# AN AUBIN CONTINUITY PATH FOR SHRINKING GRADIENT KÄHLER-RICCI SOLITONS

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Abstract. Let $D$ be a toric Kähler-Einstein Fano manifold. We show that any toric shrinking gradient Kähler-Ricci soliton on certain proper modifications of $\mathbb{C} \times D$ satisfies a complex Monge-Ampère equation. We then set up an Aubin continuity path to solve this equation and show that it has a solution at the initial value of the path parameter. This we do by implementing another continuity method.

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## 1. Introduction

1.1. **Overview.** A *Ricci soliton* $(M,\, g,\, X)$ is a complete Riemannian manifold $(M,\, g)$ endowed with a complete vector field $X$ such that

$$\mathrm{Ric}(g)+\frac{1}{2}\mathcal{L}_{X}g=\lambda g$$

for $\lambda\in\{-1,\, 0,\, 1\}$. A Ricci soliton is said to be *gradient* if the vector field $X$ is the gradient of a smooth real-valued function $f: M\to\mathbb{R}$ called the *soliton potential.* In this case, completeness of the soliton metric $g$ forces the completeness of $X$ [Zha09]. Furthermore, the soliton equation becomes

$$\mathrm{Ric}(g)+\mathrm{Hess}_{g}(f)=\lambda g.$$

If $g$ is in addition Kähler with Kähler form $\omega$, then we say that the triple $(M,\, \omega,\, X)$ is a *Kähler-Ricci soliton* if $\omega$ satisfies the equation

$$\rho_{\omega}+\frac{1}{2}\mathcal{L}_{X}\omega=\lambda\omega, \tag{1.1}$$

where $\rho_\omega$ is the Ricci form of $\omega$ and $\lambda$ is as above. Ricci solitons and Kähler-Ricci solitons are called *expanding*, *steady*, and *shrinking*, when $\lambda=-1,0,1$, respectively. We will only consider shrinking Kähler-Ricci solitons here, corresponding to when $\lambda=1$ in (1.1). As is the case for Ricci solitons, we say that a shrinking Kähler-Ricci soliton is *gradient* if $X=\nabla_{g}f$. Then (1.1) becomes

$$\rho_{\omega}+i\partial\bar{\partial}f=\lambda\omega.$$

Ricci solitons are interesting both from the perspective of canonical metrics and of the Ricci flow. On one hand, they represent one direction in which the concept of an Einstein manifold can be generalised. On compact manifolds, shrinking Ricci solitons are known to exist in several instances where there are obstructions to the existence of Einstein metrics; see for example [WZ04]. By the maximum principle, there are no nontrivial expanding or steady Ricci solitons on compact manifolds. However, there are many examples on noncompact manifolds; see for example [CD20, CDS19, Fut20] and the references therein. On the other hand, one can associate to a Ricci soliton a self-similar solution of the Ricci flow, and gradient shrinking Ricci solitons in particular provide models for finite-time Type I singularities along the flow [EMT11, Nab10]. Even in complex dimension two, it is not known which shrinking Ricci solitons arise in this way. From this point of view, it is an important problem to classify shrinking Ricci solitons in order to better understand the singularity development along the Ricci flow.

In this article, we are concerned with the construction of shrinking gradient Kähler-Ricci solitions, models for finite-time Type I singularities of the Kähler-Ricci flow. In essence, we set up an Aubin continuity path for a complex Monge-Ampère equation to construct such solitons in a particular geometric situation that allows control on the data of the equation. We then show that there is a solution to the equation for the initial value of the path parameter. This we do by implementing another continuity path.

1.2. **Main result.** In order to state the main result, recall that a complex toric manifold is a smooth $n$-dimensional complex manifold $D$ together with an effective holomorphic action of the complex torus $(\mathbb{C}^{*})^{n}$ with a compact fixed point set. In such a setting, there always exists an orbit $U\subset D$ of the $(\mathbb{C}^{*})^{n}$-action which is open and dense in $D$. The $(\mathbb{C}^{*})^{n}$-action of course determines the holomorphic action of a real torus $T^{n}\subset(\mathbb{C}^{*})^{n}$, as is easily seen for the action of the one-dimensional torus $\mathbb{C}^{*}$ on $\mathbb{P}^{1}$ via $\lambda\cdot[z_{0}:z_{1}]\mapsto[\lambda z_{1}:z_{2}]$. Our main result is stated as follows.

**Theorem A.** *Let $D^{n-1}$ be a toric Kähler-Einstein Fano manifold of complex dimension $n-1$ with Kähler form $\omega_D$ and Ricci form $\rho_{\omega_{D}}=\omega_{D}$, and consider $\mathbb{P}^{1}\times D$ with the induced product torus action acting by rotation on the $\mathbb{P}^{1}$-factor. Let $T^{n}$ denote the real torus acting on $\mathbb{P}^{1}\times D$,* *write $D_x := \{x\} \times D$, and let $\overline{M}$ be a toric Fano manifold obtained as a torus-equivariant (possibly iterated) blowup $\pi : \overline{M} \to \mathbb{P}^1 \times D$ along smooth torus-invariant subvarieties contained in $D_0$. Let $M := \overline{M} \setminus \pi^{-1}(D_\infty)$, $\widehat{M} := \mathbb{C} \times D$, write $J$ for the complex structure on $M$, $\mathfrak{t}$ for the Lie algebra of $T$, and let $z$ denote the holomorphic coordinate on the $\mathbb{C}$-factor of $\widehat{M}$. Then:*

(i) *There exists a unique complete vector field $JX \in \mathfrak{t}$ such that $X$ is the soliton vector field of any complete toric shrinking gradient Kähler-Ricci soliton on $M$.*

*Assume that the flow-lines of $JX$ are closed. Then:*

(ii) *There exists a complete Kähler metric $\omega$ on $M$ with $\mathcal{L}_{JX}\omega = 0$, $\lambda > 0$ uniquely determined by $X$, and a holomorphic isometry $\nu : (M \setminus K, \omega) \to (\widehat{M} \setminus \widehat{K}, \hat{\omega} := \omega_C + \omega_D)$, where $K \subset M$, $\widehat{K} \subset \widehat{M}$, are compact and $\omega_C := \frac{i}{2}\partial\bar{\partial}|z|^{2\lambda}$, such that $d\nu(X) = \frac{2}{\lambda} \cdot \mathrm{Re}\,(z\partial_z)$.*

(iii) *There exists a unique torus-invariant function $f \in C^\infty(M)$ such that $-\omega \lrcorner JX = df$. Moreover, $f = \nu^*\left(\frac{|z|^{2\lambda}}{2} - 1\right)$ and $\Delta_\omega f + f - \frac{X}{2} \cdot f = 0$ outside a compact subset of $M$ containing $K$.*

(iv) *Any shrinking gradient Kähler-Ricci soliton on $M$ invariant under the action of $T$ of the form $\omega + i\partial\bar{\partial}\varphi$ for some $\varphi \in C^\infty(M)$ with $\omega + i\partial\bar{\partial}\varphi > 0$ satisfies the complex Monge-Ampère equation*

$$(\omega + i\partial\bar{\partial}\varphi)^n = e^{F + \frac{X}{2}\cdot\varphi - \varphi}\omega^n, \tag{1.2}$$

*where $F \in C^\infty(M)$ is equal to a constant outside a compact subset of $M$ and is determined by the fact that*

$$\rho_\omega + \frac{1}{2}\mathcal{L}_X\omega - \omega = i\partial\bar{\partial}F \qquad \text{and} \qquad \int_M (e^F - 1)e^{-f}\omega^n = 0.$$

*Here, $\rho_\omega$ denotes the Ricci form of $\omega$.*

(v) *There exists a function $\psi \in C^\infty(M)$ invariant under the action of $T$ and with $\omega + i\partial\bar{\partial}\psi > 0$ such that*

$$(\omega + i\partial\bar{\partial}\psi)^n = e^{F + \frac{X}{2}\cdot\psi}\omega^n, \tag{1.3}$$

*where $\int_M \psi\, e^{-f}\omega^n = 0$ and outside a compact subset, $\psi = c_1 \log f + c_2 + \vartheta$ for some constants $c_1$, $c_2 \in \mathbb{R}$ and a smooth real-valued function $\vartheta : M \to \mathbb{R}$ satisfying*

$$|\nabla^i \mathcal{L}_X^j \vartheta|_\omega = O(f^{-\frac{\beta}{2}}) \qquad \text{for all } i,\, j \in \mathbb{N}, \quad \beta \in (0, \lambda^D).$$

*Here, $\nabla$ denotes the Levi-Civita connection associated to $\omega$ and $\lambda^D$ is the first non-zero eigenvalue of $-\Delta_D$ acting on $L^2$-functions on $D$.*

Note that since $M$ does not split off an $S^1$-factor, toricity implies that $M$ has finite fundamental group [CLS11], a necessary condition for the existence of a shrinking gradient Kähler-Ricci soliton [Wyl08]. Part (i) of the theorem determines the soliton vector field of any complete toric shrinking Kähler-Ricci soliton on $M$. This follows from previous results. Indeed, in [CDS19] it was shown that the soliton vector field in the Lie algebra of a torus is unique.

**Theorem 1.1** ([CDS19, Theorem D]). *Let $M$ be a non-compact complex manifold with complex structure $J$ endowed with the effective holomorphic action of a real torus $T$. Denote by $\mathfrak{t}$ the Lie algebra of $T$. Then there exists at most one element $\xi \in \mathfrak{t}$ that admits a complete shrinking gradient Kähler-Ricci soliton $(M, g, X)$ with bounded Ricci curvature with $X = \nabla^g f = -J\xi$ for a smooth real-valued function $f$ on $M$.*

The vector field $\xi$ is characterised by the fact that it is the point in a certain convex subset of $\mathfrak{t}$ at which a certain convex functional attains its minimum, if such a minimum is achieved. More precisely, for a non-compact anti-canonically polarised Kähler manifold $M$ [CDS19, Definition 7.1] with $H^1(M) = 0$ endowed with the effective holomorphic action of a real torus $T$ with Lie algebra $\mathfrak{t}$ and with compact fixed point set $\mathrm{Fix}(T)$, assume the existence of a complete shrinking gradient Kähler-Ricci soliton $\omega$ on $M$ invariant under the action of $T$ with complex structure $J$ and soliton vector field $X$. Then the action of $T$ is Hamiltonian and there exists a strictly convex functional

$\mathcal{F}_\omega : \Lambda_\omega \to \mathbb{R}_{>0}$, the "weighted volume functional" [CDS19, Definition 5.16], defined on an open convex cone $\Lambda_\omega \subset \mathfrak{t}$ uniquely determined by the image of $M$ under the moment map defined by the action of $T$ and the choice of $\omega$ [PW94, Proposition 1.4] and well-defined by the non-compact version of the Duistermaat-Heckman formula [PW94] (see also [CDS19, Theorem A.3]). By convexity, $\mathcal{F}_\omega$ attains at most one minimum $\xi \in \Lambda_\omega$ [CDS19, Lemma 5.17(i)] and under the assumption of bounded Ricci curvature, this minimum is achieved at $JX$ [CDS19, Lemma 5.17(ii)], leading to the statement of Theorem 1.1.

If $T$ in the above provides a full-dimensional torus symmetry so that $M$ is *toric* and $\omega$ is invariant under the torus action, then the domain $\Lambda_\omega$ of $\mathcal{F}_\omega$ and $\mathcal{F}_\omega$ itself only depend upon the torus action, so that both are independent of the choice of $\omega$. In addition, dropping the subscripts $\omega$, $\mathcal{F}$ is known to be proper on $\Lambda$ [Cif21, Proposition 3.1], hence attains a unique minimum. This minimum therefore defines a distinguished point in $\mathfrak{t}$, namely the only vector field in $\mathfrak{t}$ that can admit a complete toric shrinking gradient Kähler-Ricci soliton. This is the vector field of Theorem A(i). Since everything is explicit and can be computed from the torus action, one can a priori determine this vector field for a given $M$; see for example [CDS19, Section A.4].

Parts (ii) and (iii) give a reference metric on $M$ that is isometric to a model shrinking gradient Kähler-Ricci soliton outside a compact set. This requires the assumption that the flow-lines of $JX$ are closed. Indeed, this is the case for the soliton vector field on the model. With respect to this background metric, part (iv) gives a complex Monge-Ampère equation (1.2) that any complete toric shrinking Kähler-Ricci soliton on $M$ must satisfy with control on the asymptotics of the data $F$ of the equation. By [Cif21], we know that there is at most one such soliton on $M$ and we expect that this equation has a solution, resulting in a complete toric shrinking Kähler-Ricci soliton on $M$. Such a soliton should model finite time collapsing behaviour of the Kähler-Ricci flow in order to be consistent with [TZ18]. One may attempt to solve (1.2) by implementing the Aubin continuity method that was introduced for Kähler-Einstein manifolds [Aub98, Section 7.26]. Precisely in our case one may consider the path

$$\begin{cases} (\omega + i\partial\bar{\partial}\varphi_t)^n = e^{F + \frac{X}{2}\cdot\varphi_t - t\varphi_t}\omega^n, \qquad \varphi \in C^\infty(M), \quad \mathcal{L}_{JX}\varphi = 0, \quad \omega + i\partial\bar{\partial}\varphi > 0, \quad t \in [0, 1], \\ \int_M e^{F-f}\omega^n = \int_M e^{-f}\omega^n. \end{cases} \tag{$*_t$}$$

The main content of Theorem A is part (v) where we provide a solution to the equation corresponding to $t = 0$. This we do by implementing another continuity path. In the compact case, this was achieved by Zhu [Zhu00]. Further discussion involving solutions of $(*_t)$ will appear in the forthcoming [CCD].

The principal motivation behind Theorem A is that it provides a first step in the construction, via the continuity method, of a complete shrinking gradient Kähler-Ricci soliton with bounded scalar curvature on the blowup of $\mathbb{C} \times \mathbb{P}^1$ at one point, a manifold identified in [CCD22, Conjecture 1.1] as a potential new example admitting such a soliton. As shown in [CCD22], a soliton on this manifold with bounded scalar curvature must be invariant under the standard torus action, and even though the conditions on the potential soliton vector field $X$ of Theorem A are stringent, they are evidently true for the manifold in question as one can check in [CCD22, Example 2.32]. This example therefore fits into the framework of Theorem A, precisely with $D = \mathbb{P}^1$ and $\pi$ the blowup map. A solution of (1.2) for this example would complete the classification of two-dimensional complete shrinking gradient Kähler-Ricci solitons with bounded scalar curvature and in turn would identify the candidate shrinking solitons that can appear as blowup models of finite time Type I singularities of the Kähler-Ricci flow in this dimension.

Equation (1.2) a priori looks identical to the complex Monge-Ampère equation solved in [CD20], where complete steady gradient Kähler-Ricci solitons were constructed. Even though the equations appear the same and the same continuity path is used in both cases, there are important differences in the structure of both that result in several additional difficulties arising in the solution of (1.2) in contrast to the equation of [CD20]. We conclude this section by highlighting some of these differences.

- On a closed Kähler manifold, the $X$-derivative of any Kähler potential is bounded prior to any other bound; see [Zhu00]. This fact does not seem to be amenable to an arbitrary noncompact Kähler manifold and represents one of the major obstacles to adapting Tian and Zhu's work [TZ00a] to our current setting. For us, not only is the drift operator $X$ of (1.2) unbounded, in contrast to [CD20] where it is bounded, but it also has the opposite sign. This prevents us from adapting the proof of the $C^0$ a priori estimate in [CD20].
- In [Zhu00], a generalization of Calabi's conjecture is proved using a continuity path that shrinks the hypothetical soliton vector field $X$ to 0, thereby reducing the existence to Yau's original solution of the Calabi conjecture [Yau78]. In our setting, implementing such a continuity method does not preserve the weighted volume and in fact it diverges at the initial time. This explains why we work with the Aubin continuity path.
- In [CD20], the corresponding equation was solved using the continuity path with exponentially weighted function spaces. Here, we solve in polynomially weighted function spaces. The main issue comes from the fact that the linearized operator contains logarithmically growing functions in its kernel at infinity. This makes the linear theory more delicate than in the previous work [CD20].
- In obtaining an a priori $C^0$-estimate for (1.2), the toricity assumption is crucial. This was not the case in [CD20] were no toricity was required. However, a priori weighted $L^p$ estimates on the solution of (1.2) are obtained *without* requiring toricity. This applies too to the a priori estimates except for the one concerning a *lower* bound on the solution. This will all be made clear in the relevant statements.
- The order in which we obtain the a priori estimates is different to that of [CD20]. Here we first obtain an a priori *lower* bound on the radial derivative of the solution. This then allows us to derive an a priori *upper* bound on the solution. The next step is to derive an a priori *lower* bound on the solution. At this stage, we can follow the same strategy as that of [CD20] to obtain a priori *local* estimates on the solution.
- Finally, in addition to containing logarithmically growing functions, the kernel of the linearized operator contains constants, a fact that makes the a priori weighted estimate of the difference of the solution and its value at infinity more subtle in a nonlinear setting. To circumvent this issue, we apply the Bochner formula to the $X$-derivative of our solution with respect to the unknown Kähler metric.
- Our geometric setting bears some resemblance to the work [HHN15] on asymptotically cylindrical Calabi-Yau metrics. However, in the context of metric measure spaces, our setting is somewhat dissimilar to theirs, hence we take an alternative approach to obtain (weighted) a priori estimates.

1.3. **Outline of paper.** We begin in Section 2.1 by recalling the basics of shrinking Ricci and Kähler-Ricci solitons. Some important examples are discussed as well as some technical lemmas proved. We also recall the definition of a metric measure space in Section 2.2. In Section 2.3, we digress and mention some basics of polyhedrons and polyhedral cones that we need before moving on to some relevant information concerning Hamiltonian actions in Section 2.4. Section 2.5 then comprises the background material on toric geometry that we require.

In Section 3, we construct a background metric with the desired properties in Section 3, resulting in the proof of Theorem A(ii). Next, the complex Monge-Ampère equation is set-up and the normalisation of the Hamiltonian of $JX$ is obtained in Section 4, leading to the proof of Theorem A(iii)–(iv). Our background metric is isometric to a shrinking gradient Kähler-Ricci soliton compatible with $X$ outside a compact set. This is what allows us to set-up the complex Monge-Ampère equation with compactly supported data.

From Section 5 onwards, the content takes on a more analytic flavour with the proof of Theorem A(v) taking up Sections 5–7. To prove this part of Theorem A, we implement the continuity method. The specific continuity path is outlined at the beginning of Section 7 but beforehand, in Section 5, we prove a Poincaré inequality which is the content of Proposition 5.1, essential to deriving the a priori weighted energy estimate for the complex Monge-Ampère equation (1.2) with compactly supported data.

Then in Section 6, we study the properties of the drift Laplacian of our background metric acting on polynomially weighted function spaces. More precisely, we introduce polynomially weighted function spaces whose elements are invariant under the flow of $JX$ in Section 6.2 and in Section 6.3 we show that the drift Laplacian of our background metric is an isomorphism between such spaces. This latter result is the content of Theorem 6.3. Using it, we then prove Theorem 6.12 that serves as the openness part of the continuity method. The closedness part of the continuity method involves a priori estimates and these make up Section 7.

As noted previously, the presence of the unbounded vector field $X$ makes the analysis more difficult. An a priori lower bound for the radial derivative $X \cdot \psi$, where $\psi$ solves (1.2), have to be proved *before* the a priori $C^0$ bound in order to avoid a circular argument; see Section 7.4. A priori energy estimates are obtained in Section 7.5 through the use of so-called Aubin-Tian-Zhu's functionals and result in an a priori upper bound on a solution to the complex Monge-Ampère equation (1.2); cf. Proposition 7.11. As explained above, the invariance of the solution under the whole action torus is crucial in obtaining an a priori lower bound on the infimum; cf. Proposition 7.15. Then and only then an a priori upper bound on the radial derivative of a solution to (1.2) is derived; cf. Proposition 7.6. Section 7.7 is devoted to proving a local bootstrapping phenomenon for the complex Monge-Ampère equation (1.2). Finally, Section 7.8 takes care of establishing a priori weighted estimates at infinity for (1.2), leading to the completion of the proof of Theorem A(v) in Section 7.9.

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## 2. Preliminaries

### 2.1. Shrinking Ricci solitons.

The metrics we are interested in are the following.

**Definition 2.1.** A *shrinking Ricci soliton* is a triple $(M,\, g,\, X)$, where $M$ is a Riemannian manifold endowed with a complete Riemannian metric $g$ and a vector field $X$ satisfying the equation

$$\mathrm{Ric}(g) + \frac{1}{2}\mathcal{L}_X g = \frac{1}{2}g. \tag{2.1}$$

We call $X$ the *soliton vector field* and say that $(M,\, g,\, X)$ is a *gradient* Ricci soliton if $X = \nabla^g f$ for some real-valued smooth function $f$ on $M$. In this latter case, equation (2.1) reduces to

$$\mathrm{Ric}(g) + \mathrm{Hess}_g(f) = \frac{1}{2}g, \tag{2.2}$$

where $\mathrm{Hess}_g$ denotes the Hessian with respect to $g$.

If $g$ is complete and Kähler with Kähler form $\omega$, then we say that $(M,\, g,\, X)$ is a *shrinking gradient Kähler-Ricci soliton* if $X = \nabla^g f$ for some real-valued smooth function $f$ on $M$, $X$ is complete and real holomorphic, and

$$\rho_\omega + i\partial\bar{\partial}f = \omega, \tag{2.3}$$

where $\rho_\omega$ is the Ricci form of $\omega$. For gradient Ricci solitons and gradient Kähler-Ricci solitons, the function $f$ satisfying $X = \nabla^g f$ is called the *soliton potential*.

An important class of examples for us is the following.

**Example 2.2.** We have a 1-parameter family $\{\tilde{\omega}_a\}_{a>0}$ of (in-complete) shrinking gradient Kähler-Ricci soliton on $\mathbb{C}$. Indeed for each $a > 0$ , the Kähler form of the shrinking soliton is defined by $\tilde{\omega}_a := \frac{i}{2}\partial\bar{\partial}|z|^{2a}$, where $z$ is the holomorphic coordinate on $\mathbb{C}$. The soliton vector field of $\tilde{\omega}_a$ is given by $\frac{2}{a}\cdot\operatorname{Re}(z\partial_z)$. Of course when $a = 1$, $\tilde{\omega}_a$ is complete and we recover the flat shrinking soliton $\omega_{\mathbb{C}}$ on $\mathbb{C}$ with soliton vector field $2\cdot\operatorname{Re}(z\partial_z)$.

Any Kähler-Einstein manifold trivially defines a shrinking gradient Kähler-Ricci solition (with soliton vector field $X = 0$). We may then take the Cartesian product with Example 2.2 to produce many more examples. These examples provide the model at infinity for the reference metric that we will construct in Theorem A(i).

**Example 2.3.** Let $(D,\, \omega_D)$ be a Kähler-Einstein Fano manifold with Kähler form $\omega_D$. Then for each $a > 0$, the Cartesian product $\widehat{M} := \mathbb{C}\times D$ endowed with the Kähler form $\widehat{\omega}_a := \tilde{\omega}_a + \omega_D$ is an example of an (incomplete) shrinking gradient Kähler-Ricci soliton. Here, $\tilde{\omega}_a$ is as in Example 2.2. Writing $r := |z|^a$ with $z$ the complex coordinate on the $\mathbb{C}$-factor of $\widehat{M}$, the soliton vector field of this example is given by $\widehat{X} := r\partial_r = \frac{2}{a}\cdot\operatorname{Re}(z\partial_z)$. When $a = 1$, the soliton is complete and up to isometry, we obtain a complete shrinking gradient Kähler-Ricci soliton on $\mathbb{C}\times D$ with bounded scalar curvature which is unique if $D$ is moreover toric [Cif21, Corollary C]. We write $\widehat{g}_a$ and $\widehat{J}$ for the Kähler metric associated to $\widehat{\omega}_a$ and product complex structure on $\widehat{M}$ respectively.

The following lemmas concerning $(\widehat{M},\, \widehat{\omega}_a)$ will prove useful throughout.

**Lemma 2.4.** *With notation as in Example 2.3, fix $a > 0$ (and hence the function $r$) and let $\widehat{K}\subset\widehat{M}$ be a compact subset such that $\widehat{M}\setminus\widehat{K}$ is connected. If $u : \widehat{M}\setminus\widehat{K}\to\mathbb{R}$ is a smooth real-valued function defined on $\widehat{M}\setminus\widehat{K}$ that is pluriharmonic (meaning that $\partial\bar{\partial}u = 0$) and invariant under the flow of $\widehat{J}X$, then $u = c_0\log(r) + c_1$ for some $c_0,\, c_1\in\mathbb{R}$.*

*Proof.* Let $\widehat{X}^{1,0} := \frac{1}{2}(\widehat{X} - i\widehat{J}\widehat{X})$. Then since $\widehat{X}$ is real holomorphic and $\mathcal{L}_{\widehat{J}\widehat{X}}u = 0$, we see that

$$\bar{\partial}(\widehat{X}\cdot u) = \partial\bar{\partial}u\lrcorner(\widehat{X}^{1,0}) = 0,$$

i.e., $\widehat{X}\cdot u$ is holomorphic. As a real-valued holomorphic function, $\widehat{X}\cdot u$, which itself is equal to $r\partial_r u$, must be equal to a constant, $c_0$ say. Thus,

$$u = c_0\log r + c_1(x,\, \theta),$$

where $x\in D$ and for $z\in\mathbb{C}$, we write $z = \rho e^{i\theta}$ so that $\rho = |z|$. Let $\Delta_{\mathbb{C}}$ and $\Delta_D$ denote the Riemannian Laplacians with respect to the flat metric $g_{\mathbb{C}}$ on $\mathbb{C}$ and the Kähler-Einstein metric $\omega_D$ on $D$ respectively. Then $u$, being pluriharmonic, implies that $\Delta_{\mathbb{C}}u + \Delta_D u = 0$. Hence

$$\begin{aligned} 0 &= (\Delta_D + \Delta_{\mathbb{C}})(c_0\log(r) + c_1(\theta,\, x)) \\ &= \Delta_D c_1(x,\, \theta) + \Delta_{\mathbb{C}}c_1(x,\, \theta) + c_0\underbrace{\Delta_{\mathbb{C}}\log(r)}_{=0} \\ &= \Delta_D c_1(x,\, \theta) + \frac{1}{\rho^2}\frac{\partial^2 c_1}{\partial\theta^2}(x,\, \theta), \end{aligned}$$

leading to

$$\rho^2\Delta_D c_1(x,\, \theta) = -\frac{\partial^2 c_1}{\partial\theta^2}(x,\, \theta).$$

Clearly this is only possible if

$$\Delta_D c_1(x,\, \theta) = \frac{\partial^2 c_1}{\partial\theta^2}(x,\, \theta) = 0.$$

Thus, $c_1(x,\theta)=c_1(\theta)$ with the second equation then inferring that $c_1(\theta)=c_1$. This leaves us with $u=c_0\log(r)+c_1$, as desired. $\square$

We conclude this section with a gluing lemma.

**Lemma 2.5** (Gluing lemma). *With notation as in Example 2.3, fix $a>0$ (and hence the function $r$), let $\widehat{K}\subset\widehat{M}$ be a compact subset, and let $\phi\in C^\infty(\widehat{M}\setminus\widehat{K})$ be such that $\phi=O(\log(r))$, $|d\phi|_{\widehat{g}_a}=O(1)$, and $|i\partial\bar\partial\phi|_{\widehat{g}_a}=O(r^{-a})$. Then for all $R>0$ with $\widehat{K}\subseteq\{r\leq R\}$, there exists a cut-off function $\chi_R:M\to\mathbb{R}$ supported on $M\setminus\{r\leq R\}$ with $\chi_R(x)=1$ if $r(x)>2R$ such that*

$$|i\partial\bar\partial(\chi_R\cdot\phi)|_{\widehat{g}_a}\leq\frac{C}{R^{\min\{1,\,a\}}}\left(\|(\log(r))^{-1}\cdot\phi\|_{C^0(\widehat{M}\setminus\widehat{K})}+\|d\phi\|_{C^0(\widehat{M}\setminus\widehat{K},\,\widehat{g}_a)}+\|r^a\cdot i\partial\bar\partial\phi\|_{C^0(\widehat{M}\setminus\widehat{K},\,\widehat{g}_a)}\right)$$

*for some $C>0$ independent of $R$. In particular, $\chi_R\cdot\phi=\phi$ on $\{r(x)>2R\}$.*

*Proof.* Let $\chi:\mathbb{R}\to\mathbb{R}$ be a smooth function satisfying $\chi(x)=0$ for $x\leq1$, $\chi(x)=1$ for $x\geq4$, and $|\chi(x)|\leq1$ for all $x$, and with it, define a function $\chi_R:M\to\mathbb{R}$ by

$$\chi_R(x)=\chi\left(\frac{r(x)^2}{R^2}\right)\qquad\text{for } R>0 \text{ as in the statement of the lemma.}$$

Then $\chi_R$ is identically zero on $\{x\in\widehat{M}:r(x)<R\}$ and identically equal to one on the set $\{x\in\widehat{M}:r(x)>2R\}$. Define $\phi_R:=\chi_R.\phi$. Then the closed real $(1,1)$-form $i\partial\bar\partial(\chi_R.\phi)$ on $\widehat{M}$ is given by

$$\begin{aligned}i\partial\bar\partial(\chi_R.\phi)=\chi_R(r).i\partial\bar\partial\phi+\chi'\left(\frac{r^2}{R^2}\right).i\frac{\partial r^2}{R}\wedge\frac{\bar\partial\phi}{R}+\frac{\phi}{R^2}.\chi'\left(\frac{r^2}{R^2}\right).i\partial\bar\partial r^2\\+\chi'\left(\frac{r^2}{R^2}\right).\frac{i\partial\phi}{R}\wedge\frac{\bar\partial r^2}{R}+\frac{\phi}{R^2}.\chi''\left(\frac{r^2}{R^2}\right).i\frac{\partial r^2}{R}\wedge\frac{\bar\partial r^2}{R}.\end{aligned}$$

The assumptions on $\phi$ and its derivatives then imply for example that

$$|\chi_R(x).i\partial\bar\partial\phi|_{\widehat{g}_a}\leq\sup_{r\in[R,\infty)}|i\partial\bar\partial\phi|_{\widehat{g}_a}\leq\left(\sup_{r\in[R,\infty)}r^{-a}\right)\left(\sup_{r\in[R,\infty)}r^a\cdot|i\partial\bar\partial\phi|_{\widehat{g}_a}\right)\leq R^{-a}\|r^a\cdot i\partial\bar\partial\phi\|_{C^0(\widehat{M}\setminus\widehat{K},\,\widehat{g}_a)}$$

and that

$$\left|\chi'\left(\frac{r^2}{R^2}\right).i\frac{\partial r^2}{R}\wedge\frac{\bar\partial\phi}{R}\right|_{\widehat{g}_a}\leq\frac{C}{R^2}\left(\sup_{r\in[R,2R]}r\right)\left(\sup_{r\in[R,2R]}\left|i\partial r\wedge\bar\partial\phi\right|_{\widehat{g}_a}\right)\leq CR^{-1}\|d\phi\|_{C^0(\widehat{M}\setminus\widehat{K},\,\widehat{g}_a)}.$$

The estimate of the lemma is now clear. $\square$

2.2. **Basics of metric measure spaces.** We take the following from [Fut15].

A smooth metric measure space is a Riemannian manifold endowed with a weighted volume.

**Definition 2.6.** A *smooth metric measure space* is a triple $(M,g,e^{-f}dV_g)$, where $(M,g)$ is a complete Riemannian manifold with Riemannian metric $g$, $dV_g$ is the volume form associated to $g$, and $f:M\to\mathbb{R}$ is a smooth real-valued function.

A shrinking gradient Ricci soliton $(M,g,X)$ with $X=\nabla^g f$ naturally defines a smooth metric measure space $(M,g,e^{-f}dV_g)$. On such a space, we define the weighted Laplacian $\Delta_f$ by

$$\Delta_f u:=\Delta u-g(\nabla^g f,\nabla u)$$

on smooth real-valued functions $u\in C^\infty(M,\mathbb{R})$. There is a natural $L^2$-inner product $\langle\cdot,\cdot\rangle_{L^2_f}$ on the space $L^2_f$ of square-integrable smooth real-valued functions on $M$ with respect to the measure $e^{-f}dV_g$ defined by

$$\langle u,v\rangle_{L^2_f}:=\int_M uv\,e^{-f}dV_g,\qquad u,v\in L^2_f.$$

As one can easily verify, the operator $\Delta_f$ is self-adjoint with respect to $\langle\cdot,\cdot\rangle_{L^2_f}$.

In the Kähler case, we have:

**Definition 2.7.** If $(M,\, g,\, e^{-f}dV_g)$ is a smooth metric measure space and $(M,\, g)$ is Kähler, then we say that $(M,\, g,\, e^{-f}dV_g)$ is a *Kähler metric measure space.*

A shrinking gradient Kähler-Ricci soliton naturally defines such a space.

Unlike the real case, on a Kähler metric measure space we have the weighted $\bar{\partial}$-Laplacian $\Delta_f$ defined on smooth complex-valued functions $u \in C^\infty(M,\, \mathbb{C})$ by

$$\Delta_f u := \Delta_{\bar{\partial}} u - (\nabla^{1,0} u) f = g^{i\bar{\jmath}} \partial^2_{i\bar{\jmath}} u - g^{i\bar{\jmath}} (\partial_i f)(\partial_{\bar{\jmath}} u).$$

This may be a complex-valued function even if $u$ is real-valued. We define a hermitian inner product on the space $C_0^\infty(M,\, \mathbb{C})$ by

$$\langle u,\, v \rangle_{L^2_f} := \int_M u\bar{v}\, e^{-f} dV_g, \qquad u,\, v \in C_0^\infty(M,\, \mathbb{C}).$$

Then $\Delta_f$ is symmetric with respect to this inner product. In fact, we have that

$$\int_M (\Delta_f u) \bar{v}\, e^{-f} dV_g = \int_M u \overline{\Delta_f v}\, e^{-f} dV_g = -\int_M g(\bar{\partial} u, \bar{\partial} v)\, e^{-f} dV_g = -\langle \bar{\partial} u, \bar{\partial} v \rangle_{L^2_f},$$

where

$$g(\bar{\partial} u, \bar{\partial} v) = g^{i\bar{\jmath}} (\partial_{\bar{\jmath}} u)(\partial_i \bar{v}).$$

See [Fut15] and the references therein for further details.

2.3. **Polyhedrons and polyhedral cones.** We take the following from [CLS11] and [PW94, Appendix A].

Let $E$ be a real vector space of dimension $n$ and let $E^*$ denote the dual. Write $\langle \cdot\,,\, \cdot \rangle$ for the evaluation $E^* \times E \to \mathbb{R}$. Furthermore, assume that we are given a *lattice* $\Gamma \subset E$, that is, an additive subgroup $\Gamma \simeq \mathbb{Z}^n$. This gives rise to a dual lattice $\Gamma^* \subset E^*$. For any $\nu \in E$, $c \in \mathbb{R}$, let $K(\nu,\, c)$ be the (closed) half space $\{x \in E \mid \langle \nu,\, x \rangle \geq c\}$ in $E$. Then we have:

**Definition 2.8.** A *polyhedron* $P$ in $E$ is a finite intersection of half spaces, i.e.,

$$P = \bigcap_{i=1}^{r} K(\nu_i,\, c_i) \qquad \text{for } \nu_i \in E^*,\ c_i \in \mathbb{R}.$$

It is called a *polyhedral cone* if all $c_i = 0$, and moreover a *rational polyhedral cone* if all $\nu_i \in \Gamma^*$ and $c_i = 0$. In addition, a polyhedron is called *strongly convex* if it does not contain any affine subspace of $E$.

The following definition will be useful.

**Definition 2.9.** A polyhedron $P \subset E^*$ is called *Delzant* if its set of vertices is non-empty and each vertex $v \in P$ has the property that there are precisely $n$ edges $\{e_1, \ldots e_n\}$ (one-dimensional faces) emanating from $v$ and there exists a basis $\{\varepsilon_1, \ldots, \varepsilon_n\}$ of $\Gamma^*$ such that $\varepsilon_i$ lies along the ray $\mathbb{R}(e_i - v)$.

Note that any such $P$ is necessarily strongly convex.

The asymptotic cone of a polyhedron contains all the directions going off to infinity in the polyhedron.

**Definition 2.10.** Let $P$ be a polyhedron in $E$. Its *asymptotic cone*, denoted by $\mathcal{C}(P)$, is the set of vectors $\alpha \in E$ with the property that there exists $\alpha^0 \in E$ such that $\alpha^0 + t\alpha \in P$ for sufficiently large $t > 0$.

The asymptotic cone may be identified as follows.

**Lemma 2.11** ([PW94, Lemma A.3])**.** *If* $P = \bigcap_{i=1}^{r} K(\nu_i,\, c_i)$*, then* $\mathcal{C}(P) = \bigcap_{i=1}^{r} K(\nu_i,\, 0)$*.*

In particular, the asymptotic cone of a polyhedron is a polyhedral cone. We also have:

**Definition 2.12.** The *dual* of a polyhedral cone $C$ is the set $C^\vee = \{x \in E^* \mid \langle x,\, C \rangle \geq 0\}$.

2.4. **Hamiltonian actions.** Recall what it means for an action to be Hamiltonian.

**Definition 2.13.** Let $(M,\,\omega)$ be a symplectic manifold and let $T$ be a real torus acting by symplectomorphisms on $(M,\,\omega)$. Denote by $\mathfrak{t}$ the Lie algebra of $T$ and by $\mathfrak{t}^*$ its dual. Then we say that the action of $T$ is *Hamiltonian* if there exists a smooth map $\mu_\omega : M \to \mathfrak{t}^*$ such that for all $\zeta \in \mathfrak{t}$,

$$-\omega\lrcorner\zeta = du_\zeta,$$

where $u_\zeta(x) = \langle \mu_\omega(x),\, \zeta\rangle$ for all $\zeta \in \mathfrak{t}$ and $x \in M$ and $\langle \cdot\,, \cdot\rangle$ denotes the dual pairing between $\mathfrak{t}$ and $\mathfrak{t}^*$. We call $\mu_\omega$ the *moment map* of the $T$-action and we call $u_\zeta$ the *Hamiltonian (potential)* of $\zeta$.

Define

$$\Lambda_\omega := \{Y \in \mathfrak{t} \mid \mu_\omega(Y) \text{ is proper and bounded below}\} \subseteq \mathfrak{t}.$$

This set can be identified through the image of $\mu_\omega$ in the following way.

