac{2n}{3}$. To complete the proof, we compute the constant term in the above expression. If $n \not\equiv 0 \mod 3$, there is no constant term due to the factor

$$\sum q^{x^2+xy+y^2}.$$

Indeed, as $n \not\equiv 0 \mod 3$, we have $(x, y) \neq (0, 0)$ so $x^2 + xy + y^2 > 0$.

For $n \equiv 0 \mod 3$, the constant term is 1. The moduli space consists of a single rigid sheaf. Note that when $\chi(V) = 4n - 1$, a direct calculation shows that there exists a rigid sheaf W_0 on X such that $W = \pi^\star W_0 \otimes E^{\frac{2n}{3}}$ on S has the numerics determined by the exact sequence (29). Specifically the numerical assumptions on V require

$$H^2 = 4n^2 + 12n - 14$$

and we select W_0 to be in the moduli space with Chern numbers

$$\mathrm{ch}(W_0) = 3 - H + \left(\frac{2}{3} n^2 + 2n - 5 \right) [\mathrm{pt}] \, .$$

This W is then the only sheaf in its moduli space.

□

By the Lemma, if $n \not\equiv 0 \mod 3$ the extension W cannot exist. Consequently, we have

$$H^1(V \otimes I_Z) = 0$$

for all Z. This shows that the evaluation map (23) over the Hilbert scheme is surjective, and we have obtained the following result corresponding to Theorem 4 (ii):

Proposition 6. *If $V = \pi^\star V_0 \otimes E^{-(n-1)}$ with $\chi(V) = 4n-1$ and V_0 a rank 2 bundle as in (21), then*

$$\int_{S^{[n]}} s_{2n}(V^{[n]}) = 0 \ \text{ for } \ n \not\equiv 0 \mod 3.$$

We now focus on the remaining statement of Theorem 4. We show

Proposition 7. *If $V = \pi^\star V_0 \otimes E^{-(n-1)}$ with $\chi(V) = 4n-1$ and V_0 a rank 2 bundle as in (21), then*

$$\int_{S^{[n]}} s_{2n}(V^{[n]}) = 1 \ \text{ for } \ n \equiv 0 \mod 3.$$

Proof. As usual, the top Segre class is supported on the locus $\mathbb{D}$ where the evaluation map (23) is not surjective. We now identify $\mathbb{D}$ as a subvariety of the Hilbert scheme $S^{[n]}$. The nonsurjectivity locus consists of subschemes Z such that

$$H^1(V \otimes I_Z) \neq 0,$$

in other words corresponding to the existence of nontrivial extensions

$$0 \to V^\vee \otimes E \to W \to I_Z \to 0. \tag{32}$$

When $n = 3\ell$ for $\ell \in \mathbb{Z}$, as seen in Lemma 4, all such extensions have the same middle term, the unique stable rigid rank 3 vector bundle W on S with numerics specified by the lemma. A direct calculation establishes further that

$$\chi(V^\vee \otimes E, W) = 1. \tag{33}$$

We now show that

$$\mathbb{D} \simeq \mathbb{P}\mathrm{Hom}(V^\vee \otimes E, W) = \mathbb{P}H^0(V \otimes W \otimes E^{-1}).$$

To start, note the useful vanishing

$$H^1(V \otimes W) = 0$$

which will be established in Lemma 5 using Reider arguments. By Serre duality and stability

$$H^2(V \otimes W) = 0, \ H^2(V \otimes V^\vee \otimes E) = \mathbb{C}.$$

From the exact sequence (32), upon tensoring with V, these vanishings imply

$$H^1(V \otimes I_Z) = \mathbb{C}. \tag{34}$$

By (34), for any given Z in the degeneracy locus, the basic extension (32) is unique. Conversely, any non-zero morphism

$$V^\vee \otimes E \to W$$

must be injective by stability, and the cokernel sheaf Q must have rank 1 and trivial determinant. We claim that $Q = I_Z$ for some scheme Z which necessarily has length n. This is clear if Q is torsion-free. However, if Q had torsion T, then there would be a torsion-free quotient Q^{tf} of rank 1,

$$0 \to T \to Q \to Q^{\mathrm{tf}} \to 0, \text{ and } Q^{\mathrm{tf}} = I_U(-D), \text{ for a subscheme } U.$$

