CAPELLI OPERATORS FOR SPHERICAL SUPERHARMONICS AND THE DOUGALL-RAMANUJAN IDENTITY

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Abstract. Let (V,ω) be an orthosymplectic $\mathbb{Z}_2$-graded vector space and let $\mathfrak{g} := \mathfrak{gosp}(V,\omega)$ denote the Lie superalgebra of similitudes of (V,ω). It is known that as a $\mathfrak{g}$-module, the space $\mathscr{P}(V)$ of superpolynomials on V is completely reducible, unless $\dim V_{\overline{0}}$ and $\dim V_{\overline{1}}$ are positive even integers and $\dim V_{\overline{0}} \leq \dim V_{\overline{1}}$. When $\mathscr{P}(V)$ is *not* a completely reducible $\mathfrak{g}$-module, we construct a natural basis $\{D_\lambda\}_{\lambda\in\mathcal{I}}$ of "Capelli operators" for the algebra $\mathscr{PD}(V)^{\mathfrak{g}}$ of $\mathfrak{g}$-invariant superpolynomial superdifferential operators on V, where the index set $\mathcal{I}$ is the set of integer partitions of length at most two. We compute the action of the operators $\{D_\lambda\}_{\lambda\in\mathcal{I}}$ on maximal indecomposable components of $\mathscr{P}(V)$ explicitly, in terms of Knop-Sahi interpolation polynomials. Our results show that, unlike the cases where $\mathscr{P}(V)$ is completely reducible, the eigenvalues of a subfamily of the D_λ are *not* given by specializing the Knop-Sahi polynomials. Rather, the formulas for these eigenvalues involve suitably regularized forms of these polynomials. This is in contrast with what occurs for previously studied Capelli operators. In addition, we demonstrate a close relationship between our eigenvalue formulas for this subfamily of Capelli operators and the Dougall-Ramanujan hypergeometric identity.

We also transcend our results on the eigenvalues of Capelli operators to the Deligne category $\mathsf{Rep}(O_t)$. More precisely, we define categorical Capelli operators $\{\boldsymbol{D}_{t,\lambda}\}_{\lambda\in\mathcal{I}}$ that induce morphisms of indecomposable components of symmetric powers of V_t, where V_t is the generating object of $\mathsf{Rep}(O_t)$. We obtain formulas for the eigenvalue polynomials associated to the $\{\boldsymbol{D}_{t,\lambda}\}_{\lambda\in\mathcal{I}}$ that are analogous to our results for the operators $\{D_\lambda\}_{\lambda\in\mathcal{I}}$.

1. Introduction

Let $V := V_{\overline{0}} \oplus V_{\overline{1}}$ be a vector superspace equipped with a non-degenerate even supersymmetric bilinear form $\omega : V \times V \to \mathbb{C}$, and let $\mathfrak{osp}(V,\omega)$ denote the orthosymplectic Lie superalgebra that leaves ω invariant. Set

$$\mathfrak{g} := \mathfrak{gosp}(V,\omega) := \mathfrak{osp}(V,\omega) \oplus \mathbb{C}z,$$

where z is a central element of $\mathfrak{g}$. Then V has a natural $\mathfrak{g}$-module structure, where the action of z on V is defined to be $-\mathbf{1}_V$. The $\mathfrak{g}$-module structure of V induces a canonical $\mathfrak{g}$-module structure on the superalgebra $\mathscr{P}(V)$ of superpolynomials on V, and the superalgebra $\mathscr{D}(V)$ of constant-coefficient superdifferential operators on V. Indeed $\mathscr{P}(V) \cong \mathcal{S}(V^*)$ and $\mathscr{D}(V) \cong \mathcal{S}(V)$ as $\mathfrak{g}$-modules. When $\dim V_{\overline{1}} = 0$, studying $\mathscr{P}(V)$ is the subject of the classical theory of spherical harmonics. For an elegant exposition of this theory we refer the reader to [8].

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Set $d_i := \dim V_i$ for $i \in \{\overline{\mathbf{0}}, \overline{\mathbf{1}}\}$. It is known that $\mathscr{P}(V)$ is a semisimple and multiplicity-free $\mathfrak{g}$-module unless $d_{\overline{\mathbf{0}}}, d_{\overline{\mathbf{1}}} \in 2\mathbb{Z}^+$ and $d_{\overline{\mathbf{0}}} \leq d_{\overline{\mathbf{1}}}$ (see [26, 3]). Let $\mathscr{P}\mathscr{D}(V)$ denote the superalgebra of superpolynomial-coefficient superdifferential operators on V, equipped with the natural $\mathfrak{g}$-module structure defined by $x \cdot D := xD - (-1)^{|D|\cdot|x|}Dx$ for homogeneous $x \in \mathfrak{g}$ and $D \in \mathscr{P}\mathscr{D}(V)$ (for further details see for example [19, Sec. 2]). Then there is a canonical $\mathfrak{g}$-module isomorphism

$$\mathscr{P}\mathscr{D}(V) := \mathscr{P}(V) \otimes \mathscr{D}(V). \tag{1}$$

Let $\mathcal{I}$ be the set of integer partitions of length at most two, that is,

$$\mathcal{I} := \left\{(\lambda_1, \lambda_2) \in \mathbb{Z}^2 \,:\, \lambda_1 \geq \lambda_2 \geq 0\right\}.$$

In the cases that $\mathscr{P}(V)$ is a semisimple and multiplicity-free $\mathfrak{g}$-module, the irreducible components of $\mathscr{P}(V)$ are naturally indexed by elements of $\mathcal{I}$ (see [8, 3, 26]). Then by a general algebraic construction (see the discussion at the end of this section, or [18, 21]) one obtains a distinguished basis $\{D_\lambda\}_{\lambda \in \mathcal{I}}$ of *Capelli operators* for the algebra $\mathscr{P}\mathscr{D}(V)^{\mathfrak{g}}$ of $\mathfrak{g}$-invariant differential operators. By Schur's Lemma, the operators D_λ act on irreducible components of $\mathscr{P}(V)$ by scalars. The problem of computing these scalars was addressed in [21], among several other examples. We remark that the problem of computing eigenvalues of Capelli operators (which we will refer to as the *Capelli eigenvalue problem*) has a long history, and has been studied extensively in the general context of multiplicity-free actions of Lie (super)algebras [1, 10, 13, 14, 18, 22, 23, 20, 27]. In all of the previously investigated instances of the Capelli eigenvalue problem, the formulas for the eigenvalues turn out to be specializations of families of interpolation polynomials, such as Knop-Sahi polynomials, Sergeev-Veselov polynomials, Okounkov interpolation polynomials, or Ivanov polynomials. For the definition and properties of these families of polynomials, we refer the reader to [11, 17, 25, 16, 12, 9]. In particular, in [21, Theorem 1.13] we proved that the eigenvalues of the Capelli basis $\{D_\lambda\}_{\lambda \in \mathcal{I}}$ on irreducible components of $\mathscr{P}(V)$ are obtained from the two-variable interpolation polynomials previously defined by F. Knop and the first author [11] at the parameter value $\frac{1}{2}\mathrm{sdim}V - 1$, where $\mathrm{sdim}V := \dim V_{\overline{\mathbf{0}}} - \dim V_{\overline{\mathbf{1}}}$.

In this paper, we are interested in defining the Capelli operators and computing their actions on $\mathscr{P}(V)$ in the cases where $\mathscr{P}(V)$ is *not* a semisimple $\mathfrak{g}$-module. Thus, henceforth we will assume that $d_{\overline{\mathbf{0}}} = 2m$ and $d_{\overline{\mathbf{1}}} = 2n$ for $m, n \in \mathbb{N}$, where

$$k := n - m \geq 0.$$

Because of non-semisimplicity of $\mathscr{P}(V)$, the usual definition of Capelli operators (see [18, 19, 21]) needs to be tweaked slightly. Furthermore, elements of $\mathscr{P}\mathscr{D}(V)^{\mathfrak{g}}$ are not necessarily diagonalizable on $\mathscr{P}(V)$, and thus we are naturally forced to consider their Jordan decompositions.

We show that in the non-semisimple case one still has a natural basis $\{D_\lambda\}_{\lambda \in \mathcal{I}}$ of $\mathscr{P}\mathscr{D}(V)^{\mathfrak{g}}$, but a new phenomenon occurs in relation to their spectra: unlike the previous (semi-simple) instances of the Capelli eigenvalue problem, the eigenvalues of the Capelli basis are *not* always specializations of interpolation polynomials. Rather, for a subfamily of this basis, one needs polynomials that are obtained from Knop-Sahi interpolation polynomials by removing their *singular part*, that is, the part whose coefficients have poles. We provide two different formulas for the eigenvalues of this subfamily that are related to each other through a curious polynomial identity. We prove the latter polynomial identity using the classical Dougall-Ramanujan hypergeometric identity.

To explain our main results, we begin with the definition of the Knop-Sahi polynomials. We will only consider these polynomials in two variables. For the definition of these polynomials in the n-variable case, see [11]. As usual, for $m \in \mathbb{Z}^{\geq 0}$ we define the *falling factorial* $a^{\underline{m}}$ to be

$$a^{\underline{m}} := a(a-1)\cdots(a-m+1).$$

Let $\Bbbk := \mathbb{Q}(\kappa)$ be the field of rational functions in a parameter κ with coefficients in $\mathbb{Q}$. For $\lambda \in \mathcal{I}$, let $P^\kappa_\lambda \in \Bbbk[x, y]$ be defined by

$$P^\kappa_\lambda(x, y) := \sum_{i+j\leq\lambda_1-\lambda_2} \frac{(\lambda_1 - \lambda_2)!(\kappa + 1)^{\underline{\lambda_1-\lambda_2-i}}(\kappa + 1)^{\underline{\lambda_1-\lambda_2-j}}}{i!j!(\lambda_1 - \lambda_2 - i - j)!(\kappa + 1)^{\underline{\lambda_1-\lambda_2}}} x^{\underline{\lambda_2+i}} y^{\underline{\lambda_2+j}}. \tag{2}$$

The polynomial P^κ_λ is symmetric in the variables x and y, with leading term equal to $x^{\lambda_1}y^{\lambda_2}$. An important property of the polynomial P^κ_λ is the following.

Theorem 1.1. (Knop–Sahi [11]) *P^κ_λ is the unique symmetric polynomial of degree less than or equal to $|\lambda| := \lambda_1 + \lambda_2$ in $\Bbbk[x, y]$ that satisfies the following conditions:*

(i) *$P^\kappa_\lambda(\mu_1 - \kappa - 1, \mu_2) = 0$ for partitions $\mu \in \mathcal{I}$ such that $|\mu| \leq |\lambda|$ and $\mu \neq \lambda$.*
(ii) *$P^\kappa_\lambda(\lambda_1 - \kappa - 1, \lambda_2) = H_\lambda(\kappa)$, where*

$$H_\lambda(\kappa) := (\lambda_1 - \lambda_2)!\lambda_2!(\lambda_1 - 1 - \kappa)^{\underline{\lambda_2}}. \tag{3}$$

For certain $\lambda \in \mathcal{I}$, the coefficients of P^κ_λ have poles. It is straightforward to verify that these poles are always simple and occur at $\kappa \in \mathbb{Z}^{\geq 0}$. Let us now define three types of elements of $\mathcal{I}$.

Definition 1.2. Let $k_\circ \in \mathbb{Z}^{\geq 0}$. An element $\lambda \in \mathcal{I}$ is called

- $k_\circ$-*regular*, if $\lambda_1 \leq k_\circ$ or $\lambda_1 - \lambda_2 = k_\circ + 1$ or $\lambda_1 - \lambda_2 \geq 2k_\circ + 3$.
- $k_\circ$-*quasiregular*, if $\lambda_1 \geq k_\circ + 1$ and $\lambda_1 - \lambda_2 \leq k_\circ$.
- $k_\circ$-*singular*, if $k_\circ + 2 \leq \lambda_1 - \lambda_2 \leq 2k_\circ + 2$.

We denote the sets of $k_\circ$-regular, $k_\circ$-quasiregular, and $k_\circ$-singular elements of $\mathcal{I}$ by $\mathcal{I}_{k_\circ,\mathrm{reg}}$, $\mathcal{I}_{k_\circ,\mathrm{qreg}}$, and $\mathcal{I}_{k_\circ,\mathrm{sing}}$.

Remark 1.3. Here is a more concrete explanation of Definition 1.2. Recall that $k := n - m \in \mathbb{Z}^{\geq 0}$. The involution $\lambda \mapsto \lambda^\dagger$ on $\mathbb{Z}^2$, defined by

$$(\lambda_1, \lambda_2) \mapsto (\lambda_2 + k + 1, \lambda_1 - k - 1), \tag{4}$$

yields a bijection between k-quasiregular and k-singular partitions of the same size. For k-regular partitions $\lambda = (\lambda_1, \lambda_2)$ satisfying $\lambda_1 - \lambda_2 = k + 1$, we have $\lambda^\dagger = \lambda$. For all other $\lambda \in \mathcal{I}_{k,\mathrm{reg}}$ we have $\lambda^\dagger \notin \mathcal{I}$.

The following proposition is straightforward to verify using (2).

Proposition 1.4. *For $\lambda \in \mathcal{I}$ and $k_\circ \in \mathbb{Z}^{\geq 0}$, the following statements are equivalent.*

(i) *The coefficients of P^κ_λ do not have poles at $\kappa = k_\circ$.*
(ii) *$\lambda \notin \mathcal{I}_{k_\circ,\mathrm{sing}}$.*

The construction of the Capelli basis of the algebra $\mathscr{P}\mathscr{D}(V)^{\mathfrak{g}}$ relies on the structure of $\mathscr{P}(V)$ and $\mathscr{D}(V)$ as $\mathfrak{g}$-modules. The algebras $\mathscr{P}(V)$ and $\mathscr{D}(V)$ are naturally graded by degree and order respectively, so that

$$\mathscr{P}(V) \cong \bigoplus_{d\geq 0}^{\infty} \mathscr{P}^d(V) \quad \text{and} \quad \mathscr{D}(V) \cong \bigoplus_{d\geq 0}^{\infty} \mathscr{D}^d(V).$$

From now on we set

$$\mathcal{I}'_k := \mathcal{I}_{k,\mathrm{reg}} \cup \mathcal{I}_{k,\mathrm{qreg}}. \tag{5}$$

The indecomposable components of $\mathscr{P}^d(V)$ can be indexed naturally by partitions $\lambda \in \mathcal{I}'_k$ such that $|\lambda| = d$ (see Proposition 3.1 below). That is,

$$\mathscr{P}^d(V) \cong \bigoplus_{\lambda \in {}_d\mathcal{I}'_k} V_\lambda, \tag{6}$$

where ${}_d\mathcal{I}'_k := \{\lambda \in \mathcal{I}'_k : |\lambda| = d\}$, and each V_λ is an indecomposable $\mathfrak{g}$-module. Furthermore, the canonical non-degenerate pairing $\mathscr{D}^d(V) \otimes \mathscr{P}^d(V) \to \mathbb{C}$ yields a $\mathfrak{g}$-module isomorphism

$$\mathscr{D}^d(V) \cong \mathscr{P}^d(V)^* \cong \bigoplus_{\lambda \in {}_d\mathcal{I}'_k} V_\lambda^*. \tag{7}$$

From Proposition 3.1 it follows that if $\lambda \neq \mu$, then V_λ and V_μ have disjoint irreducible composition factors. Thus from (6) and (7) we obtain

$$\mathscr{P}\mathscr{D}(V)^\mathfrak{g} \cong \bigoplus_{\lambda,\mu \in \mathcal{I}'_k} (V_\lambda \otimes V_\mu^*)^\mathfrak{g} \cong \bigoplus_{\lambda,\mu \in \mathcal{I}'_k} \mathrm{Hom}_\mathfrak{g}(V_\mu, V_\lambda) \cong \bigoplus_{\lambda \in \mathcal{I}'_k} \mathrm{Hom}_\mathfrak{g}(V_\lambda, V_\lambda). \tag{8}$$

Proposition 3.1 also implies that

$$\dim \mathrm{Hom}_\mathfrak{g}(V_\lambda, V_\lambda) = \begin{cases} 1 & \text{if } \lambda \text{ is } k\text{-regular,} \\ 2 & \text{if } \lambda \text{ is } k\text{-quasiregular.} \end{cases} \tag{9}$$

Indeed when $\lambda \in \mathcal{I}_{k,\mathrm{qreg}}$, there exists a nilpotent element of $\mathrm{Hom}_\mathfrak{g}(V_\lambda, V_\lambda)$ that factors through the isomorphism $\mathrm{cosocle}(V_\lambda) \cong \mathrm{socle}(V_\lambda)$. By Corollary 3.5 the space $\mathrm{Hom}_\mathfrak{g}(V_\lambda, V_\lambda)$ has a natural direct sum decomposition into two one-dimensional subspaces, that is,

$$\mathrm{Hom}_\mathfrak{g}(V_\lambda, V_\lambda) \cong \mathbb{C}\mathbf{1}_{V_\lambda} \oplus \mathbb{C}N_\lambda, \tag{10}$$

where N_λ is the nilpotent part of the Jordan decomposition of $C\big|_{V_\lambda}$, with C denoting the Casimir operator of $\mathfrak{g}$ (note that $N_\lambda^2 = 0$). We now use (8) and (10) to define the family $\{D_\lambda\}_{\lambda \in \mathcal{I}}$.

Definition 1.5. For $\lambda \in \mathcal{I}$, we define $D_\lambda \in \mathscr{P}\mathscr{D}(V)^\mathfrak{g}$ as follows.

$$D_\lambda \sim \begin{cases} \mathbf{1}_{V_\lambda} \in \mathrm{Hom}_\mathfrak{g}(V_\lambda, V_\lambda) & \text{if } \lambda \in \mathcal{I}'_k \\ N_\lambda & \text{if } \lambda \in \mathcal{I}_{k,\mathrm{sing}}. \end{cases}$$

Here "$\sim$" means D_λ is the element of $\mathscr{P}\mathscr{D}(V)^\mathfrak{g}$ that corresponds to either $\mathbf{1}_\lambda$ or N_λ via the isomorphism (8). The operators $D_\lambda \in \mathscr{P}\mathscr{D}(V)^\mathfrak{g}$, where $\lambda \in \mathcal{I}$, are called the *Capelli operators*.

From (8) it is evident that the family $\{D_\lambda\}_{\lambda \in \mathcal{I}}$ is a basis of $\mathscr{P}\mathscr{D}(V)^\mathfrak{g}$.

2. Main results

Now let $\lambda \in \mathcal{I}$ and let $\mu \in \mathcal{I}'_k$. Then by Schur's Lemma $D_\lambda(V_\mu) \subseteq V_\mu$, and therefore the restriction $D_\lambda\big|_{V_\mu} \in \mathrm{Hom}_\mathfrak{g}(V_\mu, V_\mu)$ can be expressed as

$$D_\lambda\big|_{V_\mu} = d_{\lambda,\mu}\mathbf{1}_{V_\mu} + d'_{\lambda,\mu}N_\mu, \tag{11}$$

where $d_{\lambda,\mu}, d'_{\lambda,\mu} \in \mathbb{C}$ (note that $N_\mu = 0$ for $\mu \in \mathcal{I}_{k,\mathrm{reg}}$). Our main results in this paper address the problem of computing formulas for $d_{\lambda,\mu}$ and $d'_{\lambda,\mu}$. From Proposition 3.3 it follows that there exists a symmetric polynomial $f_\lambda \in \mathbb{C}[x,y]$ of degree $|\lambda| := \lambda_1 + \lambda_2$ such that

$$d_{\lambda,\mu} = f_\lambda(\mu_1 - k - 1, \mu_2) \text{ for all } \mu \in \mathcal{I}'_k.$$

We call f_λ the *eigenvalue polynomial* of D_λ (see Definition 3.4). It turns out (see Proposition 4.3) that

$$D_\lambda\big|_{V_\mu} = f_\lambda(\mu_1 - k - 1, \mu_2)\mathbf{1}_{V_\lambda} + \Box f_\lambda(\mu_1 - k - 1, \mu_2)N_\lambda,$$

where $f \mapsto \Box f$ is the differential operator defined by

$$\Box f(x,y) := \frac{1}{4(x-y)}\left(\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial y}(x,y)\right). \tag{12}$$

Thus, both $d_{\lambda,\mu}$ and $d'_{\lambda,\mu}$ are uniquely determined by f_λ. The problem of computing f_λ is solved by Theorems A–D below.

Theorem A. *Let $\lambda \in \mathcal{I}_{k,\mathrm{reg}}$. Then*

$$f_\lambda = \frac{1}{H_\lambda(k)}P_\lambda^k,$$

where $H_\lambda(\kappa)$ is defined in (3).

Theorem B. *Let $\lambda \in \mathcal{I}_{k,\mathrm{sing}}$. Then*

$$f_\lambda = \frac{4(\lambda_1 - \lambda_2 - k - 1)}{H'_{\lambda^\dagger}(k)}P_{\lambda^\dagger}^k,$$

where $H'_{\lambda^\dagger}(k)$ denotes the derivative of $H_{\lambda^\dagger}(\kappa)$ at $\kappa = k$.

The formulas for f_λ in Theorems A–B still follow the pattern of specializing interpolation polynomials. The new phenomenon that was described in Section 1 occurs for the formulas of f_λ when $\lambda \in \mathcal{I}_{k,\mathrm{qreg}}$.