**Proposition 2.14** ([PW94, Proposition 1.4])**.** *Let $(M,\,\omega)$ be a (possibly non-compact) symplectic manifold of real dimension $2n$ with symplectic form $\omega$ on which there is a Hamiltonian action of a real torus $T$ with moment map $\mu_\omega : M \to \mathfrak{t}^*$, where $\mathfrak{t}$ is the Lie algebra of $T$ and $\mathfrak{t}^*$ its dual. Assume that the fixed point set of $T$ is compact and that $\Lambda_\omega \neq \emptyset$. Then $\Lambda_\omega = \operatorname{int}(\mathcal{C}(\mu_\omega(M))^\vee)$.*

2.5. **Toric geometry.** In this section, we collect together some standard facts from toric geometry as well as recall those results from [Cif21] that we require. We begin with the following definition.

**Definition 2.15.** A *toric manifold* is an $n$-dimensional complex manifold $M$ endowed with an effective holomorphic action of the algebraic torus $(\mathbb{C}^*)^n$ such that the following hold true.

- The fixed point set of the $(\mathbb{C}^*)^n$-action is compact.
- There exists a point $p \in M$ with the property that the orbit $(\mathbb{C}^*)^n \cdot p \subset M$ forms a dense open subset of $M$.

We will often denote the dense orbit simply by $(\mathbb{C}^*)^n \subset M$ in what follows. The $(\mathbb{C}^*)^n$-action of course determines the action of the real torus $T^n \subset (\mathbb{C}^*)^n$.

2.5.1. *Divisors on toric varieties and fans.* Let $T^n \subset (\mathbb{C}^*)^n$ be the real torus with Lie algebra $\mathfrak{t}$ and denote the dual pairing between $\mathfrak{t}$ and the dual space $\mathfrak{t}^*$ by $\langle\cdot\,,\cdot\rangle$. There is a natural integer lattice $\Gamma \simeq \mathbb{Z}^n \subset \mathfrak{t}$ comprising all $\lambda \in \mathfrak{t}$ such that $\exp(\lambda) \in T^n$ is the identity. This then induces a dual lattice $\Gamma^* \subset \mathfrak{t}^*$. We have the following combinatorial definition.

**Definition 2.16.** A *fan* $\Sigma$ in $\mathfrak{t}$ is a finite set of rational polyhedral cones $\sigma$ satisfying:

(i) For every $\sigma \in \Sigma$, each face of $\sigma$ also lies in $\Sigma$.
(ii) For every pair $\sigma_1, \sigma_2 \in \Sigma$, $\sigma_1 \cap \sigma_2$ is a face of each.

To each fan $\Sigma$ in $\mathfrak{t}$, one can associate a toric variety $X_\Sigma$. Heuristically, $\Sigma$ contains all the data necessary to produce a partial equivariant compactification of $(\mathbb{C}^*)^n$, resulting in $X_\Sigma$. More concretely, one obtains $X_\Sigma$ from $\Sigma$ as follows. For each $n$-dimensional cone $\sigma \in \Sigma$, one constructs an affine toric variety $U_\sigma$ which we first explain. We have the dual cone $\sigma^\vee$ of $\sigma$. Denote by $S_\sigma$ the semigroup of those lattice points which lie in $\sigma^\vee$ under addition. Then one defines the semigroup ring, as a set, as all finite sums of the form

$$\mathbb{C}[S_\sigma] = \left\{ \sum \lambda_s s \;\middle|\; s \in S_\sigma \right\},$$

with the ring structure defined on monomials by $\lambda_{s_1}s_1 \cdot \lambda_{s_2}s_2 = (\lambda_{s_1}\lambda_{s_2})(s_1 + s_2)$ and extended in the natural way. The affine variety $U_\sigma$ is then defined to be $\operatorname{Spec}(\mathbb{C}[S_\sigma])$. This automatically comes endowed with a $(\mathbb{C}^*)^n$-action with a dense open orbit. This construction can also be applied to the lower dimensional cones $\tau \in \Sigma$. If $\sigma_1 \cap \sigma_2 = \tau$, then there is a natural way to map $U_\tau$ into $U_{\sigma_1}$ and $U_{\sigma_2}$ isomorphically. One constructs $X_\Sigma$ by declaring the collection of all $U_\sigma$ to be an open affine cover of $X_\Sigma$ with transition functions determined by $U_\tau$. This identification is also reversible.

**Proposition 2.17** ([CLS11, Corollary 3.1.8]). *Let $M$ be a smooth toric manifold. Then there exists a fan $\Sigma$ such that $M \simeq X_{\Sigma}$.*

**Proposition 2.18** ([CLS11, Theorem 3.2.6], Orbit-Cone Correspondence). *The $k$-dimensional cones $\sigma \in \Sigma$ are in a natural one-to-one correspondence with the $(n-k)$-dimensional orbits $O_\sigma$ of the $(\mathbb{C}^*)^n$-action on $X_\Sigma$.*

In particular, each ray $\sigma \in \Sigma$ determines a unique torus-invariant divisor $D_\sigma$. As a consequence, a torus-invariant Weil divisor $D$ on $X_\Sigma$ naturally determines a polyhedron $P_D \subset \mathfrak{t}^*$. Indeed, we can decompose $D$ uniquely as $D = \sum_{i=1}^N a_i D_{\sigma_i}$, where $\{\sigma_i\}_i \subset \Sigma$ is the collection of rays. Then by assumption, there exists a unique minimal lattice element $\nu_i \in \sigma_i \cap \Gamma$. $P_D$ is then given by

$$P_D = \{x \in \mathfrak{t}^* \mid \langle \nu_i, x\rangle \geq -a_i\} = \bigcap_{i=1}^{N} K(\nu_i, -a_i). \tag{2.4}$$

2.5.2. *Kähler metrics on toric varieties.* For a given toric manifold $M$ endowed with a Riemannian metric $g$ invariant under the action of the real torus $T^n \subset (\mathbb{C}^*)^n$ and Kähler with respect to the underlying complex structure of $M$, the Kähler form $\omega$ of $g$ is also invariant under the $T^n$-action. We call such a manifold a *toric Kähler manifold.* In what follows, we always work with a fixed complex structure on $M$.

Hamiltonian Kähler metrics have a useful characterisation due to Guillemin.

**Proposition 2.19** ([Gui94a, Theorem 4.1]). *Let $\omega$ be any $T^n$-invariant Kähler form on $M$. Then the $T^n$-action is Hamiltonian with respect to $\omega$ if and only if the restriction of $\omega$ to the dense orbit $(\mathbb{C}^*)^n \subset M$ is exact, i.e., there exists a $T^n$-invariant potential $\phi$ such that*

$$\omega = 2i\partial\bar{\partial}\phi.$$

Fix once and for all a $\mathbb{Z}$-basis $(X_1, \dots, X_n)$ of $\Gamma \subset \mathfrak{t}$. This in particular induces a background coordinate system $\xi = (\xi^1, \dots, \xi^n)$ on $\mathfrak{t}$. Using the natural inner product on $\mathfrak{t}$ to identify $\mathfrak{t} \cong \mathfrak{t}^*$, we can also identify $\mathfrak{t}^* \cong \mathbb{R}^n$. For clarity, we will denote the induced coordinates on $\mathfrak{t}^*$ by $x = (x^1, \dots, x^n)$. Let $(z_1, \dots, z_n)$ be the natural coordinates on $(\mathbb{C}^*)^n$ as an open subset of $\mathbb{C}^n$. There is a natural diffeomorphism $\mathrm{Log} : (\mathbb{C}^*)^n \to \mathfrak{t} \times T^n$ which provides a one-to-one correspondence between $T^n$-invariant smooth functions on $(\mathbb{C}^*)^n$ and smooth functions on $\mathfrak{t}$. Explicitly,

$$(z_1, \dots, z_n) \xmapsto{\mathrm{Log}} (\log(r_1), \dots, \log(r_n), \theta_1, \dots, \theta_n) = (\xi_1, \dots, \xi_n,\, \theta_1, \dots, \theta_n), \tag{2.5}$$

where $z_j = r_j e^{i\theta_j}, r_j > 0$. Given a function $H(\xi)$ on $\mathfrak{t}$, we can extend $H$ trivially to $\mathfrak{t} \times T^n$ and pull back by Log to obtain a $T^n$-invariant function on $(\mathbb{C}^*)^n$. Clearly, any $T^n$-invariant function on $(\mathbb{C}^*)^n$ can be written in this form.

Choose any branch of log and write $w = \log(z)$. Then clearly $w = \xi + i\theta$, where $\xi = (\xi^1, \dots, \xi^n)$ are real coordinates on $\mathfrak{t}$ (or, more precisely, there is a corresponding lift of $\theta$ to the universal cover with respect to which this equality holds), and so if $\phi$ is $T^n$-invariant and $\omega = 2i\partial\bar{\partial}\phi$, then we have that

$$\omega = 2i \frac{\partial^2 \phi}{\partial w^i \partial \bar{w}^j} dw_i \wedge d\bar{w}_j = \frac{\partial^2 \phi}{\partial \xi^i \partial \xi^j} d\xi^i \wedge d\theta^j. \tag{2.6}$$

In this setting, the metric $g$ corresponding to $\omega$ is given on $\mathfrak{t} \times T^n$ by

$$g = \phi_{ij}(\xi) d\xi^i d\xi^j + \phi_{ij}(\xi) d\theta^i d\theta^j, \tag{2.7}$$

and the moment map $\mu$ as a map $\mu : \mathfrak{t} \times T^n \to \mathfrak{t}^*$ is defined by the relation

$$\langle \mu(\xi, \theta), b\rangle = \langle \nabla\phi(\xi), b\rangle$$

for all $b \in \mathfrak{t}$, where $\nabla\phi$ is the Euclidean gradient of $\phi$. The $T^n$-invariance of $\phi$ implies that it depends only on $\xi$ when considered a function on $\mathfrak{t} \times T^n$ via (2.5). Since $\omega$ is Kähler, we see from (2.6) that the Hessian of $\phi$ is positive definite so that $\phi$ itself is strictly convex. In particular, $\nabla\phi$ is a diffeomorphism onto its image. Using the identifications mentioned above, we view $\nabla\phi$ as a map from $\mathfrak{t}$ into an open subset of $\mathfrak{t}^*$.

2.5.3. *Kähler-Ricci solitons on toric manifolds.* Next we define what we mean by a shrinking Kähler-Ricci soliton in the toric category.

**Definition 2.20.** A complex $n$-dimensional shrinking gradient Kähler-Ricci soliton $(M,\, g,\, X)$ with complex structure $J$ and Kähler form $\omega$ is *toric* if $(M,\, \omega)$ is a toric Kähler manifold as in Definition 2.15 and $JX$ lies in the Lie algebra $\mathfrak{t}$ of the underlying real torus $T^n$ that acts on $M$. In particular, the zero set of $X$ is compact.

It follows from [Wyl08] that $\pi_1(M) = 0$, hence the induced real $T^n$-action is automatically Hamiltonian with respect to $\omega$. Working on the dense orbit $(\mathbb{C}^*)^n \subset M$, the condition that a vector field $JY$ lies in $\mathfrak{t}$ is equivalent to saying that in the coordinate system $(\xi^1, \dots, \xi^n,\, \theta_1, \dots, \theta_n)$ from (2.5), there is a constant $b_Y = (b_Y^1, \dots, b_Y^n) \in \mathbb{R}^n$ such that

$$JY = b_Y^i \frac{\partial}{\partial\theta^i} \qquad \text{or equivalently,} \qquad Y = b_Y^i \frac{\partial}{\partial\xi^i}. \tag{2.8}$$

From Proposition 2.19, we know that $\mathcal{L}_X\omega = 2i\partial\bar{\partial}X(\phi)$. In addition, the function $X(\phi)$ on $(\mathbb{C}^*)^n$ can be written as $\langle b_X, \nabla\phi\rangle = b_X^j \frac{\partial\phi}{\partial\xi^j}$, where $b_X \in \mathbb{R}^n$ corresponds to the soliton vector field $X$ via (2.8). These observations allow us to write the shrinking soliton equation (2.3) as a real Monge-Ampère equation for $\phi$ on $\mathbb{R}^n$.

**Proposition 2.21** ([Cif21, Proposition 2.6])**.** *Let $(M,\, g,\, X)$ be a toric shrinking gradient Kähler-Ricci soliton with Kähler form $\omega$. Then there exists a unique smooth convex real-valued function $\phi$ defined on the dense orbit $(\mathbb{C}^*)^n \subset M$ such that $\omega = 2i\partial\bar{\partial}\phi$ and*

$$\det(\phi_{ij}) = e^{-2\phi + \langle b_X,\, \nabla\phi\rangle}. \tag{2.9}$$

A priori, the function $\phi$ is defined only up to addition of a linear function. However, (2.9) provides a normalisation for $\phi$ which in turn provides a normalisation for $\nabla\phi$, the moment map of the action. The next lemma shows that this normalisation coincides with that for the moment map as defined in [CDS19, Definition 5.16].

**Lemma 2.22.** *Let $(M,\, g,\, X)$ be a toric complete shrinking gradient Kähler-Ricci soliton with complex structure $J$ and Kähler form $\omega$ with soliton vector field $X = \nabla^g f$ for a smooth real-valued function $f : M \to \mathbb{R}$. Let $\phi$ be given by Proposition 2.19 and normalised by (2.9), let $JY \in \mathfrak{t}$, and let $u_Y = \langle \nabla\phi,\, b_Y\rangle$ be the Hamiltonian potential of $JY$ with $b_Y$ as in (2.8) so that $\nabla^g u_Y = Y$. Then $\mathcal{L}_{JX} u_Y = 0$ and $\Delta_\omega u_Y + u_Y - \frac{1}{2} Y \cdot f = 0$.*

To see the equivalence with [CDS19, Definition 5.16], simply replace $Y$ with $JY$ in this latter definition as here we assume that $JY \in \mathfrak{t}$, contrary to the convention in [CDS19, Definition 5.16] where it is assumed that $Y \in \mathfrak{t}$.

Given the normalisation (2.9), the next lemma identifies the image of the moment map $\mu = \nabla\phi$.

**Lemma 2.23** ([Cif21, Lemmas 4.4 and 4.5])**.** *Let $(M,\, g,\, X)$ be a complete toric shrinking gradient Kähler-Ricci soliton, let $\{D_i\}$ be the prime $(\mathbb{C}^*)^n$-invariant divisors in $M$, and let $\Sigma \subset \mathfrak{t}$ be the fan determined by Proposition 2.17. Let $\sigma_i \in \Sigma$ be the ray corresponding to $D_i$ with minimal generator $\nu_i \in \Gamma$.*

(i) *There is a distinguished Weil divisor representing the anticanonical class $-K_M$ given by*

$$-K_M = \sum_i D_i$$

*whose associated polyhedron (cf. (2.4)) is given by*

$$P_{-K_M} = \{x \mid \langle \nu_i, x\rangle \geq -1\} \tag{2.10}$$

*which is strongly convex and has full dimension in* $\mathfrak{t}^*$. *In particular, the origin lies in the interior of* $P_{-K_M}$.

(ii) *If* $\mu$ *is the moment map for the induced real* $T^n$*-action normalised by* (2.9)*, then the image of* $\mu$ *is precisely* $P_{-K_M}$.

2.5.4. *The weighted volume functional.* As a result of Lemma 2.22, we can now define the weighted volume functional.

**Definition 2.24** (Weighted volume functional, [CDS19, Definition 5.16])**.** Let $(M,\, g,\, X)$ be a complex $n$-dimensional toric shrinking gradient Kähler-Ricci soliton with Kähler form $\omega = 2i\partial\bar{\partial}\phi$ on the dense orbit with $\phi$ strictly convex with moment map $\mu = \nabla\phi$ normalised by (2.9). Assume that the fixed point set of the torus is compact and recall that

$$\Lambda_\omega := \{Y \in \mathfrak{t} \mid \langle \mu,\, Y\rangle \text{ is proper and bounded below}\} \subseteq \mathfrak{t}.$$

Then the *weighted volume functional* $F : \Lambda_\omega \to \mathbb{R}$ is defined by

$$F_\omega(v) = \int_M e^{-\langle \mu,\, v\rangle}\omega^n.$$

As the fixed point set of the torus is compact by definition, $F_\omega$ is well-defined by the non-compact version of the Duistermaat-Heckman formula [PW94] (see also [CDS19, Theorem A.3]). This leads to two important lemmas concerning the weighted volume functional in the toric category, the independence of $\Lambda_\omega$ and $F_\omega$ from the choice of shrinking soliton $\omega$.

**Lemma 2.25** ([CCD22, Lemma 2.25])**.** $\Lambda_\omega$ *is independent of the choice of toric shrinking Kähler-Ricci soliton* $\omega$ *in Definition 2.24.*

**Lemma 2.26** ([CCD22, Lemma 2.26])**.** $F_\omega$ *is independent of the choice of toric shrinking Kähler-Ricci soliton* $\omega$ *in Definition 2.24. Moreover, after identifying* $\Lambda_\omega$ *with a subset of* $\mathbb{R}^n$ *via* (2.8)*,* $F_\omega$ *is given by* $F_\omega(v) = (2\pi)^n \int_{P_{-K_M}} e^{-\langle v,\, x\rangle}\, dx$*, where* $x = (x^1, \ldots, x^n)$ *denotes coordinates on* $\mathfrak{t}^*$ *dual to the coordinates* $(\xi^1, \ldots, \xi^n)$ *on* $\mathfrak{t}$ *introduced in Section 2.5.2.*

Thus, we henceforth drop the subscript $\omega$ from $F_\omega$ and $\Lambda_\omega$ when working in the toric category. The functional $F : \Lambda \to \mathbb{R}$ is proper in this category, hence attains a critical point in $\Lambda$.

**Proposition 2.27** ([Cif21, Proof of Proposition 3.1])**.** *The functional* $F(v) = (2\pi)^n \int_{P_{-K_M}} e^{-\langle v,\, x\rangle}\, dx$ *is proper on* $\Lambda$.

In general, such a critical point turns out to be unique and characterises the soliton vector field of a complete shrinking gradient Kähler-Ricci soliton.

**Theorem 2.28** ([CDS19, Lemma 5.17], [CZ10, Theorem 1.1])**.** *Let* $(M,\, g,\, X)$ *be a complete shrinking gradient Kähler-Ricci soliton with complex structure* $J$*, Kähler form* $\omega$*, and bounded Ricci curvature. Then* $JX \in \Lambda_\omega$*,* $F_\omega$ *is strictly convex on* $\Lambda_\omega$*, and* $JX$ *is the unique critical point of* $F_\omega$ *in* $\Lambda_\omega$.

Having established in Lemmas 2.25 and 2.26 that in the toric category the weighted volume functional $F$ and its domain $\Lambda$ are determined solely by the polytope $P_{-K_M}$ which itself, by Lemma 2.23, depends only on the torus action on $M$ (i.e., is independent of the choice of shrinking soliton), and having an explicit expression for $F$ given by Lemma 2.26, after using the torus action to identify $P_{-K_M}$ via (2.10), we can determine explicitly the soliton vector field of a hypothetical toric shrinking gradient Kähler-Ricci soliton on $M$. Indeed, in light of Lemma 2.26, the unique minimimiser $b_X \in \mathfrak{t} \simeq \mathbb{R}^n$ is characterised by the fact that

$$0 = d_{b_X}\mathcal{F}(v) = \int_{P_{-K_M}} \langle x,\, v\rangle\, e^{-\langle b_X,\, x\rangle} dx \qquad \text{for all } v \in \mathbb{R}^n. \tag{2.11}$$

In the setting of Theorem A, we can also determine $\Lambda$ explicitly.

**Lemma 2.29.** *Let $\pi : \overline{M} \to \mathbb{P}^1 \times D$, $M$, $T$, and $\mathfrak{t}$ be as in Theorem A and consider the restricted map $\pi|_M : M \to \widehat{M} := D \times \mathbb{C}$. Then there exists a basis $X_1, \dots, X_n$ of $\mathfrak{t}$ such that*

$$\Lambda = \{b = (b_1, \dots, b_n) \in \mathfrak{t} \,|\, b_n > 0\}.$$

*Proof.* Since $D$ itself is toric, there exists a basis $Y_1, \dots, Y_{n-1}$ of the corresponding lattice $\Gamma_D \subset \mathfrak{t}_D$, where $\mathfrak{t}_D$ is the Lie algebra of the $(n-1)$-dimensional subtorus of $T_D$ of $T$ acting on $D$. This can then be completed to a basis $Y_1, \dots, Y_n$ for $\mathfrak{t}_D$ by including the standard rotational vector field $Y_n$ on $\mathbb{C}$. Since $M$ and $D \times \mathbb{C}$ are equivariantly biholomorphic on the complement of finitely many subvarieties of each, we can choose our basis $X_1, \dots, X_n$ for $\mathfrak{t}$ to be such that $d\pi(X_i) = Y_i$. As $D$ is Fano, by Lemma 2.23 we know that the anticanonical polyhedron $P_{-K_{D\times\mathbb{C}}}$ for $D \times \mathbb{C}$ is the "simple product", i.e.,

$$P_{-K_{D\times\mathbb{C}}} = \{(x_1, \dots, x_n) \,|\, (x_1, \dots, x_{n-1}) \in P_D \,,\, x_n \geq -1\}. \tag{2.12}$$

Moreover, it follows from the definition of $\pi$ that the normal fan $\Sigma_M$ of $P_{-K_M}$ is just a refinement of the normal fan $\Sigma_{D\times\mathbb{C}}$ of $P_{-K_{D\times\mathbb{C}}}$ (see [CLS11, Definition 3.3.17]), hence the set of defining equations for $P_{-K_M}$ is obtained from (2.12) by the addition of finitely many linear inequalities. Therefore $P_{-K_M}$ and $P_{-K_{D\times\mathbb{C}}}$ coincide outside of a sufficiently large ball $B \subset \mathfrak{t}^*$. Let $Y \in \mathfrak{t}$ and via (2.8), identify $Y$ with a point $b_Y \in \mathbb{R}^n$. Since $P_{-K_M}$ is closed, it follows that the Hamiltonian potential $\mu_Y = \langle \mu, Y \rangle = \langle x, b_Y \rangle$ is proper if and only if $|\langle x, b_Y \rangle| \to \infty$ as $|x| \to \infty$. Thus, since $D$ is compact so that $P_D$ is bounded, we see that the set of vector fields $Y \in \mathfrak{t}$ for which the Hamiltonian potential $\mu_Y$ is proper is identified with the complement of the inclusion $\mathfrak{t}_D \hookrightarrow \mathfrak{t}$ induced by our choice of basis $(X_1, \dots, X_n)$, namely $(b_1, \dots, b_{n-1}) \mapsto (b_1, \dots, b_{n-1}, 0)$. In addition, $\mu_Y$ is bounded from below if and only if $\langle x, b \rangle \to +\infty$ as $|x| \to \infty$ in $P_{-K_M}$. As we have just seen, $|x| \to \infty$ in $P_{-K_M}$ if and only if $x_n \to +\infty$, and so the condition that $\mu_Y$ be bounded from below picks out the desired component $\Lambda = \{b \in \mathfrak{t} \,|\, b_n > 0\}$ of $\mathfrak{t}$. □

2.5.5. *The Legendre transform.* Let $M$ be a toric manifold endowed of complex dimension $n$ endowed with a complete Kähler form $\omega$ invariant under the induced real $T^n$-action and with respect to which this action is Hamiltonian. Write $\omega = 2i\partial\bar{\partial}\phi$ on the dense orbit for $\phi$ strictly convex as in Proposition 2.19. Then $\nabla\phi(\mathbb{R}^n)$ is a Delzant polytope $P$. Recall that we have coordinates $\xi$ on $\mathbb{R}^n \simeq \mathfrak{t}$, $x$ on $P$, and $\theta$ on $T$. Given any smooth and strictly convex function $\psi$ on $\mathbb{R}^n$ such that $\nabla\psi(\mathbb{R}^n) = P$, there exists a unique smooth and strictly convex function $u_\psi(x)$ on $P$ defined by

$$\psi(\xi) + u_\psi(\nabla\psi) = \langle \nabla\psi,\, \xi \rangle. \tag{2.13}$$

This process is reversible; that is to say, $\psi$ is the unique function satisfying

$$\psi(\nabla u_\psi) + u_\psi(x) = \langle x,\, \nabla u_\psi \rangle,$$

where $\nabla$ now denotes the Euclidean gradient with respect to $x$. The function $u_\psi$ is called the *Legendre transform of* $\psi$ and is sometimes denoted by $L(\psi)(x)$. Clearly $L(L(\psi))(\xi) = \psi(\xi)$. The Legendre transform $u$ of $\phi$ is called the *symplectic potential* of $\omega$, as the metric $g$ associated to $\omega$ is given by

$$g = u_{ij}(x)dx^i dx^j + u^{ij}(x)d\theta^i d\theta^j.$$

The following will prove useful.

**Lemma 2.30** (cf. [Cif21, Lemma 2.10])**.** *Let $\phi$ be any smooth and strictly convex function on an open convex domain $\Omega' \subset \mathbb{R}^n$ and let $u = L(\phi)$ be the Legendre transform of $\phi$ defined on $(\nabla\phi)(\Omega') =: \Omega$. If $0 \in \Omega$, then there exists a constant $C > 0$ such that*

$$\phi(\xi) \geq C^{-1}|\xi| - C.$$

*In particular, $\phi$ is proper and bounded from below.*

If $\phi \in C^\infty(\mathbb{R}^n)$ solves (2.9), then the Legendre transform $u = L(\phi)$ satisfies

$$2\left(\langle \nabla u, x \rangle - u(x)\right) - \log\det(u_{ij}(x)) = \langle b_X, x \rangle \qquad \text{on } P_{-K_M}. \tag{2.14}$$

To study Kähler-Ricci solitons on $M$ via (2.14) on $P_{-K_M}$, we need to understand when a strictly convex function on a Delzant polytope defines a symplectic potential, i.e., is induced from a Kähler metric on $M$ via the Legendre transform. To this end, consider a Delzant polytope $P$ obtained as the image of the moment map of a toric Kähler manifold. Let $F_i$, $i = 1, \dots, d$ denote the $(n-1)$-dimensional facets of $P$ with inward-pointing normal vector $\nu_i \in \Gamma$, normalized so that $\nu_i$ is the minimal generator of $\sigma_i = \mathbb{R}_+ \cdot \nu_i$ in $\Gamma$, and let $\ell_i(x) = \langle \nu_i, x\rangle$, so that $\overline{P}$ is defined by the system of inequalities $\ell_i(x) \geq -a_i$, $i = 1, \dots, N$, $a_i \in \mathbb{R}$. Then there exists a canonical metric $\omega_P$ on $M$ [Cif21, Proposition 2.7], the symplectic potential $u_P$ of which is given explicitly by the formula [BGL08, Gui94b]

$$u_P(x) = \frac{1}{2}\sum_{i=1}^{d}(\ell_i(x) + a_i)\log(\ell_i(x) + a_i)\,. \tag{2.15}$$

In particular, the Legendre transform $\phi_P$ of $u_P$ will define the Kähler potential on the dense orbit of a globally defined Kähler metric $\omega_P$ on $M$ [BGL08, Gui94b]. More generally, it was observed by Abreu [Abr98] that the Legendre transform $L(u)$ of a strictly convex function $u$ on $P$ will define the Kähler potential on the dense orbit of a globally defined Kähler metric $\omega_P$ on $M$ if and only if $u$ has the same asymptotic behavior as $u_P$ of all orders as $x \to \partial P$. Indeed, we have the following slightly more general statement.

**Lemma 2.31** ([Abr98],[ACGTnF04],[Cif21, Proposition 2.17])**.** *A convex function $u$ on $P$ defines a Kähler metric $\omega_u$ on $M$ if and only if $u$ has the form*

$$u = u_P + v,$$

*where $v \in C^\infty(\overline{P})$ extends past $\partial P$ to all orders.*

In the case that $P = P_{-K_M}$, we read from Lemma 2.23(ii) that $a_i = -1$ for all $i$. Thus, in this case, the canonical metric on $P_{-K_M}$ has symplectic potential given by

$$u_{P_{-K_M}} = \frac{1}{2}\sum_i(\ell_i(x) + 1)\log(\ell_i(x) + 1).$$

2.5.6. *The $\hat{F}$-functional.* We next define the $\hat{F}$-functional on toric Kähler manifolds.

**Definition 2.32.** Let $(M, \omega)$ be a (possibly non-compact) toric Kähler manifold with complex structure $J$ endowed with a real holomorphic vector field $X$ such that $JX \in \Lambda_\omega$. Write $T$ for the torus acting on $M$, identify the dense orbit with $\mathbb{R}^n$, let $\xi = (\xi_1, \dots, \xi_n)$ denote coordinates on $\mathbb{R}^n$, let $b_X$ be as in (2.8), and write $\omega = 2i\partial\bar{\partial}\phi_0$ on the dense orbit as in Proposition 2.19. Write $P := (\nabla\phi_0)(\mathbb{R}^n)$ for the image of the moment map associated to $\omega$ and let $x = (x_1, \dots, x_n)$ denote coordinates on $P$. Let $\varphi \in C^\infty(M)$ be a smooth function on $M$ invariant under the action of $T$ such that $\omega + i\partial\bar{\partial}\varphi > 0$ and assume that:

(a) There exists a $C^1$-path of smooth functions $\{\varphi_s\}_{s\in[0,\,1]} \subset C^\infty(M)$ invariant under the action of $T$ such that $\varphi_0 = 0$, $\varphi_1 = \varphi$, and $\omega + i\partial\bar{\partial}\varphi_s > 0$, and $(\nabla\phi_s)(\mathbb{R}^n) = P$ for all $s \in [0,\,1]$, where $\phi_s := \phi_0 + \frac{\varphi_s}{2}$.

(b) $\int_0^1\int_{\mathbb{R}^n}|\dot{\phi}_s|\,e^{-\langle b_X,\,\nabla\phi_s\rangle}\det(\phi_{s,\,ij})\,d\xi\,ds < \infty$.

Then we define

$$\hat{F}(\varphi) := 2\int_P(L(\phi_1) - L(\phi_0))\,e^{-\langle b_X,\,x\rangle}dx.$$

The existence of the path $\{\varphi_s\}_{s\in[0,\,1]}$ satisfying conditions (a) and (b) is to guarantee that $\hat{F}(\varphi)$ is well-defined. To see this, we first note:

**Lemma 2.33.** *Under the assumptions of Definition 2.32, let $u_s := L(\phi_s)$, $\omega_s = \omega + i\partial\bar{\partial}\varphi_s$, and write $f_s := f + \frac{X}{2}\cdot\varphi_s$ for the Hamiltonian potential of $JX$ with respect to $\omega_s$, where $f$ is the Hamiltonian potential of $JX$ with respect to $\omega$. Then the following are equivalent.*

(i) $\int_0^1 \int_{\mathbb{R}^n} |\dot{\phi}_s| \, e^{-\langle b_X, \nabla\phi_s\rangle} \det(\phi_{s,ij}) \, d\xi \, ds < \infty$.
(ii) $\int_0^1 \int_P |\dot{u}_s| \, e^{-\langle b_X, x\rangle} \, dx \, ds < \infty$.
(iii) $\int_0^1 \int_M |\dot{\varphi}_s| \, e^{-f_s}\omega_s^n \, ds < \infty$.

*In particular when this is the case,* $|\hat{F}(\varphi)| < \infty$.

*Proof.* The equivalence of (i) and (iii) is clear. The equivalence of (i) and (ii) follows from [Cif21, Lemma 3.6]. Finally, for the last statement, for every $x \in P$, we have

$$|u_1 - u_0|(x) \leq \int_0^1 |\dot{u}_s|(x) \, ds.$$

Then using Fubini's theorem and noting Lemma 2.33, we estimate that

$$|\hat{F}(\varphi)| \leq 2\int_P |u_1 - u_0| \, e^{-\langle b_X, x\rangle} dx \leq 2\int_P \left(\int_0^1 |\dot{u}_s| \, ds\right) e^{-\langle b_X, x\rangle} dx = 2\int_0^1 \int_P |\dot{u}_s| \, e^{-\langle b_X, x\rangle} dx \, ds < \infty,$$

as desired. □

Under an additional assumption on the path $\{\varphi_s\}_{s\in[0,1]}$, we recover the well-known expression for the $\hat{F}$-functional given in [CTZ05, p.702].

**Lemma 2.34.** *If one (and hence all) of the conditions of Lemma 2.33 holds true and if in addition it holds true that* $\int_0^1 \int_M |\dot{\varphi}_s| \, e^{-f}\omega^n \, ds < \infty$, *then*

$$\hat{F}(\varphi) = \int_0^1 \int_M \dot{\varphi}_s \left(e^{-f}\omega^n - e^{-f_s}\omega_s^n\right) \wedge ds - \int_M \varphi \, e^{-f}\omega^n. \tag{2.16}$$

*Proof.* The extra condition implies in particular that $\int_M |\varphi| \, e^{-f}\omega^n < \infty$ since by assumption and Fubini's theorem, $\int_M |\varphi| \, e^{-f}\omega^n \leq \int_0^1 \int_M |\dot{\varphi}_s| \, e^{-f}\omega^n \, ds < +\infty$ so that the right-hand side of the above expression (2.16) is at least finite. To show that it is equivalent to the expression for $\hat{F}$ given in Definition 2.32, using the change of coordinates induced by $\nabla\phi_s : \mathbb{R}^n \to P$ and using the fact that $\dot{\phi}_s(\nabla\phi_s) = -\dot{u}_s(x)$ (cf. [Cif21, Lemma 3.6]), we compute that

$$\begin{aligned}
\hat{F}(\varphi) &= 2\int_P (u_1(x) - u_0(x)) \, e^{-\langle b_X, x\rangle} dx \\
&= 2\int_0^1 \int_P \dot{u}_s(x) \, e^{-\langle b_X, x\rangle} dx \wedge ds \\
&= -2\int_0^1 \int_P \dot{\phi}_s(\nabla\phi_s) \, e^{-\langle b_X, x\rangle} dx \wedge ds \\
&= -2\int_0^1 \int_{\mathbb{R}^n} \dot{\phi}_s \, e^{-\langle b_X, \nabla\phi_s\rangle} \det(\phi_{s,ij}) d\xi \wedge ds \\
&= -\int_0^1 \int_M \dot{\varphi}_s \, e^{-f_s}\omega_s^n \wedge ds \\
&= \int_0^1 \int_M \dot{\varphi}_s \left(e^{-f}\omega^n - e^{-f_s}\omega_s^n\right) \wedge ds - \int_M \varphi \, e^{-f}\omega^n,
\end{aligned}$$

resulting in the desired expression. Here we have used Fubini's theorem in the last equality. □

2.5.7. *Integrability and independence of the path.* In light of conditions (a) and (b) of Definition 2.32 required to define the $\hat{F}$-functional, it is useful to have sufficient conditions for when the moment polytope remains unchanged under a path of Kähler metrics. We also want to identify when the integral of each summand in the definition of $\hat{F}$ is finite separately. This will be important for achieving an a priori $C^0$-bound for our continuity path.

To this end, suppose that $(M, \omega)$ is a toric Kähler manifold, i.e., $(M, \omega)$ is Kähler with Kähler form $\omega$ with respect to a complex structure $J$, endowed with the holomorphic action of a complex torus of the same complex dimension as $(M, J)$ whose underlying real torus $T$ induces a Hamiltonian action, and let $JX \in \mathfrak{t}$. Via (2.8), we can identify $X$ with an element $b_X \in \mathbb{R}^n \simeq \mathfrak{t}$. Using Proposition 2.19, we also write $\omega = 2i\partial\bar{\partial}\phi_0$ on the dense orbit for some strictly convex function $\phi_0 : \mathbb{R}^n \to \mathbb{R}$. Assume that:

- $JX \in \Lambda_\omega$ so that the Hamiltonian potential $f$ of $JX$ is proper and bounded from below.
- There exists a smooth bounded real-valued function $F$ on $M$ so that the Ricci form $\rho_\omega$ of $\omega$ satisfies $\rho_\omega + \frac{1}{2}\mathcal{L}_X\omega - \omega = i\partial\bar{\partial}F$.

The equation in the second bullet point reads as

$$(F + \log\det(\phi_{0,\,ij}) - \langle\nabla\phi_0, b_X\rangle + 2\phi_0)_{ij} = 0 \qquad \text{on } \mathfrak{t} \simeq \mathbb{R}^n$$

so that

$$F = -\log\det(\phi_{0,\,ij}) + \langle\nabla\phi_0, b_X\rangle - 2\phi_0 + a(\xi)$$

for some linear function $a(\xi)$ defined on $\mathbb{R}^n$. By considering $2\phi_0 + a + \langle\nabla a,\, b_X\rangle$, we can therefore assume that

$$F = -\log\det(\phi_{0,\,ij}) + \langle\nabla\phi_0,\, b_X\rangle - 2\phi_0. \tag{2.17}$$

An answer to the aforementioned questions is provided by the following.

**Lemma 2.35.** *Under the above assumptions, let $\varphi \in C^\infty(M)$ be a torus-invariant smooth real-valued function on $M$ such that $\omega_\varphi := \omega + i\partial\bar{\partial}\varphi > 0$ and $|X \cdot \varphi| < C$. Define $\phi := \phi_0 + \frac{1}{2}\varphi$ so that $\omega + i\partial\bar{\partial}\varphi = 2i\partial\bar{\partial}\phi$ on the dense orbit. Then:*

(i) *The image of the moment map $\mu_{\omega_\varphi} : M \to \mathfrak{t}^*$ with respect to $\omega_\varphi$ defined by the Euclidean gradient $\nabla\phi : \mathbb{R}^n \to \mathbb{R}^n$ is equal to $P_{-K_M}$. In particular, $0 \in \operatorname{int}\big(\mu_{\omega_\varphi}(M)\big)$.*

(ii) $\int_P |L(\phi_0)|\, e^{-\langle b_X, x\rangle} dx < \infty$.

*Proof.* (i) To prove (i), we begin by noting that since $|X \cdot \varphi| < \infty$, the Hamiltonian potential $f_\varphi = f + \frac{X}{2} \cdot \varphi$ of $X$ with respect to $\omega_\varphi$ is proper and bounded from below. In particular, the image $(\nabla\phi)(\mathbb{R}^n)$ of the moment map $\mu_{\omega_\varphi} : M \to \mathfrak{t}^*$ is equal to a Delzant polyhedron $P$ [Cif21, Lemma 2.13] that a priori depends on $\varphi$. Let $u(x) := L(\phi)$ be the Legendre transform of $\phi$. Then the domain of $u$ is precisely $P$. We need to show that $P = P_{-K_M}$. To this end, let $F$ be as in (2.17). Then a computation shows that

$$-\log\det\phi_{ij} + \langle\nabla\phi, b_X\rangle - 2\phi = F + \log\left(\frac{\omega_\varphi^n}{\omega^n}\right) + \frac{X}{2} \cdot \varphi - \varphi. \tag{2.18}$$

Set $A(x) := \langle b_X,\, x\rangle$ and define

$$\rho_u(x) := 2\left(\langle\nabla u,\, x\rangle - u(x)\right) - \log\det(u_{ij}).$$

Then via the change of coordinates $x = \nabla\phi(\xi)$, we can rewrite (2.18) in terms of $u$ as

$$A(x) - \rho_u(x) = \left(F + \log\left(\frac{\omega_\varphi^n}{\omega^n}\right) + \frac{X}{2} \cdot \varphi - \varphi\right)(\nabla u(x)) \qquad \text{on } P. \tag{2.19}$$

Observe that the right-hand side of (2.19) admits a continuous extension up to the boundary $\partial P$ of $P$. Denoting the right-hand side of (2.18) by $h$ which is a function $h : M \to \mathbb{R}$, this extension is simply given by $h \circ \mu_{\omega_\varphi}^{-1}$, where $\mu_{\omega_\varphi} : M \to \overline{P}$, as the moment map, has fibers precisely the orbits of the torus action.

