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Holomorphic mappings between hyperquadrics with positive signature

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Dedicated to Professor Joseph Kohn on the occasion of his 90th birthday

Abstract: In this paper, we first give an exposition on mapping problems between indefinite hyperbolic spaces. Then we formulate a new problem along this direction, propose an approach and prove some partial results.

Keywords: Indefinite hyperbolic spaces, isometric mappings, proper holomorphic mapping.

1. Introduction

We first recall some notations and definitions. Let n, ℓ be integers such that $n \geq 2$ and $0 \leq \ell \leq n-1$. The generalized complex unit ball is defined as the following domain in $\mathbb{P}^n$:

$$\mathbb{B}_\ell^n = \{[z_0, ..., z_n] \in \mathbb{P}^n : |z_0|^2 + ... + |z_\ell|^2 > |z_{\ell+1}|^2 + ... + |z_n|^2\}.$$

In the special case of $\ell = 0$, $\mathbb{B}_0^n$ is the standard unit ball $\mathbb{B}^n$ in $\mathbb{C}^n \subset \mathbb{P}^n$. The generalized ball $\mathbb{B}_\ell^n$ carries a canonically defined indefinite metric $\omega_{\mathbb{B}_\ell^n}$ that is invariant under the action of its automorphism group $SU(\ell+1, n+1)$:

$$\omega_{\mathbb{B}_\ell^n} = -\sqrt{-1}\partial\bar{\partial}\log(\sum_{j=0}^{\ell}|z_j|^2 - \sum_{j=\ell+1}^{n}|z_j|^2).$$

The generalized ball equipped with the metric $\omega_{\mathbb{B}_\ell^n}$ is often called an indefinite hyperbolic space form. When $\ell = 0$, it is reduced to the standard hyperbolic space form (up to a normalization).

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The topological boundary $\partial\mathbb{B}^n_\ell$, is often called a generalized sphere. Its local realization is the real hyperquadric

$$\mathbb{H}^n_\ell = \{(z,w) = (z_1, \cdots, z_{n-1}, w) \in \mathbb{C}^n : \mathrm{Im} w = -\sum_{j=1}^{\ell} |z_j|^2 + \sum_{j=\ell+1}^{n-1} |z_j|^2\}$$

which serves as a basic model for Levi-nondegenerate hypersurfaces (see [4]) and plays a fundamental role in CR geometry. Note that when $\ell = 0$, $\mathbb{H}^n_0$ is the standard Heisenberg hypersurface. Due to the special geometric structure of the generalized spheres, many striking rigidity phenomena have been discovered for mappings $F : \partial\mathbb{B}^n_\ell \to \partial\mathbb{B}^N_{\ell'}$. The study of local holomorphic maps that send an open piece of $\partial\mathbb{B}^n_l$ to $\partial\mathbb{B}^N_{l'}$ with $l > 0$ was initiated by Baouendi-Huang [3]. In particular, Baouendi and the first author [3] proved a holomorphic mapping F from an open connected subset U of $\mathbb{C}^n$ to $\mathbb{C}^N (N \geq n)$, sending a piece of $\partial\mathbb{B}^n_\ell, 0 < \ell \leq \frac{n-1}{2}$, to $\partial\mathbb{B}^N_\ell$, possesses a super-rigidity property if it does not map the whole open neighborhood U into $\partial\mathbb{B}^N_\ell$. Here the mentioned super-rigidity means that the map F extends to a linear embedding of $\mathbb{P}^n$ into $\mathbb{P}^N$. This super-rigidity phenomenon in [3] contrasts with the rigidity of holomorphic mappings between Heisenberg hypersurfaces (i.e., the $0-$signature hyperquadrics) in complex spaces of different dimensions. In the $0-$signature case, the rigidity only holds when the difference in dimension is small. For instance, there is the well-known Whitney map sending $\partial\mathbb{B}^n$ to $\partial\mathbb{B}^{2n-1}$ for $n \geq 2$ (see [5]). For more results on the $0-$signature case, see [9, 10, 6] and references therein. In this paper, we concentrate on the case of $\ell > 0$. In [1], Baouendi-Ebenfelt-Huang generalized the rigidity result in [3] as follows:

Theorem 0.1 (Baouendi-Ebenfelt-Huang [1]) Let $N \geq n$, $1 \leq \ell \leq \frac{n-1}{2}$, $1 \leq \ell' \leq \frac{N-1}{2}$ and $1 \leq \ell \leq \ell' < 2l$. Let U be an open subset in $\mathbb{P}^n$ containing some $p \in \partial\mathbb{B}^n_\ell$ with $U \cap \mathbb{B}^n_\ell$ being connected, and F a holomorphic map from U into $\mathbb{P}^N$. Assume $F(U \cap \mathbb{B}^n_\ell) \subseteq \mathbb{B}^N_{\ell'}$ and $F(U \cap \partial\mathbb{B}^n_\ell) \subseteq \partial\mathbb{B}^N_{\ell'}$. Then F is an isometric embedding from $(U \cap \mathbb{B}^n_\ell, \omega_{\mathbb{B}^n_\ell})$ into $(\mathbb{B}^N_{\ell'}, \omega_{\mathbb{B}^N_{\ell'}})$.

Here we say F is isometric if it preserves the indefinite hyperbolic metrics: $F^*(\omega_{\mathbb{B}^N_{\ell'}}) = \omega_{\mathbb{B}^n_\ell}$ on $U \cap \mathbb{B}^n_\ell$. For many closely related results along these lines, the readers are referred to the papers [7, 8, 14, 15, 16, 17, 18, 19, 20, 21] and references therein. In particular, by analyzing the structure of the moduli space of linear subspaces contained in generalized balls, Ng [17] establishes the global version of Theorem 0.1.

Theorem 0.2 (Ng [17]) Let $1 \leq \ell < \frac{n}{2}, 1 \leq \ell' < \frac{N}{2}$ and $f : \mathbb{B}_\ell^n \to \mathbb{B}_{\ell'}^N$ be a proper holomorphic map. If $\ell' \leq 2\ell - 1$, then f extends to a linear embedding of $\mathbb{P}^n$ into $\mathbb{P}^N$.

In a recent paper [12], the authors and Lu-Tang induced a boundary CR invariant— geometric rank for holomorphic mappings between hyperquadrics of positive signatures. Then we gave a complete characterization for local holomorphic isometric embeddings between indefinite hyperbolic spaces in terms of this geometric rank.

Theorem 0.3 (Huang-Lu-Tang-Xiao [12]) Let $N \geq n \geq 3, 0 \leq \ell \leq n-1, \ell \leq \ell' \leq N-1$. Let U be an open subset in $\mathbb{P}^n$ containing some $p \in \partial\mathbb{B}_\ell^n$ and F be a holomorphic map from U into $\mathbb{P}^N$. Assume that $U \cap \mathbb{B}_\ell^n$ is connected and $F(U \cap \mathbb{B}_\ell^n) \subset \mathbb{B}_{\ell'}^N$, $F(U \cap \partial\mathbb{B}_\ell^n) \subset \partial\mathbb{B}_{\ell'}^N$. Then the following are equivalent.
(1) F is CR transversal and has geometric rank zero at generic points on $U \cap \partial\mathbb{B}_\ell^n$ near p.
(2) F is an isometric embedding from $(U \cap \mathbb{B}_\ell^n, \omega_{\mathbb{B}_\ell^n})$ to $(\mathbb{B}_{\ell'}^N, \omega_{\mathbb{B}_{\ell'}^N})$.