Theorem C. *Let $\lambda \in \mathcal{I}_{k,\mathrm{qreg}}$. Then*

$$f_\lambda = \lim_{\kappa\to k}\left(\frac{P_\lambda^\kappa}{H_\lambda(\kappa)} + \frac{P_{\lambda^\dagger}^\kappa}{H_{\lambda^\dagger}(\kappa)}\right). \tag{13}$$

Remark 2.1. Note that both $\frac{P_\lambda^\kappa}{H_\lambda(\kappa)}$ and $\frac{P_{\lambda^\dagger}^\kappa}{H_{\lambda^\dagger}(\kappa)}$ have poles at $\kappa = k$ (indeed $H_\lambda(k) = 0$), but the poles on the right hand side of (13) cancel out and the limit is well defined.

Since the leading term of P_λ is $x^{\lambda_1}y^{\lambda_2}$, the polynomials $\{P_\lambda^\kappa\}_{\lambda\in\mathcal{I}}$ form a basis of the algebra $\Bbbk[x,y]^{S_2}$ of symmetric polynomials in x and y with coefficients in $\Bbbk$. Indeed for any $k_\circ \in \mathbb{C}$ such that $k_\circ \notin \mathbb{Z}^{\geq 0}$, the polynomials $\{P_\lambda^{k_\circ}\}_{\lambda\in\mathcal{I}}$ form a basis of $\mathbb{C}[x,y]^{S_2}$. However, we cannot set $\kappa := k_\circ$ when $k_\circ \in \mathbb{Z}^{\geq 0}$, because the coefficients of the P_λ^κ can have poles at $\kappa = k_\circ$. In this case, one can still obtain a natural basis of $\mathbb{C}[x,y]^{S_2}$ by first suitably separating the regular part of P_λ^κ and then setting $\kappa := k_\circ$. We will describe this process more precisely below.

Definition 2.2. Let $f(x,y) \in \Bbbk[x,y]$ and let $k_\circ \in \mathbb{Q}$. Assume that the coefficients of $(\kappa - k_\circ)f(x,y)$ do not have any poles at $\kappa = k_\circ$.

(i) The *singular part* of $f(x,y)$ at $\kappa = k_\circ$ is the polynomial $\mathrm{Sing}_{k_\circ}(f) \in \mathbb{Q}[x,y]$ defined by

$$\mathrm{Sing}_{k_\circ}(f;x,y) := \lim_{\kappa\to k_\circ}(\kappa - k_\circ)f(x,y).$$

(ii) The *regular part* of $f(x,y)$ at $\kappa = k_\circ$ is the polynomial $\mathrm{Reg}_{k_\circ}(f) \in \mathbb{Q}[x,y]$ defined by

$$\mathrm{Reg}_{k_\circ}(f;x,y) := \lim_{\kappa\to k_\circ}\left(f(x,y) - \frac{1}{\kappa - k_\circ}\mathrm{Sing}_{k_\circ}(f;x,y)\right).$$

Example 2.3. Assume that $k_\circ = 1$ and $f(x,y) = x^2 + y^2 + \frac{2\kappa}{\kappa-1}xy$. Then $\mathrm{Sing}_{k_\circ}(f;x,y) = 2xy$ and $\mathrm{Reg}_{k_\circ}(f;x,y) = x^2 + y^2 + 2xy$.

For $\lambda \in \mathcal{I}$ and $k_\circ \in \mathbb{C}$ set

$$R_\lambda^{(k_\circ)} := \mathrm{Reg}_{k_\circ}(P_\lambda^\kappa).$$

Remark 2.4. Note that by Proposition 1.4, if $k_\circ \notin \mathbb{Z}^{\geq 0}$ then for all $\lambda \in \mathcal{I}$ we have

$$R_\lambda^{(k_\circ)} = \lim_{\kappa \to k_\circ} P_\lambda^\kappa = P_\lambda^{k_\circ}. \tag{14}$$

If $k_\circ \in \mathbb{Z}^{\geq 0}$ then (14) holds whenever $\lambda \notin \mathcal{I}_{k_\circ,\mathrm{sing}}$.

The following proposition is a straightforward consequence of the above discussion.

Proposition 2.5. *For $k_\circ \in \mathbb{C}$, the family $\left\{ R_\lambda^{(k_\circ)} \right\}_{\lambda \in \mathcal{I}}$ is a basis of the algebra $\mathbb{C}[x,y]^{S_2}$ of symmetric polynomials in the variables x, y.*

By analogy with the completely reducible cases, Proposition 2.5 leads to the following natural question.

Problem. Determine the coefficients $M_{\lambda,\mu} \in \mathbb{C}$ such that $f_\lambda = \sum_{\mu \in \mathcal{I}} M_{\lambda,\mu} R_\mu^{(k)}$ for $\lambda \in \mathcal{I}$.

Clearly Theorems A–B answer this problem when $\lambda \notin \mathcal{I}_{k,\mathrm{qreg}}$. Surprisingly, in the case $\lambda \in \mathcal{I}_{k,\mathrm{qreg}}$ the formulas for the coefficients $M_{\lambda,\mu}$ become much more complicated. Before we state the result (Theorem D below), we need to introduce some notation. For $d \geq 0$ set

$$\mathcal{I}(d) := \{\lambda \in \mathcal{I} \,:\, |\lambda| \leq d\}. \tag{15}$$

For $\lambda \in \mathcal{I}_{k,\mathrm{qreg}}$ set

$$\ell_\lambda := \lambda_2 - \lambda_1 + k, \tag{16}$$

so that $0 \leq \ell_\lambda \leq k$, and if $\mu \in \mathcal{I}(k - \ell_\lambda)$ then set

$$\nu(\lambda,\mu) := (\lambda_1 - \mu_1, \lambda_2 + \mu_2).$$

Note that $\nu(\lambda,\mu) \in \mathcal{I}_{k,\mathrm{qreg}}$, and in particular

$$R_{\nu(\lambda,\mu)}^{(k)} = P_{\nu(\lambda,\mu)}^k.$$

Theorem D. *Let $\lambda \in \mathcal{I}_{k,\mathrm{qreg}}$. Then*

$$f_\lambda = \frac{(\ell_\lambda+1)!}{(\lambda_1-k-1)!(\lambda_1+\ell_\lambda-k)!}\left(\frac{1}{(2k+2-\lambda_1+\lambda_2)!} R_{\lambda^\dagger}^{(k)} + \sum_{\mu \in \mathcal{I}(k-\ell_\lambda)} M_{\lambda,\mu} R_{\nu(\lambda,\mu)}^{(k)}\right), \tag{17}$$

where the $M_{\lambda,\mu}$ are defined by

$$M_{\lambda,\mu} := \frac{(-1)^{\ell_\lambda+\mu_1+\mu_2}\binom{\mu_1}{\mu_2}(\ell_\lambda+\mu_1)!}{(k-\ell_\lambda-\mu_1-\mu_2)!\ell_\lambda!(\ell_\lambda+\mu_2+1)!(\ell_\lambda+\mu_1+\mu_2)!\mu_1} \qquad \text{if } |\mu| > 0,$$

and

$$M_{\lambda,(0,0)} := \frac{(-1)^{\ell_\lambda+1}}{(k-\ell_\lambda)!(\ell_\lambda+1)!^2}\left(1 - \sum_{j=\lambda_1-k}^{\lambda_1+\ell_\lambda-k} \frac{\ell_\lambda+1}{j}\right).$$

Remark 2.6. For fixed $\lambda, \mu \in \mathcal{I}$, the formulas for the eigenvalue of $D_\lambda\big|_{V_\mu}$ given in Theorems A–D depend only on $k = n - m$ (rather than on m and n). This observation has a conceptual explanation based on the Duflo-Serganova functor [7, 24]. We briefly recall the definition of this functor. Given any Lie superalgebra $\mathfrak{g}$ and an element $x \in \mathfrak{g}_{\bar{1}}$ such that $[x,x] = 0$, we set $\mathrm{DS}_x(M) := M^x/xM$ for every $\mathfrak{g}$-module M, where $M^x := \ker(x|_M)$ and $xM := \mathrm{im}(x|_M)$. Then $\mathrm{DS}_x(M)$ is a $\mathfrak{g}_x$-module, where $\mathfrak{g}_x := \ker(\mathrm{ad}_x)/\mathrm{im}(\mathrm{ad}_x)$. Further, for every $\mathfrak{g}$-module homomorphism $h : M \to N$ we set $\mathrm{DS}_x(h) : \mathrm{DS}_x(M) \to \mathrm{DS}_x(N)$ to be the naturally induced $\mathfrak{g}_x$-module homomorphism. As shown in [7, 24], the above assignments yield a symmetric monoidal functor

$$\mathrm{DS}_x : \mathrm{Rep}(\mathfrak{g}) \to \mathrm{Rep}(\mathfrak{g}_x).$$

If $\mathfrak{g} \cong \mathfrak{gosp}(V, \omega)$, then $\mathfrak{g}_x \cong \mathfrak{gosp}(V', \omega')$ where $\mathrm{sdim}V' = \mathrm{sdim}V = -2k$. Furthermore, DS_x maps the Casimir operator of $\mathfrak{g}$, which we can consider as an element of $\hom_{\mathfrak{g}}(\mathbb{C}, \mathcal{S}^2(\mathfrak{g}))$, to the Casimir operator of $\mathfrak{g}_x$. If $\mathcal{S}^d(V) \cong \oplus I_t$ where each I_t is a generalized eigenspace of the Casimir operator of $\mathfrak{g}$ with eigenvalue t, then $\mathcal{S}^d(V_x) \cong \oplus I'_t$, where $I'_t \cong \mathrm{DS}_x(I_t)$ is the generalized eigenspace of the Casimir operator of $\mathfrak{g}_x$ with eigenvalue t. One can then show that DS_x maps Capelli operators to Capelli operators and preserves their eigenspaces. These facts imply that the eigenvalues of $D_\lambda\big|_{V_\mu}$ should only depend on k.

The proof of Theorem D is substantially more difficult than those of Theorems A–C. It relies on the following identity (in the parameter x) which, to the best of our knowledge, is new.

Theorem E. *For non-negative integers i, j, N such that $i + j \leq N$,*

$$\frac{d}{dx}\left(\frac{x^{\underline{N-i}}x^{\underline{N-j}}}{x^{\underline{N}}}\right) = \sum_{q=0}^{j}\ \sum_{p=i+j-q}^{\min\{N-q,N-1\}} \frac{(-1)^{N+p+q+1}(N-p)^{\underline{q}}\, i^{\underline{q}}\, j^{\underline{q}}\,(N-i-j)^{\underline{N-p-q}}\, x^{\underline{p-i}}\, x^{\underline{p-j}}(x-p+q)}{(N-p)\,q!\,x^{\underline{p+1}}(x-N+q)^{\underline{q}}}. \tag{18}$$

We remark that in the special case $j = 0$, Theorem E is equivalent to the formula

$$\frac{\partial}{\partial x}(x^{\underline{N}}) = \sum_{k=1}^{N} \frac{(-1)^{t+1}}{t} N^{\underline{t}} x^{\underline{N-t}}, \tag{19}$$

which can be proved by logarithmic differentiation of the binomial series for $(1 + z)^x$. However, we are unable to find a similar quick argument for the general case.

Our proof of Theorem E involves subtle computations that reduce it to a classical hypergeometric identity, usually referred to as Dougall's Theorem. Recall that a *generalized hypergeometric function* is a series of the form

$${}_p\mathbf{F}_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; z\right) := \sum_{n=0}^{\infty} \frac{a_1^{\overline{n}} \cdots a_p^{\overline{n}}}{b_1^{\overline{n}} \cdots b_q^{\overline{n}} n!} z^n, \tag{20}$$

where as usual

$$a^{\overline{n}} := a(a+1)\cdots(a+n-1) \quad \text{for } n \in \mathbb{N} \text{ and } \ a^{\overline{0}} = 1.$$

Dougall's theorem states that for $\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d} \in \mathbb{C}$ such that $\Re(\mathsf{a} + \mathsf{b} + \mathsf{c} + \mathsf{d} + 1) > 0$, we have

$${}_5\mathbf{F}_4\left(\begin{matrix} \frac{1}{2}\mathsf{a}+1, \mathsf{a}, -\mathsf{b}, -\mathsf{c}, -\mathsf{d} \\ \frac{1}{2}\mathsf{a}, \mathsf{a}+\mathsf{b}+1, \mathsf{a}+\mathsf{c}+1, \mathsf{a}+\mathsf{d}+1 \end{matrix}; 1\right) = \frac{\Gamma(\mathsf{a}+\mathsf{b}+1)\Gamma(\mathsf{a}+\mathsf{c}+1)\Gamma(\mathsf{a}+\mathsf{d}+1)\Gamma(\mathsf{a}+\mathsf{b}+\mathsf{c}+\mathsf{d}+1)}{\Gamma(\mathsf{a}+1)\Gamma(\mathsf{a}+\mathsf{b}+\mathsf{c}+1)\Gamma(\mathsf{a}+\mathsf{b}+\mathsf{d}+1)\Gamma(\mathsf{a}+\mathsf{c}+\mathsf{d}+1)}. \tag{21}$$

Identity (21) is a limit case of another identity for ${}_7\mathbf{F}_6$ that was discovered by Dougall (1907) and Ramanujan (1910). For the proof and further historical remarks on Dougall's Theorem, we refer the reader to [2, Sec. 2.2].

Theorems A, B, and D were conjectured using computations that were implemented by SageMath. Our efforts to prove Theorem D lead us to Theorems C and E.

Capelli operators in the Deligne category $\mathsf{Rep}(O_t)$. Recall from Remark 2.6 that existence of certain monoidal functors between (rigid symmetric monoidal) categories of modules implies that the formulas for f_λ should only depend on the superdimension of V. Indeed it is possible to transcend the construction of the Capelli basis $\{D_\lambda\}_{\lambda\in\mathcal{I}}$ to a universal categorical framework where the superdimension can be any complex number! More precisely, in Section 8 we show that we can define Capelli operators in the inductive completion of the Deligne category $\mathsf{Rep}(O_t)$, where $t\in\mathbb{C}$. Then we prove analogues of Theorems A–C for the corresponding eigenvalue polynomials.

The definition of the categorical Capelli operators $\{\boldsymbol{D}_{t,\lambda}\}_{\lambda\in\mathcal{I}}$ in Section 8 goes as follows. The category $\mathsf{Rep}(O_t)$ is the Karoubian rigid symmetric monoidal category generated by the self-dual object V_t of categorical dimension $t\in\mathbb{C}$. We introduce an algebra object $\mathsf{PD}_{\mathsf{V}_t}$ in the inductive completion of this category with a natural action

$$\mathsf{PD}_{\mathsf{V}_t}\otimes\mathsf{P}_{\mathsf{V}_t}\to\mathsf{P}_{\mathsf{V}_t},$$

where $\mathsf{P}_{\mathsf{V}_t}:=\bigoplus_{d\geq 0}\mathcal{S}^d(\mathsf{V}_t)$. The algebra object $\mathsf{PD}_{\mathsf{V}_t}$ is the categorical analogue of $\mathscr{P}\mathscr{D}(V)$. Moreover, $\mathrm{Hom}(\mathsf{1},\mathsf{PD}_{\mathsf{V}_t})$ can be equipped with a canonical algebra structure, and the natural action of $\mathsf{PD}_{\mathsf{V}_t}$ on $\mathsf{P}_{\mathsf{V}_t}$ yields a homomorphism of algebras $\mathrm{Hom}(\mathsf{1},\mathsf{PD}_{\mathsf{V}_t})\to\mathrm{End}(\mathsf{P}_{\mathsf{V}_t})$. The categorical Capelli operators $\boldsymbol{D}_{t,\lambda}$ that we will define in Section 8 are elements of the algebra $\mathrm{Hom}(\mathsf{1},\mathsf{PD}_{\mathsf{V}_t})$. To define these operators, first we prove that the indecomposable summands of $\mathsf{P}_{\mathsf{V}_t}$ are naturally indexed by elements of $\mathcal{I}$ if $t\notin 2\mathbb{Z}^{\leq 0}$, and by elements of $\mathcal{I}'_{\underline{k}}$ if $t\in 2\mathbb{Z}^{\leq 0}$, where

$$\underline{k}:=-\frac{t}{2}\quad\text{and}\quad\mathcal{I}'_{\underline{k}}:=\mathcal{I}_{\underline{k},\mathrm{reg}}\cup\mathcal{I}_{\underline{k},\mathrm{qreg}}.\tag{22}$$

When $t\notin 2\mathbb{Z}^{\leq 0}$, for every $\lambda\in\mathcal{I}$ the operator $\boldsymbol{D}_{t,\lambda}$ corresponds to the co-evaluation morphism

$$\mathsf{1}\xrightarrow{\epsilon_{\mathsf{V}_{t,\lambda}}}\mathsf{V}_{t,\lambda}\otimes\mathsf{V}^*_{t,\lambda}.$$

When $t\in 2\mathbb{Z}^{\leq 0}$, the definition of $\boldsymbol{D}_{t,\lambda}$ is still the same for $\underline{k}$-regular and $\underline{k}$-quasiregular λ, but for $\underline{k}$-singular λ the operator $\boldsymbol{D}_{t,\lambda}$ represents the (unique up to scaling) nilpotent element in $\mathrm{End}(\mathsf{V}_{t,\lambda^\dagger})$. See equation (63) for further details.

After defining the operator $\boldsymbol{D}_{t,\lambda}$, we can consider its restriction to each indecomposable component $\mathsf{V}_{t,\mu}$ of $\mathsf{P}_{\mathsf{V}_t}$ that is indexed by μ. This yields an element of the algebra $\mathrm{End}(\mathsf{V}_{t,\mu})$, of the form $\underline{d}_{\lambda,\mu}\mathsf{1}+\mathsf{n}_{\lambda,\mu}$ where $\underline{d}_{\lambda,\mu}\in\mathbb{C}$ and $\mathsf{n}^2_{\lambda,\mu}=0$. Furthermore

$$\underline{d}_{\lambda,\mu}=\boldsymbol{f}_\lambda(\mu_1-\underline{k}-1,\mu_2),$$

where $\boldsymbol{f}_\lambda\in\mathbb{C}[x,y]$ is a symmetric polynomial of degree $|\lambda|$. (We remark that the coefficients of $\boldsymbol{f}_\lambda$ depend on the value of $t\in\mathbb{C}$.) Theorems A′–C′ below are the extensions of Theorems A–C to the categorical setting of $\mathsf{Rep}(O_t)$.

Theorem A′. *Assume that either $t\notin 2\mathbb{Z}^{\leq 0}$, or that $t\in 2\mathbb{Z}^{\leq 0}$ and λ is $\underline{k}$-regular. Then*

$$\boldsymbol{f}_\lambda=\frac{1}{H_\lambda(\underline{k})}P^{\underline{k}}_\lambda.$$

From now on we set

$$\boldsymbol{c}_\lambda(t):=(\lambda_1-\lambda_2)(\lambda_1-\lambda_2+t-2)\quad\text{for }\lambda\in\mathcal{I}.$$

Theorem B′. *Assume that $t\in 2\mathbb{Z}^{\leq 0}$ and λ is $\underline{k}$-singular. Then*

$$\boldsymbol{f}_\lambda=\lim_{s\to t}\frac{\boldsymbol{c}_{\lambda^\dagger}(s)-\boldsymbol{c}_\lambda(s)}{H_{\lambda^\dagger}\left(-\frac{s}{2}\right)}P^{-\frac{s}{2}}_{\lambda^\dagger}.$$

Theorem C′. *Assume that $t \in 2\mathbb{Z}^{\leq 0}$ and λ is $\underline{k}$-quasiregular. Then*

$$\boldsymbol{f}_\lambda = \lim_{s\to t}\left(\frac{1}{H_\lambda\left(-\frac{s}{2}\right)}P_\lambda^{-\frac{s}{2}} + \frac{1}{H_{\lambda^\dagger}\left(-\frac{s}{2}\right)}P_{\lambda^\dagger}^{-\frac{s}{2}}\right).$$

3. Structure of $\mathscr{P}(V)$ and $\mathscr{P}\mathscr{D}(V)^{\mathfrak{g}}$

Let us begin with the description of the decomposition of the $\mathfrak{g}$-module $\mathscr{P}(V)$ as a direct sum of indecomposable submodules. As will be seen in Proposition 3.1, the indecomposable components of $\mathscr{P}(V)$ can be characterized as generalized eigenspaces of the restriction of the Casimir operator to each homogeneous component. A proof of Proposition 3.1 is given in A. Sherman's PhD thesis [26] (see also [3]).

Let $\mathfrak{b}^{\mathrm{st}}$ be the Borel subalgebra of $\mathfrak{osp}(V,\omega)$ corresponding to the fundamental system

$$\{\varepsilon_1-\varepsilon_2,\ldots,\varepsilon_{m-1}-\varepsilon_m,\varepsilon_m-\delta_1,\ldots,\delta_{n-1}-\delta_n,2\delta_n\},$$

and set $\mathfrak{b} := \mathfrak{b}^{\mathrm{st}} \oplus \mathbb{C}z$. Also, let $\mathfrak{h}^{\mathrm{st}} \subseteq \mathfrak{b}^{\mathrm{st}}$ denote the standard Cartan subalgebra of $\mathfrak{osp}(V,\omega)$, and set $\mathfrak{h} := \mathfrak{h}^{\mathrm{st}} \oplus \mathbb{C}z$. Let $\zeta \in \mathfrak{h}^*$ be the linear functional defined by $\zeta(z) = 1$ and $\zeta\big|_{\mathfrak{h}^{\mathrm{st}}} = 0$. For a $\mathfrak{b}^{\mathrm{st}}$-dominant $\mathfrak{h}^{\mathrm{st}}$-weight λ, let $V(\lambda)$ denote the irreducible finite dimensional $\mathfrak{osp}(V,\omega)$-module with highest weight λ. For any scalar $c \in \mathbb{C}$, we can consider $V(\lambda)$ as a $\mathfrak{g}$-module on which z acts by $c\mathbf{1}_{V(\lambda)}$. We denote the latter $\mathfrak{g}$-module by $V(\lambda + c\zeta)$.

