# An Aubin continuity path for shrinking gradient Kähler–Ricci solitons

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**Abstract.** Let D be a toric Kähler–Einstein Fano manifold. We show that any toric shrinking gradient Kähler–Ricci soliton on certain toric blowups of  $\mathbb{C} \times D$  satisfies a complex Monge–Ampère equation. We then set up an Aubin continuity path to solve this equation and show that it has a solution at the initial value of the path parameter. This we do by implementing another continuity method.

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### 1. Introduction

**1.1. Overview.** A *Ricci soliton* is a triple (M, g, X), where M is a Riemannian manifold endowed with a complete Riemannian metric g and a complete vector field X, such that

(1.1) 
$$\operatorname{Ric}_{g} + \frac{1}{2} \mathcal{L}_{X} g = \lambda g$$

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for some  $\lambda \in \mathbb{R}$ . The vector field X is called the *soliton vector field*. If  $X = \nabla^g f$  for some smooth real-valued function f on M, then we say that (M, g, X) is *gradient*. In this case, the soliton equation (1.1) becomes  $\mathrm{Ric}_g + \mathrm{Hess}_g(f) = \lambda g$ , and we call f the *soliton potential*. In the case of gradient Ricci solitons, the completeness of X is guaranteed by the completeness of X (see [40]).

Let (M, g, X) be a Ricci soliton. If g is Kähler and X is real holomorphic, then we say that (M, g, X) is a Kähler–Ricci soliton. Let  $\omega$  denote the Kähler form of g. If (M, g, X) is in addition gradient, then (1.1) may be rewritten as

$$\rho_{\omega} + i\partial\bar{\partial}f = \lambda\omega,$$

where  $\rho_{\omega}$  is the Ricci form of  $\omega$  and f is the soliton potential.

Finally, a Ricci soliton and a Kähler–Ricci soliton are called *steady* if  $\lambda = 0$ , *expanding* if  $\lambda < 0$ , and *shrinking* if  $\lambda > 0$  in (1.1). One can always normalise  $\lambda$ , when non-zero, to satisfy  $|\lambda| = 1$ . We henceforth assume that this is the case.

Ricci solitons are interesting both from the point of view of canonical metrics and of the Ricci flow. On one hand, they represent one direction in which the concept of an Einstein manifold can be generalised. On compact manifolds, shrinking Ricci solitons are known to exist in several instances where there are obstructions to the existence of Einstein metrics; see for example [37]. By the maximum principle, there are no nontrivial expanding or steady Ricci solitons on compact manifolds. However, there are many examples on non-compact manifolds; see for example [15, 16, 22] and the references therein. On the other hand, one can associate to a Ricci soliton a self-similar solution of the Ricci flow, and gradient shrinking Ricci solitons in particular provide models for finite-time Type I singularities of the flow [20, 29]. From this perspective, it is an important problem to classify such solitons in order to better understand singularity development along the Ricci flow.

In this article, we are concerned with the construction of shrinking gradient Kähler–Ricci solitons, models for finite-time Type I singularities of the Kähler–Ricci flow. In essence, we set up an Aubin continuity path for a complex Monge–Ampère equation to construct such solitons in a particular geometric setting that allows for control on the data of the equation. We then show that there is a solution to the equation for the initial value of the path parameter. This we do by implementing another continuity path.

**1.2. Main result.** In order to state the main result, recall that a complex toric manifold is a smooth n-dimensional complex manifold D endowed with an effective holomorphic action of the complex torus  $(\mathbb{C}^*)^n$  with a compact fixed point set. In such a setting, there always exists an orbit  $U \subset D$  of the  $(\mathbb{C}^*)^n$ -action which is open and dense in D. The  $(\mathbb{C}^*)^n$ -action of course determines the holomorphic action of a real torus  $T^n \subset (\mathbb{C}^*)^n$ , as is easily seen for the action of the one-dimensional torus  $\mathbb{C}^*$  on  $\mathbb{P}^1$  via  $\lambda \cdot [z_0 : z_1] \mapsto [\lambda z_1 : z_2]$ . This assumption is crucial for obtaining a uniform lower bound on the solution along our continuity path. Our main result is stated as follows.

**Theorem A.** Let  $D^{n-1}$  be a toric Kähler–Einstein Fano manifold of complex dimension n-1 with Kähler form  $\omega_D$  and Ricci form  $\rho_{\omega_D} = \omega_D$ , and consider  $\mathbb{P}^1 \times D$  with the induced product torus action acting by rotation on the  $\mathbb{P}^1$ -factor. Let  $T^n$  denote the real torus acting on  $\mathbb{P}^1 \times D$ , write  $D_x := \{x\} \times D$ , and let  $\overline{M}$  be a toric Fano manifold obtained as a torus-equivariant (possibly iterated) blowup  $\pi : \overline{M} \to \mathbb{P}^1 \times D$  along smooth torus-invariant

subvarieties contained in  $D_0$ . Let  $M := \overline{M} \setminus \pi^{-1}(D_\infty)$ ,  $\hat{M} := \mathbb{C} \times D$ , write J for the complex structure on M, t for the Lie algebra of  $T^n$ , and let z denote the holomorphic coordinate on the  $\mathbb{C}$ -factor of  $\hat{M}$ . Then

(i) there exists a unique complete real holomorphic vector field  $JX \in t$  such that X is the soliton vector field of any complete toric shrinking gradient Kähler–Ricci soliton on M.

Assume that the flow lines of JX are closed. Then

(ii) there exists a complete Kähler metric  $\omega$  on M invariant under the action of T,  $\lambda > 0$  uniquely determined by X, and a holomorphic isometry

$$\nu: (M \setminus K, \omega) \to (\widehat{M} \setminus \widehat{K}, \widehat{\omega} := \omega_C + \omega_D),$$

where  $K \subset M$ ,  $\hat{K} \subset \hat{M}$  are compact and  $\omega_C := \frac{i}{2}\partial \bar{\partial} |z|^{2\lambda}$ , such that

$$d\nu(X) = \frac{2}{\lambda} \cdot \operatorname{Re}(z \partial_z).$$

- (iii) There exists a unique torus-invariant function  $f \in C^{\infty}(M)$  such that  $-\omega \rfloor JX = df$ . Moreover,  $f = v^* \left(\frac{|z|^{2\lambda}}{2} - 1\right)$  and  $\Delta_{\omega} f + f - \frac{X}{2} \cdot f = 0$  outside a compact subset of M containing K.
- (iv) Any shrinking Kähler–Ricci soliton on M invariant under the action of T of the form

$$\omega + i\partial\bar{\partial}\varphi$$
 for some  $\varphi \in C^{\infty}(M)$  with  $\omega + i\partial\bar{\partial}\varphi > 0$ 

satisfies the complex Monge-Ampère equation

$$(0.3) \qquad (\omega + i\partial\bar{\partial}\varphi)^n = e^{F + \frac{X}{2}\cdot\varphi - \varphi}\omega^n,$$

where  $F \in C^{\infty}(M)$  is equal to a constant outside a compact subset of M and is determined by the fact that

$$\rho_{\omega} + \frac{1}{2} \mathcal{L}_X \omega - \omega = i \partial \bar{\partial} F$$
 and  $\int_M (e^F - 1) e^{-f} \omega^n = 0.$ 

Here,  $\rho_{\omega}$  denotes the Ricci form of  $\omega$ .

(v) There exists a function  $\psi \in C^{\infty}(M)$  invariant under the action of T and with

$$\omega + i \, \partial \bar{\partial} \psi > 0$$

such that

$$(0.4) \qquad (\omega + i\partial\bar{\partial}\psi)^n = e^{F + \frac{X}{2}\cdot\psi}\omega^n,$$

where  $\int_M \psi e^{-f} \omega^n = 0$  and, outside a compact subset,  $\psi = c_1 \log f + c_2 + \vartheta$  for some constants  $c_1, c_2 \in \mathbb{R}$  and a smooth real-valued function  $\vartheta: M \to \mathbb{R}$  satisfying

$$|\nabla^i \mathcal{L}_X^{(j)} \vartheta|_{\omega} = O(f^{-\frac{\beta}{2}}) \quad \text{for all } i, j \in \mathbb{N}, \quad \beta \in (0, \lambda^D).$$

Here,  $\nabla$  denotes the Levi-Civita connection associated to  $\omega$ ,

$$\mathcal{L}_X^{(j)} = \underbrace{\mathcal{L}_X \circ \cdots \circ \mathcal{L}_X}_{j\text{-times}},$$

and  $\lambda^D$  is the first non-zero eigenvalue of  $-\Delta_D$  acting on  $L^2$ -functions on D.

Note that, since M does not split off any  $S^1$ -factors, toricity implies that M has finite fundamental group [17], a necessary condition for the existence of a shrinking gradient Kähler–Ricci soliton on M (see [38]). Note also that, throughout, our convention for the Kähler Laplacian  $\Delta_{\omega}$  is that with respect to the Kähler form  $\omega$ ,  $\Delta_{\omega} f = \operatorname{tr}_{\omega}(i\partial\bar{\partial}f)$  for f a smooth real-valued function, so that the eigenvalues of minus the Laplacian are non-negative on a compact Riemannian manifold.

Part (i) of the theorem determines the soliton vector field of any complete toric shrinking gradient Kähler–Ricci soliton on M and follows immediately from [12, Theorem A], where it is asserted that a complete toric shrinking gradient Kähler-Ricci soliton is unique up to biholomorphism. The vector field JX is characterised by the fact that it is the point in a specific open convex subset of t at which a certain strictly convex functional attains its minimum. More precisely, because  $H^1(M,\mathbb{R})=0$  and M is toric, the action of T is Hamiltonian and there exists a strictly convex functional  $\mathcal{F}_{\omega}$ :  $\Lambda_{\omega} \to \mathbb{R}_{>0}$ , the "weighted volume functional" [16, Definition 5.16], defined on an open convex cone  $\Lambda_{\omega} \subset t$  uniquely determined by the image of M under the moment map defined by the action of T and the choice of  $\omega$  (see [31, Proposition 1.4]) and well-defined by the non-compact version of the Duistermaat-Heckman formula [31] (see also [16, Theorem A.3]). Because T provides a full-dimensional torus symmetry, the domain  $\Lambda_{\omega}$  of  $\mathcal{F}_{\omega}$  and  $\mathcal{F}_{\omega}$  itself only depend on the torus action [13] so that both are independent of the choice of  $\omega$ . Furthermore, henceforth dropping the subscripts  $\omega$ ,  $\mathcal{F}$  is known to be strictly convex [16, Lemma 5.17 (i)] and in addition proper [12, Proposition 3.1] on  $\Lambda$  in the toric case, and so it must attain a unique minimum on  $\Lambda$ . This minimum defines a distinguished point in t, namely the only vector field in t that can admit a complete toric shrinking gradient Kähler–Ricci soliton [12, Theorem 4.6]. This is precisely the vector field JX of Theorem A (i). Since everything is explicit and is determined by the torus action, one can a priori determine this vector field for a given M; see for example [16, Section A.4].

Parts (ii) and (iii) give a reference metric on M that is isometric to a model shrinking gradient Kähler–Ricci soliton outside a compact set. This requires the assumption that the flow lines of JX are closed. Indeed, this is the case for the soliton vector field on the model. With respect to this background metric, part (iv) gives a complex Monge–Ampère equation (1.3) that any complete toric shrinking gradient Kähler–Ricci soliton on M must satisfy with control on the asymptotics of the data F of the equation. By [12], we know that there is at most one such soliton on M and we expect that this equation has a solution, resulting in a complete toric shrinking gradient Kähler–Ricci soliton on M. Such a soliton should model finite time collapsing of the Kähler–Ricci flow in order to be consistent with [36]. One may attempt to solve (1.3) by implementing the Aubin continuity path that was introduced for Kähler–Einstein manifolds [4, Section 7.26]. Specifically, in our case, one may consider the path

(\*\*<sub>t</sub>) 
$$\begin{cases} (\omega + i\partial\bar{\partial}\varphi_t)^n = e^{F + \frac{X}{2} \cdot \varphi_t - t\varphi_t} \omega^n, & \varphi \in C^{\infty}(M), & \mathcal{L}_{JX}\varphi = 0, \\ \omega + i\partial\bar{\partial}\varphi > 0, & t \in [0, 1], \end{cases}$$

$$\int_M e^{F - f} \omega^n = \int_M e^{-f} \omega^n.$$

The main content of Theorem A is part (v) where we provide a solution to the equation corresponding to t = 0. This we do by implementing another continuity path. In the compact case, this was achieved by Zhu [41].

The simplest example of a toric Fano manifold D satisfying the conditions of Theorem A is  $D = \mathbb{P}^1$  with  $\pi$  the blowup map. Indeed, these choices result in M being the blowup of

 $\mathbb{C} \times \mathbb{P}^1$  at one point, a manifold for which the flow lines of JX close as one can see from Example 2.24 or [13, Example 2.33]. In [13, Conjecture 1.1], M was identified as a new manifold potentially admitting a (unique) complete shrinking gradient Kähler–Ricci soliton with bounded scalar curvature. Thanks to [6], it is now known that M admits such a soliton. However, the proof of existence in [6] is strictly dimension dependent and is *indirect* in that the soliton is constructed as a blowup limit of a specific Kähler–Ricci flow on the blowup of  $\mathbb{P}^1 \times \mathbb{P}^1$  at one point. The principal motivation behind Theorem A therefore is that it provides a first step in a *direct* construction of this soliton on M, namely via the continuity method, and is more widely applicable than the methods of [6]. It also serves to provide examples of non-compact manifolds with strictly positive Bakry–Emery tensor.

Equation (1.4) a priori looks identical to the complex Monge–Ampère equation solved in [15], where complete steady gradient Kähler–Ricci solitons were constructed. Even though the equations appear the same and the same continuity path is used in both cases, there are several important differences between the two that result in additional difficulties arising in the solution of (1.4) in contrast to the equation of [15]. We conclude this section by highlighting some of these differences.

- On a closed Kähler manifold, the *X*-derivative of any Kähler potential is bounded prior to any other bound; see [41]. This fact does not seem to be amenable to an arbitrary noncompact Kähler manifold and represents one of the major obstacles to adapting Tian and Zhu's work [35] to our current setting. For us, not only is the drift operator *X* of (1.4) unbounded, in contrast to [15] where it is bounded, but it also has the opposite sign. This prevents us from adapting the proof of the *C*<sup>0</sup> a priori estimate in [15] to the present situation.
- In [41], a generalisation of Calabi's conjecture was proved on compact Kähler manifolds using a continuity path that shrinks the hypothetical soliton vector field X to zero as the path parameter tends to zero, thereby reducing the existence at the initial value of the path parameter to Yau's solution of the Calabi conjecture [39]. In our setting, implementing such a continuity path to solve (1.3) does not preserve the weighted volume, and indeed, the weighted volume diverges at the initial value of the path parameter. This explains why the Aubin continuity path is more suited to solving (1.3), which yields (1.4) at the initial value of the path parameter (in contrast to the Calabi–Yau equation). This is precisely the equation that we provide a solution to in Theorem A (v).
- In [15], the corresponding equation was solved using the continuity path with exponentially weighted function spaces. Here, we solve (1.4) in polynomially weighted function spaces. This difference is derived from the fact that, in the present situation, the linearised operator contains logarithmically growing functions in its kernel at infinity. This makes the linear theory more delicate than in the previous work [15].
- In obtaining an a priori  $C^0$ -estimate for (1.4), the toricity assumption is crucial. This was not the case in [15] where no toricity was required. However, a priori weighted  $L^p$ -estimates on the solution of (1.4) are obtained *without* requiring toricity. The same also applies to the a priori estimates apart from the one concerning a *lower* bound on the solution. This will all be made clear in the relevant statements throughout.
- The order in which we obtain the a priori estimates differs to that of [15]. Here we first obtain an a priori *lower* bound on the radial derivative of the solution. This then allows

us to derive an a priori *upper* bound on the solution. The next step is to derive an a priori *lower* bound on the solution. At this stage, we follow the same strategy as that of [15] to obtain a priori *local* estimates on the solution.

- In addition to containing logarithmically growing functions, the kernel of the linearised operator in the present situation contains constants, a fact that makes the a priori weighted estimate of the difference of the solution and of its value at infinity more subtle in a non-linear setting. To circumvent this issue, we apply the Bochner formula to the *X*-derivative of our solution with respect to the unknown Kähler metric.
- Our geometric setting bears some resemblance to the work [25] on asymptotically cylindrical Calabi–Yau metrics. However, in the context of metric measure spaces, our setting is somewhat dissimilar to theirs in that, as metric measure spaces, our spaces have finite volume, whereas their spaces have infinite volume. This forces us to take an alternative approach to obtain (weighted) a priori estimates.

1.3. Outline of paper. We begin in Section 2.1 by recalling the basics of shrinking Ricci and Kähler–Ricci solitons. Some important examples are discussed as well as some technical lemmas proved. We also recall the definition of a metric measure space in Section 2.2. In Section 2.3, we digress and define polyhedrons and polyhedral cones before moving on to the definition of a Hamiltonian action in Section 2.4. Section 2.5 then comprises the background material on toric geometry that we require.

In Section 3, we construct a background metric with the desired properties, resulting in the proof of Theorem A (ii). Next, in Section 4, the complex Monge–Ampère equation is set up and the normalisation of the Hamiltonian of JX is obtained, leading to the proof of Theorem A (iii)–(iv). Our background metric is isometric to a shrinking gradient Kähler–Ricci soliton compatible with X outside a compact set. This is what allows us to set up the complex Monge–Ampère equation with compactly supported data.

From Section 5 onwards, the content takes on a more analytic flavour with the proof of Theorem A (v) taking up Sections 5–7. To prove this part of Theorem A, we implement the continuity method. The specific continuity path that we consider is outlined at the beginning of Section 7, but beforehand, in Section 5, we prove a Poincaré inequality which is the content of Proposition 5.1. This is essential in deriving the a priori weighted energy estimate for the complex Monge–Ampère equation (1.3) with compactly supported data.

In Section 6, we study the properties of the drift Laplacian of our background metric acting on polynomially weighted function spaces. More precisely, we introduce polynomially weighted function spaces whose elements are invariant under the flow of JX in Section 6.2. We follow this up in Section 6.3 by showing that the drift Laplacian of our background metric is an isomorphism between such spaces. This latter result is the content of Theorem 6.3. Using it, we then prove Theorem 6.12 that serves as the openness part of the continuity argument. The closedness part involves a priori estimates and these make up Section 7.

As noted previously, the presence of the unbounded vector field X makes the analysis much more involved. An a priori lower bound for the radial derivative  $X \cdot \psi$ , where  $\psi$  solves (1.3), has to be proved *before* the a priori  $C^0$  bound in order to avoid a circular argument; see Section 7.4. A priori energy estimates are obtained in Section 7.5 through the use of the so-called Aubin–Tian–Zhu functionals and result in an a priori upper bound on a solution to the complex Monge–Ampère equation (1.3); cf. Proposition 7.11. As explained above, the invar-

iance of the solution under the whole action torus is crucial in obtaining an a priori lower bound on the infimum; cf. Proposition 7.19. Then and only then an a priori upper bound on the radial derivative of a solution to (1.3) is derived; cf. Proposition 7.6. Section 7.7 is devoted to proving a local bootstrapping phenomenon for (1.3). Finally, Section 7.8 takes care of establishing a priori weighted estimates at infinity for (1.3), leading to the completion of the proof of Theorem A (v) in Section 7.9.

#### 2. Preliminaries

**2.1. Shrinking Ricci solitons.** Recall the definitions given at the beginning of Section 1.1. An important class of examples of such manifolds for us is the following.

**Example 2.1.** We have a 1-parameter family  $\{\widetilde{\omega}_a\}_{a>0}$  of (incomplete) shrinking gradient Kähler–Ricci solitons on  $\mathbb{C}$ . Indeed, for each a>0, the Kähler form of the shrinking soliton is given by  $\widetilde{\omega}_a:=\frac{i}{2}\partial\bar{\partial}|z|^{2a}$ , where z is the holomorphic coordinate on  $\mathbb{C}$ . The soliton vector field of  $\widetilde{\omega}_a$  is given by  $\frac{2}{a}\cdot \operatorname{Re}(z\partial_z)$ . Of course, when a=1,  $\widetilde{\omega}_a$  is complete and we recover the flat shrinking Gaussian soliton  $\omega_{\mathbb{C}}$  on  $\mathbb{C}$  with soliton vector field  $2\cdot \operatorname{Re}(z\partial_z)$ .

Any Kähler–Einstein manifold trivially defines a shrinking gradient Kähler–Ricci soliton (with soliton vector field X=0). We may then take the Cartesian product with Example 2.1 to produce many more examples. These examples provide the model at infinity for the reference metric that we will construct in Theorem A (i).

**Example 2.2.** Let  $(D, \omega_D)$  be a Kähler–Einstein Fano manifold with Kähler form  $\omega_D$ . Then, for each a>0, the Cartesian product  $\hat{M}:=\mathbb{C}\times D$  endowed with the Kähler form  $\hat{\omega}_a:=\tilde{\omega}_a+\omega_D$  is an example of an (incomplete) shrinking gradient Kähler–Ricci soliton. Here,  $\tilde{\omega}_a$  is as in Example 2.1. Writing  $r:=|z|^a$  with z the complex coordinate on the  $\mathbb{C}$ -factor of  $\hat{M}$ , the soliton vector field of this example is given by  $\hat{X}:=r\partial_r=\frac{2}{a}\cdot \mathrm{Re}(z\partial_z)$ . When a=1, the soliton is complete, and up to isometry, we obtain a complete shrinking gradient Kähler–Ricci soliton on  $\mathbb{C}\times D$  with bounded scalar curvature which is unique if D is moreover toric [12, Corollary C]. We write  $\hat{g}_a$  and  $\hat{J}$  for the Kähler metric associated to  $\hat{\omega}_a$  and product complex structure on  $\hat{M}$ , respectively.

The following lemma concerning  $(\hat{M}, \hat{\omega}_a)$  will prove useful throughout.

**Lemma 2.3.** With notation as in Example 2.2, fix a > 0 (and hence the function r) and let  $\widehat{K} \subset \widehat{M}$  be a compact subset such that  $\widehat{M} \setminus \widehat{K}$  is connected. If  $u : \widehat{M} \setminus \widehat{K} \to \mathbb{R}$  is a smooth real-valued function defined on  $\widehat{M} \setminus \widehat{K}$  that is pluriharmonic (meaning that  $\partial \overline{\partial} u = 0$ ) and invariant under the flow of  $\widehat{J}\widehat{X}$ , then  $u = c_0 \log(r) + c_1$  for some  $c_0, c_1 \in \mathbb{R}$ .

*Proof.* Let  $\hat{X}^{1,0}:=\frac{1}{2}(\hat{X}-i\,\hat{J}\,\hat{X})$ . Then, since  $\hat{X}$  is real holomorphic and  $\mathcal{L}_{\hat{J}}\hat{\chi}u=0$ , we see that

$$\bar{\partial}(\hat{X} \cdot u) = \partial \bar{\partial} u \, \lrcorner (\hat{X}^{1,0}) = 0,$$

i.e.,  $\widehat{X} \cdot u$  is holomorphic. As a real-valued holomorphic function,  $\widehat{X} \cdot u$ , which itself is equal to  $r \partial_r u$ , must be equal to a constant,  $c_0$  say. Thus, because u is invariant under the flow of  $\widehat{J}\widehat{X}$ ,

we can write  $u = c_0 \log r + c_1(x)$ , where  $x \in D$ . Let  $\Delta_{\mathbb{C}}$  and  $\Delta_D$  denote the Riemannian Laplacians with respect to the flat metric  $g_{\mathbb{C}}$  on  $\mathbb{C}$  and the Kähler–Einstein metric  $\omega_D$  on D, respectively. Then u being pluriharmonic implies that  $\Delta_{\mathbb{C}}u + \Delta_D u = 0$ , and so

$$0 = (\Delta_D + \Delta_{\mathbb{C}})(c_0 \log(r) + c_1(x))$$
  
=  $\Delta_D c_1(x) + \underbrace{\Delta_{\mathbb{C}} c_1(x)}_{=0} + c_0 \underbrace{\Delta_{\mathbb{C}} \log(r)}_{=0} = \Delta_D c_1(x),$ 

which leads to  $c_1(x) = c_1$ . This leaves us with  $u = c_0 \log(r) + c_1$ , as desired.

We conclude this section with a gluing lemma.

**Lemma 2.4** (Gluing lemma). With notation as in Example 2.2, fix a > 0 (and hence the function r), let  $\hat{K} \subset \hat{M}$  be a compact subset, and let  $\phi \in C^{\infty}(\hat{M} \setminus \hat{K})$  be such that

$$\phi = O(\log(r)), \quad |d\phi|_{\widehat{g}_a} = O(1), \quad and \quad |i\partial\bar{\partial}\phi|_{\widehat{g}_a} = O(r^{-a}).$$

Then, for all R > 0 with  $\hat{K} \subseteq \{r \leq R\}$ , there exists a cut-off function  $\chi_R: M \to \mathbb{R}$  supported on  $M \setminus \{r \leq R\}$  with  $\chi_R(x) = 1$  if r(x) > 2R such that

$$|i\partial\bar{\partial}(\chi_R \cdot \phi)|_{\widehat{g}_a} \leq \frac{C}{R^{\min\{1,a\}}} (\|(\log(r))^{-1} \cdot \phi\|_{C^0(\widehat{M} \setminus \widehat{K})} + \|d\phi\|_{C^0(\widehat{M} \setminus \widehat{K}, \widehat{g}_a)} + \|r^a \cdot i\partial\bar{\partial}\phi\|_{C^0(\widehat{M} \setminus \widehat{K}, \widehat{g}_a)})$$

for some C > 0 independent of R. In particular,  $\chi_R \cdot \phi = \phi$  on  $\{r(x) > 2R\}$ .

*Proof.* Let  $\chi: \mathbb{R} \to \mathbb{R}$  be a smooth function satisfying  $\chi(x) = 0$  for  $x \le 1$ ,  $\chi(x) = 1$  for  $x \ge 4$ , and  $|\chi(x)| \le 1$  for all x, and with it, define a function  $\chi_R: M \to \mathbb{R}$  by

$$\chi_R(x) = \chi\left(\frac{r(x)^2}{R^2}\right)$$
 for  $R > 0$  as in the statement of the lemma.

Then  $\chi_R$  is identically zero on  $\{x \in \widehat{M} \mid r(x) < R\}$  and identically equal to one on the set  $\{x \in \widehat{M} \mid r(x) > 2R\}$ . Define  $\phi_R := \chi_R.\phi$ . Then the closed real (1,1)-form  $i\,\partial\bar{\partial}(\chi_R.\phi)$  on  $\widehat{M}$  is given by

$$\begin{split} i\,\partial\bar{\partial}(\chi_R.\phi) &= \chi_R(r).i\,\partial\bar{\partial}\phi + \chi'\Big(\frac{r^2}{R^2}\Big).i\,\frac{\partial r^2}{R} \wedge \frac{\bar{\partial}\phi}{R} + \frac{\phi}{R^2}.\chi'\Big(\frac{r^2}{R^2}\Big).i\,\partial\bar{\partial}r^2 \\ &+ \chi'\Big(\frac{r^2}{R^2}\Big).\frac{i\,\partial\phi}{R} \wedge \frac{\bar{\partial}r^2}{R} + \frac{\phi}{R^2}.\chi''\Big(\frac{r^2}{R^2}\Big).i\,\frac{\partial r^2}{R} \wedge \frac{\bar{\partial}r^2}{R}. \end{split}$$

The assumptions on  $\phi$  and its derivatives then imply for example that

$$|\chi_{R}(x).i\partial\bar{\partial}\phi|_{\widehat{g}_{a}} \leq \sup_{r\in[R,\infty)}|i\partial\bar{\partial}\phi|_{\widehat{g}_{a}} \leq (\sup_{r\in[R,\infty)}r^{-a})(\sup_{r\in[R,\infty)}r^{a}\cdot|i\partial\bar{\partial}\phi|_{\widehat{g}_{a}})$$
$$\leq R^{-a}||r^{a}\cdot i\partial\bar{\partial}\phi||_{C^{0}(\widehat{M}\setminus\widehat{K},\widehat{g}_{a})}$$

and that

$$\left|\chi'\left(\frac{r^2}{R^2}\right).i\frac{\partial r^2}{R} \wedge \frac{\bar{\partial}\phi}{R}\right|_{\widehat{g}_a} \leq \frac{C}{R^2} \left(\sup_{r \in [R,2R]} r\right) \left(\sup_{r \in [R,2R]} |i\partial r \wedge \bar{\partial}\phi|_{\widehat{g}_a}\right)$$
$$\leq CR^{-1} \|d\phi\|_{C^0(\widehat{M} \setminus \widehat{K}, \widehat{g}_a)}.$$

The estimate of the lemma is now clear.

**2.2. Basics of metric measure spaces.** We take the following from [21]. A smooth metric measure space is a Riemannian manifold endowed with a weighted volume.

**Definition 2.5.** A smooth metric measure space is a triple  $(M, g, e^{-f} dV_g)$ , where (M, g) is a complete Riemannian manifold with Riemannian metric g,  $dV_g$  is the volume form associated to g, and  $f: M \to \mathbb{R}$  is a smooth real-valued function.

A shrinking gradient Ricci soliton (M, g, X) with  $X = \nabla^g f$  naturally defines a smooth metric measure space  $(M, g, e^{-f} dV_g)$ . On such a space, we define the weighted Laplacian  $\Delta_f$  by

$$\Delta_f u := \Delta u - g(\nabla^g f, \nabla u)$$

on smooth real-valued functions  $u \in C^{\infty}(M, \mathbb{R})$ . There is a natural  $L^2$ -inner product  $\langle \cdot, \cdot \rangle_{L^2_f}$  on the space  $L^2_f$  of square-integrable smooth real-valued functions on M with respect to the measure  $e^{-f} dV_g$  defined by

$$\langle u,v\rangle_{L^2_f}:=\int_M uve^{-f}\,dV_g,\quad u,v\in L^2_f.$$

As one can easily verify, the operator  $\Delta_f$  is self-adjoint with respect to  $\langle\,\cdot\,,\cdot\,\rangle_{L^2_f}$  .

**2.3. Polyhedrons and polyhedral cones.** We take the following from [17]. Let E be a real vector space of dimension n and let  $E^*$  denote the dual. Write  $\langle \cdot, \cdot \rangle$  for the evaluation  $E^* \times E \to \mathbb{R}$ . Furthermore, assume that we are given a *lattice*  $\Gamma \subset E$ , that is, an additive subgroup  $\Gamma \simeq \mathbb{Z}^n$ . This gives rise to a dual lattice  $\Gamma^* \subset E^*$ . For any  $\nu \in E$ ,  $c \in \mathbb{R}$ , let  $K(\nu, c)$  be the (closed) half-space  $\{x \in E \mid \langle \nu, x \rangle \geq c\}$  in E. Then we have the following definition.

**Definition 2.6.** A polyhedron P in E is a finite intersection of half-spaces, i.e.,

$$P = \bigcap_{i=1}^{r} K(\nu_i, c_i) \quad \text{for } \nu_i \in E^*, c_i \in \mathbb{R}.$$

It is called a *polyhedral cone* if all  $c_i = 0$ , and moreover a *rational polyhedral cone* if all  $v_i \in \Gamma^*$  and  $c_i = 0$ . In addition, a polyhedron is called *strongly convex* if it does not contain any affine subspace of E.

The following definition will be useful.

**Definition 2.7.** A polyhedron  $P \subset E^*$  is called *Delzant* if its set of vertices is nonempty and each vertex  $v \in P$  has the property that there are precisely n edges  $\{e_1, \ldots, e_n\}$  (one-dimensional faces) emanating from v and there exists a basis  $\{\varepsilon_1, \ldots, \varepsilon_n\}$  of  $\Gamma^*$  such that  $\varepsilon_i$  lies along the ray  $\mathbb{R}(e_i - v)$ .

Note that any such P is necessarily strongly convex. We also have the following.

**Definition 2.8.** The *dual* of a polyhedral cone C is the set

$$C^{\vee} = \{ x \in E^* \mid \langle x, C \rangle \ge 0 \}.$$

**2.4.** Hamiltonian actions. Recall what it means for an action to be Hamiltonian.

**Definition 2.9.** Let  $(M, \omega)$  be a symplectic manifold and let T be a real torus acting by symplectomorphisms on  $(M, \omega)$ . Denote by t the Lie algebra of T and by  $t^*$  its dual. Then we say that the action of T is *Hamiltonian* if there exists a smooth map  $\mu_\omega: M \to t^*$  such that, for all  $\zeta \in \mathfrak{t}$ ,  $-\omega \Box \zeta = du_{\zeta}$ , where  $u_{\zeta}(x) = \langle \mu_{\omega}(x), \zeta \rangle$  for all  $\zeta \in \mathfrak{t}$  and  $\zeta \in M$  and

**2.5. Toric geometry.** In this section, we collect together some standard facts from toric geometry as well as recall those results from [12] that we require. We begin with the following definition.

**Definition 2.10.** A *toric manifold* is an *n*-dimensional complex manifold M endowed with an effective holomorphic action of the algebraic torus  $(\mathbb{C}^*)^n$  such that the following hold true.

- The fixed point set of the  $(\mathbb{C}^*)^n$ -action is compact.
- There exists a point  $p \in M$  with the property that the orbit  $(\mathbb{C}^*)^n \cdot p \subset M$  forms a dense open subset of M.

We will often denote the dense orbit simply by  $(\mathbb{C}^*)^n \subset M$  in what follows. The  $(\mathbb{C}^*)^n$ -action of course determines the action of the real torus  $T^n \subset (\mathbb{C}^*)^n$ .

**2.5.1. Divisors on toric varieties and fans.** Let  $T^n \subset (\mathbb{C}^*)^n$  be the real torus with Lie algebra t and denote the dual pairing between t and the dual space  $t^*$  by  $\langle \cdot, \cdot \rangle$ . There is a natural integer lattice  $\Gamma \simeq \mathbb{Z}^n \subset t$  comprising all  $\lambda \in t$  such that  $\exp(\lambda) \in T^n$  is the identity. This then induces a dual lattice  $\Gamma^* \subset t^*$ . We have the following combinatorial definition.

**Definition 2.11.** A fan  $\Sigma$  in t is a finite set of rational polyhedral cones  $\sigma$  satisfying the following.

- (i) For every  $\sigma \in \Sigma$ , each face of  $\sigma$  also lies in  $\Sigma$ .
- (ii) For every pair  $\sigma_1, \sigma_2 \in \Sigma$ ,  $\sigma_1 \cap \sigma_2$  is a face of each.

To each fan  $\Sigma$  in t, one can associate a toric variety  $X_{\Sigma}$ . Heuristically,  $\Sigma$  contains all the data necessary to produce a partial equivariant compactification of  $(\mathbb{C}^*)^n$ , resulting in  $X_{\Sigma}$ . More concretely, one obtains  $X_{\Sigma}$  from  $\Sigma$  as follows. For each n-dimensional cone  $\sigma \in \Sigma$ , one constructs an affine toric variety  $U_{\sigma}$  which we first explain. We have the dual cone  $\sigma^{\vee}$  of  $\sigma$ . Denote by  $S_{\sigma}$  the semigroup of those lattice points which lie in  $\sigma^{\vee}$  under addition. Then one defines the semigroup ring, as a set, as all finite sums of the form

$$\mathbb{C}[S_{\sigma}] = \Big\{ \sum \lambda_s s \mid s \in S_{\sigma} \Big\},\,$$

with the ring structure defined on monomials by  $\lambda_{s_1} s_1 \cdot \lambda_{s_2} s_2 = (\lambda_{s_1} \lambda_{s_2})(s_1 + s_2)$  and extended in the natural way. The affine variety  $U_{\sigma}$  is then defined to be  $\operatorname{Spec}(\mathbb{C}[S_{\sigma}])$ . This automatically comes endowed with a  $(\mathbb{C}^*)^n$ -action with a dense open orbit. This construc-

tion can also be applied to the lower-dimensional cones  $\tau \in \Sigma$ . If  $\sigma_1 \cap \sigma_2 = \tau$ , then there is a natural way to map  $U_{\tau}$  into  $U_{\sigma_1}$  and  $U_{\sigma_2}$  isomorphically. One constructs  $X_{\Sigma}$  by declaring the collection of all  $U_{\sigma}$  to be an open affine cover of  $X_{\Sigma}$  with transition functions determined by  $U_{\tau}$ . This identification is also reversible.

**Proposition 2.12** ([17, Corollary 3.1.8]). Let M be a smooth toric manifold. Then there exists a fan  $\Sigma$  such that  $M \simeq X_{\Sigma}$ .

**Proposition 2.13** ([17, Theorem 3.2.6], orbit-cone correspondence). The k-dimensional cones  $\sigma \in \Sigma$  are in a natural one-to-one correspondence with the (n-k)-dimensional orbits  $O_{\sigma}$  of the  $(\mathbb{C}^*)^n$ -action on  $X_{\Sigma}$ .

In particular, each ray  $\sigma \in \Sigma$  determines a unique torus-invariant divisor  $D_{\sigma}$ . As a consequence, a torus-invariant Weil divisor D on  $X_{\Sigma}$  naturally determines a polyhedron  $P_D \subset \mathfrak{t}^*$ . Indeed, we can decompose D uniquely as  $D = \sum_{i=1}^N a_i D_{\sigma_i}$ , where  $\{\sigma_i\}_i \subset \Sigma$  is the collection of rays. Then, by assumption, there exists a unique minimal lattice element  $v_i \in \sigma_i \cap \Gamma$ . The polyhedron  $P_D$  is then given by

$$(2.1) P_D = \{x \in \mathfrak{t}^* \mid \langle v_i, x \rangle \ge -a_i\} = \bigcap_{i=1}^N K(v_i, -a_i).$$

**2.5.2.** Kähler metrics on toric varieties. For a given toric manifold M endowed with a Riemannian metric g invariant under the action of the real torus  $T^n \subset (\mathbb{C}^*)^n$  and Kähler with respect to the underlying complex structure of M, the Kähler form  $\omega$  of g is also invariant under the  $T^n$ -action. We call such a manifold a *toric Kähler manifold*. In what follows, we always work with a fixed complex structure on M.

Hamiltonian Kähler metrics have a useful characterisation due to Guillemin.

**Proposition 2.14** ([24, Theorem 4.1]). Let  $\omega$  be any  $T^n$ -invariant Kähler form on M. Then the  $T^n$ -action is Hamiltonian with respect to  $\omega$  if and only if the restriction of  $\omega$  to the dense orbit  $(\mathbb{C}^*)^n \subset M$  is exact, i.e., there exists a  $T^n$ -invariant potential  $\phi$  such that

$$\omega = 2i\partial\bar{\partial}\phi.$$

Fix once and for all a  $\mathbb{Z}$ -basis  $(X_1,\ldots,X_n)$  of  $\Gamma\subset \mathfrak{t}$ . This in particular induces a background coordinate system  $\xi=(\xi^1,\ldots,\xi^n)$  on  $\mathfrak{t}$ . Using the natural inner product on  $\mathfrak{t}$  to identify  $\mathfrak{t}\cong\mathfrak{t}^*$ , we can also identify  $\mathfrak{t}^*\cong\mathbb{R}^n$ . For clarity, we will denote the induced coordinates on  $\mathfrak{t}^*$  by  $x=(x^1,\ldots,x^n)$ . Let  $(z_1,\ldots,z_n)$  be the natural coordinates on  $(\mathbb{C}^*)^n$  as an open subset of  $\mathbb{C}^n$ . There is a natural diffeomorphism  $\mathrm{Log}:(\mathbb{C}^*)^n\to\mathfrak{t}\times T^n$  which provides a one-to-one correspondence between  $T^n$ -invariant smooth functions on  $(\mathbb{C}^*)^n$  and smooth functions on  $\mathfrak{t}$ . Explicitly,

$$(2.2) \quad (z_1,\ldots,z_n) \stackrel{\text{Log}}{\longmapsto} \left(\log(r_1),\ldots,\log(r_n),\theta_1,\ldots,\theta_n\right) = (\xi_1,\ldots,\xi_n,\theta_1,\ldots,\theta_n),$$

where  $z_j = r_j e^{i\theta_j}$ ,  $r_j > 0$ . Given a function  $H(\xi)$  on t, we can extend H trivially to  $t \times T^n$  and pull back by Log to obtain a  $T^n$ -invariant function on  $(\mathbb{C}^*)^n$ . Clearly, any  $T^n$ -invariant function on  $(\mathbb{C}^*)^n$  can be written in this form.