In a preprint [13], the authors and Lu-Tang use the above characterization to generalize the aforementioned results in [1] and [17] as follows:

Theorem 0.4 (Huang-Lu-Tang-Xiao [13]) Let $N \geq n \geq 3$, $1 \leq \ell \leq n-2, \ell \leq \ell' \leq N-1$. Let U be an open subset in $\mathbb{P}^n$ containing some $p \in \partial\mathbb{B}_\ell^n$ and F be a holomorphic map from U into $\mathbb{P}^N$. Assume $U \cap \mathbb{B}_\ell^n$ is connected and $F(U \cap \mathbb{B}_\ell^n) \subseteq \mathbb{B}_{\ell'}^N$, $F(U \cap \partial\mathbb{B}_\ell^n) \subseteq \partial\mathbb{B}_{\ell'}^N$. Assume one of the following conditions holds:

(1). $\ell' < 2\ell,\ \ell' < n-1;$
(2). $\ell' < 2\ell, N - \ell' < n;$
(3). $N - l' < 2n - 2\ell - 1,\ \ell' < n-1;$
(4). $N - l' < 2n - 2\ell - 1, N - \ell' < n.$

Then F is an isometric embedding from $(U \cap \mathbb{B}_\ell^n, \omega_{\mathbb{B}_\ell^n})$ to $(\mathbb{B}_{\ell'}^N, \omega_{\mathbb{B}_{\ell'}^N})$.

Corollary 0.5 (Huang-Lu-Tang-Xiao [13]) Let $N \geq n \geq 3$, $1 \leq \ell \leq n-2, \ell \leq \ell' \leq N-1$. Assume one of the conditions (1)–(4) in Theorem 0.4

holds. Let F be a rational proper map from $\mathbb{B}^n_\ell$ to $\mathbb{B}^N_{\ell'}$. Then F is a linear embedding from $\mathbb{P}^n$ to $\mathbb{P}^N$. Moreover, there exists $h \in Aut(\mathbb{B}^N_{\ell'})$ such that

$$h \circ F([z]) = [z_0, ..., z_\ell, 0, ..., 0, z_{\ell+1}, ..., z_n, 0, ..., 0],$$

for $[z] = [z_0, ..., z_\ell, z_{\ell+1}, ..., z_n] \in \mathbb{P}^n$, where the first zero tuple has $\ell' - l$ components.

Here for a holomorphic rational map F from $\mathbb{P}^n$ to $\mathbb{P}^N$ with $I \subseteq \mathbb{P}^n$ its set of indeterminacy, we say F is a rational proper map from $\mathbb{B}^n_\ell$ to $\mathbb{B}^N_{\ell'}$, if F maps from $\mathbb{B}^n_\ell \setminus I$ to $\mathbb{B}^N_{\ell'}$ and maps $\partial\mathbb{B}^n_\ell \setminus I$ to $\partial\mathbb{B}^N_{\ell'}$.

If none of the conditions (1)–(4) holds, then one of the following two cases must hold: (A). $\ell' \geq 2\ell$ and $N - \ell' \geq 2n - 2\ell - 1$; (B). $N - \ell' \geq n$ and $\ell' \geq n - 1$. The following examples show that Theorem 0.4 and Corollary 0.5 are in a sense optimal.

Example 1.1. *(Generalized Whitney map from $\mathbb{B}^{\ell+k}_\ell$ to $\mathbb{B}^{2\ell+2k-1}_{2\ell}$) Let $\ell \geq 1, k \geq 1$. Write $[w, z] = [w_0, w_1, \cdots, w_\ell, z_1, \cdots, z_k]$ for the homogeneous coordinates of $\mathbb{P}^{\ell+k}$ and*

$$\mathbb{B}^{\ell+k}_\ell = \{[w, z] \in \mathbb{P}^{k+\ell} : \sum_{i=0}^{\ell} |w_i|^2 > \sum_{j=1}^{k} |z_j|^2\}.$$

Write $U = \mathbb{P}^{k+\ell} \setminus \{w_0 = z_k = 0\}$. Consider the following map $G : U \to \mathbb{P}^{2k+2\ell-1}$:

$$G([w, z]) = [w_0^2, w_0w_1, \cdots, w_0w_\ell, w_1z_k, \cdots, w_\ell z_k, \\ w_0z_1, w_0z_2, \cdots, w_0z_{k-1}, z_1z_k, z_2z_k, \cdots, z_{k-1}z_k, z_k^2].$$

Notice that $|G|^2_{2\ell+1} = (|w_0|^2 + |z_k|^2)(-\sum_{i=0}^{\ell} |w_i|^2 + \sum_{j=1}^{k} |z_j|^2)$. Consequently, G maps $U \cap \mathbb{B}^{\ell+k}_\ell$ to $\mathbb{B}^{2\ell+2k-1}_{2\ell}$ and maps $U \cap \partial\mathbb{B}^{\ell+k}_\ell$ to $\partial\mathbb{B}^{2\ell+2k-1}_{2\ell}$.

Example 1.2. *(Generalized Whitney map from $\mathbb{B}^{\ell+k}_\ell$ to $\mathbb{B}^{2\ell+2k-1}_{\ell+k-1}$) Let $\ell \geq 1, k \geq 1$. Let the homogeneous coordinates $[w, z]$ of $\mathbb{P}^{\ell+k}$ and $\mathbb{B}^{\ell+k}_\ell \subseteq \mathbb{P}^{\ell+k}$ be the same as in Example 1.1. Let $V = \mathbb{P}^{\ell+k} \setminus \{w_0 = w_\ell = 0\}$ and $H : V \to \mathbb{P}^{2k+2\ell-1}$ be defined as follows:*

$$H([w, z]) = [w_0^2, w_0w_1, \cdots w_0w_{\ell-1}, w_\ell z_1, w_\ell z_2, \cdots, w_\ell z_k, \\ w_0z_1, w_0z_2, \cdots, w_0z_k, w_1w_\ell, w_2w_\ell, \cdots, w_\ell^2].$$

Notice that $|H|^2_{\ell+k} = (|w_0|^2 - |w_\ell|^2)(-\sum_{i=0}^{\ell} |w_i|^2 + \sum_{j=1}^{k} |z_j|^2)$*. Thus* H *maps* $V \cap \partial\mathbb{B}_\ell^{\ell+k}$ *to* $\partial\mathbb{B}_{\ell+k-1}^{2\ell+2k-1}$*. In particular, set* $V_+ := \{[w,z] \in V : |w_0| > |w_\ell|\}$*. Then* H *maps* $V_+ \cap \mathbb{B}_\ell^{\ell+k}$ *to* $\mathbb{B}_{\ell+k-1}^{2\ell+2k-1}$ *and maps* $V_+ \cap \partial\mathbb{B}_\ell^{\ell+k}$ *to* $\partial\mathbb{B}_{\ell+k-1}^{2\ell+2k-1}$*.*

Example 1.3. *(Generalized Whitney map from* $\mathbb{B}_\ell^{\ell+k}$ *to* $\mathbb{B}_{\ell+k-1}^{2\ell+2k-1}$*). Let* $\ell \geq 0, k \geq 2$*. Let the homogeneous coordinates* $[w,z]$ *of* $\mathbb{P}^{\ell+k}$ *and* $\mathbb{B}_\ell^{\ell+k} \subseteq \mathbb{P}^{\ell+k}$ *be the same as in Example 1.1. Let* $V = \mathbb{P}^{\ell+k} \setminus \{z_1 = z_k = 0\}$ *and* $H : V \to \mathbb{P}^{2k+2\ell-1}$ *be defined as follows:*

$$H([w,z]) = [w_0 z_k, w_1 z_k, \cdots, w_\ell z_k, z_1^2, z_1 z_2, \cdots, z_1 z_{k-1}, \\ z_2 z_k, z_3 z_k, \cdots, z_k^2, w_0 z_1, w_1 z_1, \cdots, w_\ell z_1].$$