Recall that C denotes the Casimir operator of $\mathfrak{gosp}(V,\omega)$. Then C acts on $V(\lambda+c\zeta)$ by the scalar

$$c_\lambda := (\lambda,\lambda) + 2(\lambda,\rho) = (\lambda_2-\lambda_1)(2k+2+\lambda_2-\lambda_1),$$

where $\rho := \sum_{i=1}^m(-k-i)\varepsilon_i + \sum_{i=1}^n(n-i+1)\delta_i$. For $t \in \mathbb{C}$ let $\mathscr{P}^d(V,t)$ denote the generalized t-eigenspace of the restriction of C to $\mathscr{P}^d(V)$. Note that $c_\lambda = c_{\lambda^\dagger}$ for $\lambda \in \mathcal{I}_{k,\mathrm{qreg}}$, hence $\mathscr{P}^d(V,c_\lambda) = \mathscr{P}^d(V,c_{\lambda^\dagger})$. For $\lambda \in \mathcal{I}'_k$, set

$$V_\lambda := \mathscr{P}^{|\lambda|}(V,c_\lambda).$$

The proof of the following proposition can be found in [26, Sec. 10].

Proposition 3.1. *Let $\lambda \in \mathcal{I}'_k$.*

(i) *If $\lambda \in \mathcal{I}_{k,\mathrm{reg}}$ then*

$$V_\lambda \cong V\big((\lambda_1-\lambda_2)\varepsilon_1 + |\lambda|\zeta\big).$$

In particular, V_λ is an irreducible $\mathfrak{g}$-module.

(ii) *If $\lambda \in \mathcal{I}_{k,\mathrm{qreg}}$ then V_λ is an indecomposable $\mathfrak{g}$-module with a socle filtration of length 3. When $m \geq 2$, the successive quotients of the socle filtration of V_λ are isomorphic to the modules $V(\mu^{(i)} + |\lambda|\zeta)$, $1 \leq i \leq 3$, where*

$$\mu^{(1)} = \mu^{(3)} = (\lambda_1-\lambda_2)\varepsilon_1 \quad \textit{and} \quad \mu^{(2)} = (2k+2+\lambda_2-\lambda_1)\varepsilon_1.$$

When $m = 1$, the successive quotients of the socle filtration of V_λ are isomorphic to

$$V(\mu^{(1)}+|\lambda|\zeta),\quad V(\mu^{(2)}+|\lambda|\zeta)\oplus V(\mu^{(3)}+|\lambda|\zeta),\quad \textit{and}\quad V(\mu^{(4)}+|\lambda|\zeta),$$

where

$$\mu^{(1)} = \mu^{(4)} = (\lambda_1-\lambda_2)\varepsilon_1,\ \mu^{(2)} = (2k+2+\lambda_2-\lambda_1)\varepsilon_1,\quad \textit{and}\quad \mu^{(3)} = -\varepsilon_1 + \sum_{i=1}^{\lambda_1-\lambda_2+1}\delta_i.$$

Remark 3.2. One significant difference between the non-semisimple and semi-simple cases is that in the non-semisimple cases the spaces of homogeneous harmonic polynomials of any given degree are not necessarily irreducible $\mathfrak{g}$-modules. However, for $d \leq k+1$ and $d > 2k+2$ the space of harmonic polynomials of degree d is still an irreducible $\mathfrak{g}$-module, isomorphic to $V(d\varepsilon_1 + d\zeta)$.

From now on we identify the Casimir operator C with its image in $\mathscr{PD}(V)^{\mathfrak{g}}$. Let $E \in \mathscr{PD}(V)^{\mathfrak{g}}$ denote the degree operator (which lies in the image of the center of $\mathfrak{g}$).

Proposition 3.3. *The operators C and E generate $\mathscr{PD}(V)^{\mathfrak{g}}$. Furthermore, for any differential operator $D \in \mathscr{PD}(V)^{\mathfrak{g}}$ of order d, there exists a unique symmetric polynomial $f_D(x,y)$ of degree d such that the eigenvalue of D on the indecomposable constituent V_μ is equal to $f_D(\mu_1 - k - 1, \mu_2)$.*

Proof. For $\lambda \in \mathcal{I}'_k$ set $e_\lambda := \lambda_1 + \lambda_2 = \big((\lambda_1 - (k+1)\big) + \lambda_2) + (k+1)$, and recall that

$$c_\lambda = (\lambda_2 - \lambda_1)(2k + 2 + \lambda_2 - \lambda_1) = \big((\lambda_1 - (k+1)) - \lambda_2\big)^2 - (k+1)^2.$$

Thus c_λ and e_λ are symmetric polynomials in $\lambda_1 - k - 1$ and λ_2. The restriction to V_λ of any operator of the form $D := q(C, E)$, where $q(x,y) \in \mathbb{C}[x,y]$, is of the form $q(c_\lambda, e_\lambda)\mathbf{1}_{V_\lambda} + X$, where $X \in \mathrm{End}(V_\lambda)$ is nilpotent.

Step 1. We prove that for every symmetric polynomial $h(x,y) \in \mathbb{C}[x,y]$ there exists an operator $D \in \mathscr{PD}(V)^{\mathfrak{g}}$ of order at most $\deg h$ such that for every $\lambda \in \mathcal{I}'_k$ the restriction $D\big|_{V_\lambda}$ is of the form

$$h(\lambda_1 - k - 1, \lambda_2)\mathbf{1}_{V_\lambda} + X,$$

where X is nilpotent. To prove this claim, we write h as a polynomial in $\mathsf{e}_1 = x + y$ and $\mathsf{e}_2 = xy$, that is, $h(x,y) = \sum_{i+2j \le d} a_{i,j} \mathsf{e}_1^i \mathsf{e}_2^j$, where $d := \deg h$. Writing e_1 and e_2 in terms of $x + y + k + 1$ and $(x-y)^2 - (k+1)^2$, it follows that $h(x,y)$ can also be expressed as

$$h(x,y) = \sum_{i+2j \le d} b_{i,j}(x + y + k + 1)^i \big((x-y)^2 - (k+1)^2\big)^j,$$

where $b_{i,j} \in \mathbb{C}$. It is easy to verify that the operator $D := \sum_{i+2j \le d} b_{i,j} E^i C^j$ satisfies the claimed properties.

Step 2. For $d \ge 0$ set $\mathcal{V}_d := \{D \in \mathscr{PD}(V)^{\mathfrak{g}} : \mathrm{ord}(D) \le d\}$, where $\mathrm{ord}(D)$ denotes the order of D. From (8) and (9) it follows that $\dim \mathcal{V}_d = N_d := |\mathcal{I}(d)|$. The space of symmetric polynomials of degree at most d also has dimension N_d. Furthermore, operators that correspond by Step 1 to linearly independent polynomials are also linearly independent. Thus Step 1 provides N_d linearly independent elements in $\mathcal{V}_d \cap \mathcal{A}$, where $\mathcal{A}$ is the subalgebra of $\mathscr{PD}(V)^{\mathfrak{g}}$ that is generated by C and E. This yields $\dim \mathcal{V}_d \cap \mathcal{A} \ge \dim \mathcal{V}_d$, and consequently $\mathcal{V}_d \subseteq \mathcal{A}$.

Step 3. Let $D \in \mathscr{PD}(V)^{\mathfrak{g}}$ such that $\mathrm{ord}(D) = d$. By Step 2, there exists a symmetric polynomial $f_D \in \mathbb{C}[x,y]$ such that $\deg f_D \le d$ and D is obtained from f_D by the construction of Step 1. From Step 1 it follows that $d = \mathrm{ord}(D) \le \deg f_D \le d$. Hence $\deg f_D = \mathrm{ord}(D) = d$. Finally, f_D is unique because $\mathcal{I}$ is Zariski dense in $\mathbb{C}^2$. □

Definition 3.4. For $D \in \mathscr{PD}(V)^{\mathfrak{g}}$, the polynomial $f_D(x,y)$ whose existence is guaranteed by Proposition 3.3 will be called the *eigenvalue polynomial* of D.

Corollary 3.5. *For $\lambda \in \mathcal{I}_{k,\mathrm{qreg}}$, the restriction of C to V_λ is not diagonalizable. In particular, the nilpotent part of the Jordan decomposition of $C\big|_{V_\lambda}$ is nonzero.*

Proof. Otherwise, Proposition 3.3 would imply that $D\big|_{V_\lambda}$ is diagonalizable for all $D \in \mathscr{PD}(V)^{\mathfrak{g}}$. In particular, $D_{\lambda^\dagger}\big|_{V_\lambda}$ would be diagonalizable, which is a contradiction. □

Remark 3.6. As noted in Section 1, Corollary 3.5 is crucial for being able to define the basis $\{D_\lambda\}_{\lambda \in \mathcal{I}}$ of Capelli operators for $\mathscr{PD}(V)^{\mathfrak{g}}$.

4. Vanishing properties and generalized values

Recall from (11) that $d_{\lambda,\mu}$ denotes the eigenvalue of D_λ on V_μ. The $d_{\lambda,\mu}$ satisfy the following vanishing properties which are deduced from elementary representation-theoretic arguments.

Lemma 4.1. *Let $\lambda, \mu \in \mathcal{I}$.*

(i) *Assume that $\lambda \in \mathcal{I}'_k$. Then $d_{\lambda,\mu} = 0$ for all $\mu \in \mathcal{I}'_k$ such that $|\mu| \le |\lambda|$ and $\mu \ne \lambda$. Furthermore, $d_{\lambda,\lambda} = 1$.*

(ii) *Assume that $\lambda \in \mathcal{I}_{k,\mathrm{sing}}$. Then $d_{\lambda,\mu} = 0$ for all $\mu \in \mathcal{I}'_k$ such that $|\mu| \le |\lambda|$.*

Proof. From the isomorphism (8) it follows that $D_\lambda\big|_{V_\lambda} = \mathbf{1}_{V_\lambda}$ for $\lambda \in \mathcal{I}'_k$. For $\lambda \in \mathcal{I}_{k,\mathrm{sing}}$ we have $d_{\lambda,\lambda^\dagger} = 0$ because the restriction of D_λ to $V_{\lambda^\dagger}$ is nilpotent. If $|\mu| < |\lambda|$ then we have $D_\lambda\big|_{V_\mu} = 0$ because $V_\mu \subseteq \mathscr{P}^{|\mu|}(V)$ and $\mathrm{ord}(D_\lambda) > |\mu|$. If $|\mu| = |\lambda|$, then the action of D_λ on V_μ is obtained by restriction of the $\mathfrak{g}$-equivariant map

$$\mathscr{P}(V) \otimes \mathscr{D}(V) \otimes \mathscr{P}(V) \to \mathscr{P}(V)\ ,\ p \otimes D \otimes q \mapsto pDq,$$

to a tensor product of the form $V_\eta \otimes V_\eta^* \otimes V_\mu$, where $\eta = \lambda$ or $\eta = \lambda^\dagger$ depending on wheter $\lambda \in \mathcal{I}_{k,\mathrm{qreg}}$ or $\lambda \in \mathcal{I}_{k,\mathrm{sing}}$. As $|\mu| = |\lambda|$, the map $V_\eta^* \otimes V_\mu \to \mathbb{C}$ corresponds to a $\mathfrak{g}$-invariant bilinear form $V_\eta^* \times V_\mu \to \mathbb{C}$, hence to a $\mathfrak{g}$-equivariant linear map $V_\eta^* \to V_\mu^*$. Thus, when V_μ^* and V_η^* do not have composition factors in common, we obtain $D_\lambda\big|_{V_\mu} = 0$ and in particular $d_{\lambda,\mu} = 0$. The above facts are sufficient for verifying the claims of the lemma. □

Remark 4.2. The proof of Lemma 4.1 implies that if $|\mu| \le |\lambda|$, then the nilpotent part of the Jordan decomposition of $D_\lambda\big|_{V_\mu}$ vanishes, unless λ is k-singular and $\mu = \lambda^\dagger$.

We can now write f_λ as

$$f_\lambda(x,y) = \sum_{i+j \le |\lambda|} a_{i,j}(x^i y^j + x^j y^i),$$

and interpret the constraints $f_\lambda(\mu_1 - k - 1, \mu_2) = d_{\lambda,\mu}$ for $|\mu| \le |\lambda|$ as a linear system in the coefficients $a_{i,j}$. Unfortunately, this linear system (which a priori has the same number of equations and variables) does *not* determine f_λ uniquely because of the redundancy that is caused by the coincidences

$$f_\lambda(\mu_1 - k - 1, \mu_2) = f_\lambda(\mu_1^\dagger - k - 1, \mu_2^\dagger). \tag{23}$$

But we can circumvent this issue by using the Jordan decomposition of $D_\lambda\big|_{V_\mu}$ to obtain extra conditions on f_λ.

Proposition 4.3. *Let $D \in \mathscr{P}\mathscr{D}(V)^{\mathfrak{g}}$ and assume that $f_D(x,y)$ is the eigenvalue polynomial of D. Then for $\lambda \in \mathcal{I}_{k,\mathrm{reg}} \cup \mathcal{I}_{k,\mathrm{qreg}}$ we have*

$$D\big|_{V_\lambda} = f_D(\lambda_1 - k - 1, \lambda_2)\mathbf{1}_{V_\lambda} + \Box f_D(\lambda_1 - k - 1, \lambda_2) N_\lambda,$$

where $\Box f_D$ is defined as in (12).

Proof. By Proposition 3.3 we can express D as $D = p(C, E)$ for a polynomial $p(s,t) \in \mathbb{C}[s,t]$. Note that $C\big|_{V_\lambda} = c_\lambda \mathbf{1}_{V_\lambda} + N_\lambda$ where $N_\lambda^2 = 0$, hence $C^d\big|_{V_\lambda} = c_\lambda^d \mathbf{1}_{V_\lambda} + d c_\lambda^{d-1} N_\lambda$. It follows that

$$D\big|_{V_\lambda} = p(c_\lambda, e_\lambda)\mathbf{1}_{V_\lambda} + \frac{\partial p}{\partial s}(c_\lambda, e_\lambda) N_\lambda. \tag{24}$$

By comparing the eigenvalues on both sides of (24) and noting that $\mathcal{I} \subseteq \mathbb{C}^2$ is Zariski dense, we obtain

$$f_D(x,y) = p\big((x-y)^2 - (k+1)^2, x + y + k + 1\big). \tag{25}$$

Next set

$$H(x,y) := \big((x-y)^2-(k+1)^2, x+y+k+1\big).$$

Then by the chain rule we obtain

$$\begin{cases} \dfrac{\partial f_D}{\partial x}(x,y) = \dfrac{\partial p}{\partial s}(H(x,y))(2x-2y) + \dfrac{\partial p}{\partial t}(H(x,y)), \\ \dfrac{\partial f_D}{\partial y}(x,y) = \dfrac{\partial p}{\partial s}(H(x,y))(2y-2x) + \dfrac{\partial p}{\partial t}(H(x,y)). \end{cases}$$

Taking the difference of the above relations yields

$$\frac{\partial p}{\partial s}(H(x,y)) = \square f_D(x,y). \tag{26}$$

The statement of the lemma follows from (24) and (26). □

Using Proposition 4.3, we obtain the required extra constraints that together with the vanishing conditions of Lemma 4.1 uniquely identify the polynomials f_λ. In order to give a uniform description of all of these constraints, we use the notion of the *generalized value* of a symmetric polynomial $f(x,y)$ at $\lambda \in \mathcal{I}$, denoted by $\widetilde{\mathrm{ev}}(f,\lambda)$, defined as follows.

$$\widetilde{\mathrm{ev}}(f,\lambda) := \begin{cases} f(\lambda_1-k-1,\lambda_2) & \text{if } \lambda \in \mathcal{I}_{k,\mathrm{reg}} \cup \mathcal{I}_{k,\mathrm{qreg}}, \\ \square f(\lambda_1-k-1,\lambda_2) & \text{if } \lambda \in \mathcal{I}_{k,\mathrm{sing}}. \end{cases}$$

Then Lemma 4.1 and Remark 4.2 imply the following proposition.

Proposition 4.4. *For $\lambda,\mu \in \mathcal{I}$, if $|\mu| \le |\lambda|$ then $\widetilde{\mathrm{ev}}(f_\lambda,\mu) = \delta_{\lambda,\mu}$.*

In the following corollary, $\mathcal{I}(d)$ is defined as in (15).

Corollary 4.5. *Fix a set of complex numbers $\{z_\lambda : \lambda \in \mathcal{I}(d)\}$ for some $d \ge 0$. Then there exists a unique symmetric polynomial $f(x,y)$ such that $\deg f \le d$ and $\widetilde{\mathrm{ev}}(f,\lambda) = z_\lambda$ for all $\lambda \in \mathcal{I}(d)$.*

Proof. Follows immediately from Proposition 4.4. □

5. Proofs of Theorems A, B, and C

We now proceed towards the proofs of Theorems A–C. The next lemma is a key observation.

Lemma 5.1. *Let $p(\kappa; x,y) \in \Bbbk[x,y]$ and let $k_\circ \in \mathbb{R}$ be such that the coefficients of $p(\kappa;x,y)$ do not have poles at $\kappa = k_\circ$. Further, assume that for $a,b,a',b' \in \mathbb{R}$ we have*

$$(a-k_\circ-1,b) = (a',b'-k_\circ-1) \quad \textit{and} \quad a-b-k_\circ-1 \neq 0.$$

Set

$$\alpha(\kappa) := p(\kappa; a-\kappa-1,b) \quad \textit{and} \quad \beta(\kappa) := p(\kappa; a', b'-\kappa-1). \tag{27}$$

Then

$$\square p(k_\circ; a-k_\circ-1,b) = \frac{\beta'(k_\circ)-\alpha'(k_\circ)}{4(a-b-k_\circ-1)}$$

Proof. Differentiating the equations given in (27) with respect to κ at $\kappa = k_\circ$, we obtain

$$\alpha'(k_\circ) = \frac{\partial p}{\partial \kappa}(k_\circ; a-k_\circ-1,b) - \frac{\partial p}{\partial x}(k_\circ; a-k_\circ-1,b)$$

and

$$\beta'(k_\circ) = \frac{\partial p}{\partial \kappa}(k_\circ; a',b'-k_\circ-1) - \frac{\partial p}{\partial y}(k_\circ; a',b'-k_\circ-1).$$

Taking the difference of the above relations yields the claim of the lemma. □

Lemma 5.2. *Let $p(\kappa;x,y)\in\Bbbk[x,y]$ and let $k_\circ\in\mathbb{R}$ be such that the coefficients of $(\kappa-k_\circ)p(\kappa;x,y)$ do not have poles at $\kappa=k_\circ$. Further, assume that for $a,b,a',b'\in\mathbb{R}$ we have*

$$(a-k_\circ-1,b)=(a',b'-k_\circ-1)\quad\text{and}\quad a-b-k_\circ-1\neq 0. \tag{28}$$

For $\kappa\in\mathbb{R}\backslash\{k_\circ\}$ sufficiently close to $k_\circ$, set

$$\alpha(\kappa):=p(\kappa;a-\kappa-1,b)\ \text{and}\ \beta(\kappa):=p(\kappa;a',b'-\kappa-1).$$

Then

$$\square\mathrm{Sing}_{k_\circ}(p;a-k_\circ-1,b)=\frac{1}{4(a-b-k_\circ-1)}\lim_{\kappa\to k_\circ}\left((\beta(\kappa)-\alpha(\kappa))+(\kappa-k_\circ)(\beta'(\kappa)-\alpha'(\kappa))\right).$$

Proof. Set $p_x:=\frac{\partial p}{\partial x}$ and $p_y:=\frac{\partial p}{\partial y}$. Since taking the singular part commutes with partial differentiation with respect to x and y, it suffices to prove that

$$\begin{aligned}&\lim_{\kappa\to k_\circ}\Big((\kappa-k_\circ)\big(p_x(\kappa;a-k_\circ-1,b)-p_y(\kappa;a-k_\circ-1,b)\big)\Big)\\&\qquad=\lim_{\kappa\to k_\circ}\left((\beta(\kappa)-\alpha(\kappa))+(\kappa-k_\circ)(\beta'(\kappa)-\alpha'(\kappa))\right).\end{aligned}\tag{29}$$

Set

$$h(\kappa;x,y):=(\kappa-k_\circ)p(\kappa;x,y)\in\Bbbk[x,y].$$

Note that $h(\kappa;x,y)$ is a smooth map in a neighborhood of any point of the form $(k_\circ,x_\circ,y_\circ)\in\mathbb{R}^3$. By differentiating the relation $(\kappa-k_\circ)\alpha(\kappa)=h(\kappa;a-\kappa-1,b)$ with respect to κ at $\kappa:=k_1$, with $k_1\neq k_\circ$ and k_1 sufficiently close to $k_\circ$, we obtain

$$(k_1-k_\circ)\alpha(k_1)+\alpha'(k_1)=\frac{\partial h}{\partial\kappa}(k_1,a-k_1-1,b)-(k_1-k_\circ)p_x(k_1,a-k_1-1,b). \tag{30}$$