Choose any branch of log and write  $w = \log(z)$ . Then we clearly have that  $w = \xi + i\theta$ , where  $\xi = (\xi^1, \dots, \xi^n)$  are real coordinates on t (or, more precisely, there is a corresponding lift of  $\theta$  to the universal cover with respect to which this equality holds), and so if  $\phi$  is  $T^n$ -invariant and  $\omega = 2i\partial\bar{\partial}\phi$ , then we have that

(2.3) 
$$\omega = 2i \frac{\partial^2 \phi}{\partial w^i \partial \bar{w}^j} dw_i \wedge d\bar{w}_j = \frac{\partial^2 \phi}{\partial \xi^i \partial \xi^j} d\xi^i \wedge d\theta^j.$$

In this setting, the metric g corresponding to  $\omega$  is given on  $t \times T^n$  by

$$g = \phi_{ij}(\xi) d\xi^i d\xi^j + \phi_{ij}(\xi) d\theta^i d\theta^j,$$

and the moment map  $\mu$  as a map  $\mu$ :  $t \times T^n \to t^*$  is defined by the relation

$$\langle \mu(\xi, \theta), b \rangle = \langle \nabla \phi(\xi), b \rangle$$

for all  $b \in t$ , where  $\nabla \phi$  is the Euclidean gradient of  $\phi$ . The  $T^n$ -invariance of  $\phi$  implies that it depends only on  $\xi$  when considered a function on  $t \times T^n$  via (2.2). Since  $\omega$  is Kähler, we see from (2.3) that the Hessian of  $\phi$  is positive-definite so that  $\phi$  itself is strictly convex. In particular,  $\nabla \phi$  is a diffeomorphism onto its image. Using the identifications mentioned above, we view  $\nabla \phi$  as a map from t into an open subset of  $t^*$ .

**2.5.3. Kähler–Ricci solitons on toric manifolds.** Next we define what we mean by a shrinking Kähler–Ricci soliton in the toric category.

**Definition 2.15.** A complex *n*-dimensional shrinking Kähler–Ricci soliton (M, g, X) with complex structure J and Kähler form  $\omega$  is *toric* if  $(M, \omega)$  is a toric Kähler manifold as in Definition 2.10 and JX lies in the Lie algebra t of the underlying real torus  $T^n$  that acts on M. In particular, the zero set of X is compact.

It follows from [38] that  $\pi_1(M)=0$ ; hence the induced real  $T^n$ -action is automatically Hamiltonian with respect to  $\omega$ . Working on the dense orbit  $(\mathbb{C}^*)^n \subset M$ , the condition that a vector field JY lies in t is equivalent to saying that, in the coordinate system  $(\xi^1,\ldots,\xi^n,\theta_1,\ldots,\theta_n)$  from (2.2), there is a constant  $b_Y=(b_Y^1,\ldots,b_Y^n)\in\mathbb{R}^n$  such that

(2.4) 
$$JY = b_Y^i \frac{\partial}{\partial \theta^i}, \quad \text{or equivalently,} \quad Y = b_Y^i \frac{\partial}{\partial \xi^i}.$$

From Proposition 2.14, we know that  $\mathcal{L}_X \omega = 2i \partial \bar{\partial} X(\phi)$ . In addition, the function  $X(\phi)$  on  $(\mathbb{C}^*)^n$  can be written as

$$\langle b_X, \nabla \phi \rangle = b_X^j \frac{\partial \phi}{\partial \varepsilon^j},$$

where  $b_X \in \mathbb{R}^n$  corresponds to the soliton vector field X via (2.4). These observations allow us to write the shrinking soliton equation (1.2) as a real Monge–Ampère equation for  $\phi$  on  $\mathbb{R}^n$ .

**Proposition 2.16** ([12, Proposition 2.6]). Let (M, g, X) be a toric shrinking gradient Kähler–Ricci soliton with Kähler form  $\omega$ . Then there exists a unique smooth convex real-valued function  $\phi$  defined on the dense orbit  $(\mathbb{C}^*)^n \subset M$  such that  $\omega = 2i \bar{\partial} \bar{\partial} \phi$  and

(2.5) 
$$\det(\phi_{ij}) = e^{-2\phi + \langle b_X, \nabla \phi \rangle}.$$

A priori, the function  $\phi$  is defined only up to addition of a linear function. However, (2.5) provides a normalisation for  $\phi$  which in turn provides a normalisation for  $\nabla \phi$ , the moment map of the action. The next lemma shows that this normalisation coincides with that for the moment map as defined in [16, Definition 5.16].

**Lemma 2.17.** Let (M, g, X) be a toric complete shrinking gradient Kähler–Ricci soliton with complex structure J and Kähler form  $\omega$  with soliton vector field  $X = \nabla^g f$  for a smooth real-valued function  $f: M \to \mathbb{R}$ . Let  $\phi$  be given by Proposition 2.14 and normalised by (2.5), let  $JY \in \mathfrak{t}$ , and let  $u_Y = \langle \nabla \phi, b_Y \rangle$  be the Hamiltonian potential of JY with  $b_Y$  as in (2.4) so that  $\nabla^g u_Y = Y$ . Then  $\mathcal{L}_{JX}u_Y = 0$  and  $\Delta_\omega u_Y + u_Y - \frac{1}{2}Y \cdot f = 0$ .

To see the equivalence with [16, Definition 5.16], simply replace Y with JY in this latter definition as here we assume that  $JY \in \mathfrak{t}$ , contrary to the convention in [16, Definition 5.16] where it is assumed that  $Y \in \mathfrak{t}$ .

Given the normalisation (2.5), the next lemma identifies the image of the moment map  $\mu = \nabla \phi$ .

**Lemma 2.18** ([12, Lemmas 4.4 and 4.5]). Let (M, g, X) be a complete toric shrinking gradient Kähler–Ricci soliton, let  $\{D_i\}$  be the prime  $(\mathbb{C}^*)^n$ -invariant divisors in M, and let  $\Sigma \subset \mathfrak{t}$  be the fan determined by Proposition 2.12. Let  $\sigma_i \in \Sigma$  be the ray corresponding to  $D_i$  with minimal generator  $v_i \in \Gamma$ .

(i) There is a distinguished Weil divisor representing the anticanonical class  $-K_M$  given by

$$-K_M = \sum_i D_i$$

whose associated polyhedron (cf. (2.1)) is given by

$$(2.6) P_{-K_M} = \{x \mid \langle v_i, x \rangle \ge -1\}$$

which is strongly convex and has full dimension in  $t^*$ . In particular, the origin lies in the interior of  $P_{-K_M}$ .

- (ii) If  $\mu$  is the moment map for the induced real  $T^n$ -action normalised by (2.5), then the image of  $\mu$  is precisely  $P_{-K_M}$ .
- **2.5.4.** The weighted volume functional. As a result of Lemma 2.17, we can now define the weighted volume functional.

**Definition 2.19** (Weighted volume functional [16, Definition 5.16]). Let (M, g, X) be a complex n-dimensional toric shrinking gradient Kähler–Ricci soliton with Kähler form

$$\omega = 2i\partial\bar{\partial}\phi$$

on the dense orbit with  $\phi$  strictly convex with moment map  $\mu = \nabla \phi$  normalised by (2.5). Assume that the fixed point set of the torus is compact and define the open convex cone

$$\Lambda_{\omega} := \{Y \in \mathfrak{t} \mid \langle \mu, Y \rangle \text{ is proper and bounded below} \} \subseteq \mathfrak{t}.$$

Then the *weighted volume functional*  $\mathcal{F}_{\omega}$ :  $\Lambda_{\omega} \to \mathbb{R}$  is defined by

$$\mathcal{F}_{\omega}(v) = \int_{M} e^{-\langle \mu, v \rangle} \omega^{n}.$$

As the fixed point set of the torus is compact by definition,  $\mathcal{F}_{\omega}$  is well-defined by the non-compact version of the Duistermaat–Heckman formula [31] (see also [16, Theorem A.3]). It is moreover strictly convex on  $\Lambda_{\omega}$  (see [16, Lemma 5.17 (i)]), hence has at most one critical point in this set. This leads to two important lemmas concerning the weighted volume functional in the toric category, the independence of  $\Lambda_{\omega}$  and  $\mathcal{F}_{\omega}$  from the choice of shrinking soliton  $\omega$ .

**Lemma 2.20** ([13, Lemma 2.25]).  $\Lambda_{\omega}$  is independent of the choice of toric shrinking Kähler–Ricci soliton  $\omega$  in Definition 2.19.

**Lemma 2.21** ([13, Lemma 2.26]).  $\mathcal{F}_{\omega}$  is independent of the choice of toric shrinking Kähler–Ricci soliton  $\omega$  in Definition 2.19. Moreover, after identifying  $\Lambda_{\omega}$  with a subset of  $\mathbb{R}^n$  via (2.4),  $\mathcal{F}_{\omega}$  is given by

$$\mathcal{F}_{\omega}(v) = (2\pi)^n \int_{P-K_M} e^{-\langle v, x \rangle} dx,$$

where  $x = (x^1, ..., x^n)$  denotes coordinates on  $t^*$  dual to the coordinates  $(\xi^1, ..., \xi^n)$  on t introduced in Section 2.5.2.

Thus, we henceforth drop the subscript  $\omega$  from  $\mathcal{F}_{\omega}$  and  $\Lambda_{\omega}$  when working in the toric category. The functional  $\mathcal{F}: \Lambda \to \mathbb{R}$  is in addition proper in this category [12, proof of Proposition 3.1], hence attains a unique critical point in  $\Lambda$ . This critical point characterises the soliton vector field of a complete toric shrinking gradient Kähler–Ricci soliton.

**Theorem 2.22** ([12, Theorem 4.6], [11, Theorem 1.1]). Let (M, g, X) be a complete toric shrinking gradient Kähler–Ricci soliton with complex structure J. Then  $JX \in \Lambda$  and JX is the unique critical point of  $\mathcal{F}$  in  $\Lambda$ .

Having established in Lemmas 2.20 and 2.21 that, in the toric category, the weighted volume functional F and its domain  $\Lambda$  are determined solely by the polytope  $P_{-K_M}$  which itself, by Lemma 2.18, depends only on the torus action on M (i.e., is independent of the choice of shrinking soliton), and having an explicit expression for  $\mathcal{F}$  given by Lemma 2.21, after using the torus action to identify  $P_{-K_M}$  via (2.6), we can determine explicitly the soliton vector field of a hypothetical toric shrinking gradient Kähler–Ricci soliton on M. Indeed, in light of Lemma 2.21, the unique minimiser  $b_X \in \mathfrak{t} \simeq \mathbb{R}^n$  is characterised by the fact that

$$0 = d_{b_X} \mathcal{F}(v) = \int_{P_{-K_M}} \langle x, v \rangle e^{-\langle b_X, x \rangle} \, dx \quad \text{for all } v \in \mathbb{R}^n.$$

In the setting of Theorem A, we can also determine  $\Lambda$  explicitly. To this end, with notation as in Theorem A, we make the following observation concerning the Lie algebra t of T. By assumption, the restricted map  $\pi|_M: M \to \hat{M} := \mathbb{C} \times D$  is a torus-equivariant biholomorphism on the complement of  $\pi^{-1}(D_0) \subseteq M$  and  $D_0 \subseteq \hat{M}$ ; hence  $M \setminus \pi^{-1}(D_0)$  is

 $(\mathbb{C}^*)^n$ -equivariantly biholomorphic to  $\mathbb{C}^* \times D \supseteq (\mathbb{C}^*)^n$ . It subsequently follows that t admits the splitting  $\mathfrak{t} \simeq \mathfrak{t}_{\mathbb{C}^*} \oplus \mathfrak{t}_D$ , where  $\mathfrak{t}_{\mathbb{C}^*}$  and  $\mathfrak{t}_D$  denote the Lie algebra of vector fields in t on M whose image under  $d\pi$  vanish along the D- and  $\mathbb{C}^*$ -factors of  $\widehat{M} \setminus D_0$ , respectively. With this in mind, we then have the following lemma.

**Lemma 2.23.** In the setting and notation of Theorem A and with respect to the splitting  $t \simeq t_{\mathbb{C}^*} \oplus t_D$ , the domain  $\Lambda$  of the weighted volume functional  $\mathcal{F}$  is the half-space

$$\Lambda = \{ \alpha \operatorname{Re}(z \partial_z) + Y \in \mathfrak{t}_{\mathbb{C}^*} \oplus \mathfrak{t}_D \mid \alpha > 0 \text{ and } Y \in \mathfrak{t}_D \}.$$

*Proof.* Since D is Fano, by Lemma 2.18, we know that the anticanonical polyhedron  $P_{-K_{\mathbb{C}\times D}}$  for  $\mathbb{C}\times D$  is the "simple product", i.e.,

$$(2.7) P_{-K_{\mathbb{C}\times D}} = \{(x_1, \dots, x_n) \mid x_1 \ge -1 \text{ and } (x_2, \dots, x_n) \in P_D\}.$$

Moreover, it follows from the definition of  $\pi$  that the normal fan  $\Sigma_M$  of  $P_{-K_M}$  is just a refinement of the normal fan  $\Sigma_{\mathbb{C}\times D}$  of  $P_{-K_{\mathbb{C}\times D}}$  (see [17, Definition 3.3.17]). The set of defining equations for  $P_{-K_M}$  is therefore obtained from those defining (2.7) by including finitely many linear inequalities. This in particular implies that  $P_{-K_M}$  and  $P_{-K_{\mathbb{C}\times D}}$  coincide outside a sufficiently large ball  $B\subset \mathfrak{t}^*$ .

Let  $Z \in \mathfrak{t}$  and, via (2.4), identify Z with a point  $b_Z \in \mathbb{R}^n$ . Then the distinguished vector field  $\operatorname{Re}(z\partial_z) \in \mathfrak{t}$  is identified with  $(1,0,\ldots,0) \in \mathbb{R}^n$  via the aforesaid splitting of  $\mathfrak{t}$  so that  $Z = \alpha \operatorname{Re}(z\partial_z) + Y \in \mathfrak{t}_{\mathbb{C}^*} \oplus \mathfrak{t}_D$  is identified with the point  $b_Z = (\alpha,b_2,\ldots,b_n) \in \mathbb{R}^n$  for some  $b_i \in \mathbb{R}$ ,  $i=2,\ldots,n$ . Since  $P_{-K_M}$  is closed, it follows that the Hamiltonian potential  $\mu_Z = \langle \mu, Z \rangle = \langle x, b_Z \rangle$  of Z is proper if and only if  $|\langle x, b_Z \rangle| \to +\infty$  as  $|x| \to +\infty$ . Thus, since D is compact so that  $P_D$  is bounded, we see that the set of vector fields  $Z \in \mathfrak{t}$  for which the Hamiltonian potential  $\mu_Z$  is proper is precisely the complement of the inclusion  $t_D \hookrightarrow \mathfrak{t}$ . In addition,  $\mu_Z$  is bounded from below if and only if  $\langle x, b \rangle \to +\infty$  as  $|x| \to +\infty$  in  $P_{-K_M}$ . As  $|x| \to +\infty$  in  $P_{-K_M}$  if and only if  $x_1 \to +\infty$ , the condition that  $\mu_Z$  be bounded from below picks out the desired component of  $\mathfrak{t}$  defining  $\Lambda$ .

We illustrate an application of Lemma 2.23 with the following example.

**Example 2.24.** Let  $D = \mathbb{P}^1$ , let  $\pi$  be the blowup map, and let  $([z_1 : z_2], w)$  denote coordinates on  $\mathbb{P}^1 \times \mathbb{C}$ . Then there is an action of a real two-dimensional torus  $T^2$  on  $\mathbb{P}^1 \times \mathbb{C}$  given by

$$([z_1:z_2], w) \mapsto ([e^{ib_2}z_1:z_2], e^{ib_1}w),$$

where  $(b_1,b_2)\in\mathbb{R}^2$  which we identify with the Lie algebra t of  $T^2$ . Moreover, M is the blowup of  $\mathbb{P}^1\times\mathbb{C}$  at one point which, without loss of generality, we may assume to be ([0:1],0). The action of  $T^2$  on  $\mathbb{P}^1\times\mathbb{C}$  induces a  $T^2$ -action on M in the obvious way. Lemma 2.23 then tells us that the domain  $\Lambda$  of the weighted volume functional  $\mathcal{F}$  of M is given by

$$\{(b_1, b_2) \in \mathbb{R}^2 \mid b_1 > 0 \text{ and } b_2 \in \mathbb{R}\} \subseteq \mathfrak{t}.$$

Using the Duistermaat–Heckman theorem [16, Theorem A.3], one can write  $\mathcal{F}$  as

$$\mathcal{F}(b_1, b_2) = \frac{e^{b_1}}{(b_1 - b_2)b_2} + \frac{e^{b_2}}{(b_2 - b_1)b_1} - \frac{e^{b_1 - b_2}}{b_1 b_2}.$$

Observe that this is symmetric under the transformation  $(b_1, b_2) \mapsto (b_1, b_1 - b_2)$ , a transformation that preserves  $\Lambda$ . The minimum of  $\mathcal{F}$  in  $\Lambda$  therefore lies along the line  $0 < b_1 = 2b_2$ , in which case we have, for  $b_2 > 0$ ,

$$\mathcal{F}(b_2) = \frac{e^{2b_2} - e^{b_2}}{b_2^2}.$$

We then have that

$$\mathcal{F}'(b_2) = b_2^{-3} e^{b_2} [2(b_2 - 1)e^{b_2} - (b_2 - 2)].$$

This has a zero for  $b_2 > 0$  precisely when  $2(b_2 - 1)e^{b_2} = b_2 - 2$ . Numerical approximations give the unique positive root as  $b_2 \approx 0.64$ , in agreement with [13, Example 2.33].

**2.5.5.** The Legendre transform. Let M be a toric manifold of complex dimension n endowed with a complete Kähler form  $\omega$  invariant under the induced real  $T^n$ -action and with respect to which this action is Hamiltonian. Write  $\omega = 2i\partial\bar{\partial}\phi$  on the dense orbit for  $\phi$  strictly convex as in Proposition 2.14. Then  $\nabla\phi(\mathbb{R}^n)$  is a Delzant polytope P. Recall that we have coordinates  $\xi$  on  $\mathbb{R}^n \simeq \mathfrak{t}$ , x on P, and  $\theta$  on T. Given any smooth and strictly convex function  $\psi$  on  $\mathbb{R}^n$  such that  $\nabla\psi(\mathbb{R}^n) = P$ , there exists a unique smooth and strictly convex function  $u_{\psi}(x)$  on P defined by

$$\psi(\xi) + u_{\psi}(\nabla \psi) = \langle \nabla \psi, \xi \rangle.$$

This process is reversible, that is to say,  $\psi$  is the unique function satisfying

$$\psi(\nabla u_{\psi}) + u_{\psi}(x) = \langle x, \nabla u_{\psi} \rangle,$$

where  $\nabla$  now denotes the Euclidean gradient with respect to x. The function  $u_{\psi}$  is called the *Legendre transform of*  $\psi$  and is sometimes denoted by  $L(\psi)(x)$ . Clearly,  $L(L(\psi))(\xi) = \psi(\xi)$ . The Legendre transform u of  $\phi$  is called the *symplectic potential* of  $\omega$ , as the metric g associated to  $\omega$  is given by

$$g = u_{ij}(x) dx^i dx^j + u^{ij}(x) d\theta^i d\theta^j.$$

The following will prove useful.

**Lemma 2.25** (cf. [12, Lemma 2.10]). Let  $\phi$  be any smooth and strictly convex function on an open convex domain  $\Omega' \subset \mathbb{R}^n$  and let  $u = L(\phi)$  be the Legendre transform of  $\phi$  defined on  $(\nabla \phi)(\Omega') =: \Omega$ . If  $0 \in \Omega$ , then there exists a constant C > 0 such that

$$\phi(\xi) \ge C^{-1}|\xi| - C.$$

In particular,  $\phi$  is proper and bounded from below.

If  $\phi \in C^{\infty}(\mathbb{R}^n)$  solves (2.5), then the Legendre transform  $u = L(\phi)$  satisfies

(2.8) 
$$2(\langle \nabla u, x \rangle - u(x)) - \log \det(u_{ij}(x)) = \langle b_X, x \rangle \quad \text{on } P_{-K_M}.$$

To study Kähler–Ricci solitons on M via (2.8) on  $P_{-K_M}$ , we need to understand when a strictly convex function on a Delzant polytope defines a symplectic potential, i.e., is induced from a Kähler metric on M via the Legendre transform. To this end, consider a Delzant polytope P obtained as the image of the moment map of a toric Kähler manifold. Let  $F_i$ ,  $i = 1, \ldots, d$ ,

denote the (n-1)-dimensional facets of P with inward-pointing normal vector  $v_i \in \Gamma$ , normalised so that  $v_i$  is the minimal generator of  $\sigma_i = \mathbb{R}_+ \cdot v_i$  in  $\Gamma$ , and let  $\ell_i(x) = \langle v_i, x \rangle$  so that  $\overline{P}$  is defined by the system of inequalities  $\ell_i(x) \geq -a_i$ ,  $i = 1, \ldots, N$ ,  $a_i \in \mathbb{R}$ . Then there exists a canonical metric  $\omega_P$  on M (see [12, Proposition 2.7]), the symplectic potential  $u_P$  of which is given explicitly by the formula (see [9, 24])

(2.9) 
$$u_P(x) = \frac{1}{2} \sum_{i=1}^{d} (\ell_i(x) + a_i) \log(\ell_i(x) + a_i).$$

In particular, the Legendre transform  $\phi_P$  of  $u_P$  will define the Kähler potential on the dense orbit of a globally defined Kähler metric  $\omega_P$  on M (see [9,24]). More generally, it was observed by Abreu [1] that the Legendre transform L(u) of a strictly convex function u on P will define the Kähler potential on the dense orbit of a globally defined Kähler metric  $\omega_P$  on M if and only if u has the same asymptotic behaviour as  $u_P$  of all orders as  $x \to \partial P$ . Indeed, we have the following slightly more general statement.

**Lemma 2.26** ([1], [2], [12, Proposition 2.17]). A convex function u on P defines a Kähler metric  $\omega_u$  on M if and only if u has the form

$$u = u_P + v$$
,

where  $v \in C^{\infty}(\overline{P})$  extends past  $\partial P$  to all orders.

In the case that  $P=P_{-K_M}$ , we read from Lemma 2.18 (ii) that  $a_i=-1$  for all i. Thus, in this case, the canonical metric on  $P_{-K_M}$  has symplectic potential

$$u_{P-K_M} = \frac{1}{2} \sum_{i} (\ell_i(x) + 1) \log(\ell_i(x) + 1).$$

**2.5.6.** The  $\hat{F}$ -functional. We next define the  $\hat{F}$ -functional on toric Kähler manifolds.

**Definition 2.27.** Let  $(M,\omega)$  be a (possibly non-compact) toric Kähler manifold with complex structure J endowed with a real holomorphic vector field X such that  $JX \in \Lambda_{\omega}$ . Write T for the torus acting on M, identify the dense orbit with  $\mathbb{R}^n$ , let  $\xi = (\xi_1, \ldots, \xi_n)$  denote coordinates on  $\mathbb{R}^n$ , let  $b_X$  be as in (2.4), and write  $\omega = 2i\partial\bar{\partial}\phi_0$  on the dense orbit as in Proposition 2.14. Let  $P := (\nabla\phi_0)(\mathbb{R}^n)$  denote the image of the moment map associated to  $\omega$  and let  $x = (x_1, \ldots, x_n)$  denote coordinates on P. Let  $\varphi \in C^{\infty}(M)$  be a smooth function on M invariant under the action of T such that  $\omega + i\partial\bar{\partial}\varphi > 0$  and assume that

- (a) there exists a  $C^1$ -path of smooth functions  $(\varphi_s)_{s\in[0,1]}\subset C^\infty(M)$  invariant under the action of T such that  $\varphi_0=0,\ \varphi_1=\varphi,\ \omega+i\partial\bar\partial\varphi_s>0$ , and  $(\nabla\phi_s)(\mathbb{R}^n)=P$  for all  $s\in[0,1]$ , where  $\phi_s:=\phi_0+\frac{\varphi_s}{2}$ .
- (b)  $\int_0^1 \int_{\mathbb{R}^n} |\dot{\phi}_s| e^{-\langle b_X, \nabla \phi_s \rangle} \det(\phi_{s,ij}) d\xi ds < +\infty.$

Then we define

$$\widehat{F}(\varphi) := 2 \int_{P} (L(\phi_1) - L(\phi_0)) e^{-\langle b_X, x \rangle} \, dx.$$

The existence of the path  $(\varphi_s)_{s \in [0,1]}$  satisfying conditions (a) and (b) is required so that  $\widehat{F}(\varphi)$  is well-defined. To see this, first note the following lemma.

**Lemma 2.28.** *Under the assumptions of Definition* 2.27, *let* 

$$u_s := L(\phi_s), \quad \omega_s = \omega + i \partial \bar{\partial} \varphi_s,$$

and write  $f_s := f + \frac{X}{2} \cdot \varphi_s$  for the Hamiltonian potential of JX with respect to  $\omega_s$ , where f is the Hamiltonian potential of JX with respect to  $\omega$ . Then the following are equivalent.

- (i)  $\int_0^1 \int_{\mathbb{R}^n} |\dot{\phi}_s| e^{-\langle b_X, \nabla \phi_s \rangle} \det(\phi_{s,ij}) d\xi ds < +\infty.$
- (ii)  $\int_0^1 \int_P |\dot{u}_s| e^{-\langle b_X, x \rangle} dx ds < +\infty$ .
- (iii)  $\int_0^1 \int_M |\dot{\varphi}_s| e^{-f_s} \omega_s^n ds < +\infty$ .

In particular, when this is the case,  $|\hat{F}(\varphi)| < +\infty$ .

*Proof.* The equivalence of (i) and (iii) is clear. The equivalence of (i) and (ii) follows from [12, Lemma 3.7]. Finally, for the last statement, for every  $x \in P$ , we have that

$$|u_1 - u_0|(x) \le \int_0^1 |\dot{u}_s|(x) \, ds.$$

Then, using Fubini's theorem and noting Lemma 2.28, we estimate that

$$|\widehat{F}(\varphi)| \le 2 \int_{P} |u_1 - u_0| e^{-\langle b_X, x \rangle} \, dx \le 2 \int_{P} \left( \int_{0}^{1} |\dot{u}_s| \, ds \right) e^{-\langle b_X, x \rangle} \, dx$$

$$= 2 \int_{0}^{1} \int_{P} |\dot{u}_s| e^{-\langle b_X, x \rangle} \, dx \, ds < +\infty,$$

as desired.

Under an additional assumption on the path  $(\varphi_s)_{s \in [0,1]}$ , we recover the well-known expression for the  $\hat{F}$ -functional given in [10, p. 702].

**Lemma 2.29.** If one (and hence all) of the conditions of Lemma 2.28 hold true and if in addition it holds true that  $\int_0^1 \int_M |\dot{\varphi}_s| e^{-f} \omega^n ds < +\infty$ , then

(2.10) 
$$\widehat{F}(\varphi) = \int_0^1 \int_M \dot{\varphi}_s(e^{-f}\omega^n - e^{-f_s}\omega_s^n) \wedge ds - \int_M \varphi e^{-f}\omega^n.$$

*Proof.* The extra condition implies in particular that  $\int_M |\varphi| e^{-f} \omega^n < +\infty$  since, by assumption and Fubini's theorem,

$$\int_{M} |\varphi| e^{-f} \omega^{n} \leq \int_{0}^{1} \int_{M} |\dot{\varphi}_{s}| e^{-f} \omega^{n} ds < +\infty$$

so that the right-hand side of (2.10) is at least finite. To show that it is equivalent to the expression for  $\hat{F}$  given by Definition 2.27, using the change of coordinates induced by  $\nabla \phi_s$ :  $\mathbb{R}^n \to P$  and the fact that  $\dot{\phi}_s(\nabla \phi_s) = -\dot{u}_s(x)$  (cf. [12, Lemma 3.7]), we compute that

$$\widehat{F}(\varphi) = 2 \int_{P} (u_1(x) - u_0(x)) e^{-\langle b_X, x \rangle} dx$$
$$= 2 \int_{0}^{1} \int_{P} \dot{u}_s(x) e^{-\langle b_X, x \rangle} dx \wedge ds$$

$$= -2 \int_0^1 \int_P \dot{\phi_s}(\nabla \phi_s) e^{-\langle b_X, x \rangle} dx \wedge ds$$

$$= -2 \int_0^1 \int_{\mathbb{R}^n} \dot{\phi_s} e^{-\langle b_X, \nabla \phi_s \rangle} \det(\phi_{s,ij}) d\xi \wedge ds$$

$$= -\int_0^1 \int_M \dot{\phi_s} e^{-f_s} \omega_s^n \wedge ds$$

$$= \int_0^1 \int_M \dot{\phi_s} (e^{-f} \omega^n - e^{-f_s} \omega_s^n) \wedge ds - \int_M \varphi e^{-f} \omega^n,$$

resulting in the desired expression. Here we have used Fubini's theorem in the last equality.  $\ \square$ 

**2.5.7.** Integrability and independence of the path. In light of conditions (a) and (b) of Definition 2.27 required to define the  $\hat{F}$ -functional, it remains to identify sufficient conditions for the moment polytope to remain unchanged under a path of Kähler metrics and for each summand in the integral of  $\hat{F}$  to be finite. This will be important for achieving an a priori  $C^0$ -bound along our continuity path.

To this end, suppose that  $(M, \omega)$  is a toric Kähler manifold, i.e.,  $(M, \omega)$  is Kähler with Kähler form  $\omega$  with respect to a complex structure J, endowed with the holomorphic action of a complex torus of the same complex dimension as (M, J) whose underlying real torus T induces a Hamiltonian action, and let  $JX \in \mathfrak{t}$ . Via (2.4), we can identify X with an element  $b_X \in \mathbb{R}^n \simeq \mathfrak{t}$ . Using Proposition 2.14, we can also write  $\omega = 2i \partial \bar{\partial} \phi_0$  on the dense orbit for some strictly convex function  $\phi_0 \colon \mathbb{R}^n \to \mathbb{R}$ . Assume that

- $JX \in \Lambda_{\omega}$  so that the Hamiltonian potential f of JX is proper and bounded from below.
- There exists a smooth bounded real-valued function F on M so that the Ricci form  $\rho_{\omega}$  of  $\omega$  satisfies  $\rho_{\omega} + \frac{1}{2} \mathcal{L}_X \omega \omega = i \partial \bar{\partial} F$ .

The equation in the second bullet point reads as

$$(F + \log \det(\phi_{0,ij}) - \langle \nabla \phi_0, b_X \rangle + 2\phi_0)_{ij} = 0$$
 on  $\mathfrak{t} \simeq \mathbb{R}^n$ 

so that

$$F = -\log \det(\phi_{0,ij}) + \langle \nabla \phi_0, b_X \rangle - 2\phi_0 + a(\xi) \quad \text{on } \mathbb{R}^n$$

for some affine function  $a(\xi)$  defined on  $\mathbb{R}^n$ . By considering  $2\phi_0 + a + \langle \nabla a, b_X \rangle$ , we can therefore assume that

$$(2.11) F = -\log \det(\phi_{0,ij}) + \langle \nabla \phi_0, b_X \rangle - 2\phi_0 \quad \text{on } \mathbb{R}^n.$$

The main observation of this section is the following lemma.

**Lemma 2.30.** Under the above assumptions, suppose that  $\varphi \in C^{\infty}(M)$  is a torus-invariant smooth real-valued function on M such that

$$\omega_{\varphi} := \omega + i \, \partial \bar{\partial} \varphi > 0 \quad and \quad \sup_{M} |X \cdot \varphi| < +\infty.$$

Define  $\phi := \phi_0 + \frac{1}{2}\varphi$  so that  $\omega + i\partial\bar{\partial}\varphi = 2i\partial\bar{\partial}\phi$  on the dense orbit. Then

- (i) the image of the moment map  $\mu_{\omega_{\varphi}}: M \to t^*$  with respect to  $\omega_{\varphi}$  defined by the Euclidean gradient  $\nabla \phi: \mathbb{R}^n \to \mathbb{R}^n$  is equal to  $P_{-K_M}$ . In particular,  $0 \in \text{int}(\mu_{\omega_{\varpi}}(M))$ .
- (ii)  $\int_{P} |L(\phi_0)| e^{-\langle b_X, x \rangle} dx < +\infty.$

*Proof.* (i) To prove (i), we begin by noting that, as  $\sup_M |X \cdot \varphi| < +\infty$ , the Hamiltonian potential  $f_\varphi = f + \frac{X}{2} \cdot \varphi$  of X with respect to  $\omega_\varphi$  is proper and bounded from below. In particular, the image  $(\nabla \phi)(\mathbb{R}^n)$  of the moment map  $\mu_{\omega_\varphi} \colon M \to \mathfrak{t}^*$  is equal to a Delzant polyhedron P (see [12, Lemma 2.13]) that a priori depends on  $\varphi$ . Let  $u(x) := L(\phi)$  be the Legendre transform of  $\varphi$ . Then the domain of u is precisely P. We need to show that  $P = P_{-K_M}$ . To this end, let F be as in (2.11). Then a computation shows that

$$(2.12) -\log \det \phi_{ij} + \langle \nabla \phi, b_X \rangle - 2\phi = F + \log \left(\frac{\omega_{\varphi}^n}{\omega^n}\right) + \frac{X}{2} \cdot \varphi - \varphi.$$

Set  $A(x) := \langle b_X, x \rangle$  and define

$$\rho_u(x) := 2(\langle \nabla u, x \rangle - u(x)) - \log \det(u_{ij}).$$

Then, via the change of coordinates  $x = \nabla \phi(\xi)$ , we can rewrite (2.12) in terms of u as

(2.13) 
$$A(x) - \rho_u(x) = \left(F + \log\left(\frac{\omega_{\varphi}^n}{\omega^n}\right) + \frac{X}{2} \cdot \varphi - \varphi\right) (\nabla u(x)) \quad \text{on } P.$$

Observe that the right-hand side of (2.13) admits a continuous extension up to the boundary  $\partial P$  of P. Denoting the right-hand side of (2.12) by h which is a function  $h: M \to \mathbb{R}$ , this extension is simply given by  $h \circ \mu_{\omega_{\varphi}}^{-1}$ , where  $\mu_{\omega_{\varphi}}: M \to \overline{P}$ , as the moment map, has fibres precisely the orbits of the torus action.

We now proceed as in [12, Lemma 4.5] using an argument originally due to Donaldson [19]. Suppose that P is defined by the linear inequalities  $\ell_i(x) \geq -a_i$ , where  $\ell_i(x) = \langle v_i, x \rangle$ . Since the right-hand side of (2.13) as well as A(x) has a continuous extension up to  $\partial P$ , we see that the same holds true for  $\rho_u(x)$ . Moreover, as u is the symplectic potential of the Kähler form  $\omega_{\varphi}$  on P, we read from Lemma 2.26 that there exists a function  $v \in C^{\infty}(\overline{P}')$  with  $u = u_P + v$ , where  $u_P$  is given as in (2.9), i.e.,

(2.14) 
$$u_P(x) = \frac{1}{2} \sum (\ell_i(x) + a_i) \log(\ell_i(x) + a_i).$$

Fix any facet F' of P. Without loss of generality, we may assume that F' is defined by  $\ell_1(x) = -a_1$ . Up to a change of basis in  $t^*$ , we may also assume by the Delzant condition that  $\ell_1(x) = x_1$ . Fix a point p in the interior of F'. Then, from (2.14), we see that, in a neighbourhood of p, u can be written as

$$u(x) = u_P(x) + v(x) = \frac{1}{2}(x_1 + a_1)\log(x_1 + a_1) + v_1$$

for some smooth function  $v_1$  which extends smoothly across F'. From this expression, it follows that, in a small half ball B in the interior of P containing p,  $\rho_u$  takes the form

$$\rho_u(x) = x_1 \log(x_1 + a_1) - (x_1 + a_1) \log(x_1 + a_1) + \log(x_1 + a_1) + v_2$$
  
=  $(1 - a_1) \log(x_1 + a_1) + v_2$ 

for another smooth function  $v_2$  that extends smoothly across F' in B. Thus, already knowing that  $\rho_u$  has a continuous extension across  $\partial P$ , we deduce that  $1-a_1=0$ , i.e.,  $a_1=1$ . Continuing in this manner, we see that  $a_i=1$  for all i, leading us to the conclusion that  $P=P_{-K_M}$ .

(ii) Let  $u_0 = L(\phi_0)$ . Then, as  $u_0$  is a convex function on  $P_{-K_M}$  whose gradient has image equal to all of  $\mathbb{R}^n$  by the invertibility of the Legendre transform, it is proper and bounded from below by Lemma 2.25. Let A denote the lower bound, let  $\rho_u$  be as in part (i), and write  $\rho_0 = \rho_{u_0}$ . Then F being bounded implies the existence of a constant C > 0 such that  $|\rho_0 - \langle b_X, x \rangle| < C$  on  $P_{-K_M}$ . Indeed, this is clear from (2.11). Since

$$\int_{P-K_M} u_0 e^{-\rho_0} \, dx < +\infty$$

by [12, Lemma 4.7], it follows that

$$\int_{P-K_M} u_0 e^{-\langle b_X, x \rangle} \, dx < +\infty.$$

Finiteness of the integral

$$\int_{P-K_M} e^{-\langle b_X, x \rangle} \, dx$$

together with the fact that  $u_0$  is bounded from below now yields the desired result.

#### 3. Proof of Theorem A (ii): Construction of a background metric

Given the setup and notation of Theorem A and with X determined by Theorem A (i), we henceforth assume that the flow lines of JX close. In this section, we construct a background metric on M with the properties as stated in Theorem A (ii) with a construction reminiscent of that of [25, Section 4.2]. To this end, recall for a > 0 the (incomplete) shrinking gradient Kähler–Ricci soliton ( $\hat{M} := \mathbb{C} \times D$ ,  $\hat{\omega}_a := \tilde{\omega}_a + \omega_D$ ,  $\frac{2}{a} \cdot \text{Re}(z\partial_z)$ ) of Example 2.2 with complex structure  $\hat{J}$  endowed with the product holomorphic action of the real n-torus  $\hat{T}$ , with z denoting the holomorphic coordinate on the  $\mathbb{C}$ -factor of  $\hat{M}$ , and  $r := |z|^a$ .

#### **Proposition 3.1.** There exist

- (a) a complete Kähler metric  $\omega$  on M invariant under the action of T, and
- (b) a biholomorphism  $v: M \setminus K \to \hat{M} \setminus \hat{K}$ , where  $K \subset M$ ,  $\hat{K} \subset \hat{M}$  are compact, and  $\lambda > 0$  such that
  - (i)  $d\nu(X) = \frac{2}{\lambda} \cdot \text{Re}(z\partial_z)$ ,
  - (ii)  $\omega = v^*(\widetilde{\omega}_{\lambda} + \omega_D)$ , and
- (iii) the Ricci form  $\rho_{\omega}$  of  $\omega$  satisfies

(3.1) 
$$\rho_{\omega} + \frac{1}{2} \mathcal{L}_{X} \omega - \omega = i \partial \bar{\partial} F_{1}$$

for  $F_1 \in C^{\infty}(M)$  compactly supported with  $\mathcal{L}_{JX}F_1 = 0$ .

Theorem A (ii) immediately follows from this proposition. Indeed, this is easily seen by writing  $\omega_C := \widetilde{\omega}_{\lambda}$  (cf. Example 2.1) and  $\widehat{\omega} := \widehat{\omega}_{\lambda} = \widetilde{\omega}_{\lambda} + \omega_D$  (cf. Example 2.2). With  $\lambda$  fixed in subsequent sections, this is the notation that we adopt to be consistent with that of Theorem A. Property (iii) of this proposition will be used in the next section.

Proof of Proposition 3.1. Recall that  $\pi\colon \overline{M}\to \mathbb{P}^1\times D$  is a torus-equivariant holomorphic map that restricts to a holomorphic map  $\pi\colon M\to \hat M:=\mathbb{C}\times D$  by removing the fibre  $D_\infty$  from  $\overline{M}$  and  $\mathbb{P}^1\times D$ , respectively, and that z denotes the holomorphic coordinate on the  $\mathbb{C}$ -factor of  $\hat M$ . We define the map  $\nu\colon M\setminus \pi^{-1}(D_0)\to \hat M\setminus D_0$  of (b) as the  $\mathbb{C}^*$ -equivariant map one obtains by identifying a  $\mathbb{P}^1$ -fibre in each manifold and for each point in this  $\mathbb{P}^1$ , flowing along the vector field  $X^{1,0}:=\frac12(X-i(JX))$  on M and the holomorphic vector field  $z\partial_z$  on  $\hat M$ . Since the flow lines of JX close by assumption, this map is well-defined.