Notice that $|H|^2_{\ell+k} = (|z_k|^2 - |z_1|^2)(-\sum_{i=0}^{\ell} |w_i|^2 + \sum_{j=1}^{k} |z_j|^2)$*. Thus* H *maps* $V \cap \partial\mathbb{B}_\ell^{\ell+k}$ *into* $\partial\mathbb{B}_{\ell+k-1}^{2\ell+2k-1}$*. In particular, set* $V_+ := \{[w,z] \in V : |z_k| > |z_1|\}$*. Then* H *maps* $V_+ \cap \mathbb{B}_\ell^{l+k}$ *to* $\mathbb{B}_{\ell+k-1}^{2\ell+2k-1}$ *and maps* $V_+ \cap \partial\mathbb{B}_\ell^{\ell+k}$ *to* $\partial\mathbb{B}_{\ell+k-1}^{2\ell+2k-1}$*.*

It is then a natural question to classify holomorphic maps that send a piece of $\partial\mathbb{B}_\ell^n$ to $\partial\mathbb{B}_{\ell'}^{2n-1}$. Inspired by the above results and examples, we make the following conjecture:

Conjecture 1.4. *Let* $n \geq 3, 1 \leq \ell \leq \frac{n-1}{2},\ \ell \leq \ell' \leq 2n-2$*. Let* U *be an open subset in* $\mathbb{P}^n$ *containing some* $p \in \partial\mathbb{B}_\ell^n$ *and* F *be a holomorphic map from* U *into* $\mathbb{P}^{2n-1}$*. Assume* $U \cap \mathbb{B}_\ell^n$ *is connected and* $F(U \cap \mathbb{B}_\ell^n) \subseteq \mathbb{B}_{\ell'}^{2n-1}, F(U \cap \partial\mathbb{B}_\ell^n) \subseteq \partial\mathbb{B}_{\ell'}^{2n-1}$*. Then one of the following holds:*

(1) F *is an isometric embedding from* $(U \cap \mathbb{B}_\ell^n, \omega_{\mathbb{B}_\ell^n})$ *to* $(\mathbb{B}_{\ell'}^{2n-1}, \omega_{\mathbb{B}_{\ell'}^{2n-1}})$*.*
(2) After composing appropriate automorphisms of $\mathbb{B}_\ell^n$ *and* $\mathbb{B}_{\ell'}^{2n-1}$*,* F *equals the generalized Whitney map in Example 1.1.*
(3) After composing appropriate automorphisms of $\mathbb{B}_\ell^n$ *and* $\mathbb{B}_{\ell'}^{2n-1}$*,* F *equals the generalized Whitney map in Example 1.2.*
(4) After composing appropriate automorphisms of $\mathbb{B}_\ell^n$ *and* $\mathbb{B}_{\ell'}^{2n-1}$*,* F *equals the generalized Whitney map in Example 1.3.*

In the case of (2) and (3)-(4), we have $\ell' = 2\ell$ *and* $\ell' = n-1$*, respectively.*

Note the special case of $\ell = \ell' = 0$, Conjecture 1.4 was confirmed by the work of the first author and Ji [11]. To tackle the conjecture in its full generality, motivated by the approach in [11], we propose to first understand the geometric rank of the map F (see §2.1 for the notion of the geometric rank). In this paper we prove some partial results along these lines by investigating the geometric rank of a holomorphic map F sending a piece of $\partial\mathbb{B}_\ell^n$ to $\partial\mathbb{B}_{\ell'}^N$ when the difference of ℓ and ℓ' is not too large. More precisely, we prove the following result.

Theorem 1.5. *Let $N \geq n \geq 3, 2 \leq \ell \leq \frac{n-1}{2}, \ell \leq \ell' \leq N-1$. Let U be an open subset in $\mathbb{P}^n$ containing some $q_0 \in \partial\mathbb{B}^n_\ell$ and F be a holomorphic map from U into $\mathbb{P}^N$. Assume $F(U \cap \mathbb{B}^n_\ell) \subset \mathbb{B}^N_{\ell'}, F(U \cap \partial\mathbb{B}^n_\ell) \subset \partial\mathbb{B}^N_{\ell'}$. Furthermore, assume F is CR transversal at q_0 and assume $\ell' \leq 3\ell - 2$. Then the geometric rank of F equals either 0 or 1 at every point sufficiently close to q_0.*

The paper is organized as follows. In §2.1, we recall some preliminaries and the definition of geometric rank from [12]. We establish a lemma on the Hermitian rank of real polynomials in §2.2. Then in §2.3, we use this lemma to prove Theorem 1.5.

2. Proof of Theorem 1.5

2.1. Preliminaries

We first recall some notations from [12] which will be needed in the proof. Given a fixed $\ell \geq 1$, we denote by $\delta_{j,\ell}$ the symbol which takes value -1 when $1 \leq j \leq \ell$ and 1 otherwise. For fixed integers $\ell' \geq \ell \geq 1$ and $n \geq 1$, we denote by $\delta_{j,\ell,\ell',n}$ the symbol which takes value -1 when $1 \leq j \leq \ell$ or $n \leq j \leq n + \ell' - \ell - 1$ and 1 otherwise. When $\ell' = \ell, \delta_{j,\ell,\ell',n}$ is the same as $\delta_{j,\ell}$. Let $m \geq 1$. For two m-tuples $x = (x_1, \cdots, x_m)$, $y = (y_1, \cdots, y_m)$ of complex numbers, we write $\langle x, y\rangle_\ell = \sum_{j=1}^m \delta_{j,\ell} x_j y_j$, and $|x|^2_\ell = \langle x, \bar{x}\rangle_\ell$. Also write $\langle x, y\rangle_{\ell,\ell',n} = \sum_{j=1}^m \delta_{j,\ell,\ell',n} x_j y_j$ and $|x|^2_{\ell,\ell',n} = \langle x, \bar{x}\rangle_{\ell,\ell',n}$. Note if $m \leq n-1$, the two symbols $\langle\cdot,\cdot\rangle_\ell$ and $\langle\cdot,\cdot\rangle_{\ell,\ell',n}$ are identical.

For $0 \leq \ell \leq n-1$, we define the generalized Siegel upper-half space

$$\mathbb{S}^n_\ell = \{(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C} : \mathrm{Im}(w) > -\sum_{j=1}^{\ell} |z_j|^2 + \sum_{j=\ell+1}^{n-1} |z_j|^2\}.$$

The boundary of $\mathbb{S}^n_\ell$ is the standard hyperquadrics: $\mathbb{H}^n_\ell = \{(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C} : \mathrm{Im}(w) = \sum_{j=1}^{n-1} \delta_{j,\ell} |z_j|^2\}$. We also define for $\ell \leq \ell' \leq N-1$

$$\mathbb{S}^N_{\ell,\ell',n} = \{(z, w) \in \mathbb{C}^{N-1} \times \mathbb{C} : \mathrm{Im}(w) > \sum_{j=1}^{N-1} \delta_{j,\ell,\ell',n} |z_j|^2\}.$$

We similarly define $\mathbb{S}^N_{\ell'}, \mathbb{H}^N_{\ell'}, \mathbb{H}^N_{\ell,\ell',n}$. Now for $(z, w) = (z_1, \cdots, z_{n-1}, w) \in \mathbb{C}^n$, let $\Psi_n(z, w) = [i + w, 2z, i - w] \in \mathbb{P}^n$. Then Ψ_n is the Cayley transformation which biholomorphically maps the generalized Siegel upper-half space $\mathbb{S}^n_\ell$ and its boundary $\mathbb{H}_\ell^n$ onto $\mathbb{B}_\ell^n \setminus \{[z_0, \cdots, z_n] : z_0 + z_n = 0\}$ and $\partial\mathbb{B}_\ell^n \setminus \{[z_0, \cdots, z_n] : z_0 + z_n = 0\}$, respectively.

