On minimal energy solutions to certain classes of integral equations related to soliton gases for integrable systems

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Abstract

We prove existence, uniqueness and non-negativity of solutions of certain integral equations describing the density of states $u(z)$ in the spectral theory of soliton gases for the one dimensional integrable focusing Nonlinear Schrödinger Equation (fNLS) and for the Korteweg de Vries (KdV) equation. Our proofs are based on ideas and methods of potential theory. In particular, we show that the minimizing (positive) measure for a certain energy functional is absolutely continuous and its density $u(z) \geq 0$ solves the required integral equation. In a similar fashion we show that $v(z)$, the temporal analog of $u(z)$, is the difference of densities of two absolutely continuous measures. Together, the integral equations for u, v represent nonlinear dispersion relation for the fNLS soliton gas. We also discuss smoothness and other properties of the obtained solutions. Finally, we obtain exact solutions of the above integral equations in the case of a KdV condensate and a bound state fNLS condensate. Our results is a step towards a mathematical foundation for the spectral theory of soliton and breather gases, which appeared in work of El and Tovbis, Phys. Rev. E, 2020. It is expected that the presented ideas and methods will be useful for studying similar classes of integral equation describing, for example, breather gases for the fNLS, as well as soliton gases of various integrable systems.

1 Introduction and statement of results

1.1 Introduction

Let $\mathbb{C}^+$ denote the upper half-plane and $\Gamma^+ \subset \mathbb{C}^+ \cup \mathbb{R}$ be a compact set and let $\sigma : \Gamma^+ \to [0, \infty)$ be a continuous, non-negative function on Γ^+. The motivation of this paper are two independent integral equations

$$\frac{1}{\pi} \int_{\Gamma^+} \log \left| \frac{w - \bar{z}}{w - z} \right| u(w) d\lambda(w) + \sigma(z) u(z) = \operatorname{Im} z, \tag{1.1}$$

$$\frac{1}{\pi} \int_{\Gamma^+} \log \left| \frac{w - \bar{z}}{w - z} \right| v(w) d\lambda(w) + \sigma(z) v(z) = -4 \operatorname{Im} z \operatorname{Re} z, \tag{1.2}$$

for unknown functions u and v respectively, where $z \in \Gamma^+$ and λ is some reference measure on Γ^+. For example, λ could be the area measure in a $2D$ context, or the arclength measure in the case of a contour Γ^+. The exact meaning of λ will be discussed in Assumption 1.1 below.

Before any further discussion we want to mention that a solution to any of the equations (1.1)-(1.2), if it exists, is unique. That follows from the well known properties of the Green potential, see Lemma 4.2 in Subsection 4.3 and the discussion following it.

Our goal is to prove the existence of solutions (1.1)-(1.2) and, what is especially important, the fact that the solution u of (1.1) satisfies $u(z) \geq 0$ everywhere on Γ^+. This property of u is natural from the interpretation of u as a "density of states" in the soliton gas theory, that is, the average number of waves with given spectral characteristics per unit of length and per unit of "measure" on Γ^+. Thus, the present paper is a significant step towards the mathematical foundation of the spectral theory of soliton gases for the focusing Nonlinear Schrödinger equation (fNLS) that was recently presented in [11], as well as for the Korteweg de Vries equation (KdV) that was first presented in [8].

A brief description of how equations (1.1)-(1.2), which we will call nonlinear dispersion relation (NDR), appear in the spectral theory for the fNLS soliton gas will be given in Section 2. We will also consider the case of a more general right hand side in (1.1) that we will denote by $\varphi(z)$. Finally, we are interested in the support of u, v and the smoothness of u, v under various assumptions on the smoothness of σ and the geometry of Γ^+.

If $\sigma > 0$ on Γ^+ then equations (1.1)-(1.2) are Fredholm integral equations of the second kind. In the case $\sigma \equiv 0$ on Γ^+ equations (1.1)-(1.2) are Fredholm integral equations of the first kind and the general case $\sigma \geq 0$ on Γ^+ is sometimes called Fredholm integral equations of the third kind [20]. Whereas there exists well known theory for second kind Fredholm equations that we can use to prove the existence and uniqueness of $u(z)$ when $\sigma > 0$ on Γ^+, the difficulty still lies in proving that the obtained $u(z) \geq 0$ on Γ^+. However, when it comes to the general case $\sigma \geq 0$, much less is known even about the existence of $u(z)$.

We study the NDR equations with potential theory for the upper half-plane, as the function

$$\frac{1}{\pi} \int_{\Gamma^+} \log \left| \frac{w - \bar{z}}{w - z} \right| u(w) d\lambda(w), \tag{1.3}$$

defines the Green potential for the upper half-plane $\mathbb{C}^+$ of the measure

$$d\mu = u(z) d\lambda(z). \tag{1.4}$$

Then both equations (1.1)-(1.2) can be written as

$$G\mu + \sigma u = \varphi \qquad \text{on } \Gamma^+, \tag{1.5}$$

where $\varphi(z)$ coincides with either $\operatorname{Im} z$ or with $-4 \operatorname{Im} z \operatorname{Re} z = -2 \operatorname{Im}(z^2)$ respectively, and we also write