Similarly, by differentiating the relation $(\kappa-k_\circ)\beta(\kappa)=h(\kappa,a',b'-\kappa-1)$ with respect to κ we obtain

$$(k_1-k_\circ)\beta(k_1)+\beta'(k_1)=\frac{\partial h}{\partial\kappa}(k_1,a',b'-k_1-1)-(k_1-k_\circ)p_y(k_1,a',b'-k_1-1). \tag{31}$$

By taking the difference of (30) and (31) we obtain

$$\begin{aligned}&(k_1-k_\circ)(p_x(k_1,a-k_1-1,b)-p_y(k_1,a',b'-k_1-1))\\&\qquad=\frac{\partial h}{\partial\kappa}(k_1,a-k_1-1,b)-\frac{\partial h}{\partial\kappa}(k_1,a',b'-k_1-1)+\phi(k_1),\end{aligned}\tag{32}$$

where

$$\phi(k_1):=(k_1-k_\circ)\left(\beta(k_1)-\alpha(k_1)\right)+\left(\beta'(k_1)-\alpha'(k_1)\right).$$

Note that $\lim_{k_1\to k_\circ}\phi(k_1)$ exists because

$$\phi(k_1)=\frac{d}{d\kappa}\left(h(\kappa;a-\kappa-1)-h(\kappa;a',b'-\kappa-1)\right)\big|_{\kappa=k_\circ},$$

and $h(\kappa;x,y)$ is differentiable near $(k_\circ,a-k_\circ-1,b)\in\mathbb{R}^3$. Since $h(\kappa;x,y)$ is smooth in a neighborhood of the point $(k_\circ,a-k_\circ-1,b)\in\mathbb{R}^3$, from (28) it follows that

$$\lim_{k_1\to k_\circ}\frac{\partial h}{\partial\kappa}(k_1,a-k_1-1,b)=\lim_{k_1\to k_\circ}\frac{\partial h}{\partial\kappa}(k_1,a',b'-k_1-1). \tag{33}$$

Finally the equations (32) and (33) imply (29). $\square$

For $\lambda\in\mathcal{I}_{k,\mathrm{sing}}$ set

$$r_\lambda:=-\frac{H_\lambda(k)}{H'_{\lambda^\dagger}(k)}. \tag{34}$$

Proposition 5.3. *Let $\lambda \in \mathcal{I}_{k,\mathrm{sing}}$. Then* $\mathrm{Sing}_k(P^\kappa_\lambda) = r_\lambda P^k_{\lambda^\dagger}$ *where r_λ is defined in (34).*

Proof. Set $p(x,y) := \mathrm{Sing}_k(P^\kappa_\lambda; x, y)$. By Proposition 1.4 we have $p(x,y) \neq 0$.
Step 1. We prove that $p = sP^k_{\lambda^\dagger}$ for a scalar $s \neq 0$. To this end, by Corollary 4.5 we need to verify that

$$\widetilde{\mathrm{ev}}(p,\mu) = 0 \quad \text{when } |\mu| \leq |\lambda| \text{ and } \mu \neq \lambda.$$

If μ is k-regular or k-quasiregular, then

$$\begin{aligned} p(\mu_1 - k_\circ - 1, \mu_2) &= \lim_{\kappa \to k_\circ} (\kappa - k_\circ) P^\kappa_\lambda(\mu_1 - k_\circ - 1, \mu_2) \\ &= \lim_{\kappa \to k_\circ} (\kappa - k_\circ) P^\kappa_\lambda(\mu_1 - \kappa - 1, \mu_2), \end{aligned}$$

where for the second equality we use the fact that $h(\kappa, x, y) := (\kappa - k_\circ) P^\kappa_\lambda(x,y)$ is a smooth function in a neighborhood of $(k_\circ, \mu_1 - k_\circ - 1, \mu_2)$. Therefore Theorem 1.1(i) implies that

$$\widetilde{\mathrm{ev}}(p,\mu) = p(\mu_1 - k_\circ - 1, \mu_2) = 0.$$

If μ is k-singular, then Lemma 5.2 for $(a,b) := (\mu_1, \mu_2)$, $(a', b') := (\mu_2^\dagger, \mu_1^\dagger)$, and $p(\kappa; x, y) := P^\kappa_{\lambda^\dagger}(x,y)$ implies that $\widetilde{\mathrm{ev}}(p,\mu) = 0$.

Step 2. To determine the value of s, we compare the coefficient of $x^{\underline{t_1}} y^{\underline{t_2}}$ in $P^k_{\lambda^\dagger}$ and $p(x,y)$, where $t_1 := \lambda_1^\dagger$ and $t_2 := \lambda_2^\dagger$. From (2) it is clear that $x^{\underline{t_1}} y^{\underline{t_2}}$ is the leading monomial of $P^k_{\lambda^\dagger}$ and therefore its coefficient is equal to 1. Furthermore, in the formula (2) for P^κ_λ, the monomial $x^{\underline{t_1}} y^{\underline{t_2}}$ corresponds to the term indexed by $i := k+1$ and $j := \lambda_1 - \lambda_2 - k - 1$. It is straightforward to show that the coefficient of the corresponding term in $p(x,y)$ is equal to r_λ. Note that by a direct calculation we have

$$r_\lambda = \frac{(-1)^{(k+|\lambda|)} (\lambda_1 - \lambda_2)^{\underline{k+1}}}{(2k + 2 - \lambda_1 + \lambda_2)! (\lambda_1 - \lambda_2 - k - 2)!}.$$ ☐

For $\lambda \in \mathcal{I}_{k,\mathrm{sing}}$ set

$$Q^\kappa_\lambda(x,y) := P^\kappa_\lambda - \frac{r_\lambda}{\kappa - k} P^\kappa_{\lambda^\dagger}.$$

Lemma 5.4. *For $\lambda \in \mathcal{I}_{k,\mathrm{sing}}$, the coefficients of $Q^\kappa_\lambda(x,y)$ do not have poles at $\kappa = k$. Furthermore, the polynomial $Q_\lambda(x,y) \in \mathbb{Q}[x,y]$ defined by*

$$Q_\lambda := \lim_{\kappa \to k} \left(P^\kappa_\lambda - \frac{r_\lambda}{\kappa - k} P^\kappa_{\lambda^\dagger} \right) \tag{35}$$

satisfies

$$Q_\lambda(x,y) = R^{(k)}_\lambda(x,y) - r_\lambda \frac{\partial}{\partial \kappa} P^k_{\lambda^\dagger}(x,y). \tag{36}$$

Proof. By Proposition 5.3 the coefficients of $P^\kappa_\lambda - \frac{r_\lambda}{\kappa - k} P^k_{\lambda^\dagger}$ do not have poles at $\kappa = k$. Furthermore, since the coefficients of $P^\kappa_{\lambda^\dagger}$ do not have poles at $\kappa = k$, it follows that each coefficient of the polynomial $P^\kappa_{\lambda^\dagger} - P^k_{\lambda^\dagger}$ is of the form $(\kappa - k)\varphi(\kappa)$ where $\varphi(\kappa) \in \mathbb{Q}(\kappa)$ does not have a pole at $\kappa = k$. Now

$$Q^\kappa_\lambda = P^\kappa_\lambda - \frac{r_\lambda}{\kappa - k} P^k_{\lambda^\dagger} - \frac{r_\lambda}{\kappa - k} \left(P^\kappa_{\lambda^\dagger} - R^{(k)}_{\lambda^\dagger} \right), \tag{37}$$

and therefore the coefficients of Q^κ_λ do not have poles at $\kappa = k$. Equality (36) follows from taking the limit $\kappa \to k$ in (37). ☐

For $\lambda \in \mathcal{I}_{k,\mathrm{sing}}$ set

$$\alpha_\lambda(\kappa) := -\frac{r_\lambda}{\kappa - k} H_{\lambda^\dagger}(\kappa) \quad \text{and} \quad \beta_\lambda(\kappa) := H_\lambda(\kappa). \tag{38}$$

Proposition 5.5. *Let $\lambda \in \mathcal{I}_{k,\mathrm{sing}}$ and let $Q_\lambda \in \mathbb{Q}[x,y]$ be defined as in (35). Set*

$$t_1 := \beta_\lambda(k) \quad \text{and} \quad t_2 := \frac{\beta'_\lambda(k) - \alpha'_\lambda(k)}{4(k+1-\lambda_1+\lambda_2)}, \tag{39}$$

where $\alpha_\lambda(\kappa)$ and $\beta_\lambda(\kappa)$ are defined in (38). Then

$$\widetilde{\mathrm{ev}}(Q_\lambda, \mu) = t_1 \delta_{\lambda^\dagger,\mu} + t_2 \delta_{\lambda,\mu} \quad \textit{for all } \mu \in \mathcal{I} \textit{ satisfying } |\mu| \le |\lambda|.$$

Proof. First assume that either μ is k-regular, or μ is k-quasiregular and $\mu \neq \lambda^\dagger$. By Theorem 1.1, for κ chosen sufficiently close (but not equal) to k we have

$$Q^\kappa_\lambda(\mu_1 - \kappa - 1, \mu_2) = P^\kappa_\lambda(\mu_1 - \kappa - 1, \mu_2) - \frac{r_\lambda}{\kappa - k} P^\kappa_{\lambda^\dagger}(\mu_1 - \kappa - 1, \mu_2) = 0. \tag{40}$$

By (35) we have $Q_\lambda(x,y) = \lim_{\kappa\to k} Q^\kappa_\lambda(x,y)$, so that $\widetilde{\mathrm{ev}}(Q_\lambda,\mu) = 0$.

Next assume that μ is k-singular and $\mu \neq \lambda$. We use Lemma 5.1 for $p(\kappa; x, y) := Q^\kappa_\lambda(x,y)$, $(a,b) = (\mu_1,\mu_2)$, and $(a',b') = (\mu_2^\dagger, \mu_1^\dagger)$. Note that Theorem 1.1 implies $\alpha(\kappa) = \beta(\kappa) = 0$, from which it follows that $\widetilde{\mathrm{ev}}(Q_\lambda,\mu) = 0$.

Next assume that $\mu = \lambda^\dagger$. Then symmetry of $Q_\lambda(x,y)$ implies

$$\begin{aligned} \widetilde{\mathrm{ev}}(Q_\lambda,\mu) &= Q_\lambda(\lambda_1^\dagger - k - 1, \lambda_2^\dagger) = Q_\lambda(\lambda_2^\dagger, \lambda_1^\dagger - k - 1) \\ &= Q_\lambda(\lambda_1 - k - 1, \lambda_2) = \lim_{\kappa\to k} Q^\kappa_\lambda(\lambda_1 - \kappa - 1, \lambda_2) = \lim_{\kappa\to k} H_\lambda(\kappa) = H_\lambda(k). \end{aligned}$$

Finally, assume that $\mu = \lambda$. Then

$$Q^\kappa_\lambda(\mu_1 - \kappa - 1, \mu_2) = H_\lambda(\kappa) \quad \text{and} \quad Q^\kappa_\lambda(\mu_1^\dagger - \kappa - 1, \mu_2^\dagger) = -\frac{r_\lambda}{\kappa - k} H_{\lambda^\dagger}(\kappa).$$

Lemma 5.1 for $p(\kappa; x, y) := Q^\kappa_\lambda$, $(a,b) := (\lambda_1^\dagger, \lambda_2^\dagger)$, and $(a',b') := (\lambda_2, \lambda_1)$ yields $\widetilde{\mathrm{ev}}(Q_\lambda,\mu) = t_2$. ☐

We are now ready to complete the proofs of Theorems A, B, and C.

Proof of Theorem A. Fix $\lambda \in \mathcal{I}_{k,\mathrm{reg}}$ and set $d := |\lambda|$. By Corollary 4.5 it suffices to show that

$$\widetilde{\mathrm{ev}}\left(\frac{1}{H_\lambda(k)} R^{(k)}_\lambda, \mu\right) = \delta_{\lambda,\mu} \quad \text{for } \mu \in \mathcal{I}(d).$$

Note that $R^{(k)}_\lambda = P^k_\lambda$. If μ is k-regular or k-quasiregular, this follows from taking the limit $\kappa \to k$ in Theorem 1.1. If μ is k-singular, we set $k_\circ := k$, $(a,b) := (\mu_1,\mu_2)$ and $(a',b') := (\mu_2^\dagger, \mu_1^\dagger)$ in Lemma 5.1, and note that Theorem 1.1 implies $\alpha'(k) = \beta'(k) = 0$.

Proof of Theorem B. By Corollary 4.5 it suffices to prove that

$$\widetilde{\mathrm{ev}}\left(\frac{4(\lambda_1 - \lambda_2 - k - 1)}{H'_{\lambda^\dagger}(k)} R^{(k)}_{\lambda^\dagger}, \mu\right) = \delta_{\lambda,\mu} \text{ for } \mu \in \mathcal{I},\ |\mu| \le |\lambda|.$$

The argument is based on Lemma 5.1 and is similar to the proof of Theorem A.

Proof of Theorem C. Set $h(\kappa;x,y) := \frac{1}{H_\lambda(\kappa)}P_\lambda^\kappa + \frac{1}{H_{\lambda^\dagger}(\kappa)}P_{\lambda^\dagger}^\kappa$, so that the right hand side of (13) is equal to $\lim_{\kappa\to k} h(\kappa)$. The latter limit exists because by (34) we have

$$h(\kappa;x,y) = \frac{1}{H_{\lambda^\dagger}(\kappa)}Q_{\lambda^\dagger}^\kappa(x,y) + \frac{u(\kappa)}{H_{\lambda^\dagger}(\kappa)}P_\lambda^\kappa(x,y) \quad \text{where} \quad u(\kappa) := \left(\frac{H_{\lambda^\dagger}(\kappa)}{H_\lambda(\kappa)} - \frac{H_{\lambda^\dagger}(k)}{(\kappa-k)H'_\lambda(k)}\right),$$

and

$$\lim_{\kappa\to k} u(\kappa) = \frac{d}{d\kappa}\left(\frac{H_{\lambda^\dagger}(\kappa)(\kappa-k)}{H_\lambda(\kappa)}\right)\Big|_{\kappa=k}.$$

Next we set $h_1(x,y) := \lim_{\kappa\to k} h(\kappa;x,y)$. Since $\deg f_\lambda = |\lambda|$, by Corollary 4.5 in order to prove that $f_\lambda = h_1$ it suffices to verify that $\mathrm{ev}(h_1,\mu) = \delta_{\lambda,\mu}$ for all $\mu \in \mathcal{I}$ such that $|\mu| \le |\lambda|$. If $\mu \neq \lambda^\dagger$, this follows from Proposition 5.5 and Theorem B. If $\mu = \lambda^\dagger$, this follows from Lemma 5.1 because

$$h(\kappa;\lambda_1^\dagger-\kappa-1,\lambda_2^\dagger) = h(\kappa;\lambda_2,\lambda_1-\kappa-1) = 1.$$

6. Proof of Theorem D

We begin the proof of Theorem D by the following proposition which is a variation of Theorem C.

Proposition 6.1. *For $\lambda \in \mathcal{I}_{k,\mathrm{qreg}}$, we have*

$$f_\lambda = \frac{1}{H_{\lambda^\dagger}(k)}\left(Q_{\lambda^\dagger} + \frac{\beta'_{\lambda^\dagger}(k) - \alpha'_{\lambda^\dagger}(k)}{H'_\lambda(k)}R_\lambda^{(k)}\right), \tag{41}$$

where $Q_{\lambda^\dagger}$ is defined as in Lemma 5.4.

Proof. Note that $R_\lambda^{(k)} = P_\lambda^k$. Since $\deg(Q_{\lambda^\dagger}) = \deg(f_{\lambda^\dagger}) = \deg(f_\lambda) = |\lambda|$, from Proposition 5.5 and Corollary 4.5 it follows that $Q_{\lambda^\dagger} = t_1 f_\lambda + t_2 f_{\lambda^\dagger}$, so that

$$f_\lambda = \frac{1}{t_1}\left(Q_{\lambda^\dagger} - t_2 f_{\lambda^\dagger}\right),$$

where t_1 and t_2 are defined in (39). The claim now follows from Theorem B. □

From now on we assume that $\lambda \in \mathcal{I}_{k,\mathrm{qreg}}$. Then we can express λ and $\lambda^\dagger$ as

$$\lambda = (d+k+1, d+\ell+1) \quad \text{and} \quad \lambda^\dagger = (d+k+\ell+2, d),$$

where $d \ge 0$ and $0 \le \ell \le k$. Note that $\ell = \ell_{\lambda^\dagger}$, where $\ell_{\lambda^\dagger}$ is defined in (16).

Proposition 6.2. *Suppose that $P_\lambda^\kappa = \sum \alpha_{m,n}(\kappa)x^{\underline{m}}y^{\underline{n}}$, where $\alpha_{m,n}(\kappa) \in \Bbbk$. Then*

$$f_\lambda = \frac{(\ell+1)!}{d!(d+\ell+1)!(k+\ell+2)!}R_{\lambda^\dagger}^{(k)} + \frac{(-1)^{\ell+1}(k+1)^{\underline{\ell+1}}}{d!(d+\ell+1)!(k+1)!\ell!}\sum \alpha'_{m,n}(k)x^{\underline{m}}y^{\underline{n}} + \frac{(-1)^\ell}{d!(d+\ell+1)!(k-\ell)!\ell!}\left(\sum_{i=d+1}^{d+\ell+1}\frac{1}{i} - \frac{1}{\ell+1}\right)R_\lambda^{(k)}. \tag{42}$$

Proof. From Proposition 6.1 and (36) we obtain

$$f_\lambda = \frac{1}{H_{\lambda^\dagger}(k)}\left(R_{\lambda^\dagger}^{(k)} - r_{\lambda^\dagger}\frac{\partial P_\lambda^\kappa}{\partial\kappa} + \frac{\beta'_{\lambda^\dagger}(k) - \alpha'_{\lambda^\dagger}(k)}{H'_\lambda(k)}R_\lambda^{(k)}\right). \tag{43}$$

By straightforward calculations we can verify that

$$r_{\lambda^\dagger} = \frac{(-1)^\ell(k+\ell+2)^{\underline{k+1}}}{(k-\ell)!\ell!}, \quad H_{\lambda^\dagger}(k) = \frac{(k+\ell+2)!d!(d+\ell+1)!}{(\ell+1)!}, \quad \text{and} \quad \alpha_{\lambda^\dagger}(k) = \beta_{\lambda^\dagger}(k) = H_{\lambda^\dagger}(k).$$

Next note that for a polynomial $f(\kappa) := c\prod_i (a_i - \kappa)$ we have $f'(k) = -f(k)\sum_i \frac{1}{a_i - k}$. In particular

$$\alpha'_{\lambda^\dagger}(k) = -H_{\lambda^\dagger}(k)\sum_{i=\ell+1}^{d}\frac{1}{i} \quad \text{and} \quad \beta'_{\lambda^\dagger}(k) = -H_{\lambda^\dagger}(k)\sum_{i=\ell+2}^{d+\ell+1}\frac{1}{i}.$$

Finally, $H'_\lambda(k) = \lim_{\kappa\to k}\left(\frac{H_\lambda(\kappa)}{\kappa - k}\right) = (-1)^{\ell+1}(k-\ell)!(d+\ell+1)!d!\ell!$, and the claim of the proposition follows by making substitutions in (43). □

Proof of Theorem D. By equating the right hand sides of (42) and (17), and then reparametrizing the summation in (42) in terms of $a := k-\ell-\mu_1$ and $b := \mu_2$, it follows that Theorem D is equivalent to the equation

$$\frac{\partial P^\kappa_\lambda}{\partial \kappa} = \sum_{\mu\in\mathcal{I}^*(k-\ell)} \frac{(-1)^{k-\ell+|\mu|+1}(\ell+1)!(\ell+k-\ell-\mu_1)!\binom{k-\ell-\mu_1}{\mu_2}(k-\ell)!}{(\mu_1-\mu_2)!(k-\ell-\mu_1)(\ell+\mu_2+1)!(k-\mu_1+\mu_2)!} P^\kappa_{(\mu_1+\ell+1,\mu_2+\ell+1)}, \tag{44}$$

where $\mathcal{I}^*(k-\ell) := \mathcal{I}(k-\ell)\backslash\{(k-\ell,0)\}$. Now set $N := k-\ell$. From (2) or [11, Cor. 2.3] it follows that

$$P^\kappa_{(\mu_1+\ell+1,\mu_2+\ell_1)} = x^{\underline{\ell+1}}y^{\underline{\ell+1}}P^\kappa_\mu(x-\ell-1, y-\ell-1).$$

Therefore after dividing both sides of (44) by $x^{\underline{\ell+1}}y^{\underline{\ell+1}}$ and making the substitution

$$(x,y)\mapsto(x+\ell+1, y+\ell+1),$$

we obtain that proving (44) reduces to verifying

$$\frac{\partial}{\partial\kappa}P^\kappa_{(N,0)} = \sum_{\mu\in\mathcal{I}^*(N)} \frac{(-1)^{N-\mu_1-\mu_2+1}(\ell+1)!(\ell+N-\mu_1)!\binom{N-\mu_1}{\mu_2}N!}{(\mu_1-\mu_2)!(N-\mu_1)(\ell+\mu_2+1)!(\ell+N-\mu_1+\mu_2)!}P^\kappa_\mu. \tag{45}$$

To prove (45), it suffices to verify that the coefficients of the terms $x^{\underline{i}}y^{\underline{j}}$ with $0\le i+j\le N$ on both sides are equal. These coefficients can be computed explicitly using the formula (2). After some routine algebraic computations, it follows that the equality of the coefficients of $x^{\underline{i}}y^{\underline{j}}$ on both sides of (45) is equivalent to the identity

$$\frac{d}{d\kappa}\left(\frac{\kappa^{\underline{N-i}}\,\kappa^{\underline{N-j}}}{\kappa^{\underline{N}}}\right) = \sum_\mu \frac{(-1)^{N-\mu_1-\mu_2+1}\binom{N-\mu_1}{\mu_2}i^{\underline{\mu_2}}j^{\underline{\mu_2}}(N-i-j)^{\underline{N-\mu_1-\mu_2}}}{(N-\mu_1)\kappa^{\underline{\mu_1-\mu_2}}(\kappa-\mu_1+\mu_2-1)^{\underline{\mu_2}}(\kappa-N+\mu_2)^{\underline{\mu_2}}}\kappa^{\underline{\mu_1-i}}\kappa^{\underline{\mu_1-j}}, \tag{46}$$

where the summation is on all partitions $\mu := (\mu_1,\mu_2)\neq(N,0)$ that satisfy

$$N\ge\mu_1+\mu_2\ge i+j \quad\text{and}\quad \mu_1\ge i\ge j\ge\mu_2.$$

Note that (46) is a one-variable identity in a free parameter κ. By the above discussion, Theorem D follows from (46). Note that (46) is equivalent to Theorem E after the substitutions $x := \kappa$ and $(p,q) := (\mu_1,\mu_2)$. We will prove Theorem E in the next section.