Comparing determinants, we find $D = \det T$ effective so that $D \cdot H \geq 0$. This however contradicts stability of $W \to Q^{\mathrm{tf}}$ unless $D \cdot H = 0$, in which case $D = qE$ for some $q \geq 0$. We argue that $q = 0$. Indeed, consider the kernel K of the surjection

$$W \to Q \to Q^{\mathrm{tf}} = I_U(-qE).$$

Using $K \hookrightarrow W$ and W is H-stable, it follows that K is H-stable as well. Write

$$m = \text{length } (U).$$

A direct calculation shows that

$$\begin{aligned} \chi(K,K) &= \chi(W - I_U(-qE), W - I_U(-qE)) \\ &= 2 + 4m + 4n(q-1) + 3q^2 > 2, \end{aligned}$$

which contradicts stability. The only exception is

$$q = 0,\ m \leq n \implies Q^{\mathrm{tf}} = I_U.$$

The torsion part T must have trivial determinant hence it must be supported on points. Computing Euler characteristics

$$\chi(I_U) + \chi(T) = \chi(Q) = \chi(V^\vee \otimes E) - \chi(W) = 1 - n \implies \chi(T) = m - n \leq 0.$$

This is only possible if $T = 0$ so that Q is torsion free, in fact $Q = I_U$ for some zero dimensional scheme U of length n.

Thus we have identified the degeneracy locus $\mathbb{D}$ as

$$\mathbb{D} = \mathbb{P}H^0(V \otimes W \otimes E^\vee).$$

It is straightforward to carry out the above argument in families. The crux of the matter is that (34) has constant rank, hence cohomology and base change commute over the degeneracy locus $\mathbb{D}$. We set

$$d = \dim \mathbb{D}.$$

Since $H^2(V \otimes W \otimes E^\vee) = 0$ by Serre duality and stability, it follows from (33) that

$$h^1(V \otimes W \otimes E^\vee) = d.$$

On $\mathbb{D} \times S$ we have a universal exact sequence

$$0 \to V^\vee \otimes E \otimes \mathcal{O}(-1) \to W \to \mathcal{I}_{\mathcal{Z}} \to 0. \tag{35}$$

which restricts to (32) for each point of $\mathbb{D}$. Here V, W, E are pulled back to the product from S, $\mathcal{O}(1)$ is the hyperplane bundle on $\mathbb{D}$, and $\mathcal{I}_{\mathcal{Z}}$ is the restriction to $\mathbb{D} \times S$ of the universal ideal sheaf on $S^{[n]} \times S$.

The Segre integral

$$\int_{S^{[n]}} s_{2n}(V^{[n]})$$

is calculated as an excess intersection on the degeneracy locus $\mathbb{D}$ as follows. Consider the Grassmannian $G(2n, 4n-1)$ of subspaces

$$\Lambda \hookrightarrow H^0(V) = \mathbb{C}^{4n-1}.$$

Write $\mathcal{E}$ for the tautological bundle, and let $\mathcal{F}$ denote the tautological quotient. The vector bundle

$$\mathcal{E}^\vee \otimes V^{[n]} \to G(2n, 4n-1) \times S^{[n]}$$

has a natural section s obtained as the composition

$$\Lambda \to H^0(V) \to H^0(V \otimes \mathcal{O}_Z).$$

Let

$$\mathbb{D}_0 \subset G(2n, 4n-1) \times S^{[n]}$$

be the zero locus of this section. It consists of those pairs (Z, Λ) where

$$\Lambda \subset H^0(V \otimes I_Z).$$

In particular, $h^0(V \otimes I_Z) \geq \dim \Lambda = 2n$. Note now that

$$\chi(V \otimes I_Z) = \chi(V) - 2n = 2n - 1,$$

while $H^2(V \otimes I_Z) = 0$ by Serre duality and stability. For (Z, Λ) in the degeneracy locus $\mathbb{D}_0$, we have then

$$h^1(V \otimes I_Z) \geq 1.$$

This shows that Z must be in the degeneracy locus $\mathbb{D}$, and in this case $h^1(V \otimes I_Z) = 1$ by (34). We are thus forced to have $\Lambda = H^0(V \otimes I_Z)$ so that

$$\mathbb{D}_0 = \{(Z,\, \Lambda = H^0(V \otimes I_Z)) \text{ with } Z \in \mathbb{D}\} \subset S^{[n]} \times G(2n, 4n-1).$$