Note that $\mathbb{H}_{\ell,\ell',n}^N$ is identical to $\mathbb{H}_{\ell'}^N$ when $\ell' = \ell$. When $\ell' > \ell$, $\mathbb{H}_{\ell'}^N$ is holomorphically equivalent to $\mathbb{H}_{\ell,\ell',n}^N$ by a permutation of coordinates in $\mathbb{C}^N$. We will more often work with $\mathbb{H}_{\ell,\ell',n}^N$ instead of $\mathbb{H}_{\ell'}^N$, as it makes notations simpler.

We will write $\mathrm{Aut}(\mathbb{H}_\ell^n)$ and $\mathrm{Aut}_0(\mathbb{H}_\ell^n)$ for the (holomorphic) automorphism group of $\mathbb{H}_\ell^n$ and the local isotropy group of $\mathbb{H}_\ell^n$ at 0, respectively. Write $\mathrm{Aut}^+(\mathbb{H}_\ell^n)$ and $\mathrm{Aut}_0^+(\mathbb{H}_\ell^n)$ for the automorphisms in $\mathrm{Aut}(\mathbb{H}_\ell^n)$ and $\mathrm{Aut}_0(\mathbb{H}_\ell^n)$, respectively, that in addition preserves sides (that is, maps $\mathbb{S}_\ell^n$ to $\mathbb{S}_\ell^n$). Clearly they are subgroups of $\mathrm{Aut}(\mathbb{H}_\ell^n)$ and $\mathrm{Aut}_0(\mathbb{H}_\ell^n)$, respectively. We define $\mathrm{Aut}(\mathbb{H}_{\ell,\ell',n}^N)$, $\mathrm{Aut}_0(\mathbb{H}_{\ell,\ell',n}^N)$ and $\mathrm{Aut}^+(\mathbb{H}_{\ell,\ell',n}^N)$ and $\mathrm{Aut}_0^+(\mathbb{H}_{\ell,\ell',n}^N)$ similarly.

Recall we denote by $(z, w) = (z_1, \cdots, z_{n-1}, w)$ the coordinates of $\mathbb{C}^n$. Write u for the real part of w and write

$$L_j := 2i\delta_{j,\ell}\bar{z}_j\frac{\partial}{\partial w} + \frac{\partial}{\partial z_j},\ 1 \le j \le n-1, \quad T := \frac{\partial}{\partial u}. \tag{2.1}$$

Then $\{L_1, \cdots, L_{n-1}\}$ forms a global basis for the CR tangent bundle $T^{(1,0)}\mathbb{H}_\ell^n$ of $\mathbb{H}_\ell^n$, where T is a tangent vector field of $\mathbb{H}_\ell^n$ transversal to $T^{(1,0)}\mathbb{H}_\ell^n \oplus T^{(0,1)}\mathbb{H}_\ell^n$.

Let $F = (\tilde{f}, g) = (f, \phi, g) = (f_1, \cdots, f_{n-1}, \phi_1, \cdots, \phi_{N-n}, g)$ be a holomorphic map from a neighborhood U of $p_0 \in \mathbb{H}_\ell^n$ into $\mathbb{C}^N$, satisfying $F(U \cap \mathbb{S}_\ell^n) \subset \mathbb{S}_{\ell,\ell',n}^N$ and $F(U \cap \mathbb{H}_\ell^n) \subset \mathbb{H}_{\ell,\ell',n}^N$. We additionally assume $M_1 := U \cap \mathbb{H}_\ell^n$ is connected and F is CR transversal on M_1. We will define the geometric rank for such a map F as follows:

First for each $p \in M_1$, we associate it with a map F_p defined by

$$F_p = \tau_p^F \circ F \circ \sigma_p^0 = (\tilde{f}_p, g_p) = (f_p, \phi_p, g_p). \tag{2.2}$$

Here for each $p = (z_0, w_0) \in M_1$, we write $\sigma_{(z_0,w_0)}^0 \in \mathrm{Aut}^+(\mathbb{H}_\ell^n)$ for the map

$$\sigma_{(z_0,w_0)}^0(z, w) = (z + z_0, w + w_0 + 2i\langle z, \bar{z_0}\rangle_\ell),$$

and define $\tau_{(z_0,w_0)}^F \in \mathrm{Aut}^+(\mathbb{H}_{\ell,\ell',n}^n)$ by

$$\tau_{(z_0,w_0)}^F(\xi, \eta) = (\xi - \tilde{f}(z_0, w_0), \eta - \overline{g(z_0, w_0)} - 2i\langle \xi, \overline{\tilde{f}(z_0, w_0)}\rangle_{\ell,\ell',n}).$$

Then F_p is a holomorphic map in a neighborhood of $0 \in \mathbb{C}^n$, which sends an open piece of $\mathbb{H}_\ell^n$ into $\mathbb{H}_{\ell,\ell',n}^N$ with $F_p(0) = 0$. Moreover, $F(U \cap \mathbb{S}_\ell^n) \subset \mathbb{S}_{\ell,\ell',n}^N$.

Note the fundamental commutator identities hold:

$$
\begin{aligned}
&[\bar{L}_j, L_j] = 2i\delta_{j,\ell}(\frac{\partial}{\partial w} + \frac{\partial}{\partial \bar{w}}) = 2i\delta_{j,\ell}\frac{\partial}{\partial u}, \quad 1 \le j \le n-1; \\
&[\bar{L}_j, L_k],\ [T, L_k],\ [L_j, L_k],\ [\overline{L}_k, \overline{L_k}] = 0, \quad \text{if} \quad 1 \le j \ne k \le n-1.
\end{aligned} \tag{2.3}
$$

By the assumption that $F(U \cap M_1) \subset \mathbb{H}^N_{\ell,\ell',n}$, we have

$$
\text{Im } g = \langle \tilde{f}, \bar{\tilde{f}} \rangle_{\ell,\ell',n} \quad \text{on} \quad M_1. \tag{2.4}
$$

In the following, for a holomorphic map $h = (h_1, \cdots, h_K)$ from $\mathbb{C}^n$ to $\mathbb{C}^K$, we write $h'_{z_j} = (\frac{\partial h_1}{\partial z_j}, \cdots, \frac{\partial h_K}{\partial z_j}), h''_{wz_j} = h''_{z_j w} = (\frac{\partial^2 h_1}{\partial w \partial z_j}, \cdots, \frac{\partial^2 h_K}{\partial w \partial z_j}), 1 \le j \le n-1$. The notations $h'_w, h''_{z_j z_k}, h''_{ww}$ are understood similarly. We apply $\bar{L}_j L_j$ to (2.4) and obtain

$$
\lambda(p) := (g_p)_w(0) = g_w(p) - 2i\langle \tilde{f}'_w(p), \overline{\tilde{f}(p)} \rangle_{\ell,\ell',n} = \delta_{j,\ell}\langle L_j(\tilde{f}), \overline{L_j(\tilde{f})} \rangle_{\ell,\ell',n}(p), \tag{2.5}
$$

Note this implies $\lambda(p)$ is a real number. Recall that the CR-transervsality assumption is equivalent to $\lambda(p) \ne 0$ (see for example, [3]). Furthermore, since F_p preserves the sides, we have $\lambda(p) > 0$ (see e.g. page 396 in [3]).