We now proceed as in [Cif21, Lemma 4.5] using an argument originally due to Donaldson [Don08]. Suppose that $P$ is defined by the linear inequalities $\ell_i(x) \geq -a_i$, where $\ell_i(x) = \langle\nu_i, x\rangle$. As the right-hand side of (2.19) as well as $A(x)$ has a continuous extension up to $\partial P$, we see that the same holds true for $\rho_u(x)$. Moreover, as $u$ is the symplectic potential of the Kähler form $\omega_\varphi$ on $P$, we read from Lemma 2.31 that there exists a function $v \in C^\infty(\overline{P}')$ with $u = u_P + v$, where $u_P$ is given as in (2.15) by

$$u_P(x) = \frac{1}{2}\sum(\ell_i(x) + a_i)\log(\ell_i(x) + a_i). \tag{2.20}$$

Fix any facet $F'$ of $P$. Without loss of generality, we may assume that $F'$ is defined by $\ell_1(x) = -a_1$. Up to a change of basis in $\mathfrak{t}^*$, we may also assume by the Delzant condition that $\ell_1(x) = x_1$. Fix a point $p$ in the interior of $F'$. Then from (2.20) we see that in a neighbourhood of $p$, $u$ can be written as

$$u(x) = u_P(x) + v(x) = \frac{1}{2}(x_1 + a_1)\log(x_1 + a_1) + v_1$$

for some smooth function $v_1$ which extends smoothly across $F'$. From this expression, it follows that in a small half ball $B$ in the interior of $P$ containing $p$, $\rho_u$ takes the form

$$\begin{aligned}\rho_u(x) &= x_1\log(x_1 + a_1) - (x_1 + a_1)\log(x_1 + a_1) + \log(x_1 + a_1) + v_2\\ &= (1 - a_1)\log(x_1 + a_1) + v_2\end{aligned}$$

for another smooth function $v_2$ that extends smoothly across $F'$ in $B$. Thus, already knowing that $\rho_u$ has a continuous extension across $\partial P$, we deduce that $1-a_1 = 0$, i.e, $a_1 = 1$. Continuing in this manner, we see that $a_i = 1$ for all $i$, leading us to the conclusion that $P = P_{-K_M}$.

(ii) Let $u_0 = L(\phi_0)$. Then as $u_0$ is a convex function on $P_{-K_M}$ whose gradient has image equal to all of $\mathbb{R}^n$ by the invertibility of the Legendre transform, it is proper and bounded from below by Lemma 2.30. Let $A$ denote the lower bound, let $\rho_u$ be as in part (i), and write $\rho_0 = \rho_{u_0}$. Then $F$ bounded implies the existence of a constant $C$ such that $|\rho_0 - \langle b_X,\, x\rangle| < C$ on $P_{-K_M}$. This is clear from (2.17). Since $\int_{P_{-K_M}} u_0\, e^{-\rho_0}dx < \infty$ by [Cif21, Lemma 4.7], it follows that $\int_{P_{-K_M}} u_0\, e^{-\langle b_X,\, x\rangle}dx < \infty$. Finiteness of the integral $\int_{P_{-K_M}} e^{-\langle b_X,\, x\rangle}dx$ together with the fact that $u_0$ is bounded from below now yields the result.

□

## 3. Proof of Theorem A(ii): Construction of a background metric

Given the setup as in Theorem A and with $X$ determined by Theorem A(i), we henceforth assume that the flow-lines of $JX$ close. In this section, we construct a background metric on $M$ with the properties as outlined in Theorem A(ii) with a construction reminiscent of that of [HHN15, Section 4.2]. To this end, recall for $a > 0$ the (incomplete) shrinking gradient Kähler-Ricci soliton $(\widehat{M} := \mathbb{C} \times D,\, \widehat{\omega}_a := \tilde{\omega}_a + \omega_D,\, \frac{2}{a} \cdot \operatorname{Re}(z\partial_z))$ of Example 2.3 with complex structure $\widehat{J}$ endowed with the product holomorphic action of the real $n$-torus $\widehat{T}$.

**Proposition 3.1.** *There exists*

(a) *a complete Kähler metric $\omega$ on $M$ with $\mathcal{L}_{JX}\omega = 0$, and*
(b) *a biholomorphism $\nu : M \setminus K \to \widehat{M} \setminus \widehat{K}$ where $K \subset M$, $\widehat{K} \subset \widehat{M}$, are compact,*

*and $\lambda > 0$ such that*

(i) *$d\nu(X) = \frac{2}{\lambda} \cdot \operatorname{Re}(z\partial_z)$, and*
(ii) *$\omega = \nu^*(\tilde{\omega}_\lambda + \omega_D)$,*
(iii) *the Ricci form $\rho_\omega$ of $\omega$ satisfies*

$$\rho_\omega + \frac{1}{2}\mathcal{L}_X\omega - \omega = i\partial\bar{\partial}F_1, \tag{3.1}$$

*for $F_1 \in C^\infty(M)$ compactly supported with $\mathcal{L}_{JX}F_1 = 0$.*

Theorem A(ii) immediately follows from this proposition. Indeed, this is easily seen by writing $\omega_C := \tilde{\omega}_\lambda$ (cf. Example 2.2) and $\widehat{\omega} := \widehat{\omega}_\lambda = \tilde{\omega}_\lambda + \omega_D$ (cf. Example 2.3). With $\lambda$ fixed in subsequent sections, this is the notation that we adopt to be consistent with that of Theorem A. Property (iii) of this proposition will be used in the next section.

*Proof of Proposition 3.1.* Recall that $\pi : \overline{M} \to \mathbb{P}^1 \times D$ is a torus-equivariant holomorphic map that restricts to a holomorphic map $\pi : M \to \widehat{M} := \mathbb{C} \times D$ by removing the fibre $D_\infty$ from $\overline{M}$ and $\mathbb{P}^1 \times D$ respectively, and that $z$ denotes the holomorphic coordinate on the $\mathbb{C}$-factor of $\widehat{M}$. We define the map $\nu : M\backslash\pi^{-1}(D_0) \to \widehat{M}\backslash D_0$ of (b) as the $\mathbb{C}^*$-equivariant map one obtains by identifying a $\mathbb{P}^1$-fibre in each manifold and for each point in this $\mathbb{P}^1$, flowing along the vector field $X^{1,0} := \frac{1}{2}(X - i(JX))$ on $M$ and the holomorphic vector field $z\partial_z$ on $\widehat{M}$. As the flow-lines of $JX$ close by assumption, this map is well-defined.

From the construction, it is clear that $d\nu(X^{1,0}) = \frac{2}{\lambda}\cdot z\partial_z$ for some $\lambda > 0$. This defines $\lambda$ and verifies condition (i) of the proposition. The map $\nu$ then extends to a holomorphic map $\overline{\nu} : \overline{M} \setminus \pi^{-1}(D_0) \to \widehat{M} \setminus D_0$. On $\mathbb{C} \times D$ we consider the product metric $\widehat{\omega}_\lambda$. We write $w := \frac{1}{z}$ and $r := |z|^\lambda$. Identifying $M \setminus \pi^{-1}(D_0)$ and $\widehat{M} \setminus D_0$ via $\nu$, we view these as functions, and $\widehat{\omega}_\lambda$ as a Kähler form, on the former. In this way, $w$ defines a holomorphic coordinate on $\overline{M}\backslash\pi^{-1}(D_0)$ with the divisor $D$ at infinity defined by $\{w = 0\}$.

Using $\nu$, we construct the background metric $\omega$ of (a) in the following way. As $\overline{M}$ is Fano by assumption, by the Calabi-Yau theorem [Yau78], there exists a hermitian metric $h$ on $-K_{\overline{M}}$ with strictly positive curvature form $\Theta_h$. Moreover, since the normal bundle $N_D$ of $D$ in $\overline{M}$ is trivial so that $K_D = K_{\overline{M}}|_D$ by adjunction, the Calabi-Yau theorem again guarantees the existence of a function $u \in C^\infty(D)$ such that $i\Theta_h|_D + i\partial\bar{\partial}u = \omega_D$. Extend $u$ to be constant along the $w$-direction and multiply this extension by a cut-off function depending only on $w$ to further extend $u$ to the whole of $\overline{M}$. We still denote this extension by $u$. If the support of this cut-off function is contained in a sufficiently small tubular neighbourhood of $D$, then the restriction of $i\Theta_h + i\partial\bar{\partial}u$ to any $\mathbb{P}^1$-fibre will be positive-definite. All negative components of $i\Theta_h + i\partial\bar{\partial}u$ on the total space $\overline{M}$ can be compensated for by adding a sufficiently positive "bump 2-form" of the form $\chi(|w|)dw \wedge d\bar{w}$ where $\chi$ is a bump function supported in an annulus containing the cut-off region; such a 2-form is automatically closed and (1, 1) on $\overline{M}$, and exact on $M$. This creates a Kähler form $\tau_1$ on $M$. One can verify in a sufficiently small neighbourhood of $D$ that

$$\tau_1 - \omega_D = O(|w|)\left(dw \wedge d\bar{w} + \sum_j dw \wedge d\bar{v}_{\bar{j}} + \sum_{i,j} dv_i \wedge d\bar{v}_{\bar{j}} + \sum_i dv_i \wedge d\bar{w}\right) \qquad \text{as } w \to 0, \quad (3.2)$$

for $\{v_1, \ldots, v_{n-1}\}$ local holomorphic coordinates on $D$.

Next, let $\psi : \mathbb{R} \to \mathbb{R}$ be a smooth function satisfying

$$\psi'(x),\, \psi''(x) \geq 0 \quad \text{for all } x \in \mathbb{R},$$

and

$$\psi(x) = \begin{cases} \text{const.} & \text{if } x < 1 \\ x & \text{if } x > 2, \end{cases} \qquad (3.3)$$

and consider the composition $k := \psi \circ r^2$, a real-valued smooth function on $M$. One computes that

$$\frac{i}{2}\partial\bar{\partial}k = \psi''(r^2)\,\frac{i}{2}\partial r^2 \wedge \bar{\partial}r^2 + \psi'(r^2)\,\frac{i}{2}\partial\bar{\partial}r^2 \geq 0,$$

a positive semi-definite form equal to $\frac{i}{2}\partial\bar{\partial}r^2$ on the region of $M$ where $r^2 > 2$. Define the Kähler form

$$\tau_2 := \tau_1 + \frac{i}{2}\partial\bar{\partial}k,$$

and in the holomorphic coordinates $(z,\, v)$ on $\widehat{M}$, consider the product metric $\widehat{\omega}_\lambda$ given by

$$\widehat{\omega}_\lambda := \tilde{\omega}_\lambda + \omega_D = i\partial\bar{\partial}\left(\frac{|z|^{2\lambda}}{2}\right) + \omega_D = \frac{\lambda^2 dz \wedge d\bar{z}}{2|z|^{2-2\lambda}} + \omega_D.$$

Then from (3.2), it is clear that the difference is given by

$$\tau_2 - \widehat{\omega}_\lambda = O(|w|)\left(dw \wedge d\bar{w} + \sum_j dw \wedge d\bar{v}_{\bar{j}} + \sum_{i,j} dv_i \wedge d\bar{v}_{\bar{j}} + \sum_i dv_i \wedge d\bar{w}\right) \qquad \text{as } w \to 0, \quad (3.4)$$

so that in particular,

$$|\tau_2-\widehat{\omega}_\lambda|_{\widehat{\omega}_\lambda}=O(r^{-\frac{1}{\lambda}}). \tag{3.5}$$

We now work with the hermitian metric $H$ on $-K_{\widehat{M}}$ induced by $\widehat{\omega}_\lambda$. Via the map $\nu$, this pulls back to the hermitian metric

$$H=\frac{\lambda^2\det((g_D)_{i\bar{j}})}{2|z|^{2-2\lambda}}$$

on $-K_M|_{M\setminus\pi^{-1}(D_0)}$ with respect to the local trivialisation $\partial_z\wedge\partial_{v_1}\wedge\ldots\wedge\partial_{v_{n-1}}$. The corresponding curvature form is then given by

$$-i\partial\bar{\partial}\log H=\omega_D.$$

Hence, as a difference of two curvature forms, there exists a smooth real-valued function $\phi$ defined on $M\setminus\pi^{-1}(D_0)$ such that

$$(i\Theta_h+i\partial\bar{\partial}u)-\omega_D=i\partial\bar{\partial}\phi.$$

In particular, outside a large compact subset of $M$, we have that

$$\tau_2-\widehat{\omega}_\lambda=i\partial\bar{\partial}\phi. \tag{3.6}$$

We claim that $\phi$ is in fact smooth on $\overline{M}\setminus\pi^{-1}(D_0)$. To see this, note that as

$$i\partial\bar{\partial}\phi=-i\partial\bar{\partial}\log\left(\frac{e^{-u}|\partial_z\wedge\partial_{v_1}\wedge\ldots\wedge\partial_{v_{n-1}}|^2_h}{|\partial_z\wedge\partial_{v_1}\wedge\ldots\wedge\partial_{v_{n-1}}|^2_H}\right)$$

and

$$\begin{aligned}\log&\left(\frac{e^{-u}|\partial_z\wedge\partial_{v_1}\wedge\ldots\wedge\partial_{v_{n-1}}|^2_h}{|\partial_z\wedge\partial_{v_1}\wedge\ldots\wedge\partial_{v_{n-1}}|^2_H}\right)=\log\left(\frac{e^{-u}|w|^4|\partial_w\wedge\partial_{v_1}\wedge\ldots\wedge\partial_{v_{n-1}}|^2_h}{|\partial_z\wedge\partial_{v_1}\wedge\ldots\wedge\partial_{v_{n-1}}|^2_H}\right)\\&=\log\left(\frac{2e^{-u}|\partial_w\wedge\partial_{v_1}\wedge\ldots\wedge\partial_{v_{n-1}}|^2_h|z|^{2-2\lambda}}{\lambda^2\det((g_D)_{i\bar{j}})}\right)+2\log(|w|^2)\\&=\underbrace{\log\left(\frac{2e^{-u}|\partial_w\wedge\partial_{v_1}\wedge\ldots\wedge\partial_{v_{n-1}}|^2_h}{|\partial_{v_1}\wedge\ldots\wedge\partial_{v_{n-1}}|^2_{\omega_D}}\right)}_{\text{extends smoothly over }\{w=0\}}-\underbrace{(1-\lambda)\log|w|^2+2\log(|w|^2)-\log(\lambda^2)}_{\text{pluriharmonic}},\end{aligned}$$

$\phi$ may be taken to be

$$\phi=-\log\left(\frac{2e^{-u}|\partial_w\wedge\partial_{v_1}\wedge\ldots\wedge\partial_{v_{n-1}}|^2_h}{|\partial_{v_1}\wedge\ldots\wedge\partial_{v_{n-1}}|^2_{\omega_D}}\right), \tag{3.7}$$

which, albeit defined in terms of local coordinates, is clearly globally defined on $\overline{M}\setminus\pi^{-1}(D_0)$. Thus $\phi=O(1)$ and from (3.5) and (3.6), we see that $|i\partial\bar{\partial}\phi|_{\widehat{\omega}_\lambda}=O(r^{-\frac{1}{\lambda}})$. Finally, after a computation, the expression for $\phi$ given in (3.7) gives us that $|d\phi|_{\widehat{\omega}_\lambda}=O(1)$. Now, $\widehat{\omega}_\lambda$ and $\tau_2$ are equivalent outside some large compact subset $\tilde{K}$ of $M$ by (3.5), and on the complement of $\tilde{K}$ in $M$, Lemma 2.5 implies that for all $R>0$ sufficiently large, $\phi$ admits an extension $\phi_R$ to $M$ supported on $M\setminus\tilde{K}$ such that $|i\partial\bar{\partial}\phi_R|_{\widehat{\omega}_\lambda}\leq CR^{-\min\{\lambda^{-1},1\}}$. Thus, at the expense of increasing $C$ if necessary, we can infer that $|i\partial\bar{\partial}\phi_R|_{\tau_2}\leq CR^{-\min\{\lambda^{-1},1\}}$ globally on $M$. We fix $R>0$ large enough such that $|i\partial\bar{\partial}\phi_R|_{\tau_2}<1$ and define a Kähler form on $M$ by

$$\tilde{\omega}:=\tau_2-i\partial\bar{\partial}\phi_R.$$

By what we have just said, $\tilde{\omega}$ is positive-definite everywhere on $M$ and equal to $\widehat{\omega}_\lambda$ outside a large compact subset of $M$, hence is complete. By averaging over the action of $T$, we may assume that $\mathcal{L}_{JX}\tilde{\omega}=0$ without changing the behaviour at infinity. We further modify $\tilde{\omega}$ to construct $\omega$ satisfying conditions (a) and (ii) of the proposition.

To this end, we know that since $M$ does not split off any $S^1$-factors, $\pi_1(M)=0$ by toricity [CLS11]. In particular, $H^1(M,\mathbb{R})=0$ so that the action of $T$ on $M$ is Hamiltonian with respect to $\tilde{\omega}$. Hence, there exists a smooth real-valued function $\tilde{f}$ which, after averaging, can be taken to be invariant under the action of $T$ on $M$, such that $\frac{1}{2}\mathcal{L}_X\tilde{\omega}=i\partial\bar{\partial}\tilde{f}$. It is also clear that, as $\tilde{\omega}=i\Theta_h+i\partial\bar{\partial}u_1$ for some $u_1\in C^\infty(M)$ with $i\Theta_h$ the curvature form of a hermitian metric on $-K_M$, we can write

$\rho_{\tilde{\omega}} - \tilde{\omega} = i\partial\bar{\partial}u_2$ for another function $u_2 \in C^\infty(M)$, $\rho_{\tilde{\omega}}$ here denoting the Ricci form of $\tilde{\omega}$. Thus, there exists a function $\tilde{G} \in C^\infty(M)$ such that

$$\rho_{\tilde{\omega}} - \tilde{\omega} + \frac{1}{2}\mathcal{L}_X\tilde{\omega} = i\partial\bar{\partial}\tilde{G}. \tag{3.8}$$

After averaging, we may assume that $\tilde{G}$ is invariant under the action of $T$. In particular, henceforth identifying $M$ and $\widehat{M}$ on the complement of compact subsets containing $D_0$ and $\pi^{-1}(D_0)$ respectively, we can write $\tilde{G} := \tilde{G}(r, x)$, where $r = |z|^\lambda$ is as above and $x \in D \subset \widehat{M}$. As $\tilde{\omega}$ defines a shrinking gradient Kähler-Ricci soliton on $M \setminus K$ for some $K \subset M$ compact, we see that $\tilde{G}$ is pluriharmonic on $M \setminus K$, and so it follows from Lemma 2.4 that

$$\tilde{G} = c_0 \log(r)$$

for some constant $c_0 \in \mathbb{R}$. Arguing as above, Lemma 2.5 guarantees the existence of an extension $\varphi$ of $c_0 \log(r) + \frac{c_0}{2}$ to $M$ such that $\omega := \tilde{\omega} + i\partial\bar{\partial}\varphi$ defines a Kähler metric on $M$. As $\varphi$ is pluriharmonic at infinity, it is clear that $\omega = \tilde{\omega} = \nu^*\widehat{\omega}_\lambda$ outside a large enough compact subset of $M$. Averaging over the action of $T$, we obtain our metric $\omega$ of (a) satisfying condition (ii).

Next, as in (3.8), we see that there exists a function $G \in C^\infty(M)$, invariant under the action of $T$, such that

$$\rho_\omega - \omega + \frac{1}{2}\mathcal{L}_X\omega = i\partial\bar{\partial}G. \tag{3.9}$$

Subtracting (3.8) from (3.9) yields the relation

$$\begin{aligned} i\partial\bar{\partial}G &= i\partial\bar{\partial}\tilde{G} + \rho_\omega - \rho_{\tilde{\omega}} - i\partial\bar{\partial}\varphi + i\partial\bar{\partial}\left(\frac{X}{2}\cdot\varphi\right) \\ &= i\partial\bar{\partial}\left(\tilde{G} - \log\left(\frac{\omega^n}{\tilde{\omega}^n}\right) - \varphi + \frac{X}{2}\cdot\varphi\right) \end{aligned}$$

between $G$ and $\tilde{G}$. Set

$$F_1 := \tilde{G} - \log\left(\frac{\omega^n}{\tilde{\omega}^n}\right) - \varphi + \frac{X}{2}\cdot\varphi.$$

Then $i\partial\bar{\partial}F_1 = i\partial\bar{\partial}G$ so that (3.1) holds true, and outside a large compact subset of $M$ we have that

$$F_1 = \tilde{G} - \log\left(\frac{\omega^n}{\tilde{\omega}^n}\right) - \varphi + \frac{X}{2}\cdot\varphi = c_0\log(r) - \varphi(r) + \frac{r}{2}\cdot\varphi'(r) = 0,$$

demonstrating that $F_1 \in C^\infty(M)$ and is compactly supported. As $\mathcal{L}_{JX}\tilde{G} = 0$, condition (iii) and correspondingly, the proposition, follow. □

## 4. Proof of Theorem A(iii) and (iv): Set-up of the complex Monge-Ampère equation

Returning now to the setup and notation of Theorem A, we next prove Theorem A(iii) by setting up a complex Monge-Ampère equation that any shrinking gradient Kähler-Ricci soliton on $M$ differing from our background metric by $i\partial\bar{\partial}$ of a potential must satisfy, followed by a proof of Theorem A(iii) where a normalised Hamiltonian potential of $JX$ with respect to $\omega$ is given. Throughout this section we write $r := |z|^\lambda$, where $z$ is the holomorphic coordinate on the $\mathbb{C}$-factor of $\widehat{M}$ and $\lambda > 0$ is as in Theorem A(iii) so that $d\nu(X) = r\partial_r$. Our starting point is:

**Proposition 4.1.** *Let $\omega$ be the Kähler metric in Proposition 3.1 and let $J$ denote the complex structure on $M$. Then there exists $\tilde{\varphi} \in C^\infty(M)$ with $\mathcal{L}_{JX}\tilde{\varphi} = 0$ and $\omega_{\tilde{\varphi}} := \omega + i\partial\bar{\partial}\tilde{\varphi} > 0$ such that*

$$\rho_{\omega_{\tilde{\varphi}}} + \frac{1}{2}\mathcal{L}_X\omega_{\tilde{\varphi}} = \omega_{\tilde{\varphi}} \tag{4.1}$$

*if and only if for all $a \in \mathbb{R}$, there exists $\varphi \in C^{\infty}(M)$ with $\mathcal{L}_{JX}\varphi = 0$ and $\omega + i\partial\bar{\partial}\varphi > 0$ and $F_2 \in C^{\infty}(M)$ compactly supported with $\mathcal{L}_{JX}F_2 = 0$ satisfying*

$$\rho_{\omega} + \frac{1}{2}\mathcal{L}_X\omega - \omega = i\partial\bar{\partial}F_2 \tag{4.2}$$

*such that*

$$\log\left(\frac{(\omega + i\partial\bar{\partial}\varphi)^n}{\omega^n}\right) - \frac{X}{2}\cdot\varphi + \varphi = F_2 + a. \tag{4.3}$$

*Here, $\rho_{\omega}$ and $\rho_{\omega_{\tilde{\varphi}}}$ denote the Ricci form of $\omega$ and $\omega_{\tilde{\varphi}}$ respectively.*

*Proof.* If $\varphi$ satisfies (4.3), then by taking $i\partial\bar{\partial}$ of this equation, we see that $\varphi$ satisfies (4.1) by virtue of (3.1). Conversely, assume that (4.1) holds. Then we compute:

$$\begin{aligned}
0 &= \rho_{\omega_{\tilde{\varphi}}} - \omega_{\tilde{\varphi}} + \frac{1}{2}\mathcal{L}_X\omega_{\tilde{\varphi}}\\
&= \rho_{\omega_{\tilde{\varphi}}} - \rho_{\omega} + \rho_{\omega} - \omega_{\tilde{\varphi}} + \frac{1}{2}\mathcal{L}_X\omega_{\tilde{\varphi}}\\
&= -i\partial\bar{\partial}\log\left(\frac{(\omega + i\partial\bar{\partial}\tilde{\varphi})^n}{\omega^n}\right) - i\partial\bar{\partial}\tilde{\varphi} + i\partial\bar{\partial}\left(\frac{X}{2}\cdot\tilde{\varphi}\right) + \rho_{\omega} - \omega + \frac{1}{2}\mathcal{L}_X\omega,
\end{aligned} \tag{4.4}$$

so that

$$i\partial\bar{\partial}\left(\tilde{\varphi} + \log\left(\frac{(\omega + i\partial\bar{\partial}\tilde{\varphi})^n}{\omega^n}\right) - \frac{X}{2}\cdot\tilde{\varphi}\right) = \rho_{\omega} - \omega + \frac{1}{2}\mathcal{L}_X\omega. \tag{4.5}$$

Now, as we have seen in (3.1),

$$\rho_{\omega} - \omega + \frac{1}{2}\mathcal{L}_X\omega = i\partial\bar{\partial}F_1$$

for some $JX$-invariant compactly supported $F_1 \in C^{\infty}(M)$. Plugging this into (4.5), we find that for every $a \in \mathbb{R}$,

$$i\partial\bar{\partial}\left(\tilde{\varphi} + \log\frac{(\omega + i\partial\bar{\partial}\tilde{\varphi})^n}{\omega^n} - \frac{X}{2}\cdot\tilde{\varphi} - F_1 - a\right) = 0.$$

$JX$-invariance of the sum in parentheses next implies from Lemma 2.4 that

$$\tilde{\varphi} + \log\left(\frac{(\omega + i\partial\bar{\partial}\tilde{\varphi})^n}{\omega^n}\right) - \frac{X}{2}\cdot\tilde{\varphi} = F_1 + a + H$$

for $H$ a pluriharmonic function equal to $c_0\log(r) + c_1$ outside a compact subset of $M$ for some $c_0$, $c_1 \in \mathbb{R}$. Thus,

$$\begin{aligned}
&\left(\tilde{\varphi} - H - \frac{c_0}{2}\right) + \log\left(\frac{(\omega + i\partial\bar{\partial}(\tilde{\varphi} - H - \frac{c_0}{2}))^n}{\omega^n}\right) - \frac{X}{2}\cdot\left(\tilde{\varphi} - H - \frac{c_0}{2}\right)\\
&\quad= \left(\tilde{\varphi} + \log\left(\frac{(\omega + i\partial\bar{\partial}\tilde{\varphi})^n}{\omega^n}\right) - \frac{X}{2}\cdot\tilde{\varphi}\right) - H + \frac{X}{2}\cdot H - \frac{c_0}{2}\\
&\quad= (F_1 + a + H) - H + \frac{X}{2}\cdot H - \frac{c_0}{2}\\
&\quad= F_1 + a + \frac{X}{2}\cdot H - \frac{c_0}{2}.
\end{aligned}$$

Notice that after identifying $X$ with $r\partial_r$ via $\nu$, we find that $\frac{X}{2}\cdot H - \frac{c_0}{2} = \frac{1}{2}r\partial_r(c_0\log(r) + c_1) - \frac{c_0}{2} = 0$ outside a compact set. Set $\varphi := \tilde{\varphi} - H - \frac{c_0}{2}$ and $F_2 := F_1 + \frac{X}{2}\cdot H - \frac{c_0}{2}$. Then $F_2 \in C^{\infty}(M)$, is compactly supported, both $\varphi$ and $F_2$ are $JX$-invariant, $i\partial\bar{\partial}F_2 = i\partial\bar{\partial}F_1$, and

$$\varphi + \log\left(\frac{(\omega + i\partial\bar{\partial}\varphi)^n}{\omega^n}\right) - \frac{X}{2}\cdot\varphi = F_2 + a,$$

as required. □

From Proposition 4.1, Theorem A(iv) follows as is demonstrated by the next lemma.

**Lemma 4.2.** *Let $\lambda$, $\omega$, and $\nu : (M \setminus K, \omega) \to (\widehat{M} \setminus \widehat{K}, \widehat{\omega})$, $K \subset M$, $\widehat{K} \subset \widehat{M}$ compact, be as in Proposition 3.1 respectively. Moreover, let $F_2 \in C^\infty(M)$ be as in Proposition 4.1 satisfying (4.2) and recall that $z$ denotes the holomorphic coordinate on the $\mathbb{C}$-factor of $\widehat{M}$. Let $r := |z|^\lambda$. Then there exists a unique torus-invariant smooth real-valued function $f : M \to \mathbb{R}$ such that $-\omega \lrcorner JX = df$, and $f = \nu^*\left(\frac{r^2}{2} - 1\right)$ and*

$$\Delta_\omega f + f - \frac{X}{2} \cdot f = 0 \tag{4.6}$$

*on $M \setminus K$. In particular, $f \to +\infty$ as $r \to +\infty$, hence is proper.*

*Proof.* The fact that $M$ doesn't split off any $S^1$-factors and is toric implies that $\pi_1(M) = 0$ [CLS11]. This gives us a smooth real-valued function $f \in C^\infty(M)$, defined up to a constant, with $-\omega \lrcorner JX = df$. Any such choice of $f$ is invariant under the action of $T$ by virtue of the fact that $\omega \lrcorner JX$ is invariant under this action and $T$ has fixed points so that every element of $\mathfrak{t}$ has at least one zero. Next notice that $-\widehat{\omega} \lrcorner \widehat{J} r\partial_r = d\left(\frac{r^2}{2}\right)$, where recall $\widehat{J}$ is the complex structure on $\widehat{M}$. As $\omega = \nu^*\widehat{\omega}$, it is therefore clear that $d\left(f - \frac{r^2}{2}\right) = 0$ on $M \setminus K$ so that $f$ differs from $\frac{r^2}{2}$ by a constant on this set, i.e., $f = \frac{r^2}{2} + \text{const.}$ on $M \setminus K$. Normalise $f$ so that this constant is equal to $-1$. Then $f = \nu^*\left(\frac{r^2}{2} - 1\right)$ outside a compact set. What remains to show is that with this normalisation, (4.6) holds true. To this end, using the $JX$-invariance of $F_2$ and $f$, contract (4.2) with $X^{1,0} := \frac{1}{2}(X - iJX)$ and use the Bochner formula to derive that

$$i\bar{\partial}\left(\Delta_\omega f - \frac{X}{2} \cdot f + f + \frac{X}{2} \cdot F_2\right) = 0.$$

As a real-valued holomorphic function, we must therefore have that $\Delta_\omega f - \frac{X}{2} \cdot f + f + \frac{X}{2} \cdot F_2$ is constant on $M$. But since $X \cdot F_2 = 0$ outside a compact subset of $M$, by the properties of $f$ and $\omega$ we have that outside a compact subset of $M$,

$$\Delta_\omega f - \frac{X}{2} \cdot f + f + \frac{X}{2} \cdot F_2 = \Delta_{\widehat{\omega}}\left(\frac{r^2}{2} - 1\right) - \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{r^2}{2} - 1\right) + \left(\frac{r^2}{2} - 1\right) = 0.$$

Thus, this constant is zero and we are done. ☐

Let $c_0 \in \mathbb{R}$ be such that $e^{c_0}\int_M e^{F_2 - f}\omega^n = \int_M e^{-f}\omega^n$ and define $F := F_2 + c_0$. Then

- $F \in C^\infty(M)$ and $F$ is torus-invariant,
- $F$ is equal to $c_0$ outside a compact subset of $M$, and
- $\int_M e^{F-f}\omega^n = \int_M e^{-f}\omega^n$.

Moreover, from (4.2) we have that

$$\rho_\omega - \frac{1}{2}\mathcal{L}_X\omega + \omega = i\partial\bar{\partial}F.$$

By Proposition 4.1, any shrinking gradient Kähler-Ricci soliton of the form $\omega + i\partial\bar{\partial}\varphi > 0$ on $M$ will solve the complex Monge-Ampère equation

$$\begin{cases} (\omega + i\partial\bar{\partial}\varphi)^n = e^{F + \frac{X}{2}\cdot\varphi - \varphi}\omega^n & \text{for } \varphi \in C^\infty(M) \text{ and } \varphi \text{ torus-invariant,} \\ \int_M e^{F-f}\omega^n = \int_M e^{-f}\omega^n. & \end{cases} \tag{4.7}$$

This is precisely the statement of Theorem A(iv). A strategy to solve this equation is given by considering the Aubin continuity path:

$$\begin{cases} (\omega + i\partial\bar{\partial}\varphi_t)^n = e^{F + \frac{X}{2}\cdot\varphi_t - t\varphi_t}\omega^n, & \varphi \in C^\infty(M), \quad \mathcal{L}_{JX}\varphi = 0, \quad \omega + i\partial\bar{\partial}\varphi > 0, \quad t \in [0, 1], \\ \int_M e^{F-f}\omega^n = \int_M e^{-f}\omega^n. & \end{cases} \tag{$*_t$}$$

The equation corresponding to $t = 0$ is given by

$$\begin{cases} (\omega + i\partial\bar{\partial}\psi)^n = e^{F+\frac{X}{2}\cdot\psi}\omega^n, \qquad \psi \in C^\infty(M), \qquad \mathcal{L}_{JX}\psi = 0, \qquad \omega + i\partial\bar{\partial}\psi > 0, \\ \int_M e^{F-f}\omega^n = \int_M e^{-f}\omega^n. \end{cases} \tag{$*_0$}$$

This equation we will solve by the continuity method, the particular path of which will be introduced in Section 7.1. This will yield the final part of Theorem A. Beforehand however, we prove some analytic results regarding the metric $\omega$ and those metrics asymptotic to it, beginning with a Poincaré inequality.

## 5. Poincaré inequality

In this section, we prove a Poincaré inequality for the Kähler form $\omega$ of Proposition 3.1 using the fact that it holds true on the model shrinking gradient Ricci soliton $(\widehat{M} := \mathbb{C}\times\mathbb{P}^1,\, \widehat{\omega} := \tilde{\omega}_\lambda + \omega_D,\, r\partial_r)$ [Mil09], where $r = |z|^\lambda$. This will be used in Proposition 7.9 to establish an a priori weighted $L^2$-estimate along the continuity path that we consider in deriving a solution to $(*_0)$. Recall the Hamiltonian potential $f$ of $JX$ satisfying (4.6).

We work with the Lebesgue and Sobolev spaces $L^p(e^{-f}\omega^n)$ and $W^{1,p}(e^{-f}\omega^n)$ on $M$ respectively, defined in the obvious way for $p > 1$.

**Proposition 5.1** (Poincaré inequality)**.** *For all $p > 1$, there exists a constant $C(p) > 0$ such that*

$$\left\| u - \int_M u\, e^{-f}\omega^n \right\|_{L^p(e^{-f}\omega^n)} \leq C(p)\|\nabla^g u\|_{L^p(e^{-f}\omega^n)} \qquad \text{for all } u \in W^{1,p}(e^{-f}\omega^n) \cap C^1(M).$$

*Here, $g$ is the Kähler metric associated to $\omega$.*

*Proof.* For sake of a contradiction, suppose that the assertion is not true. Then there exists a sequence of functions $(u_k)_{k\geq 1} \subset W^{1,p}(e^{-f}\omega^n)$ with the following properties:

$$\begin{cases} \|u_k\|_{L^p(e^{-f}\omega^n)} = 1, \qquad \int_M u_k\, e^{-f}\omega^n = 0, \\ \|\nabla^g u_k\|_{L^p(e^{-f}\omega^n)} \leq \frac{1}{k}. \end{cases}$$

By the Rellich-Kondrachov theorem, there exists a subsequence which we also denote by $(u_k)_{k\geq 1}$ converging to some $u_\infty \in L^p_{\text{loc}}(M)$ as $k \to +\infty$. On the other hand, for every compactly supported one-form $\alpha$ on $M$, we have that

$$\int_M u_\infty \cdot \delta^g\alpha\, \omega^n = \lim_{k\to+\infty}\int_M u_k \cdot \delta^g\alpha\, \omega^n = -\lim_{k\to+\infty}\int_M g(du_k,\, \alpha)\, \omega^n = 0,$$

where $\delta_g$ is the co-differential of $d$ with respect to $g$. Thus, $u_\infty \in W^{1,p}_{\text{loc}}(M)$ and $du_\infty = 0$ almost everywhere. In particular, $u_\infty$ is a constant.

For $R > 0$, let $D_R := f^{-1}((-\infty,\, R])$, a compact subset of $M$ by properness of $f$ (cf. Lemma 4.2). Then the fact that $\int_M u_k\, e^{-f}\omega^n = 0$ implies that for every $R > 0$,

$$\int_{D_R} u_k\, e^{-f}\omega^n = -\int_{M\setminus D_R} u_k\, e^{-f}\omega^n$$

so that by Hölder's inequality,

$$\begin{aligned}\left|\int_{D_R} u_k\, e^{-f}\omega^n\right| &\leq \int_{M\setminus D_R} |u_k|\, e^{-f}\omega^n\\ &\leq \left(\int_{M\setminus D_R} |u_k|^p\, e^{-f}\omega^n\right)^{\frac{1}{p}}\left(\int_{M\setminus D_R} e^{-f}\omega^n\right)^{1-\frac{1}{p}}\\ &\leq \|u_k\|_{L^p(e^{-f}\omega^n)}\left(\int_{M\setminus D_R} e^{-f}\omega^n\right)^{1-\frac{1}{p}}\\ &\leq \left(\int_{M\setminus D_R} e^{-f}\omega^n\right)^{1-\frac{1}{p}}.\end{aligned}$$

Furthermore, $L^p_{\mathrm{loc}}(M)$-convergence implies that

$$\int_{D_R} u_k\, e^{-f}\omega^n \to \int_{D_R} u_\infty\, e^{-f}\omega^n = u_\infty\,\mathrm{vol}_f(D_R) \qquad \text{as } k\to+\infty,$$

and so we derive that

$$|u_\infty| = \lim_{k\to+\infty}\frac{\left|\int_{D_R} u_k\, e^{-f}\omega^n\right|}{\mathrm{vol}_f(D_R)} \leq \lim_{k\to+\infty}\frac{\left(\int_{M\setminus D_R} e^{-f}\omega^n\right)^{1-\frac{1}{p}}}{\mathrm{vol}_f(D_R)} = \frac{\mathrm{vol}_f(M\setminus D_R)^{1-\frac{1}{p}}}{\mathrm{vol}_f(D_R)} \to 0 \quad \text{as } R\to+\infty,$$

where $\mathrm{vol}_f(A) := \int_A e^{-f}\omega^n$ for $A\subseteq M$. That is, $u_\infty\equiv 0$.