From the construction, it is clear that  $d\nu(X^{1,0})=\frac{2}{\lambda}\cdot z\partial_z$  for some  $\lambda>0$ . This defines  $\lambda$  and verifies condition (i) of the proposition. The map  $\nu$  then extends to a holomorphic map  $\overline{\nu}\colon \overline{M}\setminus \pi^{-1}(D_0)\to \widehat{M}\setminus D_0$ . On  $\mathbb{C}\times D$ , we consider the product metric  $\widehat{\omega}_\lambda$ . We write  $w:=\frac{1}{z}$  and  $r:=|z|^\lambda$ . Identifying  $M\setminus \pi^{-1}(D_0)$  and  $\widehat{M}\setminus D_0$  via  $\nu$ , we view these as functions, and  $\widehat{\omega}_\lambda$  as a Kähler form, both on the former. In this way, w defines a holomorphic coordinate on  $\overline{M}\setminus \pi^{-1}(D_0)$  with the divisor D at infinity defined by  $\{w=0\}$ .

Using v, we construct the background metric  $\omega$  of (a) in the following way. As  $\overline{M}$  is Fano by assumption, there exists a hermitian metric h on  $-K_{\overline{M}}$  with strictly positive curvature form  $\Theta_h$ . Moreover, since the normal bundle  $N_D$  of D in  $\overline{M}$  is trivial so that  $K_D = K_{\overline{M}}|_D$  by adjunction, the  $\partial\bar{\partial}$ -lemma guarantees the existence of a function  $u \in C^\infty(D)$  such that  $i\Theta_h|_D + i\partial\bar{\partial}u = \omega_D$ . Extend u to be constant along the w-direction and multiply this extension by a cut-off function depending only on w to further extend u to the whole of  $\overline{M}$ . We still denote this extension by u. If the support of this cut-off function is contained in a sufficiently small tubular neighbourhood of D, then the restriction of  $i\Theta_h + i\partial\bar{\partial}u$  to any of the D-fibres of the fibration will be positive-definite. All negative components of  $i\Theta_h + i\partial\bar{\partial}u$  on the total space  $\overline{M}$  can be compensated for by adding a sufficiently positive "bump 2-form" of the form  $\chi(|w|) dw \wedge d\bar{w}$ , where  $\chi$  is a bump function supported in an annulus containing the cut-off region; such a 2-form is automatically closed and (1,1) on  $\overline{M}$ , and exact on M. This creates a Kähler form  $\tau_1$  on M. One can verify in a sufficiently small neighbourhood of D that

(3.2) 
$$\tau_1 - \omega_D = O(|w|) \left( dw \wedge d\bar{w} + \sum_j dw \wedge d\bar{v}_{\bar{j}} + \sum_i dv_i \wedge d\bar{w} \right) \quad \text{as } w \to 0$$

for  $\{v_1, \ldots, v_{n-1}\}$  local holomorphic coordinates on D.

Next, let  $\psi : \mathbb{R} \to \mathbb{R}$  be a smooth function satisfying

$$\psi'(x), \psi''(x) \ge 0$$
 for all  $x \in \mathbb{R}$ ,

and

$$\psi(x) = \begin{cases} \text{const.} & \text{if } x < 1, \\ x & \text{if } x > 2, \end{cases}$$

and consider the composition  $k := \psi \circ r^2$ , a real-valued smooth function on M. One computes that

$$\frac{i}{2}\partial\bar{\partial}k = \psi''(r^2)\frac{i}{2}\partial r^2\wedge\bar{\partial}r^2 + \psi'(r^2)\frac{i}{2}\partial\bar{\partial}r^2 \geq 0,$$

a positive semi-definite form equal to  $\frac{i}{2}\partial\bar{\partial}r^2$  on the region of M where  $r^2 > 2$ . Define the Kähler form

$$\tau_2 := \tau_1 + \frac{i}{2} \partial \bar{\partial} k,$$

and in the holomorphic coordinates (z, v) on  $\widehat{M}$ , consider the product metric  $\widehat{\omega}_{\lambda}$  given by

$$\widehat{\omega}_{\lambda} := \widetilde{\omega}_{\lambda} + \omega_{D} = i \, \partial \bar{\partial} \left( \frac{|z|^{2\lambda}}{2} \right) + \omega_{D} = \frac{\lambda^{2} dz \wedge d\bar{z}}{2|z|^{2-2\lambda}} + \omega_{D}.$$

Then, from (3.2), it is clear that the difference is given by

$$\tau_{2} - \widehat{\omega}_{\lambda} = O(|w|) \left( dw \wedge d\bar{w} + \sum_{j} dw \wedge d\bar{v}_{\bar{j}} + \sum_{i} dv_{i} \wedge d\bar{w} \right) \quad \text{as } w \to 0$$

so that, in particular,

(3.3) 
$$|\tau_2 - \widehat{\omega}_{\lambda}|_{\widehat{\omega}_{\lambda}} = O(r^{-\frac{1}{\lambda}}).$$

We now work with the hermitian metric H on  $-K_{\widehat{M}}$  induced by  $\widehat{\omega}_{\lambda}$ . Via the map  $\nu$ , this pulls back to the hermitian metric

$$H = \frac{\lambda^2 \det((g_D)_{i\bar{j}})}{2|z|^{2-2\lambda}}$$

on  $-K_M|_{M\setminus\pi^{-1}(D_0)}$  with respect to the local trivialisation  $\partial_z\wedge\partial_{v_1}\wedge\cdots\wedge\partial_{v_{n-1}}$ . The corresponding curvature form is then given by  $-i\partial\bar\partial\log H=\omega_D$ . Hence, as a difference of two curvature forms, there exists a smooth real-valued function  $\phi$  defined on  $M\setminus\pi^{-1}(D_0)$  such that  $(i\,\Theta_h+i\partial\bar\partial u)-\omega_D=i\,\partial\bar\partial\phi$ . In particular, outside a large compact subset of M, we have that

(3.4) 
$$\tau_2 - \hat{\omega}_{\lambda} = i \partial \bar{\partial} \phi.$$

We claim that  $\phi$  is in fact smooth on  $\overline{M} \setminus \pi^{-1}(D_0)$ . To see this, note that, as

$$i\,\partial\bar{\partial}\phi = -i\,\partial\bar{\partial}\log\left(\frac{e^{-u}|\partial_z\wedge\partial_{v_1}\wedge\dots\wedge\partial_{v_{n-1}}|_h^2}{|\partial_z\wedge\partial_{v_1}\wedge\dots\wedge\partial_{v_{n-1}}|_H^2}\right)$$

and

$$\begin{split} \log & \left( \frac{e^{-u} |\partial_z \wedge \partial_{v_1} \wedge \dots \wedge \partial_{v_{n-1}}|_H^2}{|\partial_z \wedge \partial_{v_1} \wedge \dots \wedge \partial_{v_{n-1}}|_H^2} \right) \\ &= \log \left( \frac{e^{-u} |w|^4 |\partial_w \wedge \partial_{v_1} \wedge \dots \wedge \partial_{v_{n-1}}|_h^2}{|\partial_z \wedge \partial_{v_1} \wedge \dots \wedge \partial_{v_{n-1}}|_H^2} \right) \\ &= \log \left( \frac{2e^{-u} |\partial_w \wedge \partial_{v_1} \wedge \dots \wedge \partial_{v_{n-1}}|_H^2 |z|^{2-2\lambda}}{\lambda^2 \det((g_D)_{i\bar{J}})} \right) + 2 \log(|w|^2) \\ &= \log \left( \frac{2e^{-u} |\partial_w \wedge \partial_{v_1} \wedge \dots \wedge \partial_{v_{n-1}}|_h^2 |z|^{2-2\lambda}}{|\partial_{v_1} \wedge \dots \wedge \partial_{v_{n-1}}|_{\omega_D}^2} \right) - \underbrace{(1-\lambda) \log|w|^2 + 2 \log(|w|^2) - \log(\lambda^2)}_{\text{pluriharmonic}}, \end{split}$$

 $\phi$  may be taken to be

(3.5) 
$$\phi = -\log\left(\frac{2e^{-u}|\partial_w \wedge \partial_{v_1} \wedge \dots \wedge \partial_{v_{n-1}}|_h^2}{|\partial_{v_1} \wedge \dots \wedge \partial_{v_{n-1}}|_{\omega_D}^2}\right),$$

which, albeit defined in terms of local coordinates, is clearly globally defined on  $\overline{M} \setminus \pi^{-1}(D_0)$ . Thus,  $\phi = O(1)$ , and from (3.3) and (3.4), we see that  $|i\partial\bar\partial\phi|_{\widehat\omega_\lambda} = O(r^{-\frac{1}{\lambda}})$ . Finally, after a computation, the expression for  $\phi$  given in (3.5) gives us that  $|d\phi|_{\widehat\omega_\lambda} = O(1)$ . Now,  $\widehat\omega_\lambda$  and  $\tau_2$  are equivalent outside some large compact subset  $\widetilde K$  of M by (3.3), and on the complement of  $\widetilde K$  in M, Lemma 2.4 implies that, for all R>0 sufficiently large,  $\phi$  admits an extension  $\phi_R$  to M supported on  $M\setminus \widetilde K$  such that  $|i\partial\bar\partial\phi_R|_{\widehat\omega_\lambda}\leq CR^{-\min\{\lambda^{-1},1\}}$ . Thus, at the expense of increasing C if necessary, we can infer that  $|i\partial\bar\partial\phi_R|_{\tau_2}\leq CR^{-\min\{\lambda^{-1},1\}}$  globally on M. We fix R>0 large enough so that  $|i\partial\bar\partial\phi_R|_{\tau_2}<1$  and define a Kähler form on M by

$$\tilde{\omega} := \tau_2 - i \partial \bar{\partial} \phi_R$$
.

By what we have just said,  $\widetilde{\omega}$  is positive-definite everywhere on M and equal to  $\widehat{\omega}_{\lambda}$  outside a large compact subset, hence is complete. By averaging over the action of T, we may assume that  $\mathcal{L}_{JX}\widetilde{\omega} = 0$  without changing the behaviour at infinity. We further modify  $\widetilde{\omega}$  to construct  $\omega$  satisfying conditions (a) and (ii) of the proposition.

To this end, we know that, since M does not split off any  $S^1$ -factors,  $\pi_1(M)=0$  by toricity [17]. In particular,  $H^1(M,\mathbb{R})=0$  so that the action of T on M is Hamiltonian with respect to  $\widetilde{\omega}$ . Consequently, there exists a smooth real-valued function  $\widetilde{f}$  such that  $\frac{1}{2}\mathcal{L}_X\widetilde{\omega}=i\partial\bar{\partial}\widetilde{f}$ . By averaging,  $\widehat{f}$  can be taken to be invariant under the action of T on M. It is also clear that, as  $\widetilde{\omega}=i\Theta_h+i\partial\bar{\partial}u_1$  for some  $u_1\in C^\infty(M)$  with  $i\Theta_h$  the curvature form of a hermitian metric on  $-K_M$ , we can write  $\rho_{\widetilde{\omega}}-\widetilde{\omega}=i\partial\bar{\partial}u_2$  for another function  $u_2\in C^\infty(M)$ ,  $\rho_{\widetilde{\omega}}$  here denoting the Ricci form of  $\widetilde{\omega}$ . Thus, there exists a function  $\widetilde{G}\in C^\infty(M)$  such that

(3.6) 
$$\rho_{\widetilde{\omega}} - \widetilde{\omega} + \frac{1}{2} \mathcal{L}_X \widetilde{\omega} = i \partial \bar{\partial} \widetilde{G}.$$

After averaging, we may assume that  $\widetilde{G}$  is invariant under the action of T. In particular, henceforth identifying M and  $\widehat{M}$  on the complement of compact subsets containing  $D_0$  and  $\pi^{-1}(D_0)$ , respectively, we can write

$$\tilde{G} := \tilde{G}(r, x),$$

where  $r=|z|^{\lambda}$  is as above and  $x\in D\subset \widehat{M}$ . As  $\widetilde{\omega}$  defines a shrinking gradient Kähler–Ricci soliton on  $M\setminus K$  for some  $K\subset M$  compact, we see that  $\widetilde{G}$  is pluriharmonic on  $M\setminus K$ . It therefore follows from Lemma 2.3 that

$$\tilde{G} = c_0 \log(r)$$

for some constant  $c_0 \in \mathbb{R}$ . Arguing as above, Lemma 2.4 guarantees the existence of an extension  $\varphi$  of  $c_0 \log(r) + \frac{c_0}{2}$  to M such that  $\omega := \widetilde{\omega} + i\partial\bar{\partial}\varphi$  defines a Kähler metric on M. As  $\varphi$  is pluriharmonic at infinity, it is clear that  $\omega = \widetilde{\omega} = \nu^*\widehat{\omega}_{\lambda}$  outside a large enough compact subset of M. Averaging over the action of T, we obtain our metric  $\omega$  of (a) satisfying condition (ii).

Next, as in (3.6), we see that there exists a function  $G \in C^{\infty}(M)$  invariant under the action of T such that

(3.7) 
$$\rho_{\omega} - \omega + \frac{1}{2} \mathcal{L}_{X} \omega = i \partial \bar{\partial} G.$$

Subtracting (3.6) from (3.7) yields the relation

$$i\partial\bar{\partial}G = i\partial\bar{\partial}\tilde{G} + \rho_{\omega} - \rho_{\widetilde{\omega}} - i\partial\bar{\partial}\varphi + i\partial\bar{\partial}\left(\frac{X}{2}\cdot\varphi\right) = i\partial\bar{\partial}\left(\tilde{G} - \log\left(\frac{\omega^{n}}{\widetilde{\omega}^{n}}\right) - \varphi + \frac{X}{2}\cdot\varphi\right)$$

between G and  $\widetilde{G}$ . Set

$$F_1 := \widetilde{G} - \log\left(\frac{\omega^n}{\widetilde{\omega}^n}\right) - \varphi + \frac{X}{2} \cdot \varphi.$$

Then  $i \partial \bar{\partial} F_1 = i \partial \bar{\partial} G$  so that (3.1) holds true, and outside a large compact subset of M, we have that

$$F_1 = \widetilde{G} - \log\left(\frac{\omega^n}{\widetilde{\omega}^n}\right) - \varphi + \frac{X}{2} \cdot \varphi = c_0 \log(r) - \varphi(r) + \frac{r}{2} \cdot \varphi'(r) = 0,$$

demonstrating that  $F_1$  is smooth and is compactly supported. As  $\mathcal{L}_{JX}\tilde{G}=0$ , condition (iii) and, correspondingly, the proposition now follow.

# 4. Proof of Theorem A (iii), (iv): Set-up of the complex Monge-Ampère equation

Proof of Theorem A (iii) and (iv): Set-up of the complex Monge–Ampère equation Returning now to the setup and notation of Theorem A, we next provide a proof of Theorem A (iii) by setting up a complex Monge–Ampère equation that any shrinking Kähler–Ricci soliton on M differing from our background metric by  $i\partial\bar{\partial}$  of a potential must satisfy, followed by a proof of Theorem A (iv) where a normalised Hamiltonian potential of JX with respect to  $\omega$  is given. Throughout this section, we write  $r:=|z|^{\lambda}$ , where z is the holomorphic coordinate on the  $\mathbb{C}$ -factor of  $\hat{M}$  and  $\lambda>0$  is as in Theorem A (iii) so that  $dv(X)=r\partial_r$ . Our starting point is the following proposition.

**Proposition 4.1.** Let  $\omega$  be the Kähler metric in Proposition 3.1 and let J denote the complex structure on M. Then there exists  $\widetilde{\varphi} \in C^{\infty}(M)$  with

$$\mathcal{L}_{JX}\widetilde{\varphi} = 0$$
 and  $\omega_{\widetilde{\varphi}} := \omega + i \partial \bar{\partial} \widetilde{\varphi} > 0$ 

such that

$$\rho_{\omega_{\widetilde{\varphi}}} + \frac{1}{2} \mathcal{L}_X \omega_{\widetilde{\varphi}} = \omega_{\widetilde{\varphi}}$$

if and only if, for all  $a \in \mathbb{R}$ , there exists  $\varphi \in C^{\infty}(M)$  with  $\mathcal{L}_{JX}\varphi = 0$  and  $\omega + i \partial \bar{\partial} \varphi > 0$  and  $F_2 \in C^{\infty}(M)$  compactly supported with  $\mathcal{L}_{JX}F_2 = 0$  satisfying

(4.2) 
$$\rho_{\omega} + \frac{1}{2} \mathcal{L}_{X} \omega - \omega = i \, \partial \bar{\partial} F_{2}$$

such that

(4.3) 
$$\log\left(\frac{(\omega+i\partial\bar{\partial}\varphi)^n}{\omega^n}\right) - \frac{X}{2}\cdot\varphi + \varphi = F_2 + a.$$

Here,  $\rho_{\omega}$  and  $\rho_{\omega_{\widetilde{\omega}}}$  denote the Ricci form of  $\omega$  and  $\omega_{\widetilde{\varphi}}$ , respectively.

*Proof.* If  $\varphi$  satisfies (4.3), then by taking  $i\partial\partial$  of this equation, we see that  $\varphi$  satisfies (4.1) by virtue of (3.1). Conversely, assume that (4.1) holds. Then we compute

$$0 = \rho_{\omega_{\widetilde{\varphi}}} - \omega_{\widetilde{\varphi}} + \frac{1}{2} \mathcal{L}_X \omega_{\widetilde{\varphi}} = \rho_{\omega_{\widetilde{\varphi}}} - \rho_{\omega} + \rho_{\omega} - \omega_{\widetilde{\varphi}} + \frac{1}{2} \mathcal{L}_X \omega_{\widetilde{\varphi}}$$
$$= -i \partial \bar{\partial} \log \left( \frac{(\omega + i \partial \bar{\partial} \widetilde{\varphi})^n}{\omega^n} \right) - i \partial \bar{\partial} \widetilde{\varphi} + i \partial \bar{\partial} \left( \frac{X}{2} \cdot \widetilde{\varphi} \right) + \rho_{\omega} - \omega + \frac{1}{2} \mathcal{L}_X \omega$$

so that

$$(4.4) i\partial\bar{\partial}\Big(\widetilde{\varphi} + \log\Big(\frac{(\omega + i\partial\bar{\partial}\widetilde{\varphi})^n}{\omega^n}\Big) - \frac{X}{2}\cdot\widetilde{\varphi}\Big) = \rho_{\omega} - \omega + \frac{1}{2}\mathcal{L}_X\omega.$$

Now, as we have seen in (3.1),

$$\rho_{\omega} - \omega + \frac{1}{2} \mathcal{L}_X \omega = i \, \partial \bar{\partial} F_1$$

for some JX-invariant compactly supported  $F_1 \in C^{\infty}(M)$ . Plugging this into (4.4), we have that, for every  $a \in \mathbb{R}$ ,

$$i\,\partial\bar{\partial}\Big(\widetilde{\varphi} + \log\frac{(\omega + i\,\partial\bar{\partial}\widetilde{\varphi})^n}{\omega^n} - \frac{X}{2}\cdot\widetilde{\varphi} - F_1 - a\Big) = 0.$$

JX-invariance of the sum in parentheses next implies from Lemma 2.3 that

$$\widetilde{\varphi} + \log\left(\frac{(\omega + i\partial\overline{\partial}\widetilde{\varphi})^n}{\omega^n}\right) - \frac{X}{2} \cdot \widetilde{\varphi} = F_1 + a + H$$

for H a pluriharmonic function equal to  $c_0 \log(r) + c_1$  outside a compact subset of M for some  $c_0, c_1 \in \mathbb{R}$ . Thus,

$$\begin{split} \left(\widetilde{\varphi} - H - \frac{c_0}{2}\right) + \log\left(\frac{(\omega + i\partial\bar{\partial}(\widetilde{\varphi} - H - \frac{c_0}{2}))^n}{\omega^n}\right) - \frac{X}{2} \cdot \left(\widetilde{\varphi} - H - \frac{c_0}{2}\right) \\ &= \left(\widetilde{\varphi} + \log\left(\frac{(\omega + i\partial\bar{\partial}\widetilde{\varphi})^n}{\omega^n}\right) - \frac{X}{2} \cdot \widetilde{\varphi}\right) - H + \frac{X}{2} \cdot H - \frac{c_0}{2} \\ &= (F_1 + a + H) - H + \frac{X}{2} \cdot H - \frac{c_0}{2} = F_1 + a + \frac{X}{2} \cdot H - \frac{c_0}{2}. \end{split}$$

Notice that after identifying X with  $r\partial_r$  via  $\nu$ , we have that

$$\frac{X}{2} \cdot H - \frac{c_0}{2} = \frac{1}{2} r \partial_r (c_0 \log(r) + c_1) - \frac{c_0}{2} = 0$$

outside a compact set. Set  $\varphi:=\widetilde{\varphi}-H-\frac{c_0}{2}$  and  $F_2:=F_1+\frac{X}{2}\cdot H-\frac{c_0}{2}$ . Then  $F_2\in C^\infty(M)$  is compactly supported, both  $\varphi$  and  $F_2$  are JX-invariant,  $i\partial\bar{\partial}F_2=i\partial\bar{\partial}F_1$ , and

$$\varphi + \log\left(\frac{(\omega + i\partial\bar{\partial}\varphi)^n}{\omega^n}\right) - \frac{X}{2} \cdot \varphi = F_2 + a,$$

as required.

Theorem A (iii) is a consequence of the next lemma.

**Lemma 4.2.** Let  $\lambda$ ,  $\omega$ , and  $\nu$ :  $(M \setminus K, \omega) \to (\widehat{M} \setminus \widehat{K}, \widehat{\omega})$ ,  $K \subset M$ ,  $\widehat{K} \subset \widehat{M}$  compact, be as in Proposition 3.1. Moreover, let  $F_2 \in C^{\infty}(M)$  be as in Proposition 4.1 satisfying (4.2) and recall that z denotes the holomorphic coordinate on the  $\mathbb{C}$ -factor of  $\widehat{M}$ . Set  $r := |z|^{\lambda}$ . Then there exists a unique torus-invariant smooth real-valued function  $f: M \to \mathbb{R}$  such that  $-\omega \sqcup JX = df$ ,  $f = \nu^*(\frac{r^2}{2} - 1)$  on  $M \setminus K$ , and

(4.5) 
$$\Delta_{\omega} f + f - \frac{X}{2} \cdot f = 0 \quad \text{outside a compact subset of } M.$$

In particular,  $f \to +\infty$  as  $r \to +\infty$ , hence is proper.

To this end, using the JX-invariance of  $F_2$  and f, contract (4.2) with

$$X^{1,0} := \frac{1}{2}(X - iJX)$$

and use the Bochner formula to derive that

$$i\bar{\partial}\left(\Delta_{\omega}f - \frac{X}{2}\cdot f + f + \frac{X}{2}\cdot F_2\right) = 0.$$

As a real-valued holomorphic function, we must have that  $\Delta_{\omega} f - \frac{X}{2} \cdot f + f + \frac{X}{2} \cdot F_2$  is constant on M. But since  $X \cdot F_2 = 0$  outside a compact subset of M, by the properties of f and  $\omega$ , we have that, outside a compact subset of M,

$$\Delta_{\omega} f - \frac{X}{2} \cdot f + f + \frac{X}{2} \cdot F_2 = \Delta_{\widehat{\omega}} \left( \frac{r^2}{2} - 1 \right) - \frac{r}{2} \frac{\partial}{\partial r} \left( \frac{r^2}{2} - 1 \right) + \left( \frac{r^2}{2} - 1 \right) = 0.$$

Thus, this constant is zero and we are done.

Let  $c_0 \in \mathbb{R}$  be such that  $e^{c_0} \int_M e^{F_2 - f} \omega^n = \int_M e^{-f} \omega^n$  and define  $F := F_2 + c_0$ . Then

- $F \in C^{\infty}(M)$  and F is torus-invariant,
- F is equal to  $c_0$  outside a compact subset of M, and
- $\int_M e^{F-f} \omega^n = \int_M e^{-f} \omega^n$ .

Moreover, from (4.2), we have that

$$\rho_{\omega} - \frac{1}{2} \mathcal{L}_X \omega + \omega = i \, \partial \bar{\partial} F.$$

By Proposition 4.1, any shrinking Kähler–Ricci soliton of the form  $\omega + i \partial \bar{\partial} \varphi > 0$  on M will solve the complex Monge–Ampère equation

$$\begin{cases} (\omega + i\partial\bar{\partial}\varphi)^n = e^{F + \frac{X}{2} \cdot \varphi - \varphi} \omega^n & \text{for } \varphi \in C^{\infty}(M) \text{ and } \varphi \text{ torus-invariant,} \\ \int_M e^{F - f} \omega^n = \int_M e^{-f} \omega^n. \end{cases}$$

This is precisely the statement of Theorem A (iv). A strategy to solve this equation is given by considering the Aubin continuity path

$$\begin{cases} (\omega+i\partial\bar{\partial}\varphi_t)^n = e^{F+\frac{X}{2}\cdot\varphi_t-t\varphi_t}\omega^n, & \varphi\in C^\infty(M), & \mathcal{L}_{JX}\varphi=0, \\ & \omega+i\partial\bar{\partial}\varphi>0, & t\in[0,1], \\ \int_M e^{F-f}\omega^n = \int_M e^{-f}\omega^n. \end{cases}$$

The equation corresponding to t = 0 is given by

$$\begin{cases} (\omega + i\partial\bar{\partial}\psi)^n = e^{F + \frac{X}{2} \cdot \psi} \omega^n, & \psi \in C^{\infty}(M), \quad \mathcal{L}_{JX}\psi = 0, \\ \omega + i\partial\bar{\partial}\psi > 0, & \\ \int_M e^{F - f} \omega^n = \int_M e^{-f} \omega^n. & \end{cases}$$

This equation we will solve by the continuity method, the particular path of which will be introduced in Section 7.1. This will yield the final part of Theorem A. Beforehand however, we prove some analytic results regarding the metric  $\omega$  and those metrics that are asymptotic to it, beginning with a Poincaré inequality.

## 5. Poincaré inequality

In this section, we prove a Poincaré inequality for the Kähler form  $\omega$  of Proposition 3.1 using the fact that it holds true on the model shrinking gradient Kähler–Ricci soliton

$$(\widehat{M} := \mathbb{C} \times \mathbb{P}^1, \widehat{\omega} := \widetilde{\omega}_{\lambda} + \omega_D, r\partial_r)$$

(see [27]), where  $r = |z|^{\lambda}$ . This will be used in Proposition 7.9 to establish an a priori weighted  $L^2$ -estimate along the continuity path that we consider in deriving a solution to  $(*_0)$ . Recall the Hamiltonian potential f of JX satisfying (4.5).

We work with the Lebesgue and Sobolev spaces  $L^p(e^{-f}\omega^n)$  and  $W^{1,p}(e^{-f}\omega^n)$  on M, respectively, defined in the obvious way for p>1, and we denote

$$\int_{M}ue^{-f}\omega^{n}:=\frac{1}{\int_{M}e^{-f}\omega^{n}}\int_{M}ue^{-f}\omega^{n}\quad\text{for all }u\in L^{p}(e^{-f}\omega^{n}).$$

By Hölder's inequality and the finiteness of  $\int_M e^{-f} \omega^n$ , the integral  $\int_M u e^{-f} \omega^n$  is finite.

**Proposition 5.1** (Poincaré inequality). For all p > 1, there exists a constant C(p) > 0 such that

$$\left\| u - \int_M u e^{-f} \omega^n \right\|_{L^p(e^{-f} \omega^n)} \le C(p) \|\nabla^g u\|_{L^p(e^{-f} \omega^n)}$$

for all  $u \in W^{1,p}(e^{-f}\omega^n) \cap C^1(M)$ . Here, g is the Kähler metric associated to  $\omega$ .

*Proof.* For sake of a contradiction, suppose that the assertion is not true. Then there exists a sequence of functions  $(u_k)_{k>1} \subset W^{1,p}(e^{-f}\omega^n)$  with the following properties:

$$\begin{cases} \|u_k\|_{L^p(e^{-f}\omega^n)} = 1, & \int_M u_k e^{-f}\omega^n = 0, \\ \|\nabla^g u_k\|_{L^p(e^{-f}\omega^n)} \le \frac{1}{k}. \end{cases}$$

Indeed, since  $\int_M e^{-f} \omega^n < +\infty$ , an application of Hölder's inequality demonstrates that we can normalise the sequence  $(u_k)_{k\geq 1}$  so that the weighted integral of each function in the sequence is zero. By the Rellich-Kondrachov theorem, there exists a subsequence which we

also denote by  $(u_k)_{k\geq 1}$  converging to some  $u_\infty\in L^p_{loc}(M)$  as  $k\to +\infty$ . On the other hand, for every compactly supported one-form  $\alpha$  on M, we have that

$$\int_{M}u_{\infty}\cdot\delta^{g}\alpha\omega^{n}=\lim_{k\rightarrow+\infty}\int_{M}u_{k}\cdot\delta^{g}\alpha\omega^{n}=-\lim_{k\rightarrow+\infty}\int_{M}g(du_{k},\alpha)\omega^{n}=0,$$

where  $\delta_g$  is the co-differential of d with respect to g. Thus,  $u_{\infty} \in W^{1,p}_{loc}(M)$  and  $du_{\infty} = 0$  almost everywhere. In particular,  $u_{\infty}$  is constant.

For R>0, let  $D_R:=f^{-1}((-\infty,R])$ , a compact subset of M by properness of f (cf. Lemma 4.2). Then the fact that  $\int_M u_k e^{-f} \omega^n = 0$  implies that, for every R>0,

$$\int_{D_R} u_k e^{-f} \omega^n = -\int_{M \setminus D_R} u_k e^{-f} \omega^n.$$

It then follows from Hölder's inequality that

$$\left| \int_{D_R} u_k e^{-f} \omega^n \right| \leq \int_{M \setminus D_R} |u_k| e^{-f} \omega^n$$

$$\leq \left( \int_{M \setminus D_R} |u_k|^p e^{-f} \omega^n \right)^{\frac{1}{p}} \left( \int_{M \setminus D_R} e^{-f} \omega^n \right)^{1 - \frac{1}{p}}$$

$$\leq ||u_k||_{L^p(e^{-f} \omega^n)} \left( \int_{M \setminus D_R} e^{-f} \omega^n \right)^{1 - \frac{1}{p}}$$

$$= \left( \int_{M \setminus D_R} e^{-f} \omega^n \right)^{1 - \frac{1}{p}}.$$

Furthermore,  $L^p_{loc}(M)$ -convergence implies that

$$\int_{D_R} u_k e^{-f} \omega^n \to \int_{D_R} u_\infty e^{-f} \omega^n = u_\infty \operatorname{vol}_f(D_R) \quad \text{as } k \to +\infty.$$

This allows us to derive that

$$|u_{\infty}| = \lim_{k \to +\infty} \frac{|\int_{D_R} u_k e^{-f} \omega^n|}{\operatorname{vol}_f(D_R)} \le \lim_{k \to +\infty} \frac{(\int_{M \setminus D_R} e^{-f} \omega^n)^{1 - \frac{1}{p}}}{\operatorname{vol}_f(D_R)}$$
$$= \frac{\operatorname{vol}_f(M \setminus D_R)^{1 - \frac{1}{p}}}{\operatorname{vol}_f(D_R)} \to 0 \quad \text{as } R \to +\infty,$$

where  $\operatorname{vol}_f(A) := \int_A e^{-f} \omega^n$  for  $A \subseteq M$ . That is,  $u_\infty \equiv 0$ .

Next, choose C > 0 such that f + C > 0 on M, something that is possible to do by Lemma 4.2, and let  $\eta: \mathbb{R} \to \mathbb{R}$  be a smooth function satisfying  $\eta(x) = 0$  for  $x \le 1$ ,  $\eta(x) = 1$  for  $x \ge 2$ , and  $|\eta(x)| \le 1$  for all x. Define  $\eta_R: M \to \mathbb{R}$  by

$$\eta_R(x) = \eta \left( \frac{\sqrt{f(x) + C}}{R} \right)$$
 for  $R > 0$  a positive constant to be chosen later.

Then, with  $\frac{1}{p} + \frac{1}{q} = 1$ , we have that, for some positive constant C(p) > 0 that may vary from line to line,

(5.1) 
$$1 = \|u_k\|_{L^p(e^{-f}\omega^n)}^p$$

$$\leq C(p) (\|(1 - \eta_R)u_k\|_{L^p(e^{-f}\omega^n)}^p + \|\eta_R u_k\|_{L^p(e^{-f}\omega^n)}^p)$$

$$\begin{split} &\leq C(p) \bigg( \int_{D_R} |u_k|^p e^{-f} \omega^n + \int_M |\eta_R u_k|^p e^{-f} \omega^n \bigg) \\ &\leq C(p) \bigg( \int_{D_R} |u_k|^p e^{-f} \omega^n + \int_M \bigg| \eta_R u_k - \int_M \eta_R u_k e^{-f} \omega^n \bigg|^p e^{-f} \omega^n \\ &\quad + \bigg| \int_M \eta_R u_k e^{-f} \omega^n \bigg|^p \bigg) \\ &\leq C(p) \bigg( \int_{D_R} |u_k|^p e^{-f} \omega^n + \int_M \bigg| \eta_R u_k - \int_M \eta_R u_k e^{-f} \omega^n \bigg|^p e^{-f} \omega^n \\ &\quad + \|u_k\|_{L^p(e^{-f} \omega^n)}^p \|\eta_R\|_{L^q(e^{-f} \omega^n)}^p \bigg) \\ &\leq C(p) \bigg( \int_{D_R} |u_k|^p e^{-f} \omega^n + \int_M \bigg| \eta_R u_k - \int_M \eta_R u_k e^{-f} \omega^n \bigg|^p e^{-f} \omega^n \\ &\quad + \operatorname{vol}_f (M \setminus D_{\frac{R^2}{2}})^{\frac{p}{q}} \bigg). \end{split}$$

Now, for R > 0 sufficiently large,  $\eta_R u_k$  is supported on the set where  $\omega$  is isometric to  $\widehat{\omega}$  via the biholomorphism  $\nu$  of Proposition 3.1, a manifold on which we know that the assertion already holds true [27]. Applying this observation to the middle term in the last line of (5.1), we arrive at the fact that, for R > 0 sufficiently large,

$$\begin{split} 1 &\leq C(p) \bigg( \int_{D_R} |u_k|^p e^{-f} \omega^n + \|\nabla^g (\eta_R u_k)\|_{L^p(e^{-f} \omega^n)}^p + \operatorname{vol}_f (M \setminus D_{\frac{R^2}{2}})^{\frac{p}{q}} \bigg) \\ &\leq C(p) \bigg( \int_{D_R} |u_k|^p e^{-f} \omega^n + \|\nabla^g \eta_R\|_{L^{\infty}(M)}^p \|u_k\|_{L^p(e^{-f} \omega^n)}^p \\ &\qquad \qquad + \|\nabla^g u_k\|_{L^p(e^{-f} \omega^n)}^p + \operatorname{vol}_f (M \setminus D_{\frac{R^2}{2}})^{\frac{p}{q}} \bigg) \\ &\leq C(p) \bigg( \int_{D_R} |u_k|^p e^{-f} \omega^n + \frac{1}{R^p} + \frac{1}{k^p} + \operatorname{vol}_f (M \setminus D_{\frac{R^2}{2}})^{\frac{p}{q}} \bigg). \end{split}$$

As  $u_k \to 0$  in  $L^p_{loc}(M)$  as  $k \to +\infty$ , we see upon letting  $k \to +\infty$  that, for all R > 0 sufficiently large,

$$1 \le C(p) \left( \frac{1}{R^p} + \operatorname{vol}_f(M \setminus D_{\frac{R^2}{2}})^{\frac{p}{q}} \right).$$

Letting  $R \to +\infty$  now yields the desired contradiction.

#### 6. Linear theory

Working again in the setting and notation of Theorem A, we set up the linear theory for metrics asymptotic to  $\omega$ . Openness along the continuity path that we apply to solve  $(*_0)$  will automatically follow. Although Theorem A holds true for torus-invariant functions, in order to remain as broad as possible, we present the linear theory under minimal assumptions, namely for JX-invariant functions.

**6.1. Main setting.** Let  $\tilde{g}$  be any JX-invariant Kähler metric on M with Kähler form  $\tilde{\omega}$  and Levi-Civita connection  $\nabla^{\tilde{g}}$  satisfying

(6.1) 
$$|(\nabla^g)^i \mathcal{L}_X^{(j)}(\widetilde{\omega} - \omega)|_g = O(r^{-\gamma}) \quad \text{for all } i, j \ge 0,$$

for some  $\gamma \in (0, \lambda^D)$ , where  $r = |z|^{\lambda}$  and  $\lambda^D$  is the first non-zero eigenvalue of  $-\Delta_D$  acting on  $L^2$ -functions on D. Write  $X = \nabla^{\widetilde{g}} \widetilde{f}$  for some smooth function  $\widetilde{f} \colon M \to \mathbb{R}$ , a function defined up to an additive constant that is guaranteed to exist because, as noted previously,  $H^1(M,\mathbb{R}) = 0$  by toricity. We use  $\nu$  to identify M and  $\widehat{M}$  so that  $X = r\partial_r$  outside a compact set. Since  $\nabla^g f = X = \nabla^{\widetilde{g}} \widetilde{f}$ , it follows from (6.1) that  $|f - \widetilde{f}| = O(r^{-\gamma + 2})$  as  $r \to +\infty$ . Throughout, we denote  $\Delta_{\widetilde{g},X} := \Delta_{\widetilde{g}} - X$ .

We begin by identifying a good barrier function for this particular geometric setup.

**Lemma 6.1.** For all  $\delta \in (0,1)$ , there exists  $R(\delta) > 0$  such that the function  $e^{\delta f}$  is a subsolution of the following equation:

$$\Delta_{\tilde{g},X}e^{\delta f} \leq 0$$
 on  $f \geq R(\delta)$ .

Moreover, the logarithm and polynomial powers of f (which equals  $\frac{|z|^{2\lambda}}{2} - 1$  outside a compact subset of M) satisfy, for all  $\delta > 0$ ,

$$\Delta_{\tilde{g},X} f^{-\delta} = 2\delta f^{-\delta} + O(f^{-\delta-1})$$
 and  $\Delta_{\tilde{g},X} \log(f+1) = -2$ 

outside a compact subset of M.

*Proof.* Using (6.1) and the fact that

$$(\Delta_{\widetilde{g}} - \Delta_{g})f = 2(\Delta_{\widetilde{\omega}} - \Delta_{\omega})f = (\widetilde{\omega} - \omega) * i\partial\bar{\partial}f = O(|\widetilde{g} - g|_{\widetilde{g}}),$$

the last equality because the Hessian of f is bounded on M, we compute that

$$\Delta_{\widetilde{g},X} e^{\delta f} = (\delta \Delta_{\widetilde{g},X} f + \delta^2 | \nabla^{\widetilde{g}} f|_{\widetilde{g}}^2) e^{\delta f}$$

$$= \delta \left( \Delta_{g,X} f + (\Delta_{\widetilde{g},X} - \Delta_{g,X}) f + \delta | \nabla^{\widetilde{g}} f|_{\widetilde{g}}^2 \right) e^{\delta f}$$

$$= \delta \left( -2f + \delta | \nabla^{\widetilde{g}} f|_{\widetilde{g}}^2 + O(|\widetilde{g} - g|_{\widetilde{g}}) \right) e^{\delta f}$$

$$= \delta \left( -2f + \delta | X|_g^2 (1 + o(1)) + o(1) \right) e^{\delta f}$$

$$\leq 0$$

outside a sufficiently large compact subset of M. Here we have also used that  $|X|_g^2 = 2f + 2$  and  $\delta \in (0, 1)$  in the last line.

A similar computation based on the asymptotics of  $\tilde{g}$  given by (6.1) shows that

$$\begin{split} \Delta_{\widetilde{g},X} f^{-\delta} &= (\Delta_{\widetilde{g}} - X)(f^{-\delta}) \\ &= -\delta f^{-\delta-1} (\Delta_{\widetilde{g}} f - X \cdot f) + \delta(\delta + 1) f^{-\delta-2} |\nabla^{\widetilde{g}} f|_{\widetilde{g}}^2 \\ &= -\delta f^{-\delta-1} (\Delta_{g} f - X \cdot f) - \delta f^{-\delta-1} (\Delta_{\widetilde{g}} f - \Delta_{g} f) + \delta(\delta + 1) f^{-\delta-2} |\nabla^{\widetilde{g}} f|_{\widetilde{g}}^2 \\ &= 2\delta f^{-\delta} - \delta f^{-\delta-1} \underbrace{(\Delta_{\widetilde{g}} f - \Delta_{g} f)}_{=O(|\widetilde{g} - g|_{g})} + \delta(\delta + 1) f^{-\delta-2} |\nabla^{\widetilde{g}} f|_{\widetilde{g}}^2 \\ &= O(|X|_{g}^{2}) = O(f) \end{split}$$

$$= 2\delta f^{-\delta} + O(f^{-\delta-1}).$$

As  $\log(r^2)$  is pluriharmonic outside a compact set, the fact that  $X = r\partial_r$  outside a compact set gives us that

$$\Delta_{\tilde{g},X} \log(f+1) = \Delta_{\tilde{g},X} \log(r^2) = -2$$

outside a compact subset of M, as claimed.