Note that the projection onto $S^{[n]}$ induces an isomorphism

$$\mathbb{D}_0 \simeq \mathbb{D}.$$

By [F], Section 14.4, we have

$$\int_{S^{[n]}} s_{2n}(V^{[n]}) = \int_{\mathbb{D}_0} c_d(\mathbb{E}),$$

where the excess virtual bundle $\mathbb{E}$ is

$$\mathbb{E} = \mathcal{E}^\vee \otimes V^{[n]} - N,$$

with N the normal bundle of $\mathbb{D}_0 \subset S^{[n]} \times G(2n, 4n-1)$. We have in K-theory,

$$N = \mathcal{E}^\vee \otimes \mathcal{F} + TS^{[n]} - T\mathbb{D}_0.$$

Putting all together,

$$\mathbb{E} = \mathcal{E}^\vee \otimes (V^{[n]} - \mathcal{F}) + T\mathbb{D}_0 - TS^{[n]}. \tag{36}$$

It remains to identify $\mathbb{E}$ explicitly in terms of the universal sequence (35). For simplicity we write identities in K-theory on fibers. To start, note that on $\mathbb{D}_0$ we have

$$\begin{aligned} \mathcal{E}|_Z &= H^0(V \otimes I_Z), \\ \mathcal{F}|_Z &= H^0(V) - H^0(V \otimes I_Z), \\ TS^{[n]}|_Z &= \mathrm{Ext}^1(I_Z, I_Z), \end{aligned}$$

and from the Euler sequence

$$T\mathbb{D}_0 = H^0(V \otimes W \otimes E^\vee) \otimes \mathcal{O}(1) - \mathbb{C}. \tag{37}$$

The evaluation sequence

$$0 \to H^0(V \otimes I_Z) \to H^0(V) \to H^0(V \otimes \mathcal{O}_Z) \to H^1(V \otimes I_Z) \to 0$$

gives

$$V^{[n]} - \mathcal{F} = R^1\pi_\star(V \otimes \mathcal{I}_{\mathcal{Z}}),$$

a line bundle on $\mathbb{D}_0$. Here $\pi : \mathbb{D} \times S \to \mathbb{D}$ is the projection. From the sequence (35), after tensoring with V and pushing forward to $\mathbb{D}$ via π we obtain

$$0 \to \mathcal{O}(-1) \to H^0(V \otimes W) \to R^0\pi_\star(V \otimes \mathcal{I}_{\mathcal{Z}}) \to 0,$$

$$0 \to R^1\pi_\star(V \otimes \mathcal{I}_{\mathcal{Z}}) \to \mathcal{O}(-1) \to 0.$$

Therefore in K-theory

$$\begin{aligned} R^0\pi_\star(V \otimes \mathcal{I}_{\mathcal{Z}}) &= H^0(V \otimes W) - \mathcal{O}(-1) \\ R^1\pi_\star(V \otimes \mathcal{I}_{\mathcal{Z}}) &= \mathcal{O}(-1). \end{aligned}$$

We obtain

$$\mathcal{E}^\vee \otimes (V^{[n]} - \mathcal{F}) = H^0(V \otimes W)^\vee \otimes \mathcal{O}(-1) - \mathbb{C}. \tag{38}$$

Noting from (35) that

$$\mathcal{I}_{\mathcal{Z}} = W - V^\vee \otimes E \otimes \mathcal{O}(-1),$$

it follows that

$$\mathcal{I}_{\mathcal{Z}}^\vee \otimes \mathcal{I}_{\mathcal{Z}} = V \otimes V^\vee + W \otimes W^\vee - V \otimes W \otimes E^\vee \otimes \mathcal{O}(1) - V^\vee \otimes W^\vee \otimes E \otimes \mathcal{O}(-1).$$