7. Proof of Theorem E

In this section we prove Theorem E, which completes the proof of Theorem D. Set $d := N-i$ and $\psi_L(x) := \frac{x^{\underline{d}}}{(x-N+1)\cdots(x-N+j)}$. Then identity (18) is equivalent to the relation

$$\frac{d}{dx}\psi_L(x) = \psi_R(x), \tag{47}$$

where

$$\psi_R(x) := \sum_{q=0}^{j} \sum_{p=N-d+j-q}^{\min\{N-q,N-1\}} \frac{(-1)^{N+p+q+1}(N-p)^{\underline{q}}\,(N-d)^{\underline{q}}\,j^{\underline{q}}\,(d-j)^{\underline{N-p-q}}\,x^{\underline{p+d-N}}\,x^{\underline{p-j}}(x-p+q)}{(N-p)\,q!\,x^{\underline{p+1}}(x-N+q)^{\underline{q}}}.$$

Our strategy is to prove a *two*-variable identity that implies (47) as a special case. For integers $q, r \geq 0$ such that $d \geq r \geq q \geq 0$, let $E(q,r)$ be the rational function in variables x, y defined by

$$E(q,r) := (-1)^{r+q+1} r^{\underline{q}}(x-y-d)^{\underline{q}} j^{\underline{q}} (d-j)^{\underline{r-q}} \left(\frac{x^{\underline{d-r}}(y+r+q)}{rq!(y+r+j)^{\underline{j+r}}}\right)\left(\frac{(y+r-1)^{\underline{r-q}}}{y+q}\right).$$

Lemma 7.1. *Set*

$$\psi_1(x,y) := \sum_{q=0}^{j} \sum_{r=\max\{1,q\}}^{d-j+q} E(q,r).$$

Then $\psi_R(x) = \psi_1(x, x-N)$.

Proof. This is a straightforward computation. Note that $r = N - p$, where p is as in the definition of $\psi_R(x)$. □

Lemma 7.2. *Set*

$$\psi_2(x,y) := -\frac{x^{\underline{d}}}{\prod_{t=1}^{j}(y+t)}\left(-\sum_{t=1}^{j}\frac{1}{y+t}\right) + \frac{\partial x^{\underline{d}}}{\partial x}\left(\frac{1}{\prod_{t=1}^{j}(y+t)}\right).$$

Then $\frac{d}{dx}\psi_L(x) = \psi_2(x, x-N)$.

Proof. This follows from computing $\frac{d}{dx}\psi_L(x)$ using the Leibniz rule. □

Lemma 7.1 and Lemma 7.2 imply that in order to verify (47), it suffices to prove that

$$\psi_1(x,y) = \psi_2(x,y). \tag{48}$$

The rest of this section is devoted to the proof of (48). Set

$$\check{E}(q,r) := (y+1)\cdots(y+j)E(q,r).$$

Then (48) is equivalent to

$$-\left(\sum_{t=1}^{j}\frac{1}{y+t}\right)x^{\underline{d}} + \frac{\partial x^{\underline{d}}}{\partial x} = \sum_{q=0}^{j}\sum_{r=\max\{1,q\}}^{d-j+q}\check{E}(q,r). \tag{49}$$

Next set $\ell := d - j$ and $s := r - q$. For q and r in the range of indices on the right hand side of (49) we have $0 \leq s \leq \ell$ when $q \geq 1$, and $1 \leq s \leq \ell$ when $q = 0$. Thus, the right hand side of (49) can be written as a double sum over the indices $(q,s) \in \mathcal{T}$, where

$$\mathcal{T} := \left\{(a_1,a_2) \in \mathbb{Z}^2 \ :\ 0 \leq a_1 \leq j,\ 0 \leq a_2 \leq \ell,\ (a_1,a_2) \neq (0,0)\right\}.$$

Now define $\delta_{s,0} := 1$ if $s = 0$, and $\delta_{s,0} := 0$ if $s \geq 1$. After substituting x by $x + j + \ell$ and dividing both sides of (49) by $(j+\ell)!$, it follows that (49) is equivalent to the identity

$$-\frac{(x+j+\ell)^{\underline{j+\ell}}}{(j+\ell)!}\left(\sum_{t=1}^{j}\frac{1}{y+t}\right) + \frac{1}{(j+\ell)!}\left(\frac{\partial}{\partial x}(x+j+\ell)^{\underline{j+\ell}}\right) = \sum_{s=0}^{\ell}(-1)^{s+1}F(s), \tag{50}$$

where

$$F(s) := \sum_{q=\delta_{s,0}}^{j}(y+2q+s)\binom{j}{q}\binom{\ell}{s}\left(\frac{(q+s-1)!}{(j+\ell)!}\right)\frac{(y+j)^{\underline{j-q}}}{(y+q+s+j)^{\underline{j+1}}}(x-y)^{\underline{q}}(x+j+\ell)^{\underline{j+\ell-q-s}}.$$

Consequently, to complete the proof of Theorem E, it suffices to verify (50). We will prove (50) after the proof of Proposition 7.3 below, which yields explicit formulas for $F(s)$.

Proposition 7.3. *Let $F(s)$ be defined as above. Then*

$$F(s) = \frac{(x+j+\ell)^{\underline{\ell-s}}(x+j)^{\underline{j}}}{s(\ell-s)!(j+\ell)^{\underline{j}}} \qquad \text{for } 1 \le s \le \ell, \tag{51}$$

and

$$F(0) = \frac{(x+j+\ell)^{\underline{j+\ell}}}{(j+\ell)!}\sum_{t=1}^{j}\left(\frac{1}{y+t} - \frac{1}{x+t}\right). \tag{52}$$

Proof. Set

$$H(s) := \frac{(\ell-s)!(j+\ell)^{\underline{j}}}{(x+j+\ell)^{\underline{j+\ell-s}}}F(s),$$

so that $H(s) = \sum_{q=\delta_{s,0}}^{j} E_1(q,s)$, where

$$E_1(q,s) := \frac{1}{s!}\binom{j}{q}(y+2q+s)(q+s-1)!\frac{(y+j)^{\underline{j-q}}}{(y+q+s+j)^{\underline{j+1}}}(x-y)^{\underline{q}}\frac{(x+j+s)^{\underline{s}}}{(x+q+s)^{\underline{q+s}}}.$$

It suffices to prove that

$$H(s) = \frac{1}{s} \text{ for } s \in \mathbb{Z} \text{ such that } 1 \le s \le \ell, \tag{53}$$

and

$$H(0) = \sum_{k=1}^{j}\frac{1}{y+k} - \sum_{k=1}^{j}\frac{1}{x+k}. \tag{54}$$

Our strategy is to relate $H(s)$ to Dougall's Theorem. First note that

$$\frac{E_1(q+1,s)}{E_1(q,s)} = \frac{h_1(q)}{h_2(q)} \text{ for } 0 \le q \le j \text{ and } E_1(q,s) = 0 \text{ for } q > j, \tag{55}$$

where

$$h_1(q) = \left(q + \frac{1}{2}y + \frac{1}{2}s + 1\right)(q+y+s)(q+y+s)(q-j)(q+s)(q+y-x),$$

and

$$h_2(q) = \left(q + \frac{1}{2}y + \frac{1}{2}s\right)(q+y+s+j+1)(q+y+1)(q+x+s+1)(q+1).$$

Furthermore,

$$E_1(0,s) = \frac{(y+j)^{\underline{j}}(x+j+s)^{\underline{s}}}{s(y+s+j)^{\underline{j}}(x+s)^{\underline{s}}} \qquad \text{for } s \in \mathbb{Z}^+. \tag{56}$$

We can write (56) as $E_1(0,s) = \frac{1}{s}\phi(s)$, where

$$\phi(s) := \frac{(y+j)^{\underline{j}}\Gamma(x+j+s+1)\Gamma(x+1)}{(y+s+j)^{\underline{j}}\Gamma(x+s+1)\Gamma(x+j+1)}.$$

Thus we can extend $E_1(0,s)$ to a meromorphic function of s for any choice of $x,y \in \mathbb{C}$. Note that if $x,y > 0$ then $E_1(0,s)$ does not have any poles for $s \in \mathbb{R}^+$. Using (55) we can extend $E_1(q,s)$ for $1 \le q \le j$ to a continuous function for $s \ge 0$ as long as $x, y, x-y-j > 0$. In particular, under the same conditions on x and y, we can extend $H(s)$ to a continuous function of the parameter $s \in \mathbb{R}^+$ by setting $H(s) := \sum_{q=0}^{j} E_1(q,s)$. From (55) it follows that

$$H(s) = E_1(0,s)\left(1 + \sum_{q=1}^{j}\left(\prod_{0\leq q'\leq q-1}\frac{h_1(q')}{h_2(q')}\right)\right). \tag{57}$$

The products that appear in the summands of (57) simplify, and by comparing with (20) we obtain

$$H(s) = E_1(0,s)\left[{}_5\mathbf{F}_4\left(\begin{matrix}\frac{1}{2}y+\frac{1}{2}s+1\,,\ y+s\,,\ -j\,,\ s\,,\ y-x\\ \frac{1}{2}y+\frac{1}{2}s\,,\ y+s+j+1\,,\ y+1\,,\ x+s+1\end{matrix};1\right)\right] \quad \text{for } s\in\mathbb{R}^+. \tag{58}$$

Note that in the hypergeometric series on the right hand side of (58), only the first $j+1$ terms are nonzero (because of the $-j$ appearing in the top row of parameters). From Dougall's Theorem for $\mathsf{a} = y+s$, $\mathsf{b} = j$, $\mathsf{c} = -s$, and $\mathsf{d} = x-y$, we obtain

$$\begin{aligned} H(s) &= E_1(0,s)\frac{\Gamma(y+s+j+1)\Gamma(y+1)\Gamma(x+s+1)\Gamma(x+j+1)}{\Gamma(y+s+1)\Gamma(y+j+1)\Gamma(x+1)\Gamma(x+j+s+1)}\\ &= E_1(0,s)\frac{(x+j)^{\underline{j}}}{(y+j)^{\underline{j}}}\frac{(y+s+j)^{\underline{j}}}{(x+s+j)^{\underline{j}}}. \end{aligned}$$

If $s\in\mathbb{Z}$ and $1\leq s\leq \ell$, then from (56) it follows that

$$H(s) = \frac{1}{s}\left(\frac{(y+j)^{\underline{j}}(x+j+s)^{\underline{s}}}{(y+s+j)^{\underline{j}}(x+s)^{\underline{s}}}\right)\frac{(x+j)^{\underline{j}}}{(y+j)^{\underline{j}}}\frac{(y+s+j)^{\underline{j}}}{(x+s+j)^{\underline{j}}} = \frac{1}{s}.$$

This completes the proof of (53). For (54), set

$$\psi_3(s) := \frac{(x+j)^{\underline{j}}}{(y+j)^{\underline{j}}}\frac{(y+s+j)^{\underline{j}}}{(x+s+j)^{\underline{j}}},$$

so that $H(s) = E_1(0,s)\psi_3(s)$. Then

$$\begin{aligned} H(0) = \sum_{q=1}^{j}E_1(q,0) &= \lim_{s\to 0^+}\sum_{q=1}^{j}E_1(q,s) = \lim_{s\to 0^+}\left(H(s)-E_1(0,s)\right)\\ &= \lim_{s\to 0}E_1(0,s)\left(\psi_3(s)-1\right) = \lim_{s\to 0^+}\phi(s)\left(\frac{1}{s}(\psi_3(s)-1)\right) = \psi_3'(0)\lim_{s\to 0^+}\phi(s). \end{aligned}$$

It is straightforward to check that $\lim_{s\to 0^+}\phi(s) = 1$ and $\psi_3'(0) = \sum_{k=1}^{j}\frac{1}{y+k} - \sum_{k=1}^{j}\frac{1}{x+k}$. □

We now return to the proof of (50). Using the Leibniz rule and (19) we have

$$\begin{aligned} \frac{\partial}{\partial x}(x+j+\ell)^{\underline{j+\ell}} &= \frac{\partial}{\partial x}\left((x+j+\ell)^{\underline{\ell}}(x+j)^{\underline{j}}\right)\\ &= (x+j)^{\underline{j}}\frac{\partial}{\partial x}(x+j+\ell)^{\underline{\ell}} + (x+j+\ell)^{\underline{\ell}}\frac{\partial}{\partial x}(x+j)^{\underline{j}}\\ &= (x+j)^{\underline{j}}\left(\sum_{t=1}^{\ell}(-1)^{t+1}\frac{\ell^{\underline{t}}}{t}(x+j+\ell)^{\underline{\ell-t}}\right) + (x+j+\ell)^{\underline{j+\ell}}\left(\sum_{t=1}^{j}\frac{1}{x+t}\right). \end{aligned}$$

Identity (50) follows from substituting the latter formula in its left hand side, and rewriting its right hand side using Proposition 7.3.

8. Capelli operators in Deligne's Category $\mathsf{Rep}(O_t)$

In this section, we define the categorical Capelli operators $\boldsymbol{D}_{t,\lambda}$ and prove Theorems A′–C′. We begin by defining general categorical analogues of the algebras $\mathscr{P}(V)$ and $\mathscr{PD}(V)$. Let C be a Karoubian $\mathbb{F}$-linear symmetric monoidal category, where $\mathbb{F}$ is a field of characteristic zero. Given an object X of C, set $\mathsf{P}_\mathsf{X}^d := \mathcal{S}^d(\mathsf{X})$ for $d \geq 0$ and $\mathsf{P}_\mathsf{X} := \bigoplus_{d\geq 0} \mathsf{P}_\mathsf{X}^d$, where we consider P_X^d as an object of the inductive completion of C. Then P_X is a commutative algebra object when equipped with the multiplication morphism $\boldsymbol{\mu}_\mathsf{X} : \mathsf{P}_\mathsf{X} \otimes \mathsf{P}_\mathsf{X} \to \mathsf{P}_\mathsf{X}$ that is induced from the monoidal structure of C. If X is left rigid and X^* denotes the left dual of X, then we set

$$\mathsf{PD}_\mathsf{X} := \mathsf{P}_\mathsf{X} \otimes \mathsf{P}_{\mathsf{X}^*} \cong \bigoplus_{p,q\geq 0} \mathcal{S}^p(\mathsf{X}) \otimes \mathcal{S}^q(\mathsf{X}^*).$$

For $q \geq p \geq 0$ the evaluation morphism $\boldsymbol{\epsilon}_{\mathcal{S}^p(\mathsf{X})} : \mathcal{S}^p(\mathsf{X}^*) \otimes \mathcal{S}^p(\mathsf{X}) \to \mathbb{1}$ yields a morphism

$$\mathbf{tr}_{p,q} : \mathcal{S}^p(\mathsf{X}^*) \otimes \mathcal{S}^q(\mathsf{X}) \to \mathcal{S}^{q-p}(\mathsf{X}),$$

and we set

$$\boldsymbol{\gamma}_{p,q} : \mathsf{P}_\mathsf{X} \otimes \mathcal{S}^p(\mathsf{X}^*) \otimes \mathcal{S}^q(\mathsf{X}) \to \mathsf{P}_\mathsf{X} \quad , \quad \boldsymbol{\gamma}_{p,q} := \boldsymbol{\mu}_\mathsf{X} \circ (1 \otimes \mathbf{tr}_{p,q}).$$

For $p > q \geq 0$ we set $\boldsymbol{\gamma}_{p,q} := 0$. Then $\boldsymbol{\gamma} := \oplus_{p,q\geq 0}\boldsymbol{\gamma}_{p,q}$ is a morphism $\boldsymbol{\gamma} : \mathsf{PD}_\mathsf{X} \otimes \mathsf{P}_\mathsf{X} \to \mathsf{P}_\mathsf{X}$. Moreover, there exists a unique morphism $\tilde{\boldsymbol{\mu}} : \mathsf{PD}_\mathsf{X} \otimes \mathsf{PD}_\mathsf{X} \to \mathsf{PD}_\mathsf{X}$ satisfying

$$\boldsymbol{\gamma}(\tilde{\boldsymbol{\mu}} \otimes 1) = \boldsymbol{\gamma}(1 \otimes \boldsymbol{\gamma}).$$

Thus PD_X is an associative algebra object and P_X is a PD_X-module in the inductive completion of C. The "order" filtration of PD_X is given by setting

$$\mathsf{PD}_\mathsf{X}^i := \mathsf{P}_\mathsf{X} \otimes \mathcal{S}^i(\mathsf{X}^*) \quad \text{for } i \geq 0.$$

There is also a $\mathbb{Z}$-grading on PD_X given by

$$\mathsf{PD}_{\mathsf{X},i} := \bigoplus_{p-q=i} \mathcal{S}^p(\mathsf{X}) \otimes \mathcal{S}^q(\mathsf{X}^*),$$

so that $\mathsf{PD}_\mathsf{X} \cong \oplus_{i\in\mathbb{Z}}\mathsf{PD}_{\mathsf{X},i}$. Note that $\mathsf{PD}_{\mathsf{X},0}$ is a subalgebra object of PD_X. If $\boldsymbol{\iota}_\mathsf{X} : \mathbb{1} \to \mathsf{X} \otimes \mathsf{X}^*$ is the co-evaluation of X then clearly $\boldsymbol{\iota}_X \in \mathrm{Hom}(\mathbb{1}, \mathsf{PD}_0)$.

Next suppose that there exists an isomorphism $\mathsf{X} \xrightarrow{\boldsymbol{\beta}} \mathsf{X}^*$, and set

$$\boldsymbol{\omega}_\mathsf{X} := (1 \otimes \boldsymbol{\beta}^{-1})\boldsymbol{\iota}_\mathsf{X} \quad \text{and} \quad \boldsymbol{\omega}_\mathsf{X}^* := (\boldsymbol{\beta} \otimes 1)\boldsymbol{\iota}_\mathsf{X}. \tag{59}$$

It is straightforward to verify that $\boldsymbol{\omega}_\mathsf{X} \in \mathrm{Hom}(\mathbb{1}, \mathsf{PD}_2)$ and $\boldsymbol{\omega}_\mathsf{X}^* \in \mathrm{Hom}(\mathbb{1}, \mathsf{PD}_{-2})$.

We now return to the Deligne category $\mathsf{Rep}(O_t)$. Recall that $\mathsf{Rep}(O_t)$ is the Karoubian $\mathbb{C}$-linear rigid symmetric monoidal category generated by the self-dual object V_t of categorical dimension $t \in \mathbb{C}$. We denote the identity object of $\mathsf{Rep}(O_t)$ by $\mathbb{1}$ and the braiding of $\mathsf{Rep}(O_t)$ by

$$\boldsymbol{\sigma} : \mathsf{M} \otimes \mathsf{N} \to \mathsf{N} \otimes \mathsf{M}.$$

Since V_t is self-dual, we have evaluation and co-evaluation morphisms

$$\boldsymbol{\epsilon} : \mathsf{V}_t \otimes \mathsf{V}_t \to \mathbb{1} \quad \text{and} \quad \boldsymbol{\iota} : \mathbb{1} \to \mathsf{V}_t \otimes \mathsf{V}_t,$$

that satisfy the usual duality axioms. Furthermore, these morphisms satisfy the relations

$$\boldsymbol{\sigma\iota} = \boldsymbol{\iota} \quad , \quad \boldsymbol{\epsilon\sigma} = \boldsymbol{\epsilon} \quad , \quad \boldsymbol{\epsilon\iota} = t.$$

By definition, for $d \geq 0$ the $\mathbb{C}$-algebra $\mathrm{End}(\mathsf{V}_t^{\otimes d})$ is generated by the morphisms $1^{\otimes(i-1)}\otimes\boldsymbol{\sigma}\otimes 1^{\otimes(d-i-1)}$ and $\boldsymbol{\iota\epsilon} \otimes 1^{\otimes(d-2)}$. The category $\mathsf{Rep}(O_t)$ satisfies the following properties (see [4, 5, 15]).

Proposition 8.1. *The following statements hold in the category* $\mathsf{Rep}(O_t)$.