**6.2. Function spaces.** We next define the function spaces within which we will work. For  $\beta \in \mathbb{R}$  and k a non-negative integer, define  $C_{X,\beta}^{2k}(M)$  to be the space of JX-invariant continuous functions u on M with 2k continuous derivatives such that

$$\|u\|_{C^{2k}_{X,\beta}}:=\sum_{i+2j\leq 2k}\sup_{M}|f^{\frac{\beta}{2}}(\nabla^{\widetilde{g}})^{i}(\mathcal{L}_{X}^{(j)}u)|_{\widetilde{g}}<+\infty.$$

Thanks to (6.1), this norm is equivalent to that defined with respect to the background metric g; hence we may use either  $\tilde{g}$  or g with our particular choice depending on the context. Similarly, as f and  $\tilde{f}$  are equivalent at infinity, these function spaces can be defined in terms of either of these two potential functions. Define  $C_{X,\beta}^{\infty}(M)$  to be the intersection of the spaces  $C_{X,\beta}^{2k}(M)$ over all  $k \in \mathbb{N}_0$ .

Notice in the definition of the above norm that the number of spatial derivatives that appear in each summand is no more than twice the number of Lie derivative terms that appear. This is because, when solving the Poisson equation for the weighted Laplacian as defined in (6.1), the weighted Laplacian can be treated as a second-order parabolic operator with the time derivative corresponding to the X-derivative. These heuristics are used in the proof of Theorem 6.3 below.

Let  $\delta(\tilde{g})$  denote the injectivity radius of  $\tilde{g}$ , write  $d_{\tilde{g}}(x,y)$  for the distance with respect to  $\widetilde{g}$  between two points  $x, y \in M$ , and let  $\varphi_t^X$  denote the flow of X for time t. A tensor T on M is said to be in  $C^{0,2\alpha}_{\beta}(M)$ ,  $\alpha \in (0,\frac{1}{2})$ , if

$$\begin{split} [T]_{C^{0,2\alpha}_{\beta}} &:= \sup_{\substack{x \neq y \in M \\ d_{\widetilde{g}}(x,y) < \delta(\widetilde{g})}} \left[ \min(f(x),f(y))^{\frac{\beta}{2}} \frac{|T(x) - P_{x,y}T(y)|_h}{d_{\widetilde{g}}(x,y)^{2\alpha}} \right] \\ &+ \sup_{\substack{x \in M \\ t \neq s \geq 1}} \left[ \min(t,s)^{\frac{\beta}{2}} \frac{\left| (\varphi^X_t)_* T(x) - \left(\hat{P}_{\varphi^X_s(x),\varphi^X_t(x)}((\varphi^X_s)_* T(x))\right) \right|_h}{|t-s|^{\alpha}} \right] < +\infty, \end{split}$$

where  $P_{x,y}$  denotes parallel transport along the unique geodesic joining x, y, and  $\hat{P}_{\varphi_s^X(x),\varphi_t^X(x)}$ denotes parallel transport along the unique flow line of X joining  $\varphi_s^X(x)$ ,  $\varphi_t^X(x)$ .

For  $\beta \in \mathbb{R}$ , k a non-negative integer, and  $\alpha \in (0, \frac{1}{2})$ , define the Hölder space  $C_{X,\beta}^{2k,2\alpha}(M)$  with polynomial weight  $f^{\frac{\beta}{2}}$  to be the set of  $u \in C_{X,\beta}^{2k}(M)$  for which the norm

$$||u||_{C^{2k,2\alpha}_{X,\beta}} := ||u||_{C^{2k}_{X,\beta}} + \sum_{i+2j=2k} [(\nabla^{\tilde{g}})^i (\mathcal{L}_X^{(j)} u)]_{C^{0,2\alpha}_{\beta}}$$

is finite. It is straightforward to check that the space  $C_{X,\beta}^{2k,2\alpha}(M)$  is a Banach space. The intersection  $\bigcap_{k\geq 0} C_{X,\beta}^{2k}(M)$  we denote by  $C_{X,\beta}^{\infty}(M)$ .

We now consider a smooth cut-off function  $\chi\colon M\to [0,1]$  which equals 1 outside a compact set. The source function  $\chi\colon M\to [0,1]$  which equals 1 outside a compact set. The source function  $\chi\colon M\to [0,1]$ 

pact set. The source function space  $\mathcal{D}_{X,\beta}^{2k+2,2\alpha}(M)$  is defined as

$$\mathcal{D}_{X,\beta}^{2k+2,2\alpha}(M) := \left( \mathbb{R} \chi \log r \oplus \mathbb{R} \oplus C_{X,\beta}^{2k+2,2\alpha}(M) \right),$$

endowed with the norm

$$||u||_{\mathcal{D}_{X,\beta}^{2k+2,2\alpha}} := |c_1| + |c_2| + ||\widetilde{u}||_{C_{X,\beta}^{2k+2,2\alpha}}, \quad u := c_1 \chi \log r + c_2 + \widetilde{u}.$$

The target function space is defined as

$$\mathcal{C}^{2k,2\alpha}_{X,\beta}(M) := (\mathbb{R} \oplus C^{2k,2\alpha}_{X,\beta}(M)),$$

endowed with a norm defined in a similar manner to above. We define

$$\mathcal{C}^{\infty}_{X,\beta}(M) := \bigcap_{k \ge 0} \mathcal{C}^{2k,2\alpha}_{X,\beta}(M).$$

Finally, we define the spaces

$$\mathcal{M}^{2k+2,2\alpha}_{X,\beta}(M):=\{\varphi\in C^2_{\mathrm{loc}}(M)\mid \widetilde{\omega}+i\,\partial\bar{\partial}\varphi>0\}\cap\mathcal{D}^{2k+2,2\alpha}_{X,\beta}(M),$$

and we will work with the following convex set of Kähler potentials:

$$\mathcal{M}_{X,\beta}^{\infty}(M) = \bigcap_{k>0} \mathcal{M}_{X,\beta}^{2k+2,2\alpha}(M).$$

Notice that, for each  $k \geq 0$ , the spaces  $\mathcal{M}_{X,\beta}^{2k+2,2\alpha}(M)$  depend on the choice of a background metric  $\widetilde{\omega}$ . However, these spaces are all equivalent as soon as  $\widetilde{\omega}$  satisfies (6.1).

**6.3. Preliminaries and Fredholm properties of the linearised operator.** We proceed with the same set-up as in Section 6.1, beginning with the following useful observation.

**Lemma 6.2.** Let  $(\varphi_t)_{t \in [0,1]}$  be a  $C^1$ -path of smooth functions in  $\mathcal{M}_{X,\beta}^{\infty}(M)$  for some  $\beta > 0$  and write

$$\widetilde{\omega}_t := \widetilde{\omega} + i \partial \overline{\partial} \varphi_t > 0$$
 and  $\widetilde{f}_t := \widetilde{f} + \frac{X}{2} \cdot \varphi_t$ 

so that  $-d\widetilde{\omega}_t \rfloor JX = d\widetilde{f}_t$ .

(i) Let  $G: \mathbb{R} \to \mathbb{R}$  be a  $C^1$ -function such that, for some  $-\infty < \alpha < 1$ ,

$$|G(x)| + |G'(x)| \le e^{\alpha x}, \quad x \ge -C.$$

Then

$$\int_{M} G(\widetilde{f_{t}}) e^{-\widetilde{f_{t}}} \widetilde{\omega}_{t}^{n} = \int_{M} G(\widetilde{f_{0}}) e^{-\widetilde{f_{0}}} \widetilde{\omega}_{0}^{n}, \quad t \in [0, 1].$$

(ii) 
$$\int_0^1 \int_M |\dot{\varphi}_t| e^{-\tilde{f}_t} \widetilde{\omega}_t^n dt < +\infty \text{ and } \int_0^1 \int_M |\dot{\varphi}_t| e^{-\tilde{f}} \widetilde{\omega}^n dt < +\infty.$$

*Proof.* (i) By differentiating, one sees that

$$\begin{split} \frac{d}{dt} & \left( \int_{M} G(\tilde{f}_{t}) e^{-\tilde{f}_{t}} \tilde{\omega}_{t}^{n} \right) \\ & = \int_{M} G'(\tilde{f}_{t}) \frac{X}{2} \cdot \dot{\varphi}_{t} e^{-\tilde{f}_{t}} \tilde{\omega}_{t}^{n} + \int_{M} G(\tilde{f}_{t}) \left( \Delta_{\tilde{\omega}_{t}} \dot{\varphi}_{t} - \frac{X}{2} \cdot \dot{\varphi}_{t} \right) e^{-\tilde{f}_{t}} \tilde{\omega}_{t}^{n} \\ & = \int_{M} G'(\tilde{f}_{t}) \frac{X}{2} \cdot \dot{\varphi}_{t} e^{-\tilde{f}_{t}} \tilde{\omega}_{t}^{n} - \frac{1}{2} \int_{M} G'(\tilde{f}_{t}) \nabla^{g_{\varphi_{t}}} \tilde{f}_{t} \cdot \dot{\varphi}_{t} e^{-\tilde{f}_{t}} \tilde{\omega}_{t}^{n} = 0. \end{split}$$

Here, we have used integration by parts together with the fact that  $X = \nabla^{\tilde{g}_t} \tilde{f}_t$  for all  $t \in [0, 1]$ , where  $\tilde{g}_t$  denotes the Kähler metric associated to  $\tilde{\omega}_t$ .

(ii) First note that, by definition of the function space, the weighted measures  $e^{-\widetilde{f}_t}\widetilde{\omega}_t^n$  and  $e^{-\widetilde{f}}\widetilde{\omega}_t^n$  are equivalent to each other. Therefore, it suffices to verify only that

$$\int_0^1 \int_M |\dot{\varphi}_t| e^{-\widetilde{f}} \widetilde{\omega}^n \, dt < +\infty.$$

But from the definition of the function space and  $\tilde{\omega}$ , this is trivially satisfied.

Next, define the following map as in [33]:

$$MA_{\widetilde{\omega}}: \psi \in \{\varphi \in C^2_{\text{loc}}(M) \mid \widetilde{\omega}_{\varphi} := \widetilde{\omega} + i \partial \bar{\partial} \varphi > 0\} \mapsto \log \left(\frac{\widetilde{\omega}_{\psi}^n}{\widetilde{\omega}^n}\right) - \frac{X}{2} \cdot \psi \in \mathbb{R}.$$

For any  $\psi \in C^2_{loc}(M)$ , let  $\tilde{g}_{\psi}$  (respectively  $\tilde{g}_{t\psi}$ ) denote the Kähler metric associated to the Kähler form  $\tilde{\omega}_{\psi}$  (respectively  $\tilde{\omega}_{t\psi}$  for any  $t \in [0, 1]$ ). Brute force computations show that

$$MA_{\widetilde{\omega}}(0) = 0,$$

$$D_{\psi}MA_{\widetilde{\omega}}(u) = \Delta_{\widetilde{\omega}_{\psi}}u - \frac{X}{2} \cdot u, \quad u \in C^{2}_{loc}(M),$$

$$(6.2) \qquad \frac{d^{2}}{dt^{2}}(MA_{\widetilde{\omega}}(t\psi)) = \frac{d}{dt}(\Delta_{\widetilde{\omega}_{t\psi}}\psi) = -|\partial\bar{\partial}\psi|^{2}_{\widetilde{g}_{t\psi}} \quad \text{for } t \in [0, 1],$$

$$(6.3) \qquad MA_{\widetilde{\omega}}(\psi) = MA_{\widetilde{\omega}}(0) + \frac{d}{dt}\Big|_{t=0} MA_{\widetilde{\omega}}(t\psi)$$

$$+ \int_{0}^{1} \int_{0}^{u} \frac{d^{2}}{dt^{2}}(MA_{\widetilde{\omega}}(t\psi)) dt du$$

$$= \Delta_{\widetilde{\omega}}\psi - \frac{X}{2} \cdot \psi - \int_{0}^{1} \int_{0}^{u} |\partial\bar{\partial}\psi|^{2}_{\widetilde{g}_{t\psi}} dt du.$$

The main result of this section is that the drift Laplacian of  $\tilde{g}$  is an isomorphism between polynomially weighted function spaces with zero mean value.

**Theorem 6.3.** Let  $\alpha \in (0, \frac{1}{2})$ ,  $k \in \mathbb{N}$ , and  $\beta \in (0, \lambda^D)$ . Then the drift Laplacian

$$\Delta_{\widetilde{g},X} \colon \mathcal{D}_{X,\beta}^{2k+2,2\alpha}(M) \cap \left\{ \int_{M} u e^{-\widetilde{f}} \widetilde{\omega}^{n} = 0 \right\} \to \mathcal{C}_{X,\beta}^{2k,2\alpha}(M) \cap \left\{ \int_{M} v e^{-\widetilde{f}} \widetilde{\omega}^{n} = 0 \right\}$$

is an isomorphism of Banach spaces.

**Remark 6.4.** In the statement of Theorem 6.3, if  $D = \mathbb{P}^1$  endowed with its metric of constant sectional curvature 1, then  $\lambda^{\mathbb{P}^1} = 2$  and correspondingly  $\beta \in (0, \lambda^{\mathbb{P}^1}) = (0, 2)$ . In general, Lichnerowicz's estimate implies that  $\lambda^D \geq 2$ ; see [5, Theorem 6.14] for a proof. The rate  $\gamma$  from (6.1) can take any value in the interval  $(0, \lambda^D)$ . In Section 6.4, we will apply Theorem 6.3 with  $\gamma = \beta$ .

*Proof of Theorem* 6.3. First observe that the drift Laplacian  $\Delta_{\tilde{g},X}$  is symmetric with respect to the weighted measure  $e^{-\tilde{f}}\tilde{\omega}^n$ , a measure with finite volume. Set

$$\begin{split} &H^1_{\widetilde{f}}(M) := \{u \in H^1_{\mathrm{loc}}(M) \ JX\text{-invariant} \mid u \in L^2(e^{-\widetilde{f}}\widetilde{\omega}^n), \ \nabla^{\widetilde{g}}u \in L^2(e^{-\widetilde{f}}\widetilde{\omega}^n)\}, \\ &W^2_{\widetilde{f}}(M) := \{u \in H^1_{\widetilde{f}}(M) \mid \Delta_{\widetilde{\omega},X}u \in L^2(e^{-\widetilde{f}}\widetilde{\omega}^n)\}, \end{split}$$

endowed with the obvious norms induced by that of  $L^2(e^{-\widetilde{f}}\widetilde{\omega}^n)$ . It can be shown that the operator  $\Delta_{\widetilde{g},X}$  restricted to compactly supported smooth JX-invariant functions admits a unique self-adjoint extension to  $W^2_{\widetilde{f}}(M)$ , with domain contained in  $H^1_{\widetilde{f}}(M)$  and with discrete  $L^2(e^{-\widetilde{f}}\widetilde{\omega}^n)$ -spectrum; see [18, Proposition 6.13] and [23, Theorem 4.6] in the context of expanding gradient Ricci solitons, but whose proofs can be adapted to the current situation. Observe also that the kernel (and hence the cokernel) of this operator is the constant

functions. By considering any function F in the codomain as an element of the weighted  $L^2$ space  $L^2(e^{-\tilde{f}}\widetilde{\omega}^n)$ , we can therefore find a unique weak solution  $u \in H^1(e^{-\tilde{f}}\widetilde{\omega}^n)$  with zero
weighted mean value of the equation

$$(6.4) \Delta_{\widetilde{g},X} u = F.$$

In addition, we have the estimate

$$(6.5) ||u||_{L^{2}(e^{-\tilde{f}}\widetilde{\omega}^{n})} + ||\nabla^{\tilde{g}}u||_{L^{2}(e^{-\tilde{f}}\widetilde{\omega}^{n})} \le C||F||_{L^{2}(e^{-\tilde{f}}\widetilde{\omega}^{n})} \le C||F||_{C^{0}}$$

for some positive constant C independent of u and F that may vary from line to line. This estimate essentially follows from the weighted  $L^2$ -Poincaré inequality with respect to the drift Laplacian  $\Delta_{\widetilde{g}} - X$ . We improve on the regularity of u through a series of claims.

**Claim 6.5.** There exists a positive constant  $C = C(\widetilde{\omega}, n)$  such that

$$|u(x)| \le Ce^{\frac{\widetilde{f}(x)}{2}} ||F||_{C^0}, \quad x \in M.$$

*Proof of Claim* 6.5. By conjugating (6.4) with a suitable weight, notice that the function  $v := e^{-\tilde{f}/2}u$  satisfies

$$\Delta_{\widetilde{g}}v = e^{-\frac{\widetilde{f}}{2}}F + \left(\frac{1}{4}|X|_{\widetilde{g}}^2 - \frac{1}{2}\Delta_{\widetilde{g}}\widetilde{f}\right)v.$$

This implies that |v| satisfies the following differential inequality in the weak sense:

(6.6) 
$$\Delta_{\widetilde{g}}|v| \ge -C|v| - C||F||_{C^0}.$$

Here we have made use of the non-negativity of  $|X|_{\tilde{g}}^2$  together with the boundedness of  $\Delta_{\tilde{g}}\tilde{f}$  given by (6.1).

We perform a local Nash-Moser iteration on (6.6) in  $B_{\tilde{g}}(x,r)$ . More precisely, since  $(M^{2n}, \tilde{g})$  is a Riemannian manifold with Ricci curvature bounded from below, the results of [32] yield the following local Sobolev inequality:

(6.7) 
$$\left(\frac{1}{\operatorname{vol}_{\widetilde{g}}(B_{\widetilde{g}}(x,r))} \int_{B_{\widetilde{g}}(x,r)} |\varphi|^{\frac{2n}{n-1}} \widetilde{\omega}^{n}\right)^{\frac{n-1}{n}} \\ \leq \left(\frac{C(r_{0})r^{2}}{\operatorname{vol}_{\widetilde{g}}(B_{\widetilde{g}}(x,r))} \int_{B_{\widetilde{g}}(x,r)} |\widetilde{\nabla}\varphi|^{2}_{\widetilde{g}} \widetilde{\omega}^{n}\right)$$

for any  $\varphi \in H_0^1(B_{\widetilde{g}}(x,r))$  and for all  $x \in M$  and  $0 < r < r_0$ , where  $r_0$  is some fixed positive radius.

A Nash–Moser iteration proceeds in several steps. First, one multiplies (6.4) across by  $\eta_{s,s'}^2 v |v|^{2(p-1)}$  with  $p \ge 1$ , where  $\eta_{s,s'}$ , with 0 < s+s' < r and s,s' > 0, is a Lipschitz cut-off function with compact support in  $B_{\widetilde{g}}(x,s+s')$  equal to 1 on  $B_{\widetilde{g}}(x,s)$  and with

$$|\widetilde{\nabla}\eta_{s,s'}|_{\widetilde{g}} \leq \frac{1}{s'}$$

almost everywhere. One then integrates by parts and uses the Sobolev inequality of (6.7) to obtain a so-called "reversed Hölder inequality" which, after iteration, leads to the bound

$$\sup_{B_{\widetilde{g}}(x,\frac{r}{2})} |v| \le C(\|v\|_{L^{2}(B_{\widetilde{g}}(x,r))} + \|F\|_{L^{\infty}(B_{\widetilde{g}}(x,r))})$$

$$\le C(\|u\|_{L^{2}(e^{-\widetilde{f}}\widetilde{\omega}^{n})} + \|F\|_{C^{0}(M)}) \le C\|F\|_{C^{0}(M)}$$

for  $r \leq r_0$ , where  $C = C(r_0, \tilde{\omega}, n)$ . Here we have made use of (6.5) in the last line. This estimate yields an a priori local  $C^0$ -estimate which is uniform on the centre of the ball  $B_{\tilde{g}}(x, \frac{r}{2})$ . In particular, unravelling the definition of the function v, one obtains the expected a priori uniform exponential growth, namely

$$|u(x)| \le Ce^{\frac{\tilde{f}(x)}{2}} ||F||_{C^0}, \quad x \in M.$$

Thanks to Claim 6.5, by local Schauder elliptic estimates, we actually see that u lies in  $C_{loc}^{2k+2,2\alpha}$  and that we have the estimates

$$(6.8) ||u||_{C^{2k+2\alpha}(\{\tilde{f} < R\})} \le C ||F||_{C^{2k,2\alpha}(\{\tilde{f} < 2R\})} \le C ||F||_{\mathcal{C}^{2k,2\alpha}_{X,\beta}}$$

for some positive constant  $C = C(R, \tilde{\omega}, n)$ . We now proceed to prove the expected a priori weighted estimates on u and on its derivatives.

**Claim 6.6.** There exists a positive constant  $A = A(\tilde{\omega}, n)$  such that

$$|u(x)| \le A \log \widetilde{f}(x) ||F||_{C^0}$$
 for all  $x \in M$  with  $\widetilde{f}(x) \ge 2$ .

*Proof of Claim* 6.6. Let  $\varepsilon > 0$  and let  $\delta \in (0, 1)$  be such that

$$\lim_{\widetilde{f}\to +\infty} (u - \varepsilon e^{\delta \widetilde{f}}) = -\infty,$$

parameters that we can choose by Claim 6.5. For A>0 a constant to be determined later, we have outside a compact set  $\{\tilde{f}\geq R(\delta)\}$  the inequality

$$\Delta_{\widetilde{g},X}(u - A\log(f+1) - \varepsilon e^{\delta \widetilde{f}}) \ge -\|F\|_{C^0} + 2A > 0,$$

so long as  $A > \frac{1}{2} ||F||_{C^0}$ . Here Lemma 6.1 has been applied. Appealing to the maximum principle then yields the bound

$$\max_{\{\tilde{f} \geq R(\delta)\}} \left( u - A \log(f+1) - \varepsilon e^{\delta \tilde{f}} \right) = \max_{\{\tilde{f} = R(\delta)\}} \left( u - A \log(f+1) - \varepsilon e^{\delta \tilde{f}} \right).$$

Next, letting  $\varepsilon \to 0$ , we see that

$$u - A\log(f+1) \le \max_{\{\tilde{f} = R(\delta)\}} \left(u - A\log(f+1)\right) \le 0$$

if we set  $A := C \max_{\{\widetilde{f} = R(\delta)\}} u \le C \|F\|_{C^0}$  with  $C := C(\delta, \widetilde{\omega}, n)$ . This we can do thanks to (6.8). Applying the same argument to -u concludes the proof of the claim.

Observe that  $\widetilde{u} := u + c \chi \log r$ , where  $F - c \in C_{X,\beta}^{2k,2\alpha}(M)$ , satisfies the equation

(6.9) 
$$\Delta_{\widetilde{g},X}\widetilde{u} = F + c\Delta_{\widetilde{g},X}(\chi \log r)$$

$$= F - c + \underbrace{c + c\Delta_{\widetilde{g},X}(\chi \log r)}_{\text{compactly supported}} := \widetilde{F} \in C_{X,\beta}^{2k,2\alpha}(M).$$

The next claim estimates the  $C_{\text{loc}}^{2k+2,2\alpha}$ -norms of  $\tilde{u}$  in terms of the data F and of its local  $C^0$ -norm. For this purpose, define the corresponding solution to the Ricci flow

$$g(\tau) := (-\tau)(\phi_{\tau}^X)^* g$$
 for  $\tau < 0$ ,

where

$$\partial_{\tau}\phi_{\tau}^{X} = \frac{X}{2(-\tau)} \circ \phi_{\tau}^{X} \quad \text{and} \quad \phi_{\tau=-1}^{X} = \operatorname{Id}_{\mathbb{C} \times D}.$$

Here.

$$\phi_{\tau}^{X}(z,\theta) = \left(\frac{z}{\sqrt{-\tau}},\theta\right) \text{ for } (z,\theta) \in \mathbb{C} \times D.$$

In particular, if  $A_{r_1,r_2} := \{(z,\theta) \in \mathbb{C} \times D \mid r_1 \leq |z| \leq r_2\}$  for  $0 \leq r_1 < r_2$ , then

$$\phi_{\tau}^{X}(A_{r_1,r_2}) = A_{\frac{r_1}{\sqrt{-\tau}},\frac{r_2}{\sqrt{-\tau}}}.$$

**Claim 6.7.** There exist a radius  $r_0 > 0$  and a positive constant C such that if  $r \ge r_0$ , then

(6.10) 
$$\|\widetilde{u}\|_{C_{X,0}^{2k+2,2\alpha}(A_{r(x)-C,r(x)+C})}$$

$$\leq C(\|\widetilde{u}\|_{C^{0}(A_{r(x)}^{\underline{r(x)}},C_{r(x)})} + \|\widetilde{F}\|_{C_{X,0}^{2k,2\alpha}(A_{r(x)}^{\underline{r(x)}},C_{r(x)})}).$$

Moreover,

$$(6.11) |X \cdot \widetilde{u}| + |\nabla^{\widetilde{g}} \widetilde{u}|_{\widetilde{g}} + |\nabla^{\widetilde{g},2} \widetilde{u}|_{\widetilde{g}} \le C \log r \|F\|_{\mathcal{C}^{2k,2\alpha}_{X,B}}, \quad r \ge r_0.$$

*Proof of Claim* 6.7. Since (6.9) is expressed in terms of the Riemannian metric  $\tilde{g}$ , we define analogously the family of metrics  $\tilde{g}(\tau) := (-\tau)(\phi_{\tau}^{X})^{*}\tilde{g}$  for  $\tau < 0$ , where

$$\partial_{\tau}\phi_{\tau}^{X} = \frac{X}{2(-\tau)} \circ \phi_{\tau}^{X}$$
 and  $\phi_{\tau=-1}^{X} = \operatorname{Id}_{\mathbb{C} \times D}$ .

For  $-\tau \in [\frac{1}{2}, 2]$ , the metrics  $\tilde{g}(\tau)$  are uniformly equivalent and their covariant derivatives (with respect to g) and time derivatives are bounded by (6.1). Now,  $\tilde{u}(\tau) := (\phi_{\tau}^{X})^{*}\tilde{u}$  satisfies

(6.12) 
$$\partial_{\tau} \widetilde{u} = \Delta_{\widetilde{g}(\tau)} \widetilde{u} + \widetilde{F}(\tau), \quad \widetilde{F}(\tau) := -(-\tau)^{-1} (\phi_{\tau}^{X})^{*} F.$$

Standard parabolic Schauder estimates applied to (6.12) on a ball  $B_g(x, r_0)$ ,  $2r_0 < \text{inj}(g)$ , then ensure the existence of a uniform positive constant C such that

$$\begin{split} \|\widetilde{u}(\tau)\|_{C^{2k+2,2\alpha}(B_g(x,r_0)\times[-\frac{3}{2},-1])} \\ &\leq C\left(\|\widetilde{u}(\tau)\|_{C^0(B_g(x,2r_0)\times[-2,-\frac{1}{2}])} + \|\widetilde{F}(\tau)\|_{C^{2k,2\alpha}(B_g(x,2r_0)\times[-2,-\frac{1}{2}])}\right). \end{split}$$

Unravelling the definition of the function  $\tilde{u}(\tau)$  and that of the metrics  $\tilde{g}(\tau)$  then yields (6.10) after observing that

$$\bigcup_{\tau \in [-2, -\frac{1}{2}]} \phi_{\tau}^{X}(B_{g}(x, 2r_{0})) \subset A_{\frac{r(x)}{\sqrt{2}} - \sqrt{2}r_{0}, \sqrt{2}r(x) + 2\sqrt{2}r_{0}}.$$

The final estimate (6.11) is a straightforward combination of (6.10) together with the a priori bound from Claim 6.6.

Now we are in a position to linearise equation (6.4) outside a compact set with respect to the background metric. Namely, we write

(6.13) 
$$\Delta_{g,X}\widetilde{u} = \widetilde{F} + (\Delta_g - \Delta_{\widetilde{g}})\widetilde{u} := G,$$

where G satisfies pointwise estimate

(6.14) 
$$G - \tilde{F} = (g^{-1} - \tilde{g}^{-1}) * \partial \bar{\partial} u = O(r^{-\gamma}) |\partial \bar{\partial} u|_g,$$

here \* denoting any linear combination of contractions of tensors with respect to the metric g. Indeed, this estimate holds true by virtue of (6.1). We rewrite (6.13) (outside a compact set) as follows:

$$(6.15) \Delta_C \widetilde{u} - X \cdot \widetilde{u} + \Delta_D \widetilde{u} = G.$$

Here  $\Delta_C$  and  $\Delta_D$  denote the Riemannian Laplacian of the metric  $\omega_C$  on  $\mathbb{C}$  and  $\omega_D$  on D, respectively. Integrating this equation over D at a sufficiently large height r, we find that

(6.16) 
$$\Delta_{C,X}\overline{u}(r) = \overline{G}(r), \quad r \ge r_0,$$

where

$$\overline{u}(r) := \int_D \widetilde{u}(r,\cdot)\omega_D^{n-1}$$
 and  $\overline{G}(r) = \int_D G(r,\cdot)\omega_D^{n-1}$ ,

both functions in the r-variable only because both are JX-invariant by definition. We next derive some estimates on  $\overline{u}(r)$ .

## Claim 6.8. One has

$$|\overline{u}(r)| \leq C \|\widetilde{F}\|_{C^{2k,2\alpha}_{X,\beta}}, \quad r \geq r_0.$$

Moreover,  $\lim_{r\to+\infty} \overline{u}(r) =: u_{\infty}$  exists, is finite, and

$$|\overline{u}(r) - u_{\infty}| \le C \left( r^{-\beta} \| \widetilde{F} \|_{C_{X,\beta}^{2k,2\alpha}} + r^{-\gamma} \sup_{\{f \ge \frac{r^2}{2}\}} |\partial \bar{\partial} u| \right), \quad r \ge r_0.$$

Proof of Claim 6.8. Equation (6.16) can be rewritten as

$$(6.17) \left| \frac{X \cdot X \cdot \overline{u}(r)}{r^2} - X \cdot \overline{u}(r) \right| \le C \left( r^{-\beta} \| \widetilde{F} \|_{C_{X,\beta}^{2k,2\alpha}} + r^{-\gamma} \sup_{\{f \ge \frac{r^2}{2}\}} |\partial \bar{\partial} u| \right), \quad r \ge r_0,$$

by virtue of (6.14). This is a first-order differential inequality for  $X \cdot \overline{u}(r)$ . Now, estimate (6.11) from Claim 6.7 implies a first rough estimate, namely

$$\left|\frac{X\cdot X\cdot \overline{u}(r)}{r^2} - X\cdot \overline{u}(r)\right| \leq C r^{-\min\{\beta,\gamma\}} (1+\log r) \|\widetilde{F}\|_{C^{2k,2\alpha}_{X,\beta}}, \quad r \geq r_0.$$

Grönwall's inequality then leads to the bound

$$|X \cdot \overline{u}(r)| \le C \|\widetilde{F}\|_{C_{X,\beta}^{2k,2\alpha}} e^{\frac{r^2}{2}} \int_{r}^{+\infty} s^{-\min\{\beta,\gamma\}} (1 + \log s) s e^{-\frac{s^2}{2}} ds$$
  
$$\le C \|\widetilde{F}\|_{C_{X,\beta}^{2k,2\alpha}} r^{-\min\{\beta,\gamma\}} \log r, \quad r \ge r_0,$$

for some uniform positive constant C independent of  $r \ge r_0$ . Integrating once more in r, Claim 6.6 ensures that  $\overline{u}(r)$  admits a limit  $u_{\infty}$  as  $r \to +\infty$  and that, for  $r \ge r_0$ ,

$$|\overline{u}(r)| \leq |\overline{u}(r_0)| + C \|\widetilde{F}\|_{C_{X,\beta}^{2k,2\alpha}} \int_{r_0}^r s^{-\min\{\beta,\gamma\}-1} \log s \, ds \leq C \|\widetilde{F}\|_{C_{X,\beta}^{2k,2\alpha}}$$

for some positive constant C which is independent of r (and of the data F) that may vary from line to line. This concludes the proof of the first part of the claim.

Returning to inequality (6.17), another application Grönwall's inequality leads to the bound

$$\begin{split} |X \cdot \overline{u}(r)| &\leq C e^{\frac{r^2}{2}} \Biggl( \int_r^{+\infty} s^{-\beta} s e^{-\frac{s^2}{2}} \, ds \, \|\widetilde{F}\|_{C^{2k,2\alpha}_{X,\beta}} + \int_r^{+\infty} s^{-\gamma} s e^{-\frac{s^2}{2}} \, ds \, \sup_{\{f \geq \frac{r^2}{2}\}} |\partial \bar{\partial} u| \Biggr) \\ &\leq C \Bigl( r^{-\beta} \, \|\widetilde{F}\|_{C^{2k,2\alpha}_{X,\beta}} + r^{-\gamma} \, \sup_{\{f \geq \frac{r^2}{2}\}} |\partial \bar{\partial} u| \Bigr), \quad r \geq r_0. \end{split}$$

Integrating this inequality once more between r and  $r = +\infty$  yields the second part of the claim.

The next claim concerns the uniform boundedness of the projection of u onto the orthogonal complement of the kernel of  $\Delta_D$ , D being interpreted as embedded in each level set  $\{f = \frac{r^2}{2}\}$ .

**Claim 6.9.** Given  $\delta \in (0, \min\{\beta, \gamma\})$ , there exists  $r_0 = r_0(\delta, \widetilde{\omega}, n)$  such that

$$\|\widetilde{u} - \overline{u}(r)\|_{L^2(D)} \le C \|\widetilde{F}\|_{C_Y^{2k,2\alpha}r^{-\delta}}, \quad r \ge r_0.$$

*Proof of Claim* 6.9. Recall that, by (6.15) and (6.16),  $\Delta_{g,X}\tilde{u} = G$  so that

(6.18) 
$$\Delta_{g,X}(\widetilde{u} - \overline{u}(r)) = G - \overline{G}(r)$$

outside a compact set. Since, for any function v, we have

$$2v\Delta_{C,X}v = \Delta_{C,X}(v^2) - 2|\nabla^{\mathbb{C}}v|_{g_{\mathbb{C}}}^2,$$

multiplying (6.18) across by  $\tilde{u} - \overline{u}(r)$  and integrating over D gives rise to the lower bound

$$(6.19) \qquad \Delta_{C,X} (\|\widetilde{u} - \overline{u}(r)\|_{L^{2}(D)}^{2})$$

$$\geq \Delta_{C,X} (\|\widetilde{u} - \overline{u}(r)\|_{L^{2}(D)}^{2}) - 2 \int_{\mathbb{P}^{1}} |\nabla^{\mathbb{C}} (\widetilde{u} - \overline{u}(r))|_{g_{\mathbb{C}}}^{2} \frac{\omega_{D}^{n-1}}{(n-1)!}$$

$$= 2 \int_{D} (\widetilde{u} - \overline{u}(r)) \Delta_{C,X} (\widetilde{u} - \overline{u}(r)) \frac{\omega_{D}^{n-1}}{(n-1)!}$$

$$= 2 \int_{D} (\widetilde{u} - \overline{u}(r)) (G - \overline{G}(r) - \Delta_{D} (\widetilde{u} - \overline{u}(r))) \frac{\omega_{D}^{n-1}}{(n-1)!}$$

$$= 2 \|\nabla^{g_{D}} (\widetilde{u} - \overline{u}(r))\|_{L^{2}(D)}^{2} + 2 \langle G - \overline{G}(r), \widetilde{u} - \overline{u}(r) \rangle_{L^{2}(D)}$$

$$\geq 2\lambda^{D} \|\widetilde{u} - \overline{u}(r)\|_{L^{2}(D)}^{2} - 2 \|G - \overline{G}(r)\|_{L^{2}(D)} \|\widetilde{u} - \overline{u}(r)\|_{L^{2}(D)},$$

where we have made use of the Poincaré inequality on  $(D, g_D)$  in the last line. Young's inequality then implies for  $\varepsilon \in (0, \lambda^D)$  that

$$\Delta_{C,X} \left( \|\widetilde{u} - \overline{u}(r)\|_{L^2(D)}^2 \right) \ge 2(\lambda^D - \varepsilon) \|\widetilde{u} - \overline{u}(r)\|_{L^2(D)}^2 - C_{\varepsilon} \|G - \overline{G}(r)\|_{L^2(D)}^2.$$

Therefore, invoking estimate (6.14) and Claim 6.7 together with the previous inequality, we find that

$$\Delta_{C,X}(\|\widetilde{u} - \overline{u}(r)\|_{L^{2}(D)}^{2}) \ge 2(\lambda^{D} - \varepsilon)\|\widetilde{u} - \overline{u}(r)\|_{L^{2}(D)}^{2}$$
$$- C_{\varepsilon}\|\widetilde{F}\|_{C_{X,\beta}^{2k,2\alpha}}^{2k,2\alpha}r^{-2\min\{\beta,\gamma\}}\log^{2}r, \quad r \ge 2.$$

By Lemma 6.1 applied to  $\tilde{g} := g$ , we see that

$$\Delta_{C,X}(r^{-2\delta}) = 2\delta r^{-2\delta} + O(r^{-2\delta-2}),$$

which, for A > 0 and  $\delta \in (0, \min\{\beta, \gamma\})$ , implies that

$$\begin{split} & \Delta_{C,X} \big( \| \widetilde{u} - \overline{u}(r) \|_{L^{2}(D)}^{2} - Ar^{-2\delta} \big) \\ & \geq 2(\lambda^{D} - \varepsilon) \| \widetilde{u} - \overline{u}(r) \|_{L^{2}(D)}^{2} - C_{\varepsilon} \| \widetilde{F} \|_{C_{X,\beta}^{2k,2\alpha}}^{2k,2\alpha} r^{-2\min\{\beta,\gamma\}} \log^{2} r \\ & - 2A\delta r^{-2\delta} - ACr^{-2\delta-2} \\ & \geq 2(\lambda^{D} - \varepsilon) \big( \| \widetilde{u} - \overline{u}(r) \|_{L^{2}(D)}^{2} - Ar^{-2\delta} \big) + 2A(\lambda^{D} - \varepsilon - \delta) r^{-2\delta} \\ & - C_{\varepsilon} \| \widetilde{F} \|_{C_{X,\beta}^{2k,2\alpha}}^{2k,2\alpha} r^{-2\min\{\beta,\gamma\}} \log^{2} r - ACr^{-2\delta-2} \\ & \geq 2(\lambda^{D} - \varepsilon) \big( \| \widetilde{u} - \overline{u}(r) \|_{L^{2}(D)}^{2} - Ar^{-2\delta} \big), \end{split}$$

provided that  $\varepsilon \in (0, \lambda^D - \delta)$ ,  $r \geq r_0 = r_0(\delta, n, \widetilde{\omega})$ , and  $A \geq C \|\widetilde{F}\|_{C_{X,B}^{2k,2\alpha}}$ .

Now, since  $\|\widetilde{u} - \overline{u}(r)\|_{L^2(D)}$  is growing at most logarithmically by Claim 6.6, given B > 0, we compute that

$$\Delta_{C,X} \left( \left\| \widetilde{u} - \overline{u}(r) \right\|_{L^2(D)}^2 - A r^{-2\delta} - B r \right) \geq 2 (\lambda^D - \varepsilon) \left( \left\| \widetilde{u} - \overline{u}(r) \right\|_{L^2(D)}^2 - A r^{-2\delta} - B r \right)$$

if  $\varepsilon \in (0, \lambda^D - \delta)$ ,  $r \ge r_0 = r_0(\delta, n, \widetilde{\omega})$ , and  $A \ge C \|\widetilde{F}\|_{C^{2k, 2\alpha}_{X, \beta}}$ . In particular, the maximum principle applied to the function  $\|\widetilde{u} - \overline{u}(r)\|_{L^2(D)}^2 - Ar^{-2\delta} - Br$  outside a compact set of the form  $r \ge r_0$  leads to the equality

$$\max_{\{r \ge r_0\}} \left( \|\widetilde{u} - \overline{u}(r)\|_{L^2(D)}^2 - Ar^{-2\delta} - Br \right) = \max \left\{ 0, \max_{\{r = r_0\}} \left( \|\widetilde{u} - \overline{u}(r)\|_{L^2(D)}^2 - Ar^{-2\delta} - Br \right) \right\}.$$

Letting  $B \to 0$  and setting  $A = C \| \tilde{F} \|_{C^{2k,2\alpha}_{X,\beta}}$  with C sufficiently large but uniform in the data F and the radius r, one arrives at the expected bound

$$\|\widetilde{u} - \overline{u}(r)\|_{L^2(D)} \le C \|\widetilde{F}\|_{C^{2k,2\alpha}_{X,\beta}} r^{-\delta}, \quad r \ge r_0.$$

The next claim proves a quantitative almost sharp weighted  $C^0$ -estimate on  $\tilde{u} - u_{\infty}$  in terms of the data F.