Next, let $\eta:\mathbb{R}\to\mathbb{R}$ be a smooth function satisfying $\eta(x)=0$ for $x\leq 1$, $\eta(x)=1$ for $x\geq 2$, and $|\eta(x)|\leq 1$ for all $x$, and define $\eta_R : M\to\mathbb{R}$ by

$$\eta_R(x) = \eta\left(\frac{\sqrt{f(x)}}{R}\right) \qquad \text{for } R>0 \text{ a positive constant to be chosen later.}$$

Then with $\frac{1}{p}+\frac{1}{q}=1$, we have that

$$\begin{aligned}1 &= \|u_k\|^p_{L^p(e^{-f}\omega^n)} \leq C(p)\left(\|(1-\eta_R)u_k\|^p_{L^p(e^{-f}\omega^n)} + \|\eta_R u_k\|^p_{L^p(e^{-f}\omega^n)}\right)\\ &\leq C(p)\left(\int_{D_R}|u_k|^p\, e^{-f}\omega^n + \int_M |\eta_R u_k|^p\, e^{-f}\omega^n\right)\\ &\leq C(p)\left(\int_{D_R}|u_k|^p\, e^{-f}\omega^n + \int_M\left|\eta_R u_k - \int_M \eta_R u_k\, e^{-f}\omega^n\right|^p e^{-f}\omega^n + \left|\int_M \eta_R u_k\, e^{-f}\omega^n\right|^p\right)\\ &\leq C(p)\left(\int_{D_R}|u_k|^p\, e^{-f}\omega^n + \int_M\left|\eta_R u_k - \int_M \eta_R u_k\, e^{-f}\omega^n\right|^p e^{-f}\omega^n + \|u_k\|_{L^p(e^{-f}\omega^n)}\|\eta_R\|^p_{L^q(e^{-f}\omega^n)}\right)\\ &\leq C(p)\left(\int_{D_R}|u_k|^p\, e^{-f}\omega^n + \int_M\left|\eta_R u_k - \int_M \eta_R u_k\, e^{-f}\omega^n\right|^p e^{-f}\omega^n + \mathrm{vol}_f\left(M\setminus D_{\frac{R^2}{2}}\right)^{\frac{p}{q}}\right).\end{aligned} \tag{5.1}$$

Now, for $R>0$ sufficiently large, $\eta_R u_k$ is supported on the set where $\omega$ is isometric to $\widehat{\omega}$ via the biholomorphism $\nu$ of Proposition 3.1, a manifold on which we know that the assertion already holds true [Mil09]. Applying this observation to the middle term in the last line of (5.1), we arrive at the fact that for $R > 0$ sufficiently large,

$$\begin{aligned}1 &\leq C(p)\left(\int_{D_R}|u_k|^p\,e^{-f}\omega^n + \|\nabla^g(\eta_R u_k)\|^p_{L^p(e^{-f}\omega^n)} + \mathrm{vol}_f\left(M\setminus D_{\frac{R^2}{2}}\right)^{\frac{p}{q}}\right)\\ &\leq C(p)\left(\int_{D_R}|u_k|^p\,e^{-f}\omega^n + \|\nabla^g\eta_R\|^p_{L^\infty(M)}\|u_k\|^p_{L^p(e^{-f}\omega^n)} + \|\nabla^g u_k\|^p_{L^p(e^{-f}\omega^n)} + \mathrm{vol}_f\left(M\setminus D_{\frac{R^2}{2}}\right)^{\frac{p}{q}}\right)\\ &\leq C(p)\left(\int_{D_R}|u_k|^p\,e^{-f}\omega^n + \frac{1}{R^p} + \frac{1}{k^p} + \mathrm{vol}_f\left(M\setminus D_{\frac{R^2}{2}}\right)^{\frac{p}{q}}\right).\end{aligned}$$

As $u_k \to 0$ in $L^p_{\mathrm{loc}}(M)$ as $k \to +\infty$, we see upon letting $k \to +\infty$ that for all $R > 0$ sufficiently large,

$$1 \leq C(p)\left(\frac{1}{R^p} + \mathrm{vol}_f\left(M\setminus D_{\frac{R^2}{2}}\right)^{\frac{p}{q}}\right).$$

Letting $R \to +\infty$ now yields the desired contradiction. □

## 6. Linear theory

Working in the setting and notation of Theorem A, we set up the linear theory for metrics asymptotic to $\omega$. This will prove openness in the continuity path that we apply to solve $(*_0)$. Although Theorem A works with toric-invariant functions, in order to be as broadly applicable as possible, we present the linear theory under minimal assumptions, namely for $JX$-invariant functions.

6.1. **Main setting.** Let $\tilde{g}$ be any $JX$-invariant Kähler metric on $M$ with Kähler form $\tilde{\omega}$ and Levi-Civita connection $\nabla^{\tilde{g}}$ satisfying

$$|(\nabla^g)^i\mathcal{L}_X^{(j)}(\tilde{\omega}-\omega)|_g = O(r^{-\gamma}) \qquad \text{for all } i,\, j \geq 0, \tag{6.1}$$

for some $\gamma \in (0, \lambda^D)$, where $\lambda^D$ is the first non-zero eigenvalue of $-\Delta_D$ acting on $L^2$-functions on $D$ and $r = |z|^\lambda$. We write $X = \nabla^{\tilde{g}}\tilde{f}$ for some smooth function $\tilde{f} : M \to \mathbb{R}$, a function which is guaranteed to exist because, as noted previously $H^1(M) = 0$ by toricity, and which is defined up to an additive constant. We use $\nu$ to identify $M$ and $\widehat{M}$ so that $X = r\partial_r$ outside a compact set. Since $\nabla^g f = X = \nabla^{\tilde{g}}\tilde{f}$, it follows from (6.1) that $|f - \tilde{f}| = O(r^{-\gamma+2})$ as $r \to +\infty$.

We begin by identifying a good barrier function for our particular geometric setup.

**Lemma 6.1.** *For all $\delta \in (0, 1)$, there exists $R(\delta) > 0$ such that the function $e^{\delta f}$ is a sub-solution of the following equation:*

$$\Delta_{\tilde{g},X}e^{\delta f} \leq 0 \qquad \text{on } f \geq R(\delta).$$

*Moreover, the logarithm and polynomial powers of $f := \frac{|z|^{2\lambda}}{2\lambda} - 1$ satisfy for all $\delta > 0$,*

$$\Delta_{\tilde{g},X}f^{-\delta} = 2\delta f^{-\delta} + O(f^{-\delta-1}),\quad \Delta_{\tilde{g},X}\log(f+1) = -2, \qquad \text{outside a compact subset of } M.$$

*Proof.* Using (6.1), we compute that

$$\begin{aligned}\Delta_{\tilde{g},X}e^{\delta f} &= \left(\delta\Delta_{\tilde{g},X}f + \delta^2|\nabla^{\tilde{g}}f|^2_{\tilde{g}}\right)e^{\delta f}\\ &= \delta\left(\Delta_{g,X}f + (\Delta_{\tilde{g},X}-\Delta_{g,X})f + \delta|\nabla^{\tilde{g}}f|^2_{\tilde{g}}\right)e^{\delta f}\\ &= \delta\left(-2f + \delta|\nabla^{\tilde{g}}f|^2_{\tilde{g}} + O(|\tilde{g}-g|_{\tilde{g}})\right)e^{\delta f}\\ &= \delta\left(-2f + \delta|X|^2_g(1+o(1)) + o(1)\right)e^{\delta f}\\ &\leq 0\end{aligned}$$

outside a sufficiently large compact set $K$ of $M$. Here we have also used the fact that $|X|^2_g = 2f + 2$ and $\delta \in (0, 1)$ in the last line.

A similar computation based on the asymptotics of $\tilde{g}$ dictated by (6.1) shows that

$$\begin{aligned}\Delta_{\tilde{g},X}f^{-\delta} &= (\Delta_{\tilde{g}} - X)(f^{-\delta})\\ &= -\delta f^{-\delta-1}(\Delta_{\tilde{g}}f - X\cdot f) + \delta(\delta+1)f^{-\delta-2}|\nabla^{\tilde{g}}f|^2_{\tilde{g}}\\ &= -\delta f^{-\delta-1}(\Delta_{g}f - X\cdot f) - \delta f^{-\delta-1}(\Delta_{\tilde{g}}f - \Delta_g f) + \delta(\delta+1)f^{-\delta-2}|\nabla^{\tilde{g}}f|^2_{\tilde{g}}\\ &= 2\delta f^{-\delta} - \delta f^{-\delta-1}\underbrace{(\Delta_{\tilde{g}}f - \Delta_g f)}_{=O(|\tilde{g}-g|_g)} + \delta(\delta+1)f^{-\delta-2}\underbrace{|\nabla^{\tilde{g}}f|^2_{\tilde{g}}}_{=O(|X|^2_{\tilde{g}})=O(f)}\\ &= 2\delta f^{-\delta} + O(f^{-\delta-1}).\end{aligned}$$

As $\log(r^2)$ is pluriharmonic outside a compact set, the fact that $X = r\partial_r$ outside a compact set implies that

$$\Delta_{\tilde{g},X}\log(f+1) = \Delta_{\tilde{g},X}\log\left(r^2\right) = -2,$$

as claimed. □

6.2. **Function spaces.** We next define the function spaces within which we will work.

- For $\beta\in\mathbb{R}$ and $k$ a non-negative integer, define $C^{2k}_{X,\beta}(M)$ to be the space of $JX$-invariant continuous functions $u$ on $M$ with $2k$ continuous derivatives such that
$$\|u\|_{C^{2k}_{X,\beta}} := \sum_{i+2j\leq 2k}\sup_M\left|f^{\frac{\beta}{2}}(\nabla^{\tilde{g}})^i\left(\mathcal{L}_X^{(j)}u\right)\right|_{\tilde{g}} < \infty.$$
This norm is equivalent to that defined with respect to the background metric $g$ thanks to (6.1). This allows us to use either $\tilde{g}$ or $g$ depending on the context. Similarly, since $f$ and $\tilde{f}$ are equivalent at infinity, these spaces can be arbitrarily defined in terms of one of these potential functions. Define $C^{\infty}_{X,\beta}(M)$ to be the intersection of the spaces $C^{2k}_{X,\beta}(M)$ over all $k\in\mathbb{N}_0$.

- Let $\delta(\tilde{g})$ denote the injectivity radius of $\tilde{g}$, write $d_{\tilde{g}}(x,y)$ for the distance with respect to $\tilde{g}$ between two points $x,y\in M$, and let $\varphi^X_t$ denote the flow of $X$ for time $t$. A tensor $T$ on $M$ is said to be in $C^{0,2\alpha}_{\beta}(M)$, $\alpha\in\left(0,\frac{1}{2}\right)$, if
$$\begin{aligned}[T]_{C^{0,2\alpha}_{\beta}} := &\sup_{\substack{x\neq y\in M\\ d_{\tilde{g}}(x,y)<\delta(\tilde{g})}}\left[\min(f(x),f(y))^{\frac{\beta}{2}}\frac{|T(x)-P_{x,y}T(y)|_h}{d_{\tilde{g}}(x,y)^{2\alpha}}\right]\\ &+\sup_{\substack{x\in M\\ t\neq s\geq 1}}\left[\min(t,s)^{\frac{\beta}{2}}\frac{|(\varphi^X_t)_*T(x) - (\widehat{P}_{\varphi^X_s(x),\varphi^X_t(x)}((\varphi^X_s)_*T(x)))|_h}{|t-s|^{\alpha}}\right] < \infty,\end{aligned}$$
where $P_{x,y}$ denotes parallel transport along the unique geodesic joining $x$ and $y$, and $\widehat{P}_{\varphi^X_s(x),\varphi^X_t(x)}$ denotes parallel transport along the unique flow-line of $X$ joining $\varphi^X_s(x)$ and $\varphi^X_t(x)$.
- For $\beta\in\mathbb{R}$, $k$ a non-negative integer, and $\alpha\in\left(0,\frac{1}{2}\right)$, define the Hölder space $C^{2k,2\alpha}_{X,\beta}(M)$ with polynomial weight $f^{\frac{\beta}{2}}$ to be the set of $u\in C^{2k}_{X,\beta}(M)$ for which the norm
$$\|u\|_{C^{2k,2\alpha}_{X,\beta}} := \|u\|_{C^{2k}_{X,\beta}} + \sum_{i+2j=2k}\left[(\nabla^{\tilde{g}})^i\left(\mathcal{L}^{(j)}_X u\right)\right]_{C^{0,2\alpha}_{\beta}}$$
is finite. It is straightforward to check that the space $C^{2k,2\alpha}_{X,\beta}(M)$ is a Banach space. The intersection $\bigcap_{k\geq 0}C^{2k}_{X,\beta}(M)$ we denote by $C^{\infty}_{X,\beta}(M)$.
- We now consider a smooth cut-off function $\chi : M\to[0,1]$ which equals 1 outside a compact set. The source function space $\mathcal{D}^{2k+2,2\alpha}_{X,\beta}(M)$ is defined as:
$$\mathcal{D}^{2k+2,2\alpha}_{X,\beta}(M) := \left(\mathbb{R}\chi\log r\oplus\mathbb{R}\oplus C^{2k+2,2\alpha}_{X,\beta}(M)\right),$$

endowed with the norm

$$\|u\|_{\mathcal{D}^{2k+2,\,2\alpha}_{X,\,\beta}} := |c_1| + |c_2| + \|\tilde{u}\|_{C^{2k+2,\,2\alpha}_{X,\,\beta}},$$
$$u := c_1\chi\log r + c_2 + \tilde{u}.$$

The target function space is defined as

$$\mathcal{C}^{2k,\,2\alpha}_{X,\,\beta}(M) := \left(\mathbb{R}\oplus C^{2k,\,2\alpha}_{X,\,\beta}(M)\right),$$

endowed with the norm defined analogously as above. Define

$$\mathcal{C}^{\infty}_{X,\,\beta}(M) := \bigcap_{k\geq 0}\mathcal{C}^{2k,\,2\alpha}_{X,\,\beta}(M).$$

- Finally, we define the spaces

$$\mathcal{M}^{2k+2,\,2\alpha}_{X,\,\beta}(M) := \left\{\varphi\in C^2_{\mathrm{loc}}(M)\,|\,\tilde{\omega}+i\partial\bar{\partial}\varphi>0\right\}\bigcap\mathcal{D}^{2k+2,\,2\alpha}_{X,\,\beta}(M),$$

and one considers the following convex set of Kähler potentials:

$$\mathcal{M}^{\infty}_{X,\,\beta}(M) = \bigcap_{k\geq 0}\mathcal{M}^{2k+2,\,2\alpha}_{X,\,\beta}(M).$$

Notice that for each $k\geq 0$, the spaces $\mathcal{M}^{2k+2,\,2\alpha}_{X,\,\beta}(M)$ depend on the choice of a background metric $\tilde{\omega}$. However, these spaces are equivalent whenever $\tilde{\omega}$ satisfies (6.1).

6.3. **Preliminaries and Fredholm properties of the linearised operator.** We proceed with the same set-up as in Section 6.1, beginning with the following useful observation.

**Lemma 6.2.** *Let $(\varphi_t)_{t\in[0,\,1]}$ be a $C^1$-path of smooth functions in $\mathcal{M}^{\infty}_{X,\,\beta}(M)$ for some $\beta>0$ and write $\tilde{\omega}_t := \tilde{\omega}+i\partial\bar{\partial}\varphi_t>0$ and $\tilde{f}_t := \tilde{f}+\frac{X}{2}\cdot\varphi_t$ so that $-d\tilde{\omega}_t\lrcorner JX = d\tilde{f}_t$.*

(i) *Let $G:\mathbb{R}\to\mathbb{R}$ be a $C^1$-function such that for some $-\infty<\alpha<1$, $|G(x)|+|G'(x)|\leq e^{\alpha x}$, $x\geq -C$. Then*

$$\int_M G(\tilde{f}_t)\,e^{-\tilde{f}_t}\tilde{\omega}_t^n = \int_M G(f_{\tilde{\omega}_0})\,e^{-f_{\tilde{\omega}_0}}\tilde{\omega}_0^n,\qquad t\in[0,\,1]. \tag{6.2}$$

(ii) $\int_0^1\int_M|\dot{\varphi}_t|\,e^{-\tilde{f}_t}\tilde{\omega}_t^n\,dt<\infty$ *and* $\int_0^1\int_M|\dot{\varphi}_t|\,e^{-\tilde{f}}\tilde{\omega}^n\,dt<\infty$.

*Proof.* (i) By differentiating, one obtains

$$\begin{aligned}\frac{d}{dt}\left(\int_M G(\tilde{f}_t)\,e^{-\tilde{f}_t}\tilde{\omega}_t^n\right) &= \int_M G'(\tilde{f}_t)\frac{X}{2}\cdot\dot{\varphi}_t\,e^{-\tilde{f}_t}\tilde{\omega}_t^n + \int_M G(\tilde{f}_t)\left(\Delta_{\tilde{\omega}_t}\dot{\varphi}_t-\frac{X}{2}\cdot\dot{\varphi}_t\right)e^{-\tilde{f}_t}\tilde{\omega}_t^n\\ &= \int_M G'(\tilde{f}_t)\frac{X}{2}\cdot\dot{\varphi}_t\,e^{-\tilde{f}_t}\tilde{\omega}_t^n - \frac{1}{2}\int_M G'(\tilde{f}_t)\nabla^{g_{\varphi_t}}\tilde{f}_t\cdot\dot{\varphi}_t\,e^{-\tilde{f}_t}\tilde{\omega}_t^n\\ &= 0.\end{aligned}$$

Here, we have used integration by parts together with the fact that $X=\nabla^{\tilde{g}_t}\tilde{f}_t$ for all $t\in[0,\,1]$, where $\tilde{g}_t$ denotes the Kähler metric associated to $\tilde{\omega}_t$.

(ii) First note that by definition of the function space, the weighted measures $e^{-\tilde{f}_t}\tilde{\omega}_t^n$ and $e^{-\tilde{f}}\tilde{\omega}^n$ are equivalent to each other. Therefore it suffices to verify only that $\int_0^1\int_M|\dot{\varphi}_t|\,e^{-\tilde{f}}\tilde{\omega}^n\,dt<\infty$. But from the definition of the function space and $\tilde{\omega}$, this is trivially satisfied.

□

Next, define the following map as in [Sie13]:

$$MA_{\tilde{\omega}}:\psi\in\left\{\varphi\in C^2_{\mathrm{loc}}(M)\,|\,\tilde{\omega}_\varphi:=\tilde{\omega}+i\partial\bar{\partial}\varphi>0\right\}\mapsto\log\left(\frac{\tilde{\omega}_\psi^n}{\tilde{\omega}^n}\right)-\frac{X}{2}\cdot\psi\in\mathbb{R}.$$

For any $\psi \in C^2_{\mathrm{loc}}(M)$, let $\tilde{g}_\psi$ (respectively $\tilde{g}_{t\psi}$) denote the Kähler metric associated to the Kähler form $\tilde{\omega}_\psi$ (resp. $\tilde{\omega}_{t\psi}$ for any $t \in [0,\,1]$). Brute force computations show that

$$\begin{aligned} MA_{\tilde{\omega}}(0) &= 0,\\ D_\psi MA_{\tilde{\omega}}(u) &= \Delta_{\tilde{\omega}_\psi} u - \frac{X}{2}\cdot u, \quad u \in C^2_{\mathrm{loc}}(M),\\ \frac{d^2}{dt^2}\left(MA_{\tilde{\omega}}(t\psi)\right) &= \frac{d}{dt}(\Delta_{\tilde{\omega}_{t\psi}}\psi) = -|\partial\bar{\partial}\psi|^2_{\tilde{g}_{t\psi}} \quad \text{for } t \in [0,\,1],\\ MA_{\tilde{\omega}}(\psi) &= MA_{\tilde{\omega}}(0) + \frac{d}{dt}\Big|_{t=0} MA_{\tilde{\omega}}(t\psi) + \int_0^1\int_0^u \frac{d^2}{dt^2}(MA_{\tilde{\omega}}(t\psi))\,dt\,du\\ &= \Delta_{\tilde{\omega}}\psi - \frac{X}{2}\cdot\psi - \int_0^1\int_0^u |\partial\bar{\partial}\psi|^2_{\tilde{g}_{t\psi}}\,dt\,du. \end{aligned} \tag{6.3}$$

Our main result of this section is that the drift Laplacian of $\tilde{g}$ is an isomorphism between polynomially weighted function spaces with zero mean value.

**Theorem 6.3.** *Let $\alpha \in \left(0,\,\frac{1}{2}\right)$, $k \in \mathbb{N}$, and $\beta \in (0, \lambda^D)$. Then the drift Laplacian*

$$\Delta_{\tilde{g},\,X} : \mathcal{D}^{2k+2,\,2\alpha}_{X,\,\beta}(M) \cap \left\{\int_M u\, e^{-\tilde{f}}\tilde{\omega}^n = 0\right\} \to \mathcal{C}^{2k,\,2\alpha}_{X,\,\beta}(M) \cap \left\{\int_M v\, e^{-\tilde{f}}\tilde{\omega}^n = 0\right\}$$

*is an isomorphism of Banach spaces.*

**Remark 6.4.** In the statement of Theorem 6.3, if $D = \mathbb{P}^1$ endowed with its metric of constant sectional curvature 1, the upper bound of the range $(0, \lambda^D)$ allowed for $\beta$ is $\lambda^D = 2$. In general, Lichnerowicz's estimate implies that $\lambda^D \geq 2$: see [Bal06, Theorem 6.14] for a proof. The polynomial rate $\gamma \in (0, \lambda^D)$ from assumption (6.1) can be any such number. In Section 6.4 we apply Theorem 6.3 with $\gamma = \beta$.

*Proof of Theorem 6.3.* Observe that given such data $F$, if we interpret this function as an element of the weighted $L^2$-space $L^2(e^{-\tilde{f}}\tilde{\omega}^n)$, then there exists a unique weak solution $u \in H^1(e^{-\tilde{f}}\tilde{\omega}^n)$ to the equation

$$\Delta_{\tilde{g},\,X} u = F, \tag{6.4}$$

with zero weighted-mean value. This can be proved by observing that the drift Laplacian $\Delta_{\tilde{g},\,X}$ is symmetric with respect to the weighted measure $e^{-\tilde{f}}\tilde{\omega}^n$, a measure with finite volume, and the fact that this operator has discrete $L^2(e^{-\tilde{f}}\tilde{\omega}^n)$-spectrum. Moreover, we have the estimate

$$\|u\|_{L^2(e^{-\tilde{f}}\tilde{\omega}^n)} + \|\nabla^{\tilde{g}} u\|_{L^2(e^{-\tilde{f}}\tilde{\omega}^n)} \leq C\|F\|_{L^2(e^{-\tilde{f}}\tilde{\omega}^n)} \leq C\|F\|_{C^0}, \tag{6.5}$$

for some positive constant $C$ independent of $u$ and $F$ that may vary from line to line. This estimate essentially follows from the weighted $L^2$-Poincaré inequality with respect to the drift Laplacian $\Delta_{\tilde{g}} - X\cdot$. We improve on the regularity of $u$ through a series of claims.

**Claim 6.5.** There exists a positive constant $C = C(\tilde{\omega}, n)$ such that

$$|u(x)| \leq C e^{\frac{\tilde{f}(x)}{2}}\|F\|_{C^0}, \qquad x \in M.$$

*Proof.* By conjugating (6.4) with a suitable weight, notice that the function $v := e^{-\frac{\tilde{f}}{2}}u$ satisfies

$$\Delta_{\tilde{g}} v = e^{-\frac{\tilde{f}}{2}}F + \left(\frac{1}{4}|X|^2_{\tilde{g}} - \frac{1}{2}\Delta_{\tilde{g}}\tilde{f}\right)v.$$

This implies that $|v|$ satisfies the following differential inequality in the weak sense:

$$\Delta_{\tilde{g}}|v| \geq -C|v| - C\|F\|_{C^0}. \tag{6.6}$$

Here we have made use of the non-negativity of $|X|^2_{\tilde{g}}$ together with the boundedness of $\Delta_{\tilde{g}}\tilde{f}$ given by (6.1).

We perform a local Nash-Moser iteration on (6.6) in $B_{\tilde{g}}(x, r)$. More precisely, since $(M^{2n}, \tilde{g})$ is a Riemannian manifold with Ricci curvature bounded from below, the results of [SC92] yield the following local Sobolev inequality:

$$\left(\frac{1}{\mathrm{vol}_{\tilde{g}}(B_{\tilde{g}}(x, r))}\int_{B_{\tilde{g}}(x,r)}|\varphi|^{\frac{2n}{n-1}}\,\tilde{\omega}^n\right)^{\frac{n-1}{n}} \leq \left(\frac{C(r_0)r^2}{\mathrm{vol}_{\tilde{g}}(B_{\tilde{g}}(x, r))}\int_{B_{\tilde{g}}(x,r)}|\widetilde{\nabla}\varphi|^2_{\tilde{g}}\,\tilde{\omega}^n\right) \tag{6.7}$$

for any $\varphi \in H^1_0(B_{\tilde{g}}(x, r))$ and for all $x \in M$ and $0 < r < r_0$, where $r_0$ is some fixed positive radius.

A Nash-Moser iteration proceeds in several steps. First, one multiplies (6.4) across by $\eta^2_{s,\,s'}v|v|^{2(p-1)}$ with $p \geq 1$, where $\eta_{s,\,s'}$, with $0 < s + s' < r$ and $s, s' > 0$, is a Lipschitz cut-off function with compact support in $B_{\tilde{g}}(x,\, s + s')$ equal to 1 on $B_{\tilde{g}}(x,\, s)$ and with $|\widetilde{\nabla}\eta_{s,\,s'}|_{\tilde{g}} \leq \frac{1}{s'}$ almost everywhere. One then integrates by parts and uses the Sobolev inequality of (6.7) to obtain a so-called "reversed Hölder inequality" which, after iteration, leads to the bound

$$\begin{aligned}\sup_{B_{\tilde{g}}(x,\,\frac{r}{2})}|v| &\leq C\left(\|v\|_{L^2(B_{\tilde{g}}(x,r))} + \|F\|_{L^\infty(B_{\tilde{g}}(x,r))}\right)\\ &\leq C\left(\|u\|_{L^2(e^{-\tilde{f}}\tilde{\omega}^n)} + \|F\|_{C^0(M)}\right)\\ &\leq C\|F\|_{C^0(M)}\end{aligned} \tag{6.8}$$

for $r \leq r_0$, where $C = C(r_0, \tilde{\omega}, n)$. Here we have made use of (6.5) in the last line. This estimate yields an a priori local $C^0$-estimate which is uniform on the center of the ball $B_{\tilde{g}}(x,\, \frac{r}{2})$. In particular, unravelling the definition of the function $v$, one obtains the expected a priori uniform exponential growth, namely

$$|u(x)| \leq Ce^{\frac{\tilde{f}(x)}{2}}\|F\|_{C^0}, \qquad x \in M.$$

□

Thanks to Claim 6.5, by local Schauder elliptic estimates, $u$ actually lies in $C^{2k+2,2\alpha}_{loc}$ and we have the estimates

$$\|u\|_{C^{2k+2\alpha}(\{\tilde{f}\,<\,R\})} \leq C\|F\|_{C^{2k,2\alpha}(\{\tilde{f}\,<\,2R\})} \leq C\|F\|_{C^{2k,\,2\alpha}_{X,\,\beta}} \tag{6.9}$$

for some positive constant $C = C(R, \tilde{\omega}, n)$.

We now move on to prove the expected a priori weighted estimates on $u$ and its derivatives.

**Claim 6.6.** There exists a positive constant $A = A(\tilde{\omega}, n)$ such that

$$|u(x)| \leq A\log\tilde{f}(x)\|F\|_{C^0}, \quad \tilde{f}(x) \geq 2.$$

*Proof.* Let $\varepsilon > 0$ and let $\delta \in (0, 1)$ be such that $\lim_{\tilde{f}\to+\infty}\left(u - \varepsilon e^{\delta\tilde{f}}\right) = -\infty$, parameters we can choose by Claim 6.5. For $A > 0$ to be determined later, we have outside a compact set $\{\tilde{f} \geq R(\delta)\}$,

$$\Delta_{\tilde{g},\,X}\left(u - A\log(f + 1) - \varepsilon e^{\delta\tilde{f}}\right) \geq -\|F\|_{C^0} + 2A > 0,$$

so long as $A > \frac{1}{2}\|F\|_{C^0}$. Here Lemma 6.1 has been applied. Appealing to the maximum principle then gives us the bound

$$\max_{\{\tilde{f}\geq R(\delta)\}}\left(u - A\log(f + 1) - \varepsilon e^{\delta\tilde{f}}\right) = \max_{\{\tilde{f}=R(\delta)\}}\left(u - A\log(f + 1) - \varepsilon e^{\delta\tilde{f}}\right).$$

Next, by letting $\varepsilon$ go to 0, we see that

$$u - A\log(f + 1) \leq \max_{\{\tilde{f}=R(\delta)\}}(u - A\log(f + 1)) \leq 0$$

if we set $A := C\max_{\{\tilde{f}=R(\delta)\}}u \leq C\|F\|_{C^0}$ with $C := C(\delta, \tilde{\omega}, n)$, something we can do thanks to (6.9).

The same argument applies to $-u$ which concludes the proof of the claim. □

Observe that $\tilde{u} := u - c\chi \log r$, where $F - c \in C^{2k,2\alpha}_{X,\beta}(M)$, satisfies the equation

$$\Delta_{\tilde{g},\,X}\tilde{u} = \tilde{F} \in C^{2k,2\alpha}_{X,\beta}(M). \tag{6.10}$$

The next claim estimates the $C^{2k+2,2\alpha}_{loc}$-norms of $\tilde{u}$ in terms of the data $F$ and its local $C^0$-norm. For this purpose, define the corresponding solution to the Ricci flow $g(\tau) := (-\tau)(\phi^X_\tau)^*g$, defined for $\tau < 0$ where $\partial_\tau \phi^X_\tau = \frac{X}{2(-\tau)} \circ \phi^X_\tau$ and $\phi^X_{\tau=-1} = \mathrm{Id}_{\mathbb{C}\times D}$. Here, $\phi^X_\tau(z,\theta) = (\frac{z}{\sqrt{-\tau}}, \theta)$ for $(z,\theta) \in \mathbb{C}\times D$. In particular, if $A_{r_1,r_2} := \{(z,\theta) \in \mathbb{C}\times D \,|\, r_1 \leq |z| \leq r_2\}$ for $0 \leq r_1 < r_2$, then $\phi^X_\tau(A_{r_1,r_2}) = A_{\frac{r_1}{\sqrt{-\tau}},\frac{r_2}{\sqrt{-\tau}}}$.

**Claim 6.7.** There exists a radius $r_0 > 0$ and a positive constant $C$ such that if $r \geq r_0$, then

$$\|\tilde{u}\|_{C^{2k+2,2\alpha}_{X,0}\left(A_{r(x)-C,r(x)+C}\right)} \leq C\left(\|\tilde{u}\|_{C^0\left(A_{\frac{r(x)}{C},Cr(x)}\right)} + \|\tilde{F}\|_{C^{2k,2\alpha}_{X,0}\left(A_{\frac{r(x)}{C},Cr(x)}\right)}\right). \tag{6.11}$$

Moreover,

$$|X\cdot\tilde{u}| + \left|\nabla^{\tilde{g}}\tilde{u}\right|_{\tilde{g}} + \left|\nabla^{\tilde{g},2}\tilde{u}\right|_{\tilde{g}} \leq C\log r\|F\|_{C^{2k,\,2\alpha}_{X,\,\beta}}, \quad r \geq r_0. \tag{6.12}$$

*Proof of Claim 6.7.* Since (6.10) is expressed in terms of the Riemannian metric $\tilde{g}$, we analogously define the family $\tilde{g}(\tau) := (-\tau)(\phi^X_\tau)^*\tilde{g}$ for $\tau < 0$, where $\partial_\tau\phi^X_\tau = \frac{X}{2(-\tau)} \circ \phi^X_\tau$ and $\phi^X_{\tau=-1} = \mathrm{Id}_{\mathbb{C}\times D}$.

For $-\tau \in \left[\frac{1}{2},\,2\right]$, the metrics $\tilde{g}(\tau)$ are uniformly equivalent and their covariant derivatives (with respect to $g$) and time derivatives are bounded by (6.1). Now, $\tilde{u}(\tau) := (\phi^X_\tau)^*\tilde{u}$ satisfies

$$\partial_\tau\tilde{u} = \Delta_{\tilde{g}(\tau)}\tilde{u} + \tilde{F}(\tau), \quad \tilde{F}(\tau) := -(-\tau)^{-1}(\phi^X_\tau)^*F. \tag{6.13}$$

Standard parabolic Schauder estimates applied to (6.13) on a ball $B_g(x,r_0)$, $2r_0 < \mathrm{inj}(g)$, then ensure the existence of a uniform positive constant $C$ such that

$$\|u(\tau)\|_{C^{2k+2,\,2\alpha}(B_g(x,r_0)\times[-\frac{3}{2},-1])} \leq C\left(\|u(\tau)\|_{C^0(B_g(x,2r_0)\times[-2,-\frac{1}{2}])} + \|\tilde{F}(\tau)\|_{C^{2k,2\alpha}(B_g(x,2r_0)\times[-2,-\frac{1}{2}])}\right).$$

Unravelling the definition of the function $\tilde{u}(\tau)$ and that of the metrics $\tilde{g}(\tau)$ then yields (6.11) after observing that

$$\bigcup_{\tau\in\left[-2,-\frac{1}{2}\right]} \phi^X_\tau\left(B_g(x,\,2r_0)\right) \subset A_{\frac{r(x)}{\sqrt{2}}-\sqrt{2}r_0,\sqrt{2}r(x)+2\sqrt{2}r_0}.$$

The final estimate (6.12) is a straightforward combination of (6.11) together with the a priori bound from Claim 6.6. ☐

Now we are in a position to linearize equation (6.4) outside a compact set, namely

$$\Delta_{g,\,X}\tilde{u} = \tilde{F} + (\Delta_g - \Delta_{\tilde{g}})u := G, \tag{6.14}$$

where $G$ satisfies pointwise

$$G - \tilde{F} = (g^{-1} - \tilde{g}^{-1}) * \partial\bar{\partial}u = O(r^{-\gamma})|\partial\bar{\partial}u|_g, \tag{6.15}$$

with $*$ denoting any linear combination of contractions of tensors with respect to the metric $g$. Indeed, this holds true by (6.1). We rewrite (6.14) (outside a compact set) as follows:

$$\Delta_C\tilde{u} - X\cdot\tilde{u} + \Delta_D\tilde{u} = G. \tag{6.16}$$

Here $\Delta_C$ and $\Delta_D$ denote the Riemannian Laplacian of the metric $\omega_C$ on $\mathbb{C}$ and $\omega_D$ on $D$ respectively. Integrating this equation over $D$ at a sufficiently large height $r$, we find that

$$\Delta_{C,\,X}\overline{u}(r) = \overline{G}(r), \quad r \geq r_0, \tag{6.17}$$

where

$$\overline{u}(r) := \fint_D \tilde{u}(r,\cdot)\,\omega_D^{n-1} \qquad \text{and} \qquad \overline{G}(r) = \fint_D G(r,\cdot)\,\omega_D^{n-1},$$

both functions in the $r$-variable only because both are $JX$-invariant by definition. We next derive some estimates on $\overline{u}(r)$.

**Claim 6.8.** One has
$$|\overline{u}(r)|\leq C\|\tilde{F}\|_{C^{2k,2\alpha}_{X,\beta}},\quad r\geq r_0.$$
Moreover, $\lim_{r\to+\infty}\overline{u}(r)=:u_\infty$ exists, is finite, and
$$|\overline{u}(r)-u_\infty|\leq C\left(r^{-\beta}\|\tilde{F}\|_{C^{2k,2\alpha}_{X,\beta}}+r^{-\gamma}\sup_{\{f\geq\frac{r^2}{2}\}}|\partial\bar{\partial}u|\right),\qquad r\geq r_0.$$

*Proof.* Equation (6.17) can be rewritten as
$$\left|\frac{X\cdot X\cdot\overline{u}(r)}{r^2}-X\cdot\overline{u}(r)\right|\leq C\left(r^{-\beta}\|\tilde{F}\|_{C^{2k,2\alpha}_{X,\beta}}+r^{-\gamma}\sup_{\{f\geq\frac{r^2}{2}\}}|\partial\bar{\partial}u|\right),\qquad r\geq r_0,\tag{6.18}$$
based on (6.15). This is a first order differential inequality for $X\cdot\overline{u}(r)$. Now, estimate (6.12) from Claim 6.7 implies a first rough estimate, namely
$$\left|\frac{X\cdot X\cdot\overline{u}(r)}{r^2}-X\cdot\overline{u}(r)\right|\leq Cr^{-\min\{\beta,\gamma\}}\left(1+\log r\right)\|\tilde{F}\|_{C^{2k,2\alpha}_{X,\beta}},\quad r\geq r_0.\tag{6.19}$$
Grönwall's inequality then leads to the bound
$$\begin{aligned}|X\cdot\overline{u}(r)|&\leq C\|\tilde{F}\|_{C^{2k,2\alpha}_{X,\beta}}e^{\frac{r^2}{2}}\int_r^{+\infty}s^{-\min\{\beta,\gamma\}}\left(1+\log s\right)se^{-\frac{s^2}{2}}ds\\&\leq C\|\tilde{F}\|_{C^{2k,2\alpha}_{X,\beta}}r^{-\min\{\beta,\gamma\}}\log r,\quad r\geq r_0,\end{aligned}\tag{6.20}$$
for some uniform positive constant $C$ independent of $r\geq r_0$. Integrating once more in $r$, Claim 6.6 ensures that $\overline{u}(r)$ admits a limit $u_\infty$ as $r\to+\infty$ and that for $r\geq r_0$,
$$\begin{aligned}|\overline{u}(r)|&\leq|\overline{u}(r_0)|+C\|\tilde{F}\|_{C^{2k,2\alpha}_{X,\beta}}\int_{r_0}^{r}s^{-\min\{\beta,\gamma\}-1}\log s\,ds\\&\leq C\|\tilde{F}\|_{C^{2k,2\alpha}_{X,\beta}}\end{aligned}$$
for some positive constant $C$ that may vary from line to line which is independent of $r$ (and of the data $F$). This concludes the proof of the first part of the claim.

Returning to inequality (6.18), another application Grönwall's inequality leads to
$$\begin{aligned}|X\cdot\overline{u}(r)|&\leq Ce^{\frac{r^2}{2}}\left(\int_r^{+\infty}s^{-\beta}\,se^{-\frac{s^2}{2}}ds\|\tilde{F}\|_{C^{2k,2\alpha}_{X,\beta}}+\int_r^{+\infty}s^{-\gamma}\,se^{-\frac{s^2}{2}}ds\sup_{\{f\geq\frac{r^2}{2}\}}|\partial\bar{\partial}u|\right)\\&\leq C\left(r^{-\beta}\|\tilde{F}\|_{C^{2k,2\alpha}_{X,\beta}}+r^{-\gamma}\sup_{\{f\geq\frac{r^2}{2}\}}|\partial\bar{\partial}u|\right),\quad r\geq r_0.\end{aligned}$$
Integrating this inequality once more between $r$ and $r=+\infty$ yields the latter part of the claim. □

The next claim concerns the uniform boundedness of the projection of $u$ onto the orthogonal complement of the kernel of $\Delta_D$, $D$ being interpreted as embedded in each level set $\{f=\frac{r^2}{2}\}$.