(i) *For $d \geq 0$, the algebra* $\mathrm{End}(\mathsf{V}_t^{\otimes d})$ *is isomorphic to the Brauer algebra* $Br_d(t)$.
(ii) *Every indecomposable object of* $\mathsf{Rep}(O_t)$ *is isomorphic to the image of a primitive idempotent in* $\mathrm{End}(\mathsf{V}_t^{\otimes d})$ *for some* $d \geq 0$.
(iii) $\mathrm{Hom}(\mathsf{V}_t^{\otimes p}, \mathsf{V}_t^{\otimes q}) = \{0\}$ *if $p-q$ is odd. If $p-q$ is even, then* $\mathrm{Hom}(\mathsf{V}_t^{\otimes p}, \mathsf{V}_t^{\otimes q})$ *is generated as a* $(Br_q(t), Br_p(t))$*-bimodule by* $\boldsymbol{\epsilon}^{\otimes \frac{p-q}{2}} \otimes 1^{\otimes q}$ *if* $p > q$, *and by* $\boldsymbol{\iota}^{\otimes \frac{q-p}{2}} \otimes 1^{\otimes p}$ *if* $q > p$.
(iv) *If $t \notin \mathbb{Z}$, then* $\mathsf{Rep}(O_t)$ *is an abelian semisimple tensor category and in particular* $Br_d(t)$ *is a semisimple algebra.*
(vi) *If $t \in \mathbb{Z}$ and $p, q \geq \mathbb{Z}^{\geq 0}$ such that $p - 2q = t$, then there exists a symmetric monoidal full functor* $\mathrm{F}_{p|q} : \mathsf{Rep}(O_t) \to \mathsf{Rep}(\mathfrak{osp}(p|2q))$ *such that* $\mathrm{F}_{p|q}(\mathsf{V}_t) = \mathbb{C}^{p|2q}$, *where* $\mathbb{C}^{p|2q}$ *is the defining representation of* $\mathfrak{osp}(p|2q)$.

Our next goal is to define categorical analogues of invariant differential operators, and in particular the Euler and Casimir operators. To this end, we set

$$\mathsf{A}_t := \mathrm{Hom}\left(1, \mathsf{PD}_{\mathsf{V}_t,0}\right) \quad \text{and} \quad \mathsf{B}_t := \mathrm{Hom}\left(1, \mathsf{PD}_{\mathsf{V}_t}\right).$$

Then A_t and B_t are algebras with the products defined by

$$\boldsymbol{\alpha}_1 \otimes \boldsymbol{\alpha}_2 \mapsto \tilde{\boldsymbol{\mu}} \circ (\boldsymbol{\alpha}_1 \otimes \boldsymbol{\alpha}_2) \circ \boldsymbol{\iota}_\circ,$$

where $\boldsymbol{\iota}_\circ : 1 \to 1 \otimes 1$ is the co-evaluation of 1. One can interpret B_t as the algebra of O_t-invariant differential operators acting on $\mathsf{P}_{\mathsf{V}_t}$. Similarly, A_t can be interpreted as the algebra of GO_t-invariant differential operators on $\mathsf{P}_{\mathsf{V}_t}$.

The morphism $\boldsymbol{\gamma} : \mathsf{PD}_{\mathsf{V}_t} \otimes \mathsf{P}_{\mathsf{V}_t} \to \mathsf{P}_{\mathsf{V}_t}$ induces homomorphisms of associative algebras

$$\gamma_{\mathsf{A}_t} : \mathsf{A}_t \to \mathrm{End}(\mathsf{P}_{\mathsf{V}_t}) \quad \text{and} \quad \gamma_{\mathsf{B}_t} : \mathsf{B}_t \to \mathrm{End}(\mathsf{P}_{\mathsf{V}_t}).$$

Then $\boldsymbol{E}_t := \gamma_{\mathsf{A}_t}(\boldsymbol{\iota})$ acts by the scalar d on $\mathsf{P}^d_{\mathsf{V}_t}$. Set $\boldsymbol{\Delta}_t := \frac{1}{2}\gamma_{\mathsf{B}_t}\left(\boldsymbol{\omega}_{\mathsf{V}_t}\right)$ and $\boldsymbol{\Theta}_t := \frac{1}{2}\gamma_{\mathsf{B}_t}\left(\boldsymbol{\omega}^*_{\mathsf{V}_t}\right)$, where $\boldsymbol{\omega}_{\mathsf{V}_t}$ and $\boldsymbol{\omega}^*_{\mathsf{V}_t}$ are defined as in (59), with $\boldsymbol{\beta} := 1_{\mathsf{V}_t}$. It is straightforward to verify the relations

$$[\mathbf{E}_t, \boldsymbol{\Delta}_t] = -2\boldsymbol{\Delta}_t,\ [\boldsymbol{E}_t, \boldsymbol{\Theta}_t] = 2\boldsymbol{\Theta}_t,\ [\boldsymbol{\Delta}_t, \boldsymbol{\Theta}_t] = \boldsymbol{E}_t + \frac{t}{2}. \tag{60}$$

Now set

$$\boldsymbol{C}_t := (\boldsymbol{E}_t + \frac{t}{2})^2 - 2\boldsymbol{\Theta}_t\boldsymbol{\Delta}_t - 2\boldsymbol{\Delta}_t\boldsymbol{\Theta}_t - \frac{t^2}{4} + t = \boldsymbol{E}_t^2 + t\boldsymbol{E}_t - 4\boldsymbol{\Theta}_t\boldsymbol{\Delta}_t - 2\boldsymbol{E}_t. \tag{61}$$

One can check that $\boldsymbol{C}_t$ is indeed the Casimir element for the Lie algebra object $\mathfrak{g}_t \simeq \Lambda^2(\mathsf{V}_t)$.

Proposition 8.2. *Let $t \in \mathbb{C}$ and let A_t, B_t, γ_{A_t}, and γ_{B_t} be as above. Then the following statements hold.*

(i) $\gamma_{\mathsf{B}_t}(\mathsf{B}_t)$ *is generated by $\boldsymbol{\Delta}_t$ and $\boldsymbol{\Theta}_t$.*
(ii) $\gamma_{\mathsf{B}_t}(\mathsf{B}_t)$ *is isomorphic to the universal enveloping algebra* $U(\mathfrak{sl}_2)$.
(iii) $\gamma_{\mathsf{A}_t}(\mathsf{A}_t)$ *is generated by $\boldsymbol{C}_t$ and $\boldsymbol{E}_t$.*

Proof. (i) Fix $b \in \gamma_{\mathsf{B}_t}(\mathsf{B}_t)$ and choose $d \in \mathbb{N}$ sufficiently large such that the restriction of b on $\mathsf{P}^{\leq d}_{\mathsf{V}_t} := \bigoplus_{p \leq d} \mathsf{P}^p_{\mathsf{V}_t}$ uniquely determines b among elements of $\gamma_{\mathsf{B}_t}(\mathsf{B}_t)$. Since projections from $\mathsf{P}^{\leq d}_{\mathsf{V}_t}$ onto $\mathsf{P}^p_{\mathsf{V}_t}$ for $0 \leq p \leq d$ can be expressed as polynomials in $\boldsymbol{E}_t$, we can write b as

$$b = \sum_{0 \leq p,q \leq d} f_q(\boldsymbol{E}_t) b g_p(\boldsymbol{E}_t),$$

where $f_i, g_i \in \mathbb{C}[x]$. Each summand $f_q(\boldsymbol{E}_t) b g_p(\boldsymbol{E}_t)$ can be identified with an element of $\mathrm{Hom}(\mathsf{P}^p_{\mathsf{V}_t}, \mathsf{P}^q_{\mathsf{V}_t})$ for some $p, q \geq 0$. To complete the proof of (i), it suffices to express element of $\mathrm{Hom}(\mathsf{P}^p_{\mathsf{V}_t}, \mathsf{P}^q_{\mathsf{V}_t})$ in terms of $\boldsymbol{\Delta}_t$ and $\boldsymbol{\Theta}_t$. To prove the latter claim, first assume $p = q$. Recall that the algebra $\mathrm{End}(\mathsf{V}_t^{\otimes p})$

is generated by the symmetric group S_p and the morphism $\iota\epsilon \otimes 1^{\otimes(p-2)}$. Since $\mathsf{P}^p_{\mathsf{V}_t} = \mathcal{S}^p(\mathsf{V}_t)$ is a direct summand of $\mathsf{V}_t^{\otimes p}$, the canonical restriction $\mathrm{End}(\mathsf{V}_t^{\otimes p}) \to \mathrm{End}(\mathsf{P}^p_{\mathsf{V}_t})$ is a surjection. But the action of S_p on $\mathcal{S}^p(\mathsf{V}_t)$ is trivial, hence $\mathrm{End}(\mathsf{P}^p_{\mathsf{V}_t}) = \mathrm{End}(\mathcal{S}^p(\mathsf{V}_t))$ is generated by $\boldsymbol{\Theta}_t\boldsymbol{\Delta}_t$. Next assume that $p \neq q$. Then by Proposition 8.1(iii), for $q > p$ the homomorphism

$$\mathbb{C}[\boldsymbol{\Theta}_t\boldsymbol{\Delta}_t]\boldsymbol{\Theta}_t^{\frac{q-p}{2}}\mathbb{C}[\boldsymbol{\Theta}_t\boldsymbol{\Delta}_t] \to \mathrm{Hom}(\mathcal{S}^p(\mathsf{V}_t), \mathcal{S}^q(\mathsf{V}_t))$$

is surjective. Similarly, for $p > q$ the homomorphism

$$\mathbb{C}[\boldsymbol{\Theta}_t\boldsymbol{\Delta}_t]\boldsymbol{\Delta}_t^{\frac{p-q}{2}}\mathbb{C}[\boldsymbol{\Theta}_t\boldsymbol{\Delta}_t] \to \mathrm{Hom}(\mathcal{S}^p(V_t), \mathcal{S}^q(V_t))$$

is surjective.

(ii) Since $\boldsymbol{\Delta}_t, \boldsymbol{E}_t + \frac{t}{2}, -\boldsymbol{\Theta}_t$ form a standard $\mathfrak{sl}_2$-triple, we obtain a surjection $U(\mathfrak{sl}_2) \to \gamma_{\mathsf{B}_t}(\mathsf{B}_t)$. Next we prove that the latter homomorphism is injective.

First, we assume that $t \notin 2\mathbb{Z}$. For every simple object X of $\mathsf{Rep}(O_t)$ the space $\mathsf{M}_\mathsf{X} := \mathrm{Hom}(\mathsf{X}, \mathsf{P}_{\mathsf{V}_t})$ is a $\gamma_{\mathsf{B}_t}(\mathsf{B}_t)$-module and hence a $U(\mathfrak{sl}_2)$-module. It suffices to show that $\mathsf{M} := \oplus_\mathsf{X}\mathsf{M}_\mathsf{X}$ is a faithful $U(\mathfrak{sl}_2)$-module, where the direct sum is taken over isomorphism classes of simple objects of $\mathsf{Rep}(O_t)$. Note that each M_X is a weight module with weights in $\mathbb{Z}^{\geq 0} + \frac{t}{2}$. From the theory of Verma modules for $\mathfrak{sl}_2$ it follows that if d is such that $\dim \mathrm{Hom}(\mathsf{X}, \mathsf{P}^{d-2}_{\mathsf{V}_t}) < \dim \mathrm{Hom}(\mathsf{X}, \mathsf{P}^d_{\mathsf{V}_t})$, then M contains a Verma module with lowest weight $d + \frac{t}{2}$ as a subrepresentation. Next we show that for all but finitely many $d \geq 0$, the latter inequality holds for some X. Indeed since $\boldsymbol{\Theta}_t$ induces a monomorphism $\mathsf{P}^{d-2}_{\mathsf{V}_t} \to \mathsf{P}^d_{\mathsf{V}_t}$, it suffices to show that $\mathsf{P}^{d-2}_{\mathsf{V}_t}$ and $\mathsf{P}^d_{\mathsf{V}_t}$ are not isomorphic objects. The latter follows from comparing the categorical dimensions, which is given by the formula

$$\dim \mathsf{P}^d_{\mathsf{V}_t} = \frac{t(t+1)\ldots(t+d-1)}{d!}.$$

Hence M contains $\mathfrak{sl}_2$-submodules which are Verma modules with lowest weights $d + \frac{t}{2}$ for all but finitely many $d \in \mathbb{N}$. The intersections of the annihilators of these Verma modules is the trivial ideal of $U(\mathfrak{sl}_2)$ (see [6, Sec. 8.4]), hence M is a faithful $U(\mathfrak{sl}_2)$-module.

If $t \in 2\mathbb{Z}$ the result follows from the analogous result for $\mathfrak{osp}(2m|2n)$ with $2m - 2n = t$, where $m, n \in \mathbb{N}$ (see for instance [26]) using the functor $\mathrm{F}_{m|n}$ defined in Proposition 8.1(vi).

(iii) Note that $\gamma_{\mathsf{A}_t}(\mathsf{A}_t)$ is the centralizer of $\boldsymbol{E}_t$ inside $\gamma_{\mathsf{B}_t}(\mathsf{B}_t)$. Thus (ii) implies that (iii) is equivalent to the well-known fact that the centralizer of the Cartan subalgebra in $U(\mathfrak{sl}_2)$ is generated by the Cartan subalgebra and the Casimir operator. □

Lemma 8.3. *Let $d \in \mathbb{Z}^{\geq 0}$ and let $\boldsymbol{C}_{t,d}$ be the image of $\boldsymbol{C}_t$ in $\mathrm{End}(\mathsf{P}^d_{\mathsf{V}_t})$. Let $p^d_t(x) \in \mathbb{C}[x]$ be the minimal degree monic polynomial such that $p^d_t(\boldsymbol{C}_{t,d}) = 0$. Then*

$$p^d_t(x) = \prod_{\substack{0 \leq a \leq d \\ d \equiv a \bmod 2}} (x - a(a+t-2)).$$

Proof. For $d \leq 1$ the statement is trivial since $\boldsymbol{C}_{t,0} = 0$ and $\boldsymbol{C}_{t,1} = (t-1)1_{\mathsf{V}_t}$. We will prove the statement by induction on d.

First we assume that $t \notin \mathbb{Z}$. We claim that

$$\mathsf{P}^d_{\mathsf{V}_t} \cong \mathsf{P}^{d-2}_{\mathsf{V}_t} \oplus \ker \boldsymbol{\Delta}_t\big|_{\mathsf{P}^d_{\mathsf{V}_t}}.$$

Indeed, from representation theory of $\mathfrak{sl}_2$ (see the proof of Proposition 8.2(ii)) it follows that $\boldsymbol{\Delta}_t$ is surjective. Semisimplicity of $\mathsf{Rep}(O_t)$ implies the claim.

By (61), the operator $\boldsymbol{C}_{t,d}$ acts on $\ker \boldsymbol{\Delta}_t$ by the scalar $d(d+t-2)$. Since $d(d+t-2)$ is not a root of $p_t^{d-2}(x)$, we have $p_t^d(x) = (x - d(d+t-2))p_t^{d-2}(x)$. The statement now follows by induction.

Next assume that $t \in \mathbb{Z}$. Choose positive integers a, b such that $a - 2b = t$. Then from Proposition 3.1 it follows that $p_t^r(x)$ is the minimal polynomial for $\mathrm{F}_{a|b}(\boldsymbol{C}_{t,d})$, where $\mathrm{F}_{a|b}$ is the functor given in Proposition 8.1(iv). The homomorphism $\mathrm{F}_{a|b} : \mathrm{End}(\mathsf{P}^d) \to \mathrm{End}(\mathscr{P}^d(V))$ is surjective since $\mathrm{F}_{a|b}$ is full. On the other hand, $\mathrm{End}(\mathsf{P}^d)$ is spanned by $\left\{\boldsymbol{\Theta}_t^p \boldsymbol{\Delta}_t^p : 0 \leq p \leq \lfloor \frac{d}{2} \rfloor\right\}$ (see the proof of Proposition 8.2(i)). Hence

$$\dim \mathrm{End}(\mathsf{P}_{\mathsf{V}_t}^d) \leq 1 + \left\lfloor \frac{d}{2} \right\rfloor = \dim \mathrm{End}(\mathscr{P}^d(V)),$$

and thus $\mathrm{F}_{a|b}$ is an isomorphism. Consequently, the minimal polynomials of $\mathrm{F}_{a|b}(\boldsymbol{C}_{t,d})$ and $\boldsymbol{C}_{t,d}$ are identical. The statement now follows from the decomposition of $\mathscr{P}^d(V)$ as a $\mathfrak{gosp}(a|2b)$-module (see Proposition 3.1 and [26, Sec. 10]). □

Remark 8.4. If $t \notin 2\mathbb{Z}^{\leq 0}$ then $p_t^d(x)$ does not have multiple roots. If $t \in 2\mathbb{Z}^{\leq 0}$ then the multiplicity of each root of $p_t^d(x)$ is at most 2.

Lemma 8.5. *Let $p_t^d(x)$ be as in Lemma 8.3. Let u_i, $1 \leq i \leq e$, be the distinct roots of $p_t^d(x)$ with corresponding multiplicities $m_{u_i} \in \{1, 2\}$. Set $\mathsf{W}_{u_i} := \ker\left((\boldsymbol{C}_{t,d} - u_i)^{m_{u_i}}\right)$. Then $\mathsf{P}_{\mathsf{V}_t}^d \cong \bigoplus_{i=1}^e \mathsf{W}_{u_i}$ and every W_{u_i} is an indecomposable object of $\mathsf{Rep}(O_t)$.*

Proof. The proof of the decomposition $\mathsf{P}_{\mathsf{V}_t}^d \cong \bigoplus_{i=1}^e \mathsf{W}_{u_i}$ is similar to that of the Primary Decomposition Theorem in linear algebra. Set $q_i(x) := \frac{p_t^d(x)}{(x-u_i)^{m_{u_i}}}$ and choose $g_i, h_i \in \mathbb{C}[x]$ such that $q_i(x)g_i(x) + (x - u_i)^{m_{u_i}} h_i(x) = 1$. Then the morphisms $\boldsymbol{\pi}_i := q_i(\boldsymbol{C}_{t,d})g_i(\boldsymbol{C}_{t,d})$ are the projections onto the W_{u_i}.

Next we show that each W_{u_i} is indecomposable. Recall that $\boldsymbol{C}_{t,d}$ generates the algebra $\mathrm{End}(\mathsf{P}_{\mathsf{V}_t}^d)$ (see the proof of Proposition 8.2(i)). Since W_{u_i} is a direct summand, it follows that $\mathrm{End}(\mathsf{W}_{u_i})$ is also generated by the restriction of $\boldsymbol{C}_{t,d}$. Consequently, $\mathrm{End}(\mathsf{W}_{u_i}) \cong \mathbb{C}[x]/\langle (x-u_i)^{m_{u_i}} \rangle$, hence W_{u_i} is indecomposable. □

Recall from (22) that $\underline{k} := -\frac{t}{2}$. For $\lambda \in \mathcal{I}$ and $t \in \mathbb{C}$ set

$$\boldsymbol{c}_\lambda(t) := (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_2 + t - 2) = (\lambda_2 - \lambda_1)(\lambda_2 - \lambda_1 + 2\underline{k} + 2).$$

Now let $d \in \mathbb{Z}^{\geq 0}$, and let $p_t^d(x)$ be as in Lemma 8.3. Given $\lambda \in \mathcal{I}$ such that $|\lambda| = d$, we denote the multiplicity of the root $\boldsymbol{c}_\lambda(t)$ of $p_t^d(x)$ by m_λ. Lemma 8.5 immediately implies the following corollary, which is the categorical analogue of (6). Recall that $\mathcal{I}_{\underline{k}}'$ is defined as in (22).

Corollary 8.6. *For $\lambda \in \mathcal{I}$ set $\mathsf{V}_{t,\lambda} := \ker\left((\boldsymbol{C}_{t,d} - \boldsymbol{c}_\lambda(t))^{m_\lambda}\right)$ where $d := |\lambda|$. Then the following statements hold.*

(i) *If $t \notin 2\mathbb{Z}^{\leq 0}$ then $\mathsf{P}_{\mathsf{V}_t}^d \cong \bigoplus_{\lambda \in \mathcal{I}, |\lambda| = d} \mathsf{V}_{t,\lambda}$.*

(ii) *If $t \in 2\mathbb{Z}^{\leq 0}$ then $\mathsf{P}_{\mathsf{V}_t}^d \cong \bigoplus_{\lambda \in {}_d\mathcal{I}_{\underline{k}}'} \mathsf{V}_{t,\lambda}$.*

We are now going to define the *eigenvalue polynomial* $\boldsymbol{f}_{\boldsymbol{D}}$ for an element $\boldsymbol{D} \in \mathsf{A}_\mathsf{t}$. From now on, for $\boldsymbol{D} \in \mathsf{A}_t$ we denote the restriction of $\boldsymbol{\gamma}_{\mathsf{A}_t}(\boldsymbol{D})$ to $\mathsf{V}_{t,\mu}$ by $\boldsymbol{D}\big|_{\mathsf{V}_{t,\mu}}$.