**Claim 6.10.** Given  $\delta \in (0, \min\{\beta, \gamma\})$ , there exists  $r_0 = r_0(\delta, \widetilde{\omega}, n) > 0$  independent of F (and the solution u) such that

$$\sup_{r\geq r_0} r^{\delta} |\widetilde{u} - u_{\infty}| \leq C \|\widetilde{F}\|_{C^{2k,2\alpha}_{X,\beta}}.$$

Proof of Claim 6.10. It suffices to prove that, for all  $\delta \in (0, \min\{\beta, \gamma\})$ , there exists  $r_0 = r_0(\delta, n, \widetilde{\omega}) > 0$  such that

(6.20) 
$$\sup_{r \ge r_0} r^{\delta} |\widetilde{u} - \overline{u}(r)| \le C \|\widetilde{F}\|_{C_{X,\beta}^{2k,2\alpha}}.$$

Indeed, the triangle inequality together with Claims 6.7 and 6.8 already yield such a uniform  $C^0$ -polynomial rate on the difference  $\overline{u}(r) - u_{\infty}$ .

In order to prove (6.20), we apply a local parabolic Nash–Moser iteration to the following heat equation with a source term (see for instance [26, Theorem 6.17]) by recalling that, for  $\tau < 0$ ,  $\widetilde{u}(\tau) := (\phi_{\tau}^X)^* \widetilde{u}$  and  $\overline{u}(r,\tau) := (\phi_{\tau}^X)^* \overline{u}(r) = \overline{u}(\frac{r}{\sqrt{-\tau}})$ ,

$$\begin{split} \partial_{\tau} \big( \widetilde{u} - \overline{u}(\,\cdot\,,\cdot\,) \big) (\tau) &= \Delta_{(-\tau) \cdot g_D} \big( \widetilde{u} - \overline{u}(\,\cdot\,,\cdot\,) \big) (\tau) \\ &+ \underbrace{\Delta_C \big( \widetilde{u} - \overline{u}(\,\cdot\,,\cdot\,) \big) (\tau) - (G - \overline{G})(\tau)}_{:=S(\tau), \text{ source term}}, \quad (-\tau) \in \Big[\frac{1}{2}, 2\Big]. \end{split}$$

Here we have used (6.12), (6.13), and (6.16). Also, the notation  $(-\tau) \cdot g_D$  denotes the metric on D rescaled by  $(-\tau)$ . In particular, there exists C > 0 such that if  $r \ge r_0$ ,

(6.21) 
$$\sup_{f = \frac{r^2}{2}} |\widetilde{u} - \overline{u}(r)| = \sup_{f = \frac{r^2}{2}} |\widetilde{u}(\tau) - \overline{u}(r,\tau)|_{\tau = -1}$$

$$\leq C \sup_{(-\tau) \in [1/2,2]} (\|\widetilde{u}(\tau) - \overline{u}(r,\tau)\|_{L^2(D)} + |S(\tau)|)$$

$$\leq C \sup_{s \in [r/\sqrt{2},\sqrt{2}r]} (\|\widetilde{u}(1) - \overline{u}(s,1)\|_{L^2(D)} + |S(1)|).$$

The source term can be estimated as follows: if  $k \ge 1$ ,  $(-\tau) \in [\frac{1}{2}, 2]$  and  $r \ge r_0$ , then

$$\begin{split} \left| \Delta_C \left( \widetilde{u} - \overline{u}(\cdot, \cdot) \right) (\tau) - (G - \overline{G})(\tau) \right| &\leq C \left( r^{-\beta} \| \widetilde{F} \|_{C_{X,\beta}^{2k,2\alpha}} + r^{-\gamma} \sup_{f \geq \frac{r^2}{4}} |\partial \overline{\partial} u| + r^{-2} \sup_{f \geq \frac{r^2}{4}} |u| \right) \\ &\leq C \| \widetilde{F} \|_{C_{X,\beta}^{2k,2\alpha}} r^{-\min\{\beta,\gamma\}} (1 + \log r), \end{split}$$

where we have applied Claim 6.7 to  $X \cdot \tilde{u}$  and  $X \cdot X \cdot \tilde{u}$  in order to estimate  $\Delta_C \tilde{u}$ .

Finally, thanks to (6.21), Claim 6.9 combined with the above estimate on the source term implies that

$$\sup_{f = \frac{r^2}{2}} |\widetilde{u} - \overline{u}(r)| \le C \|\widetilde{F}\|_{C_{X,\beta}^{2k,2\alpha}} r^{-\delta} + C \|\widetilde{F}\|_{C_{X,\beta}^{2k,2\alpha}} r^{-\min\{\beta,\gamma\}} (1 + \log r)$$

$$\le C \|\widetilde{F}\|_{C_{X,\beta}^{2k,2\alpha}} r^{-\delta}, \quad r \ge r_0,$$

as claimed.

The next claim provides a quantitative sharp weighted  $C^0$ -estimate on  $\widetilde{u} - u_{\infty}$  in terms of the data.

**Claim 6.11.** Given  $\beta \in (0, \lambda^D)$ , there exists  $r_0 = r_0(\beta, \widetilde{\omega}, n) > 0$  independent of F (and the solution u) such that

$$\sup_{r\geq r_0} r^{\beta} |\widetilde{u} - u_{\infty}| \leq C \|\widetilde{F}\|_{C^{2k,2\alpha}_{X,\beta}}.$$

*Proof of Claim* 6.11. Applying (6.10) to  $\tilde{u} - u_{\infty}$  together with Claim 6.10 demonstrates that, for  $\delta \in (0, \min\{\beta, \gamma\})$ ,

$$|X \cdot \widetilde{u}|(x) + |\nabla^{\widetilde{g}} \widetilde{u}|(x) + |\nabla^{\widetilde{g},2} \widetilde{u}|(x) \le C \|\widetilde{F}\|_{C^{2k,2\alpha}_{X,B}} r^{-\delta}, \quad r \ge r_0.$$

Recalling (6.14), the previous estimate implies in turn the following one:

(6.22) 
$$|G - \tilde{F}| \le C \|\tilde{F}\|_{C_{V,\beta}^{2k,2\alpha}} r^{-\gamma - \delta}, \quad r \ge r_0.$$

On one hand, thanks to Claim 6.8, one obtains an improved decay on  $\overline{u}(r) - u_{\infty}$ , namely

$$|\overline{u}(r) - u_{\infty}| \le C \|\widetilde{F}\|_{C_{Y,\beta}^{2k,2\alpha}}(r^{-\min\{\beta,\gamma+\delta\}}), \quad r \ge r_0.$$

On the other hand, (6.22) can then be inserted into the proof of Claim 6.9 to establish an improved  $L^2(D)$ -decay on  $\tilde{u} - \overline{u}(r)$ . Indeed, from inequality (6.19), we deduce that, for  $r \ge r_0$ ,

$$\Delta_{C,X}(\|\widetilde{u} - \overline{u}(r)\|_{L^{2}(D)}^{2}) 
\geq 2\lambda^{D} \|\widetilde{u} - \overline{u}(r)\|_{L^{2}(D)}^{2} - C \|\widetilde{u} - \overline{u}(r)\|_{L^{2}(D)} \|\widetilde{F}\|_{C_{X,\beta}^{2k,2\alpha}} r^{-\min\{\beta,\gamma+\delta\}} 
\geq 2(\lambda^{D} - \varepsilon) \|\widetilde{u} - \overline{u}(r)\|_{L^{2}(D)}^{2} - C_{\varepsilon} \|\widetilde{F}\|_{C_{X,\beta}^{2k,2\alpha}}^{2k,2\alpha} r^{-2\min\{\beta,\gamma+\delta\}}$$

for any  $\varepsilon \in (0, \lambda^D)$ . Using a barrier function of the form  $r^{-2\delta'}$  with

$$0 < \delta' \le \min\{\beta, \gamma + \delta\} < \lambda^D$$

and by choosing  $\varepsilon > 0$  carefully, one arrives at an improved  $L^2(D)$ -decay of the form above, specifically

$$\|\widetilde{u} - \overline{u}(r)\|_{L^2(D)} \le C \|\widetilde{F}\|_{C_{Y,\beta}^{2k,2\alpha}} r^{-\delta'}, \quad r \ge r_0.$$

The proof of Claim 6.10 can now be adapted to give a corresponding improved pointwise decay. By applying this reasoning a finite number of times, one arrives at the desired sharp decay on  $\tilde{u} - u_{\infty}$ .

Theorem 6.3 now follows by combining Claim 6.7 (after multiplying by the weight  $r^{\beta}$ ) and Claim 6.11.

**6.4.** Small perturbations along the continuity path. In this section, we show, using the implicit function theorem, that the invertibility of the drift Laplacian given by Theorem 6.3 allows for small perturbations in polynomially weighted function spaces of solutions to the complex Monge–Ampère equation that we wish to solve. This forms the openness part of the continuity method as will be explained later in Section 7.1.

In notation reminiscent of that of [34, Chapter 5], we consider the space  $(\mathcal{C}^{2,2\alpha}_{X,\beta}(M))_{\widetilde{\omega},0}$  of functions  $F \in \mathcal{C}^{2,2\alpha}_{X,\beta}(M)$  with

$$\int_{M} (e^{F} - 1)e^{-\tilde{f}} \tilde{\omega}^{n} = 0.$$

This function space is a hypersurface of the Banach space  $\mathcal{C}_{X,\beta}^{2,2\alpha}(M)$ . Notice that the tangent space at a function  $F_0$  is the set of functions  $u \in \mathcal{C}_{X,\beta}^{2,2\alpha}(M)$  with

$$\int_{M} u e^{F_0 - \tilde{f}} \widetilde{\omega}^n = 0.$$

We have the following theorem.

#### Theorem 6.12. Let

$$F_0 \in (\mathcal{C}^{2,2\alpha}_{X,\beta}(M))_{\widetilde{\omega},0} \cap \mathcal{C}^{\infty}_{X,\beta}(M)$$
 for some  $\beta \in (0,\lambda^D)$ 

and let  $\psi_0 \in \mathcal{M}_{X,B}^{\infty}(M)$  be a solution of the complex Monge-Ampère equation

$$\log\left(\frac{\widetilde{\omega}_{\psi_0}^n}{\widetilde{\omega}^n}\right) - \frac{X}{2} \cdot \psi_0 = F_0.$$

Then, for any  $\alpha \in (0, \frac{1}{2})$ , there exists a neighbourhood  $U_{F_0} \subset (C^{2,2\alpha}_{X,\beta}(M))_{\widetilde{\omega},0}$  of  $F_0$  such that, for all  $F \in U_{F_0}$ , there exists a unique function  $\psi \in \mathcal{M}^{4,2\alpha}_{X,\beta}(M)$  such that

(6.23) 
$$\log\left(\frac{\widetilde{\omega}_{\psi}^{n}}{\widetilde{\omega}^{n}}\right) - \frac{X}{2} \cdot \psi = F.$$

Moreover, if  $F \in U_{F_0}$  lies in  $\mathcal{C}_{X,\beta}^{\infty}(M)$ , then the unique solution  $\psi \in \mathcal{M}_{X,\beta}^{4,2\alpha}(M)$  to (6.23) lies in  $\mathcal{M}_{X,\beta}^{\infty}(M)$ .

**Remark 6.13.** Consideration of only finite regularity of the difference  $\omega - \tilde{\omega}$  (which lowers the assumptions on the regularity of the coefficients of the drift Laplacian  $\Delta_{\tilde{e},X}$ ) and of the data  $(\psi_0, F_0)$  would lead to a more refined version of Theorem 6.12.

*Proof of Theorem* 6.12. To apply the implicit function theorem for Banach spaces, we must reformulate the statement of Theorem 6.12 in terms of the map  $MA_{\widetilde{\omega}}$  introduced formally at the beginning of Section 6.3. To this end, consider the mapping

$$MA_{\widetilde{\omega}} \colon \psi \in \mathcal{M}^{4,2\alpha}_{X,\beta}(M) \mapsto \log \left(\frac{\widetilde{\omega}^n_{\psi}}{\widetilde{\omega}^n}\right) - \frac{X}{2} \cdot \psi \in (\mathcal{C}^{2,2\alpha}_{X,\beta}(M))_{\widetilde{\omega},0}, \quad \alpha \in \left(0,\frac{1}{2}\right).$$

Notice that the function spaces above can be defined by either using the metric  $\tilde{g}$  or  $\tilde{g}_{t\psi_0}$ for any  $t \in [0, 1]$ . To see that  $MA_{\widetilde{\omega}}$  is well-defined, apply the Taylor expansion (6.3) to the background metric  $\widetilde{\omega}$  to obtain

(6.24) 
$$MA_{\widetilde{\omega}}(\psi) = \log\left(\frac{\widetilde{\omega}_{\psi}^{n}}{\widetilde{\omega}^{n}}\right) - \frac{X}{2} \cdot \psi$$
$$= \Delta_{\widetilde{\omega}}\psi - \frac{X}{2} \cdot \psi - \int_{0}^{1} \int_{0}^{u} |\partial \bar{\partial}\psi|_{\widetilde{g}_{t\psi}}^{2} dt du.$$

Then, by the very definition of  $\mathcal{D}_{X,\beta}^{4,2\alpha}(M)$ , the first two terms of the last line of (6.24) lie in

 $C_{X,\gamma}^{2,2\alpha}(M)$ .

Now, if S and T are tensors in  $C_{X,\gamma_1}^{2k,2\alpha}(M)$  and  $C_{X,\gamma_2}^{2k,2\alpha}(M)$ , respectively, with  $\gamma_i \geq 0$ , i=1,2, then S\*T lies in  $C_{X,\gamma_1+\gamma_2}^{2k,2\alpha}(M)$ , where \* denotes any linear combination of contraction  $C_{X,\gamma_1+\gamma_2}^{2k,2\alpha}(M)$ , where \* denotes any linear combination of contraction  $C_{X,\gamma_1+\gamma_2}^{2k,2\alpha}(M)$ .

Next notice that

$$|i\,\partial\bar\partial\psi|^2_{\widetilde g_{t\psi}}=\widetilde g_{t\psi}^{-1}*\widetilde g_{t\psi}^{-1}*(\nabla^{\widetilde g})^2\psi*(\nabla^{\widetilde g})^2\psi$$

and that

$$\widetilde{g}_{t\psi}^{-1}-\widetilde{g}^{-1}\in C^{2,2\alpha}_{X,\beta}(M).$$

Thus, applying (6.25) twice to

$$S = T = (\nabla^{\widetilde{g}})^2 \psi$$

and to the inverse  $\tilde{g}_{tyl}^{-1}$  with weights  $\gamma_1 = \gamma_2 = \beta$  and k = 1, one finds that

$$|i\,\partial\bar{\partial}\psi|^2_{\widetilde{g}_{t\psi}}\in C^{2,2\alpha}_{X,2\beta}(M)\subset C^{2,2\alpha}_{X,\beta}(M)\quad\text{for each }t\in[0,1]$$

and that

$$\left\|\int_0^1 \int_0^u |i\,\partial\bar{\partial}\psi|^2_{\widetilde{g}_{t\psi}}\,dt\,du\right\|_{C^{2,2\alpha}_{X,\beta}} \leq C(k,\alpha,\widetilde{g})\|\psi\|_{\mathcal{D}^{4,2\alpha}_{X,\beta}},$$

as long as  $\|\psi\|_{\mathcal{D}^{4,2\alpha}_{X,\beta}} \le 1$ . Finally, the JX-invariance of the right-hand side of (6.24) is clear and Lemma 6.2 (i) ensures that the function

$$\exp MA_{\tilde{\omega}}(\psi) - 1$$

has zero mean value with respect to the weighted measure  $e^{-\tilde{f}}\tilde{\omega}^n$ . Indeed, Lemma 6.2 (i) applied to the linear path  $\tilde{\omega}_{\tau} := \tilde{\omega} + i\partial\bar{\partial}(\tau\psi)$  for  $\tau \in [0,1]$  gives us that

$$\int_{M} \left( \exp M A_{\widetilde{\omega}}(\psi) - 1 \right) e^{-\widetilde{f}} \widetilde{\omega}^{n} = \int_{M} e^{-\widetilde{f}_{\psi}} \widetilde{\omega}_{\psi}^{n} - \int_{M} e^{-\widetilde{f}} \widetilde{\omega}^{n} = 0.$$

By (6.2), we have that

$$\begin{split} D_{\psi_0} M A_{\widetilde{\omega}} \colon \psi \in \mathcal{M}_{X,\beta}^{4,2\alpha}(M) \cap \left\{ \int_M u e^{-\widetilde{f}_{\psi_0}} \widetilde{\omega}_{\psi_0}^n = 0 \right\} \\ \mapsto \Delta_{\widetilde{\omega}_{\psi_0}} \psi - \frac{X}{2} \cdot \psi \in T_{F_0}(\mathcal{C}_{X,\beta}^{2,2\alpha}(M))_{\widetilde{\omega},0}, \end{split}$$

where the tangent space of  $(\mathcal{C}^{2,2\alpha}_{X,\beta}(M))_{\widetilde{\omega},0}$  at  $F_0$  is equal to the set of functions  $u\in\mathcal{C}^{2,2\alpha}_{X,\beta}(M)$  with 0 mean value with respect to the weighted measure  $e^{-\widetilde{f}_{\psi_0}}\widetilde{\omega}^n_{\psi_0}$ . Therefore, after applying Theorem 6.3 to the background metric  $\widetilde{\omega}_{\psi_0}$  in place of  $\widetilde{\omega}$ , we conclude that  $D_{\psi_0}MA_{\widetilde{\omega}}$  is an isomorphism of Banach spaces. The result now follows by applying the implicit function theorem to the map  $MA_{\widetilde{\omega}}$  in a neighbourhood of

$$\psi_0 \in \mathcal{M}_{X,\beta}^{4,2\alpha}(M) \cap \left\{ \int_M u e^{-\widetilde{f}_{\psi_0}} \widetilde{\omega}_{\psi_0}^n = 0 \right\}.$$

The proof of the regularity at infinity of the solution  $\psi$  in case the data  $F \in \mathcal{C}^{\infty}_{X,\beta}(M)$  follows by a standard bootstrapping and will therefore be omitted; see Propositions 7.32 and 7.34 for the nonlinear setting.

## 7. Proof of Theorem A (v): A priori estimates

**7.1. The continuity path.** Recall the setup and notation of Theorem A: J denotes the complex structure on M, z the holomorphic coordinate on the  $\mathbb{C}$ -component of  $\widehat{M}$ , and we write  $r = |z|^{\lambda}$ , treating both r and z as functions on M via v. It is clear then that  $X = r\partial_r$  on  $M \setminus K$ .

Recall from (1.3) that the complex Monge–Ampère equation we wish to solve is

$$\begin{cases} (\omega+i\partial\bar{\partial}\psi)^n = e^{F+\frac{X}{2}\cdot\psi}\omega^n, & \psi\in C^\infty(M), \quad \mathcal{L}_{JX}\psi=0, \\ & \omega+i\partial\bar{\partial}\psi>0, \end{cases}$$
 
$$\begin{cases} \int_M e^{F-f}\omega^n = \int_M e^{-f}\omega^n, \end{cases}$$

where  $F: M \to \mathbb{R}$  is a JX-invariant smooth function equal to a constant  $c_0$  outside a compact subset V of M and  $f: M \to \mathbb{R}$  is the Hamiltonian potential of X with respect to  $\omega$ , i.e.,  $-\omega J X = df$ , normalised so that  $\Delta_{\omega} f - f + \frac{X}{2} \cdot f = 0$  outside a compact set. Define  $F_s := \log(1 + s(e^F - 1))$ . In this section, we prove Theorem A(v) by providing a solution to  $(*_0)$  by implementing the continuity path

$$(\star_s) \begin{cases} (\omega + i\partial\bar{\partial}\psi_s)^n = e^{F_s + \frac{X}{2} \cdot \psi_s} \omega^n, & \psi_s \in \mathcal{M}_{X,\beta}^{\infty}(M), & \mathcal{L}_{JX}\psi_s = 0, \\ s \in [0,1], & \\ \int_M e^{F - f} \omega^n = \int_M e^{-f} \omega^n, & \int_M \psi_s e^{-f} \omega^n = 0. \end{cases}$$

When s = 0,  $(\star_0)$  admits the trivial solution, namely  $\psi_0 \equiv 0$ . When s = 1,  $(\star_1)$  corresponds to  $(\star_0)$ , that is, the equation that we wish to solve. Via the a priori estimates to follow, we will show that the set  $s \in [0,1]$  for which  $(\star_s)$  has a solution is closed. As we have just seen, this set is non-empty. Openness of this set follows from the isomorphism properties of the drift Laplacian given by Theorem 6.12. Connectedness of [0,1] then implies that  $(\star_s)$  has a solution for s = 1, resulting in the desired solution of  $(\star_0)$ .

**7.2.** The continuity path re-parametrised. To obtain certain localisation results and, in turn, a priori estimates for  $(\star_s)$ , we need to consider a reformulation of  $(\star_s)$  in the following way. Identify  $(M \setminus K, \omega)$  and  $(\widehat{M} \setminus \widehat{K}, \widehat{\omega})$  using  $\nu$ , where  $K \subset M$ ,  $\widehat{K} \subset \widehat{M}$  are compact, and define  $F_s := \log(1 + s(e^F - 1))$ . Then there exists a compact subset  $K \subset V \subset M$  such that, for all  $s \in [0, 1]$ ,  $F_s$  is equal to a constant  $c_s$  on  $M \setminus V$ . Explicitly,  $c_s = \log(1 + s(e^{c_0} - 1))$ . Note that  $c_s$  varies continuously as a function of s and that  $(\star_s)$  takes the form

$$(\omega + i\partial\bar{\partial}\psi_s)^n = e^{F_s + \frac{X}{2} \cdot \psi_s} \omega^n.$$

Let  $\eta_s := -2c_s \log(r)$ , a real-valued function defined on  $M \setminus K$ . Then, with g denoting the Kähler metric associated to  $\omega$ , it is clear that

$$\|(\log(r))^{-1} \cdot \eta_s\|_{C^0(M \setminus K)} + \|d\eta_s\|_{C^0(M \setminus K,g)} + \|r \cdot i \partial \bar{\partial} \eta_s\|_{C^0(M \setminus K,g)}$$

$$\leq 2|c_s| \left(1 + \sup_{M \setminus K} r^{-1}\right) \leq C(K),$$

and so Lemma 2.4 yields the existence of a bump function  $\chi \colon M \to \mathbb{R}$  supported on  $M \setminus V$  and a compact subset  $W \supset V$ , both independent of s, such that  $\chi = 1$  on  $M \setminus W$  and such that, for all  $s \in [0,1]$ ,  $\omega_s := \omega + i \partial \bar{\partial} (\chi \cdot \eta_s) > 0$  on M. Define  $\Phi_s := \chi \cdot \eta_s$ . Then  $\omega_s = \omega + i \partial \bar{\partial} \Phi_s$ , and since  $\Phi_s = -2c_s \log r$  on  $M \setminus W$ , that is, a pluriharmonic function,  $\omega_s$  is isometric to  $\omega$  on this set. Furthermore, we find that

$$\log\left(\frac{(\omega_{s}+i\partial\bar{\partial}(\psi_{s}-\Phi_{s}))^{n}}{\omega_{s}^{n}}\right) - \frac{X}{2} \cdot (\psi_{s}-\Phi_{s})$$

$$= \log\left(\frac{(\omega+i\partial\bar{\partial}\psi_{s})^{n}}{(\omega+i\partial\bar{\partial}\Phi_{s})^{n}}\right) - \frac{X}{2} \cdot (\psi_{s}-\Phi_{s})$$

$$= \log\left(\frac{(\omega+i\partial\bar{\partial}\psi_{s})^{n}}{(\omega+i\partial\bar{\partial}\psi_{s})^{n}}\right) - \frac{X}{2} \cdot \psi_{s} - \log\left(\frac{(\omega+i\partial\bar{\partial}\Phi_{s})^{n}}{(\omega^{n})}\right) + \frac{X}{2} \cdot \Phi_{s}$$

$$= F_{s} - \left(\log\left(\frac{(\omega+i\partial\bar{\partial}\Phi_{s})^{n}}{(\omega^{n})}\right) - \frac{X}{2} \cdot \Phi_{s}\right) =: G_{s},$$

with  $G_s$  vanishing on  $M \setminus W$ . Set  $\vartheta_s := \psi_s - \Phi_s$ . Then  $\vartheta_s \in \mathbb{R} \oplus C^{\infty}_{X,\beta}(M)$  and we can rewrite  $(\star_s)$  in terms of  $\vartheta_s$  as

$$(\star \star_{s}) \log \left( \frac{(\omega_{s} + i \partial \bar{\partial} \vartheta_{s})^{n}}{\omega_{s}^{n}} \right) - \frac{X}{2} \cdot \vartheta_{s} = G_{s}, \quad \vartheta_{s} \in \mathbb{R} \oplus C_{X,\beta}^{\infty}(M), \quad \mathcal{L}_{JX} \vartheta_{s} = 0,$$
$$\omega_{s} + i \partial \bar{\partial} \vartheta_{s} > 0, \quad s \in [0, 1],$$

with the support of  $G_s$  contained in W and  $\omega_s = \omega$  on  $M \setminus W$ . We derive a priori estimates for  $(\star \star_s)$ , the advantage over  $(\star_s)$  being that it allows for a localisation of the infimum and supremum of  $|\vartheta_s|$ , essentially because the unbounded log term has been absorbed into the background metric  $\omega_s$  in  $(\star \star_s)$ . As we have control on  $\Phi_s$ , the a priori estimates we derive for  $\vartheta_s$  will translate into the desired a priori estimates for  $\psi_s$ , thereby allowing us to complete the closedness part of the continuity method for  $(\star_s)$ .

Define  $\sigma_s := \omega_s + i \partial \bar{\partial} \vartheta_s$ . Then, in terms of the Ricci forms  $\rho_{\sigma_s}$  and  $\rho_{\omega_s}$  of  $\sigma_s$  and  $\omega_s$ , respectively,  $(\star \star_s)$  yields

(7.1) 
$$\rho_{\sigma_s} + \frac{1}{2} \mathcal{L}_X \sigma_s = \rho_{\omega_s} + \frac{1}{2} \mathcal{L}_X \omega_s - i \partial \bar{\partial} G_s.$$

We will write  $h_s$  for the Kähler metric associated to  $\sigma_s$ .

We will need the following lemma regarding the Hamiltonian potential  $f_{\omega_s}$  of X with respect to  $\omega_s$ .

**Lemma 7.1.** Let  $f_{\omega_s} := f + \frac{X}{2} \cdot \Phi_s$ . Then  $-\omega_s \, \lrcorner \, JX = df_{\omega_s}$  and there exists a compact subset  $U \subset M$  containing W such that, for all  $s \in [0,1]$ , there exists  $H_s \in C^{\infty}(M)$  varying smoothly in s and equal to  $-c_s$  on  $M \setminus U$  so that

(7.2) 
$$\Delta_{\omega_s} f_{\omega_s} - \frac{X}{2} \cdot f_{\omega_s} + f_{\omega_s} = H_s.$$

*Proof.* The first assertion is clear. Regarding the normalisation condition (7.2), a computation shows that, for the Ricci forms  $\rho_{\omega}$  and  $\rho_{\omega_s}$  of  $\omega$  and  $\omega_s$ , respectively,

$$\rho_{\omega_s} + \frac{1}{2} \mathcal{L}_X \omega_s - \omega_s = \rho_\omega + \frac{1}{2} \mathcal{L}_X \omega - \omega - i \partial \bar{\partial} \left( \log \left( \frac{\omega_s^n}{\omega^n} \right) - \frac{X}{2} \cdot \Phi_s + \Phi_s \right)$$
$$= i \partial \bar{\partial} (F_2 + G_s - F_s - \Phi_s),$$

where we have used (4.2). Write  $Q_s := F_2 + G_s - F_s - \Phi_s$ . Then  $Q_s$  is JX-invariant and it is easy to see that  $Q_s$  is equal to  $2c_s \log(r) - c_s$  outside a compact subset  $U \supseteq W$  of M independent of s. Contracting the identity

$$\rho_{\omega_s} + \frac{1}{2} \mathcal{L}_X \omega_s - \omega_s = i \, \partial \bar{\partial} Q_s$$

with  $X^{1,0} := \frac{1}{2}(X - iJX)$  and arguing as in Lemma 4.2 using the JX-invariance of the functions involved, we find that

$$\Delta_{\omega_s} f_{\omega_s} - \frac{X}{2} \cdot f_{\omega_s} + f_{\omega_s} + \frac{X}{2} \cdot Q_s$$

is constant on M. But since, on  $M \setminus W$ ,  $\omega_s = \omega$ ,  $f_{\omega_s} = f - c_s$ , and  $\frac{X}{2} \cdot Q_s = c_s$ , this constant must be zero. Hence the result follows with  $H_s := -\frac{X}{2} \cdot Q_s$ .

This allows for a normalisation for the Hamiltonian potential  $f_{\sigma_s} := f_{\omega_s} + \frac{X}{2} \cdot \vartheta_s$  of X with respect to  $\sigma_s$ .

**Lemma 7.2.** Let  $f_{\sigma_s} := f_{\omega_s} + \frac{X}{2} \cdot \vartheta_s$  and let U be as in Lemma 7.1. Then  $-\sigma_{s} \lrcorner JX = df_{\sigma_s}$ ,

and for all  $s \in [0, 1]$ , there exists a compactly supported function  $P_s \in C^{\infty}(M)$  varying smoothly in s with supp  $P_s \subseteq U$  such that

$$\Delta_{\sigma_s} f_{\sigma_s} - \frac{X}{2} \cdot f_{\sigma_s} = -f + P_s.$$

*Proof.* Again, the first assertion is clear. As for (7.2), we have that

$$\begin{split} \frac{X}{2} \cdot \log \left( \frac{\sigma_s^n}{\omega_s^n} \right) &= \frac{1}{2} \operatorname{tr}_{\sigma_s} \mathcal{L}_X \sigma_s - \frac{1}{2} \operatorname{tr}_{\omega_s} \mathcal{L}_X \omega_s \\ &= \operatorname{tr}_{\sigma_s} (i \, \partial \bar{\partial} f_{\sigma_s}) - \operatorname{tr}_{\omega} (i \, \partial \bar{\partial} f_{\omega_s}) \\ &= \Delta_{\sigma_s} f_{\sigma_s} - \Delta_{\omega_s} f_{\omega_s}. \end{split}$$

Thus, contracting both sides of  $(\star \star_s)$  with  $\frac{X}{2}$ , we obtain

$$\Delta_{\sigma_s} f_{\sigma_s} - \Delta_{\omega_s} f_{\omega_s} = \frac{X}{2} \cdot G_s + \frac{X}{2} \cdot \left( f_{\omega_s} + \frac{X}{2} \cdot \vartheta_s \right) - \frac{X}{2} \cdot f_{\omega_s},$$

i.e.,

$$\Delta_{\sigma_s} f_{\sigma_s} - \frac{X}{2} \cdot f_{\sigma_s} = \Delta_{\omega_s} f_{\omega_s} - \frac{X}{2} \cdot f_{\omega_s} + \frac{X}{2} \cdot G_s.$$

Hence we derive from (7.2) that

$$\Delta_{\sigma_s} f_{\sigma_s} - \frac{X}{2} \cdot f_{\sigma_s} = H_s + \frac{X}{2} \cdot G_s - f_{\omega_s}.$$

With  $P_s := H_s + \frac{X}{2} \cdot G_s - \frac{X}{2} \cdot \Phi_s$ , the result is now clear.

- **7.3. Summary of notation.** For clarity, in this section, we provide a summary of our notation regarding the various Kähler forms in play.
  - F is the data in  $(*_0)$  equal to a constant  $c_0$  outside a compact set.
  - $\omega$  is the background Kähler form given in  $(*_0)$  isometric to  $\omega_C + \omega_D$  outside a fixed compact subset  $K \subset M$ .
  - g is the Kähler metric associated to  $\omega$ .
  - f is the Hamiltonian potential of JX with respect to  $\omega$  given in Theorem A (iii). It is equal to  $\frac{|z|^{2\lambda}}{2} 1$  outside the compact subset  $K \subset M$  and normalised so that

$$\Delta_{\omega}f - f + \frac{X}{2} \cdot f = 0$$

outside a compact set.

- $c_s := \log(1 + s(e^{c_0} 1)).$
- $F_s$  is the data in  $(\star_s)$  equal to  $c_s$  outside a fixed compact subset  $V \subset M$  with  $V \supset K$ .

- $\psi_s$  is the solution to the original continuity path  $(\star_s)$ .
- $\Phi_s = -2\chi \cdot c_s \log r$ , where  $0 \le \chi \le 1$  is a bump function identically equal to 1 outside a fixed compact subset  $W \supset V \supset K$  of M. In particular, notice that

$$\Phi_s = -c_s \log(2(f+1))$$
 on  $M \setminus W$ .

- $\omega_s := \omega + i \partial \bar{\partial} \Phi_s$  is the 1-parameter family of background metrics isometric to  $\omega$  outside a compact set independent of s appearing in  $(\star \star_s)$ .
- $g_s$  is the Kähler metric associated to  $\omega_s$ .
- $f_s := f + \frac{X}{2} \cdot \Phi_s$  is the Hamiltonian potential of JX with respect to  $\omega_s$ .
- $\vartheta_s = \psi_s \Phi_s$  is the solution of the re-parametrised continuity path  $(\star \star_s)$ .
- $\sigma_s := \omega_s + i \partial \bar{\partial} \vartheta_s$  is the associated Kähler metric.
- $f_{\sigma_s}$  is the Hamiltonian potential of JX with respect to  $\sigma_s$ . It is normalised by the equation

$$\Delta_{\sigma_s} f_{\sigma_s} - \frac{X}{2} \cdot f_{\sigma_s} = -f + P_s,$$

where  $P_s$  is compactly supported.

- $h_s$  is the Kähler metric associated to  $\sigma_s$ .
- **7.4.** A priori lower bound on the radial derivative. The fact that the data  $G_s$  of  $(\star\star_s)$  is compactly supported allows us to localise the extrema of  $X\cdot\vartheta_s$  using the maximum principle. This leads to a uniform lower bound on  $X\cdot\vartheta_s$  and in particular on  $X\cdot\psi_s$ .

**Lemma 7.3** (Localising the supremum and infimum of the radial derivative). Suppose that  $(\vartheta_s)_{0 \le s \le 1}$  is a path of solutions in  $\mathbb{R} \oplus C^{\infty}_{X,\beta}(M)$  to  $(\star \star_s)$ . Then

$$\sup_{M} X \cdot \vartheta_{s} = \max\{0, \max_{W} X \cdot \vartheta_{s}\} \quad and \quad \inf_{M} X \cdot \vartheta_{s} = \min\{0, \min_{W} X \cdot \vartheta_{s}\}.$$

*Proof.* First, using  $\nu$  to identify  $(M, \omega)$  and  $(\widehat{M}, \widehat{\omega})$  on  $M \setminus W$ , notice that

$$\frac{X}{2} \cdot \left( \log \left( \frac{\sigma_s^n}{\omega_s^n} \right) \right) = \operatorname{tr}_{\sigma_s} \mathcal{L}_{\frac{X}{2}} \sigma_s - \operatorname{tr}_{\omega_s} \mathcal{L}_{\frac{X}{2}} \omega_s$$

$$= \operatorname{tr}_{\sigma_s} \mathcal{L}_{\frac{X}{2}} (\omega_s + i \partial \bar{\partial} \vartheta_s) - \operatorname{tr}_{\omega} \mathcal{L}_{\frac{X}{2}} \omega$$

$$= \operatorname{tr}_{\sigma_s} \omega_C + \frac{1}{2} \Delta_{\sigma_s} (X \cdot \vartheta_s) - \operatorname{tr}_{\omega} \omega_C$$

$$= \operatorname{tr}_{\sigma_s} \omega_C + \frac{1}{2} \Delta_{\sigma_s} (X \cdot \vartheta_s) - 1.$$

Thus, upon differentiating  $(\star \star_s)$  along X, we obtain on  $M \setminus W$  the equation

(7.3) 
$$\Delta_{\sigma_s, X}\left(\frac{X \cdot \vartheta_s}{2}\right) := \Delta_{\sigma_s}\left(\frac{X \cdot \vartheta_s}{2}\right) - \frac{X}{2} \cdot \left(\frac{X \cdot \vartheta_s}{2}\right) = 1 - \operatorname{tr}_{\sigma_s} \omega_C.$$

Now, on  $M \setminus V$ , we have that

$$\operatorname{tr}_{\sigma_{S}} \omega_{C} = \frac{n\sigma_{S}^{n-1} \wedge \omega_{C}}{\sigma_{S}^{n}} = \frac{ne^{-\frac{X \cdot \vartheta_{S}}{2}}\sigma_{S}^{n-1} \wedge \omega_{C}}{\omega^{n}};$$

hence

(7.4) 
$$1 - \operatorname{tr}_{\sigma_{s}} \omega_{C} = e^{-\frac{X \cdot \vartheta_{s}}{2}} \left( e^{\frac{X \cdot \vartheta_{s}}{2}} - \frac{n\sigma_{s}^{n-1} \wedge \omega_{C}}{\omega^{n}} \right)$$
$$= e^{-\frac{X \cdot \vartheta_{s}}{2}} \left( \frac{\sigma_{s}^{n} - n\sigma_{s}^{n-1} \wedge \omega_{C}}{\omega^{n}} \right).$$

For k = 1, ..., n, we have for dimensional reasons that

$$\omega^k = (\omega_D + \omega_C)^k = \omega_D^k + k\omega_D^{k-1} \wedge \omega_C.$$

Thus,

$$\begin{split} \sigma_s^n &= (\omega + i \partial \bar{\partial} \vartheta_s)^n \\ &= \sum_{k=0}^n \binom{n}{k} \omega^k \wedge (i \partial \bar{\partial} \vartheta_s)^{n-k} \\ &= (i \partial \bar{\partial} \vartheta_s)^n + \sum_{k=1}^n \binom{n}{k} \omega^k \wedge (i \partial \bar{\partial} \vartheta_s)^{n-k} \\ &= (i \partial \bar{\partial} \vartheta_s)^n + \sum_{k=1}^n \binom{n}{k} (\omega_D^k + k \omega_D^{k-1} \wedge \omega_C) \wedge (i \partial \bar{\partial} \vartheta_s)^{n-k} \\ &= (i \partial \bar{\partial} \vartheta_s)^n + \sum_{k=1}^n \binom{n}{k} (\omega_D^k + k \omega_D^{k-1} \wedge \omega_C) \wedge (i \partial \bar{\partial} \vartheta_s)^{n-k} \\ &= (i \partial \bar{\partial} \vartheta_s)^n + \sum_{k=1}^n \binom{n}{k} \omega_D^k \wedge (i \partial \bar{\partial} \vartheta_s)^{n-k} + \sum_{k=1}^n k \binom{n}{k} \omega_D^{k-1} \wedge (i \partial \bar{\partial} \vartheta_s)^{n-k} \wedge \omega_C \\ &= \sum_{k=0}^n \binom{n}{k} \omega_D^k \wedge (i \partial \bar{\partial} \vartheta_s)^{n-k} + \sum_{k=1}^n k \binom{n}{k} \omega_D^{k-1} \wedge (i \partial \bar{\partial} \vartheta_s)^{n-k} \wedge \omega_C \end{split}$$

and

$$\begin{split} n\sigma_s^{n-1} \wedge \omega_C &= n \sum_{j=0}^{n-1} \binom{n-1}{j} \omega^j \wedge (i\partial\bar{\partial}\vartheta_s)^{n-1-j} \wedge \omega_C \\ &= ni\partial\bar{\partial}\vartheta_s^{n-1} \wedge \omega_C + n \sum_{j=1}^{n-1} \binom{n-1}{j} \omega^j \wedge (i\partial\bar{\partial}\vartheta_s)^{n-1-j} \wedge \omega_C \\ &= ni\partial\bar{\partial}\vartheta_s^{n-1} \wedge \omega_C + n \sum_{j=1}^{n-1} \binom{n-1}{j} (\omega_D^j + j\omega_D^{j-1} \wedge \omega_C) \\ &= ni\partial\bar{\partial}\vartheta_s^{n-1} \wedge \omega_C + n \sum_{j=1}^{n-1} \binom{n-1}{j} \omega_D^j \wedge (i\partial\bar{\partial}\vartheta_s)^{n-1-j} \wedge \omega_C \\ &= ni\partial\bar{\partial}\vartheta_s^{n-1} \wedge \omega_C + n \sum_{j=1}^{n} \binom{n-1}{j} \omega_D^j \wedge (i\partial\bar{\partial}\vartheta_s)^{n-1-j} \wedge \omega_C \\ &= ni\partial\bar{\partial}\vartheta_s^{n-1} \wedge \omega_C + n \sum_{k=2}^{n} \binom{n-1}{k-1} \omega_D^{k-1} \wedge (i\partial\bar{\partial}\vartheta_s)^{n-k} \wedge \omega_C \\ &= n \sum_{k=1}^{n} \binom{n-1}{k-1} \omega_D^{k-1} \wedge (i\partial\bar{\partial}\vartheta_s)^{n-k} \wedge \omega_C. \end{split}$$

Consequently,

$$\begin{split} \sigma_s^n - n\sigma_s^{n-1} \wedge \omega_C &= (i\,\partial\bar{\partial}\vartheta_s)^n + \sum_{k=1}^n \binom{n}{k} \omega_D^k \wedge (i\,\partial\bar{\partial}\vartheta_s)^{n-k} \\ &+ \sum_{k=1}^n k \binom{n}{k} \omega_D^{k-1} \wedge (i\,\partial\bar{\partial}\vartheta_s)^{n-k} \wedge \omega_C \\ &- n \sum_{k=1}^n \binom{n-1}{k-1} \omega_D^{k-1} \wedge (i\,\partial\bar{\partial}\vartheta_s)^{n-k} \wedge \omega_C \\ &= (i\,\partial\bar{\partial}\vartheta_s)^n + \sum_{k=1}^n \binom{n}{k} \omega_D^k \wedge (i\,\partial\bar{\partial}\vartheta_s)^{n-k} \\ &+ \sum_{k=1}^n \underbrace{\left[k\binom{n}{k} - n\binom{n-1}{k-1}\right]}_{=0} \omega_D^{k-1} \wedge (i\,\partial\bar{\partial}\vartheta_s)^{n-k} \wedge \omega_C \\ &= \sum_{k=0}^n \binom{n}{k} \omega_D^k \wedge (i\,\partial\bar{\partial}\vartheta_s)^{n-k} = (\omega_D + i\,\partial\bar{\partial}\vartheta_s)^n. \end{split}$$

Combining (7.3) and (7.4), we find that

(7.5) 
$$\Delta_{\sigma_{S},X}\left(\frac{X\cdot\vartheta_{S}}{2}\right) = e^{-\frac{X\cdot\vartheta_{S}}{2}} \frac{(\omega_{D} + i\partial\bar{\partial}\vartheta_{S})^{n}}{\omega^{n}}.$$
first-order operator acting on  $X\cdot\vartheta_{S}$ 

Indeed, the right-hand side of (7.5) can be written schematically as

$$(7.6) \quad \frac{(\omega_D + i\partial\bar{\partial}\vartheta_s)^n}{\omega^n} = \frac{1}{r^2} (X \cdot (X \cdot \vartheta_s)\alpha_1 + \nabla^{g_D}(X \cdot \vartheta_s) * \nabla^{g_D}(X \cdot \vartheta_s) * \alpha_2),$$

where  $\alpha_1$  and  $\alpha_2$  are tensors on  $M \setminus V$  depending polynomially on  $i \partial \bar{\partial} \vartheta_s$  and where \* denotes any linear combination of tensors with respect to the background metric  $\omega$ . This can be seen, for example, by noting that, on  $M \setminus V$ ,

$$\frac{(\omega_D + i\partial\bar{\partial}\vartheta_s)^n}{\omega^n} = \frac{(i\partial\bar{\partial}\vartheta_s)^n}{\omega^n} + \sum_{k=1}^{n-1} \binom{n}{k} \frac{\omega_D^{n-k} \wedge (i\partial\bar{\partial}\vartheta_s)^k}{\omega^n},$$

together with an application of the following claim.