We apply $\bar{L}_k, L_j, j \ne k$ to (2.4) and get $\langle L_j(\tilde{f}), \overline{L_k(\tilde{f})} \rangle_{\ell,\ell',n} |_p = 0$. Let for $1 \le j \le n-1$,

$$
\begin{aligned}
E_j(p) &:= (\frac{\partial \tilde{f}_p}{\partial z_j}) |_0 = (\frac{\partial f_{p,1}}{\partial z_j}, \cdots, \frac{\partial f_{p,n-1}}{\partial z_j}, \frac{\partial \phi_{p,1}}{\partial z_j} \cdots, \frac{\partial \phi_{p,N-n}}{\partial z_j}) |_0 = L_j(\tilde{f})(p); \\
E_w(p) &:= (\frac{\partial \tilde{f}_p}{\partial w}) |_0 = (\frac{\partial f_{p,1}}{\partial w}, \cdots, \frac{\partial f_{p,n-1}}{\partial w}, \frac{\partial \phi_{p,1}}{\partial w} \cdots, \frac{\partial \phi_{p,N-n}}{\partial w}) |_0 = T(\tilde{f})(p).
\end{aligned}
$$

Then

$$
\langle E_j(p), \overline{E_j(p)} \rangle_{l,l',n} = \delta_{j,l}\lambda(p), \quad \langle E_j(p), \overline{E_k(p)} \rangle_{\ell,\ell',n} = 0, \quad 1 \le j \ne k \le n-1. \tag{2.6}
$$

Write E for the $(n-1)\times(N-1)$ matrix whose j^{th} row is $\frac{E_j(p)}{\sqrt{\lambda(p)}}, 1 \le j \le n-1$. Then E satisfies $E I_{\ell,\ell',n,N-1} \bar{E}^t = I_{\ell,n-1}$. Here $I_{\ell,m}$ denotes the $m \times m$ diagonal matrix whose j^{th} diagonal element equals to $\delta_{j,\ell}, 1 \le j \le m$. Similarly, $I_{\ell,\ell',n,m}$ denotes the $m \times m$ diagonal matrix whose j^{th} diagonal element equals to $\delta_{j,\ell,\ell',n}, 1 \le j \le m$.

As in [3], we can choose $(N-1)$-dimensional row vectors $C_1(p), \cdots, C_{N-n}(p)$

such that if we write

$$A(p) = \begin{pmatrix} \frac{E_1(p)}{\sqrt{\lambda(p)}} \\ \cdots \\ \frac{E_{n-1}(p)}{\sqrt{\lambda(p)}} \\ C_1(p) \\ \cdots \\ C_{N-n}(p) \end{pmatrix}$$

then

$$A(p) I_{\ell,\ell',n,N-1} \overline{A(p)}^t = I_{\ell,\ell',n,N-1}, \text{ i.e., } A(p) \in U(\ell,\ell',n,N-1). \tag{2.7}$$

Here recall $U(\ell,\ell',n,m) = \{T \in GL(m,\mathbb{C}) : T I_{\ell,\ell',n,m} \bar{T}^t = I_{\ell,\ell',n,m}\}$. Note that one can choose $C_j(p)$'s in such a way that $A(p)$ is smooth in p for $p \approx p_0$ by the standard Gram-Schmidt process.

Next note $B(p) := A^{-1}(p) = I_{\ell,\ell',n,N-1} \overline{A(p)}^t I_{\ell,\ell',n,N-1}$ is also in $U(\ell,\ell',n,N-1)$. Write

$$B(p) = (B_1(p), \cdots, B_{n-1}(p), \hat{B}_n(p), \cdots, \hat{B}_{N-1}(p)),$$

where $B_j(p)'$s and $\hat{B}_i(p)'$s are $(N-1)$-dimensional column vectors. Note $B_1(p), \cdots, B_{n-1}(p)$ only depend on $E_1(p), \cdots, E_{n-1}(p)$. Indeed,we have

$$(B_1(p), \cdots, B_{n-1}(p)) = I_{\ell,\ell',n,N-1} \left(\frac{\overline{E_1(p)}^t}{\sqrt{\lambda(p)}}, \cdots, \frac{\overline{E_{n-1}(p)}^t}{\sqrt{\lambda(p)}} \right) I_{\ell,n-1}. \tag{2.8}$$

Define $F_p^* = (\tilde{f}_p^*, g_p^*) = ((f_p^*)_1, \cdots, (f_p^*)_{n-1}, (\phi_p^*)_1, \cdots, (\phi_p^*)_{N-n}, g_p^*)$ by

$$F_p^* = \frac{1}{\sqrt{\lambda(p)}} F_p \begin{pmatrix} B(p) & 0 \\ 0 & \frac{1}{\sqrt{\lambda(p)}} \end{pmatrix}. \tag{2.9}$$

Then F_p^* is a holomorphic map in a neighborhood of $0 \in \mathbb{C}^n$, which sends an open piece of $\mathbb{H}_\ell^n$ into $\mathbb{H}_{\ell,\ell',n}^N$ with $F_p^*(0) = 0$ and the following holds (See [3], [1] for more details).

$$\begin{cases} f_p^* = z + O(|w| + |(z,w)|^2) \\ \phi_p^* = O(|w| + |(z,w)|^2) \\ g_p^* = w + O(|(z,w)|^2). \end{cases}$$

Let

(2.10)

$$a(p)=(a_1(p),\cdots,a_{n-1}(p),a_n(p),\cdots,a_{N-1}(p)):=\frac{\partial \tilde{f}_p^*}{\partial w}(0)=\frac{1}{\sqrt{\lambda(p)}}E_w(p)B(p).$$

Note

(2.11)

$$a_k(p)=\frac{1}{\sqrt{\lambda(p)}}E_w(p)B_k(p)\ \text{ for }\ 1\le k\le n-1,\ \text{and}\ |a(p)|^2_{\ell,\ell',n}=\frac{1}{\lambda(p)}|E_w(p)|^2_{\ell,\ell',n}.$$

Set for $1\le k,j\le n-1$,

$$d_{kj}(p):=\frac{\partial^2(f_p^*)_k}{\partial z_j\partial w}\ |_0=\frac{1}{\sqrt{\lambda(p)}}(\tilde{f}_p)''_{wz_j}(0)B_k(p)=\frac{1}{\sqrt{\lambda(p)}}L_j(\tilde{f}'_w)(p)B_k(p),$$

$$c_k(p):=\frac{\partial^2 g_p^*}{\partial z_k\partial w}\ |_0=\frac{1}{\lambda(p)}(g_p)''_{wz_k}(0)=\frac{1}{\lambda(p)}L_k(g'_w-2i\langle\tilde{f}'_w,\overline{\tilde{f}(p)}\rangle_{\ell,\ell',n,N})\ |_p,$$

$$r(p):=\frac{1}{2}\mathrm{Re}\Big(\frac{\partial^2 g_p^*}{\partial w^2}\Big)\ |_0=\frac{1}{2\lambda(p)}\mathrm{Re}((g_p)''_{ww}(0))=\frac{1}{2\lambda(p)}\mathrm{Re}(g''_{ww}-2i\langle\tilde{f}''_{ww},\overline{\tilde{f}(p)}\rangle_{\ell,\ell',n,N})\ |_p\ .$$

Write $(\xi,\eta)=(\xi_1,\cdots,\xi_{N-1},\eta)$ for the coordinates of $\mathbb{C}^N$ and define

$$G_p(\xi,\eta)=\Big(\frac{\xi-a(p)\eta}{Q_p(\xi,\eta)},\ \frac{\eta}{Q_p(\xi,\eta)}\Big), \tag{2.12}$$

where $Q_p(\xi,\eta)=1+2i\langle\xi,\overline{a(p)}\rangle_{\ell,\ell',n}+\big(r(p)-i\langle a(p),\overline{a(p)}\rangle_{\ell,\ell',n}\big)\eta$. Then $G_p\in \mathrm{Aut}_0^+(\mathbb{H}^N_{\ell,\ell',n})$. Let F_p^{**} be the composition of F_p^* with G_p :

$$F_p^{**}=(\tilde{f}_p^{**},g_p^{**})=(f_p^{**},\phi_p^{**},g_p^{**}):=G_p\circ F_p^*. \tag{2.13}$$

Here f_p^{**} has $n-1$ components, and ϕ_p^{**} has $N-n$ components. Next we recall some notations (from [9, 10] and [3]) for functions of weighted degree that will be used in the remaining context of the paper. We assign the weight of z to be 1, and assign the weight of u and w to be 2. We say a smooth function $h(z,\bar{z},u)$ on $U\cap\mathbb{H}^n_\ell$ is of quantity $O_{wt}(s)$ for $0\le s\in\mathbb{N}$, if $\left|\frac{h(tz,t\bar{z},t^2u)}{t^s}\right|$ is bounded for (z,u) on any compact subset of $U\cap\mathbb{H}^n_l$ and t close to 0. Similarly, we say h is of quantity $o_{wt}(s)$ for $0\le s\in\mathbb{N}$, if $\left|\frac{h(tz,t\bar{z},t^2u)}{t^s}\right|$ converges to 0 uniformly for (z,u) on any compact subset of $U\cap\mathbb{H}^n_\ell$ as t goes to 0.