Here all tensor products are derived. We therefore finally calculate

$$\begin{aligned} TS^{[n]} &= \mathcal{E}xt^1_\pi(\mathcal{I}_{\mathcal{Z}}, \mathcal{I}_{\mathcal{Z}}) \\ &= -\mathcal{E}xt^\bullet_\pi(\mathcal{I}_{\mathcal{Z}}, \mathcal{I}_{\mathcal{Z}}) + \mathbb{C} + \mathbb{C} \\ &= H^\bullet(V \otimes W \otimes E^\vee) \otimes \mathcal{O}(1) + H^0(V \otimes W)^\vee \otimes \mathcal{O}(-1) - \mathbb{C} - \mathbb{C}. \end{aligned} \tag{39}$$

This last equality uses the fact that V, W are stable and rigid, and that $V \otimes W$ has no higher cohomology.

Collecting the expressions (37), (38), (39) in the excess bundle $\mathbb{E}$ given by (36), we find

$$\mathbb{E} = H^1(V \otimes W \otimes E^\vee) \otimes \mathcal{O}(1) = \mathbb{C}^d \otimes \mathcal{O}(1).$$

We conclude

$$\int_{X^{[n]}} s_{2n}(V^{[n]}) = \int_{\mathbb{D}_0} c_d(\mathbb{E}) = \int_{\mathbb{D}_0} h^d = 1.$$

This completes the proof. □

Lemma 5. *Let $n = 3\ell$. Assume*

$$V = \pi^\star V_0 \otimes E^{-3\ell+1},\ W = \pi^\star W_0 \otimes E^{2\ell}$$

where $\chi(V) = 4n-1$ and V_0, W_0 are two rigid bundles satisfing conditions (21) *and* (31)*. We have*

$$H^1(V \otimes W) = 0.$$

Proof. The argument is an application of Reider's method. Assume that

$$H^1(V \otimes W) = \mathrm{Ext}^1(W, V^\vee \otimes E)^\vee \neq 0.$$

We construct a nontrivial extension

$$0 \to V^\vee \otimes E \to F \to W \to 0.$$

The middle sheaf F is H-stable by an argument similar to that of Proposition 2. Indeed, if F is not H-stable, let $G \hookrightarrow F$ be the maximal semistable destabilizing subsheaf. Then

$$\mathrm{rk}\, G < \mathrm{rk}\, F \ \text{ and } \ \mu_H(G) \geq \mu_H(F) > \mu_H(V^\vee \otimes E)\,.$$

We see that G cannot be a subsheaf of the kernel $V^\vee \otimes E$ since this would contradict the H-stability of the latter. Therefore, we have a nonzero morphism

$$\phi : G \to W \implies \mu_H(G) < \mu_H(W).$$

Writing $c_1(G) = aH + bE$ we see that

$$\mu_H(F) \leq \mu_H(G) < \mu_H(W) \implies -\frac{2}{5} \leq \frac{a}{\mathrm{rk}\, G} < -\frac{1}{3}$$

which is impossible.

Finally, a direct calculation shows

$$\begin{aligned} \chi(F,F) &= \chi(W,W) + \chi(V^\vee \otimes E, V^\vee \otimes E) + \chi(V^\vee \otimes E, W) + \chi(W, V^\vee \otimes E) \\ &= 2 + 2 + 1 + (2n+1) > 2 \end{aligned}$$

contradicting the stability of F. □

4.4. **Proof of Theorem 2.** We now prove Theorem 2. The statement will follow combinatorially using the geometric input provided by Theorem 4.

Define first the following combination of the basic power series A_i:

$$f(z) = A_0(z)^5 \cdot A_1(z)^{20} \cdot A_3(z)^2 \tag{40}$$

$$g(z) = A_0(z)^{-4} \cdot A_1(z)^{-22} \cdot A_2(z)^2 \cdot A_3(z)^{-4} \cdot A_4(z)^{-1}$$

$$h(z) = A_0(z)^{-3} \cdot A_1(z)^{-18} \cdot A_2(z)^2 \cdot A_3(z)^{-2} \cdot A_4(z)^{-1}.$$

We will derive identities between the functions f, g, h using the two calculations provided by Theorem 4.