**Claim 7.4.** Let Y and Z be real holomorphic vector fields such that [Y, Z] = 0. Then, for any smooth real-valued function v on M with  $\mathcal{L}_{JY}v = \mathcal{L}_{JZ}v = 0$ , we have

$$\frac{i}{2}\partial\bar{\partial}v(Y,Z) = \frac{i}{2}\partial\bar{\partial}v(JY,JZ) = 0 \quad and \quad Z\cdot(Y\cdot v) = Y\cdot(Z\cdot v) = 2i\,\partial\bar{\partial}v(Z,JY).$$

*Proof of Claim* 7.4. The first equality follows from the fact that

$$2i\partial\bar{\partial}v(Y,Z) = 2i\partial\bar{\partial}v(JY,JZ) = dd^{c}v(JY,JZ)$$
$$= JY \cdot (d^{c}v(JZ)) - JZ \cdot (d^{c}v(JY)) - d^{c}v([JY,JZ]).$$

As for the second, the vanishing of [Y, Z] implies that  $Z \cdot (Y \cdot v) = Y \cdot (Z \cdot v)$ , whereas with

$$Y^{1,0} := \frac{1}{2}(Y - iJY)$$
 and  $Z^{1,0} := \frac{1}{2}(Z - iJZ)$ ,

the invariance of v and the fact that  $JY \cdot (Z \cdot v) = 0$  implies that

$$\frac{1}{4}Y \cdot (Z \cdot v) = Y^{1,0} \cdot (Z^{1,0} \cdot v) = \overline{Y^{1,0}} \cdot (Z^{1,0} \cdot v) = \partial \bar{\partial} v(Z^{1,0}, \overline{Y^{1,0}}) 
= \frac{i}{2} \partial \bar{\partial} v(Z, JY) - \frac{1}{2} \underbrace{\partial \bar{\partial} v(JY, JZ)}_{=0}.$$

The strong maximum principle combined with the fact that  $X \cdot \vartheta_s \to 0$  at infinity now implies the result.

From this, we can derive a lower bound on  $X \cdot \vartheta_s$ , and hence on  $X \cdot \psi_s$ .

**Proposition 7.5.** There exists a positive constant C such that, for all  $s \in [0, 1]$ , we have  $X \cdot \vartheta_s \ge -C$ . In particular,  $X \cdot \psi_s > -C$  for all  $s \in [0, 1]$ .

*Proof.* In order to prove that  $X \cdot \vartheta_s$  is uniformly bounded from below, first note that, since  $X \cdot \Phi_s$  is bounded and  $X \cdot \vartheta_s$  tends to zero at infinity,  $f_{\sigma_s} := f + \frac{X}{2} \cdot \Phi_s + \frac{X}{2} \cdot \vartheta_s$  is a proper function bounded from below by virtue of the fact that f is by Lemma 4.2. Then, since  $X = \nabla^{h_s} f_{\sigma_s}$ ,  $f_{\sigma_s}$  must attain its global minimum at a point lying in the zero set of X and hence must coincide with the global minimum of f on this set, that is to say,

$$f_{\sigma_s} \ge \min_{\{X=0\}} f_{\sigma_s} = \min_{\{X=0\}} f \ge -C.$$

The lower bound on  $X \cdot \vartheta_s$  then follows from the previous localisation of the minimum of this function given by Lemma 7.3.

**7.5.** A priori  $C^0$ -estimate. We proceed with the a priori estimate on the  $C^0$ -norm of  $(\vartheta_s)_{0 \le s \le 1}$  which is uniform in  $s \in [0, 1]$ . We begin with two crucial observations, the first a localisation result for the global extrema of  $\vartheta_s$ .

**Lemma 7.6** (Localising the supremum and infimum of a solution of  $(\star \star_s)$ ). Suppose that  $(\vartheta_s)_{0 \le s \le 1}$  is a path of solutions in  $\mathbb{R} \oplus C^{\infty}_{X,\beta}(M)$  to  $(\star \star_s)$ . Then  $\sup_M \vartheta_s = \max_W \vartheta_s$  (respectively  $\inf_M \vartheta_s = \min_W \vartheta_s$ ).

*Proof.* We first prove the assertion of Lemma 7.6 concerning the supremum of a solution  $\vartheta_s$ . To this end, observe from  $(\star \star_s)$  and the basic inequality  $\log(1+x) \le x$  for all x > -1 that  $\vartheta_s$  is a subsolution of the following differential inequality:

$$\Delta_{\omega_s}\vartheta_s - \frac{X}{2}\cdot\vartheta_s \geq G_s,$$

where we recall that  $G_s$  is compactly supported. Let  $\varepsilon > 0$  and consider any smooth function  $u_{\varepsilon}$  on M identically equal to  $2\varepsilon \log(r)$  on  $M \setminus W$  such that  $\lim_{\varepsilon \to 0} u_{\varepsilon} = 0$  uniformly on compact sets of M. This function will serve as a barrier function. Indeed, since  $\log(r)$  is pluriharmonic,

one has that, on  $M \setminus W$ ,

(7.7) 
$$\Delta_{\omega_s} \left( \vartheta_s - 2\varepsilon \log(r) \right) - \frac{X}{2} \cdot \left( \vartheta_s - 2\varepsilon \log(r) \right) \ge \varepsilon > 0.$$

Now  $\vartheta_s$  being bounded on M implies that the function  $\vartheta_s - 2\varepsilon \log(r)$  tends to  $-\infty$  as  $r \to +\infty$ . In particular, this latter function must attain its maximum on M. The maximum principle applied to (7.7) then ensures that it must be attained in W, i.e.,

$$\max_{M}(\vartheta_{s}-u_{\varepsilon})=\max_{W}(\vartheta_{s}-u_{\varepsilon}).$$

In conclusion, we have that

$$\vartheta_s(x) \le u_{\varepsilon}(x) + \max_W (\vartheta_s - u_{\varepsilon}), \quad x \in M,$$

which leads to the bound  $\vartheta_s(x) \le \max_W \vartheta_s$  by letting  $\varepsilon \to 0$  and making use of the assumption on  $u_{\varepsilon}$ . Since this holds true for any  $x \in M$ , the desired estimate follows.

The statement involving the infimum of  $\vartheta_s$  can be proved in a manner similar to the above beginning with the differential inequality

$$\Delta_{\sigma_s}\vartheta_s - \frac{X}{2}\cdot\vartheta_s \leq G_s,$$

which itself follows from the arithmetic mean-geometric mean inequality.

**7.5.1. Aubin–Tian–Zhu's functionals.** We now introduce two functionals that have been defined and used by Aubin [3], Bando and Mabuchi [7], and Tian [34, Chapter 6] in the study of Fano manifolds, and by Tian and Zhu [35] in the study of shrinking gradient Kähler–Ricci solitons on compact Kähler manifolds.

**Definition 7.7.** Let  $(\varphi_t)_{0 \le t \le 1}$  be a  $C^1$ -path in  $\mathcal{M}_{X,\beta}^{\infty}(M)$  from  $\varphi_0 = 0$  to  $\varphi_1 = \varphi$ . We define the following two generalised weighted energies:

$$\begin{split} I_{\omega,X}(\varphi) &:= \int_{M} \varphi(e^{-f} \omega^{n} - e^{-f - \frac{X}{2} \cdot \varphi} \omega_{\varphi}^{n}), \\ J_{\omega,X}(\varphi) &:= \int_{0}^{1} \int_{M} \dot{\varphi_{s}}(e^{-f} \omega^{n} - e^{-f - \frac{X}{2} \cdot \varphi_{s}} \omega_{\varphi_{s}}^{n}) \wedge ds. \end{split}$$

At first sight, these two functionals resemble relative weighted mean values of a potential  $\varphi$  in  $\mathcal{M}_{X,\beta}^{\infty}(M)$  or of a path  $(\varphi_t)_{0 \le t \le 1}$  in  $\mathcal{M}_{X,\beta}^{\infty}(M)$ , respectively. When  $X \equiv 0$  and  $(M,\omega)$  is a compact Kähler manifold, an integration by parts together with some algebraic manipulations (see Aubin's seminal paper [3] or Tian's book [34, Chapter 6]) show that

(7.8) 
$$I_{\omega,0}(\varphi) = \sum_{k=0}^{n-1} \int_{M} i \, \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega^{k} \wedge \omega_{\varphi}^{n-1-k},$$

$$J_{\omega,0}(\varphi) = \sum_{k=0}^{n-1} \frac{k+1}{n+1} \int_{M} i \, \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega^{k} \wedge \omega_{\varphi}^{n-1-k}.$$

This justifies the description of  $I_{\omega,0}(\varphi)$  and  $J_{\omega,0}(\varphi)$  as modified energies. Moreover, it demonstrates that, on a compact Kähler manifold,  $J_{\omega,0}$  is a true functional, that is to say, it does not depend on the choice of path.

Such formulae (7.8) for  $I_{\omega,X}$  and  $J_{\omega,X}$  for a non-vanishing vector field X and a non-compact Kähler manifold  $(M,\omega)$  do not seem to be readily available for a good reason: the exponential function is not algebraic. However, following Tian and Zhu's work [35], one can prove that the essential properties shared by both  $I_{\omega,0}$  and  $J_{\omega,0}$  hold true for a non-vanishing vector field X in a non-compact setting. The proof follows exactly as in [15, Theorem 7.5].

**Theorem 7.8.**  $I_{\omega,X}(\varphi)$  and  $J_{\omega,X}(\varphi)$  are well-defined for  $\varphi \in \mathcal{M}_{X,\beta}^{\infty}(M)$ . Moreover,  $J_{\omega,X}$  does not depend on the choice of a  $C^1$  path  $(\varphi_t)_{0 \leq t \leq 1}$  in  $\mathcal{M}_{X,\beta}^{\infty}(M)$  from  $\varphi_0 = 0$  to  $\varphi_1 = \varphi$ , hence defines a functional on  $\mathcal{M}_{X,\beta}^{\infty}(M)$ . Finally, the first variation of the difference  $(I_{\omega,X} - J_{\omega,X})$  is given by

$$\frac{d}{dt}(I_{\omega,X} - J_{\omega,X})(\varphi_t) = -\int_{M} \varphi_t \Big( \Delta_{\omega_{\varphi_t}} \dot{\varphi}_t - \frac{X}{2} \cdot \dot{\varphi}_t \Big) e^{-f_{\varphi_t}} \omega_{\varphi_t}^n,$$

where  $f_{\varphi_t} := f + \frac{X}{2} \cdot \varphi_t$  satisfies  $X = \nabla^{\omega_{\varphi_t}} f_{\varphi_t}$  and where  $(\varphi_t)_{0 \le t \le 1}$  is any  $C^1$ -path in  $\mathcal{M}_{X,\beta}^{\infty}(M)$  from  $\varphi_0 = 0$  to  $\varphi_1 = \varphi$ .

Recall that the equation we wish to solve is  $(\star_s)$ , namely

$$e^{-f_{\psi_S}}\omega_{\psi_S}^n = e^{F_S - f}\omega^n.$$

**Proposition 7.9** (A priori energy estimates). Let  $(\psi_s)_{0 \le s \le 1}$  be a path of solutions in  $\mathcal{M}_{X,B}^{\infty}(M)$  to  $(\star_s)$ . Then, for  $p \in (1,2)$ , there exists a positive constant

$$C = C(n, p, \omega, \sup_{s \in [0,1]} ||F_s||_{C^0})$$

such that

$$\sup_{0 \le s \le 1} \int_{M} |\psi_{s} - \overline{\psi}_{s}|^{p} e^{-f} \omega^{n} \le C,$$

where  $\overline{\psi}_s := \int_M \psi_s e^{-f} \omega^n$ . In particular, if  $\overline{\psi}_s = 0$ , then

$$\sup_{0 \le s \le 1} \int_{M} |\vartheta_{s}|^{p} e^{-f} \omega^{n} \le C.$$

*Proof.* By Theorem 7.8, we can use any  $C^1$ -path  $(\varphi_t)_{0 \le t \le 1}$  in  $\mathcal{M}_{X,\beta}^{\infty}(M)$  from  $\varphi_0 = 0$  to  $\varphi_1 = \varphi \in \mathcal{M}_{X,\beta}^{\infty}(M)$  to compute  $J_{\omega,X}(\varphi)$ . As in [35], we choose two different paths to compute  $J_{\omega,X}(\psi)$ , the first being the linear path defined by  $\varphi_t := t\psi, t \in [0,1]$ , for  $\psi \in \mathcal{M}_{X,\beta}^{\infty}(M)$  a solution to  $(\star_s)$ . For this path, Theorem 7.8 asserts that

$$(I_{\omega,X} - J_{\omega,X})(\psi) = -\int_0^1 \int_M t \psi \Big( \Delta_{\omega_{t\psi}} \psi - \frac{X}{2} \cdot \psi \Big) e^{-f - t \frac{X}{2} \cdot \psi} \omega_{t\psi}^n \wedge dt.$$

Integration by parts with respect to the weighted volume form  $e^{-f-t\frac{X}{2}\cdot\psi}\omega_{t\psi}^n$  then leads to

$$(7.9) (I_{\omega,X} - J_{\omega,X})(\psi)$$

$$= n \int_0^1 \int_M ti \partial \psi \wedge \bar{\partial} \psi \wedge (e^{-f - t\frac{X}{2} \cdot \psi} \omega_{t\psi}^{n-1}) \wedge dt$$

$$= n \int_0^1 \int_M ti \partial \psi \wedge \bar{\partial} \psi \wedge (e^{-f - t\frac{X}{2} \cdot \psi} ((1 - t)\omega + t\omega_{\psi})^{n-1}) \wedge dt$$

$$= n \sum_{k=0}^{n-1} \binom{n-1}{k} \left( \int_0^1 t^{k+1} (1-t)^{n-1-k} \times \int_M i \partial \psi \wedge \bar{\partial} \psi \wedge (e^{-f-t\frac{X}{2} \cdot \psi} \omega^{n-1-k} \wedge \omega_\psi^k) \right) \wedge dt$$

$$\geq n \int_0^1 t (1-t)^{n-1} \int_M i \partial \psi \wedge \bar{\partial} \psi \wedge (e^{-f-t\frac{X}{2} \cdot \psi} \omega^{n-1}) \wedge dt$$

$$= n \int_M \left( \int_0^1 t (1-t)^{n-1} e^{-t\frac{X}{2} \cdot \psi} dt \right) i \partial \psi \wedge \bar{\partial} \psi \wedge e^{-f} \omega^{n-1}.$$

From this, the following claim will allow us to obtain a lower bound.

**Claim 7.10.** There exist positive uniform constants A, c such that

$$\int_0^1 t (1-t)^{n-1} e^{-t\frac{X}{2} \cdot \psi} dt \ge \frac{c}{(\frac{X}{2} \cdot \psi + A)^2}.$$

*Proof of Claim* 7.10. For  $k \ge k_n := 2n(n-1)$ , we find using integration by parts and a change of variable that

$$\int_{0}^{1} t(1-t)^{n-1}e^{-kt} dt = \int_{0}^{1} (1-s)s^{n-1}e^{-k(1-s)} ds$$

$$= e^{-k} \left\{ \left( 1 + \frac{n}{k} \right) \int_{0}^{1} s^{n-1}e^{ks} ds - \frac{e^{k}}{k} \right\}$$

$$= e^{-k} \left\{ \left( 1 + \frac{n}{k} \right) \left( \frac{e^{k}}{k} - \frac{(n-1)}{k} \int_{0}^{1} s^{n-2}e^{ks} ds \right) - \frac{e^{k}}{k} \right\}$$

$$\geq \left( 1 + \frac{n}{k} \right) \left( \frac{1}{k} - \frac{(n-1)}{k^{2}} (1 - e^{-k}) \right) - \frac{1}{k}$$

$$= \frac{k - n(n-1)}{k^{3}} + e^{-k} \frac{(n+k)(n-1)}{k^{3}}$$

$$\geq \frac{1}{2k^{2}}.$$

Here we have bounded  $s^{n-2}$  from above by 1 in the fourth inequality. Set  $A := k_n - \inf_M \frac{X}{2} \cdot \psi$  and let  $k = \frac{X}{2} \cdot \psi + A$ . Then  $k \ge k_n$ , A is uniformly bounded from above by Proposition 7.5, and it follows from what we have just derived that

$$\int_0^1 t(1-t)^{n-1} e^{-t(\frac{X}{2}\cdot\psi+A)} dt \ge \frac{1}{2(\frac{X}{2}\cdot\psi+A)^2},$$

resulting in the desired bound.

Applying Claim 7.10 to (7.9) yields the lower bound

$$(7.10) (I_{\omega,X} - J_{\omega,X})(\psi) \ge c \int_{M} i \, \partial \psi \wedge \bar{\partial} \psi \wedge \frac{e^{-f} \, \omega^{n-1}}{(\frac{X}{2} \cdot \psi + A)^{2}}$$

$$\ge c \int_{M} \frac{|\nabla^{g} \, \psi|_{g}^{2}}{(\frac{X}{2} \cdot \psi + A)^{2}} e^{-f} \, \omega^{n}$$

for some positive constant c. We also require an upper bound on  $(I_{\omega,X} - J_{\omega,X})(\psi)$  to complete the proof of the proposition.

To this end, we consider the continuity path of solutions  $\varphi_s := \psi_s$ ,  $s \in [0, 1]$ , to  $(\star_s)$  to compute  $(I_{\omega,X} - J_{\omega,X})(\psi)$ . First observe that the first variations  $(\dot{\psi}_s)_{0 \le s \le 1}$  satisfy the following PDE obtained from  $(\star_s)$  by differentiating with respect to the parameter s:

$$\Delta_{\omega_{\psi_s}}\dot{\psi}_s - \frac{X}{2}\cdot\dot{\psi}_s = \dot{F}_s, \quad 0 \le s \le 1.$$

Combined with  $(\star_s)$  and Theorem 7.8, we see that

$$(I_{\omega,X} - J_{\omega,X})(\psi) = \int_0^1 \int_M \psi_t \cdot (-\dot{F}_t) e^{-f_{\psi_t}} \omega_{\psi_t}^n \wedge dt$$
$$= \int_0^1 \int_M \psi_t \cdot (-\dot{F}_t) e^{F_t - f} \omega^n \wedge dt$$

so that, from (7.10), for some c > 0,

(7.11) 
$$\int_0^1 \int_M \psi_t \cdot (-\dot{F}_t) e^{F_t - f} \omega^n \wedge dt \ge c \int_M \frac{|\nabla^g \psi|_g^2}{(\frac{X}{2} \cdot \psi + A)^2} e^{-f} \omega^n.$$

Now, as

$$\frac{d}{ds} \left( \int_{M} e^{-f_{\psi_s}} \omega_{\psi_s}^n \right) = 0$$

by Lemma 6.2 (i) with  $G \equiv 1$ , we derive from  $(\star_s)$  that

$$\int_{M} \dot{F}_{t} e^{F_{t} - f} \omega^{n} = 0.$$

This allows us to rewrite (7.11) as

$$\int_0^1 \int_M (\psi_t - \overline{\psi}_t) \cdot (-\dot{F}_t) e^{F_t - f} \omega^n \wedge dt \ge c \int_M \frac{|\nabla^g \psi|_g^2}{(\frac{X}{2} \cdot \psi + A)^2} e^{-f} \omega^n,$$

with  $\overline{\psi}_t$  as in the statement of the proposition. Applying the Poincaré inequality of Proposition 5.1, we then see that, for any  $p \in (1,2)$  and  $\frac{1}{q} = 1 - \frac{1}{p}$ ,

$$(7.12) \quad \left(\int_{M} |\psi - \overline{\psi}|^{p} e^{-f} \omega^{n}\right)^{\frac{2}{p}}$$

$$\leq C \left(\int_{M} |\nabla^{g} \psi|_{g}^{p} e^{-f} \omega^{n}\right)^{\frac{2}{p}}$$

$$\leq C \left(\int_{M} \frac{|\nabla^{g} \psi|_{g}^{2}}{(\frac{X}{2} \cdot \psi + A)^{2}} e^{-f} \omega^{n}\right) \left(\int_{M} \left(\frac{X}{2} \cdot \psi + A\right)^{\frac{2p}{2-p}} e^{-f} \omega^{n}\right)^{\frac{2-p}{p}}$$

$$\leq C \left(\int_{0}^{1} \int_{M} |\psi_{t} - \overline{\psi}_{t}| |\dot{F}_{t}| e^{F_{t} - f} \omega^{n} \wedge dt\right)$$

$$\times \left(\int_{M} \left(\frac{X}{2} \cdot \psi + A\right)^{\frac{2p}{2-p}} e^{-f} \omega^{n}\right)^{\frac{2-p}{p}}$$

$$\leq C \int_0^1 \left( \int_M |\psi_t - \overline{\psi}_t|^p e^{-f} \omega^n \right)^{\frac{1}{p}} \left( \int_M |\dot{F}_t|^q e^{qF_t} e^{-f} \omega^n \right)^{\frac{1}{q}} dt$$

$$\times \left( \int_M \left( \frac{X}{2} \cdot \psi + A \right)^{\frac{2-p}{2-p}} e^{-f} \omega^n \right)^{\frac{1}{p}}$$

$$\leq C \int_0^1 \left( \int_M |\psi_t - \overline{\psi}_t|^p e^{-f} \omega^n \right)^{\frac{1}{p}} dt$$

$$\times \left( \int_M \left( \frac{X}{2} \cdot \psi + A \right)^{\frac{2-p}{2-p}} e^{-f} \omega^n \right)^{\frac{2-p}{p}}.$$

Here we have used Hölder's inequality in the second and fourth lines with respect to the weighted measure  $e^{-f}\omega^n$ .

Next, observe from Lemma 6.2 (i) that, for all  $r \in \mathbb{N}$ ,

$$c \int_{M} (f_{\psi_s} + A)^r e^{-f} \omega^n \le \int_{M} (f_{\psi_s} + A)^r e^{F_s} e^{-f} \omega^n$$

$$= \int_{M} (f_{\psi_s} + A)^r e^{-f_{\psi_s}} \omega^n_{\psi_s}$$

$$= \int_{M} (f + A)^r e^{-f} \omega^n \le C(r).$$

By induction on r, using the fact that  $\frac{X}{2} \cdot \psi + A \ge 0$  and that  $A \le C$  by Proposition 7.5, one can prove directly from this that

$$\int_{M} \left(\frac{X}{2} \cdot \psi + A\right)^{r} e^{-f} \omega^{n} \le C(r) \quad \text{for all } r \in \mathbb{N}.$$

It then follows from Hölder's inequality that this statement holds true for all  $r \ge 1$ . Applying this to (7.12), we arrive at the fact that, for all  $p \in (1, 2)$ ,

$$\left(\int_{M} |\psi - \overline{\psi}|^{p} e^{-f} \omega^{n}\right)^{\frac{2}{p}} \leq C(p) \int_{0}^{1} \left(\int_{M} |\psi_{t} - \overline{\psi_{t}}|^{p} e^{-f} \omega^{n}\right)^{\frac{1}{p}} dt,$$

i.e.,

$$\|\psi - \overline{\psi}\|_{L^{p}(e^{-f}\omega^{n})}^{2} \leq C(p) \int_{0}^{1} \|\psi_{t} - \overline{\psi}_{t}\|_{L^{p}(e^{-f}\omega^{n})} dt \quad \text{for any } p \in (1,2).$$

This last inequality applies to any truncated path of the one-parameter family of solutions  $(\psi_s)_{0 \le s \le 1}$  of  $(\star_s)$ . Thus,

(7.13) 
$$\|\psi_{s} - \overline{\psi}_{s}\|_{L^{p}(e^{-f}\omega^{n})}^{2} \leq C \int_{0}^{1} \|\psi_{st} - \overline{\psi}_{st}\|_{L^{p}(e^{-f}\omega^{n})} dt$$
$$= \frac{C}{s} \int_{0}^{s} \|\psi_{t} - \overline{\psi}_{t}\|_{L^{p}(e^{-f}\omega^{n})} dt.$$

This is a Grönwall-type differential inequality and can be integrated as follows. Let

$$H(s) := \int_0^s \|\psi_t - \overline{\psi}_t\|_{L^p(e^{-f}\omega^n)} \, dt$$

and observe that (7.13) can be rewritten as

$$H'(s) \le \frac{C}{s^{\frac{1}{2}}} (H(s))^{\frac{1}{2}}, \quad s \in (0, 1].$$

Integrating then implies that  $H(s) \le C(n, \omega, \sup_{s \in [0,1]} ||F_s||_{C^0}) \cdot s$  for all  $s \in [0,1]$  which, after applying (7.13) once more, yields the desired upper bound.

**7.5.2.** A priori estimate on  $\sup_M \vartheta_s$ . Let  $\vartheta_s$  be a solution to  $(\star \star_s)$  for some fixed value of the parameter  $s \in [0, 1]$ . We next obtain an upper bound for  $\sup_M \vartheta_s$  uniform in s. To obtain such a bound, it suffices by Lemma 7.6 to only bound  $\max_W \vartheta_s$  from above. We do this by implementing a local Nash–Moser iteration using the fact that  $\vartheta_s$  is a supersolution of the linearised complex Monge–Ampère equation of which the drift Laplacian with respect to the known metric  $\omega_s$  forms a part.

**Proposition 7.11** (A priori upper bound on  $\sup_{M} \vartheta$ ). Let  $(\vartheta_s)_{0 \le s \le 1}$  be a path of solutions in  $\mathbb{R} \oplus C^{\infty}_{X,\beta}(M)$  to  $(\star \star_s)$ . Then there exists a positive constant

$$C = C(n, \omega, \sup_{s \in [0,1]} \|G_s\|_{C^0})$$
 such that  $\sup_{0 \le s \le 1} \sup_{W} \vartheta_s \le C$ .

*Proof.* Let  $s \in [0, 1]$  and let  $(\vartheta_s)_+ := \max\{\vartheta_s, 0\}$ . This is a non-negative Lipschitz function. The strategy of proof is standard: we use a Nash–Moser iteration to obtain an a priori upper bound on  $\sup_W (\vartheta_s)_+$  in terms of the (weighted) energy of  $(\vartheta_s)_+$  on a tubular neighbourhood of W. The result then follows by invoking Proposition 7.9.

To this end, notice that, since  $\log(1+x) \le x$  for all x > -1 and since  $\vartheta_s$  is a solution to  $(\star \star_s)$ ,  $\vartheta_s$  satisfies the differential inequality

(7.14) 
$$\Delta_{\omega_s} \vartheta_s - \frac{X}{2} \cdot \vartheta_s \ge -|G_s| \quad \text{on } M.$$

$$\begin{split} \sup_{B_{gs}(x,\frac{\varepsilon}{2})} (\vartheta_s)_+ &\leq C(n,p,\omega,\varepsilon) \left( \|(\vartheta_s)_+\|_{L^p(B_{gs}(x,\varepsilon),e^{-f_{\omega_s}}\omega_s^n)}^p + \|G_s\|_{C^0}^p \right)^{\frac{1}{p}} \\ &\leq C(n,p,\omega,\varepsilon) \left( \int_{\{f < R\}} (\vartheta_s)_+^p e^{-f_{\omega_s}}\omega_s^n + \|G_s\|_{C^0}^p \right)^{\frac{1}{p}} \\ &\leq C(n,p,\omega,\varepsilon) \left( \int_{\{f < R\}} |\vartheta_s|^p e^{-f}\omega^n + \|G_s\|_{C^0}^p \right)^{\frac{1}{p}} \\ &\leq C(n,p,\omega,\varepsilon,\sup_{s \in [0,1]} \|G_s\|_{C^0}). \end{split}$$

Here, we have made use of Proposition 7.9 in the last line.

**7.5.3.** A priori estimate on  $\inf_M \vartheta_s$ . Recall that the equation we wish to solve is  $(\star_s)$ , that is,

$$e^{-f_{\psi_S}}\omega_{\psi_S}^n = e^{F_S - f}\omega^n$$

where  $\omega_{\psi_s} := \omega + i \partial \bar{\partial} \psi_s > 0$  and  $f_{\psi_s} := f + \frac{X}{2} \cdot \psi_s$ . This pair satisfies  $-\omega_{\psi_s} \bot X = df_{\psi_s}$ . We work under the assumption that  $\int_M \psi_s e^{-f} \omega^n = 0$ .

An upper bound on the  $I_{\omega,X}$ -functional. We first show that the  $I_{\omega,X}$ -functional is bounded along the continuity path.

**Lemma 7.12.**  $\sup_{s \in [0,1]} I_{\omega,X}(\psi_s) \leq C(\sup_M (\vartheta_s)_+).$ 

*Proof.* By assumption,  $\int_{M} \psi_{s} e^{-f} \omega^{n} = 0$  so that

$$\int_{\{\psi_s \ge 0\}} \psi_s e^{-f} \omega^n = -\int_{\{\psi_s \le 0\}} \psi_s e^{-f} \omega^n.$$

We therefore have that

$$I_{\omega,X}(\psi_{s}) = \int_{M} \psi_{s}(e^{-f}\omega^{n} - e^{-f\psi_{s}}\omega_{\psi_{s}}^{n}) = -\int_{M} \psi_{s}e^{-f\psi_{s}}\omega_{\psi_{s}}^{n}$$

$$= -\int_{\{\psi_{s} \geq 0\}} \psi_{s}e^{-f\psi_{s}}\omega_{\psi_{s}}^{n} + \int_{\{\psi_{s} \leq 0\}} (-\psi_{s})e^{-f\psi_{s}}\omega_{\psi_{s}}^{n}$$

$$\leq \int_{\{\psi_{s} \leq 0\}} (-\psi_{s})e^{-f\psi_{s}}\omega_{\psi_{s}}^{n} = \int_{\{\psi_{s} \leq 0\}} (-\psi_{s})e^{F_{s}}e^{-f}\omega^{n}$$

$$\leq C\int_{\{\psi_{s} \leq 0\}} (-\psi_{s})e^{-f}\omega^{n} = C\int_{\{\psi_{s} \geq 0\}} \psi_{s}e^{-f}\omega^{n}$$

$$= C\left(\int_{\{\vartheta_{s} \geq -\Phi_{s}\}} (\vartheta_{s} + \Phi_{s})e^{-f}\omega^{n}\right)$$

$$\leq C\left(\int_{M} |\Phi_{s}|e^{-f}\omega^{n} + \int_{\{\vartheta_{s} \geq -\Phi_{s}\}} \vartheta_{s}e^{-f}\omega^{n}\right)$$

$$\leq C + C\sup_{M} \vartheta_{s}^{+}\int_{\{\vartheta_{s} \geq -\Phi_{s}\}} e^{-f}\omega^{n}$$

$$\leq C\left(1 + \sup_{M} (\vartheta_{s})_{+}\right).$$

From this, the result follows.

An upper bound on the weighted  $L^p$ -norm of the gradient of the Legendre transform. Recall the continuity path  $(\star_s)$ ,

$$(\omega + i\partial\bar{\partial}\psi_s)^n = e^{F_s + \frac{X}{2} \cdot \psi_s} \omega^n, \quad s \in [0, 1],$$

where

$$F_s := \log(se^F + (1-s))$$
 and  $i\partial\bar{\partial}F = \rho_\omega + \frac{1}{2}\mathcal{L}_X\omega - \omega$ .

Here,  $\rho_{\omega}$  denotes the Ricci form of  $\omega$  and  $F \in C^{\infty}(M)$  is bounded. On  $\mathfrak{t} \simeq \mathbb{R}^n$ , we have coordinates  $\xi := (\xi_1, \ldots, \xi_n)$ , induced coordinates  $x = (x_1, \ldots, x_n)$  on  $\mathfrak{t}^*$  which contains the image of the moment map, and we can write  $\omega = 2i\partial\bar{\partial}\phi_0$  for a convex function  $\phi_0$  on  $\mathbb{R}^n \simeq \mathfrak{t}$  up to the addition of a linear function (cf. Section 2.5). Let  $b_X \in \mathbb{R}^n$  denote the vector field  $JX \in \mathfrak{t}$  as in (2.4), write  $\nabla$  for the Levi-Civita connection of the flat metric on  $\mathbb{R}^n$ , and  $\langle \cdot, \cdot \rangle$  for the corresponding inner product. As in (2.11), we normalise  $\phi_0$  so that

$$F = -\log \det(\phi_{0,ij}) + \langle \nabla \phi_0, b_X \rangle - 2\phi_0.$$

Set  $\phi_s := \phi_0 + \frac{1}{2}\psi_s$ . Then, in the coordinates  $\xi$  on  $\mathbb{R}^n$ , equation  $(\star_s)$  becomes

$$\det(\phi_{s,ij}) = (se^F + (1-s))e^{\langle \nabla \phi_s, b_X \rangle - \langle \nabla \phi_0, b_X \rangle} \det(\phi_{0,ij}), \quad s \in [0,1].$$

Plugging in the definition of F, this becomes

$$\det(\phi_{s,ij}) = \left(se^{-2\phi_0 - \log\det(\phi_{0,ij})} + (1-s)e^{-\langle\nabla\phi_0,b_X\rangle}\right)e^{\langle\nabla\phi_s,b_X\rangle}\det(\phi_{0,ij})$$
$$= \left(se^{-2\phi_0} + (1-s)e^{-\langle\nabla\phi_0,b_X\rangle}\det(\phi_{0,ij})\right)e^{\langle\nabla\phi_s,b_X\rangle}, \quad s \in [0,1],$$

or equivalently,

$$(7.15) \quad e^{-\langle \nabla \phi_s, b_X \rangle} \det(\phi_{s,ij}) = se^{-2\phi_0} + (1-s)e^{-\langle \nabla \phi_0, b_X \rangle} \det(\phi_{0,ij}), \quad s \in [0, 1].$$

Let  $u_s = L(\phi_s)$ . Then we have the following uniform integral bound on  $|\nabla u_s|^p$ ,  $p \ge 1$ .

**Lemma 7.13.** *For all* p > 1,

$$\sup_{s\in[0,1]}\int_{P-K_M}|\nabla u_s|^p e^{-\langle b_X,x\rangle}\,dx\leq C.$$

Proof. First note that

$$\int_{\mathbb{R}^n} |\xi|^p e^{-\langle b_X, \nabla \phi_0 \rangle} \det(\phi_{0,ij}) \, d\xi \le C.$$

Indeed, since F is equal to a constant  $c_0$  off of a compact subset of M, we see that F is globally bounded on M. This means that

$$\sup_{\mathbb{R}^n} |-\log \det(\phi_{0,ij}) + \langle \nabla \phi_0, b_X \rangle - 2\phi_0| \le C,$$

resulting in the fact that

$$\int_{\mathbb{R}^n} |\xi|^p e^{-\langle b_X, \nabla \phi_0 \rangle} \det(\phi_{0,ij}) \, d\xi \le C \int_{\mathbb{R}^n} |\xi|^p e^{-2\phi_0} \, d\xi \le C,$$

where we have used Lemma 2.25 in the last inequality. Therefore, using Lemma 2.25 once again and (7.15), we find that

$$\begin{split} \int_{P-K_M} & |\nabla u_s|^p e^{-\langle b_X, x \rangle} \, dx \\ &= \int_{\mathbb{R}^n} |\xi|^p e^{-\langle b_X, \nabla \phi_s \rangle} \det(\phi_{s,ij}) \, d\xi \\ &= s \int_{\mathbb{R}^n} |\xi|^p e^{-2\phi_0} \, d\xi + (1-s) \int_{\mathbb{R}^n} |\xi|^p e^{-\langle b_X, \nabla \phi_0 \rangle} \det(\phi_{0,ij}) \, d\xi \\ &\leq C, \end{split}$$

as desired.

An upper bound on the  $\hat{F}$ -functional. Now, our background metric  $\omega$  satisfies the two bullet points above Lemma 2.30 as demonstrated in the already proved Theorem A (ii)–(iv). As a consequence, it is clear from Lemma 2.30 (i) that condition (a) of Definition 2.27 holds true. The hypothesis of Lemma 2.29 as well as condition (b) of Definition 2.27 via Lemma 2.28 also hold true thanks to Lemma 6.2 (ii). Thus, the  $\hat{F}$ -functional from Definition 2.27 is finite and therefore well-defined along the continuity path  $(\star_s)$  and moreover, by Lemma 2.29, may be expressed along in terms of the  $J_{\omega,X}$ -functional as

$$\widehat{F}(\psi_s) = J_{\omega,X}(\psi_s) - \int_M \psi_s e^{-f} \omega^n.$$

We next show that  $\hat{F}$  is bounded above along the continuity path  $(\star_s)$  using Lemma 7.12. This will in turn provide an a priori estimate on the weighted integral of the Legendre transform  $u_s := L(\phi_s)$  of  $\phi_s$ . From this, we derive an a priori estimate on the weighted  $L^1$ -norm of  $u_s$ . Via the Sobolev inequality, we then obtain local control on  $u_s$ , and as a result, on  $\psi_s$ . This eventually leads to the desired uniform lower bound on  $\inf_M \vartheta_s$ .