Proposition 8.7. *Let $\boldsymbol{D} \in \mathsf{A}_t$. Set $\mathcal{S} := \mathcal{I}$ if $t \notin 2\mathbb{Z}^{\leq 0}$ and $\mathcal{S} := \mathcal{I}_{\underline{k}}'$ otherwise. Then there exists a unique symmetric polynomial $\boldsymbol{f}_{\boldsymbol{D}}(x, y)$ such that for every $\lambda \in \mathcal{S}$, we have*

$$\boldsymbol{D}\big|_{\mathsf{V}_{t,\lambda}} = \boldsymbol{f}_{\boldsymbol{D}}(\lambda_1 - \underline{k} - 1, \lambda_2) \cdot 1_{\mathsf{V}_{t,\lambda}} + \boldsymbol{\eta}_\lambda,$$

where $\boldsymbol{\eta}_\lambda \in \mathrm{End}(\mathsf{V}_{t,\lambda})$ satisfies $\boldsymbol{\eta}_\lambda^2 = 0$.

Proof. Existence follows from Corollary 8.6 and Proposition 8.2(iii), and the argument is similar to the proof of Proposition 3.3. Uniqueness follows from the fact that $\mathcal{P}$ is Zariski dense in $\mathbb{C}^2$. Note that when $t \in 2\mathbb{Z}^{\leq 0}$, the coincidence relation (23) implies that two symmetric polynomials that agree on $\mathcal{S}$ also agree on $\mathcal{I}$. □

Definition 8.8. Let $\boldsymbol{D} \in \mathsf{A}_t$. The *eigenvalue polynomial* of $\boldsymbol{D}$ is the polynomial $\boldsymbol{f}_{\boldsymbol{D}}(x, y) \in \mathbb{C}[x, y]$ that is given in Proposition 8.7.

The construction of the Capelli operators in $\mathsf{Rep}(O_t)$. Our next task is to define the Capelli operators $\{\boldsymbol{D}_{t,\lambda}\}_{\lambda\in\mathcal{I}}$.

- If $t \notin 2\mathbb{Z}^{\leq 0}$ then for all $\lambda \in \mathcal{I}$ we define $\boldsymbol{D}_{t,\lambda} \in \mathsf{A}_t$ as the element corresponding to the co-evaluation morphism

$$1 \xrightarrow{\epsilon_{\mathsf{V}_{t,\lambda}}} \mathsf{V}_{t,\lambda} \otimes \mathsf{V}^*_{t,\lambda}. \tag{62}$$

- If $t \in 2\mathbb{Z}_{\leq 0}$ and λ is $\underline{k}$-regular or $\underline{k}$-quasiregular, we define $\boldsymbol{D}_{t,\lambda}$ as in (62).
- If $t \in 2\mathbb{Z}_{\leq 0}$ and λ is $\underline{k}$-singular, we define $\boldsymbol{D}_{t,\lambda}$ as the element of A_t corresponding to the morphism

$$1 \xrightarrow{\boldsymbol{C}_{t,|\lambda|}-\boldsymbol{c}_\lambda(t)} \mathsf{V}_{t,\lambda^\dagger} \otimes \mathsf{V}^*_{t,\lambda^\dagger}, \tag{63}$$

  Using the fact that $(\boldsymbol{C}_{t,|\lambda|} - \boldsymbol{c}_\lambda(t)) \in \mathrm{End}(\mathsf{V}_{t,\lambda^\dagger}) \cong \mathrm{Hom}(1, \mathsf{V}_{t,\lambda^\dagger} \otimes \mathsf{V}^*_{t,\lambda^\dagger})$. Here and in the rest of the paper $\lambda^\dagger$ is defined as in Remark 1.3, but with k replaced by $\underline{k}$. Lemma 8.5 implies that $\boldsymbol{C}_{t,|\lambda|} - \boldsymbol{c}_\lambda(t)$ is a nilpotent element of order two in $\mathrm{End}(\mathsf{V}_{t,\lambda^\dagger})$.

For $d \geq 0$ let J_t^d denote the annihilator of $\mathsf{P}^{\leq d}_{\mathsf{V}_t} := \bigoplus_{p\leq d} \mathsf{P}^p_{\mathsf{V}_t}$ in A_t. Since A_t is commutative, J^d is a two-sided ideal of A_t. Moreover, we have a decomposition

$$\mathsf{A}_t = \mathsf{A}_t^d \oplus \mathsf{J}_t^d,$$

where $\mathsf{A}_t^d := \mathsf{PD}^d_{\mathsf{V}_t} \cap \mathsf{A}_t$. Let

$$\boldsymbol{\pi}_{t,d} : \mathsf{A}_t \to \mathsf{A}_t^d$$

denote the projection with kernel J_t^d. Our next task is to show that as t varies, the projections $\boldsymbol{\pi}_{t,d}$ deform with polynomials coefficients. From Proposition 8.2 it follows that $\boldsymbol{\gamma}_{\mathsf{A}_t}$ is an injection. Thus, from now on we identify $\boldsymbol{C}_t$ and $\boldsymbol{E}_t$ with their images under $\boldsymbol{\gamma}_{\mathsf{A}_t}$.

Lemma 8.9. *For $i, j, d \geq 0$, there exist polynomials $\phi_{i,j,d,i',j'} \in \mathbb{C}[x]$ such that*

$$\boldsymbol{\pi}_{t,d}(\boldsymbol{C}_t^i \boldsymbol{E}_t^j) = \sum_{2i'+j'\leq d} \phi_{i,j,d,i',j'}(t) \boldsymbol{C}_t^{i'} \boldsymbol{E}_t^{j'} \quad \textit{for all } t \in \mathbb{C}.$$

Proof. We will describe a recursive procedure for finding the $\phi_{i,j,d,i',j'}$ with the desired properties.

Step 1. Using one-variable interpolation, for every $j \geq 0$ we can find scalars a_p for $0 \leq p \leq d$ such that $\boldsymbol{E}_t^j - \sum_{p=0}^d a_p \boldsymbol{E}_t^p \in \mathsf{J}_t^d$. This proves the statement for the special case $i = 0$.

Step 2. We show that for every $i \geq 0$ there exist polynomials $\psi_{p,q}(t)$ for $1 \leq p \leq N_i$ and $0 \leq q \leq \lfloor \frac{d}{2} \rfloor$, where $N_i \in \mathbb{N}$, such that the element $\boldsymbol{L} \in \mathsf{A}_t$ defined by

$$\boldsymbol{L} := \sum_{p=0}^{\lfloor \frac{d}{2} \rfloor} \sum_{q=1}^{N_i} \psi_{p,q}(t) \boldsymbol{C}_t^p \boldsymbol{E}_t^q, \tag{64}$$

satisfies $\boldsymbol{C}_t^i - \boldsymbol{L} \in \mathsf{J}_t^d$. Indeed by (60) and (61) we can write $\boldsymbol{C}_t^i$ as a linear combination of monomials of the form $\boldsymbol{E}_t^p \boldsymbol{\Theta}_t^q \boldsymbol{\Delta}_t^q$ with coefficients that are polynomial in t. Furthermore we can discard the monomials for which $q > \lfloor \frac{d}{2} \rfloor$, because $\boldsymbol{\Delta}_t^q\big|_{\mathcal{S}^d(\mathsf{V}_t)} = 0$. Next we use (60) and (61) again to rewrite $\boldsymbol{\Theta}_t^q \boldsymbol{\Delta}_t^q$ first in terms of powers of $\boldsymbol{\Theta}_t \boldsymbol{\Delta}_t$ and then in terms of powers of $\boldsymbol{C}_t$. The latter process will only add extra powers of $\boldsymbol{E}_t$ and coefficients that are polynomial in t. This completes the proof of existence of $\boldsymbol{L}$.

Step 3. Fix a pair (i, j) of exponents. Assume that the statement of the lemma holds for all $\boldsymbol{C}_t^{i'} \boldsymbol{E}_t^{j'}$ such that either $i' < i$, or $i' = i$ and $2i' + j' < 2i + j$. We now verify the lemma for $\boldsymbol{C}_t^i \boldsymbol{E}_t^j$. If $2i + j \leq d$ there is nothing to prove, and therefore we assume that $2i + j \geq d + 1$. If $j = 0$ then using Step 2 we can reduce the problem to monomials of the form $\boldsymbol{C}_t^{i'} \boldsymbol{E}_t^{j'}$ where $i' < i$. If $j > 0$ then set $\boldsymbol{L}' := \boldsymbol{C}_t^i \prod_{r=2i}^{2i+j-1} (\boldsymbol{E}_t - r)$. Note that $\boldsymbol{C}_t - \boldsymbol{L}'$ is a linear combination of monomials $\boldsymbol{C}_t^i \boldsymbol{E}_t^{j'}$ satsifying $2i + j' < 2i + j$. Furthermore, by Step 2 the restriction of $\boldsymbol{C}_t^i$ to $\bigoplus_{p=0}^{2i-1} \mathcal{S}^p(\mathsf{V}_t)$ is equal to a linear combination of monomials $\boldsymbol{C}_t^p \boldsymbol{E}_t^q$ where $p \leq i - 1$, with polynomial coefficients. It follows that the restriction of $\boldsymbol{L}'$ to $\bigoplus_{p=0}^{d} \mathcal{S}^p(\mathsf{V}_t)$ is also equal to a linear combination of monomials $\boldsymbol{C}_t^p \boldsymbol{E}_t^q$ where $p \leq i - 1$, with polynomial coefficients. □

The next lemma is the categorical incarnation of Lemma 4.1.

Lemma 8.10. *Let $\lambda \in \mathcal{I}$ and set $d := |\lambda|$.*

(i) *If $t \notin 2\mathbb{Z}^{\leq 0}$, then $\boldsymbol{D}_{t,\lambda}$ is the unique element of A_t^d such that $\boldsymbol{D}_{t,\lambda}\big|_{\mathsf{V}_{t,\lambda}} = 1$ and $\boldsymbol{D}_{t,\lambda}\big|_{\mathsf{V}_{t,\mu}} = 0$ for all $\mu \in \mathcal{I}$ satisfying $|\mu| \leq |\lambda|$ and $\mu \neq \lambda$.*

(ii) *If $t \in 2\mathbb{Z}^{\leq 0}$ and λ is $\underline{k}$-regular, then $\boldsymbol{D}_{t,\lambda}$ is the unique element of A_t^d such that $\boldsymbol{D}_{t,\lambda}\big|_{\mathsf{V}_{t,\lambda}} = 1$ and $\boldsymbol{D}_{t,\lambda}\big|_{\mathsf{V}_{t,\mu}} = 0$ for all $\mu \in \mathcal{I}'_{\underline{k}}$ satisfying $|\mu| \leq |\lambda|$ and $\mu \neq \lambda$.*

(iii) *If $t \in 2\mathbb{Z}^{\leq 0}$ and λ is $\underline{k}$-singular, then $\boldsymbol{D}_{t,\lambda}$ is the unique element of A_t^d such that*

$$\boldsymbol{D}_{t,\lambda}\big|_{\mathsf{V}_{t,\lambda^\dagger}} = \boldsymbol{C}_{t,d} - \boldsymbol{c}_\lambda(t),$$

and $\boldsymbol{D}_{t,\lambda}\big|_{\mathsf{V}_{t,\mu}} = 0$ for all $\mu \in \mathcal{I}'_{\underline{k}}$ satisfying $|\mu| \leq |\lambda|$ and $\mu \neq \lambda^\dagger$.

(iv) *If $t \in 2\mathbb{Z}^{\leq 0}$ and λ is $\underline{k}$-quasiregular, then $\boldsymbol{D}_{t,\lambda}$ is the unique element of A_t^d such that*

$$\boldsymbol{D}_{t,\lambda}\big|_{\mathsf{V}_{t,\lambda}} = 1,$$

and $\boldsymbol{D}_{t,\lambda}\big|_{\mathsf{V}_{t,\mu}} = 0$ for all $\mu \in \mathcal{I}'_{\underline{k}}$ satisfying $|\mu| \leq |\lambda|$ and $\mu \neq \lambda$.

Proof. The stated properties of $\boldsymbol{D}_{t,\lambda}$ are straightforward from the definition. Uniqueness follows from the fact that any element of A_t^d is uniquely determined by its restriction to a morphism of $\mathsf{P}_{\mathsf{V}_t}^{\leq d}$. □

Let $\lambda \in \mathcal{I}$ and set $d := |\lambda|$. For $s \in \mathbb{C}$ such that $s \notin 2\mathbb{Z}^{\leq 0}$ we define $\boldsymbol{L}_{s,t,\lambda} \in \mathsf{A}_t$ by

$$\boldsymbol{L}_{s,t,\lambda} := \frac{\prod_{i=0}^{d-1} (\boldsymbol{E}_t - i) \prod_{|\nu|=d,\nu\neq\lambda} (\boldsymbol{C}_t - \boldsymbol{c}_\nu(s))}{d! \prod_{|\nu|=d,\nu\neq\lambda} (\boldsymbol{c}_\lambda(s) - \boldsymbol{c}_\nu(s))}. \tag{65}$$

We remark that $\boldsymbol{L}_{s,t,\lambda}$ is well-defined because the factors $(\boldsymbol{c}_\lambda(s) - \boldsymbol{c}_\nu(s))$ in the denominator of (65) vanish only if $s \in 2\mathbb{Z}^{\leq 0}$ and λ and ν are a pair of k_s-quasiregular and k_s-singular partitions, where $k_s := -\frac{s}{2}$. We can now expand the right hand side of (65) and express $\boldsymbol{L}_{s,t,\lambda}$ as

$$\boldsymbol{L}_{s,t,\lambda} = \sum_{i,j \geq 0} \eta_{\lambda,i,j}(s) \boldsymbol{C}_t^i \boldsymbol{E}_t^j, \tag{66}$$

where the $\eta_{\lambda,i,j}$ are rational functions of s. Note that the $\eta_{\lambda,i,j}$ are independent of t and do not have poles in $\mathbb{C}$ outside the set $2\mathbb{Z}^{\leq 0}$.

Definition 8.11. For $s \in \mathbb{C}$ such that $s \notin 2\mathbb{Z}^{\leq 0}$, we define $\boldsymbol{D}_{s,t,\lambda} \in \mathsf{A}_t$ as follows.

(i) If either $t \notin 2\mathbb{Z}^{\leq 0}$, or $t \in 2\mathbb{Z}^{\leq 0}$ and λ is $\underline{k}$-regular, then we set $\boldsymbol{D}_{s,t,\lambda} := \boldsymbol{L}_{s,t,\lambda}$.
(ii) If $t \in 2\mathbb{Z}^{\leq 0}$ and λ is $\underline{k}$-singular, then we set $\boldsymbol{D}_{s,t,\lambda} := (\boldsymbol{c}_{\lambda^\dagger}(s) - \boldsymbol{c}_\lambda(s))\boldsymbol{L}_{s,t,\lambda^\dagger}$.
(iii) If $t \in 2\mathbb{Z}^{\leq 0}$ and λ is $\underline{k}$-quasiregular, then we set $\boldsymbol{D}_{s,t,\lambda} := \boldsymbol{L}_{s,t,\lambda} + \boldsymbol{L}_{s,t,\lambda^\dagger}$.

Using (66) we can express $\boldsymbol{D}_{s,t,\lambda}$ as

$$\boldsymbol{D}_{s,t,\lambda} = \sum_{i,j} \eta_{i,j}(s) \boldsymbol{C}_t^i \boldsymbol{E}_t^j, \tag{67}$$

where $\eta_{i,j}(s)$ is equal to $\eta_{\lambda,i,j}(s)$ or $(\boldsymbol{c}_{\lambda^\dagger}(s) - \boldsymbol{c}_\lambda(s))\eta_{\lambda^\dagger,i,j}(s)$ or $\eta_{\lambda,i,j}(s) + \eta_{\lambda^\dagger,i,j}(s)$ in cases (i), (ii), and (iii) of Definition 8.11, respectively.

For $\underline{k}$-quasiregular λ we define

$$\tilde{\phi}_{1,\lambda}(s) := \prod_{|\nu|=|\lambda|,\nu\neq\lambda,\lambda^\dagger} (\boldsymbol{c}_\lambda(s) - \boldsymbol{c}_\nu(s))^{-1} \quad \text{and} \quad \tilde{\phi}_{2,\lambda}(s) := \prod_{|\nu|=|\lambda|,\nu\neq\lambda,\lambda^\dagger} (\boldsymbol{c}_{\lambda^\dagger}(s) - \boldsymbol{c}_\nu(s))^{-1}.$$

The next proposition is a key step in the proofs of Theorems A′–C′.

Proposition 8.12. *The rational functions $\eta_{i,j}(s)$ in (67) do not have any poles at $s = t$.*

Proof. For $\boldsymbol{D}_{s,t,\lambda}$ as in Definition 8.11(i)-(ii) this follows from the fact that for $\lambda, \nu \in \mathcal{I}$ such that $|\lambda| = |\nu|$, we have $\boldsymbol{c}_\lambda(s) = \boldsymbol{c}_\nu(s)$ if and only if $s \in 2\mathbb{Z}^{\leq 0}$ and λ and ν are a pair of $(-\frac{s}{2})$-quasiregular and $(-\frac{s}{2})$-singular partitions. For $\boldsymbol{D}_{s,t,\lambda}$ as in Definition 8.11(iii) note that

$$\boldsymbol{D}_{s,t,\lambda} = \prod_{|\nu|=d,\nu\neq\lambda,\lambda^\dagger} (\boldsymbol{C}_t - \boldsymbol{c}_\nu(s)) \frac{\prod_{i=0}^{d-1}(\boldsymbol{E}_t - i)}{d!} \boldsymbol{D}', \tag{68}$$

where $d := |\lambda|$ and

$$\boldsymbol{D}' := \left(\tilde{\phi}_{1,\lambda}(s) \frac{\boldsymbol{C}_t - \boldsymbol{c}_{\lambda^\dagger}(s)}{\boldsymbol{c}_\lambda(s) - \boldsymbol{c}_{\lambda^\dagger}(s)} - \tilde{\phi}_{2,\lambda}(s) \frac{\boldsymbol{C}_t - \boldsymbol{c}_\lambda(s)}{\boldsymbol{c}_\lambda(s) - \boldsymbol{c}_{\lambda^\dagger}(s)} \right).$$

Then $\boldsymbol{D}' = \gamma_0(s) + \gamma_1(s)\boldsymbol{C}_t$ where

$$\gamma_0(s) = -\frac{\tilde{\phi}_{1,\lambda}(s)\boldsymbol{c}_{\lambda^\dagger}(s) - \tilde{\phi}_{2,\lambda}(s)\boldsymbol{c}_\lambda(s)}{\boldsymbol{c}_\lambda(s) - \boldsymbol{c}_{\lambda^\dagger}(s)} \quad \text{and} \quad \gamma_1(s) = \frac{\tilde{\phi}_{1,\lambda}(s) - \tilde{\phi}_{2,\lambda}(s)}{\boldsymbol{c}_\lambda(s) - \boldsymbol{c}_{\lambda^\dagger}(s)}.$$

From the remark about vanishing of the differences $(\boldsymbol{c}_\lambda(s) - \boldsymbol{c}_\nu(s))$ it follows that $\tilde{\phi}_{1,\lambda}(s)$ and $\tilde{\phi}_{2,\lambda}(s)$ do not have poles at $s = t$. Furthermore, from $\boldsymbol{c}_\lambda(t) = \boldsymbol{c}_{\lambda^\dagger}(t)$ it follows that $\tilde{\phi}_{1,\lambda}(t) = \tilde{\phi}_{2,\lambda}(t)$. This implies that $\gamma_0(s)$ and $\gamma_1(s)$ do not have poles at $s = t$. Hence the coefficients $\eta_{i,j}(s)$ of $\boldsymbol{D}_{s,t,\lambda}$ are also regular at $s = t$. □

Because of Proposition 8.12, for $t \in 2\mathbb{Z}^{\leq 0}$ and $\lambda \in \mathcal{I}$ we can define

$$\boldsymbol{D}_{t,t,\lambda} := \sum_{i,j} \eta_{i,j}(t) \boldsymbol{C}_t^i \boldsymbol{E}_t^j = \lim_{s\to t} \boldsymbol{D}_{s,t,\lambda}. \tag{69}$$

Proposition 8.13. *$\boldsymbol{D}_{t,\lambda} = \boldsymbol{\pi}_{t,d}(\boldsymbol{D}_{t,t,\lambda})$.*