**Claim 6.9.** Given $\delta\in(0,\min\{\beta,\gamma\})$, there exists $r_0=r_0(\delta,\tilde{\omega},n)$ such that
$$\|\tilde{u}-\overline{u}(r)\|_{L^2(D)}\leq C\|\tilde{F}\|_{C^{2k,2\alpha}_{X,\beta}}r^{-\delta},\quad r\geq r_0.\tag{6.21}$$

*Proof.* Recall that $\Delta_{g,X}\tilde{u}=G$ by (6.16) and (6.17) so that
$$\Delta_{g,X}(\tilde{u}-\overline{u}(r))=G-\overline{G}(r),\tag{6.22}$$
outside a compact set. Since for any function $v$,
$$2v\Delta_{C,X}v=\Delta_{C,X}(v^2)-2|\nabla^{\mathbb{C}}v|^2_{g_{\mathbb{C}}},$$

multiplying (6.22) across by $\tilde{u}-\overline{u}(r)$ and integrating over $D$, we obtain

$$\begin{aligned}\Delta_{C,X}\left(\|\tilde{u}-\overline{u}(r)\|^2_{L^2(D)}\right)&\geq\Delta_{C,X}\left(\|\tilde{u}-\overline{u}(r)\|^2_{L^2(D)}\right)-2\int_{\mathbb{P}^1}|\nabla^{\mathbb{C}}(\tilde{u}-\overline{u}(r))|^2_{g_{\mathbb{C}}}\frac{\omega_D^{n-1}}{(n-1)!}\\&=2\int_D(\tilde{u}-\overline{u}(r))\Delta_{C,X}(\tilde{u}-\overline{u}(r))\,\frac{\omega_D^{n-1}}{(n-1)!}\\&=2\int_D(\tilde{u}-\overline{u}(r))(G-\overline{G}(r)-\Delta_D(\tilde{u}-\overline{u}(r)))\,\frac{\omega_D^{n-1}}{(n-1)!}\\&=2\|\nabla^{g_D}(\tilde{u}-\overline{u}(r))\|^2_{L^2(D)}+2\langle G-\overline{G}(r),\,\tilde{u}-\overline{u}(r)\rangle_{L^2(D)}\\&\geq2\lambda^D\|\tilde{u}-\overline{u}(r)\|^2_{L^2(D)}-2\|G-\overline{G}(r)\|_{L^2(D)}\|\tilde{u}-\overline{u}(r)\|_{L^2(D)},\end{aligned}\tag{6.23}$$

where we have made use of the Poincaré inequality on $(D,\,g_D)$ in the last line. Young's inequality then implies for $\varepsilon\in(0,\lambda^D)$ that

$$\Delta_{C,X}\left(\|\tilde{u}-\overline{u}(r)\|^2_{L^2(D)}\right)\geq2(\lambda^D-\varepsilon)\|\tilde{u}-\overline{u}(r)\|^2_{L^2(D)}-C_\varepsilon\|G-\overline{G}(r)\|^2_{L^2(D)}.$$

Therefore, invoking estimate (6.15) and Claim 6.7 together with the previous inequality, we find that

$$\Delta_{C,X}\left(\|\tilde{u}-\overline{u}(r)\|^2_{L^2(D)}\right)\geq2(\lambda^D-\varepsilon)\|\tilde{u}-\overline{u}(r)\|^2_{L^2(D)}-C_\varepsilon\|\tilde{F}\|^2_{C^{2k,2\alpha}_{X,\beta}}r^{-2\min\{\beta,\gamma\}}\log^2 r,\quad r\geq2.$$

By Lemma 6.1 applied to $\tilde{g}:=g$, we see that

$$\Delta_{C,X}(r^{-2\delta})=2\delta r^{-2\delta}+O(r^{-2\delta-2}),$$

which, for $A>0$ and $\delta\in(0,\min\{\beta,\gamma\})$, implies that

$$\begin{aligned}\Delta_{C,X}\left(\|\tilde{u}-\overline{u}(r)\|^2_{L^2(D)}-Ar^{-2\delta}\right)&\geq2(\lambda^D-\varepsilon)\|\tilde{u}-\overline{u}(r)\|^2_{L^2(D)}-C_\varepsilon\|\tilde{F}\|^2_{C^{2k,2\alpha}_{X,\beta}}r^{-2\min\{\beta,\gamma\}}\log^2 r\\&\quad-2A\delta r^{-2\delta}-ACr^{-2\delta-2}\\&\geq2(\lambda^D-\varepsilon)\left(\|\tilde{u}-\overline{u}(r)\|^2_{L^2(D)}-Ar^{-2\delta}\right)+2A(\lambda^D-\varepsilon-\delta)r^{-2\delta}\\&\quad-C_\varepsilon\|\tilde{F}\|^2_{C^{2k,2\alpha}_{X,\beta}}r^{-2\min\{\beta,\gamma\}}\log^2 r-ACr^{-2\delta-2}\\&\geq2(\lambda^D-\varepsilon)\left(\|\tilde{u}-\overline{u}(r)\|^2_{L^2(D)}-Ar^{-2\delta}\right),\end{aligned}$$

so long as $\varepsilon\in(0,\lambda^D-\delta)$, $r\geq r_0=r_0(\delta,n,\tilde{\omega})$, and $A\geq C\|\tilde{F}\|_{C^{2k,2\alpha}_{X,\beta}}$.

Now, since $\|\tilde{u}-\overline{u}(r)\|_{L^2(D)}$ is growing at most logarithmically by Claim 6.6, given $B>0$, we compute that

$$\Delta_{C,X}\left(\|\tilde{u}-\overline{u}(r)\|^2_{L^2(D)}-Ar^{-2\delta}-Br\right)\geq2(\lambda^D-\varepsilon)\left(\|\tilde{u}-\overline{u}(r)\|^2_{L^2(D)}-Ar^{-2\delta}-Br\right),$$

if $\varepsilon\in(0,\lambda^D-\delta)$, $r\geq r_0=r_0(\delta,n,\tilde{\omega})$, and $A\geq C\|\tilde{F}\|_{C^{2k,2\alpha}_{X,\beta}}$. In particular, the maximum principle applied to the function $\|\tilde{u}-\overline{u}(r)\|^2_{L^2(D)}-Ar^{-2\delta}-Br$ outside a compact set of the form $r\geq r_0$ leads to the equality

$$\max_{\{r\geq r_0\}}\left(\|\tilde{u}-\overline{u}(r)\|^2_{L^2(D)}-Ar^{-2\delta}-Br\right)=\max\left\{0,\max_{\{r=r_0\}}\left(\|\tilde{u}-\overline{u}(r)\|^2_{L^2(D)}-Ar^{-2\delta}-Br\right)\right\}.$$

Letting $B\to0$ and setting $A=C\|\tilde{F}\|_{C^{2k,2\alpha}_{X,\beta}}$ with $C$ sufficiently large but uniform in the data $F$ and the radius $r$, one arrives at the expected bound:

$$\|\tilde{u}-\overline{u}(r)\|_{L^2(D)}\leq C\|\tilde{F}\|_{C^{2k,2\alpha}_{X,\beta}}r^{-\delta},\quad r\geq r_0.$$

□

**Claim 6.10.** Given $\delta \in (0, \min\{\beta, \gamma\})$, there exists $r_0 = r_0(\delta, \tilde{\omega}, n) > 0$ independent of $F$ (and the solution $u$) such that:

$$\sup_{r \geq r_0} r^{\delta} |\tilde{u} - u_{\infty}| \leq C \|\tilde{F}\|_{C^{2k,2\alpha}_{X,\beta}}.$$

*Proof.* It suffices to prove that for all $\delta \in (0, \min\{\beta, \gamma\})$, there exists $r_0 = r_0(\delta, n, \tilde{\omega}) > 0$ such that

$$\sup_{r \geq r_0} r^{\delta} |\tilde{u} - \overline{u}(r)| \leq C \|\tilde{F}\|_{C^{2k,2\alpha}_{X,\beta}}. \tag{6.24}$$

Indeed, the triangle inequality together with Claim 6.7 and Claim 6.8 already yield such a uniform $C^0$-polynomial rate on the difference $\overline{u}(r) - u_{\infty}$.

In order to prove (6.24), we apply a local parabolic Nash-Moser iteration to the following heat equation with a source term (see for instance [Lie96, Theorem 6.17] for a proof) by recalling that for $\tau < 0$, $\tilde{u}(\tau) := (\phi^X_{\tau})^* \tilde{u}$ and $\overline{u}(r, \tau) := (\phi^X_{\tau})^* \overline{u}(r) = \overline{u}\left(\frac{r}{\sqrt{-\tau}}\right)$:

$$\partial_{\tau}\left(\tilde{u} - \overline{u}(\cdot, \cdot)\right)(\tau) = \Delta_{(-\tau)\cdot g_D}\left(\tilde{u} - \overline{u}(\cdot, \cdot)\right)(\tau) + \underbrace{\Delta_C\left(\tilde{u} - \overline{u}(\cdot, \cdot)\right)(\tau) - \left(G - \overline{G}\right)(\tau)}_{:=S(\tau) \quad \text{source term}}, \qquad (-\tau) \in \left[\frac{1}{2}, 2\right].$$

Here we have used (6.13), (6.14) and (6.17). Here, the notation $(-\tau) \cdot g_D$ denotes the metric on $D$ rescaled by $(-\tau)$. In particular, there exists $C > 0$ such that if $r \geq r_0$,

$$\begin{aligned} \sup_{f = \frac{r^2}{2}} |\tilde{u} - \overline{u}(r)| &= \sup_{f = \frac{r^2}{2}} |\tilde{u}(\tau) - \overline{u}(r, \tau)|_{\tau = -1} \\ &\leq C \sup_{(-\tau) \in [1/2, 2]} \left(\|\tilde{u}(\tau) - \overline{u}(r, \tau)\|_{L^2(D)} + |S(\tau)|\right) \\ &\leq C \sup_{s \in [r/\sqrt{2}, \sqrt{2} r]} \left(\|\tilde{u}(1) - \overline{u}(s, 1)\|_{L^2(D)} + |S(1)|\right). \end{aligned} \tag{6.25}$$

The source term can be estimated as follows: if $k \geq 1$, $(-\tau) \in \left[\frac{1}{2}, 2\right]$ and $r \geq r_0$,

$$\begin{aligned} \left|\Delta_C\left(\tilde{u} - \overline{u}(\cdot, \cdot)\right)(\tau) - \left(G - \overline{G}\right)(\tau)\right| &\leq C \left( r^{-\beta} \|\tilde{F}\|_{C^{2k,2\alpha}_{X,\beta}} + r^{-\gamma} \sup_{f \geq \frac{r^2}{4}} |\partial\bar{\partial} u| + r^{-2} \sup_{f \geq \frac{r^2}{4}} |u| \right) \\ &\leq C \|\tilde{F}\|_{C^{2k,2\alpha}_{X,\beta}} r^{-\min\{\beta, \gamma\}} (1 + \log r), \end{aligned}$$

where we have applied Claim 6.7 to $X \cdot \tilde{u}$ and $X \cdot X \cdot \tilde{u}$ in order to estimate $\Delta_C \tilde{u}$.

Finally, thanks to (6.25), Claim 6.9 combined with the above estimate on the source term implies that

$$\begin{aligned} \sup_{f = \frac{r^2}{2}} |\tilde{u} - \overline{u}(r)| &\leq C \|\tilde{F}\|_{C^{2k,2\alpha}_{X,\beta}} r^{-\delta} + C \|\tilde{F}\|_{C^{2k,2\alpha}_{X,\beta}} r^{-\min\{\beta, \gamma\}} (1 + \log r) \\ &\leq C \|\tilde{F}\|_{C^{2k,2\alpha}_{X,\beta}} r^{-\delta}, \quad r \geq r_0, \end{aligned}$$

as expected. □

**Claim 6.11.** Given $\beta \in (0, \lambda^D)$, there exists $r_0 = r_0(\beta, \tilde{\omega}, n) > 0$ independent of $F$ (and the solution $u$) such that

$$\sup_{r \geq r_0} r^{\beta} |\tilde{u} - u_{\infty}| \leq C \|\tilde{F}\|_{C^{2k,2\alpha}_{X,\beta}}.$$

*Proof.* Applying (6.11) from Claim 6.7 to $\tilde{u} - u_{\infty}$ together with Claim 6.10 demonstrate that if $\delta \in (0, \min\{\beta, \gamma\})$, then

$$|X \cdot \tilde{u}|(x) + |\nabla^{\tilde{g}} \tilde{u}|(x) + |\nabla^{\tilde{g}, 2} \tilde{u}|(x) \leq C \|\tilde{F}\|_{C^{2k,2\alpha}_{X,\beta}} r^{-\delta}, \quad r \geq r_0.$$

This estimate implies in turn the following one based on (6.15):

$$|G - \tilde{F}| \leq C \|\tilde{F}\|_{C^{2k,2\alpha}_{X,\beta}} r^{-\gamma - \delta}, \quad r \geq r_0. \tag{6.26}$$

On one hand, thanks to Claim 6.8, one obtains an improved decay on $\overline{u}(r)-u_\infty$, namely

$$|\overline{u}(r)-u_\infty|\leq C\|\tilde{F}\|_{C^{2k,2\alpha}_{X,\beta}}\left(r^{-\min\{\beta,\gamma+\delta\}}\right),\quad r\geq r_0.$$

On the other hand, (6.26) can then be re-inserted into the proof of Claim 6.9 to establish an improved $L^2(D)$-decay on $\tilde{u}-\overline{u}(r)$. Indeed, inequality (6.23) gives for $r\geq r_0$,

$$\begin{aligned}\Delta_{C,X}\left(\|\tilde{u}-\overline{u}(r)\|^2_{L^2(D)}\right)&\geq 2\lambda^D\|\tilde{u}-\overline{u}(r)\|^2_{L^2(D)}-C\|\tilde{u}-\overline{u}(r)\|_{L^2(D)}\|\tilde{F}\|_{C^{2k,2\alpha}_{X,\beta}}r^{-\min\{\beta,\gamma+\delta\}}\\&\geq 2(\lambda^D-\varepsilon)\|\tilde{u}-\overline{u}(r)\|^2_{L^2(D)}-C_\varepsilon\|\tilde{F}\|^2_{C^{2k,2\alpha}_{X,\beta}}r^{-2\min\{\beta,\gamma+\delta\}}\end{aligned}$$

for any $\varepsilon\in(0,\lambda^D)$. Using a barrier function of the form $r^{-2\delta'}$ with $0<\delta'\leq\min\{\beta,\gamma+\delta\}<\lambda^D$ and by choosing $\varepsilon$ carefully, one arrives at an improved $L^2(D)$-decay of the form above, namely

$$\|\tilde{u}-\overline{u}(r)\|_{L^2(D)}\leq C\|\tilde{F}\|_{C^{2k,2\alpha}_{X,\beta}}r^{-\delta'},\qquad r\geq r_0.$$

The proof of Claim 6.10 can now be adapted to give a corresponding improved pointwise decay. By applying this reasoning a finite number of times, one arrives at the expected sharp decay on $\tilde{u}-u_\infty$. □

Theorem 6.3 now follows by combining Claim 6.7 (once multiplied by the weight $r^\beta$) and Claim 6.11. □

6.4. **Small perturbations along the continuity path.** In this section we show, using the implicit function theorem, that the invertibility of the drift Laplacian given by Theorem 6.3 allows for small perturbations in polynomially weighted function spaces of solutions to the complex Monge-Ampère equation that we wish to solve. This forms the openness part of the continuity method as will be explained later in Section 7.1.

In notation reminiscent of that of [Tia00a, Chapter 5], we consider the space $\left(\mathcal{C}^{2,2\alpha}_{X,\beta}(M)\right)_{\tilde{\omega},0}$ of functions $F\in\mathcal{C}^{2,2\alpha}_{X,\beta}(M)$ with

$$\int_M\left(e^F-1\right)e^{-\tilde{f}}\tilde{\omega}^n=0.$$

This function space is a non-linear Banach space. Notice that the tangent space at a function $F_0$ is the set of functions $u\in\mathcal{C}^{2,2\alpha}_{X,\beta}(M)$ with

$$\int_M u\,e^{F_0-\tilde{f}}\tilde{\omega}^n=0.$$

We have:

**Theorem 6.12.** *Let $F_0\in\left(\mathcal{C}^{2,2\alpha}_{X,\beta}(M)\right)_{\tilde{\omega},0}\cap\mathcal{C}^\infty_{X,\beta}(M)$ for some $\beta\in(0,\lambda^D)$ and let $\psi_0\in\mathcal{M}^\infty_{X,\beta}(M)$ be a solution of the complex Monge-Ampère equation*

$$\log\left(\frac{\tilde{\omega}^n_{\psi_0}}{\tilde{\omega}^n}\right)-\frac{X}{2}\cdot\psi_0=F_0.$$

*Then for any $\alpha\in\left(0,\frac{1}{2}\right)$, there exists a neighbourhood $U_{F_0}\subset\left(C^{2,2\alpha}_{X,\beta}(M)\right)_{\tilde{\omega},0}$ of $F_0$ such that for all $F\in U_{F_0}$, there exists a unique function $\psi\in\mathcal{M}^{4,2\alpha}_{X,\beta}(M)$ such that*

$$\log\left(\frac{\tilde{\omega}^n_{\psi}}{\tilde{\omega}^n}\right)-\frac{X}{2}\cdot\psi=F.\tag{6.27}$$

*Moreover, if $F\in U_{F_0}$ lies in $\mathcal{C}^\infty_{X,\beta}(M)$ then the unique solution $\psi\in\mathcal{M}^{4,2\alpha}_{X,\beta}(M)$ to (6.27) lies in $\mathcal{M}^\infty_{X,\beta}(M)$.*

**Remark 6.13.** Theorem 6.12 does not assume any finite regularity on the data $(\psi_0,\,F_0)$ in the relevant function spaces. This essentially follows from Theorem 6.3 where the closeness of $\tilde{\omega}$ to $\omega$ in derivatives is assumed.

*Proof of Theorem 6.12.* In order to apply the implicit function theorem for Banach spaces, we must reformulate the statement of Theorem 6.12 in terms of the map $MA_{\tilde{\omega}}$ introduced formally at the beginning of Section 6.3. To this end, consider the mapping

$$\begin{aligned} MA_{\tilde{\omega}} : \psi \in \mathcal{M}^{4,\,2\alpha}_{X,\,\beta}(M) \\ \mapsto \log\left(\frac{\tilde{\omega}^n_{\psi}}{\tilde{\omega}^n}\right) - \frac{X}{2}\cdot\psi \in \left(\mathcal{C}^{2,\,2\alpha}_{X,\,\beta}(M)\right)_{\tilde{\omega},0}, \qquad \alpha\in\left(0,\,\frac{1}{2}\right). \end{aligned}$$

Notice that the function spaces above can be defined by either using the metric $\tilde{g}$ or $\tilde{g}_{t\psi_0}$ for any $t\in[0,\,1]$. To see that $MA_{\tilde{\omega}}$ is well-defined, apply the Taylor expansion (6.3) to the background metric $\tilde{\omega}$ to obtain

$$\begin{aligned} MA_{\tilde{\omega}}(\psi) &= \log\left(\frac{\tilde{\omega}^n_{\psi}}{\tilde{\omega}^n}\right) - \frac{X}{2}\cdot\psi \\ &= \Delta_{\tilde{\omega}}\psi - \frac{X}{2}\cdot\psi - \int_0^1\int_0^u |\partial\bar{\partial}\psi|^2_{\tilde{g}_{t\psi}}\,dt\,du. \end{aligned} \tag{6.28}$$

Then by the very definition of $\mathcal{D}^{4,\,2\alpha}_{X,\,\beta}(M)$, the first two terms of the last line of (6.28) lie in $\mathcal{C}^{2,\,2\alpha}_{X,\,\beta}(M)$.

Now, if $S$ and $T$ are tensors in $C^{2k,\,2\alpha}_{X,\,\gamma_1}(M)$ and $C^{2k,\,2\alpha}_{X,\gamma_2}(M)$ respectively, with $\gamma_i\geq 0$, $i=1,2$, then observe that $S*T$ lies in $C^{2k,\,2\alpha}_{X,\,\gamma_1+\gamma_2}(M)$, where $*$ denotes any linear combination of contractions of tensors with respect to the metric $\tilde{g}$. Moreover,

$$\|S*T\|_{C^{2k,\,2\alpha}_{X,\,\gamma_1+\gamma_2}} \leq C(k,\,\alpha)\|S\|_{C^{2k,\,2\alpha}_{X,\,\gamma_1}}\cdot\|T\|_{C^{2k,\,2\alpha}_{X,\,\gamma_2}}. \tag{6.29}$$

Next notice that

$$|i\partial\bar{\partial}\psi|^2_{\tilde{g}_{t\psi}} = \tilde{g}^{-1}_{t\psi} * (\nabla^{\tilde{g}})^2\psi * (\nabla^{\tilde{g}})^2\psi$$

and that

$$\tilde{g}^{-1}_{t\psi} - \tilde{g}^{-1} \in C^{2,\,2\alpha}_{X,\,\beta}(M).$$

Thus, applying (6.29) twice to $S=T=(\nabla^{\tilde{g}})^2\psi$ and to the inverse $\tilde{g}^{-1}_{t\psi}$ with weights $\gamma_1=\gamma_2=\beta$ and $k=1$, one finds that $|i\partial\bar{\partial}\psi|^2_{\tilde{g}_{\psi}}\in C^{2,\,2\alpha}_{X,\,2\beta}(M)\subset C^{2,\,2\alpha}_{X,\,\beta}(M)$ for each $t\in[0,\,1]$ and that

$$\left\|\int_0^1\int_0^u |i\partial\bar{\partial}\psi|^2_{\tilde{g}_{t\psi}}\,dt\,du\right\|_{C^{2,\,2\alpha}_{X,\,\beta}} \leq C\left(k,\alpha,\tilde{g}\right)\|\psi\|_{\mathcal{D}^{4,\,2\alpha}_{X,\,\beta}},$$

so long as $\|\psi\|_{\mathcal{D}^{4,\,2\alpha}_{X,\,\beta}}\leq 1$. Finally, the $JX$-invariance of the right-hand side of (6.28) is clear and Lemma 6.2(i) ensures that $\exp MA_{\tilde{\omega}}(\psi)-1$ has zero mean value with respect to the weighted measure $e^{-\tilde{f}}\tilde{\omega}$.

By (6.3), we have that

$$\begin{aligned} D_{\psi_0}MA_{\tilde{\omega}} : \psi \in \mathcal{M}^{4,\,2\alpha}_{X,\,\beta}(M)\cap\left\{\int_M u\,e^{-\tilde{f}_{\psi_0}}\tilde{\omega}^n_{\psi_0}=0\right\} \\ \mapsto \Delta_{\tilde{\omega}_{\psi_0}}\psi - \frac{X}{2}\cdot\psi \in T_{F_0}\left(\mathcal{C}^{2,\,2\alpha}_{X,\,\beta}(M)\right)_{\tilde{\omega},0}, \end{aligned}$$

where the tangent space of $\left(\mathcal{C}^{2,\,2\alpha}_{X,\,\beta}(M)\right)_{\tilde{\omega},0}$ at $F_0$ is equal to the set of functions $u\in\mathcal{C}^{2,2\alpha}_{X,\,\beta}(M)$ with 0 mean value with respect to the weighted measure $e^{-\tilde{f}_{\psi_0}}\tilde{\omega}^n_{\psi_0}$. Therefore, after applying Theorem 6.3 to the background metric $\tilde{\omega}_{\psi_0}$ in place of $\tilde{\omega}$, we conclude that $D_{\psi_0}MA_{\tilde{\omega}}$ is an isomorphism of Banach spaces. The result now follows by applying the inverse function theorem to the map $MA_{\tilde{\omega}}$ in a neighbourhood of $\psi_0\in\mathcal{M}^{4,\,2\alpha}_{X,\,\beta}(M)\cap\left\{\int_M u\,e^{-\tilde{f}_{\psi_0}}\tilde{\omega}^n_{\psi_0}=0\right\}$.

We postpone the proof of the regularity at infinity of the solution $\psi$ in case the data $F\in\mathcal{C}^{\infty}_{X,\beta}(M)$ to Proposition 7.30. □

## 7. Proof of Theorem A(v): A priori estimates

7.1. **The continuity path.** Recall the setup of Theorem A. Recall that $J$ denotes the complex structure on $M$ and $z$ the holomorphic coordinate on the $\mathbb{C}$-component of $\widehat{M}$. We write $r = |z|^{\lambda}$, treating both $r$ and $z$ as functions on $M$ via $\nu$. It is clear then that $X = r\partial_r$ on $M \setminus K$.

Recall from (1.2) that the complex Monge-Ampère equation we wish to solve is

$$\begin{cases} (\omega + i\partial\bar{\partial}\psi)^n = e^{F+\frac{X}{2}\cdot\psi}\omega^n, \qquad \psi \in C^\infty(M), \qquad \mathcal{L}_{JX}\psi = 0, \qquad \omega + i\partial\bar{\partial}\psi > 0, \\ \int_M e^{F-f}\omega^n = \int_M e^{-f}\omega^n, \end{cases} \tag{$*_0$}$$

where $F : M \to \mathbb{R}$ is a $JX$-invariant smooth function equal to a constant $c_0$ outside a compact subset $V$ of $M$ and $f : M \to \mathbb{R}$ is the Hamiltonian potential of $X$ with respect to $\omega$, i.e., $-\omega \lrcorner JX = df$, normalised so that

$$\Delta_\omega f - f + \frac{X}{2} \cdot f = 0$$

outside a compact set. Define $F_s := \log(1 + s(e^F - 1))$. In this section, we prove Theorem A(v) by providing a solution to $(*_0)$ by implementing the continuity path

$$\begin{cases} (\omega + i\partial\bar{\partial}\psi_s)^n = e^{F_s+\frac{X}{2}\cdot\psi_s}\omega^n, \qquad \psi_s \in \mathcal{M}^\infty_{X,\beta}(M), \qquad \mathcal{L}_{JX}\psi_s = 0, \qquad s \in [0, 1], \\ \int_M e^{F-f}\omega^n = \int_M e^{-f}\omega^n, \qquad \int_M \psi_s\, e^{-f}\omega^n = 0. \end{cases} \tag{$\star_s$}$$

When $s = 0$, $(\star_0)$ admits the trivial solution, namely $\psi_0 \equiv 0$. When $s = 1$, $(\star_1)$ corresponds to $(*_0)$, that is, the equation that we wish to solve. Via the a priori estimates to follow, we will show that the set $s \in [0, 1]$ for which $(\star_s)$ has a solution is closed. As we have just seen, this set is non-empty. Openness of this set follows from the isomorphism properties of the drift Laplacian given by Theorem 6.12. Connectedness of $[0, 1]$ then implies that $(\star_s)$ has a solution for $s = 1$, resulting in the desired solution of $(*_0)$.

7.2. **The continuity path re-parametrised.** To obtain certain localisation results and in turn, a priori estimates for $(\star_s)$, we need to consider a reformulation of $(\star_s)$ in the following way. Identify $(M \setminus K, \omega)$ and $(\widehat{M} \setminus \widehat{K}, \widehat{\omega})$ using $\nu$, where $K \subset M$, $\widehat{K} \subset \widehat{M}$ are compact, and define $F_s := \log(1 + s(e^F - 1))$. Then there exists a compact subset $K \subset V \subset M$ such that for all $s \in [0, 1]$, $F_s$ is equal to a constant $c_s$ on $M \setminus V$. Explicitly, $c_s = \log(1 + s(e^{c_0} - 1))$. Note that $c_s$ varies continuously as a function of $s$ and that $(\star_s)$ can be rewritten as

$$(\omega + i\partial\bar{\partial}\psi_s)^n = e^{F_s+\frac{X}{2}\cdot\psi_s}\omega^n.$$

Let $\eta_s := -2c_s \log(r)$, a real-valued function defined on $M \setminus K$. Then, with $g$ denoting the Kähler metric associated to $\omega$, it is clear that

$$\|(\log(r))^{-1} \cdot \eta_s\|_{C^0(M\setminus K)} + \|d\eta_s\|_{C^0(M\setminus K,\, g)} + \|r^\lambda \cdot i\partial\bar{\partial}\eta_s\|_{C^0(M\setminus K,\, g)} \leq 2|c_s| \left(1 + \sup_{M\setminus K} r^{-1}\right) \leq C(K),$$

and so Lemma 2.5 infers the existence of a bump function $\chi : M \to \mathbb{R}$ supported on $M \setminus V$ and a compact subset $W \supset V$, both independent of $s$, such that $\chi = 1$ on $M \setminus W$ and such that for all $s \in [0, 1]$, $\omega_s := \omega + i\partial\bar{\partial}(\chi \cdot \eta_s) > 0$ on $M$. Define $\Phi_s := \chi \cdot \eta_s$. Then $\omega_s = \omega + i\partial\bar{\partial}\Phi_s$ and since $\Phi_s = -2c_s \log r$ on $M \setminus W$, that is, a pluriharmonic function, $\omega_s$ is isometric to $\omega$ on this set. Furthermore, we find that

$$\begin{aligned} \log\left(\frac{(\omega_s + i\partial\bar{\partial}(\psi_s - \Phi_s))^n}{\omega_s^n}\right) - \frac{X}{2} \cdot (\psi_s - \Phi_s) &= \log\left(\frac{(\omega + i\partial\bar{\partial}\psi_s)^n}{(\omega + i\partial\bar{\partial}\Phi_s)^n}\right) - \frac{X}{2} \cdot (\psi_s - \Phi_s) \\ &= \log\left(\frac{(\omega + i\partial\bar{\partial}\psi_s)^n}{\omega^n}\right) - \frac{X}{2} \cdot \psi_s - \log\left(\frac{(\omega + i\partial\bar{\partial}\Phi_s)^n}{\omega^n}\right) + \frac{X}{2} \cdot \Phi_s \\ &= F_s - \left(\log\left(\frac{(\omega + i\partial\bar{\partial}\Phi_s)^n}{\omega^n}\right) - \frac{X}{2} \cdot \Phi_s\right) =: G_s, \end{aligned}$$

with $G_s$ vanishing on $M \setminus W$. Set $\vartheta_s := \psi_s - \Phi_s$. Then $\vartheta_s \in \mathbb{R} \oplus C^{\infty}_{X,\beta}(M)$ and we can rewrite $(\star_s)$ in terms of $\vartheta_s$ as

$$\log\left(\frac{(\omega_s + i\partial\bar{\partial}\vartheta_s)^n}{\omega_s^n}\right) - \frac{X}{2}\cdot\vartheta_s = G_s, \quad \vartheta_s \in \mathbb{R} \oplus C^{\infty}_{X,\beta}(M), \quad \mathcal{L}_{JX}\vartheta_s = 0, \quad \omega_s + i\partial\bar{\partial}\vartheta_s > 0,\ s \in [0,\,1], \tag{$\star\star_s$}$$

with the support of $G_s$ contained in $W$ and $\omega_s = \omega$ on $M \setminus W$. We derive a priori estimates for $(\star\star_s)$, the advantage over $(\star_s)$ being that it allows for a localisation of the infimum and supremum of $|\vartheta_s|$, essentially because the unbounded log term has been absorbed into the background metric $\omega_s$ in $(\star\star_s)$. As we have control on $\Phi_s$, the a priori estimates we derive for $\vartheta_s$ will translate into the desired a priori estimates for $\psi_s$, thereby allowing us to complete the closedness part of the continuity method for $(\star_s)$.

Define $\sigma_s := \omega_s + i\partial\bar{\partial}\vartheta_s$. Then in terms of the Ricci forms $\rho_{\sigma_s}$ and $\rho_{\omega_s}$ of $\sigma_s$ and $\omega_s$ respectively, $(\star\star_s)$ yields

$$\rho_{\sigma_s} + \frac{1}{2}\mathcal{L}_X\sigma_s = \rho_{\omega_s} + \frac{1}{2}\mathcal{L}_X\omega_s - i\partial\bar{\partial}G_s. \tag{7.1}$$

We will write $h_s$ for the Kähler metric associated to $\sigma_s$.

We will need the following regarding the Hamiltonian potential $f_{\omega_s}$ of $X$ with respect to $\omega_s$.

**Lemma 7.1.** *Let $f_{\omega_s} := f + \frac{X}{2}\cdot\Phi_s$. Then $-\omega_s \lrcorner JX = df_{\omega_s}$ and there exists a compact subset $U \subset M$ containing $W$ such that for all $s \in [0,\,1]$, there exists $H_s \in C^{\infty}(M)$ varying smoothly in $s$ and equal to $-c_s$ on $M \setminus U$ such that*

$$\Delta_{\omega_s} f_{\omega_s} - \frac{X}{2}\cdot f_{\omega_s} + f_{\omega_s} = H_s. \tag{7.2}$$

*Proof.* The first assertion is clear. Regarding the normalisation condition (7.2), a computation shows that for the Ricci forms $\rho_\omega$ and $\rho_{\omega_s}$ of $\omega$ and $\omega_s$ respectively,

$$\begin{aligned}\rho_{\omega_s} + \frac{1}{2}\mathcal{L}_X\omega_s - \omega_s &= \rho_\omega + \frac{1}{2}\mathcal{L}_X\omega - \omega - i\partial\bar{\partial}\left(\log\left(\frac{\omega_s^n}{\omega^n}\right) - \frac{X}{2}\cdot\Phi_s + \Phi_s\right)\\ &= i\partial\bar{\partial}(F_2 + G_s - F_s - \Phi_s),\end{aligned}$$

where we have used (4.2). Write $Q_s := F_2 + G_s - F_s - \Phi_s$. Then $Q_s$ is $JX$-invariant and it is easy to see that $Q_s$ is equal to $2c_s\log(r) - c_s$ outside a compact subset $U \supseteq W$ of $M$ independent of $s$. Contracting the identity

$$\rho_{\omega_s} + \frac{1}{2}\mathcal{L}_X\omega_s - \omega_s = i\partial\bar{\partial}Q_s$$

with $X^{1,0} := \frac{1}{2}(X - iJX)$ and arguing as in Lemma 4.2 using the $JX$-invariance of the functions involved, we find that

$$\Delta_{\omega_s} f_{\omega_s} - \frac{X}{2}\cdot f_{\omega_s} + f_{\omega_s} + \frac{X}{2}\cdot Q_s$$

is constant on $M$. But since on $M \setminus W$, $\omega_s = \omega$, $f_{\omega_s} = f - c_s$, and $\frac{X}{2}\cdot Q_s = c_s$, this constant must be zero. Hence the result follows with $H_s := -\frac{X}{2}\cdot Q_s$. □

This allows for a normalisation for the Hamiltonian potential $f_{\sigma_s} := f_{\omega_s} + \frac{X}{2}\cdot\vartheta_s$ of $X$ with respect to $\sigma_s$.

**Lemma 7.2.** *Let $f_{\sigma_s} := f_{\omega_s} + \frac{X}{2}\cdot\vartheta_s$ and let $U$ be as in Lemma 7.1. Then $-\sigma_s \lrcorner JX = df_{\sigma_s}$ and for all $s \in [0,\,1]$, there exists a compactly supported function $P_s \in C^{\infty}(M)$ varying smoothly in $s$ with* $\operatorname{supp} P_s \subseteq U$ *such that*

$$\Delta_{\sigma_s} f_{\sigma_s} - \frac{X}{2}\cdot f_{\sigma_s} = -f + P_s.$$

*Proof.* Again, the first assertion is clear. As for (7.2), we have that

$$\begin{aligned}\frac{X}{2}\cdot\log\left(\frac{\sigma_s^n}{\omega_s^n}\right)&=\frac{1}{2}\operatorname{tr}_{\sigma_s}\mathcal{L}_X\sigma_s-\frac{1}{2}\operatorname{tr}_{\omega_s}\mathcal{L}_X\omega_s\\&=\operatorname{tr}_{\sigma_s}(i\partial\bar{\partial}f_{\sigma_s})-\operatorname{tr}_{\omega}(i\partial\bar{\partial}f_{\omega_s})\\&=\Delta_{\sigma_s}f_{\sigma_s}-\Delta_{\omega_s}f_{\omega_s}.\end{aligned}$$

Thus, contracting both sides of $(\star\star_s)$ with $\frac{X}{2}$, we obtain

$$\Delta_{\sigma_s}f_{\sigma_s}-\Delta_{\omega_s}f_{\omega_s}=\frac{X}{2}\cdot G_s+\frac{X}{2}\cdot\left(f_{\omega_s}+\frac{X}{2}\cdot\vartheta_s\right)-\frac{X}{2}\cdot f_{\omega_s},$$

i.e.,

$$\Delta_{\sigma_s}f_{\sigma_s}-\frac{X}{2}\cdot f_{\sigma_s}=\Delta_{\omega_s}f_{\omega_s}-\frac{X}{2}\cdot f_{\omega_s}+\frac{X}{2}\cdot G_s.$$

Hence we derive from (7.2) that

$$\Delta_{\sigma_s}f_{\sigma_s}-\frac{X}{2}\cdot f_{\sigma_s}=H_s+\frac{X}{2}\cdot G_s-f_{\omega_s}.$$

With $P_s:=H_s+\frac{X}{2}\cdot G_s-\frac{X}{2}\cdot\Phi_s$, the result is now clear. □

7.3. **Summary of notation.** For clarity, in this section we provide a summary of our notation regarding the various Kähler forms in play.

- $F$ is the data in $(*_0)$ equal to $c_0$ outside a compact set.
- $\omega$ is the background Kähler form given in $(*_0)$ isometric to $\omega_C+\omega_D$ outside a compact set.
- $g$ is the Kähler metric associated to $\omega$.
- $f$ is the Hamiltonian potential of $JX$ with respect to $\omega$ given in Theorem A(iii). It is equal to $|z|^{2\lambda}-1$ outside a compact set and normalised such that

$$\Delta_\omega f-f+\frac{X}{2}\cdot f=0$$

  outside a compact set.
- $c_s:=\log(1+s(e^{c_0}-1))$.
- $F_s$ is the data in $(\star_s)$ equal to $c_s$ outside a fixed compact subset $V\subset M$.
- $\psi_s$ is the solution to the original continuity path $(\star_s)$.
- $\Phi_s$ is a function equal to $-2c_s\log r$ outside a fixed compact set $W\supset V$ of $M$.
- $\omega_s:=\omega+i\partial\bar{\partial}\Phi_s$ is the 1-parameter family of background metrics isometric to $\omega$ outside a compact set independent of $s$ appearing in $(\star\star_s)$.
- $g_s$ is the Kähler metric associated to $\omega_s$.
- $f_s:=f+\frac{X}{2}\cdot\Phi_s$ is the Hamiltonian potential of $JX$ with respect to $\omega_s$.
- $\vartheta_s=\psi_s-\Phi_s$ is the solution of the re-parametrised continuity path $(\star\star_s)$.
- $\sigma_s:=\omega_s+i\partial\bar{\partial}\vartheta_s$ is the associated Kähler metric.
- $f_{\sigma_s}$ is the Hamiltonian potential of $JX$ with respect to $\sigma_s$. It is normalised by

$$\Delta_{\sigma_s}f_{\sigma_s}-\frac{X}{2}\cdot f_{\sigma_s}=-f+P_s,$$

  where $P_s$ is compactly supported.
- $h_s$ is the Kähler metric associated to $\sigma_s$.