**Lemma 7.14.** 
$$\hat{F}(\psi_s) \leq C(\sup_{M}(\vartheta_s)_+).$$

*Proof.* By assumption, we have that  $\int_M \psi_s e^{-f} \omega^n = 0$  so that  $\widehat{F}(\psi_s) = J_{\omega,X}(\psi_s)$ . Moreover, from (7.10), we read that  $(I_{\omega,X} - J_{\omega,X})(\psi_s) \ge 0$ . Thus, Lemma 7.12 implies that

$$\widehat{F}(\psi_s) = J_{\omega,X}(\psi_s) = I_{\omega,X}(\psi_s) - (I_{\omega,X} - J_{\omega,X})(\psi_s) \le I_{\omega,X}(\psi_s) + 0 \le C\left(\sup_{M}(\vartheta_s)_+\right),$$

as claimed.

An upper bound on the weighted integral of the Legendre transform. We know that

$$\begin{split} \int_{P-K_{M}} |u_{s}| e^{-\langle b_{X}, x \rangle} \, dx &\leq \int_{P-K_{M}} |u_{s} - u_{0}| e^{-\langle b_{X}, x \rangle} \, dx + \int_{P-K_{M}} |u_{0}| e^{-\langle b_{X}, x \rangle} \, dx \\ &\leq \int_{P-K_{M}} \left( \int_{0}^{1} |\dot{u}_{st}| \, dt \right) e^{-\langle b_{X}, x \rangle} \, dx + \int_{P-K_{M}} |u_{0}| e^{-\langle b_{X}, x \rangle} \, dx, \end{split}$$

and these last two integrals are finite by Lemma 6.2 (ii) via Lemma 2.28, and Lemma 2.30 (ii), respectively. By definition, the  $\hat{F}$ -functional along  $(\star_s)$  is given by

(7.16) 
$$\widehat{F}(\psi_s) = 2 \int_{P_{-KM}} (u_s - u_0) e^{-\langle b_X, x \rangle} dx.$$

Therefore with

$$\int_{P-K_M} |u_0| e^{-\langle b_X, x \rangle} \, dx \quad \text{and} \quad \int_{P-K_M} |u_1| e^{-\langle b_X, x \rangle} \, dx$$

convergent, we can split the integral in (7.16). Together with the integral bound given in Lemma 2.30 (ii), this leads to the following consequence of Lemma 7.14.

Corollary 7.15.

$$\sup_{s \in [0,1]} \int_{P_{-K_M}} u_s e^{-\langle b_X, x \rangle} \, dx \le C.$$

An upper bound on the weighted  $L^1$ -norm of the Legendre transform. We now use Corollary 7.15 to derive a uniform weighted  $L^1$ -norm on  $u_s$ . Notice that we must make use of the already obtained uniform upper bound on  $\vartheta_s$ .

### Lemma 7.16.

$$\sup_{s \in [0,1]} \int_{P_{-K_M}} |u_s| e^{-\langle b_X, x \rangle} \, dx \le C.$$

*Proof.* Recall from the definition of the Legendre transform that, for all  $x \in P_{-K_M}$ ,

$$u_s(x) - u_0(x) = \sup_{\xi \in \mathbb{R}^n} \left( \langle x, \xi \rangle - \phi_s(\xi) \right) - u_0(x)$$

$$\geq \langle x, \nabla u_0(x) \rangle - \phi_s(\nabla u_0(x)) - u_0(x)$$

$$= \phi_0(\nabla u_0(x)) - \phi_s(\nabla u_0(x))$$

$$= -\frac{1}{2} \psi_s(\nabla u_0(x))$$

$$= -\frac{1}{2} \Phi_s(\nabla u_0(x)) - \frac{1}{2} \vartheta_s(\nabla u_0(x))$$

$$\geq -\frac{1}{2} \Phi_s(\nabla u_0(x)) - C$$

for some uniform positive constant C. Here we have used the a priori upper bound on  $\vartheta_s$  given by Proposition 7.11 in the last line. With this, we estimate that

$$(7.17) \int_{P_{-K_{M}}} |u_{s}|e^{-\langle b_{X},x\rangle} dx$$

$$\leq \int_{P_{-K_{M}}} \left(u_{s} - u_{0} + \frac{1}{2}\Phi_{s}(\nabla u_{0}(x)) + C\right) e^{-\langle b_{X},x\rangle} dx$$

$$+ \int_{P_{-K_{M}}} \left|u_{0} - \frac{1}{2}\Phi_{s}(\nabla u_{0}(x)) - C\right| e^{-\langle b_{X},x\rangle} dx$$

$$\leq \int_{P_{-K_{M}}} u_{s}e^{-\langle b_{X},x\rangle} dx + 2 \int_{P_{-K_{M}}} |u_{0}|e^{-\langle b_{X},x\rangle} dx$$

$$+ 2C \int_{P_{-K_{M}}} e^{-\langle b_{X},x\rangle} dx + \int_{P_{-K_{M}}} |\Phi_{s}(\nabla u_{0}(x))|e^{-\langle b_{X},x\rangle} dx$$

$$\leq C' + \int_{P_{-K_{M}}} |\Phi_{s}(\nabla u_{0}(x))|e^{-\langle b_{X},x\rangle} dx$$

for a uniform positive constant C'. Here we have used Corollary 7.15, Lemma 2.30 (ii), and the fact that

$$\int_{P_{-K_M}} e^{-\langle b_X, x \rangle} dx = (2\pi)^n \int_M e^{-f} \omega^n < +\infty$$

to bound each of the terms in the third line, respectively. The final integral we bound in the following way.

Choose a compact subset  $U \subset M$  strictly containing W and  $f^{-1}((-\infty, 1])$ . This we can do because f is proper and bounded below. Next, choose R > 0 sufficiently large so that  $(\nabla \phi_0)(U) \subset B_R(0)$ . Then, in particular,  $(\nabla \phi_0)(W) \subset B_R(0)$  and

$$\langle b_X, x \rangle > 1$$
 for all  $x \in P_{-K_M} \setminus (B_R(0) \cap P_{-K_M})$ ,

the latter being true because  $\langle b_X, x \rangle = f(\nabla u_0(x))$  for all  $x \in P_{-K_M}$ . Then, recalling that  $\Phi_s = -c_s \log(2(f+1))$  on  $M \setminus W$ , which in particular holds on  $P_{-K_M} \setminus (B_R(0) \cap P_{-K_M})$ , and using the fact that  $0 < \log(x) < x$  for all x > 1, we estimate that

$$\int_{P_{-K_M}} |\Phi_s(\nabla u_0(x))| e^{-\langle b_X, x \rangle} dx$$

$$= \int_{B_R(0) \cap P_{-K_M}} |\Phi_s(\nabla u_0(x))| e^{-\langle b_X, x \rangle} dx$$

$$+ \int_{P_{-K_M} \setminus (B_R(0) \cap P_{-K_M})} |\Phi_s(\nabla u_0(x))| e^{-\langle b_X, x \rangle} dx$$

$$\leq C \left( 1 + \int_{P_{-K_M} \setminus (B_R(0) \cap P_{-K_M})} |\log(2(f(\nabla u_0(x)) + 1))| e^{-\langle b_X, x \rangle} dx \right)$$

$$= C \left( 1 + \int_{P_{-K_M} \setminus (B_R(0) \cap P_{-K_M})} \log(2(\langle b_X, x \rangle + 1)) e^{-\langle b_X, x \rangle} dx \right)$$

$$\leq C \left( 1 + \int_{P_{-K_M} \setminus (B_R(0) \cap P_{-K_M})} (1 + \langle b_X, x \rangle) e^{-\langle b_X, x \rangle} dx \right) \leq C'$$

for a uniform positive constant C'. Combined with (7.17), this yields the desired bound.

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**Local control on u\_s.** Lemmas 7.13 and 7.16, combined with an application of the Sobolev inequality, now give us local control on  $u_s$ .

**Proposition 7.17.** There exists C > 0 such that, for all  $x \in P_{-K_M}$  and  $s \in [0, 1]$ ,

$$|u_s(x) - u_0(x)| \le Ce^{\langle b_X, x \rangle}$$
.

*Proof.* From the first paragraph of the proof of Lemma 2.23, we know that, outside a compact subset,  $P_{-K_M}$  coincides with the Cartesian product of the half line and  $P_D$ , the polytope associated to D. More precisely, in light of (2.7),  $P_{-K_M}$  coincides with

$$[a, \infty) \times P_D \subseteq \mathbb{R} \times \mathbb{R}^{n-1}$$
 for some  $a \in \mathbb{R}$ 

outside a convex compact subset. Suppose that  $x \in P_{-K_M}$  lies in the region  $[a+1,\infty) \times P_D$ . Then there exists  $b \in [a+1,\infty)$  such that  $x \in \{b\} \times P_D$ . Let

$$\Omega := [b-1, b+1] \times P_D \subseteq [a, \infty) \times P_D \subseteq \mathbb{R} \times \mathbb{R}^{n-1}.$$

Set  $U_s := u_s - u_0$  and let q > n. Then, since  $U_s$  is smooth up to  $\partial P_{-K_M}$  by Lemma 2.26 (i), we can apply the Sobolev inequality from [28, Theorem 3.4] (which in particular states that the Sobolev constant depends only on the Euclidean diameter and measure of  $\Omega$ ), together with Lemmas 7.13 and 7.16, to determine that, for a uniform constant C > 0,

$$|U_{s}(x)| \leq ||U_{s}||_{C^{0}(\Omega)} \leq \left||U_{s} - \frac{1}{|\Omega|} \int_{\Omega} U_{s} dx\right||_{C^{0}(\Omega)} + \frac{1}{|\Omega|} \int_{\Omega} |U_{s}| dx$$

$$\leq C ||\nabla U_{s}||_{L^{q}(\Omega)} + \frac{1}{|\Omega|} \int_{\Omega} |U_{s}| dx$$

$$\leq C \left(\sup_{y \in \Omega} e^{\langle b_{X}, y \rangle} + \left(\sup_{y \in \Omega} e^{\langle b_{X}, y \rangle}\right)^{\frac{1}{q}}\right) \leq C \sup_{y \in \Omega} e^{\langle b_{X}, y \rangle},$$

because  $0 < \frac{1}{q} < 1$ . Continuing, we find that

$$|U_s(x)| \le C \sup_{y \in \Omega} e^{\langle b_X, y \rangle} = C e^{\langle b_X, x \rangle} \cdot \sup_{y \in \Omega} e^{\langle b_X, y - x \rangle} \le C e^{\langle b_X, x \rangle}.$$

A slight modification of this argument also shows that

$$|U_s(x)| \le Ce^{\langle b_X, x \rangle}$$
 for all  $x \in P_{-K_M} \setminus ([a+1, \infty) \times P_D)$ 

which, as noted above, is a compact convex subset of  $\mathbb{R}^n$ . In sum, we arrive at the bound

$$|U_s(x)| \le Ce^{\langle b_X, x \rangle}$$
 for all  $x \in P_{-K_M}$ ,

as required.

**Local control on \psi\_s.** The previous proposition can be reformulated to give local control on  $\psi_s$ .

**Proposition 7.18.** There exists C > 0 such that, for all  $x \in M$  and  $s \in [0, 1]$ ,

$$\psi_{\mathfrak{s}}(x) \ge -Ce^{f(x)}$$
.

*Proof.* The definition of the Legendre transform and Proposition 7.17 gives us that, for all  $\xi \in \mathbb{R}^n$  and  $s \in [0, 1]$ ,

$$\begin{split} \psi_{s}(\xi) &= 2(\phi_{s}(\xi) - \phi_{0}(\xi)) = 2 \Big( \sup_{x \in P_{-K_{M}}} \{ \langle \xi, x \rangle - u_{s}(x) \} - \phi_{0}(\xi) \Big) \\ &\geq 2 \Big( \langle \xi, \nabla \phi_{0}(\xi) \rangle - u_{s}(\nabla \phi_{0}(\xi)) - \phi_{0}(\xi) \Big) \\ &= 2 \Big( u_{0}(\nabla \phi_{0}(\xi)) - u_{s}(\nabla \phi_{0}(\xi)) \Big) \\ &> - C e^{\langle b_{X}, \nabla \phi_{0}(\xi) \rangle} = - C e^{f(\xi)} \end{split}$$

for some uniform C > 0, as claimed.

A priori lower bound on  $\inf_M \vartheta_s$ . This brings us to the concluding bound of this section. Proposition 7.18 yields a uniform lower bound on  $\min_W \psi_s$ . By Lemma 7.6, this results in a uniform lower bound on  $\inf_M \vartheta_s$ . This is demonstrated in the following proposition.

**Proposition 7.19** (A priori lower bound on  $\inf_M \vartheta_s$ ). Let  $(\vartheta_s)_{0 \le s \le 1}$  be a path of solutions in  $\mathbb{R} \oplus C^{\infty}_{X,\beta}(M)$  to  $(\star \star_s)$ . Then there exists a uniform constant C > 0 such that

$$\inf_{0 \le s \le 1} \inf_{M} \vartheta_s \ge -C.$$

*Proof.* Combining Lemma 7.6 and Proposition 7.18, we find that, for all  $s \in [0, 1]$ ,

$$\inf_{M} \vartheta_{s} = \min_{W} \vartheta_{s} = \min_{W} (\psi_{s} - \Phi_{s}) \ge \min_{W} (-Ce^{f} - \Phi_{s}) \ge -C.$$

7.6. A priori upper bound on the radial derivative. The  $C^0$ -bound on  $\vartheta_s$  allows us to derive an a priori upper bound on  $X \cdot \vartheta_s$ .

**Proposition 7.20.** Let  $(\vartheta_s)_{0 \le s \le 1}$  be a path of solutions in  $\mathbb{R} \oplus C_{X,\beta}^{\infty}(M)$  to  $(\star \star_s)$ . Then there exists a positive constant  $C = C(n, \omega, \sup_{s \in [0,1]} \|G_s\|_{C^0})$  such that

$$\sup_{0 \le s \le 1} \sup_{M} X \cdot \vartheta_s \le C.$$

In particular,  $X \cdot \vartheta_s < C$  for all  $s \in [0, 1]$ .

*Proof.* Our proof is based on that of Siepmann in the case of an expanding gradient Kähler–Ricci soliton; see [33, Lemma 5.4.14]. We adapt his proof here to our particular setting. We begin with Claim 7.4, which gives

(7.18) 
$$X \cdot X \cdot \vartheta_{s} = 2i \partial \bar{\partial} \vartheta_{s}(X, JX) = 2(\sigma_{s}(X, JX) - \omega_{s}(X, JX))$$
$$\geq -2\omega_{s}(X, JX) = -2|X|_{\sigma_{s}}^{2}.$$

To get an upper bound for  $X \cdot \vartheta_s$ , we introduce the flow  $(\varphi_t^X)_{t \in \mathbb{R}}$  generated by the vector field  $\frac{X}{2}$ . This flow is complete since X grows linearly at infinity. Define  $\vartheta_x^s(t) := \vartheta_s(\varphi_t^X(x))$  for  $(x,t) \in M \times \mathbb{R}$ . Then, for any cut-off function  $\eta: \mathbb{R}_+ \to [0,1]$  such that  $\eta(0) = 1$  and  $\eta'(0) = 0$ , we have that

$$\int_0^{+\infty} \eta''(t)\vartheta_x^s(t)\,dt = -\int_0^{+\infty} \eta'(t)(\vartheta_x^s)'(t)\,dt = (\vartheta_x^s)'(0) + \int_0^{+\infty} \eta(t)(\vartheta_x^s)''(t)\,dt.$$

Using (7.18), it then follows that

$$\begin{split} \frac{X}{2} \cdot \vartheta_{s}(x) &= (\vartheta_{x}^{s})'(0) \\ &\leq -\int_{\text{supp}(\eta)} \frac{X}{2} \cdot \left(\frac{X}{2} \cdot \vartheta_{s}\right) (\varphi_{t}^{X}(x)) \, dt + \sup_{t \in \text{supp}(\eta'')} |\vartheta_{x}^{s}(t)| \int_{\text{supp}(\eta'')} |\eta''(t)| \, dt \\ &\leq \frac{1}{2} \int_{\text{supp}(\eta)} |X|_{g_{s}}^{2} (\varphi_{t}^{X}(x)) \, dt + \sup_{t \in \text{supp}(\eta'')} |\vartheta_{s}(\varphi_{t}^{X}(x))| \int_{\text{supp}(\eta'')} |\eta''(t)| \, dt. \end{split}$$

Choose  $\eta$  such that  $\operatorname{supp}(\eta) \subset [0,1]$  and let x now be the point where  $X \cdot \vartheta_s$  attains its maximum value. By Lemma 7.3 (i), we know that x is contained in W. Hence, we deduce from the above that

$$\frac{X}{2} \cdot \vartheta_s(x) \le C \left( \sup_{s \in [0,1]} \left( \sup_{t \in [0,1]} \varphi_t^X(W) |X|_{g_s}^2 \right) + \|\vartheta_s\|_{C^0} \right).$$

The result now follows from the uniform upper bound on  $\|\vartheta_s\|_{C^0}$ .

7.7. A priori estimates on higher derivatives. We next derive a priori global bounds on higher derivatives of solutions to the complex Monge–Ampère equation  $(\star \star_s)$ , beginning with the  $C^2$ -estimate. The a priori bounds we derive hold everywhere on the manifold M, not just on a given fixed compact subset. The unboundedness of the vector field X prevents us from applying standard local estimates to higher derivatives of solutions to  $(\star \star_s)$ .

# 7.7.1. $C^2$ a priori estimate.

**Proposition 7.21** (A priori  $C^2$ -estimate). Suppose that  $(\vartheta_s)_{0 \le s \le 1}$  is a path of solutions in  $\mathbb{R} \oplus C^{\infty}_{X,B}(M)$  to  $(\star \star_s)$ . Then there exists a positive constant

$$C = C(n, \omega, \sup_{s \in [0,1]} ||G_s||_{C^2})$$

such that the following  $C^2$  a priori estimate holds true:

$$\sup_{0 \le s \le 1} \|i \partial \bar{\partial} \vartheta_s\|_{C^0} \le C.$$

In particular,

$$\sup_{0 \le s \le 1} \|i \, \partial \bar{\partial} \psi_s\|_{C^0} \le C.$$

*Proof.* Following closely [14, Proposition 6.6] where the approach taken is based on standard computations performed in Yau's seminal paper [39, pp. 347–351] (see also [33, Lemma 5.4.16] and [34, pp. 52–55]), we let  $\Delta_s$  denote the Laplacian with respect to  $\sigma_s$  and first estimate the drift Laplacian  $\Delta_s - \frac{X}{2}$  of  $\text{tr}_{\omega_s} \sigma_s$  to obtain

$$(7.19) \quad \left(\Delta_{s} - \frac{X}{2} \cdot\right) \operatorname{tr}_{\omega_{s}} \sigma_{s} \geq \frac{(\vartheta_{s})_{i\bar{j}k}(\vartheta_{s})_{\bar{i}j\bar{k}}}{(1 + (\vartheta_{s})_{i\bar{i}})(1 + (\vartheta_{s})_{k\bar{k}})} + \Delta_{s} G_{s} \\ - C \operatorname{tr}_{\omega_{s}} \sigma_{s} \cdot \operatorname{tr}_{\sigma_{s}} \omega_{s} \cdot \left(1 + \inf_{M} \operatorname{Rm}(g_{s})\right) - C(n, \omega).$$

Let  $u_s := e^{-\lambda \vartheta_s} (n + \Delta_s \vartheta_s)$ , where  $\lambda > 0$  will be specified later. Then one estimates the drift Laplacian  $\Delta_s - \frac{X}{2}$  of  $u_s$  with respect to  $\sigma_s$  in the following way using the fact that  $\vartheta_s$  satisfies  $(\star \star_s)$ :

$$\begin{split} \left(\Delta_{s} - \frac{X}{2} \cdot \right) u_{s} &\geq e^{-\lambda \vartheta_{s}} \Delta_{s} G_{s} + e^{-\lambda \vartheta_{s}} g_{s} \left(\nabla^{s} \left(\frac{X}{2}\right), i \partial \bar{\partial} \vartheta_{s}\right) - C_{s} n^{2} e^{-\lambda \vartheta_{s}} \\ &+ \lambda \left(\frac{X}{2} \cdot \vartheta_{s}\right) u_{s} - \lambda n u_{s} + (\lambda + C_{s}) e^{\frac{\lambda \vartheta_{s} - G_{s} - \frac{X}{2} \cdot \vartheta_{s}}{n-1}} u_{s}^{\frac{n}{n-1}}, \end{split}$$

where  $\nabla^s$  is the Levi-Civita connection of  $g_s$  and  $C_s := \inf_{i \neq k} \operatorname{Rm}_{i\bar{i}k\bar{k}}^s$ ,  $\operatorname{Rm}^s$  here denoting the complex linear extension of the curvature operator of the metric  $g_s$ . As  $C_s$  is uniformly bounded below in s by a constant A (which we may assume is at most 1), we may choose  $\lambda > 0$  sufficiently large so that  $\lambda + A = 1$ . Moreover, as

$$\left| g_s \left( \nabla^s \left( \frac{X}{2} \right), i \partial \bar{\partial} \vartheta_s \right) \right| \le C \| \nabla^s X \|_{C^0} (1 + u)$$

for some generic constant C > 0, we deduce that u satisfies the differential inequality

$$\left(\Delta_s - \frac{X}{2}\right)u_s \ge -C_1(1+u_s) + C_2 u_s^{\frac{n}{n-1}},$$

where  $C_1$  and  $C_2$  depend only on n, A,

$$\sup_{s \in [0,1]} \|\vartheta_s\|_{C^0}, \quad \sup_{s \in [0,1]} \|X \cdot \vartheta_s\|_{C^0}, \quad \sup_{s \in [0,1]} \|G_s\|_{C^2}, \quad \text{and} \quad \sup_{s \in [0,1]} \|\nabla^s X\|_{C^0}.$$

The combination of Propositions 7.5, 7.11, 7.19, and 7.20 shows that  $C_1$  and  $C_2$  depend only on n, A and  $\sup_{s \in [0,1]} ||G_s||_{C^2}$ .

Since  $u_s$  is non-negative and converges to n at infinity as  $\vartheta_s \in \mathbb{R} \oplus C_{X,\beta}^{\infty}(M)$ , an application of the maximum principle to an exhausting sequence of domains of M yields an upper bound for  $n + \Delta_s \vartheta_s$  and, consequently, the desired bound on  $i \partial \bar{\partial} \vartheta_s$ .

A useful consequence of Proposition 7.21 is that the Kähler metrics induced by  $\sigma_s$  and  $\omega_s$  are uniformly equivalent.

**Corollary 7.22.** Let  $(\vartheta_s)_{0 \le s \le 1}$  be a path of solutions in  $\mathbb{R} \oplus C_{X,\beta}^{\infty}(M)$  to  $(\star \star_s)$ , and for  $s \in [0,1]$ , let  $g_s, h_s$  denote the Kähler metrics induced by  $\omega_s, \sigma_s$ , respectively. Then the tensors  $g_s^{-1}h_s$  and  $h_s^{-1}g_s$  satisfy the following uniform estimate:

$$\sup_{0 \le t \le 1} \|g_s^{-1} h_s\|_{C^0} + \sup_{0 \le t \le 1} \|h_s^{-1} g_s\|_{C^0} \le C$$

for some positive constant  $C = C(n, \omega, \sup_{s \in [0,1]} ||G_s||_{C^2})$ . In particular, the metrics g and  $(h_s)_{0 \le s \le 1}$  are uniformly equivalent.

*Proof.* The estimate follows as in [15, Corollary 7.15] using Propositions 7.5, 7.20, and 7.21. The fact that  $\omega$  and  $\sigma_s$  differ by a (1, 1)-form whose norm is controlled uniformly in s yields the last claim of the corollary.

**7.7.2.**  $C^3$  a priori estimate. We now present the  $C^3$ -estimate.

**Proposition 7.23** (A priori  $C^3$ -estimate). Suppose that  $(\vartheta_s)_{0 \le s \le 1}$  is a path of solutions in  $\mathbb{R} \oplus C^{\infty}_{X,\beta}(M)$  to  $(\star \star_s)$  and let  $g_s$  be the Kähler metric induced by  $\omega_s$  with Levi-Civita connection  $\nabla^{g_s}$ . Then

$$\sup_{0\leq s\leq 1} \|\nabla^{g_s} \partial \bar{\partial} \vartheta_s\|_{C^0} \leq C(n, \omega, \sup_{s\in[0,1]} \|G_s\|_{C^3}).$$

In particular,

(7.20) 
$$\sup_{0 \le s \le 1} \|\nabla^{g_s} (X \cdot \vartheta_s)\|_{C^0} \le C(n, \omega, \sup_{s \in [0, 1]} \|G_s\|_{C^3}).$$

*Proof.* We follow closely the proof given in [14, Proposition 6.9] which itself is based on [30]. Set

$$S(h_s, g_s) := |\nabla^{g_s} h_s|_{h_s}^2$$

Then, from the definition of S, we see that

$$S(h_s, g_s) = h_s^{i\bar{J}} h_s^{k\bar{l}} h_s^{p\bar{q}} \nabla_i^{g_s} (h_s)_{kp} \overline{\nabla_i^{g_s} (h_s)_{lq}} = |\Psi|_{h_s}^2,$$

where

$$\Psi_{ij}^{k}(h_{s},g_{s}) := \Gamma(h_{s})_{ij}^{k} - \Gamma(g_{s})_{ij}^{k} = h_{s}^{k\bar{l}} \nabla_{i}^{g_{s}}(h_{s})_{i\bar{l}}.$$

Now, since  $\vartheta_s$  solves  $(\star\star_s)$ ,  $(M,h_s,X)$  is an "approximate" steady gradient Kähler–Ricci soliton in the following precise sense: if  $h_s(t) := (\varphi_t^X)^*h_s$  and  $g_s(t) := (\varphi_t^X)^*g_s$ , where  $(\varphi_t^X)_{t\in\mathbb{R}}$  is the one-parameter family of diffeomorphisms generated by  $\frac{X}{2}$ , then  $(h_s(t))_{t\in\mathbb{R}}$  is a solution of the following perturbed Kähler–Ricci flow with initial condition  $h_s$ :

$$\partial_t h_s(t) = -\operatorname{Ric}(h_s(t)) + (\varphi_t^X)^* \left( \mathcal{L}_{\frac{X}{2}} g_s + \operatorname{Ric}(g_s) + \nabla^{g_s} \bar{\nabla}^{g_s} G_s \right), \quad t \in \mathbb{R},$$
  
$$h_s(0) = h_s.$$

In particular,  $\partial_t h_s = -\operatorname{Ric}(h_s) + (\varphi_t^X)^* \Lambda$ , where  $\Lambda := \mathcal{L}_{\frac{X}{2}} g_s + \operatorname{Ric}(g_s) + \nabla^{g_s} \bar{\nabla}^{g_s} G_s$  has uniformly controlled  $C^1$ -norm as  $g_s$  is isometric to g and  $G_s$  is equal to zero, all outside a compact set independent of s.

Define  $S(t) := S(h_s(s), g_s(t))$  and correspondingly  $\Psi(t) := \Psi(h_s(t), g_s(t))$ . We adapt [8, Proposition 3.2.8] to our setting. By a brute force computation, we have that

$$\begin{split} \Delta_{\sigma_s} S &= 2\operatorname{Re} \left(h_s^{i\bar{\jmath}} h_s^{p\bar{q}} (h_s)_{k\bar{l}} (\Delta_{\sigma_s,1/2} \Psi_{ip}^k) \overline{\Psi_{jq}^l} \right) + |\nabla^{h_s} \Psi|_{h_s}^2 + |\overline{\nabla}^{h_s} \Psi|_{h_s}^2 \\ &+ \operatorname{Ric} (h_s)^{i\bar{\jmath}} h_s^{p\bar{q}} (h_s)_{k\bar{l}} \Psi_{ip}^k \overline{\Psi_{jq}^l} + h_s^{i\bar{\jmath}} \operatorname{Ric} (h_s)^{p\bar{q}} (h_s)_{k\bar{l}} \Psi_{ip}^k \overline{\Psi_{jq}^l} \\ &- h_s^{i\bar{\jmath}} h_s^{p\bar{q}} \operatorname{Ric} (h_s)_{k\bar{l}} \Psi_{ip}^k \overline{\Psi_{jq}^l}, \end{split}$$

where

$$\Delta_{\sigma_{\mathcal{S}},1/2} := h_{\mathcal{S}}^{i\bar{\jmath}} \nabla_{i}^{h_{\mathcal{S}}} \nabla_{\bar{\imath}}^{h_{\mathcal{S}}}, \quad T^{i\bar{\jmath}} := h_{\mathcal{S}}^{i\bar{k}} h_{\mathcal{S}}^{l\bar{\jmath}} T_{k\bar{l}}$$

for  $T_{k\bar{l}} \in \Lambda^{1,0} M \otimes \Lambda^{0,1} M$ . We also have that

$$\begin{split} \partial_{u}\Psi(u)_{ip}^{k}|_{u=0} &= \partial_{u}|_{u=0} \left(\Gamma(h_{s}(u)) - \Gamma(g_{s}(u))\right)_{ip}^{k} \\ &= \nabla_{i}^{h_{s}} \left(-\operatorname{Ric}(h_{s})_{p}^{k} + \Lambda_{p}^{k}\right) - \nabla_{i}^{g_{s}} \left(\mathcal{L}_{\frac{X}{2}}(g_{s})_{p}^{k}\right), \\ \partial_{u}h_{s}^{i\bar{j}}|_{u=0} &= \operatorname{Ric}(h_{s})^{i\bar{j}} - \Lambda^{i\bar{j}}. \end{split}$$

Finally, using the second Bianchi identity, we compute that

$$\Delta_{\sigma_s,1/2}\Psi_{ip}^k = h_s^{a\bar{b}}\nabla_a^{h_s}\operatorname{Rm}(g_s)_{i\bar{b}\,p}^k - \nabla_i^{h_s}\operatorname{Ric}(h_s)_p^k,$$

which in turn implies that the following evolution equation is satisfied by  $\Psi$ :

$$\partial_u \Psi_{ip}^k(u)|_{u=0} = \Delta_{\sigma_s,1/2} \Psi_{ip}^k + T_{ip}^k$$

for a tensor T of the form

$$T = h_s^{-1} * \nabla^{h_s} \operatorname{Rm}(g_s) + \nabla^{h_s} \Lambda - \nabla^{g_s} (\mathcal{L}_{\frac{X}{2}} g_s)$$
  
=  $h_s^{-1} * \nabla^{g_s} \operatorname{Rm}(g_s) + h_s^{-1} * h_s^{-1} * \operatorname{Rm}(g_s) * \Psi + h_s^{-1} * \Psi * \Lambda + \nabla^{g_s} (\Lambda - \mathcal{L}_{\frac{X}{2}} g_s).$ 

Notice the simplification here regarding the "bad" term  $-\nabla^{h_s} \operatorname{Ric}(h_s)$ . Since this flow is evolving only by diffeomorphism, we know that

$$S(t) = (\varphi_t^X)^* S(h_s, g_s), \quad \partial_u S|_{u=0} = \frac{X}{2} \cdot S(h_s, g_s).$$

Hence Young's inequality, together with the boundedness of  $||h_s^{-1}g_s||_{C^0}$  and  $||h_sg_s^{-1}||_{C^0}$  ensured by Corollary 7.22 and the boundedness of the covariant derivatives of the tensors  $\text{Rm}(g_s)$  and  $\Lambda$ , imply that

$$\Delta_{\sigma_S} S - \frac{X}{2} \cdot S \ge -C(S+1)$$

for some positive uniform constant C.

We use as a barrier function the trace  $\operatorname{tr}_{\omega_s} \sigma_s$  which, by (7.19) and the uniform equivalence of the metrics  $g_s$  and  $h_s$  provided by Corollary 7.22, satisfies

$$\Delta_{\sigma_s} \operatorname{tr}_{\omega_s} \sigma_s - \frac{X}{2} \cdot \operatorname{tr}_{\omega_s} \sigma_s \ge C^{-1} S - C,$$

where C is a uniform positive constant that may vary from line to line. By applying the maximum principle to  $\varepsilon S + \operatorname{tr}_{\omega_s} \sigma_s$  for some sufficiently small  $\varepsilon > 0$ , one arrives at the desired a priori estimate.

The proof of (7.20) is a consequence of the previously proved a priori bound on  $\nabla^{g_s} \partial \overline{\partial} \vartheta_s$ , once we differentiate ( $\star\star_s$ ).

We next establish Hölder regularity of  $g_s^{-1}h_s$  and  $h_s^{-1}g_s$ , an improvement on Corollary 7.22.

**Corollary 7.24.** Let  $(\vartheta_s)_{0 \le s \le 1}$  be a path of solutions in  $\mathbb{R} \oplus C_{X,\beta}^{\infty}(M)$  to  $(\star \star_s)$ , and for  $s \in [0, 1]$ , let  $h_s$  be the Kähler metric induced by  $\sigma_s$ . Then, for any  $\alpha \in (0, \frac{1}{2})$ , the tensors  $g_s^{-1}h_s$  and  $h_s^{-1}g_s$  satisfy the following uniform estimate:

$$\sup_{0 \le s \le 1} (\|g_s^{-1} h_s\|_{C^{0,2\alpha}_{loc}} + \|h_s^{-1} g_s\|_{C^{0,2\alpha}_{loc}}) \le C(n,\alpha,\omega,\sup_{s \in [0,1]} \|G_s\|_{C^3}).$$

*Proof.* By standard local interpolation inequalities applied to Propositions 7.21 and 7.23, we see that

$$\|g_s^{-1}h_s\|_{C^{0,2\alpha}_{loc}} \le C(n,\alpha,\omega, \sup_{s\in[0,1]} \|G_s\|_{C^3}).$$

Combining the previous estimate with Corollary 7.22, it suffices to prove a uniform bound on the local  $2\alpha$ -Hölder norm of  $h_s^{-1}g_s$ . We conclude with the following observation: if u is a positive function on M in  $C_{loc}^{0,2\alpha}(M)$  uniformly bounded from below by a positive constant, then  $[u^{-1}]_{2\alpha} \leq [u]_{2\alpha} (\inf_M u)^{-2}$ . By invoking Corollary 7.22 once more, this last remark applied to  $h_s^{-1}g_s$  implies that

$$||h_s^{-1}g_s||_{C^{0,2\alpha}_{loc}} \le C(n,\alpha,\omega,\sup_{s\in[0,1]}||G_s||_{C^3})$$

as well.

**7.7.3. Local bootstrapping.** We now improve the local regularity of our continuity path of solutions to  $(\star\star_s)$ . This estimate will be used in deriving the subsequent weighted a priori estimates.

**Proposition 7.25.** Let  $(\vartheta_s)_{0 \le s \le 1}$  be a path of solutions in  $\mathbb{R} \oplus C^{\infty}_{X,\beta}(M)$  to  $(\star \star_s)$ . Then, for any  $\alpha \in (0, \frac{1}{2})$  and for any compact subset  $K \subset M$ ,

$$\sup_{0\leq s\leq 1}\|\vartheta_s\|_{C^{3,2\alpha}(K)}\leq C(n,\alpha,\omega,\sup_{s\in[0,1]}\|G_s\|_{C^3},K).$$

*Proof.* From the standard computations involved in the proof of the a priori  $C^2$ -estimate, we derive that

$$(7.21) \qquad \Delta_{\sigma_{s}} \left( \Delta_{\omega_{s}} \vartheta_{s} - \frac{X}{2} \cdot \vartheta_{s} \right)$$

$$= \Delta_{\sigma_{s}} G_{s} + h_{s}^{-1} * g_{s}^{-1} * \operatorname{Rm}(g_{s}) + \operatorname{Rm}(g_{s}) * \nabla^{h_{s}} \bar{\nabla}^{h_{s}} \vartheta_{s} * h_{s}^{-1} + g_{s}^{-1} * g_{s}^{-1} * \operatorname{Rm}(g_{s}) + g_{s}^{-1} * h_{s}^{-1} * h_{s}^{-1} * \bar{\nabla}^{h_{s}} \nabla^{h_{s}} \bar{\nabla}^{h_{s}} \vartheta_{s} * \nabla^{h_{s}} \bar{\nabla}^{h_{s}} \vartheta_{s}$$

$$+ g_{s}^{-1} * h_{s}^{-1} * h_{s}^{-1} * \bar{\nabla}^{h_{s}} \nabla^{h_{s}} \bar{\nabla}^{h_{s}} \bar{\nabla}^{h_{s}} \vartheta_{s} * \nabla^{h_{s}} \bar{\nabla}^{h_{s}} \vartheta_{s}$$

$$- (\Delta_{\sigma_{s}} - \Delta_{\omega_{s}}) \left( \frac{X \cdot \vartheta_{s}}{2} \right),$$

where \* denotes the ordinary contraction of two tensors. Now, since X is real holomorphic and  $\vartheta_s$  being JX-invariant, we see that

$$(7.22) i\partial\overline{\partial}(X\cdot\vartheta_s) = \mathcal{L}_X(i\partial\overline{\partial}\vartheta_s) = \nabla_X^{g_s}(i\partial\overline{\partial}\vartheta_s) + i\partial\overline{\partial}\vartheta_s * \nabla^{g_s}X.$$

Therefore, thanks to (7.22), we have the following pointwise estimate:

$$(7.23) \quad |(\Delta_{\sigma_{s}} - \Delta_{\omega_{s}})(X \cdot \vartheta_{s})|$$

$$= |h_{s}^{-1} * i \partial_{\bar{\sigma}} \vartheta_{s} * i \partial_{\bar{\sigma}} (X \cdot \vartheta_{s})|_{g_{s}}$$

$$\leq |h_{s}^{-1} g_{s}|_{g_{s}} \cdot |i \partial_{\bar{\sigma}} \vartheta_{s}|_{g_{s}} \cdot (|i \partial_{\bar{\sigma}} \vartheta_{s}|_{g_{s}}) \nabla^{g_{s}} X|_{g_{s}} + |\nabla^{g_{s}} i \partial_{\bar{\sigma}} \vartheta_{s}|_{g_{s}} |X|_{g_{s}}).$$

By Propositions 7.21 and 7.23 together with (7.23), the  $C^0$ -norm of the right-hand side of (7.21) is uniformly bounded on compact subsets and, thanks to Corollary 7.24, so too are the coefficients of  $\Delta_{\sigma_s}$  in the  $C^{0,2\alpha}_{loc}$ -sense. As a result, by applying the Morrey–Schauder  $C^{1,2\alpha}$ -estimates, we see that, for any  $x \in M$  and for  $\delta < \inf_{g_s}(M)$ ,

$$\left\| \Delta_{\omega_s} \vartheta_s - \frac{X}{2} \cdot \vartheta_s \right\|_{C^{1,2\alpha}(B_{g_s}(x,\delta))} \le C(x,\delta,\alpha).$$

Finally, applying standard interior Schauder estimates for elliptic equations once again with respect to  $\Delta_{\omega_s,X}$ , we deduce that

$$\|\vartheta_{s}\|_{C^{3,2\alpha}(B_{g_{s}}(x,\frac{\delta}{2}))} \leq C(x,\delta,\alpha) \Big( \|\Delta_{\omega_{s}}\vartheta_{s} - \frac{X}{2} \cdot \vartheta_{s} \|_{C^{1,2\alpha}(B_{g_{s}}(x,\delta))} + \|\vartheta_{s}\|_{C^{1,2\alpha}(B_{g_{s}}(x,\delta))} \Big)$$

$$\leq C(x,\delta,\alpha).$$

We next establish the following well-known local regularity result for solutions to  $(\star \star_s)$ .