Proof. It suffices to check that $\boldsymbol{D}_{t,t,\lambda}$ satisfies the vanishing properties given in Lemma 8.10. If $t \notin 2\mathbb{Z}^{\leq 0}$ then $\mathsf{Rep}(O_t)$ is semisimple, and in particular the $\mathsf{V}_{t,\nu}$ are simple objects. It is then straightforward to check the action of $\boldsymbol{D}_{t,t,\lambda} = \boldsymbol{L}_{t,t,\lambda}$ on each $\mathsf{V}_{t,\nu}$ using (65). If $t \in 2\mathbb{Z}^{\leq 0}$ then from Lemma 8.9 and Proposition 8.12 it follows that

$$\boldsymbol{\pi}_{t,d}(\boldsymbol{D}_{t,t,\lambda}) = \lim_{s\to t} \boldsymbol{\pi}_{t,d}(\boldsymbol{D}_{s,t,\lambda}),$$

and again we can compute the action of $\boldsymbol{D}_{s,t,\lambda}$ using (65). The argument is by a case by case consideration, and we will only give the details for the most difficult case, i.e., when λ is $\underline{k}$-quasiregular. In this case $\boldsymbol{D}_{s,t,\lambda} = \boldsymbol{L}_{s,t,\lambda} + \boldsymbol{L}_{s,t,\lambda^\dagger}$ for s sufficiently close but not equal to t. Now choose $\nu \in \mathcal{I}'_{\underline{k}}$. If $|\nu| < |\lambda|$ then both $\boldsymbol{L}_{s,t,\lambda}$ and $\boldsymbol{L}_{s,t,\lambda^\dagger}$ vanish on $\mathsf{V}_{t,\nu}$ because they contain the factor $(\boldsymbol{E}_t - |\nu|)$. This implies that $\boldsymbol{D}_{s,t,\lambda}\big|_{\mathsf{V}_{t,\nu}} = 0$, hence $\boldsymbol{D}_{t,t,\lambda}\big|_{\mathsf{V}_{t,\nu}} = 0$. Next assume that $|\nu| = |\lambda|$ and set $d := |\lambda|$. If ν is $\underline{k}$-regular, then $\mathsf{V}_{t,\nu}$ is a simple object and both $\boldsymbol{L}_{s,t,\lambda}$ and $\boldsymbol{L}_{s,t,\lambda^\dagger}$ contain the factor $(\boldsymbol{C}_t - \boldsymbol{c}_\nu(s))$, which acts on $\mathsf{V}_{t,\nu}$ by $(\boldsymbol{c}_\nu(t) - \boldsymbol{c}_\nu(s))1_{\mathsf{V}_{t,\nu}}$. Since $\lim_{s\to t}(\boldsymbol{c}_\nu(t) - \boldsymbol{c}_\nu(s)) = 0$, we obtain $\boldsymbol{\pi}_{t,d}(\boldsymbol{D}_{t,t,\lambda})\big|_{\mathsf{V}_{t,\nu}} = 0$. If ν is $\underline{k}$-quasiregular and $\nu \neq \lambda$, then from (68) it follows that

$$\boldsymbol{D}_{s,t,\lambda} = (\boldsymbol{C}_t - \boldsymbol{c}_\nu(s))(\boldsymbol{C}_t - \boldsymbol{c}_{\nu^\dagger}(s))\boldsymbol{D}$$

for some $\boldsymbol{D} \in \mathsf{A}_t$ of the form $\boldsymbol{D} = \sum_{i,j} \psi_{i,j}(s)\boldsymbol{C}_t^i\boldsymbol{E}_t^j$, where the $\psi_{i,j}$ are rational functions without poles at $s = t$. Now set $\boldsymbol{N} := (\boldsymbol{C}_t - \boldsymbol{c}_{\nu^\dagger}(t))\big|_{\mathsf{V}_{t,\nu}}$. Then

$$(\boldsymbol{C}_t - \boldsymbol{c}_\nu(s))(\boldsymbol{C}_t - \boldsymbol{c}_{\nu^\dagger}(s))\big|_{\mathsf{V}_{t,\nu}} = (\boldsymbol{c}_{\nu^\dagger}(t) - \boldsymbol{c}_\nu(s) + \boldsymbol{N})(\boldsymbol{c}_{\nu^\dagger}(t) - \boldsymbol{c}_{\nu^\dagger}(s) + \boldsymbol{N}).$$

As $\boldsymbol{N}^2 = 0$ and $\lim_{s\to t}(\boldsymbol{c}_{\nu^\dagger}(t) - \boldsymbol{c}_{\nu^\dagger}(s)) = \lim_{s\to t}(\boldsymbol{c}_{\nu^\dagger}(t) - \boldsymbol{c}_\nu(s)) = 0$, we obtain $\lim_{s\to t}\boldsymbol{D}_{s,t,\lambda}\big|_{\mathsf{V}_{t,\nu}} = 0$. Finally, if $\nu = \lambda$ then from (68) it follows that $\boldsymbol{D}_{s,t,\lambda}\big|_{\mathsf{V}_{t,\nu}} = \boldsymbol{D}^{(1)}\boldsymbol{D}^{(2)}$ for

$$\boldsymbol{D}^{(1)} := \prod_{|\eta|=d,\eta\neq\lambda,\lambda^\dagger} (\boldsymbol{c}_\lambda(t) - \boldsymbol{c}_\eta(s) + \boldsymbol{N})$$

and

$$\boldsymbol{D}^{(2)} := \left(\frac{\tilde{\phi}_{1,\lambda}(s)}{\boldsymbol{c}_\lambda(s) - \boldsymbol{c}_{\lambda^\dagger}(s)}(\boldsymbol{c}_\lambda(t) - \boldsymbol{c}_{\lambda^\dagger}(s) + \boldsymbol{N}) - \frac{\tilde{\phi}_{2,\lambda}(s)}{\boldsymbol{c}_\lambda(s) - \boldsymbol{c}_{\lambda^\dagger}(s)}(\boldsymbol{c}_\lambda(t) - \boldsymbol{c}_\lambda(s) + \boldsymbol{N})\right),$$

Since $\boldsymbol{N}^2 = 0$, we have $\boldsymbol{D}^{(i)}\big|_{\mathsf{V}_{t,\nu}} = \gamma_0^{(i)}(s) + \gamma_1^{(i)}(s)\boldsymbol{N}$ for $i \in \{1, 2\}$, so that

$$\boldsymbol{D}_{s,t,\lambda}\big|_{\mathsf{V}_{t,\nu}} = \gamma_0(s) + \gamma_1(s)\boldsymbol{N}, \tag{70}$$

where $\gamma_0(s) = \gamma_0^{(1)}(s)\gamma_0^{(2)}(s)$ and $\gamma_1(s) = \gamma_0^{(1)}(s)\gamma_1^{(2)}(s) + \gamma_1^{(1)}(s)\gamma_0^{(2)}(s)$. To complete the proof we need to verify that $\lim_{s\to t}\gamma_0(s) = 1$ and $\lim_{s\to t}\gamma_1(s) = 0$.

To prove $\lim_{s\to t}\gamma_0(s) = 1$ first note that $\lim_{s\to t}\gamma_0^{(1)}(s) = \tilde{\phi}_{1,\lambda}(t)^{-1}$. Furthermore,

$$\gamma_0^{(2)}(s) = \tilde{\phi}_{1,\lambda}(s)\frac{\boldsymbol{c}_\lambda(t) - \boldsymbol{c}_{\lambda^\dagger}(s)}{\boldsymbol{c}_\lambda(s) - \boldsymbol{c}_{\lambda^\dagger}(s)} - \tilde{\phi}_{2,\lambda}(s)\frac{\boldsymbol{c}_\lambda(t) - \boldsymbol{c}_\lambda(s)}{\boldsymbol{c}_\lambda(s) - \boldsymbol{c}_{\lambda^\dagger}(s)},$$

and from $\tilde{\phi}_{1,\lambda}(t) = \tilde{\phi}_{2,\lambda}(t)$ it follows that $\lim_{s\to t}\gamma_0^{(2)}(s) = \tilde{\phi}_{1,\lambda}(t)$.

To prove $\lim_{s\to t}\gamma_1(s) = 0$, note that

$$\gamma_1(s) = \tilde{\phi}_{1,\lambda}(s)\tilde{\phi}_{2,\lambda}(s)\left(\prod_{|\nu|=d\,,\,\nu\neq\lambda,\lambda^\dagger} (\boldsymbol{c}_\lambda(t) - \boldsymbol{c}_\nu(s))\right)\left(\tilde{\phi}_3(s) + \tilde{\phi}_4(s)\right)$$

where $\tilde{\phi}_3(s) := \frac{\tilde{\phi}_{2,\lambda}(s)^{-1} - \tilde{\phi}_{1,\lambda}(s)^{-1}}{\boldsymbol{c}_\lambda(s) - \boldsymbol{c}_{\lambda^\dagger}(s)}$ and

$$\tilde{\phi}_4(s) := \left(\sum_{|\nu|=d\,,\,\nu\neq\lambda,\lambda^\dagger} \frac{1}{\boldsymbol{c}_\lambda(t) - \boldsymbol{c}_\nu(s)}\right)\left(\frac{\tilde{\phi}_{2,\lambda}(s)^{-1}(\boldsymbol{c}_\lambda(t) - \boldsymbol{c}_{\lambda^\dagger}(s)) - \tilde{\phi}_{1,\lambda}(s)^{-1}(\boldsymbol{c}_\lambda(t) - \boldsymbol{c}_\lambda(s))}{\boldsymbol{c}_\lambda(s) - \boldsymbol{c}_{\lambda^\dagger}(s)}\right).$$

From $\tilde{\phi}_{1,\lambda}(t) = \tilde{\phi}_{2,\lambda}(t)$ it follows that

$$\lim_{s\to t} \tilde{\phi}_4(s) = \tilde{\phi}_{1,\lambda}(t)^{-1} \sum_{|\nu|=d\,,\,\nu\neq\lambda,\lambda^\dagger} \frac{1}{\boldsymbol{c}_\lambda(t)-\boldsymbol{c}_\nu(t)}. \tag{71}$$

Furthermore $\boldsymbol{c}_\lambda(s) - \boldsymbol{c}_{\lambda^\dagger}(s) = (s-t)\,(2(\lambda_1-\lambda_2)+t-2)$, so that

$$\begin{aligned}\lim_{s\to 0}\tilde{\phi}_3(s) &= \lim_{s\to t}\frac{(\tilde{\phi}_{2,\lambda}(s)^{-1}-\tilde{\phi}_{2,\lambda}(t)^{-1})-(\tilde{\phi}_{1,\lambda}(s)^{-1}-\tilde{\phi}_{1,\lambda}^{-1}(t))}{\boldsymbol{c}_\lambda(s)-\boldsymbol{c}_{\lambda^\dagger}(s)}\\ &= \frac{1}{2(\lambda_1-\lambda_2)+(t-2)}\left(\frac{d}{ds}\tilde{\phi}_{2,\lambda}(s)^{-1}\big|_{s=t}-\frac{d}{ds}\tilde{\phi}_{1,\lambda}(s)^{-1}\big|_{s=t}\right).\end{aligned}$$

Now

$$\frac{d}{ds}\tilde{\phi}_{1,\lambda}(s)^{-1}\big|_{s=t} = \tilde{\phi}_{1,\lambda}(t)^{-1}\sum_{|\nu|=d\,,\,\nu\neq\lambda,\lambda^\dagger}\frac{(\lambda_1-\lambda_2)-(\nu_1-\nu_2)}{\boldsymbol{c}_\lambda(t)-\boldsymbol{c}_\nu(t)} \tag{72}$$

and

$$\frac{d}{ds}\tilde{\phi}_{2,\lambda}(s)^{-1}\big|_{s=t} = \tilde{\phi}_{2,\lambda}(t)^{-1}\sum_{|\nu|=d\,,\,\nu\neq\lambda,\lambda^\dagger}\frac{(-\lambda_1+\lambda_2-t+2)-(\nu_1-\nu_2)}{\boldsymbol{c}_\lambda(t)-\boldsymbol{c}_\nu(t)}. \tag{73}$$

From (71), (72) and (73) it follows that $\lim_{s\to t}(\tilde{\phi}_3(s)+\tilde{\phi}_4(s))=0$, hence $\lim_{s\to t}\gamma_1(s)=0$. ☐

Remark 8.14. There is a more conceptual argument for proving $\lim_{s\to t}\gamma_1(s)=0$ in (70) as follows. The construction of $\mathsf{Rep}(O_t)$ is valid over the field $\mathbb{C}(\xi)$ of rational functions in a parameter ξ, yielding a Karoubian rigid symmetric monoidal category generated by a self-dual object V_ξ of dimension ξ. Let us denote the latter category by $\mathsf{Rep}(O_\xi)$. The algebra A_t and the operators $\boldsymbol{C}_t$, $\boldsymbol{E}_t$, and $\boldsymbol{D}_{t,\lambda}$ have counterparts A_ξ, $\boldsymbol{C}_\xi$, $\boldsymbol{E}_\xi$, and $\boldsymbol{D}_{\xi,\lambda}$ in the inductive completion of $\mathsf{Rep}(O_\xi)$. For $t\in\mathbb{C}$, let $\mathscr{O}_t\subseteq\mathbb{C}(\xi)$ denote the local ring of rational functions without a pole at $\xi=t$, and let $\overline{\mathsf{A}}_\xi\subseteq\mathsf{A}_\xi$ be the $\mathscr{O}_t$-subalgebra of A_ξ generated by $\boldsymbol{C}_\xi$ and $\boldsymbol{E}_\xi$. Further, let $\mathrm{ev}_{\xi=t}:\overline{\mathsf{A}}_\xi\to\mathsf{A}_t$ be the ring homomorphism obtained by naturally extending $\mathrm{ev}_{\xi=t}(\boldsymbol{C}_\xi):=\boldsymbol{C}_t$ and $\mathrm{ev}_{\xi=t}(\boldsymbol{E}_\xi):=\boldsymbol{E}_t$. One can show that $\mathsf{Rep}(O_\xi)$ is semisimple, and it follows that the restriction of $\boldsymbol{D}_{\xi,\lambda}$ to $\mathsf{P}^{\leq d}_{\mathsf{V}_\xi}$ is an idempotent morphism. One can then use the fact that $\mathrm{ev}_{\xi=t}$ is a ring homomorphism to prove that $\mathrm{ev}_{\xi=t}(\boldsymbol{D}_{\xi,\lambda})$ is an idempotent when restricted to $\mathsf{P}^{\leq d}_{\mathsf{V}_t}$, and therefore it does not have a nonzero nilpotent part.

Lemma 8.15. *Assume that $t\in 2\mathbb{Z}^{\leq 0}$. Let $U_t\subseteq\mathbb{C}$ be an open set such that $U_t\cap\mathbb{Z}=\{t\}$. For $s\in U_t$ let $\boldsymbol{L}_s\in\mathsf{A}_s$ be defined by*

$$\boldsymbol{L}_s := \sum_{i,j=0}^{p}\psi_{i,j}(s)\boldsymbol{C}_s^i\boldsymbol{E}_s^j, \tag{74}$$

where the $\psi_{i,j}$ are rational functions without poles in U_t. Let $\boldsymbol{f}_{\boldsymbol{L}_s}$ be defined as in Definition 8.8 for $s\in U_t$. Then $\boldsymbol{f}_{\boldsymbol{L}_t}=\lim_{s\to t}\boldsymbol{f}_{\boldsymbol{L}_s}$ as elements of $\mathbb{C}[x,y]$.

Proof. Let $s\in U_t\backslash\{t\}$. Then the category $\mathsf{Rep}(O_s)$ is semisimple and by (74) we have

$$\boldsymbol{f}_{\boldsymbol{L}_s}(\mu_1+\tfrac{s}{2}-1,\mu_2)=\textstyle\sum_{i,j=0}^{p}\psi_{i,j}(s)\boldsymbol{c}_\mu(s)^i(\mu_1+\mu_2)^j\quad\text{for }\mu\in\mathcal{I}.$$

Since $\mathcal{I}$ is Zariski dense in $\mathbb{C}^2$, it follows that

$$\boldsymbol{f}_{\boldsymbol{L}_s}(x,y)=\textstyle\sum_{i,j=0}^{p}\psi_{i,j}(s)((x-y)^2-(\frac{s}{2}-1)^2)(x+y)^j.$$

In particular, the coefficients of $\boldsymbol{f}_{\boldsymbol{L}_s}(x,y)$ are rational functions without poles in U_t. Thus, the limit $\lim_{s\to t}\boldsymbol{f}_{\boldsymbol{L}_s}$ exists.

The action of $\boldsymbol{L}_t$ on $\mathsf{V}_{t,\mu}$, where $\mu \in \mathcal{I}'_{\underline{k}}$, is equal to $\sum_{i,j=0}^{p} \psi_{i,j}(t)(\boldsymbol{c}_\mu(t)+\boldsymbol{N})^i(\mu_1+\mu_2)^j$, where $\boldsymbol{N}$ is the nilpotent part of $\boldsymbol{C}_t\big|_{\mathsf{V}_{t,\mu}}$ (recall that $\boldsymbol{N}^2=0$). Thus

$$\boldsymbol{f}_{\boldsymbol{L}_t}(\mu_1-\underline{k}-1,\mu_2)=\sum_{i,j=0}^{p}\psi_{i,j}(t)\boldsymbol{c}_\mu(t)^i(\mu_1+\mu_2)^j.$$

Consequently for all $\mu \in \mathcal{I}'_{\underline{k}}$,

$$\boldsymbol{f}_{\boldsymbol{L}_t}(\mu_1-\underline{k}-1,\mu_2)=\lim_{s\to t}\boldsymbol{f}_{\boldsymbol{L}_s}\left(\mu_1+\frac{s}{2}-1,\mu_2\right)=\lim_{s\to t}\boldsymbol{f}_{\boldsymbol{L}_s}(\mu_1-\underline{k}-1,\mu_2), \tag{75}$$

where for the second equality we use the fact that the coefficients of $\boldsymbol{f}_{\boldsymbol{L}_s}(x,y)$ do not have poles in U_t. But then (75) also holds for all $\mu\in\mathcal{I}$ since both $\boldsymbol{f}_{\boldsymbol{L}_t}$ and $\lim_{s\to t}\boldsymbol{f}_{\boldsymbol{L}_s}$ are symmetric polynomials. Since $\mathcal{I}$ is Zariski dense in $\mathbb{C}^2$, we obtain $\boldsymbol{f}_{\boldsymbol{L}_t}=\lim_{s\to t}\boldsymbol{f}_{\boldsymbol{L}_s}$. □

We are now ready to prove Theorems A′–C′. Recall that $\boldsymbol{f}_\lambda := \boldsymbol{f}_{\boldsymbol{D}_{t,\lambda}}$, where the right hand side is defined as in Definition 8.8. Since $\boldsymbol{D}_{t,\lambda}\in\mathsf{A}_t^d$, Lemma 8.9 implies that $\deg \boldsymbol{f}_\lambda \le |\lambda|$.

Proof of Theorems A′–C′. First assume that $t\notin 2\mathbb{Z}^{\le 0}$. Then by Lemma 8.10(i) the polynomial $\boldsymbol{f}_\lambda$ satisfies vanishing conditions analogous to the hypotheses of Theorem 1.1, and the claim follows. Next assume that $t\in 2\mathbb{Z}^{\le 0}$. Our strategy is to reduce this case to the case $t\notin 2\mathbb{Z}^{\le 0}$. Let $\eta_{i,j}(s)$ be as in (67). Set $U_t:=\{t\}\cup(\mathbb{C}\backslash\mathbb{Z})$. By Lemma 8.9,

$$\boldsymbol{\pi}_{t,d}(\boldsymbol{D}_{s,t,\lambda})=\sum_{i,j,i',j'}\eta_{i,j}(s)\phi_{i,j,d,i',j'}(t)\boldsymbol{C}_s^i\boldsymbol{E}_s^j \quad \text{for } s\in U_t.$$

Using Proposition 8.13 and Proposition 8.12 we obtain

$$\begin{aligned}\boldsymbol{D}_{t,\lambda}&=\boldsymbol{\pi}_{t,d}(\boldsymbol{D}_{t,t,\lambda})=\lim_{s\to t}\boldsymbol{\pi}_{t,d}(\boldsymbol{D}_{s,t,\lambda})\\&=\lim_{s\to t}\sum_{i,j,i',j'}\eta_{i,j}(s)\phi_{i,j,d,i',j'}(t)\boldsymbol{C}_t^i\boldsymbol{E}_t^j=\lim_{s\to t}\sum_{i,j,i',j'}\eta_{i,j}(s)\phi_{i,j,d,i',j'}(s)\boldsymbol{C}_t^i\boldsymbol{E}_t^j,\end{aligned}$$

where in the last step we use Proposition 8.12. Set

$$\boldsymbol{L}_s:=\sum_{i,j,i',j'}\eta_{i,j}(s)\phi_{i,j,d,i',j'}(s)\boldsymbol{C}_s^i\boldsymbol{E}_s^j,$$

so that $\boldsymbol{L}_t=\boldsymbol{D}_{t,\lambda}$. Next fix $s\in U_t\backslash\{t\}$. We use the special case of Theorem A′ that was proved above to compute $\boldsymbol{f}_{\boldsymbol{L}_s}$. If λ is $\underline{k}$-regular, then $\boldsymbol{L}_s=\boldsymbol{\pi}_{s,d}(\boldsymbol{L}_{s,s,\lambda})$ and thus $\boldsymbol{f}_{\boldsymbol{L}_s}=\frac{1}{H_\lambda(-\frac{s}{2})}P_\lambda^{-\frac{s}{2}}$. If λ is $\underline{k}$-singular, then $\boldsymbol{L}_s=(\boldsymbol{c}_{\lambda^\dagger}(s)-\boldsymbol{c}_\lambda(s))\boldsymbol{\pi}_{s,d}(\boldsymbol{L}_{s,s,\lambda^\dagger})$ and thus

$$\boldsymbol{f}_{\boldsymbol{L}_s}=\frac{\boldsymbol{c}_{\lambda^\dagger}(s)-\boldsymbol{c}_\lambda(s)}{H_\lambda\left(-\frac{s}{2}\right)}P_{\lambda^\dagger}^{-\frac{s}{2}}.$$

Finally, if λ is $\underline{k}$-quasiregular, then $\boldsymbol{L}_s=\boldsymbol{\pi}_{s,d}(\boldsymbol{L}_{s,s,\lambda}+\boldsymbol{L}_{s,s,\lambda^\dagger})$ and thus

$$\boldsymbol{f}_{\boldsymbol{L}_s}=\frac{1}{H_\lambda\left(-\frac{s}{2}\right)}P_\lambda^{-\frac{s}{2}}+\frac{1}{H_{\lambda^\dagger}\left(-\frac{s}{2}\right)}P_{\lambda^\dagger}^{-\frac{s}{2}}.$$

Now Lemma 8.15 implies Theorems A′–C′.

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