In general, for a smooth function $h(z,\bar{z},u)$ on $U\cap\mathbb{H}^n_\ell$, we denote $h^{(k)}(z,\bar{z},u)$ the sum of terms of weighted degree k in the Taylor expansion of h at 0. And

$h^{(k)}(z,\bar{z},u)$ also sometimes denotes a weighted homogeneous polynomial of degree k, if h is not specified. When $h^{(k)}(z,\bar{z},u)$ extends to a holomorphic polynomial of weighted degree k, we write it as $h^{(k)}(z,w)$ or $h^{(k)}(z)$ if it depends only on z.

Under the notations above, by Lemma 2.2 in [3], we have the following normalization and CR Gauss-Codazzi equation. Here recall $(z,w) = (z_1,\cdots,z_{n-1},w)$ denotes the coordinates in $\mathbb{C}^n = \mathbb{C}^{n-1}\times\mathbb{C}$.

Lemma 2.1. *For each $p\in M$, F_p^{**} satisfies the normalization condition:*

$$\begin{cases} f_p^{**} = z + \frac{i}{2}a_p^{**(1)}(z)w + O_{wt}(4) \\ \phi_p^{**} = \phi_p^{**(2)}(z) + O_{wt}(3) \\ g_p^{**} = w + O_{wt}(5), \end{cases}$$

with

$$\langle \bar{z}, a_p^{**(1)}(z)\rangle_\ell |z|_\ell^2 = |\phi_p^{**(2)}(z)|_\tau^2,\ \tau = \ell' - l. \tag{2.14}$$

By [12], if we write $a_p^{**(1)}(z) = z\mathcal{A}(p)$ for any $(n-1)\times(n-1)$ matrix $\mathcal{A}(p)$, then the geometric rank of F at p is defined as the rank of the matrix $\mathcal{A}(p)$. See more details of the definition in [12].

We next recall the definition of geometric rank for maps between generalized spheres. Let F be a holomorphic map from a small neighborhood U of $q\in\partial\mathbb{B}_\ell^n$ to $\mathbb{C}^N$. Assume $F(U\cap\mathbb{B}_\ell^n)\subset\mathbb{B}_{\ell'}^N$ and $F(U\cap\partial\mathbb{B}_\ell^n)\subset\partial\mathbb{B}_{\ell'}^N$, and in addition F is CR-transversal along $U\cap\partial\mathbb{B}_\ell^n$. We can find some Cayley transformations Φ_q that biholomorphically maps $\mathbb{S}_\ell^n$ and $\mathbb{H}_\ell^n$ to $\mathbb{B}_\ell^n\setminus V$ and $\partial\mathbb{B}_\ell^n\setminus V$, respectively, for some variety V with $q\notin V$. Write $p=\Phi_q^{-1}(q)\in\mathbb{H}_\ell^n$.

Similarly, we can find some Cayley transformation $\Psi_{F(q)}$ that biholomorphically maps $\mathbb{S}_{\ell,\ell',n}^N$ and $\mathbb{H}_{\ell,\ell',n}^N$ to $\mathbb{B}_{\ell'}^N\setminus W$ and $\partial\mathbb{B}_{\ell'}^N\setminus W$, respectively, for some variety W with $F(q)\notin W$. Set $\hat{F}=\Psi_{F(q)}^{-1}\circ F\circ\Phi_q$ and regard it as a germ of map at $p\in\mathbb{H}_\ell^n$. We then define the geometric rank of F at q, denoted by $Rk_F(q)$, to be the geometric rank $Rk_{\hat{F}}(p)$ of $\hat{F}$ at p. By [12], $Rk_F(q)$ is independent of the choices of Φ_q and $\Psi_{F(q)}$, and thus it is well-defined.

2.2. Proof of a lemma

In the paper [9], where the first author first introduced the ideas of normal form and moving point trick to study mappings between hyperquadrics, a lemma (Lemma 3.2 in [9]) played a fundamental role. After the work [9], the lemma has been widely used in the study of mapping problems in CR geometry, as it provides an effective tool in determining the rank of Hermitian polynomials. Here we recall the definition of the rank of a real polynomial or more generally a real-valued real analytic function $R(z,\overline{z})$ at some point $z_0 \in \mathbb{C}$. Suppose $R(z,\overline{z})$ can be written as $R(z,\overline{z}) = \sum_{i=1}^{p} |f_i(z)|^2 - \sum_{j=1}^{q} |g_j(z)|^2, p, q \in \mathbb{Z}^{\geq 0}$, where f_i's and g_j's are holomorphic functions near z_0, and $f_1, \cdots, f_p, g_1, \cdots, g_q$ are linearly independent over $\mathbb{C}$. Then we say $R(z,\overline{z})$ is of finite rank and $r = p+q$ is called the rank of $R(z,\overline{z})$. We remark that the rank of $R(z,\overline{z})$ is independent of the choices of f_i's and g_j's. The rank of $R(z,\overline{z})$ is zero if and only if $R(z,\overline{z})$ is identically zero. Lemma 3.2 in [9] can be stated as follows:

Write $z = (z_1, \cdots, z_m)$ for the coordinates in $\mathbb{C}^m, m \geq 2$. Write $|z|$ for the Euclidean norm of z. Let $A(z,\overline{z})$ be a real analytic function near 0 such that

$$A(z,\overline{z})|z|^2 = \sum_{j=1}^{m-1} \psi_j(z)\overline{\phi_j(z)}, \tag{2.15}$$

where $\psi_j(z)$ and $\phi_j(z)$ are holomorphic functions near $0 \in \mathbb{C}^m$. Then $A(z,\overline{z})$ must have rank zero, that is, $A(z,\overline{z})$ must be identically zero.

In this section, we prove a lemma of similar flavor, and will use it to study the geometric rank of holomorphic mappings sending a piece of $\partial\mathbb{B}^n_\ell$ into $\partial\mathbb{B}^N_{\ell'}$.

Lemma 2.2. *Let ℓ, m, a, b be nonnegative integers such that $2 \leq \ell \leq \frac{m}{2}$ and $0 \leq a \leq 2\ell - 2$. Let $\varphi_1, ..., \varphi_a, \psi_1, ..., \psi_b$ be homogeneous holomorphic polynomials of the same degree in $\mathbb{C}^m$ such that*

$$-\sum_{j=1}^{a} |\varphi_j(z)|^2 + \sum_{j=1}^{b} |\psi_j(z)|^2 = A(z,\bar{z})|z|^2_\ell, \quad z \in \mathbb{C}^m, \tag{2.16}$$

where $A(z,\bar{z})$ is a real polynomial. Then $A(z,\bar{z}) = \pm|h(z)|^2$ for some holomorphic polynomial h.