7.4. **A priori lower bound on the radial derivative.** The fact that the data $G_s$ of $(\star\star_s)$ is compactly supported allows us to localise the extrema of $X\cdot\vartheta_s$ using the maximum principle. This leads to a uniform lower bound on $X\cdot\vartheta_s$ and in particular, on $X\cdot\psi_s$.

**Lemma 7.3** (Localising the supremum and infimum of the radial derivative)**.** *Let $\vartheta_s$ be a path of solutions in $\mathbb{R}\oplus C^\infty_{X,\beta}(M)$ to $(\star\star_s)$. Then $\sup_M X\cdot\vartheta_s=\max_W X\cdot\vartheta_s$ and $\inf_M X\cdot\vartheta_s=\min_W X\cdot\vartheta_s$.*

*Proof.* First, using $\nu$ to identify $(M, \omega)$ and $(\widehat{M}, \widehat{\omega})$ on $M \setminus W$, notice that

$$\begin{aligned}\frac{X}{2}\cdot\left(\log\left(\frac{\sigma_s^n}{\omega_s^n}\right)\right) &= \operatorname{tr}_{\sigma_s}\mathcal{L}_{\frac{X}{2}}\sigma_s - \operatorname{tr}_{\omega_s}\mathcal{L}_{\frac{X}{2}}\omega_s\\ &= \operatorname{tr}_{\sigma_s}\mathcal{L}_{\frac{X}{2}}(\omega_s + i\partial\bar{\partial}\vartheta_s) - \operatorname{tr}_{\omega}\mathcal{L}_{\frac{X}{2}}\omega\\ &= \operatorname{tr}_{\sigma_s}\omega_C + \frac{1}{2}\Delta_{\sigma_s}(X\cdot\vartheta_s) - \operatorname{tr}_{\omega}\omega_C\\ &= \operatorname{tr}_{\sigma_s}\omega_C + \frac{1}{2}\Delta_{\sigma_s}(X\cdot\vartheta_s) - 1\end{aligned}$$

Thus, upon differentiating $(\star\star_s)$ along $X$, we obtain on $M \setminus W$ the equation

$$\Delta_{\sigma_s, X}\left(\frac{X\cdot\vartheta_s}{2}\right) := \Delta_{\sigma_s}\left(\frac{X\cdot\vartheta_s}{2}\right) - \frac{X}{2}\cdot\left(\frac{X\cdot\vartheta_s}{2}\right) = 1 - \operatorname{tr}_{\sigma_s}\omega_C. \tag{7.3}$$

Now on $M \setminus V$, we have

$$\begin{aligned}\operatorname{tr}_{\sigma_s}\omega_C &= \frac{n\sigma_s^{n-1}\wedge\omega_C}{\sigma_s^n}\\ &= \frac{ne^{-\frac{X\cdot\vartheta_s}{2}}\sigma_s^{n-1}\wedge\omega_C}{\omega^n},\end{aligned}$$

hence

$$\begin{aligned}1 - \operatorname{tr}_{\sigma_s}\omega_C &= e^{-\frac{X\cdot\vartheta_s}{2}}\left(e^{\frac{X\cdot\vartheta_s}{2}} - \frac{n\sigma_s^{n-1}\wedge\omega_C}{\omega^n}\right)\\ &= e^{-\frac{X\cdot\vartheta_s}{2}}\left(\frac{\sigma_s^n - n\sigma_s^{n-1}\wedge\omega_C}{\omega^n}\right).\end{aligned} \tag{7.4}$$

For $k = 1, \ldots, n$, we have for dimensional reasons

$$\omega^k = (\omega_D + \omega_C)^k = \omega_D^k + k\omega_D^{k-1}\wedge\omega_C.$$

Thus,

$$\begin{aligned}\sigma_s^n &= (\omega + i\partial\bar{\partial}\vartheta_s)^n\\ &= \sum_{k=0}^{n}\binom{n}{k}\omega^k\wedge(i\partial\bar{\partial}\vartheta_s)^{n-k}\\ &= (i\partial\bar{\partial}\vartheta_s)^n + \sum_{k=1}^{n}\binom{n}{k}\omega^k\wedge(i\partial\bar{\partial}\vartheta_s)^{n-k}\\ &= (i\partial\bar{\partial}\vartheta_s)^n + \sum_{k=1}^{n}\binom{n}{k}(\omega_D^k + k\omega_D^{k-1}\wedge\omega_C)\wedge(i\partial\bar{\partial}\vartheta_s)^{n-k}\\ &= (i\partial\bar{\partial}\vartheta_s)^n + \sum_{k=1}^{n}\binom{n}{k}\omega_D^k\wedge(i\partial\bar{\partial}\vartheta_s)^{n-k} + \sum_{k=1}^{n}k\binom{n}{k}\omega_D^{k-1}\wedge(i\partial\bar{\partial}\vartheta_s)^{n-k}\wedge\omega_C,\\ &= \sum_{k=0}^{n}\binom{n}{k}\omega_D^k\wedge(i\partial\bar{\partial}\vartheta_s)^{n-k} + \sum_{k=1}^{n}k\binom{n}{k}\omega_D^{k-1}\wedge(i\partial\bar{\partial}\vartheta_s)^{n-k}\wedge\omega_C,\end{aligned}$$

and

$$\begin{aligned}
n\sigma_s^{n-1}\wedge\omega_C &= n\sum_{j=0}^{n-1}\binom{n-1}{j}\omega^j\wedge(i\partial\bar{\partial}\vartheta_s)^{n-1-j}\wedge\omega_C\\
&= ni\partial\bar{\partial}\vartheta_s^{n-1}\wedge\omega_C + n\sum_{j=1}^{n-1}\binom{n-1}{j}\omega^j\wedge(i\partial\bar{\partial}\vartheta_s)^{n-1-j}\wedge\omega_C\\
&= ni\partial\bar{\partial}\vartheta_s^{n-1}\wedge\omega_C + n\sum_{j=1}^{n-1}\binom{n-1}{j}(\omega_D^j + j\omega_D^{j-1}\wedge\omega_C)\wedge(i\partial\bar{\partial}\vartheta_s)^{n-1-j}\wedge\omega_C\\
&= ni\partial\bar{\partial}\vartheta_s^{n-1}\wedge\omega_C + n\sum_{j=1}^{n-1}\binom{n-1}{j}\omega_D^j\wedge(i\partial\bar{\partial}\vartheta_s)^{n-1-j}\wedge\omega_C\\
&= ni\partial\bar{\partial}\vartheta_s^{n-1}\wedge\omega_C + n\sum_{k=2}^{n}\binom{n-1}{k-1}\omega_D^{k-1}\wedge(i\partial\bar{\partial}\vartheta_s)^{n-k}\wedge\omega_C\\
&= n\sum_{k=1}^{n}\binom{n-1}{k-1}\omega_D^{k-1}\wedge(i\partial\bar{\partial}\vartheta_s)^{n-k}\wedge\omega_C.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\sigma_s^n - n\sigma_s^{n-1}\wedge\omega_C &= (i\partial\bar{\partial}\vartheta_s)^n + \sum_{k=1}^{n}\binom{n}{k}\omega_D^k\wedge(i\partial\bar{\partial}\vartheta_s)^{n-k} + \sum_{k=1}^{n}k\binom{n}{k}\omega_D^{k-1}\wedge(i\partial\bar{\partial}\vartheta_s)^{n-k}\wedge\omega_C\\
&\quad - n\sum_{k=1}^{n}\binom{n-1}{k-1}\omega_D^{k-1}\wedge(i\partial\bar{\partial}\vartheta_s)^{n-k}\wedge\omega_C\\
&= (i\partial\bar{\partial}\vartheta_s)^n + \sum_{k=1}^{n}\binom{n}{k}\omega_D^k\wedge(i\partial\bar{\partial}\vartheta_s)^{n-k}\\
&\quad + \sum_{k=1}^{n}\underbrace{\left[k\binom{n}{k} - n\binom{n-1}{k-1}\right]}_{=0}\omega_D^{k-1}\wedge(i\partial\bar{\partial}\vartheta_s)^{n-k}\wedge\omega_C\\
&= \sum_{k=0}^{n}\binom{n}{k}\omega_D^k\wedge(i\partial\bar{\partial}\vartheta_s)^{n-k}\\
&= (\omega_D + i\partial\bar{\partial}\vartheta_s)^n.
\end{aligned}$$

Combining (7.3) and (7.4), we find that

$$\Delta_{\sigma_s, X}\left(\frac{X\cdot\vartheta_s}{2}\right) = \underbrace{e^{-\frac{X\cdot\vartheta_s}{2}}\frac{(\omega_D + i\partial\bar{\partial}\vartheta_s)^n}{\omega^n}}_{\text{first order operator acting on } X\cdot\vartheta_s}. \tag{7.5}$$

Indeed, the right-hand side of (7.5) can be written schematically as:

$$\frac{(\omega_D + i\partial\bar{\partial}\vartheta_s)^n}{\omega^n} = \frac{1}{r^2}\left(X\cdot(X\cdot\vartheta_s)\alpha_1 + \nabla^{g_D}(X\cdot\vartheta_s)*\nabla^{g_D}(X\cdot\vartheta_s)*\alpha_2\right), \tag{7.6}$$

where $\alpha_1$ and $\alpha_2$ are tensors on $M\setminus V$ depending polynomially on $i\partial\bar{\partial}\vartheta_s$ and where $*$ denotes any linear combination of tensors with respect to the background metric $\omega$. This can be seen, for example, by noting that on $M\setminus V$,

$$\frac{(\omega_D + i\partial\bar{\partial}\vartheta_s)^n}{\omega^n} = \frac{(i\partial\bar{\partial}\vartheta_s)^n}{\omega^n} + \sum_{k=1}^{n-1}\binom{n}{k}\frac{\omega_D^{n-k}\wedge(i\partial\bar{\partial}\vartheta_s)^k}{\omega^n},$$

together with an application of the following claim.

**Claim 7.4.** Let $Y$ and $Z$ be real holomorphic vector fields such that $[Y,\,Z]=0$. Then for any smooth real-valued function $v$ on $M$ with $\mathcal{L}_{JY}v=\mathcal{L}_{JZ}v=0$, we have $\frac{i}{2}\partial\bar{\partial}v(Y,\,Z)=\frac{i}{2}\partial\bar{\partial}v(JY,\,JZ)=0$ and $Z\cdot(Y\cdot v)=Y\cdot(Z\cdot v)=2i\partial\bar{\partial}v(Z,\,JY)$.

*Proof of Claim 7.4.* The first equality follows from the fact that

$$2i\partial\bar{\partial}v(Y,\,Z)=2i\partial\bar{\partial}v(JY,\,JZ)=dd^{c}v(JY,\,JZ)=JY\cdot(d^{c}v(JZ))-JZ\cdot(d^{c}v(JY))-d^{c}v([JY,\,JZ]).$$

As for the second, the vanishing of $[Y,\,Z]$ implies that $Z\cdot(Y\cdot v)=Y\cdot(Z\cdot v)$, whereas with $Y^{1,\,0}:=\frac{1}{2}(Y-iJY)$ and $Z^{1,\,0}:=\frac{1}{2}(Z-iJZ)$, the invariance of $v$ and the fact that $JY\cdot(Z\cdot v)=0$ implies that

$$\frac{1}{4}Y\cdot(Z\cdot v)=Y^{1,\,0}\cdot(Z^{1,\,0}\cdot v)=\overline{Y^{1,\,0}}\cdot(Z^{1,\,0}\cdot v)=\partial\bar{\partial}v(Z^{1,\,0},\,\overline{Y^{1,\,0}})=\frac{i}{2}\partial\bar{\partial}v(Z,\,JY)-\frac{1}{2}\underbrace{\partial\bar{\partial}v(JY,\,JZ)}_{=0}.$$

□

The strong maximum principle combined with the fact that $X\cdot\vartheta_{s}\to 0$ at infinity now implies the result. □

From this, we can derive a lower bound on $X\cdot\vartheta_{s}$, and hence on $X\cdot\psi_{s}$.

**Proposition 7.5.** *There exists a positive constant $C$ such that for all $s\in[0,\,1]$, $X\cdot\vartheta_{s}\geq -C$. In particular, $X\cdot\psi_{s}>-C$ for all $s\in[0,\,1]$.*

*Proof.* In order to prove that $X\cdot\vartheta_{s}$ is uniformly bounded from below, first note that since $X\cdot\Phi_{s}$ is bounded and $X\cdot\vartheta_{s}$ tends to zero at infinity, $f_{\sigma_{s}}:=f+\frac{X}{2}\cdot\Phi_{s}+\frac{X}{2}\cdot\vartheta_{s}$ is a proper function bounded from below by virtue of the fact that $f$ is by Lemma 4.2. Then since $X=\nabla^{h_{s}}f_{\sigma_{s}}$, $f_{\sigma_{s}}$ must attain its global minimum at a point lying in the zero set of $X$ and hence must coincide with the global minimum of $f$ on this set; that is to say,

$$f_{\sigma_{s}}\geq\min_{\{X\,=\,0\}}f_{\sigma_{s}}=\min_{\{X\,=\,0\}}f\geq -C.$$

The lower bound on $X\cdot\vartheta_{s}$ then follows from the previous localisation of the minimum of this function given by Lemma 7.3. □

7.5. **A priori $C^{0}$-estimate.** We begin with the a priori estimate on the $C^{0}$-norm of $(\vartheta_{s})_{0\leq s\leq 1}$ which is uniform in $s\in[0,\,1]$. We begin with two crucial observations, the first a localisation result for the global extrema of $\vartheta_{s}$.

**Lemma 7.6** (Localising the supremum and infimum of a solution of $(\star\star_{s})$)**.** *Let $\vartheta_{s}$ be a path of solutions in $\mathbb{R}\oplus C^{\infty}_{X,\,\beta}(M)$ to $(\star\star_{s})$. Then $\sup_{M}\vartheta_{s}=\max_{W}\vartheta_{s}$ (resp. $\inf_{M}\vartheta_{s}=\min_{W}\vartheta_{s}$).*

*Proof.* We prove the assertions of Lemma 7.6 that concern the supremum of a solution $\vartheta_{s}$ only. The statements on the infimum of $\vartheta_{s}$ can be proved in a similar manner.

Observe from $(\star\star_{s})$ that $\vartheta_{s}$ is a subsolution of the following differential inequality:

$$\Delta_{\omega_{s}}\vartheta_{s}-\frac{X}{2}\cdot\vartheta_{s}\geq G_{s}. \tag{7.7}$$

Let $\varepsilon>0$ and consider any smooth function $u_{\varepsilon}$ on $M$ identically equal to $2\varepsilon\log(r)$ on $M\setminus W$ such that $\lim_{\varepsilon\to 0}u_{\varepsilon}=0$ uniformly on compact sets of $M$. This function will serve as a barrier function. Indeed, one has that on $M\setminus W$,

$$\Delta_{\omega_{s}}\left(\vartheta_{s}-2\varepsilon\log(r)\right)-\frac{X}{2}\cdot(\vartheta_{s}-2\varepsilon\log(r))\geq\varepsilon>0. \tag{7.8}$$

Now $\vartheta_{s}$ being bounded on $M$ implies that the function $\vartheta_{s}-2\varepsilon\log(r)$ tends to $-\infty$ as $r\to+\infty$. In particular, this latter function must attain its maximum on $M$. The maximum principle applied to (7.8) then ensures that it must be attained in $W$, i.e., $\max_{M}(\vartheta_{s}-u_{\varepsilon})=\max_{W}(\vartheta_{s}-u_{\varepsilon})$. In conclusion, we have that

$$\vartheta_s(x) \leq u_\varepsilon(x) + \max_W(\vartheta_s - u_\varepsilon), \quad x \in M,$$

which leads to the bound $\vartheta_s(x) \leq \max_W \vartheta_s$ by letting $\varepsilon \to 0$ and making use of the assumption on $u_\varepsilon$. As this holds true for any $x \in M$, the desired estimate follows. □

7.5.1. *Aubin-Tian-Zhu's functionals.* We now introduce two functionals that have been defined and used by Aubin [Aub84], Bando and Mabuchi [BM87], and Tian [Tia00b, Chapter 6] in the study of Fano manifolds, and by Tian and Zhu [TZ00b] in the study of shrinking gradient Kähler-Ricci solitons on compact Kähler manifolds.

**Definition 7.7.** Let $(\varphi_t)_{0 \leq t \leq 1}$ be a $C^1$-path in $\mathcal{M}^\infty_{X,\beta}(M)$ from $\varphi_0 = 0$ to $\varphi_1 = \varphi$. We define the following two generalised weighted energies:

$$I_{\omega, X}(\varphi) := \int_M \varphi \left( e^{-f}\omega^n - e^{-f - \frac{X}{2}\cdot\varphi}\omega^n_\varphi \right),$$
$$J_{\omega, X}(\varphi) := \int_0^1 \int_M \dot{\varphi}_s \left( e^{-f}\omega^n - e^{-f - \frac{X}{2}\cdot\varphi_s}\omega^n_{\varphi_s} \right) \wedge ds.$$

At first sight, these two functionals resemble relative weighted mean values of a potential $\varphi$ in $\mathcal{M}^\infty_{X,\beta}(M)$ or of a path $(\varphi_t)_{0 \leq t \leq 1}$ in $\mathcal{M}^\infty_{X,\beta}(M)$ respectively. When $X \equiv 0$ and $(M, \omega)$ is a compact Kähler manifold, an integration by parts together with some algebraic manipulations (see Aubin's seminal paper [Aub84] or Tian's book [Tia00b, Chapter 6]) show that

$$\begin{aligned} I_{\omega, 0}(\varphi) &= \sum_{k=0}^{n-1} \int_M i\partial\varphi \wedge \bar{\partial}\varphi \wedge \omega^k \wedge \omega_\varphi^{n-1-k}, \\ J_{\omega, 0}(\varphi) &= \sum_{k=0}^{n-1} \frac{k+1}{n+1} \int_M i\partial\varphi \wedge \bar{\partial}\varphi \wedge \omega^k \wedge \omega_\varphi^{n-1-k}. \end{aligned} \tag{7.9}$$

This justifies the description of $I_{\omega,0}(\varphi)$ and $J_{\omega,0}(\varphi)$ as modified energies. Moreover, it demonstrates that on a compact Kähler manifold $J_{\omega,0}$ is a true functional, that is to say, it does not depend on the choice of path.

Such formulae (7.9) for $I_{\omega,X}$ and $J_{\omega,X}$ for a non-vanishing vector field $X$ and a non-compact Kähler manifold $(M, \omega)$ do not seem to be readily available for a good reason; the exponential function is not algebraic. However, following Tian and Zhu's work [TZ00b], one can prove that the essential properties shared by both $I_{\omega,0}$ and $J_{\omega,0}$ hold true for a non-vanishing vector field $X$ in a non-compact setting. The proof follows exactly as in [CD20, Theorem 7.5].

**Theorem 7.8.** *$I_{\omega,X}(\varphi)$ and $J_{\omega,X}(\varphi)$ are well-defined for $\varphi \in \mathcal{M}^\infty_{X,\beta}(M)$. Moreover, $J_{\omega,X}$ does not depend on the choice of a $C^1$ path $(\varphi_t)_{0 \leq t \leq 1}$ in $\mathcal{M}^\infty_{X,\beta}(M)$ from $\varphi_0 = 0$ to $\varphi_1 = \varphi$, hence defines a functional on $\mathcal{M}^\infty_{X,\beta}(M)$. Finally, the first variation of the difference $(I_{\omega,X} - J_{\omega,X})$ is given by*

$$\frac{d}{dt}\left(I_{\omega,X} - J_{\omega,X}\right)(\varphi_t) = -\int_M \varphi_t \left( \Delta_{\omega_{\varphi_t}} \dot{\varphi}_t - \frac{X}{2} \cdot \dot{\varphi}_t \right) e^{-f_{\varphi_t}} \omega^n_{\varphi_t}, \tag{7.10}$$

*where $f_{\varphi_t} := f + \frac{X}{2} \cdot \varphi_t$ satisfies $X = \nabla^{\omega_{\varphi_t}} f_{\varphi_t}$ and where $(\varphi_t)_{0 \leq t \leq 1}$ is any $C^1$-path in $\mathcal{M}^\infty_{X,\beta}(M)$ from $\varphi_0 = 0$ to $\varphi_1 = \varphi$.*

Recall that the equation we wish to solve is $(\star_s)$, namely

$$e^{-f_{\psi_s}}\omega^n_{\psi_s} = e^{F_s - f}\omega^n.$$

**Proposition 7.9** (A priori energy estimates)**.** *Let $(\psi_s)_{0 \leq s \leq 1}$ be a path of solutions in $\mathcal{M}^\infty_{X,\beta}(M)$ to $(\star_s)$. Then for $p \in (1, 2)$, there exists a positive constant $C = C\left(n, p, \omega, \sup_{s \in [0,1]} \|F_s\|_{C^0}\right)$ such that*

$$\sup_{0 \leq s \leq 1} \int_M |\psi_s - \overline{\psi}_s|^p \, e^{-f}\omega^n \leq C,$$

*where* $\overline{\psi}_s := \int_M \psi_s\, e^{-f}\omega^n$. *In particular, if* $\overline{\psi}_s = 0$, *then*

$$\sup_{0\,\leq\, s\,\leq\, 1}\int_M |\vartheta_s|^p\, e^{-f}\omega^n \leq C.$$

*Proof.* As a consequence of Theorem 7.8, we can use any $C^1$-path $(\varphi_t)_{0\,\leq\, t\,\leq\, 1}$ in $\mathcal{M}^{\infty}_{X,\,\beta}(M)$ from $\varphi_0 = 0$ to $\varphi_1 = \varphi \in \mathcal{M}^{\infty}_{X,\,\beta}(M)$ to compute $J_{\omega,\,X}(\varphi)$. As in [TZ00b], we choose two different paths to compute $J_{\omega,\,X}(\psi)$, the first being the linear path defined by $\varphi_t := t\psi$, $t \in [0,\,1]$, for $\psi \in \mathcal{M}^{\infty}_{X,\,\beta}(M)$ a solution to $(\star_s)$. For this path, Theorem 7.8 asserts that

$$(I_{\omega,\,X} - J_{\omega,\,X})\,(\psi) = -\int_0^1\int_M t\psi\left(\Delta_{\omega_{t\psi}}\psi - \frac{X}{2}\cdot\psi\right)\, e^{-f-t\frac{X}{2}\cdot\psi}\omega^n_{t\psi}\wedge dt.$$

Integration by parts with respect to the weighted volume form $e^{-f-t\frac{X}{2}\cdot\psi}\omega^n_{t\psi}$ then leads to

$$\begin{aligned}
(I_{\omega,\,X} - J_{\omega,\,X})\,(\psi) &= n\int_0^1\int_M t\, i\partial\psi\wedge\bar{\partial}\psi\wedge\left(e^{-f-t\frac{X}{2}\cdot\psi}\omega^{n-1}_{t\psi}\right)\wedge dt\\
&= n\int_0^1\int_M t\, i\partial\psi\wedge\bar{\partial}\psi\wedge\left(e^{-f-t\frac{X}{2}\cdot\psi}\left((1-t)\omega + t\omega_{\psi}\right)^{n-1}\right)\wedge dt\\
&= n\sum_{k\,=\,0}^{n-1}\binom{n-1}{k}\left(\int_0^1 t^{k+1}(1-t)^{n-1-k}\int_M i\partial\psi\wedge\bar{\partial}\psi\wedge\left(e^{-f-t\frac{X}{2}\cdot\psi}\omega^{n-1-k}\wedge\omega^k_{\psi}\right)\right)\wedge dt\\
&\geq n\int_0^1 t(1-t)^{n-1}\int_M i\partial\psi\wedge\bar{\partial}\psi\wedge\left(e^{-f-t\frac{X}{2}\cdot\psi}\omega^{n-1}\right)\wedge dt\\
&= n\int_M\left(\int_0^1 t(1-t)^{n-1}e^{-t\frac{X}{2}\cdot\psi}\, dt\right) i\partial\psi\wedge\bar{\partial}\psi\wedge e^{-f}\omega^{n-1}.
\end{aligned} \tag{7.11}$$

From this, the following claim will allow us to obtain a lower bound.

**Claim 7.10.** There exists positive uniform constants $A$, $c$, such that

$$\int_0^1 t(1-t)^{n-1}e^{-t\frac{X}{2}\cdot\psi}\, dt \geq \frac{c}{\left(\frac{X}{2}\cdot\psi + A\right)^2}.$$

*Proof.* For $k \geq k_n$, for some $k_n > 0$ to be defined later, we find using integration by parts and a change of variable that

$$\begin{aligned}
\int_0^1 t(1-t)^{n-1}e^{-kt}\, dt &= \int_0^1 (1-s)s^{n-1}e^{-k(1-s)}\, ds\\
&= e^{-k}\left\{\left(1+\frac{n}{k}\right)\int_0^1 s^{n-1}e^{ks}\, ds - \frac{e^k}{k}\right\}\\
&= e^{-k}\left\{\left(1+\frac{n}{k}\right)\left(\frac{e^k}{k} - \frac{(n-1)}{k}\int_0^1 s^{n-2}e^{ks}\, ds\right) - \frac{e^k}{k}\right\}\\
&\geq \left(1+\frac{n}{k}\right)\left(\frac{1}{k} - \frac{(n-1)}{k^2}(1-e^{-k})\right) - \frac{1}{k}\\
&= \frac{k-n(n-1)}{k^3} + e^{-k}\frac{(n+k)(n-1)}{k^3}\\
&\geq \frac{1}{2k^2},
\end{aligned}$$

if $k \geq k_n := 2n(n-1)$. Here we have bounded $s^{n-2}$ from above by 1 in the fourth inequality.

Set $A := k_n - \inf_M \frac{X}{2}\cdot\psi$ and let $k = \frac{X}{2}\cdot\psi + A$. Then $k \geq k_n$ and the above tells us that

$$\int_0^1 t(1-t)^{n-1}e^{-t\left(\frac{X}{2}\cdot\psi + A\right)}\, dt \geq \frac{1}{2\left(\frac{X}{2}\cdot\psi + A\right)^2},$$

resulting in the desired bound. □

Applying Claim 7.10 to (7.11) results in the lower bound

$$(I_{\omega,X}-J_{\omega,X})(\psi)\geq c\int_M i\partial\psi\wedge\bar\partial\psi\wedge\frac{e^{-f}\omega^{n-1}}{\left(\frac{X}{2}\cdot\psi+A\right)^2}\geq c\int_M\frac{|\nabla^g\psi|_g^2}{\left(\frac{X}{2}\cdot\psi+A\right)^2}\,e^{-f}\omega^n, \tag{7.12}$$

for some positive constant $c$.

We also require an upper bound on $(I_{\omega,X}-J_{\omega,X})(\psi)$ to complete the proof of the proposition. To this end, we consider the continuity path of solutions $\varphi_s:=\psi_s$, $s\in[0,1]$, to $(\star_s)$ to compute $(I_{\omega,X}-J_{\omega,X})(\psi)$. First observe that the first variations $(\dot\psi_s)_{0\leq s\leq 1}$ satisfy the following PDE obtained from $(\star_s)$ by differentiating with respect to the parameter $s$:

$$\Delta_{\omega_{\psi_s}}\dot\psi_s-\frac{X}{2}\cdot\dot\psi_s=\dot F_s,\qquad 0\leq s\leq 1.$$

Combined with $(\star_s)$ and Theorem 7.8, this leads to

$$\begin{aligned}(I_{\omega,X}-J_{\omega,X})(\psi)&=\int_0^1\int_M\psi_t\cdot(-\dot F_t)\,e^{-f_{\psi_t}}\omega_{\psi_t}^n\wedge dt\\&=\int_0^1\int_M\psi_t\cdot(-\dot F_t)\,e^{F_t-f}\omega^n\wedge dt\end{aligned}\tag{7.13}$$

so that, from (7.12), for some $c>0$,

$$\int_0^1\int_M\psi_t\cdot(-\dot F_t)\,e^{F_t-f}\omega^n\wedge dt\geq c\int_M\frac{|\nabla^g\psi|_g^2}{\left(\frac{X}{2}\cdot\psi+A\right)^2}\,e^{-f}\omega^n. \tag{7.14}$$

Now, as

$$\frac{d}{ds}\left(\int_M e^{-f_{\psi_s}}\omega_{\psi_s}^n\right)=0$$

by Lemma 6.2(i) with $G\equiv 1$, we derive from $(\star_s)$ that

$$\int_M\dot F_t e^{F_t-f}\omega^n=0.$$

This allows us to rewrite (7.14) as

$$\int_0^1\int_M(\psi_t-\overline{\psi}_t)\cdot(-\dot F_t)\,e^{F_t-f}\omega^n\wedge dt\geq c\int_M\frac{|\nabla^g\psi|_g^2}{\left(\frac{X}{2}\cdot\psi+A\right)^2}\,e^{-f}\omega^n,$$

with $\overline{\psi}_t$ as in the statement of the proposition. Applying the Poincaré inequality of Proposition 5.1, we then see that for any $p \in (1,\,2)$ and $\frac{1}{q} = 1 - \frac{1}{p}$,

$$\begin{aligned}\left(\int_M |\psi - \overline{\psi}|^p\, e^{-f}\omega^n\right)^{\frac{2}{p}} &\leq C\left(\int_M |\nabla^g\psi|_g^p\, e^{-f}\omega^n\right)^{\frac{2}{p}}\\ &\leq C\left(\int_M \frac{|\nabla^g\psi|_g^2}{\left(\frac{X}{2}\cdot\psi + A\right)^2}\, e^{-f}\omega^n\right)\left(\int_M\left(\frac{X}{2}\cdot\psi + A\right)^{\frac{2p}{2-p}} e^{-f}\omega^n\right)^{\frac{2-p}{p}}\\ &\leq C\left(\int_0^1\int_M |\psi_t - \overline{\psi}_t||\dot{F}_t|\, e^{F_t - f}\omega^n\wedge dt\right)\left(\int_M\left(\frac{X}{2}\cdot\psi + A\right)^{\frac{2p}{2-p}} e^{-f}\omega^n\right)^{\frac{2-p}{p}}\\ &\leq C\int_0^1\left(\int_M |\psi_t - \overline{\psi}_t|^p\, e^{-f}\omega^n\right)^{\frac{1}{p}}\left(\int_M |\dot{F}_t|^q\, e^{qF_t}\, e^{-f}\omega^n\right)^{\frac{1}{q}} dt\left(\int_M\left(\frac{X}{2}\cdot\psi + A\right)^{\frac{2p}{2-p}} e^{-f}\omega^n\right)^{\frac{2-p}{p}}\\ &\leq C\int_0^1\left(\int_M |\psi_t - \overline{\psi}_t|^p\, e^{-f}\omega^n\right)^{\frac{1}{p}} dt\left(\int_M\left(\frac{X}{2}\cdot\psi + A\right)^{\frac{2p}{2-p}} e^{-f}\omega^n\right)^{\frac{2-p}{p}}.\end{aligned} \tag{7.15}$$

Here we have used Hölder's inequality in the second and fourth lines with respect to the weighted measure $e^{-f}\omega^n$.

Next, observe from Lemma 6.2(i) that for all $r \in \mathbb{N}$,

$$c\int_M (f_{\psi_s}{+}A)^r\, e^{-f}\omega^n \leq \int_M (f_{\psi_s}{+}A)^r e^{F_s}\, e^{-f}\omega^n = \int_M (f_{\psi_s}{+}A)^r\, e^{-f_{\psi_s}}\omega_{\psi_s}^n = \int_M (f{+}A)^r\, e^{-f}\omega^n \leq C(r).$$

By induction, using the fact that $\frac{X}{2}\cdot\psi + A \geq 0$ and that $A \leq C$ by Corollary 7.5, one can prove directly from this that

$$\int_M\left(\frac{X}{2}\cdot\psi + A\right)^r e^{-f}\omega^n \leq C(r) \qquad \text{for all } r \in \mathbb{N}.$$

It then follows from Hölder's inequality that this statement holds true for all $r \geq 1$. Applying this to (7.15), we arrive at the fact that for all $p \in (1,\,2)$,

$$\left(\int_M |\psi_t - \overline{\psi}_t|^p\, e^{-f}\omega^n\right)^{\frac{2}{p}} \leq C(p)\int_0^1\left(\int_M |\psi - \overline{\psi}|^p\, e^{-f}\omega^n\right)^{\frac{1}{p}} dt,$$

i.e.,

$$\|\psi - \overline{\psi}\|^2_{L^p(e^{-f}\omega^n)} \leq C(p)\int_0^1 \|\psi_t - \overline{\psi}_t\|_{L^p(e^{-f}\omega^n)}\, dt \qquad \text{for any } p \in (1,\,2). \tag{7.16}$$

This last inequality applies to any truncated path of the one-parameter family of solutions $(\psi_s)_{0\,\leq\,s\,\leq\,1}$ of $(\star_s)$. Thus,

$$\begin{aligned}\|\psi_s - \overline{\psi}_s\|^2_{L^p(e^{-f}\omega^n)} &\leq C\int_0^1 \|\psi_{st} - \overline{\psi}_{st}\|_{L^p(e^{-f}\omega^n)}\, dt\\ &= \frac{C}{s}\int_0^s \|\psi_t - \overline{\psi}_t\|_{L^p(e^{-f}\omega^n)}\, dt.\end{aligned} \tag{7.17}$$

This is a Grönwall-type differential inequality and can be integrated as follows. Let

$$H(s) := \int_0^s \|\psi_t - \overline{\psi}_t\|_{L^p(e^{-f}\omega^n)}\, dt,$$

and observe that (7.17) may be rewritten as

$$H'(s) \leq \frac{C}{s^{\frac{1}{2}}}(H(s))^{\frac{1}{2}}, \quad s \in (0,1].$$

Integrating then implies that $H(s) \leq C\left(n, \omega, \sup_{s\in[0,1]}\|F_s\|_{C^0}\right) \cdot s$ for all $s \in [0, 1]$ which, after applying (7.17) once more, yields to the desired upper bound. □

7.5.2. *A priori estimate on* $\sup_M \vartheta$. Let $\vartheta_s$ be a solution to $(\star\star_s)$ for some fixed value of the parameter $s \in [0, 1]$. We next obtain an upper bound for $\sup_M \vartheta_s$ uniform in $s$. To obtain such a bound, it suffices by Lemma 7.6 to only bound $\max_W \vartheta_s$ from above. We do this by implementing a local Nash-Moser iteration using the fact that $\vartheta_s$ is a super-solution of the linearised complex Monge-Ampère equation of which the drift Laplacian with respect to the known metric $\omega_s$ forms a part.

**Proposition 7.11** (A priori upper bound on $\sup_M \vartheta$)**.** *Let* $(\vartheta_s)_{0\leq s\leq 1}$ *be a path of solutions in* $\mathbb{R}\oplus C^{\infty}_{X,\beta}(M)$ *to* $(\star\star_s)$*. Then there exists a positive constant* $C = C\left(n, \omega, \sup_{s\in[0,1]}\|G_s\|_{C^0}\right)$ *such that*

$$\sup_{0\leq s\leq 1}\sup_W \vartheta_s \leq C.$$

*Proof.* Let $s \in [0, 1]$ and let $(\vartheta_s)_+ := \max\{\vartheta_s, 0\}$. This is a non-negative Lipschitz function. The strategy of proof is standard; we use a Nash-Moser iteration to obtain an a priori upper bound on $\sup_W(\vartheta_s)_+$ in terms of the (weighted) energy of $(\vartheta_s)_+$ on a tubular neighbourhood of $W$. The result then follows by invoking Proposition 7.9.

To this end, notice that since $\log(1 + x) \leq x$ for all $x > -1$ and since $\vartheta_s$ is a solution to $(\star\star_s)$, $\vartheta_s$ satisfies the differential inequality

$$\Delta_{\omega_s}\vartheta_s - \frac{X}{2}\cdot\vartheta_s \geq -|G_s| \qquad \text{on } M. \tag{7.18}$$

Let $g_s$ denote the Kähler metric associated to $\omega_s$ and let $f_{\omega_s} := f + \frac{X}{2}\cdot\Phi_s$. Then these metrics are all equivalent to $g$ uniformly in $s$ and $-\omega_s\lrcorner X = df_{\omega_s}$. Let $x \in \{f < R\}$ and $\varepsilon > 0$ be such that $B_{g_s}(x, \varepsilon) \Subset \{f < R\}$ and multiply (7.18) across by $\eta^2_{t,t'}(\vartheta_s)_+|(\vartheta_s)_+|^{2(p-1)}$ with $p \geq 1$, where $\eta_{t,t'}$, with $0 < t+t' < \varepsilon$ and $t, t' > 0$, is a Lipschitz cut-off function with compact support in $B_{g_s}(x, t+t')$ equal to 1 on $B_{g_s}(x, t)$ and with $|\nabla^{g_s}\eta_{t,t'}|_{g_s} \leq \frac{1}{t'}$ almost everywhere. Next, integrate by parts and use a local Sobolev inequality for the pair $(\omega_s, f_{\omega_s})$ to obtain a so-called "reversed Hölder inequality" which after iteration leads to the bound for $p \in (1, 2)$,

$$\begin{aligned}
\sup_{B_{g_s}(x,\frac{\varepsilon}{2})}(\vartheta_s)_+ &\leq C(n,p,\omega,\varepsilon)\left(\|(\vartheta_s)_+\|^p_{L^p(B_{g_s}(x,\varepsilon),\,e^{-f_{\omega_s}}\omega_s^n)} + \|G_s\|^p_{C^0}\right)^{\frac{1}{p}}\\
&\leq C(n,p,\omega,\varepsilon)\left(\int_{\{f<R\}}(\vartheta_s)_+^p\,e^{-f_{\omega_s}}\omega_s^n + \|G_s\|^p_{C^0}\right)^{\frac{1}{p}}\\
&\leq C(n,p,\omega,\varepsilon)\left(\int_{\{f<R\}}|\vartheta_s|^p\,e^{-f}\omega^n + \|G_s\|^p_{C^0}\right)^{\frac{1}{p}}\\
&\leq C\left(n,p,\omega,\varepsilon,\sup_{s\in[0,1]}\|G_s\|_{C^0}\right).
\end{aligned}$$

Here, we have made use of Proposition 7.9 in the last line. □

7.5.3. *A priori estimate on* $\inf_M \vartheta$. Recall that the equation we wish to solve is

$$e^{-f_{\psi_s}}\omega^n_{\psi_s} = e^{F_s - f}\omega^n, \tag{$(\star_s)$}$$

where $\omega_{\psi_s} := \omega + i\partial\bar{\partial}\psi_s > 0$ and $f_{\psi_s} := f + \frac{X}{2}\cdot\psi_s$. This pair satisfies $-\omega_{\psi_s}\lrcorner X = df_{\psi_s}$. We work under the assumption that $\int_M \psi_s\, e^{-f}\omega^n = 0$.