First, over the $K3$ blowup, let $V = V_0 \otimes E^{-(n-2)}$ with V_0 rigid H-stable of rank 2 so that $\chi(V) = 4n-1$. [5] One checks that

$$c_1(V)^2 = 20n - 22,\ c_2(V) = 5n - 4,\ c_1(V) \cdot K_S = 2(n-2),\ K_S^2 = -1.$$

Using (1) for the vector bundle V, we obtain that

$$\int_{S^{[n]}} s(V^{[n]}) = [z^n]\, A_0(z)^{5n-4} \cdot A_1(z)^{20n-22} \cdot A_2(z)^2 \cdot A_3(z)^{2n-4} \cdot A_4(z)^{-1} = f(z)^n \cdot g(z).$$

With the aid of Proposition 5 this rewrites as

$$[z^n]\, f(z)^n \cdot g(z) = (-1)^n(2n+1),$$

where the brackets denote the suitable coefficient in the given power series. Therefore

$$\sum_{n=0}^{\infty} [z^n]\, f(z)^n \cdot g(z) = \sum_{n=0}^{\infty} z^n (-1)^n (2n+1) = \frac{1-z}{(1+z)^2}.$$

[5] As noted in Lemma 3, this numerical setup only exists over $K3$s of certain genus determined by n. The argument here strongly uses the universality of the Segre series.

Write

$$z = \frac{w}{f(w)}. \tag{41}$$

The Lagrange-Bürmann inversion formula [WW] is the general identity

$$\sum_{n=0}^{\infty} \left([z^n]\, f(z)^n \cdot g(z)\right) \cdot z^n = \frac{g(w)}{f(w)} \cdot \frac{dw}{dz}.$$

In our situation, it gives

$$\frac{1-z}{(1+z)^2} = \frac{g(w)}{f(w)} \cdot \frac{dw}{dz}. \tag{42}$$

In similar fashion, for $V = V_0 \otimes E^{-(n-1)}$, by making use of Proposition 6 and Proposition 7, we obtain

$$\frac{1}{1-z^3} = \frac{h(w)}{f(w)} \cdot \frac{dw}{dz}. \tag{43}$$

The expression

$$\frac{1}{1-z^3} = \sum_{k=0}^{\infty} z^{3k}$$

encodes the fact that the Segre integrals are 0 for $n \not\equiv 0 \mod 3$ and equal to 1 for $n \equiv 0 \mod 3$.

We now explain how equations (42) and (43) give the remaining functions A_3 and A_4. Dividing the two equations we obtain

$$\frac{h(w)}{g(w)} = \frac{(1+z)^2}{(1-z)(1-z^3)}.$$

This gives via (41)

$$\frac{w}{f(w)} \cdot \frac{h(w)}{g(w)} = z \cdot \frac{(1+z)^2}{(1-z)(1-z^3)}.$$

Let us write

$$w = t(1+3t)^3.$$

We compute

$$\frac{w}{f(w)} \cdot \frac{h(w)}{g(w)} = w \cdot A_0(w)^{-4} \cdot A_1(w)^{-16} = \frac{t}{1+3t},$$

where the first equality follows by (40), and the second equality uses the expressions for A_0, A_1 given by Theorem 1. Therefore

$$\frac{z(1+z)^2}{(1-z)(1-z^3)} = \frac{t}{1+3t} \implies z = \mathsf{y}(t),$$

for the function $\mathsf{y}(t)$ of the introduction. With this understood, we find via (41)

$$f(w) = \frac{w}{z} = \frac{t(1+3t)^3}{\mathsf{y}}.$$

Using (40), we obtain

$$A_3(w) = f(w)^{\frac{1}{2}} \cdot A_0(w)^{-\frac{5}{2}} \cdot A_1(w)^{-10} = \frac{1}{1+3t} \cdot \left(\frac{t}{\mathsf{y}}\right)^{1/2}$$

where in the last equality we used the expressions for A_0, A_1 in Theorem 1.

Similarly, from equation (42) we compute

$$g(w) = f(w) \cdot \frac{1-z}{(1+z)^2} \cdot \frac{dz/dt}{dw/dt} = \frac{t(1+3t)^3}{\mathsf{y}} \cdot \frac{1-\mathsf{y}}{(1+\mathsf{y})^2} \cdot \frac{\mathsf{y}'}{(1+3t)^2(1+12t)}.$$

Combined with (40) we obtain the expression for A_4 claimed in Theorem 2. □

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