Remark 2.3. *The above lemma is optimal in the sense that the conclusion fails if $a > 2\ell - 2$. See the following example which corresponds to the case where $\ell = 2, m = 4$ and $a = 3$.*

Example 2.4. *Let $z = (z_1, \cdots, z_4) \in \mathbb{C}^4$ and thus $|z|^2_2 = -|z_1|^2 - |z_2|^2 + |z_3|^2 + |z_4|^2$. Let $A(z,\overline{z}) = |z_1|^2 + |z_2|^2$. Then we have*

$$A(z,\overline{z})|z|^2_2 = -|z_1|^4 - 2|z_1|^2|z_2|^2 - |z_2|^4 + |z_1|^2|z_3|^2 + |z_1|^2|z_4|^2 + |z_2|^2|z_3|^2 + |z_2|^2|z_4|^2.$$

Notice that $A(z,\overline{z})$ is of rank two and cannot be written as $|h|^2$ or $-|h|^2$ for any holomorphic function h.

Proof of Lemma 2.2: We assume $\varphi_j(z)$ and $\psi_j(z)$ are not all identically zero, for otherwise the conclusion is trivial. Also the conclusion is easy by checking the zero locus of the two sides of (2.16), if each $\varphi_j = 0$ or each $\psi_j = 0$. We will therefore assume $a \geq 1$ and $b \geq 1$. We can also make $a \leq b$ by adding zero components to ψ_j's. Write $\xi = [\xi_0, ..., \xi_{m-1}]$ for the homogeneous coordinates in $\mathbb{P}^{m-1}$. Define a rational map from $\mathbb{P}^{m-1}$ to $\mathbb{P}^{a+b-1}$:

$$[F]([\xi]) = [\varphi_1(\xi), ..., \varphi_a(\xi), \psi_1(\xi), ..., \psi_b(\xi)].$$

Note $[F]$ is a well-defined holomorphic map on $\mathbb{P}^{m-1} \setminus V$, where the variety $V = \{[\xi] \in \mathbb{P}^{m-1} : \varphi_1(\xi) = ... = \varphi_a(\xi) = \psi_1(\xi) = ... = \psi_b(\xi)\}$. Recall that

$$\partial\mathbb{B}_k^{N-1} = \{[w_0, ..., w_{N-1}] \in \mathbb{P}^{N-1} : |w_0|^2 + ... + |w_k|^2 = |w_{k+1}|^2 + ... + |w_{N-1}|^2\}.$$

Note $|F(z)|_a^2 := -\sum_{j=1}^{a} |\varphi_j(z)|^2 + \sum_{j=1}^{b} |\psi_j(z)|^2 = 0$ when $|z|_\ell^2 = 0$ for $z \in \mathbb{C}^m$. Consequently, $[F]([\xi])$ gives a holomorphic map that sends an open piece of $\partial\mathbb{B}_{\ell-1}^{m-1}$ into $\partial\mathbb{B}_{a-1}^{a+b-1}$. We will make use of a transversality result, Theorem 1.1 of [2]. For that, we first verify the condition (1.2) in Theorem 1.1 of [2] holds.

Note the numbers of the negative and positive eigenvalues of the Levi form of $\partial\mathbb{B}_{a-1}^{a+b-1}$ are $a-1$ and $b-1$. Notice by assumption, $a-1 \leq 2\ell-3 \leq m-3$. Hence Condition (1.2) in Theorem 1.1 of [2] holds. Then it follows from Theorem 1.1 in [2] (see also Lemma 4.1 in [3]) that one of the following two mutually exclusive statements must hold:

(I). There exits a neighborhood $V \subseteq \mathbb{P}^{m-1}$ of some open piece of $\partial\mathbb{B}_{\ell-1}^{m-1}$ such that $[F](V) \subseteq \partial\mathbb{B}_{a-1}^{a+b-1}$;

(II). $[F]$ is transversal to $\partial\mathbb{B}_{a-1}^{a+b-1}$ at $[F](p)$ for a generic point $p \in \partial\mathbb{B}_{\ell-1}^{m-1}$.

If (I) holds, then $|F(\xi)|_a^2 \equiv 0$. In this case the quantity in (2.16) equals zero, and consequently, $A(z,\overline{z}) \equiv 0$. It then remains to consider the case where (II) holds. In this case, by moving to a generic point p, we assume $[F]$ is CR transversal along $V \cap \partial\mathbb{B}_{\ell-1}^{m-1}$ for a small neighborhood V of p. By the transversailty, $[F]$ either preserves or interchanges the sides of $\partial\mathbb{B}_{\ell-1}^{m-1}$ and $\partial\mathbb{B}_{a-1}^{a+b-1}$. We will apply Theorem 1.1 of [1] (we can also use Theorem 0.4) to the two cases separately:

Case (A). Suppose $[F]$ preserves sides of $\partial\mathbb{B}^{m-1}_{\ell-1}$ and $\partial\mathbb{B}^{a+b-1}_{a-1}$ (i.e., $[F]$ maps $V\cap\mathbb{B}^{m-1}_{\ell-1}$ to $\mathbb{B}^{a+b-1}_{a-1}$). Recall from §2.1 the map $(z,w)\to[i+w,2z,i-w]$ from $\mathbb{C}^{m-1}$ to $\mathbb{P}^{m-1}$ gives the Cayley transformation which biholomorphically maps the generalized Siegel upper-half space $\mathbb{S}^{m-1}_{\ell-1}$ to $\mathbb{B}^{m-1}_{\ell-1}\setminus\{[\xi_0,\cdots,\xi_{m-1}]:\xi_0+\xi_{m-1}=0\}$. Denote this map by ρ. Notice that

$$\rho^{-1}([\xi_0,\xi',\xi_{m-1}])=(\frac{i\xi'}{\xi_0+\xi_{m-1}},\frac{i\xi_0-i\xi_{m-1}}{\xi_0+\xi_{m-1}}).$$

Likewise, denote by r the Cayley transformation from $\mathbb{S}^{a+b-1}_{a-1}$ to $\mathbb{B}^{a+b-1}_{a-1}$. Composing $[F]$ with an automorphism of $\mathbb{B}^{a+b-1}_{a-1}$ and shrinking V if necessary, we can assume $\widehat{F}=r^{-1}\circ[F]\circ\rho$ is a well-defined holomorphic map in a neighborhood of some piece of $\mathbb{H}^{m-1}_{\ell-1}$. Moreover it sends the piece of $\mathbb{H}^{m-1}_{\ell-1}$ to $\mathbb{H}^{a+b-1}_{a-1}$, and maps the $\mathbb{S}^{m-1}_{\ell-1}$ side to $\mathbb{S}^{a+b-1}_{a-1}$.

Applying part (a) of Theorem 1.1 in [1] to $\widehat{F}$, we get $\ell\le a$ and $m-\ell\le b$. Furthermore, since $a-1<2(\ell-1)$, there exist a local biholomorphism γ of $\mathbb{H}^{a+b-1}_{a-1}$ and an automorphism τ of $\mathbb{H}^{m-1}_{\ell-1}$ such that

(2.17)
$$\gamma\circ\widehat{F}\circ\tau(z_1,\cdots,z_{m-2},w)=\gamma\circ r^{-1}\circ[F]\circ\rho\circ\tau=(z_1,\cdots,z_{\ell-1},\Phi,z_\ell,\cdots,z_{m-2},\Psi,w).$$

Here Φ and Ψ are holomorphic maps with $a-\ell$ and $b+\ell-m$ components (In our case, we know they are rational maps), respectively. And they satisfy $|\Phi|=|\Psi|$.