*An upper bound on the* $I$*-functional.* We first show that the $I$-functional is bounded along the continuity path.

**Lemma 7.12.** $\sup_{s\in[0,1]} I(\psi_s) \le C(\sup_M(\vartheta_s)_+)$.

*Proof.* By assumption, $\int_M \psi_s\, e^{-f}\omega^n = 0$ so that $\int_{\{\psi_s\ge 0\}} \psi_s\, e^{-f}\omega^n = -\int_{\{\psi_s\le 0\}} \psi_s\, e^{-f}\omega^n$. Therefore we have that

$$\begin{aligned}
I(\psi_s) &= \int_M \psi_s\left(e^{-f}\omega^n - e^{-f_{\psi_s}}\omega^n_{\psi_s}\right) = -\int_M \psi_s\, e^{-f_{\psi_s}}\omega^n_{\psi_s}\\
&= -\int_{\{\psi_s\ge 0\}} \psi_s\, e^{-f_{\psi_s}}\omega^n_{\psi_s} + \int_{\{\psi_s\le 0\}} (-\psi_s)\, e^{-f_{\psi_s}}\omega^n_{\psi_s} \le \int_{\{\psi_s\le 0\}} (-\psi_s)\, e^{-f_{\psi_s}}\omega^n_{\psi_s}\\
&= \int_{\{\psi_s\le 0\}} (-\psi_s)\, e^{F_s}\, e^{-f}\omega^n \le C\int_{\{\psi_s\le 0\}} (-\psi_s)\, e^{-f}\omega^n = C\int_{\{\psi_s\ge 0\}} \psi_s\, e^{-f}\omega^n\\
&= C\left(\int_{\{\vartheta_s\ge -\Phi_s\}} (\vartheta_s+\Phi_s)\, e^{-f}\omega^n\right) \le C\Bigg(\underbrace{\int_M |\Phi_s|\, e^{-f}\omega^n}_{\text{bounded}} + \int_{\{\vartheta_s\ge -\Phi_s\}} \vartheta_s\, e^{-f}\omega^n\Bigg)\\
&\le C + C\sup_M \vartheta_s^+ \int_{\{\vartheta_s\ge -\Phi_s\}} e^{-f}\omega^n \le C + C\sup_M(\vartheta_s)_+ \int_M e^{-f}\omega^n\\
&\le C(1+\sup_M(\vartheta_s)_+).
\end{aligned}$$

From this, the result follows. □

*An upper bound on the $\hat{F}$-functional.* Recall the continuity path $(\star_s)$:

$$(\omega + i\partial\bar{\partial}\psi_s)^n = e^{F_s + \frac{X}{2}\cdot\psi_s}\omega^n, \qquad s\in[0,\,1], \tag{7.19}$$

where

$$F_s := \log\left(se^F + (1-s)\right) \qquad \text{and} \qquad i\partial\bar{\partial}F = \rho_\omega + \frac{1}{2}\mathcal{L}_X\omega - \omega.$$

Here, $\rho_\omega$ denotes the Ricci form of $\omega$ and $F\in C^\infty(M)$ is bounded. On $\mathfrak{t}\simeq\mathbb{R}^n$ we have coordinates $\xi := (\xi_1,\ldots,\xi_n)$, induced coordinates $x = (x_1,\ldots,x_n)$ on $\mathfrak{t}^*$ which contains the image of the moment map, and we can write $\omega = 2i\partial\bar{\partial}\phi_0$ for a convex function $\phi_0$ on $\mathbb{R}^n\simeq\mathfrak{t}$ up to the addition of a linear function (cf. Section 2.5). Let $b_X\in\mathbb{R}^n$ denote the vector field $JX\in\mathfrak{t}$ as in (2.8), write $\nabla$ for the Levi-Civita connection of the flat metric on $\mathbb{R}^n$, and $\langle\cdot\,,\,\cdot\rangle$ for the corresponding inner product. As in (2.17), we normalize $\phi_0$ so that

$$F = -\log\det(\phi_{0,\,ij}) + \langle\nabla\phi_0,\, b_X\rangle - 2\phi_0.$$

Set $\phi_s := \phi_0 + \frac{1}{2}\psi_s$. Our background metric $\omega$ satisfies the two bullet points above Lemma 2.35 as demonstrated in the already proved Theorem A(ii)–(iv). As a consequence, it is clear from Lemma 2.35(i) that condition (a) of Definition 2.32 holds true. The hypothesis of Lemma 2.34 as well as condition (b) of Definition 2.32 via Lemma 2.33 also hold true thanks to Lemma 6.2(ii). Thus, the $\hat{F}$-functional from Definition 2.32 is finite and therefore well-defined along the continuity path $(\star_s)$ and moreover, by Lemma 2.34, may be expressed along in terms of the $J$-functional as

$$\hat{F}(\psi_s) = J(\psi_s) - \int_M \psi_s\, e^{-f}\omega^n.$$

We next show that $\hat{F}$ is bounded above along this continuity path. Together with Lemma 7.12, this will in turn provide an a priori weighted estimate on the integral of the Legendre transform $u_s := L(\phi_s)$ of $\phi_s$ which will lead to the desired lower bound on $\inf_M\vartheta_s$.

**Lemma 7.13.** $\hat{F}(\psi_s) \le C(\sup_M(\vartheta_s)_+)$.

*Proof.* By assumption we have that $\int_M \psi_s\, e^{-f}\omega^n = 0$ so that $\hat{F}(\psi_s) = J(\psi_s)$. Moreover, from (7.12) we read that $(I-J)(\psi_s)\ge 0$. Thus, Lemma 7.12 implies that

$$\hat{F}(\psi_s) = J(\psi_s) = I(\psi_s) - (I-J)(\psi_s) \le I(\psi_s) + 0 \le C\left(\sup_M(\vartheta_s)_+\right),$$

as claimed. □

*An upper bound on the weighted integral of the Legendre transform.* We know that

$$\begin{aligned}\int_{P_{-K_M}} |u_s|\, e^{-\langle b_X,\, x\rangle} dx &\leq \int_{P_{-K_M}} |u_s - u_0|\, e^{-\langle b_X,\, x\rangle} dx + \int_{P_{-K_M}} |u_0|\, e^{-\langle b_X,\, x\rangle} dx \\ &\leq \int_{P_{-K_M}} \left(\int_0^1 |\dot{u}_{st}|\, dt\right) e^{-\langle b_X,\, x\rangle} dx + \int_{P_{-K_M}} |u_0|\, e^{-\langle b_X,\, x\rangle} dx,\end{aligned}$$

and these last two integrals are finite by Lemma 6.2(ii) via Lemma 2.33, and Lemma 2.35(ii), respectively. By definition, the $\hat{F}$-functional along $(\star_s)$ is given by

$$\hat{F}(\psi_s) = 2 \int_{P_{-K_M}} (u_s - u_0)\, e^{-\langle b_X,\, x\rangle} dx. \tag{7.20}$$

Therefore with $\int_{P_{-K_M}} |u_0|\, e^{-\langle b_X,\, x\rangle} dx$ and $\int_{P_{-K_M}} |u_1|\, e^{-\langle b_X,\, x\rangle} dx$ convergent, we can split the integral in (7.20). Together the integral bound given in Lemma 2.35(ii), this leads to the following consequence of Lemma 7.13.

**Corollary 7.14.**

$$\sup_{s\,\in\,[0,\,1]} \int_{P_{-K_M}} u_s\, e^{-\langle b_X,\, x\rangle}\, dx < C.$$

*A priori lower bound on* $\inf_M \vartheta_s$. This bound together with the uniform upper bound already obtained for $\sup_M \vartheta_s$ now yields a uniform lower bound on $\inf_W \vartheta_s$. By Lemma 7.6, this results in a uniform lower bound on $\inf_M \vartheta_s$.

**Proposition 7.15** (A priori lower bound on $\inf_M \vartheta_s$)**.** *Let $(\vartheta_s)_{0\,\leq\,s\,\leq\,1}$ be a path of solutions in $\mathbb{R} \oplus C^{\infty}_{X,\,\beta}(M)$ to $(\star\star_s)$. Then there exists a positive constant $C = C\left(n, \omega, \sup_{s\in[0,1]} \|G_s\|_{C^0}\right)$ such that*

$$\inf_{0\,\leq\,s\,\leq\,1}\, \inf_W \vartheta_s \geq -C.$$

*Proof.* As $0 \in P_{-K_M}$, Lemma 2.30 stipulates that $\phi_s$ attains a unique minimum. Denote the point where this minimum is achieved by $\xi^s_{\min} \in \mathbb{R}^n \simeq \mathfrak{t}$ and define

$$\widehat{\phi}_s(\xi) := \phi_s(\xi + \xi^s_{\min}) - \inf_{\mathbb{R}^n} \phi_s.$$

Then $\widehat{\phi}_s \geq 0$ and attains a unique minimum at the origin with the value of 0 at this point. Let $u_s = L(\phi_s)$ and $\widehat{u}_s := L(\widehat{\phi}_s)$. Then by general properties of the Legendre transform, $\widehat{u}_s \geq 0$ and in terms of $u_s$, $\widehat{u}_s$ is given by

$$\widehat{u}_s = u_s + \inf_{\mathbb{R}^n} \phi_s - \langle \xi^s_{\min},\, x\rangle.$$

Using the uniform upper bound on $\int_{P_{-K_M}} u_s\, e^{-\langle b_X,\, x\rangle}\, dx$ given by Corollary 7.14 together with property (2.11) of $b_X$, we compute that

$$\begin{aligned}0 \leq \int_{P_{-K_M}} \widehat{u}_s\, e^{-\langle b_X,\, x\rangle}\, dx &= \int_{P_{-K_M}} \left(u_s + \inf_{\mathbb{R}^n} \phi_s - \langle \xi^s_{\min},\, x\rangle\right) e^{-\langle b_X,\, x\rangle}\, dx \\ &= \int_{P_{-K_M}} u_s\, e^{-\langle b_X,\, x\rangle}\, dx + \left(\inf_{\mathbb{R}^n} \phi_s\right) \int_{P_{-K_M}} e^{-\langle b_X,\, x\rangle}\, dx \\ &\leq C\left(1 + \inf_{\mathbb{R}^n} \phi_s\right).\end{aligned}$$

Thus, $\inf_{\mathbb{R}^n} \phi_s \geq -C$ for some $C > 0$ uniformly in $s$ and we deduce that

$$\phi_0(\xi) + \frac{1}{2}\left(\Phi_s(\xi) + \vartheta_s\right)(\xi) = \phi_0(\xi) + \frac{1}{2}\psi_s(\xi) = \phi_s(\xi) \geq \inf_{\mathbb{R}^n} \phi_s \geq -C \qquad \text{for all } \xi \in \mathbb{R}^n.$$

Given that $\phi_0 + \frac{1}{2}\Phi_s$ is uniformly bounded from above on any compact subset of $M$, the result follows. □

7.6. **A priori upper bound on the radial derivative.** The $C^0$-bound on $\vartheta_s$ allows us to derive an a priori upper bound on $X \cdot \vartheta_s$.

**Proposition 7.16.** *Let $(\vartheta_s)_{0\,\leq\,s\,\leq\,1}$ be a path of solutions in $\mathbb{R} \oplus C^{\infty}_{X,\,\beta}(M)$ to $(\star\star_s)$. Then there exists a positive constant $C = C\left(n, \tau, \sup_{s\in[0,1]} \|G_s\|_{C^0}\right)$ such that*

$$\sup_{0\,\leq\,s\,\leq\,1}\,\sup_{M} X \cdot \vartheta_s \leq C.$$

*In particular, $X \cdot \vartheta_s < C$ for all $s \in [0,\,1]$.*

*Proof.* Our proof is based on that of Siepmann in the case of an expanding gradient Kähler-Ricci soliton; see [Sie13, Lemma 5.4.14]. We adapt his proof here to our particular setting.

Applying Claim 7.4, we have that

$$X \cdot X \cdot \vartheta_s = 2i\partial\bar{\partial}\vartheta_s(X,\,JX) = 2\left(\sigma_s(X,\,JX) - \omega_s(X,\,JX)\right) \geq -2\omega_s(X,\,JX) = -2|X|^2_{g_s}. \tag{7.21}$$

To get an upper bound for $X \cdot \vartheta_s$, we introduce the flow $(\varphi^X_t)_{t\in\mathbb{R}}$ generated by the vector field $\frac{X}{2}$. This flow is complete since $X$ grows linearly at infinity. Define $\vartheta^s_x(t) := \vartheta_s(\varphi^X_t(x))$ for $(x,\,t) \in M\times\mathbb{R}$. Then for any cut-off function $\eta : \mathbb{R}_+ \to [0,\,1]$ such that $\eta(0) = 1$ and $\eta'(0) = 0$ we have that

$$\begin{aligned}\int_0^{+\infty} \eta''(t)\vartheta^s_x(t)dt &= -\int_0^{+\infty} \eta'(t)(\vartheta^s_x)'(t)dt \\ &= (\vartheta^s_x)'(0) + \int_0^{+\infty} \eta(t)(\vartheta^s_x)''(t)dt.\end{aligned}$$

Using (7.21), it then follows that

$$\begin{aligned}\frac{X}{2}\cdot\vartheta_s(x) = (\vartheta^s_x)'(0) &\leq -\int_{\mathrm{supp}(\eta)} \frac{X}{2}\cdot\left(\frac{X}{2}\cdot\vartheta_s\right)(\varphi^X_t(x))\,dt + \sup_{t\,\in\,\mathrm{supp}(\eta'')} |\vartheta^s_x(t)| \int_{\mathrm{supp}(\eta'')} |\eta''(t)|\,dt \\ &\leq \frac{1}{2}\int_{\mathrm{supp}(\eta)} |X|^2_{g_s}(\varphi^X_t(x))\,dt + \sup_{t\,\in\,\mathrm{supp}(\eta'')} |\vartheta_s(\varphi^X_t(x)| \int_{\mathrm{supp}(\eta'')} |\eta''(t)|\,dt.\end{aligned}$$

Choose $\eta$ such that $\mathrm{supp}(\eta) \subset [0,\,1]$ and let $x$ now be the point where $X\cdot\vartheta_s$ attains its maximum value. By Lemma 7.3(i), we know that $x$ is contained in $W$. Hence, the above gives us that

$$\frac{X}{2}\cdot\vartheta_s(x) \leq C\left(\sup_{s\,\in\,[0,\,1]}\left(\sup_{\cup_{t\in[0,\,1]}\varphi^X_t(W)} |X|^2_{g_s}\right) + \|\vartheta_s\|_{C^0}\right).$$

The result now follows from the uniform upper bound on $\|\vartheta_s\|_{C^0}$. □

7.7. **A priori estimates on higher derivatives.** We next derive a priori *local* bounds on higher derivatives of solutions to the complex Monge-Ampère equation $(\star\star_s)$, beginning with the $C^2$-estimate.

7.7.1. *$C^2$ a priori estimate.*

**Proposition 7.17** (A priori $C^2$-estimate)**.** *Let $(\vartheta_s)_{0\,\leq\,s\,\leq\,1}$ be a path of solutions in $\mathbb{R}\oplus C^{\infty}_{X,\,\beta}(M)$ to $(\star\star_s)$. Then there exists a positive constant $C = C\left(n, \omega, \sup_{s\in[0,1]} \|G_s\|_{C^2}\right)$ such that the following $C^2$ a priori estimate holds true:*

$$\sup_{0\,\leq\,s\,\leq\,1} \|i\partial\bar{\partial}\vartheta_s\|_{C^0} \leq C.$$

*In particular,*

$$\sup_{0\,\leq\,s\,\leq\,1} \|i\partial\bar{\partial}\psi_s\|_{C^0} \leq C.$$

*Proof.* Following closely [CD16, Proposition 6.6] where the approach taken is based on standard computations performed in Yau's seminal paper [Yau78, pp.347–351] (see also [Sie13, Lemma 5.4.16] and [Tia00b, pp.52–55]), let $\Delta_s$ be the Laplacian with respect to $\sigma_s$, one estimates the drift Laplacian $\Delta_s - \frac{X}{2}\cdot$ of $\operatorname{tr}_{\omega_s}\sigma_s$ as follows:

$$\begin{aligned}\left(\Delta_s - \frac{X}{2}\cdot\right)\operatorname{tr}_{\omega_s}\sigma_s \geq \frac{(\vartheta_s)_{i\bar{j}k}(\vartheta_s)_{\bar{i}j\bar{k}}}{(1+(\vartheta_s)_{i\bar{i}})(1+(\vartheta_s)_{k\bar{k}})} + \Delta_s G_s \\ - C\operatorname{tr}_{\omega_s}\sigma_s\cdot\operatorname{tr}_{\sigma_s}\omega_s\cdot(1+\inf_M \operatorname{Rm}(g_s)) - C(n,\omega).\end{aligned} \tag{7.22}$$

Let $u_s := e^{-\lambda\vartheta_s}(n+\Delta_s\vartheta_s)$, where $\lambda > 0$ will be specified later. Then one estimates the drift Laplacian $\Delta_s - \frac{X}{2}\cdot$ of $u_s$ with respect to $\sigma_s$ in the following way using the fact that $\vartheta_s$ satisfies $(\star\star_s)$:

$$\begin{aligned}\left(\Delta_s - \frac{X}{2}\cdot\right)u_s \geq e^{-\lambda\vartheta_s}\Delta_s G_s + e^{-\lambda\vartheta_s}g_s\left(\nabla^s\left(\frac{X}{2}\right), i\partial\bar{\partial}\vartheta_s\right) - C_s n^2 e^{-\lambda\vartheta_s} + \lambda\left(\frac{X}{2}\cdot\vartheta_s\right)u_s - \lambda n u_s \\ + (\lambda + C_s)e^{\frac{\lambda\vartheta_s - G_s - \frac{X}{2}\cdot\vartheta_s}{n-1}}u_s^{\frac{n}{n-1}},\end{aligned}$$

where $\nabla^s$ is the Levi-Civita connection of $g_s$ and $C_s := \inf_{i\neq k}\operatorname{Rm}^s_{i\bar{i}k\bar{k}}$, $\operatorname{Rm}^s$ here denoting the complex linear extension of the curvature operator of the metric $g_s$. As $C_s$ is uniformly bounded below in $s$ by a constant $A$ (which may assume is $\leq 1$), we may choose $\lambda > 0$ sufficiently large so that $\lambda + A = 1$. Moreover, as

$$\left|g_s\left(\nabla^s\left(\frac{X}{2}\right), i\partial\bar{\partial}\vartheta_s\right)\right| \leq C\|\nabla^s X\|_{C^0}(1+u),$$

for some generic constant $C > 0$, we deduce that $u$ satisfies the following differential inequality:

$$\left(\Delta_s - \frac{X}{2}\cdot\right)u_s \geq -C_1(1+u_s) + C_2 u_s^{\frac{n}{n-1}},$$

where $C_1$ and $C_2$ depend only on $n$, $A$, $\sup_{s\in[0,1]}\|\vartheta_s\|_{C^0}$, $\sup_{s\in[0,1]}\|X\cdot\vartheta_s\|_{C^0}$, $\sup_{s\in[0,1]}\|G_s\|_{C^2}$, and $\sup_{s\in[0,1]}\|\nabla^s X\|_{C^0}$. The combination of Propositions 7.5, 7.11, 7.15 and 7.16 shows that $C_1$ and $C_2$ depend only on $n$, $A$ and $\sup_{s\in[0,1]}\|G_s\|_{C^2}$.

As $u_s$ is non-negative and converges to $n$ at infinity as $\vartheta_s \in \mathbb{R}\oplus C^\infty_{X,\beta}(M)$, an application of the maximum principle to an exhausting sequence of domains of $M$ yields an upper bound for $n + \Delta_s\vartheta_s$ and consequently, the desired bound on $i\partial\bar{\partial}\vartheta_s$. □

A useful consequence of Proposition 7.17 is that the Kähler metrics induced by $\sigma_s$ and $\omega_s$ are uniformly equivalent.

**Corollary 7.18.** *Let $(\vartheta_s)_{0\leq s\leq 1}$ be a path of solutions in $\mathbb{R}\oplus C^\infty_{X,\beta}(M)$ to $(\star\star_s)$ and for $s\in[0,1]$, let $g_s$, $h_s$ denote the Kähler metrics induced by $\omega_s$, $\sigma_s$ respectively. Then the tensors $g_s^{-1}h_s$ and $h_s^{-1}g_s$ satisfy the following uniform estimate:*

$$\sup_{0\leq t\leq 1}\|g_s^{-1}h_s\|_{C^0} + \sup_{0\leq t\leq 1}\|h_s^{-1}g_s\|_{C^0} \leq C$$

*for some positive constant $C = C\left(n,\omega,\sup_{s\in[0,1]}\|G_s\|_{C^2}\right)$. In particular, the metrics $g$ and $(h_s)_{0\leq s\leq 1}$ are uniformly equivalent.*

*Proof.* The estimate follows as in [CD20, Corollary 7.15] using Proposition 7.17 and Corollary 7.5. The fact that $\omega$ and $\sigma_s$ differ by a (1, 1)-form whose norm is controlled uniformly in $s$ yields the last claim of the corollary. □

7.7.2. $C^3$ *a priori estimate.* We now present the $C^3$-estimate.

**Proposition 7.19** (A priori $C^3$-estimate)**.** *Let $(\vartheta_s)_{0\leq s\leq 1}$ be a path of solutions in $\mathbb{R}\oplus C^{\infty}_{X,\beta}(M)$ to $(\star\star_s)$ and let $g_s$ be the Kähler metric induced by $\omega_s$ with Levi-Civita connection $\nabla^{g_s}$. Then*

$$\sup_{0\leq s\leq 1}\|\nabla^{g_s}\partial\bar{\partial}\vartheta_s\|_{C^0}\leq C\left(n,\omega,\,\sup_{s\in[0,1]}\|G_s\|_{C^3}\right).$$

*In particular,*

$$\sup_{0\leq s\leq 1}\|\nabla^{g_s}\left(X\cdot\vartheta_s\right)\|_{C^0}\leq C\left(n,\omega,\,\sup_{s\in[0,1]}\|G_s\|_{C^3}\right). \tag{7.23}$$

*Proof.* We follow closely the proof given in [CD16, Proposition 6.9] which itself is based on [PSS07].

Set

$$S(h_s,\,g_s):=|\nabla^{g_s}h_s|^2_{h_s}.$$

Then from the definition of $S$, we see that

$$\begin{aligned}S(h_s,\,g_s)=&h_s^{i\bar{j}}h_s^{k\bar{l}}h_s^{p\bar{q}}\nabla_i^{g_s}(h_s)_{kp}\overline{\nabla_j^{g_s}(h_s)_{lq}}\\=&|\Psi|^2_{h_s},\end{aligned}$$

where

$$\begin{aligned}\Psi^k_{ij}(h_s,\,g_s)&:=\Gamma(h_s)^k_{ij}-\Gamma(g_s)^k_{ij}\\&=h_s^{k\bar{l}}\nabla_i^{g_s}(h_s)_{j\bar{l}}.\end{aligned}$$

Now, since $\vartheta_s$ solves $(\star\star_s)$, $(M,\,h_s,\,X)$ is an "approximate" steady gradient Kähler-Ricci soliton in the following precise sense: if $h_s(t):=(\varphi_t^X)^*h_s$ and $g_s(t):=(\varphi_t^X)^*g_s$, where $(\varphi_t^X)_{t\in\mathbb{R}}$ is the one-parameter family of diffeomorphisms generated by $\frac{X}{2}$, then $(h_s(t))_{t\in\mathbb{R}}$ is a solution of the following perturbed Kähler-Ricci flow with initial condition $h_s$:

$$\begin{aligned}&\partial_t h_s(t)=-\operatorname{Ric}(h_s(t))+(\varphi_t^X)^*\left(\mathcal{L}_{\frac{X}{2}}g_s+\operatorname{Ric}(g_s)+\nabla^{g_s}\bar{\nabla}^{g_s}G_s\right),\qquad t\in\mathbb{R},\\&h_s(0)=h_s.\end{aligned}$$

In particular, $\partial_t h_s=-\operatorname{Ric}(h_s)+(\varphi_t^X)^*\Lambda$, where $\Lambda:=\mathcal{L}_{\frac{X}{2}}g_s+\operatorname{Ric}(g_s)+\nabla^{g_s}\bar{\nabla}^{g_s}G_s$ has uniformly controlled $C^1$-norm as $g_s$ is isometric to $g$ and $G_s$ is equal to zero, all outside a compact set independent of $s$.

Define $S(t):=S(h_s(s),\,g_s(t))$ and correspondingly set $\Psi(t):=\Psi(h_s(t),\,g_s(t))$. We adapt [BEG13, Proposition 3.2.8] to our setting. By a brute force computation, we have that

$$\begin{aligned}\Delta_{\sigma_s}S=&\,2\operatorname{Re}\left(h_s^{i\bar{j}}h_s^{p\bar{q}}(h_s)_{k\bar{l}}\left(\Delta_{\sigma_s,1/2}\Psi^k_{ip}\right)\overline{\Psi^l_{jq}}\right)+|\nabla^{h_s}\Psi|^2_{h_s}+|\overline{\nabla}^{h_s}\Psi|^2_{h_s}\\&+\operatorname{Ric}(h_s)^{i\bar{j}}h_s^{p\bar{q}}(h_s)_{k\bar{l}}\Psi^k_{ip}\overline{\Psi^l_{jq}}+h_s^{i\bar{j}}\operatorname{Ric}(h_s)^{p\bar{q}}(h_s)_{k\bar{l}}\Psi^k_{ip}\overline{\Psi^l_{jq}}-h_s^{i\bar{j}}h_s^{p\bar{q}}\operatorname{Ric}(h_s)_{k\bar{l}}\Psi^k_{ip}\overline{\Psi^l_{jq}},\end{aligned}$$

where

$$\begin{aligned}\Delta_{\sigma_s,1/2}&:=h_s^{i\bar{j}}\nabla_i^{h_s}\nabla_{\bar{j}}^{h_s},\\T^{i\bar{j}}&:=h_s^{i\bar{k}}h_s^{l\bar{j}}T_{k\bar{l}},\end{aligned}$$

for $T_{k\bar{l}}\in\Lambda^{1,0}M\otimes\Lambda^{0,1}M$. We also have that

$$\begin{aligned}\partial_u\Psi(u)^k_{ip}|_{u=0}&=\partial_u|_{u=0}(\Gamma(h_s(u))-\Gamma(g_s(u)))^k_{ip}\\&=\nabla_i^{h_s}(-\operatorname{Ric}(h_s)^k_p+\Lambda^k_p)-\nabla_i^{g_s}(\mathcal{L}_{\frac{X}{2}}(g_s)^k_p),\\\partial_u h_s^{i\bar{j}}|_{u=0}&=\operatorname{Ric}(h_s)^{i\bar{j}}-\Lambda^{i\bar{j}}.\end{aligned}$$

Finally, using the second Bianchi identity, we compute that

$$\Delta_{\sigma_s,1/2}\Psi^k_{ip}=h_s^{a\bar{b}}\nabla_a^{h_s}\operatorname{Rm}(g_s)^k_{i\bar{b}p}-\nabla_i^{h_s}\operatorname{Ric}(h_s)^k_p,$$

which in turn implies that the following evolution equation is satisfied by $\Psi$:

$$\partial_u \Psi_{ip}^k(u)|_{u=0} = \Delta_{\sigma_s,1/2}\Psi_{ip}^k + T_{ip}^k$$

for a tensor $T$ of the form

$$\begin{aligned} T &= h_s^{-1} * \nabla^{h_s}\operatorname{Rm}(g_s) + \nabla^{h_s}\Lambda - \nabla^{g_s}(\mathcal{L}_{\frac{X}{2}} g_s) \\ &= h_s^{-1} * \nabla^{g_s}\operatorname{Rm}(g_s) + h_s^{-1} * h_s^{-1} * \operatorname{Rm}(g_s) * \Psi + h_s^{-1} * \Psi * \Lambda + \nabla^{g_s}(\Lambda - \mathcal{L}_{\frac{X}{2}} g_s). \end{aligned}$$

Notice the simplification here regarding the "bad" term $-\nabla^{h_s}\operatorname{Ric}(h_s)$. Since this flow is evolving only by diffeomorphism, we know that

$$\begin{aligned} S(t) &= (\varphi_t^X)^* S(h_s, g_s), \\ \partial_u S|_{u=0} &= \frac{X}{2}\cdot S(h_s, g_s). \end{aligned}$$

Hence Young's inequality, together with the boundedness of $\|h_s^{-1}g_s\|_{C^0}$ and $\|h_s g_s^{-1}\|_{C^0}$ ensured by Corollary 7.18 and the boundedness of the covariant derivatives of the tensors $\operatorname{Rm}(g_s)$ and $\Lambda$, imply that

$$\Delta_{\sigma_s} S - \frac{X}{2}\cdot S \geq -C(S+1),$$

for some positive uniform constant $C$.

We use as a barrier function the trace $\operatorname{tr}_{\omega_s}\sigma_s$ which, by (7.22) and the uniform equivalence of the metrics $g_s$ and $h_s$ provided by Corollary 7.18, satisfies

$$\Delta_{\sigma_s}\operatorname{tr}_{\omega_s}\sigma_s - \frac{X}{2}\cdot\operatorname{tr}_{\omega_s}\sigma_s \geq C^{-1}S - C,$$

where $C$ is a uniform positive constant that may vary from line to line. By applying the maximum principle to $\varepsilon S + \operatorname{tr}_{\omega_s}\sigma_s$ for some sufficiently small $\varepsilon > 0$, one arrives at the desired a priori estimate.

The proof of (7.23) is a consequence of the previously proved a priori bound on $\nabla^{g_s}\partial\bar{\partial}\vartheta_s$, once we differentiate $(\star\star_s)$. □

We next establish Hölder regularity of $g_s^{-1}h_s$ and $h_s^{-1}g_s$, an improvement on Corollary 7.18.

**Corollary 7.20.** *Let $(\vartheta_s)_{0\leq s\leq 1}$ be a path of solutions in $\mathbb{R}\oplus C^\infty_{X,\beta}(M)$ to $(\star\star_s)$ and for $s\in[0,1]$, let $h_s$ be the Kähler metric induced by $\sigma_s$. Then for any $\alpha\in\left(0,\frac{1}{2}\right)$, the tensors $g_s^{-1}h_s$ and $h_s^{-1}g_s$ satisfy the following uniform estimate:*

$$\sup_{0\leq s\leq 1}\left(\|g_s^{-1}h_s\|_{C^{0,2\alpha}_{\mathrm{loc}}} + \|h_s^{-1}g_s\|_{C^{0,2\alpha}_{\mathrm{loc}}}\right) \leq C\left(n,\alpha,\omega,\sup_{s\in[0,1]}\|G_s\|_{C^3}\right).$$

*Proof.* By standard local interpolation inequalities applied to Propositions 7.17 and 7.19, we see that

$$\|g_s^{-1}h_s\|_{C^{0,2\alpha}_{\mathrm{loc}}} \leq C\left(n,\alpha,\omega,\sup_{s\in[0,1]}\|G_s\|_{C^3}\right).$$

Combining the previous estimate with Corollary 7.18, it suffices to prove a uniform bound on the local $2\alpha$-Hölder norm of $h_s^{-1}g_s$. We conclude with the following observation: if $u$ is a positive function on $M$ in $C^{0,2\alpha}_{\mathrm{loc}}(M)$ uniformly bounded from below by a positive constant, then $[u^{-1}]_{2\alpha}\leq [u]_{2\alpha}(\inf_M u)^{-2}$. By invoking Corollary 7.18 once more, this last remark applied to $h_s^{-1}g_s$ implies that

$$\|h_s^{-1}g_s\|_{C^{0,2\alpha}_{\mathrm{loc}}} \leq C\left(n,\alpha,\omega,\sup_{s\in[0,1]}\|G_s\|_{C^3}\right),$$

as well. □

7.7.3. *Local bootstrapping.* We now improve the local regularity of our continuity path of solutions to $(\star\star_s)$. This estimate will be used in deriving the subsequent weighted a priori estimates.

**Proposition 7.21.** *Let $(\vartheta_s)_{0\leq s\leq 1}$ be a path of solutions in $\mathbb{R}\oplus C^{\infty}_{X,\beta}(M)$, $\alpha\in\left(0,\,\frac{1}{2}\right)$, to $(\star\star_s)$. Then for any $\alpha\in\left(0,\,\frac{1}{2}\right)$ and for any compact subset $K\subset M$,*

$$\sup_{0\leq s\leq 1}\|\vartheta_s\|_{C^{3,\,2\alpha}(K)}\leq C\left(n,\alpha,\omega,\,\sup_{s\in[0,1]}\|G_s\|_{C^3},K\right).$$

*Proof.* From the standard computations involved in the proof of the a priori $C^2$-estimate, one can derive that

$$\begin{aligned}\Delta_{\sigma_s}\left(\Delta_{\omega_s}\vartheta_s-\frac{X}{2}\cdot\vartheta_s\right)=&\Delta_{\sigma_s}G_s+h_s^{-1}*g_s^{-1}*\mathrm{Rm}(g_s)+\mathrm{Rm}(g_s)*\nabla^{h_s}\bar{\nabla}^{h_s}\vartheta_s*h_s^{-1}\\&+g_s^{-1}*g_s^{-1}*\mathrm{Rm}(g_s)+g_s^{-1}*h_s^{-1}*h_s^{-1}*\bar{\nabla}^{h_s}\nabla^{h_s}\bar{\nabla}^{h_s}\vartheta_s*\nabla^{h_s}\bar{\nabla}^{h_s}\nabla^{h_s}\vartheta_s\\&-(\Delta_{\sigma_s}-\Delta_{\omega_s})\left(\frac{X\cdot\vartheta_s}{2}\right),\end{aligned}\tag{7.24}$$

where $*$ denotes the ordinary contraction of two tensors. Now, since $X$ is real holomorphic and $\vartheta_s$ being $JX$-invariant,

$$i\partial\bar{\partial}(X\cdot\vartheta_s)=\mathcal{L}_X(i\partial\bar{\partial}\vartheta_s)=\nabla^{g_s}_X(i\partial\bar{\partial}\vartheta_s)+i\partial\bar{\partial}\vartheta_s*\nabla^{g_s}X.\tag{7.25}$$

Therefore, thanks to (7.25), one gets the following pointwise estimate:

$$\begin{aligned}|(\Delta_{\sigma_s}-\Delta_{\omega_s})(X\cdot\vartheta_s)|&=\left|h_s^{-1}*i\partial\bar{\partial}\vartheta_s*i\partial\bar{\partial}(X\cdot\vartheta_s)\right|_{g_s}\\&\leq|h_s^{-1}g_s|_{g_s}\cdot|i\partial\bar{\partial}\vartheta_s|_{g_s}\cdot\left(|i\partial\bar{\partial}\vartheta_s|_{g_s}|\nabla^{g_s}X|_{g_s}+|\nabla^{g_s}i\partial\bar{\partial}\vartheta_s|_{g_s}|X|_{g_s}\right).\end{aligned}\tag{7.26}$$

By Propositions 7.17 and 7.19 together with (7.26), the $C^0$-norm of the right-hand side of (7.24) is uniformly bounded on compact subsets and, thanks to Corollary 7.20, so too are the coefficients of $\Delta_{\sigma_s}$ in the $C^{0,\,2\alpha}_{\mathrm{loc}}$-sense. As a result, by applying the Morrey-Schauder $C^{1,\,2\alpha}$-estimates, we see that for any $x\in M$ and for $\delta<\mathrm{inj}_{g_s}(M)$,

$$\left\|\Delta_{\omega_s}\vartheta_s-\frac{X}{2}\cdot\vartheta_s\right\|_{C^{1,\,2\alpha}(B_{g_s}(x,\,\delta))}\leq C(x,\,\delta,\,\alpha).$$

Finally, applying standard interior Schauder estimates for elliptic equations once again with respect to $\Delta_{\omega_s,\,X}$ leads to the bound:

$$\begin{aligned}\|\vartheta_s\|_{C^{3,\,2\alpha}(B_{g_s}(x,\,\frac{\delta}{2}))}&\leq C(x,\,\delta,\,\alpha)\left(\left\|\Delta_{\omega_s}\vartheta_s-\frac{X}{2}\cdot\vartheta_s\right\|_{C^{1,\,2\alpha}(B_{g_s}(x,\,\delta))}+\|\vartheta_s\|_{C^{1,\,2\alpha}(B_{g_s}(x,\,\delta))}\right)\\&\leq C(x,\,\delta,\,\alpha).\end{aligned}$$

□

We next establish the following well-known local regularity result for solutions to $(\star\star_s)$.