**Proposition 7.26.** Let  $G_s \in C^{k,\alpha}_{loc}(M)$  for some  $k \geq 1$  and  $\alpha \in (0,1)$  and suppose that  $\vartheta_s \in C^{3,\alpha}_{loc}(M)$  is a solution to  $(\star \star_s)$  with data  $G_s$ . Then  $\vartheta_s \in C^{k+2,\alpha}_{loc}(M)$ . Moreover, for all  $k \geq 1$ ,  $\alpha \in (0,1)$ , and compact subset  $K \subset M$ ,

$$\|\vartheta_s\|_{C^{k+2,\alpha}(K)} \leq C(n,\alpha,\omega, \sup_{s\in[0,1]} \|G_s\|_{C^{\max\{k,3\},\alpha}}, K).$$

*Proof.* We prove this proposition by induction on  $k \ge 1$ . The case k = 1 is true by Proposition 7.25, so let  $G_s \in C^{k+1,\alpha}_{loc}(M)$  and let  $\vartheta_s \in C^{3,\alpha}_{loc}(M)$  be a solution of  $(\star \star_s)$ . Then, by induction,  $\vartheta_s \in C^{k+2,\alpha}_{loc}(M)$ . Let  $x \in M$  and choose local holomorphic coordinates defined on  $B_{g_s}(x,\delta)$  for some  $0 < \delta < \inf_{g_s}(M)$ . Then, since  $\vartheta_s$  satisfies

$$G_s = \log\left(\frac{\sigma_s^n}{\omega_s^n}\right) - \frac{X}{2} \cdot \vartheta_s,$$

we know that, for j = 1, ..., 2n, the derivative  $\partial_i \vartheta_s$  satisfies

$$\Delta_{\sigma_s}(\partial_j \vartheta_s) = \partial_j \left( G_s + \frac{X}{2} \cdot \vartheta_s \right) \in C^{k,\alpha}_{loc}(M).$$

As the coefficients of  $\Delta_{\sigma_s}$  are in  $C_{\text{loc}}^{k,\alpha}(M)$ , an application of the standard interior Schauder estimates for elliptic equations now gives us the desired local regularity result, namely

$$\partial_j \vartheta_s \in C^{k+2,\alpha}_{loc}(M)$$
 for all  $j = 1, \dots, 2n$ ,

or equivalently,  $\vartheta_s \in C^{k+3,\alpha}_{loc}(M)$  together with the expected estimate.

**7.8. Weighted a priori estimates.** Our first proposition establishes an a priori decay estimate on the gradient of the X-derivative of solutions to  $(\star \star_s)$ . Its proof uses the Bochner formula in an essential way.

**Proposition 7.27.** Let  $(\vartheta_s)_{0 \le s \le 1}$  be a path of solutions in  $\mathbb{R} \oplus C^{\infty}_{X,\beta}(M)$  to  $(\star \star_s)$ . Then there exist positive constants C,  $R_0$ , and  $\varepsilon > 0$  such that, for all  $s \in [0, 1]$ ,

$$|\nabla^g (X \cdot \vartheta_s)|_g \le \frac{C}{f^{\varepsilon}}, \quad f \ge R_0.$$

*Proof.* Let  $u := X \cdot \vartheta_s$ , write  $\Delta_{h_s,X} := \Delta_{h_s} - X \cdot$ , where  $\Delta_{h_s}$  denotes the Riemannian Laplacian with respect to  $h_s$ , and recall from (7.5) the differential equation satisfied by u outside a sufficiently large compact set W of M,

(7.24) 
$$\frac{1}{2}\Delta_{h_s,X}u = 2e^{-\frac{X\cdot\vartheta_s}{2}}\frac{(\omega_D + i\,\partial\bar{\partial}\vartheta_s)^n}{\omega^n}.$$

Applying the Bochner formula for the drift Laplacian to the function u, we obtain

$$\begin{split} &\frac{1}{2}\Delta_{h_{s},X}|\nabla^{h_{s}}u|_{h_{s}}^{2} \\ &= |\operatorname{Hess}_{h_{s}}(u)|_{h_{s}}^{2} + \operatorname{Ric}(h_{s})(\nabla^{h_{s}}u,\nabla^{h_{s}}u) + \operatorname{Hess}_{h_{s}}(f_{\sigma_{s}})(\nabla^{h_{s}}u,\nabla^{h_{s}}u) \\ &+ \langle \nabla^{h_{s}}\Delta_{h_{s},X}u,\nabla^{h_{s}}u\rangle_{h_{s}} \\ &= |\operatorname{Hess}_{h_{s}}(u)|_{h_{s}}^{2} + \operatorname{Ric}(g_{s})(\nabla^{h_{s}}u,\nabla^{h_{s}}u) + \operatorname{Hess}_{g_{s}}(f_{\omega_{s}})(\nabla^{h_{s}}u,\nabla^{h_{s}}u) \\ &- i\partial\bar{\partial}G_{s}(\nabla^{h_{s}}u,\nabla^{h_{s}}u) + 4\Big\langle \nabla^{h_{s}}\Big(e^{-\frac{X\cdot\partial_{s}}{2}}\frac{(\omega_{D}+i\partial\bar{\partial}\partial_{s})^{n}}{\omega^{n}}\Big),\nabla^{h_{s}}u\Big\rangle_{h_{s}}, \end{split}$$

where we have used (7.1) and (7.24) in the second equality. As  $G_s$  is supported in W and  $g_s$  is isometric to g on  $M \setminus W$ , on this latter set, this equation reads as

$$\begin{split} \frac{1}{2}\Delta_{h_s,X}|\nabla^{h_s}u|_{h_s}^2 &= |\mathrm{Hess}_{h_s}(u)|_{h_s}^2 + \mathrm{Ric}(g)(\nabla^{h_s}u,\nabla^{h_s}u) + \mathrm{Hess}_g(f)(\nabla^{h_s}u,\nabla^{h_s}u) \\ &+ 4\Big\langle\nabla^{h_s}\Big(e^{-\frac{X\cdot\vartheta_s}{2}}\frac{(\omega_D+i\partial\bar\partial\vartheta_s)^n}{\omega^n}\Big),\nabla^{h_s}u\Big\rangle_{h_s} \end{split}$$

which, using the properties of g, then becomes

(7.25) 
$$\Delta_{h_s,X} |\nabla^{h_s} u|_{h_s}^2 = 2|\operatorname{Hess}_{h_s}(u)|_{h_s}^2 + 2|\nabla^{h_s} u|_g^2 + 8\left|\nabla^{h_s} \left(e^{-\frac{X\cdot\vartheta_s}{2}} \frac{(\omega_D + i\partial\bar{\partial}\vartheta_s)^n}{\omega^n}\right), \nabla^{h_s} u\right|_{h_s}\right|$$

on  $M \setminus W$ . Henceforth working on  $M \setminus W$ , we analyse the last term of this equation in the following claim.

**Claim 7.28.** On  $M \setminus W$ , we have that

$$\begin{split} \left| \left\langle \nabla^{h_{s}} \left( e^{-\frac{X \cdot \vartheta_{s}}{2}} \frac{(\omega_{D} + i \partial \bar{\partial} \vartheta_{s})^{n}}{\omega^{n}} \right), \nabla^{h_{s}} u \right\rangle_{h_{s}} \right| \\ & \leq \frac{C}{r} \left( |\operatorname{Hess}_{h_{s}}(u)|_{h_{s}} + |\nabla^{h_{s}} u|_{h_{s}} \right) |\nabla^{h_{s}} u|_{h_{s}}. \end{split}$$

*Proof of Claim* 7.28. By the pointwise Cauchy–Schwarz inequality together with the a priori  $C^2$  estimate from Proposition 7.21, it suffices to prove that, on  $M \setminus W$ ,

$$\left| \nabla^{g} \left( e^{-\frac{X \cdot \vartheta_{s}}{2}} \frac{(\omega_{D} + i \partial \bar{\partial} \vartheta_{s})^{n}}{\omega^{n}} \right) \right|_{g} \leq \frac{C}{r} \left( |\text{Hess}_{h_{s}}(u)|_{h_{s}} + |\nabla^{h_{s}} u|_{h_{s}} \right).$$

Now, thanks to (7.6), the a priori bounds on  $X \cdot \vartheta_s$  (Propositions 7.5 and 7.20) and its gradient (Proposition 7.23), one gets schematically

$$\begin{split} \left| \nabla^{g} \left( e^{-\frac{X \cdot \vartheta_{s}}{2}} \frac{(\omega_{D} + i \partial \bar{\partial} \vartheta_{s})^{n}}{\omega^{n}} \right) \right|_{g} &\leq C \left( \frac{1}{r} |\nabla^{g} u|_{g} + \frac{1}{r^{2}} |\nabla^{g} u|_{g}^{2} + \frac{1}{r} |\text{Hess}_{g}(u)|_{g} \right) \\ &\leq \frac{C}{r} \left( |\nabla^{g} u|_{g} + |\text{Hess}_{g}(u)|_{g} \right), \end{split}$$

where we have used implicitly the a priori  $C^3$  bound (Proposition 7.23). In order to conclude, it suffices to observe that

$$|\operatorname{Hess}_{h_s}(u) - \operatorname{Hess}_g(u)|_g \le C |\nabla^g \partial \overline{\partial} \vartheta_s|_g |\nabla^g u|_g \le C |\nabla^g u|_g$$

where C is a positive constant independent of  $s \in [0, 1]$  that may vary from line to line. Here we have used Proposition 7.23 again in the last line.

Combining (7.25) with Claim 7.28 and using Proposition 7.22 to deal with the term  $|\nabla^{h_s}u|_g^2$  of (7.25), all in all, we end up with the following differential inequality satisfied by  $|\nabla^{h_s}u|_{h_s}^2$ :

$$\begin{split} \Delta_{h_s,X} |\nabla^{h_s} u|_{h_s}^2 &\geq 2|\mathrm{Hess}_{h_s}(u)|_{h_s}^2 + C^{-1} |\nabla^{h_s} u|_{h_s}^2 \\ &- \frac{C}{r} \big(|\mathrm{Hess}_{h_s}(u)|_{h_s} + |\nabla^{h_s} u|_{h_s} \big) |\nabla^{h_s} u|_{h_s}. \end{split}$$

Next, upon applying Young's inequality, we derive that, on the set  $\{r > R\}$  for some R > 0 with  $W \subset \{r \le R\}$  chosen sufficiently large,

(7.26) 
$$\Delta_{h_s,X} |\nabla^{h_s} u|_{h_s}^2 \ge \frac{1}{2} C^{-1} |\nabla^{h_s} u|_{h_s}^2.$$

Now, Lemma 7.2 ensures that  $f_{\sigma_s}^{-\beta}$  for  $\beta > 0$  satisfies outside a sufficiently large uniform compact set of M the differential inequality

$$\Delta_{h_{s},X} f_{\sigma_{s}}^{-\beta} = -\beta f_{\sigma_{s}}^{-\beta-1} \left( \Delta_{h_{s},X} f_{\sigma_{s}} - (\beta + 1) |X|_{h_{s}}^{2} f_{\sigma_{s}}^{-1} \right)$$

$$= \beta \left( 2f_{\sigma_{s}} - X \cdot \vartheta_{s} + (\beta + 1) |X|_{h_{s}}^{2} f_{\sigma_{s}}^{-1} \right) f_{\sigma_{s}}^{-\beta-1}$$

$$\leq 2\beta (1 + Cf_{\sigma_{s}}^{-1}) f_{\sigma_{s}}^{-\beta} \leq 3\beta f_{\sigma_{s}}^{-\beta}$$

for some uniform positive constant C. Here we have used Proposition 7.5 in the last line to bound  $-X \cdot \vartheta_s$  uniformly from above. We have also used (7.20) from Proposition 7.23 to bound  $|X|_{h_s}^2$  from above, since  $2|X|_{h_s}^2 = 2X \cdot f_{\sigma_s} = 2X \cdot f + X \cdot X \cdot \vartheta_s = r^2 + O(r)$ , where  $O(\cdot)$  is uniform in  $s \in [0,1]$ . Recalling (7.26), one can then use  $f_{\sigma_s}^{-\beta}$  for some  $\beta > 0$  to be specified as a barrier function. Indeed, if A > 0, then outside a sufficiently large compact subset of M, we have that

$$(7.27) \Delta_{h_s,X}(|\nabla^{h_s}u|_{h_s}^2 - Af_{\sigma_s}^{-\beta}) \ge \frac{1}{2}C^{-1}(|\nabla^{h_s}u|_{h_s}^2 - Af_{\sigma_s}^{-\beta})$$

whenever  $6\beta \le C^{-1}$ . The maximum principle applied to (7.27) now yields the desired estimate.

This leads to the following weighted estimate.

**Corollary 7.29.** Let  $(\vartheta_s)_{0 \le s \le 1}$  be a path of solutions in  $\mathbb{R} \oplus C^{\infty}_{X,\beta}(M)$  to  $(\star \star_s)$  and let C,  $R_0$ , and  $\varepsilon > 0$  be as in Proposition 7.27. Then, for all  $s \in [0,1]$ , there exists  $\vartheta_s^{\infty} \in \mathbb{R}$  such that

$$|\vartheta_s - \vartheta_s^{\infty}| + |X \cdot \vartheta_s| + |\nabla^g \vartheta_s|_g \le \frac{C}{f^{\frac{\varepsilon}{2}}}, \quad f \ge R_0.$$

*Proof.* First observe that, since  $X = \nabla^g f$ , for any vector field Y on M, we have that

$$\begin{split} g(\nabla^g(X \cdot \vartheta_s), Y) &= \operatorname{Hess}_g(f)(\nabla^g \vartheta_s, Y) + \operatorname{Hess}_g(\vartheta_s)(X, Y) \\ &= \frac{1}{2} (\mathcal{L}_X g)(\nabla^g \vartheta_s, Y) + \operatorname{Hess}_g(\vartheta_s)(X, Y). \end{split}$$

In particular, upon setting  $Y := \nabla^g \vartheta_s$ , using the JX-invariance of  $\vartheta_s$  and the fact that

$$\frac{X}{2} \cdot |\nabla^g \vartheta_s|_g^2 = \operatorname{Hess}_g(\vartheta_s)(X, \nabla^g \vartheta_s)$$

and  $\frac{1}{2}\mathcal{L}_X g = g_C$  on  $M \setminus W$ , we see that, on this set,

$$g(\nabla^{g}(X \cdot \vartheta_{s}), \nabla^{g} \vartheta_{s}) = |\nabla^{C} \vartheta_{s}|_{gC}^{2} + \frac{X}{2} \cdot |\nabla^{g} \vartheta_{s}|_{g}^{2}$$

$$= r^{-2} \underbrace{|X \cdot \vartheta_{s}|^{2}}_{\leq C} + r^{-2} \underbrace{|JX \cdot \vartheta_{s}|^{2}}_{=0} + \frac{X}{2} \cdot |\nabla^{g} \vartheta_{s}|_{g}^{2}$$

$$\leq \frac{C}{r^{2}} + \frac{X}{2} \cdot |\nabla^{g} \vartheta_{s}|_{g}^{2},$$

where we have also used the boundedness of  $|X \cdot \vartheta_s|$  given by Propositions 7.5 and 7.20 in the last line. Therefore by Young's inequality together with Proposition 7.27, we find that

$$\begin{split} \frac{X}{2} \cdot |\nabla^g \vartheta_s|_g^2 &\geq -|\nabla^g (X \cdot \vartheta_s)|_g |\nabla^g \vartheta_s|_g - \frac{C}{r^2} \\ &\geq -\frac{C}{r^{2\varepsilon}} |\nabla^g \vartheta_s|_g - \frac{C}{r^2} \\ &\geq -\frac{C}{r^{2\varepsilon}} |\nabla^g \vartheta_s|_g^2 - \frac{C}{r^{\min\{2\varepsilon,2\}}}, \end{split}$$

where C is a positive constant that may vary from line to line. The previous differential inequality can be reformulated as follows:

$$\partial_r (e^{-Cr^{-2\varepsilon}} |\nabla^g \vartheta_s|_g^2) \ge -\frac{Ce^{-Cr^{-2\varepsilon}}}{r^{1+\min\{2\varepsilon,2\}}}$$

Integrating from r to  $r=+\infty$  and using the assumption that the covariant derivatives of  $\vartheta_s$  decay to 0 at infinity, we subsequently deduce that

$$0 \le e^{-Cr^{-2\varepsilon}} |\nabla^g \vartheta_s|_g^2 \le C \int_r^{+\infty} s^{-1-\min\{2\varepsilon,2\}} e^{-Cs^{-2\varepsilon}} ds$$

so that

$$0 \leq |\nabla^g \vartheta_s|_g^2 \leq C e^{Cr^{-2\varepsilon}} \int_r^{+\infty} s^{-1-\min\{2\varepsilon,2\}} \underbrace{e^{-Cs^{-2\varepsilon}}}_{\leq 1} ds \leq C r^{-\min\{2\varepsilon,2\}} e^{Cr^{-2\varepsilon}}.$$

As  $e^{Cr^{-2\varepsilon}}$  is bounded at infinity, we arrive at the estimate  $|\nabla^g \vartheta_s|_g \leq Cr^{-\min\{\varepsilon,1\}}$ .

Next note from the mean value theorem on D that, at height r,

$$\left| \vartheta_{s}(r,\cdot) - \int_{D} \vartheta_{s}(r,\cdot) \omega_{D}^{n-1} \right| \leq \sup_{D \times \{r\}} |\nabla^{g} \vartheta_{s}|_{g} \operatorname{diam}_{g} D \leq \frac{C}{r^{\varepsilon}},$$

and thanks to Proposition 7.27 that

(7.29) 
$$\left| X \cdot \vartheta_s(r, \cdot) - \int_D X \cdot \vartheta_s(r, \cdot) \omega_D^{n-1} \right| \le \frac{C}{r^{\varepsilon}}.$$

These inequalities we will make use of later.

Linearising  $(\star \star_s)$  around the background metric g on  $M \setminus W$ , we can write

$$(7.30) \Delta_{g,X}\vartheta_s = \int_0^1 \int_0^u |\partial\bar{\partial}\vartheta_s|_{h_{s,\tau}}^2 d\tau du, h_{s,\tau} := (1-\tau)g + \tau h_s.$$

Integrating over  $D \times \{r\}$  then yields the equation

$$\Delta_{C,X}\overline{\vartheta_s}(r) = \int_D \int_0^1 \int_0^u |\partial \bar{\partial} \vartheta_s|_{h_{s,\tau}}^2 d\tau du \,\omega_D^{n-1},$$

where we recall that

$$\overline{\vartheta_s}(r) := \int_{D \times \{r\}} \vartheta_s(r, \cdot) \omega_D^{n-1}.$$

By Corollary 7.22, we therefore have that

(7.31) 
$$0 \le \Delta_{C,X} \overline{\vartheta_s}(r) \le C \int_D |i \, \partial \bar{\partial} \vartheta_s|_g^2 \omega_D^{n-1}$$

for some uniform constant C > 0.

Now, since  $\nabla^g X = \nabla^{g,2} f = g_C$ , one gets the following pointwise estimate obtained by considering an orthonormal frame of the form  $(r^{-1}X, r^{-1}JX, (e_i, Je_i)_{1 \le i \le n-1})$ , where  $(e_i, Je_i)_{1 \le i \le n-1}$  is an orthonormal frame with respect to  $g_D$ :

$$|i\partial\bar{\partial}\vartheta_s|_g^2 \leq C|\nabla^{g,2}\vartheta_s|_g^2 \leq C\big(r^{-2}|\nabla^g(X\cdot\vartheta_s)|_g^2 + r^{-2}|\nabla^g\vartheta_s|_g^2 + |\nabla^{g_D,2}\vartheta_s|_{g_D}^2\big)$$

for some uniform positive constant C. Integrating over D, using integration by parts together with Proposition 7.27, we next derive that

(7.32) 
$$\int_{D} |i \partial \bar{\partial} \vartheta_{s}|_{g}^{2} \omega_{D}^{n-1} \leq \frac{C}{r^{4\varepsilon+2}} + \int_{D} |\nabla^{g_{D},2} \vartheta_{s}|_{g_{D}}^{2} \omega_{D}^{n-1}.$$

Now, by Bochner formula applied to  $(D, g_D)$  and the function  $\vartheta_s$ , we have that

(7.33) 
$$\Delta_{D} |\nabla^{g_{D}} \vartheta_{s}|_{g_{D}}^{2} = 2|\nabla^{g_{D},2} \vartheta_{s}|_{g_{D}}^{2} + 2\operatorname{Ric}(g_{D})(\nabla^{g_{D}} \vartheta_{s}, \nabla^{g_{D}} \vartheta_{s}) + 2g_{D}(\nabla^{g_{D}} \Delta_{D} \vartheta_{s}, \nabla^{g_{D}} \vartheta_{s}) \\ \geq 2|\nabla^{g_{D},2} \vartheta_{s}|_{g_{D}}^{2} + 2g_{D}(\nabla^{g_{D}} \Delta_{D} \vartheta_{s}, \nabla^{g_{D}} \vartheta_{s}),$$

where we have used that  $g_D$  has non-negative Ricci curvature. (Ricci curvature bounded from below would be enough to complete the argument thanks to the decay on the gradient of  $\vartheta_s$  that we have just proved above.) Integrating (7.33) on D and noticing that  $\Delta_D \vartheta_s = 2 \operatorname{tr}_{\omega_D} (i \partial \bar{\partial} \vartheta_s)$ 

then leads to the bound

$$(7.34) \qquad \int_{D} |\nabla^{g_{D},2} \vartheta_{s}|_{g_{D}}^{2} \omega_{D}^{n-1} \leq \int_{D} |\nabla^{g_{D}} \Delta_{D} \vartheta_{s}|_{g_{D}} |\nabla^{g_{D}} \vartheta_{s}|_{g_{D}} \omega_{D}^{n-1}$$

$$\leq C \sup_{D \times \{r\}} |\nabla^{g_{D}} (i \partial \bar{\partial} \vartheta_{s})|_{g_{D}} |\nabla^{g_{D}} \vartheta_{s}|_{g_{D}} \leq \frac{C}{r^{\varepsilon}},$$

where C denotes a uniform positive constant that may vary from line to line. Here we have used Proposition 7.23 and the decay on the gradient of  $\vartheta_s$  previously proved in the last line. Combining (7.31), (7.32), and (7.34), we can now infer that

$$0 \le \Delta_{C,X} \overline{\vartheta_s}(r) \le \frac{C}{r^{4\varepsilon + 2}} + \frac{C}{r^{\varepsilon}}.$$

We then have that

$$0 \le \frac{\partial}{\partial r} (e^{-\frac{r^2}{2}} X \cdot \overline{\vartheta}_s) \le C r^{1-\varepsilon} e^{-\frac{r^2}{2}}.$$

After integrating this differential inequality from r to  $r = +\infty$ , we find that

$$-C\int_{r}^{+\infty} s^{1-\varepsilon} e^{-\frac{s^2}{2}} ds \le e^{-\frac{r^2}{2}} X \cdot \overline{\vartheta}_s(r) \le 0.$$

Now,  $\int_r^{+\infty} s^{1-\varepsilon} e^{-\frac{s^2}{2}} ds \leq C r^{-\varepsilon} e^{-\frac{r^2}{2}}$  for r large enough, which can be proved using integration by parts. In particular, we have that  $-C r^{-\varepsilon} \leq X \cdot \overline{\vartheta_s}(r) \leq 0$ . Integrating once more yields the existence of a constant  $\vartheta_s^{\infty} \in \mathbb{R}$  such that  $\vartheta_s^{\infty} \leq \overline{\vartheta_s}(r) \leq \vartheta_s^{\infty} + C r^{-\varepsilon}$ . The triangle inequality applied to the oscillation estimates (7.28) and (7.29) then imply the desired estimates for  $\vartheta_s$  and  $X \cdot \vartheta_s$ , respectively.

As an intermediate step, we obtain a first rough decay estimate of the difference between the background metric and the metric resulting from the solution to  $(\star \star_s)$ . More precisely, we have the following corollary.

**Corollary 7.30.** Let  $(\vartheta_s)_{0 \le s \le 1}$  be a path of solutions in  $\mathbb{R} \oplus C^{\infty}_{X,\beta}(M)$  to  $(\star \star_s)$ . If  $\alpha \in (0, \frac{1}{2})$ , then there exist C > 0 and  $\varepsilon > 0$  such that, for all  $s \in [0, 1]$ ,

$$\|f^{\frac{\varepsilon}{2}} \cdot i \partial \bar{\partial} \vartheta_{s}\|_{C^{0,2\alpha}_{loc}} \leq C.$$

*Proof.* It suffices to prove this estimate outside a compact set W such that  $\omega_s = \omega$  on  $M \setminus W$ . To this end, let  $x \in M \setminus W$  and choose normal holomorphic coordinates in a ball  $B_g(x,\iota)$  for some  $\iota > 0$  uniform in  $x \in M$ . Let  $g_{\tau\vartheta_s}^{i\bar{\jmath}}$  denote the components of the inverse of the Kähler metric associated to  $\omega + i\partial\bar{\partial}(\tau\vartheta_s)$  in these coordinates and set

$$a^{i\bar{j}} := \int_0^1 g_{\tau\vartheta_s}^{i\bar{j}} d\tau.$$

Then we have that

$$0 = \log\left(\frac{\sigma_s^n}{\omega^n}\right) - \frac{X}{2} \cdot \vartheta_s = \int_0^1 \frac{d}{d\tau} \log\left(\frac{\omega_{\tau\vartheta_s}^n}{\omega^n}\right) d\tau - \frac{X}{2} \cdot \vartheta_s$$
$$= \left(\int_0^1 g_{\tau\vartheta_s}^{i\bar{\jmath}} d\tau\right) \partial_i \partial_{\bar{\jmath}} \vartheta_s - \frac{X}{2} \cdot \vartheta_s = a^{i\bar{\jmath}} \partial_i \partial_{\bar{\jmath}} \vartheta_s - \frac{X}{2} \cdot \vartheta_s.$$

Now, by Corollary 7.24,  $\|a^{i\bar{j}}\|_{C^{0,2\alpha}_{loc}}$  is uniformly bounded from above and  $a^{i\bar{j}} \geq \Lambda^{-1}\delta^{i\bar{j}}$  on  $B_g(x,\iota)$  for some uniform constant  $\Lambda > 0$ . Therefore, by considering  $\frac{X}{2} \cdot \vartheta_s$  as a source term,

the Schauder estimates imply that

$$\begin{split} \|\vartheta_{s} - \vartheta_{s}^{\infty}\|_{C^{2,2\alpha}(B_{g}(x,\iota/2))} &\leq C(\|X \cdot \vartheta_{s}\|_{C^{0,2\alpha}(B_{g}(x,\iota))} + \|\vartheta_{s} - \vartheta_{s}^{\infty}\|_{C^{0}(B_{g}(x,\iota))}) \\ &\leq Cf(x)^{-\frac{\varepsilon}{2}} \end{split}$$

for some uniform positive constant  $C = C(n, \alpha, \omega)$ . Here we have used Proposition 7.27 and Corollary 7.29 in the last line. The desired rough a priori decay estimate on  $i\partial\bar{\partial}\vartheta_s$  and its Hölder semi-norm now follow.

The next result proves a sharp decay at infinity on the  $C^0$ -norm of the difference between a solution to  $(\star \star_s)$  and its limit at infinity.

**Theorem 7.31.** Let  $(\vartheta_s)_{0 \le s \le 1}$  be a path of solutions in  $\mathbb{R} \oplus C^{\infty}_{X,\beta}(M)$  to  $(\star \star_s)$ . Then there exist  $R_0 > 0$  and C > 0 such that, for  $s \in [0,1]$ ,

$$|\vartheta_s - \vartheta_s^{\infty}| \le \frac{C}{f^{\frac{\beta}{2}}}, \quad f \ge R_0,$$

where  $\vartheta_s^{\infty} \in \mathbb{R}$  is as in Corollary 7.29 and  $\beta$  is as in Theorem A(v). Moreover, there exists C > 0 such that  $\|\vartheta_s\|_{\mathcal{D}_V^{2,2\alpha}} \leq C$ .

*Proof.* Linearising  $(\star \star_s)$  around g outside a compact set to obtain (7.30) and using the uniform equivalence of the metrics  $h_s$  and g given by Corollary 7.22 together with the bounds of Corollary 7.30, we obtain the improved estimate

$$0 \le \Delta_{g,X} \vartheta_s \le C r^{-2\varepsilon}$$
.

Akin to the proof of Claims 6.8 and 6.9, one estimates  $X \cdot \overline{\vartheta_s}$  and  $\vartheta_s - \overline{\vartheta_s}$  separately. Estimating the former can be reduced to an ODE which gives  $X \cdot \overline{\vartheta_s} = O(r^{-2\varepsilon})$  uniformly in  $s \in [0,1]$ , and by integrating from r to  $r = +\infty$ , we obtain  $\overline{\vartheta_s} - \vartheta_s^\infty = O(r^{-2\varepsilon})$ . The latter estimate uses the Poincaré inequality on D endowed with its metric  $g_D$ . By assumption,  $\lambda^D > \beta > 0$  is the first non-zero eigenvalue of the spectrum of the Laplacian on D, and so one has that  $\vartheta_s - \overline{\vartheta_s} = O(r^{-\min\{\beta,2\varepsilon\}})$ . Combining these two estimates, one arrives at the fact that  $\vartheta_s - \vartheta_s^\infty = O(r^{-\min\{\beta,2\varepsilon\}})$ , which is a strict improvement of Corollary 7.29, provided that  $\varepsilon < \beta$ .

Next, invoking local parabolic Schauder estimates established in [(6.10), Claim 6.7] with k = 0 applied to the linearisation of  $(\star \star_s)$  around the background metric g outside a compact set as in (7.30) yields the existence of a positive constant C such that, for  $R \ge R_0$ ,

$$\begin{split} &\|\vartheta_{s}-\vartheta_{s}^{\infty}\|_{C^{2,2\alpha}_{X,\min\{\beta,2\varepsilon\}}}\\ &\leq C(\|\vartheta_{s}-\vartheta_{s}^{\infty}\|_{C^{0}_{X,\min\{\beta,2\varepsilon\}}}+\|i\partial\overline{\partial}\vartheta_{s}\|_{C^{0,2\alpha}_{X,\min\{\beta,2\varepsilon\}}}\|i\partial\overline{\partial}\vartheta_{s}\|_{C^{0}(r\geq R)})+C(R)\\ &\leq C\|\vartheta_{s}-\vartheta_{s}^{\infty}\|_{C^{0}_{X,\min\{\beta,2\varepsilon\}}}+C\|\vartheta_{s}-\vartheta_{s}^{\infty}\|_{C^{2,2\alpha}_{X,\min\{\beta,2\varepsilon\}}}R^{-\min\{\beta,2\varepsilon\}}+C(R), \end{split}$$

where we have invoked local uniform estimates given by Propositions 7.21 and 7.23. By choosing R large enough and absorbing the relevant terms, one finds in particular that

$$\|\vartheta_{s} - \vartheta_{s}^{\infty}\|_{C^{2,2\alpha}_{X,\min\{\beta,2\varepsilon\}}} \leq C$$

for some uniform positive constant C. This implies that  $|i \partial \bar{\partial} \vartheta_s|_g = O(r^{-\min\{\beta, 2\varepsilon\}})$ .

By iterating the previous steps a finite number of times, the decay on  $\vartheta_s$  is multiplied by 2 with each iteration until it eventually reaches the threshold decay  $r^{-\beta}$ .

We now present the weighted  $C^4$ -estimate.

**Proposition 7.32** (Weighted  $C^4$  a priori estimate). Let  $(\vartheta_s)_{0 \le s \le 1}$  be a path of solutions in  $\mathbb{R} \oplus C^{\infty}_{X,\beta}(M)$  to  $(\star \star_s)$ . If  $\alpha \in (0,\frac{1}{2})$ , then there exists C > 0 such that, for all  $s \in [0,1]$ ,

*Proof.* In order to prove the a priori bound on the  $C_{X,2}^{4,2\alpha}$ -norm of  $\vartheta_s - \vartheta_s^{\infty}$ , we first establish the following uniform decay on the third derivatives of  $\vartheta_s - \vartheta_s^{\infty}$ .

**Claim 7.33.** There exists C > 0 such that, for all  $s \in [0, 1]$ ,

$$\|\nabla^g \vartheta_s\|_{C^{2,2\alpha}_{X,\beta}} \leq C.$$

In particular,

$$|\nabla^g \partial \overline{\partial} \vartheta_s|_g \leq \frac{C}{r^{\beta}}.$$

*Proof of Claim* 7.33. We differentiate the linearisation of  $(\star \star_s)$  around the background metric g outside a compact set as given in (7.30) to get schematically, on  $\{r \geq R\}$  with R sufficiently large,

(7.36) 
$$\Delta_{g,X}(\nabla^{g}\vartheta_{s}) = \nabla^{g}\vartheta_{s} + Q(\partial\bar{\partial}\vartheta_{s}, \nabla^{g}\partial\bar{\partial}\vartheta_{s}),$$

$$\|Q(\partial\bar{\partial}\vartheta_{s}, \nabla^{g}\partial\bar{\partial}\vartheta_{s})\|_{C^{0,2\alpha}_{X,\beta}} \leq C \|\nabla^{g}\partial\bar{\partial}\vartheta_{s}\|_{C^{0,2\alpha}_{X,\beta}} \|\partial\bar{\partial}\vartheta_{s}\|_{C^{0,2\alpha}_{X,\beta}},$$

$$\leq \frac{C}{R^{\beta}} \|\nabla^{g}\partial\bar{\partial}\vartheta_{s}\|_{C^{0,2\alpha}_{X,\beta}}.$$

Here we have used Theorem 7.31 in the last inequality. In particular, as in the proof of Theorem 7.31, by choosing *R* large enough and absorbing the nonlinear term on the right-hand side of (7.36), thanks to Proposition 7.26 together with Theorem 7.31, one is led to the bound

$$\|\nabla^g \vartheta_s\|_{C^{2,2\alpha}_{X,\beta}} \leq C.$$

In particular, the desired decay on  $|\nabla^g \partial \overline{\partial} \vartheta_s|_g$  holds true.

By Proposition 7.26, in order to establish (7.35), it suffices to estimate the  $C_{X,2}^{2,2\alpha}$ -norm of the right-hand side of the linearisation of  $(\star \star_s)$  around the background metric g as given in (7.30) once it is localised on  $\{r > R\}$  for R sufficiently large. As in the proof of Claim 7.33, the linearisation of  $(\star \star_s)$  around the background metric g outside a compact set as given in (7.30) gives schematically, on  $\{r > R\}$ ,

$$\begin{split} \Delta_{g,X}\vartheta_s &= Q(\partial\bar{\partial}\vartheta_s),\\ \|Q(\partial\bar{\partial}\vartheta_s)\|_{C^{2,2\alpha}_{X,\beta}} &\leq C(\|\vartheta_s - \vartheta_s^\infty\|_{C^{2,2\alpha}_{X,\beta}}^2 + \|\partial\bar{\partial}\vartheta_s\|_{C^{2,2\alpha}_{X,\beta}}\|\partial\bar{\partial}\vartheta_s\|_{C^{0,2\alpha}(r>R)}\\ &+ \|\nabla^g\partial\bar{\partial}\vartheta_s\|_{C^{0,2\alpha}_{X,\beta}}\|\nabla^g\partial\bar{\partial}\vartheta_s\|_{C^0(r>R)})\\ &\leq C(1 + R^{-\beta}\|\vartheta_s - \vartheta_s^\infty\|_{C^{4,2\alpha}_{X,\beta}} + \|\vartheta_s - \vartheta_s^\infty\|_{C^{4,2\alpha}_{X,\beta}}\|\nabla^g\partial\bar{\partial}\vartheta_s\|_{C^0(r>R)})\\ &\leq C(1 + R^{-\beta}\|\vartheta_s - \vartheta_s^\infty\|_{C^{4,2\alpha}_{X,\beta}}) \end{split}$$

for some positive uniform constant that may vary from line to line. Here we have used Theorem 7.31 in the second and third inequalities together with Claim 7.33 in the last inequality. In particular, Theorem 6.3 applied to  $\vartheta_s - \vartheta_s^{\infty}$  and k = 2 and  $\alpha \in (0, \frac{1}{2})$  gives for some constant C independent of R the following bound:

$$\|\vartheta_s - \vartheta_s^{\infty}\|_{C_{Y,\beta}^{4,2\alpha}} \le C(R) + CR^{-\beta} \|\vartheta_s - \vartheta_s^{\infty}\|_{C_{Y,\beta}^{4,2\alpha}}.$$

This yields the expected a priori estimate after absorbing the last term on the right-hand side of the previous estimates into the left-hand side.

The next proposition gives the a priori higher-order weighted estimates. Since its proof is along the same lines as that of Proposition 7.32, we omit it.

**Proposition 7.34** (Higher-order weighted estimates). Let  $(\vartheta_s)_{0 \le s \le 1}$  be a path of solutions in  $\mathbb{R} \oplus C^{2k+2,2\alpha}_{X,\beta}(M)$  to  $(\star\star_s)$  for  $k \ge 1$ . If  $\alpha \in (0,\frac{1}{2})$  and if there exists  $C_{k,\alpha} > 0$  such that, for all  $s \in [0,1]$ ,

$$\|\vartheta_{s}\|_{\mathcal{D}_{X,\beta}^{2k+2,2\alpha}} \leq C_{k,\alpha},$$

then there exists  $C_{k+1,\alpha} > 0$  such that, for all  $s \in [0, 1]$ ,

$$\|\vartheta_{s}\|_{\mathcal{D}_{X,\beta}^{2(k+1)+2,2\alpha}} \leq C_{k+1,\alpha}.$$

**7.9.** Completion of the proof of Theorem A (v). We finally prove Theorem A (v). Set

$$S := \{ s \in [0, 1] \mid \text{there exists } \psi_s \in \mathcal{M}^{\infty}_{X, \beta}(M) \text{ satisfying } (\star_s) \}.$$

Note that  $S \neq \emptyset$  since  $0 \in S$  (take  $\psi_0 = 0$ ).

We first claim that S is open. Indeed, this follows from Theorem 6.12: if  $s_0 \in S$ , then by Theorem 6.12, there exists  $\varepsilon_0 > 0$  such that, for all  $s \in (s_0 - \varepsilon_0, s_0 + \varepsilon_0)$ , there exists a solution  $\psi_s \in \mathcal{M}_{X,\beta}^{4,2\alpha}(M)$  to  $(\star_s)$  with data  $F_s \in (\mathcal{C}_{X,\beta}^{2,2\alpha}(M))_{\omega,0}$ . Since the data  $F_s$  lies in  $\mathcal{C}_{X,\beta}^{\infty}(M)$ , Theorem 6.12 ensures that, for each s in this interval,  $\psi_s \in \mathcal{M}_{X,\beta}^{\infty}(M)$ . It follows that  $(s_0 - \varepsilon_0, s_0 + \varepsilon_0) \cap [0, 1] \subseteq S$ .

We next claim that S is closed. To see this, take a sequence  $(s_k)_{k\geq 0}$  in S converging to some  $s_\infty \in S$ . Then, for  $F_k := F_{s_k}$ ,  $k \geq 0$ , the corresponding solutions  $\psi_{s_k} =: \psi_k$ ,  $k \geq 0$ , of  $(\star_s)$  satisfy

(7.37) 
$$(\omega + i\partial \bar{\partial}\psi_k)^n = e^{F_k + \frac{X}{2} \cdot \psi_k} \omega^n, \quad k \ge 0.$$

It is straightforward to check that the sequence  $(F_k)_{k\geq 0}$  is uniformly bounded in  $\mathcal{C}^{2,2\alpha}_{X,\beta}(M)$ . As a consequence, the sequence  $(\psi_k)_{k\geq 0}$  is uniformly bounded in  $\mathcal{M}^{4,2\alpha}_{X,\beta}(M)$  by Proposition 7.32. Indeed, recall the correspondence between solutions of  $(\star_s)$  and  $(\star\star_s)$ :  $\psi_k$  is a solution to  $(\star_s)$  if and only if  $\vartheta_{s_k} = \psi_{s_k} - \Phi_{s_k}$  is a solution to  $(\star\star_s)$ . The Arzelà-Ascoli theorem therefore allows us to pull out a subsequence of  $(\psi_k)_{k\geq 0}$  that converges to some  $\psi_\infty \in C^{4,2\alpha'}_{loc}(M)$ ,  $\alpha' \in (0,\alpha)$ . As  $(\psi_k)_{k\geq 0}$  is uniformly bounded in  $\mathcal{M}^{4,2\alpha}_{X,\beta}(M)$ ,  $\psi_\infty$  will also lie in  $\mathcal{M}^{4,2\alpha}_{X,\beta}(M)$ . We need to show that  $(\omega+i\partial\bar\partial\psi_\infty)(x)>0$  at every point  $x\in M$ . For this, it suffices to show that  $(\omega+i\partial\bar\partial\psi_\infty)^n(x)>0$  for every  $x\in M$ . This is seen to hold true by letting k tend to  $+\infty$  (up to a subsequence) in (7.37). The fact that  $\psi_\infty\in\mathcal{M}^\infty_{X,\beta}(M)$  follows from Proposition 7.34.

Finally, as an open and closed non-empty subset of [0, 1], connectedness of [0, 1] implies that S = [0, 1]. This completes the proof of the Theorem A (v).

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