Next note there exist an automorphism g of $\mathbb{B}^{m-1}_{\ell-1}$ and an automorphism G of $\mathbb{B}^{a+b-1}_{a-1}$ such that $\gamma\circ r^{-1}=r^{-1}\circ G$ and $\rho\circ\tau=g\circ\rho$. Then (2.17) is reduced to

$$G\circ[F]\circ g\big([\xi_0,\cdots,\xi_{m-1}]\big)=r\circ(z_1,\cdots,z_{\ell-1},\Phi,z_\ell,\cdots,z_{m-2},\Psi,w)\circ\rho^{-1}.$$

Using the explicit formulas of r and ρ^{-1}, the above is reduced to

$$G\circ[F]\circ g\big([\xi_0,\xi',\xi_{m-1}]\big)=[\xi_0,\xi_1,\cdots,\xi_{\ell-1},\widetilde{\Phi},\xi_\ell,\cdots,\xi_{m-2},\widetilde{\Psi},\xi_{m-1}].$$

Here $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are rational maps with $a-\ell$ and $b+\ell-m$ components, respectively, and they satisfy $|\widetilde{\Phi}|=|\widetilde{\Psi}|$.

Finally since G and g preserve the indefinite norms $|\cdot|^2_a$ and $|\cdot|^2_\ell$, respectively, we have in case (A) that $|F(\xi)|^2_a=|h(\xi)|^2|\xi|^2_\ell$ for some holomorphic polynomial h.

Case (B). Suppose $[F]$ change sides of $\partial\mathbb{B}^{m-1}_{\ell-1}$ and $\partial\mathbb{B}^{a+b-1}_{a-1}$ (i.e., $[F]$ maps $V \cap \mathbb{B}^{m-1}_{\ell-1}$ to $\mathbb{P}^{a+b-1} \setminus \overline{\mathbb{B}^{a+b-1}_{a-1}} \approx \mathbb{B}^{a+b-1}_{b-1}$). Again composing $[F]$ with an automorphism of $\mathbb{B}^{a+b-1}_{a-1}$ and shrinking V if necessary, we can assume $\widehat{F} = r^{-1} \circ [F] \circ \rho$ is a well-defined holomorphic map in a neighborhood of some piece of $\mathbb{H}^{m-1}_{\ell-1}$, where ρ and r are as above. Moreover $\widehat{F}$ sends the piece of $\mathbb{H}^{m-1}_{\ell-1}$ to $\mathbb{H}^{a+b-1}_{a-1}$, and maps the $\mathbb{S}^{m-1}_{\ell-1}$ side to $\mathbb{C}^{a+b-1} \setminus \overline{\mathbb{S}^{a+b-1}_{a-1}}$.

Applying part (b) of Theorem 1.1 in [1] to $\widehat{F}$, we have $a \geq m - \ell$ and $b \geq \ell$. Moreover since $a - 1 \leq 2\ell - 3 \leq m - 3 < m - 2$, there exist a local biholomorphism γ of $\mathbb{H}^{a+b-1}_{a-1}$ and an automorphism τ of $\mathbb{H}^{m-1}_{\ell-1}$ such that

$$\gamma \circ \widehat{F} \circ \tau(z_1, \cdots, z_{m-2}, w) = \gamma \circ r^{-1} \circ F \circ \rho \circ \tau = (z_\ell, \cdots, z_{m-2}, \Phi, z_1, \cdots, z_{\ell-1}, \Psi, -w)$$

Here Φ and Ψ are rational maps with $a + \ell - m$ and $b - \ell$ components, respectively. Moreover, they satisfy $|\Phi| = |\Psi|$.

Similarly as above, we see there exist an automorphism g of $\mathbb{B}^{m-1}_{\ell-1}$ and an automorphism G of $\mathbb{B}^{a+b-1}_{a-1}$ such that

$$G \circ [F] \circ g([\xi_0, \cdots, \xi_{m-1}]) = [\xi_{m-1}, \xi_\ell, \cdots, \xi_{m-2}, \widetilde{\Phi}, \xi_1, \cdots, \xi_{\ell-1}, \widetilde{\Psi}, \xi_0].$$

Here $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are rational maps with $a + \ell - m$ and $b - \ell$ components, respectively. Moreover, they satisfy $|\widetilde{\Phi}| = |\widetilde{\Psi}|$. As above, we have in case (B) that $|F(\xi)|^2_a = -|h(\xi)|^2 |\xi|^2_\ell$ for some holomorphic polynomial h.

This proves Lemma 2.2. $\square$

2.3. Geometric rank of the map

In this section, we use the set up in §2.1 and Lemma 2.2 to give a proof for Theorem 1.5.

Proof of Theorem 1.5: Composing F with automorphisms of $\mathbb{B}^n_\ell$ and $\mathbb{B}^N_{\ell'}$ if necessary, we assume that F is well-defined in a neighborhood of $q_0 = [1, 0, ..., 0, 1] \in \partial\mathbb{B}^n_\ell$ with $F(q_0) = [1, 0, ..., 0, 1] \in \partial\mathbb{B}^N_{\ell'}$. Denote by Ψ_n the Cayley transformation from $\mathbb{S}^n_\ell$ to $\mathbb{B}^n_\ell$ as described in §2.1, and Φ_N the Cayley transformation from $\mathbb{S}^N_{\ell,\ell',n}$ to $\mathbb{B}^N_{\ell'}$. Then $\widetilde{F} := \Phi_N^{-1} \circ F \circ \Psi_n$ is well-defined in a small neighborhood of $0 \in \mathbb{H}^n_\ell$ (Recall $\Psi_n(0) = q_0$). Note $\tilde{F}$ is side-preserving (i.e., it maps $\mathbb{S}^n_\ell$ to $\mathbb{S}^N_{\ell,\ell',n}$ near 0). Moreover, by the definition of the geometric rank (see Section 3 in [12]), the geometric rank of F at $q \approx q_0$ is equal to that of $\widetilde{F}$ at $\Psi_n^{-1}(q)$ near 0. Thus it suffices to prove the new map $\widetilde{F}$ has geometric rank 0 or 1 near 0. To keep notations simple, we will still write the new map as F instead of $\widetilde{F}$. That is, F is now a holomorphic map from a neighborhood V of $0 \in \mathbb{H}_\ell^n$ to $\mathbb{C}^N$, satisfying

$$F(V \cap \mathbb{S}_\ell^n) \subseteq \mathbb{S}_{\ell,\ell',n}^N \qquad \textit{and} \qquad F(V \cap \mathbb{H}_\ell^n) \subseteq \mathbb{H}_{\ell,\ell',n}^N.$$

By shrinking V if necessary, we can additionally assume $M_1 := V \cap \mathbb{H}_\ell^n$ is connected and F is CR transversal along M_1. Fix p near 0 on M_1. We define F_p as in (2.2). We run the normalization process in §2.1 to F_p and obtain F_p^*, F_p^{**} in (2.9) and (2.13), respectively. Write $F_p^{**} = (\tilde{f}_p^{**}, g_p^{**}) = (f_p^{**}, \phi_p^{**}, g_p^{**})$ as in (2.13). Then Lemma 2.1 holds, and in particular (2.14) holds. Write $a_p^{**(1)}(z) = z\mathcal{A}(p)$ and

$$A(z, \overline{z}) = \langle \bar{z}, a_p^{**(1)}(z) \rangle_\ell = z\mathcal{A}(p) I_{\ell,n-1} \bar{z}^t.$$

Let $m = n-1, a = \tau = \ell' - \ell, b = (N - \ell') - (n - \ell)$ and

$$(\varphi_1(z), \cdots, \varphi_a(z), \psi_1(z), \cdots, \psi_b(z)) = \phi_p^{**(2)}(z).$$

Then we have (2.16) holds:

$$-\sum_{j=1}^{a} |\varphi_j(z)|^2 + \sum_{j=1}^{b} |\psi_j(z)|^2 = A(z, \bar{z}) |z|_\ell^2.$$

Note by assumption $a \leq 2\ell - 2$. By Lemma 2.2, $A(z, \bar{z}) = \pm |h(z)|^2$ for some holomorphic polynomial h. Consequently, the Hermitian matrix $\mathcal{A}(p) I_{\ell,n-1}$ has rank either 0 or 1; and so is $\mathcal{A}(p)$. The conclusion of the theorem then follows by the definition of geometric rank. □

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