**Proposition 7.22.** *Let $G_s\in C^{k,\,\alpha}_{\mathrm{loc}}(M)$ for some $k\geq 1$ and $\alpha\in(0,\,1)$ and let $\vartheta_s\in C^{3,\,\alpha}_{\mathrm{loc}}(M)$ be a solution to $(\star\star_s)$ with data $G_s$. Then $\vartheta_s\in C^{k+2,\alpha}_{\mathrm{loc}}(M)$. Moreover, for all $k\geq 1$, $\alpha\in(0,\,1)$, and compact subset $K\subset M$,*

$$\|\vartheta_s\|_{C^{k+2,\alpha}(K)}\leq C\left(n,\alpha,\omega,\,\sup_{s\in[0,1]}\|G_s\|_{C^{\max\{k,3\},\alpha}},K\right).$$

*Proof.* We prove this proposition by induction on $k\geq 1$. The case $k=1$ is true by Proposition 7.21, so let $G_s\in C^{k+1,\alpha}_{\mathrm{loc}}(M)$ and let $\vartheta_s\in C^{3,\,\alpha}_{\mathrm{loc}}(M)$ be a solution of $(\star\star_s)$. Then by induction, $\vartheta_s\in C^{k+2,\alpha}_{\mathrm{loc}}(M)$. Let $x\in M$ and choose local holomorphic coordinates defined on $B_{g_s}(x,\,\delta)$ for some $0<\delta<\mathrm{inj}_{g_s}(M)$. Then since $\vartheta_s$ satisfies

$$G_s=\log\left(\frac{\sigma_s^n}{\omega_s^n}\right)-\frac{X}{2}\cdot\vartheta_s,$$

we know that for $j = 1, ..., 2n$, the derivative $\partial_j \vartheta_s$ satisfies

$$\Delta_{\sigma_s}(\partial_j \vartheta_s) = \partial_j \left( G_s + \frac{X}{2} \cdot \vartheta_s \right) \in C^{k,\alpha}_{\mathrm{loc}}(M).$$

As the coefficients of $\Delta_{\sigma_s}$ are in $C^{k,\alpha}_{\mathrm{loc}}(M)$, an application of the standard interior Schauder estimates for elliptic equations now gives us the desired local regularity result, namely $\partial_j \vartheta_s \in C^{k+2,\alpha}_{\mathrm{loc}}(M)$ for all $j = 1, ..., 2n$, or equivalently, $\vartheta_s \in C^{k+3,\alpha}_{\mathrm{loc}}(M)$ together with the expected estimate. □

### 7.8. **Weighted a priori estimates.**

**Proposition 7.23.** *Let $(\vartheta_s)_{0 \leq s \leq 1}$ be a path of solutions in $\mathbb{R} \oplus C^{\infty}_{X,\beta}(M)$ to $(\star\star_s)$. Then there exist positive constants $C$, $R_0$ and $\varepsilon > 0$ such that for all $s \in [0, 1]$,*

$$|\nabla^g (X \cdot \vartheta_s)|_g \leq \frac{C}{f^{\varepsilon}}, \quad f \geq R_0. \tag{7.27}$$

*Proof.* Let $u := X \cdot \vartheta_s$, write $\Delta_{h_s, X} := \Delta_{h_s} - X \cdot$ where $\Delta_{h_s}$ denotes the Riemannian laplacian associated to the metric $h_s$, and recall from (7.5) the differential equation satisfied by $u$ outside a sufficiently large compact set $W$ of $M$:

$$\frac{1}{2}\Delta_{h_s, X} u = 2e^{-\frac{X \cdot \vartheta_s}{2}} \frac{(\omega_D + i\partial\bar{\partial}\vartheta_s)^n}{\omega^n}. \tag{7.28}$$

Applying the Bochner formula for the drift Laplacian to the function $u$, we obtain

$$\begin{aligned}
\frac{1}{2}\Delta_{h_s, X} |\nabla^{h_s} u|^2_{h_s} &= |\operatorname{Hess}_{h_s}(u)|^2_{h_s} + \operatorname{Ric}(h_s)(\nabla^{h_s} u, \nabla^{h_s} u) + \operatorname{Hess}_{h_s}(f_{\sigma_s})(\nabla^{h_s} u, \nabla^{h_s} u) \\
&\quad + \langle \nabla^{h_s} \Delta_{h_s, X} u, \nabla^{h_s} u \rangle_{h_s} \\
&= |\operatorname{Hess}_{h_s}(u)|^2_{h_s} + \operatorname{Ric}(g_s)(\nabla^{h_s} u, \nabla^{h_s} u) + \operatorname{Hess}_{g_s}(f_{\omega_s})(\nabla^{h_s} u, \nabla^{h_s} u) \\
&\quad - i\partial\bar{\partial} G_s(\nabla^{h_s} u, \nabla^{h_s} u) + 4 \left\langle \nabla^{h_s} \left( e^{-\frac{X \cdot \vartheta_s}{2}} \frac{(\omega_D + i\partial\bar{\partial}\vartheta_s)^n}{\omega^n} \right), \nabla^{h_s} u \right\rangle_{h_s},
\end{aligned}$$

where we have used (7.1) and (7.28) in the second equality. As $G_s$ is supported in $W$ and $g_s$ is isometric to $g$ on $M \setminus W$, on this latter set this equation reads as

$$\begin{aligned}
\frac{1}{2}\Delta_{h_s, X} |\nabla^{h_s} u|^2_{h_s} &= |\operatorname{Hess}_{h_s}(u)|^2_{h_s} + \operatorname{Ric}(g)(\nabla^{h_s} u, \nabla^{h_s} u) + \operatorname{Hess}_g(f)(\nabla^{h_s} u, \nabla^{h_s} u) \\
&\quad + 4 \left\langle \nabla^{h_s} \left( e^{-\frac{X \cdot \vartheta_s}{2}} \frac{(\omega_D + i\partial\bar{\partial}\vartheta_s)^n}{\omega^n} \right), \nabla^{h_s} u \right\rangle_{h_s},
\end{aligned}$$

which, using the properties of $g$, then becomes

$$\Delta_{h_s, X} |\nabla^{h_s} u|^2_{h_s} = 2|\operatorname{Hess}_{h_s}(u)|^2_{h_s} + 2|\nabla^{h_s} u|^2_g + 8 \left\langle \nabla^{h_s} \left( e^{-\frac{X \cdot \vartheta_s}{2}} \frac{(\omega_D + i\partial\bar{\partial}\vartheta_s)^n}{\omega^n} \right), \nabla^{h_s} u \right\rangle_{h_s}, \tag{7.29}$$

on $M \setminus W$. Henceforth working on $M \setminus W$, we analyse the last term of this equation.

**Claim 7.24.** On $M \setminus W$,

$$\left| \left\langle \nabla^{h_s} \left( e^{-\frac{X \cdot \vartheta_s}{2}} \frac{(\omega_D + i\partial\bar{\partial}\vartheta_s)^n}{\omega^n} \right), \nabla^{h_s} u \right\rangle_{h_s} \right| \leq \frac{C}{r} \left( |\operatorname{Hess}_{h_s}(u)|_{h_s} + |\nabla^{h_s} u|_{h_s} \right) |\nabla^{h_s} u|_{h_s}.$$

*Proof of Claim 7.24.* By the pointwise Cauchy-Schwarz inequality together with the a priori $C^2$ estimate from Proposition 7.17, it is sufficient to prove that on $M \setminus W$,

$$\left| \nabla^g \left( e^{-\frac{X \cdot \vartheta_s}{2}} \frac{(\omega_D + i\partial\bar{\partial}\vartheta_s)^n}{\omega^n} \right) \right|_g \leq \frac{C}{r} \left( |\operatorname{Hess}_{h_s}(u)|_{h_s} + |\nabla^{h_s} u|_{h_s} \right).$$

Now, thanks to (7.6), the a priori bounds on $X \cdot \vartheta_s$ (Propositions 7.5 and 7.16) and its gradient (Proposition 7.19), one gets schematically:

$$\begin{aligned}\left|\nabla^g\left(e^{-\frac{X\cdot\vartheta_s}{2}}\frac{(\omega_D+i\partial\bar{\partial}\vartheta_s)^n}{\omega^n}\right)\right|_g &\leq C\left(\frac{1}{r}|\nabla^g u|_g+\frac{1}{r^2}|\nabla^g u|_g^2+\frac{1}{r}|\operatorname{Hess}_g(u)|_g\right)\\ &\leq \frac{C}{r}\left(|\nabla^g u|_g+|\operatorname{Hess}_g(u)|_g\right),\end{aligned}$$

where we have used implicitly the a priori $C^3$ bound (Proposition 7.19).

In order to conclude, it is sufficient to observe that:

$$\begin{aligned}\left|\operatorname{Hess}_{h_s}(u)-\operatorname{Hess}_g(u)\right|_g &\leq C|\nabla^g\partial\bar{\partial}\vartheta_s|_g|\nabla^g u|_g\\ &\leq C|\nabla^g u|_g,\end{aligned}$$

where $C$ is a positive constant independent of $s\in[0,1]$ that may vary from line to line. Here we have used Proposition 7.19 again in the last line.

□

Combining (7.29) with Claim 7.24, and using Proposition 7.18 to deal with the term $|\nabla^{h_s}u|^2_g$ of (7.29), all in all we end up with the following differential inequality satisfied by $|\nabla^{h_s}u|^2_{h_s}$:

$$\Delta_{h_s,X}|\nabla^{h_s}u|^2_{h_s}\geq 2|\operatorname{Hess}_{h_s}(u)|^2_{h_s}+C^{-1}|\nabla^{h_s}u|^2_{h_s}-\frac{C}{r}\left(|\operatorname{Hess}_{h_s}(u)|_{h_s}+|\nabla^{h_s}u|_{h_s}\right)|\nabla^{h_s}u|_{h_s}.$$

Hence, upon applying Young's inequality, one can see that on the set $\{r>R\}$ for some $R>0$ with $W\subset\{r\leq R\}$ chosen sufficiently large ,

$$\Delta_{h_s,X}|\nabla^{h_s}u|^2_{h_s}\geq\frac{1}{2}C^{-1}|\nabla^{h_s}u|^2_{h_s}. \tag{7.30}$$

Now, Lemma 7.2 ensures that $f_{\sigma_s}^{-\beta}$ for $\beta>0$ satisfies outside a sufficiently large uniform compact set of $M$,

$$\begin{aligned}\Delta_{h_s,X}f_{\sigma_s}^{-\beta} &= -\beta f_{\sigma_s}^{-\beta-1}\left(\Delta_{h_s,X}f_{\sigma_s}-(\beta+1)|X|^2_{h_s}f_{\sigma_s}^{-1}\right)\\ &=\beta\left(2f_{\sigma_s}-X\cdot\vartheta_s+(\beta+1)|X|^2_{h_s}f_{\sigma_s}^{-1}\right)f_{\sigma_s}^{-\beta-1}\\ &\leq 2\beta\left(1+Cf_{\sigma_s}^{-1}\right)f_{\sigma_s}^{-\beta}\leq 3\beta f_{\sigma_s}^{-\beta},\end{aligned}$$

for some uniform positive constant $C$. Here we have used Proposition 7.5 in the last line to bound $-X\cdot\vartheta_s$ from above uniformly. We have also used (7.23) from Proposition 7.19 to bound $|X|^2_{h_s}$ from above since $2|X|^2_{h_s}=2X\cdot f_{\sigma_s}=2X\cdot f+X\cdot X\cdot\vartheta_s=r^2+O(r)$ where $O(\cdot)$ is uniform in $s\in[0,1]$. Recalling (7.30), one can then use $f_{\sigma_s}^{-\beta}$ for some $\beta>0$ to be specified as a barrier function. Indeed, if $A>0$, then outside a sufficiently large compact subset of $M$ we have that

$$\Delta_{h_s,X}\left(|\nabla^{h_s}u|^2_{h_s}-Af_{\sigma_s}^{-\beta}\right)\geq\frac{1}{2}C^{-1}\left(|\nabla^{h_s}u|^2_{h_s}-Af_{\sigma_s}^{-\beta}\right) \tag{7.31}$$

whenever $6\beta\leq C^{-1}$. The maximum principle applied to (7.31) now yields the desired estimate. □

**Corollary 7.25.** *Let $(\vartheta_s)_{0\leq s\leq 1}$ be a path of solutions in $\mathbb{R}\oplus C^\infty_{X,\beta}(M)$ to $(\star\star_s)$ and let $C$, $R_0$, and $\varepsilon>0$ be as in Proposition 7.23. Then for all $s\in[0,1]$, there exists $\vartheta_s^\infty\in\mathbb{R}$ such that*

$$|\vartheta_s-\vartheta_s^\infty|+|X\cdot\vartheta_s|+|\nabla^g\vartheta_s|_g\leq\frac{C}{f^{\frac{\varepsilon}{2}}},\quad f\geq R_0. \tag{7.32}$$

*Proof.* First observe that since $X=\nabla^g f$, for any vector field $Y$ on $M$ we have that

$$\begin{aligned}g(\nabla^g(X\cdot\vartheta_s),Y) &=\operatorname{Hess}_g(f)(\nabla^g\vartheta_s,Y)+\operatorname{Hess}_g(\vartheta_s)(X,Y)\\ &=\frac{1}{2}(\mathcal{L}_Xg)(\nabla^g\vartheta_s,Y)+\operatorname{Hess}_g(\vartheta_s)(X,Y).\end{aligned}$$

In particular, upon setting $Y := \nabla^g\vartheta_s$, using the $JX$-invariance of $\vartheta_s$ and the fact that $\frac{X}{2}\cdot|\nabla^g\vartheta_s|^2_g = \mathrm{Hess}_g(\vartheta_s)(X,\,\nabla^g\vartheta_s)$ and $\frac{1}{2}\mathcal{L}_Xg = g_C$ on $M\setminus W$, we see that on this set,

$$\begin{aligned}g(\nabla^g(X\cdot\vartheta_s),\nabla^g\vartheta_s) &= |\nabla^C\vartheta_s|^2_{g_C} + \frac{X}{2}\cdot|\nabla^g\vartheta_s|^2_g\\ &= r^{-2}\underbrace{|X\cdot\vartheta_s|^2}_{\leq C} + r^{-2}\underbrace{|JX\cdot\vartheta_s|^2}_{=0} + \frac{X}{2}\cdot|\nabla^g\vartheta_s|^2_g\\ &\leq \frac{C}{r^2} + \frac{X}{2}\cdot|\nabla^g\vartheta_s|^2_g,\end{aligned}$$

where we have also used the boundedness of $|X\cdot\vartheta_s|$ given by Propositions 7.5 and 7.16 in the last line. Thus, by Young's inequality together with Proposition 7.23, we find that

$$\begin{aligned}\frac{X}{2}\cdot|\nabla^g\vartheta_s|^2_g &\geq -|\nabla^g(X\cdot\vartheta_s)|_g|\nabla^g\vartheta_s|_g - \frac{C}{r^2}\\ &\geq -\frac{C}{r^{2\varepsilon}}|\nabla^g\vartheta_s|_g - \frac{C}{r^2}\\ &\geq -\frac{C}{r^{2\varepsilon}}|\nabla^g\vartheta_s|^2_g - \frac{C}{r^{\min\{2\varepsilon,\,2\}}},\end{aligned}$$

where $C$ is a positive constant that may vary from line to line. The previous differential inequality can be reformulated as follows:

$$\partial_r\left(e^{-Cr^{-2\varepsilon}}|\nabla^g\vartheta_s|^2_g\right) \geq -\frac{Ce^{-Cr^{-2\varepsilon}}}{r^{1+\min\{2\varepsilon,\,2\}}}.$$

Integrating from $r$ to $r=+\infty$, using the assumption that the covariant derivatives of $\vartheta_s$ decay to 0 at infinity, leads to the bound

$$0 \leq e^{-Cr^{-2\varepsilon}}|\nabla^g\vartheta_s|^2_g \leq C\int_r^{+\infty} s^{-1-\min\{2\varepsilon,\,2\}}e^{-Cs^{-2\varepsilon}}\,ds,$$

so that

$$0 \leq |\nabla^g\vartheta_s|^2_g \leq Ce^{Cr^{-2\varepsilon}}\int_r^{+\infty} s^{-1-\min\{2\varepsilon,\,2\}}\underbrace{e^{-Cs^{-2\varepsilon}}}_{\leq 1}\,ds \leq Cr^{-\min\{2\varepsilon,\,2\}}e^{Cr^{-2\varepsilon}}.$$

As $e^{Cr^{-2\varepsilon}}$ is bounded at infinity, we arrive at the estimate $|\nabla^g\vartheta_s|_g \leq Cr^{-\min\{\varepsilon,\,1\}}$.

Next note from the mean value theorem on $D$ that at height $r$,

$$\left|\vartheta_s(r,\,\cdot) - \fint_D \vartheta_s(r,\,\cdot)\,\omega_D^{n-1}\right| \leq \sup_{D\times\{r\}}|\nabla^g\vartheta_s|_g\,\mathrm{diam}_g D \leq \frac{C}{r^{\varepsilon}}, \tag{7.33}$$

and that

$$\left|X\cdot\vartheta_s(r,\,\cdot) - \fint_D X\cdot\vartheta_s(r,\,\cdot)\,\omega_D^{n-1}\right| \leq \frac{C}{r^{\varepsilon}}, \tag{7.34}$$

thanks to Proposition 7.23. These inequalities we will make use of later.

Linearising $(\star\star_s)$ around the background metric $g$ on $M\setminus W$ yields

$$\Delta_{g,\,X}\vartheta_s = \int_0^1\int_0^u |\partial\bar{\partial}\vartheta_s|^2_{h_{s,\tau}}\,d\tau du, \qquad h_{s,\tau} := (1-\tau)g + \tau h_s. \tag{7.35}$$

Integrating over $D\times\{r\}$ then results in

$$\Delta_{C,X}\overline{\vartheta_s}(r) = \int_D\int_0^1\int_0^u |\partial\bar{\partial}\vartheta_s|^2_{h_{s,\tau}}\,d\tau du\,\omega_D^{n-1},$$

where recall that

$$\overline{\vartheta_s}(r) := \fint_{D\times\{r\}} \vartheta_s(r,\,\cdot)\,\omega_D^{n-1}.$$

By Corollary 7.18, we therefore have that

$$0 \leq \Delta_{C,X}\overline{\vartheta_s}(r) \leq C\int_D |i\partial\bar{\partial}\vartheta_s|^2_g\,\omega_D^{n-1}, \tag{7.36}$$

for some uniform constant $C > 0$.

Now, since $\nabla^g X = \nabla^{g,2} f = g_C$, one gets the following pointwise estimate obtained by considering an orthonormal frame of the form $(r^{-1}X, r^{-1}JX, (e_i, Je_i)_{1\leq i\leq n-1})$ where $(e_i, Je_i)_{1\leq i\leq n-1}$ is an orthonormal frame with respect to $g_D$:

$$\begin{aligned} |i\partial\bar{\partial}\vartheta_s|^2_g &\leq C|\nabla^{g,2}\vartheta_s|^2_g \\ &\leq C\left(r^{-2}|\nabla^g(X\cdot\vartheta_s)|^2_g + r^{-2}|\nabla^g\vartheta_s|^2_g + |\nabla^{g_D,2}\vartheta_s|^2_{g_D}\right) \end{aligned} \tag{7.37}$$

for some uniform positive constant $C$. Integrating over $D$, using integration by parts together with Proposition 7.23, we derive that

$$\int_D |i\partial\bar{\partial}\vartheta_s|^2_g\,\omega_D^{n-1} \leq \frac{C}{r^{4\varepsilon+2}} + \int_D |\nabla^{g_D,2}\vartheta_s|^2_{g_D}\,\omega_D^{n-1}. \tag{7.38}$$

Now, by Bochner formula applied to $(D,\, g_D)$ and the function $\vartheta_s$:

$$\begin{aligned} \Delta_D|\nabla^{g_D}\vartheta_s|^2_{g_D} &= 2|\nabla^{g_D,2}\vartheta_s|^2_{g_D} + 2\operatorname{Ric}(g_D)(\nabla^{g_D}\vartheta_s, \nabla^{g_D}\vartheta_s) + 2g_D\left(\nabla^{g_D}\Delta_D\vartheta_s, \nabla^{g_D}\vartheta_s\right) \\ &\geq 2|\nabla^{g_D,2}\vartheta_s|^2_{g_D} + 2g_D\left(\nabla^{g_D}\Delta_D\vartheta_s, \nabla^{g_D}\vartheta_s\right), \end{aligned} \tag{7.39}$$

where we have used that $g_D$ has nonnegative Ricci curvature (Ricci curvature bounded from below would be enough to complete the argument thanks to the decay on the gradient of $\vartheta_s$ we have just proved above).

Integrating (7.39) on $D$ and noticing that $\Delta_D\vartheta_s = 2\operatorname{tr}_{\omega_D}(i\partial\bar{\partial}\vartheta_s)$ leads to:

$$\begin{aligned} \int_D |\nabla^{g_D,2}\vartheta_s|^2_{g_D}\,\omega_D^{n-1} &\leq \int_D |\nabla^{g_D}\Delta_D\vartheta_s|_{g_D}|\nabla^{g_D}\vartheta_s|_{g_D}\,\omega_D^{n-1} \\ &\leq C\sup_{D\times\{r\}}|\nabla^{g_D}(i\partial\bar{\partial}\vartheta_s)|_{g_D}|\nabla^{g_D}\vartheta_s|_{g_D} \\ &\leq \frac{C}{r^{\varepsilon}}, \end{aligned} \tag{7.40}$$

where $C$ denotes a uniform positive constant that may vary from line to line. Here we have used Proposition 7.19 and the decay on the gradient of $\vartheta_s$ previously proved in the last line. The combination of (7.36), (7.38) and (7.40) lead to:

$$0 \leq \Delta_{C,X}\overline{\vartheta_s}(r) \leq \frac{C}{r^{4\varepsilon+2}} + \frac{C}{r^{\varepsilon}}. \tag{7.41}$$

We then have that

$$0 \leq \frac{\partial}{\partial r}\left(e^{-\frac{r^2}{2}}X\cdot\overline{\vartheta_s}\right) \leq Cr^{1-\varepsilon}e^{-\frac{r^2}{2}}.$$

After integration of this differential inequality from $r$ to $r = +\infty$, one gets

$$-C\int_r^{+\infty} s^{1-\varepsilon}e^{-\frac{s^2}{2}}\,ds \leq e^{-\frac{r^2}{2}}X\cdot\overline{\vartheta_s}(r) \leq 0.$$

Now, $\int_r^{+\infty} s^{1-\varepsilon}e^{-\frac{s^2}{2}}\,ds \leq Cr^{-\varepsilon}e^{-\frac{r^2}{2}}$ for $r$ large enough which can be proved using integration by parts. In particular,

$$-Cr^{-\varepsilon} \leq X\cdot\overline{\vartheta_s}(r) \leq 0.$$

Integrating once more yields the existence of a constant $\vartheta_s^\infty \in \mathbb{R}$ such that $\vartheta_s^\infty \leq \overline{\vartheta_s}(r) \leq \vartheta_s^\infty + Cr^{-\varepsilon}$. The triangle inequality applied to the oscillation estimates (7.33) and (7.34) then imply the desired estimates for $\vartheta_s$ and $X\cdot\vartheta_s$, respectively.

□

As a first intermediate result, we obtain a first rough decay of the difference of the solution and the background metric. More precisely:

**Corollary 7.26.** *Let $(\vartheta_s)_{0\leq s\leq 1}$ be a path of solutions in $\mathbb{R}\oplus C^{\infty}_{X,\beta}(M)$ to $(\star\star_s)$. Then there exists $C$ and $\varepsilon>0$ such that for all $s\in[0,\,1]$ and $\alpha\in\left(0,\frac{1}{2}\right)$,*

$$\|f^{\frac{\varepsilon}{2}}\cdot i\partial\bar{\partial}\vartheta_s\|_{C^{0,2\alpha}_{\mathrm{loc}}}\leq C.$$

*Proof.* It is sufficient to prove this estimate outside a compact set $W$ such that $\omega_s=\omega$ on $M\setminus W$. Let $x\in M\setminus W$ and choose normal holomorphic coordinates in a ball $B_g(x,\iota)$ for some $\iota>0$ uniform in $x\in M$. Let $g^{i\bar{\jmath}}_{\tau\vartheta_s}$ denote the components of the inverse of the Kähler metric associated to $\omega+i\partial\bar{\partial}(\tau\vartheta_s)$ in these coordinates and set

$$a^{i\bar{\jmath}}:=\int_0^1 g^{i\bar{\jmath}}_{\tau\vartheta_s}\,d\tau.$$

Then we have that

$$\begin{aligned}0&=\log\left(\frac{\sigma_s^n}{\omega^n}\right)-\frac{X}{2}\cdot\vartheta_s\\&=\int_0^1\frac{d}{d\tau}\log\left(\frac{\omega^n_{\tau\vartheta_s}}{\omega^n}\right)d\tau-\frac{X}{2}\cdot\vartheta_s\\&=\left(\int_0^1 g^{i\bar{\jmath}}_{\tau\vartheta_s}\,d\tau\right)\partial_i\partial_{\bar{\jmath}}\vartheta_s-\frac{X}{2}\cdot\vartheta_s\\&=a^{i\bar{\jmath}}\partial_i\partial_{\bar{\jmath}}\vartheta_s-\frac{X}{2}\cdot\vartheta_s.\end{aligned}$$

Now, by Corollary 7.20, $\|a^{i\bar{\jmath}}\|_{C^{0,2\alpha}_{\mathrm{loc}}}$ is uniformly bounded from above and $a^{i\bar{\jmath}}\geq\Lambda^{-1}\delta^{i\bar{\jmath}}$ on $B_g(x,\iota)$ for some uniform constant $\Lambda>0$. Therefore, by considering $\frac{X}{2}\cdot\vartheta_s$ as a source term, the Schauder estimates imply that

$$\begin{aligned}\|\vartheta_s-\vartheta_s^{\infty}\|_{C^{2,2\alpha}(B_g(x,\iota/2))}&\leq C\Bigg(\|X\cdot\vartheta_s\|_{C^{0,2\alpha}(B_g(x,\iota))}+\|\vartheta_s-\vartheta_s^{\infty}\|_{C^0(B_g(x,\iota))}\Bigg)\\&\leq Cf(x)^{-\frac{\varepsilon}{2}},\end{aligned}$$

for some uniform positive constant $C=C\,(n,\alpha,\omega)$. Here we have used Proposition 7.23 and Corollary 7.25 in the last line. The desired rough a priori decay estimate on $i\partial\bar{\partial}\vartheta_s$ and its Hölder semi-norm now follows. □

**Theorem 7.27.** *Let $(\vartheta_s)_{0\leq s\leq 1}$ be a path of solutions in $\mathbb{R}\oplus C^{\infty}_{X,\beta}(M)$ to $(\star\star_s)$. Then there exist $R_0>0$ and $C>0$ such that for $s\in[0,\,1]$:*

$$|\vartheta_s-\vartheta_s^{\infty}|\leq\frac{C}{f^{\frac{\beta}{2}}},\quad f\geq R_0,$$

*where $\vartheta_s^{\infty}\in\mathbb{R}$ is as in Corollary 7.25.*

*Moreover, there exists $C>0$ such that $\|\vartheta_s\|_{\mathcal{D}^{2,2\alpha}_{X,\beta}}\leq C$.*

*Proof.* Linearising $(\star\star_s)$ around $g$ outside a compact set to obtain (7.35) and using the uniform equivalence of the metrics $h_s$ and $g$ given by Corollary 7.18 together with the bounds of Corollary 7.26, we obtain the improved estimate

$$0\leq\Delta_{g,\,X}\vartheta_s\leq Cr^{-2\varepsilon}.$$

Similar to the proof of Claims 6.8 and 6.9, one estimates separately $X\cdot\overline{\vartheta_s}$ from $\vartheta_s-\overline{\vartheta_s}$. The former estimate can be reduced to an ODE which gives $X\cdot\overline{\vartheta_s}=O(r^{-2\varepsilon})$ uniformly in $s\in[0,\,1]$ and by integrating from $r$ to $r=+\infty$, $\overline{\vartheta_s}-\vartheta_s^{\infty}=O(r^{-2\varepsilon})$. The latter estimate is based on the use of the

Poincaré inequality on $D$ endowed with its metric $g_D$. By assumption, $\lambda^D > \beta > 0$ is the first non-zero eigenvalue of the spectrum of the laplacian on $D$, therefore one gets $\vartheta_s - \overline{\vartheta_s} = O(r^{-\min\{\beta,2\varepsilon\}})$. Combining these two estimates, one arrives at $\vartheta_s - \vartheta_s^\infty = O(r^{-\min\{\beta,2\varepsilon\}})$ which is a strict improvement of Corollary 7.25 as long as $\varepsilon < \beta/2$. Invoking local parabolic Schauder estimates established in [(6.11), Claim 6.7] with $k = 0$ applied to the linearisation of $(\star\star_s)$ around the background metric $g$ outside a compact set as in (7.35) set yields some positive constant $C$ such that for $R \geq R_0$,

$$\begin{aligned}\|\vartheta_s - \vartheta_s^\infty\|_{C^{2,2\alpha}_{X,\min\{\beta,2\varepsilon\}}} &\leq C\left(\|\vartheta_s - \vartheta_s^\infty\|_{C^0_{X,\min\{\beta,2\varepsilon\}}} + \|i\partial\overline{\partial}\vartheta_s\|_{C^{0,2\alpha}_{X,\min\{\beta,2\varepsilon\}}}\|i\partial\overline{\partial}\vartheta_s\|_{C^0(r\geq R)}\right) + C(R)\\ &\leq C\|\vartheta_s - \vartheta_s^\infty\|_{C^0_{X,\min\{\beta,2\varepsilon\}}} + C\|\vartheta_s - \vartheta_s^\infty\|_{C^{2,2\alpha}_{X,\min\{\beta,2\varepsilon\}}} R^{-\min\{\beta,2\varepsilon\}} + C(R),\end{aligned}$$

where we have invoked local uniform estimates ensured by Propositions 7.17 and 7.19. By absorption, i.e. by choosing $R$ large enough, one gets in particular that $\|\vartheta_s - \vartheta_s^\infty\|_{C^{2,2\alpha}_{X,\min\{\beta,2\varepsilon\}}} \leq C$ for some uniform positive constant $C$.

This implies that $|i\partial\overline{\partial}\vartheta_s|_g = O(r^{-\min\{\beta,2\varepsilon\}})$. By iterating the previous reasoning a finite number of times, the decay on $\vartheta_s$ is eventually multiplied by 2 at each step unless it has reached the threshold decay $r^{-\beta}$ as expected.

□

**Proposition 7.28** (Weighted $C^4$ a priori estimate). *Let $(\vartheta_s)_{0\leq s\leq 1}$ be a path of solutions in $\mathbb{R} \oplus C^\infty_{X,\beta}(M)$ to $(\star\star_s)$. Then if $\alpha \in (0, \frac{1}{2})$, there exists $C > 0$ such that for all $s \in [0, 1]$,*

$$\|\vartheta_s - \vartheta_s^\infty\|_{C^{4,2\alpha}_{X,\beta}} \leq C. \tag{7.42}$$

*Proof.* In order to prove the a priori bound on the $C^{4,2\alpha}_{X,2}$-norm of $\vartheta_s - \vartheta_s^\infty$, we first establish the following uniform decay on the third derivatives of $\vartheta_s - \vartheta_s^\infty$:

**Claim 7.29.** There exists $C > 0$ such that for all $s \in [0, 1]$,

$$\|\nabla^g\vartheta_s\|_{C^{2,2\alpha}_{X,\beta}} \leq C.$$

In particular,

$$|\nabla^g\partial\overline{\partial}\vartheta_s|_g \leq \frac{C}{r^\beta}.$$

*Proof of Claim 7.29.* Let us differentiate the linearisation of $(\star\star_s)$ around the background metric $g$ outside a compact set as given in (7.35) to get schematically on $\{r \geq R\}$ with $R$ sufficiently large:

$$\begin{aligned}&\Delta_{g,X}(\nabla^g\vartheta_s) = \nabla^g\vartheta_s + Q(\partial\overline{\partial}\vartheta_s, \nabla^g\partial\overline{\partial}\vartheta_s),\\ &\|Q(\partial\overline{\partial}\vartheta_s, \nabla^g\partial\overline{\partial}\vartheta_s)\|_{C^{0,2\alpha}_{X,\beta}} \leq C\|\nabla^g\partial\overline{\partial}\vartheta_s\|_{C^{0,2\alpha}_{X,\beta}}\|\partial\overline{\partial}\vartheta_s\|_{C^{0,2\alpha}(r>R)} \leq \frac{C}{R^\beta}\|\nabla^g\partial\overline{\partial}\vartheta_s\|_{C^{0,2\alpha}_{X,\beta}}.\end{aligned} \tag{7.43}$$

Here we have used Theorem 7.27 in the last inequality.

In particular, similar to the proof of Theorem 7.27, by absorbing the non-linear term on the righthand side of (7.43) by choosing $R$ large enough, one is led thanks to Proposition 7.22 together with Theorem 7.27 to:

$$\|\nabla^g\vartheta_s\|_{C^{2,2\alpha}_{X,\beta}} \leq C.$$

In particular, the desired decay on the $|\nabla^g\partial\overline{\partial}\vartheta_s|_g$ holds true.

□

By Proposition 7.22, it is sufficient to estimate the $C^{2,2\alpha}_{X,2}$-norm of the righthand side of the linearisation of $(\star\star_s)$ around the background metric $g$ as given in (7.35) once it is localized on $\{r > R\}$, $R$ sufficiently large, in order to establish (7.42). As in the proof of Claim 7.29, the linearisation of $(\star\star_s)$ around the background metric $g$ outside a compact set as given in (7.35) together with Claim 7.29 gives schematically on $\{r > R\}$:

$$\begin{aligned}\Delta_{g,X}\vartheta_s &= Q(\partial\overline{\partial}\vartheta_s),\\ \|Q(\partial\overline{\partial}\vartheta_s)\|_{C^{2,2\alpha}_{X,\beta}} &\leq C\left(\|\vartheta_s-\vartheta_s^{\infty}\|^2_{C^{2,2\alpha}_{X,\beta}}+\|\partial\overline{\partial}\vartheta_s\|_{C^{2,2\alpha}_{X,\beta}}\|\partial\overline{\partial}\vartheta_s\|_{C^{0,2\alpha}(r>R)}+\|\nabla^g\partial\overline{\partial}\vartheta_s\|_{C^{0,2\alpha}_{X,\beta}}\|\nabla^g\partial\overline{\partial}\vartheta_s\|_{C^0(r>R)}\right)\\ &\leq C\left(1+R^{-\beta}\|\vartheta_s-\vartheta_s^{\infty}\|_{C^{4,2\alpha}_{X,\beta}}+\|\vartheta_s-\vartheta_s^{\infty}\|_{C^{4,2\alpha}_{X,\beta}}\|\nabla^g\partial\overline{\partial}\vartheta_s\|_{C^0(r>R)}\right)\\ &\leq C\left(1+R^{-\beta}\|\vartheta_s-\vartheta_s^{\infty}\|_{C^{4,2\alpha}_{X,\beta}}\right),\end{aligned}$$

for some positive uniform constant that may vary from line to line. Here we have used Theorem 7.27 in the second and third inequalities together with Claim 7.29 in the last inequality. In particular, Theorem 6.3 applied to $\vartheta_s-\vartheta_s^{\infty}$ and $k=2$ and $\alpha\in\left(0,\frac{1}{2}\right)$ gives for some constant $C$ independent of $R$:

$$\|\vartheta_s-\vartheta_s^{\infty}\|_{C^{4,2\alpha}_{X,\beta}}\leq C(R)+CR^{-\beta}\|\vartheta_s-\vartheta_s^{\infty}\|_{C^{4,2\alpha}_{X,\beta}},$$

which gives the expected a priori estimate by absorption of the last term on the righthand side of the previous estimates by the lefthand side, i.e. the norm $C^{4,2\alpha}_{X,\beta}$ of $\vartheta_s-\vartheta_s^{\infty}$. ☐

The next proposition establishes a priori higher order weighted estimates. Since the proof is along the same lines as that of Proposition 7.28, we omit it.

**Proposition 7.30** (Higher order weighted estimates)**.** *Let $(\vartheta_s)_{0\leq s\leq 1}$ be a path of solutions in $\mathbb{R}\oplus C^{2k+2,\,2\alpha}_{X,\,\beta}(M)$ to $(\star\star_s)$ for $k\geq 1$. If $\alpha\in(0,\frac{1}{2})$ and if there exists $C_{k,\alpha}>0$ such that for all $s\in[0,\,1]$, $\|\vartheta_s\|_{\mathcal{D}^{2k+2,2\alpha}_{X,\,\beta}}\leq C_{k,\alpha}$, then there exists $C_{k+1,\alpha}>0$ such that for all $s\in[0,\,1]$,*

$$\|\vartheta_s\|_{\mathcal{D}^{2(k+1)+2,2\alpha}_{X,\,\beta}}\leq C_{k+1,\alpha}. \tag{7.44}$$

7.9. **Completion of the proof of Theorem A(v).** We finally prove Theorem A(v). Set

$$S:=\left\{s\in[0,\,1]\,|\,\text{there exists }\psi_s\in\mathcal{M}^{\infty}_{X,\,\beta}(M)\text{ satisfying }(\star_s)\right\}.$$

Note that $S\neq\emptyset$ since $0\in S$ (take $\psi_0=0$).

We first claim that $S$ is open. Indeed, this follows from Theorem 6.12; if $s_0\in S$, then by Theorem 6.12, there exists $\varepsilon_0>0$ such that for all $s\in(s_0-\varepsilon_0,\,s_0+\varepsilon_0)$, there exists a solution $\psi_s\in\mathcal{M}^{4,\,2\alpha}_{X,\,\beta}(M)$ to $(\star_s)$ with data $F_s\in\left(\mathcal{C}^{2,2\alpha}_{X,\,\beta}(M)\right)_{\omega,0}$. Since the data $F_s$ lies in $\mathcal{C}^{\infty}_{X,\,\beta}(M)$, Theorem 6.12 ensures that for each $s$ in this interval, $\psi_s\in\mathcal{M}^{\infty}_{X,\,\beta}(M)$. It follows that $(s_0-\varepsilon_0,\,s_0+\varepsilon_0)\cap[0,\,1]\subseteq S$.

We next claim that $S$ is closed. To see this, take a sequence $(s_k)_{k\geq 0}$ in $S$ converging to some $s_\infty\in S$. Then for $F_k:=F_{s_k}$, $k\geq 0$, the corresponding solutions $\psi_{s_k}=:\psi_k$, $k\geq 0$, of $(\star_s)$ satisfy

$$(\omega+i\partial\bar{\partial}\psi_k)^n=e^{F_k+\frac{X}{2}\cdot\psi_k}\omega^n,\qquad k\geq 0. \tag{7.45}$$

It is straightforward to check that the sequence $(F_k)_{k\geq 0}$ is uniformly bounded in $\mathcal{C}^{2,\,2\alpha}_{X,\,\beta}(M)$. As a consequence, the sequence $(\psi_k)_{k\geq 0}$ is uniformly bounded in $\mathcal{M}^{4,\,2\alpha}_{X,\,\beta}(M)$ by Proposition 7.28. Indeed, recall the correspondence between solutions of $(\star_s)$ and $(\star\star_s)$: $\psi_k$ is a solution to $(\star_s)$ if and only if $\vartheta_{s_k}=\psi_{s_k}-\Phi_{s_k}$ is a solution to $(\star\star_s)$. The Arzelà-Ascoli theorem therefore allows us to pull out a subsequence of $(\psi_k)_{k\geq 0}$ that converges to some $\psi_\infty\in C^{4,\,2\alpha'}_{\mathrm{loc}}(M)$, $\alpha'\in(0,\alpha)$. As $(\psi_k)_{k\geq 0}$ is uniformly bounded in $\mathcal{M}^{4,\,2\alpha}_{X,\,\beta}(M)$, $\psi_\infty$ will also lie in $\mathcal{M}^{4,\,2\alpha}_{X,\,\beta}(M)$. We need to show that $(\omega+i\partial\bar{\partial}\psi_\infty)(x)>0$ at every point $x\in M$. For this, it suffices to show that $(\omega+i\partial\bar{\partial}\psi_\infty)^n(x)>0$ for every $x\in M$. This is seen to hold true by letting $k$ tend to $+\infty$ (up to a subsequence) in (7.45). The fact that $\psi_\infty\in\mathcal{M}^{\infty}_{X,\,\beta}(M)$ follows from Proposition 7.30.

Finally, as an open and closed non-empty subset of $[0,\,1]$, connectedness of $[0,\,1]$ implies that $S=[0,\,1]$. This completes the proof of the Theorem A(v).

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