$$G\mu(z) = \frac{1}{\pi} \int \log \left| \frac{z - \overline{w}}{z - w} \right| d\mu(w) \tag{1.6}$$

for the Green potential of an in general signed Borel measure μ in $\mathbb{C}^+$. When dealing with (1.1), i.e. in the most of the paper, we will assume that μ is a non-negative measure. Throughout the paper a measure will mean a non-negative Borel measure. We will always indicate it clearly if we allow signed measures. Sometimes we will write non-negative measure in order to emphasize that we want $\mu \geq 0$. For a measure, $G\mu$ is superharmonic on $\mathbb{C}^+$, harmonic on $\mathbb{C}^+ \setminus \operatorname{supp}(\mu)$ and $G\mu = 0$ on the real line and at infinity (provided that μ has compact support in $\mathbb{C}^+$).

Consider first the case of $\sigma \equiv 0$ on Γ^+, which corresponds to the soliton condensate, [11]. Then (1.5) becomes

$$G\mu = \varphi. \tag{1.7}$$

Our first observation is that (1.7) is the Euler-Lagrange equation for the Green energy functional

$$J_0(\mu) = \int G\mu d\mu - 2\int \varphi d\mu, \tag{1.8}$$

which we want to minimize among all the (non-negative) Borel measures μ with $\operatorname{supp}\mu \subset \Gamma^+$. It is well known [15] [22], that (if Γ^+ is a compact subset of $\mathbb{C}^+$ of positive capacity, and φ is continuous) the minimizing measure (the minimizer) μ^* exists and is unique. Moreover, μ^* satisfies equation (1.7) quasi everywhere (q.e.), i.e., up to a possible set of zero capacity, on $\operatorname{supp}\mu^*$ and the inequality

$$G\mu^* \geq \varphi \tag{1.9}$$

holds q.e. on $\Gamma^+ \setminus \operatorname{supp}\mu^*$. Thus, our goals are:

- to modify the energy functional J_0 into J_σ so that the corresponding Euler-Lagrange equation for the minimizer μ^* will be (1.5) instead of (1.7);
- to prove that under suitable conditions on φ the Euler-Lagrange equation (1.5) for μ^* holds not only on $\operatorname{supp}\mu^*$ but on the full Γ^+;
- to clarify the meaning of the density u^* of μ^* with respect to the reference measure λ.

1.2 Minimization of modified energy functional

We are able to satisfactorily answer these questions in the case $\Gamma^+ \subset \mathbb{C}^+$ by studing the minimization problem for J_σ among all the (non-negative) Borel measures μ with $\operatorname{supp}\mu \subset \Gamma^+$. Additional complications arise in the case $\Gamma^+ \cap \mathbb{R} \neq \emptyset$ (which is a very relevant situation) that we cannot overcome at this moment. Thus we restrict to Γ^+ being a compact subset of $\mathbb{C}^+$. The measure λ is assumed to satisfy the following mild condition.

Assumption 1.1. We assume that $\operatorname{supp}(\lambda) = \Gamma^+$ with $0 < \int_{\Gamma^+} d\lambda < +\infty$, and its Green potential $G\lambda$ is bounded and continuous on $\mathbb{C}^+$.

It follows from Assumption 1.1 that Γ^+ has positive logarithmic capacity [22]. As typical examples we may think of Γ^+ as a finite union of piece-wise smooth contours and closed 2D regions (closure of connected open set), where λ is arclength measure on a smooth contour and λ is the Lebesgue area measure on a 2D domain.

Now we introduce the energy functional that is a modification of (1.8).

Definition 1.2. For a continuous $\varphi : \Gamma^+ \to \mathbb{R}$ we define

$$J_\sigma(\mu) := \begin{cases} J_0(\mu) + \displaystyle\int \sigma u^2 d\lambda, & \text{if } \sigma\mu = \sigma u\lambda \text{ is absolutely} \\ & \text{continuous with respect to } \lambda, \\ +\infty, & \text{otherwise.} \end{cases} \tag{1.10}$$

The first main result of the paper is the following Theorem 1.3, which is proven in Section 3.

Theorem 1.3. *Let $\Gamma^+ \subset \mathbb{C}^+$ be a compact set with a measure λ that satisfies Assumption 1.1. Suppose the functions $\varphi : \Gamma^+ \to \mathbb{R}$ and $\sigma : \Gamma^+ \to [0, \infty)$ are continuous. Then the following hold.*

(a) *There is unique minimizing measure μ^* on Γ^+ for the energy functional J_σ that is defined in Definition 1.2. The measure $\sigma\mu^*$ is absolutely continuous with respect to λ, that is, $\sigma u^*\lambda = \sigma\mu^*$ for some density $\sigma u^* \in L^1(\lambda)$.*

(b) *If u^* is such that $\sigma u^*\lambda = \sigma\mu^*$, then we have*

$$G\mu^* + \sigma u^* = \varphi \qquad \mu^*\text{-a.e.} \quad \text{on } \Gamma^+. \tag{1.11}$$

(c) *If φ is defined everywhere on $\mathbb{C}^+$ and is positive, continuous, and superharmonic there, then also*

$$G\mu^* = \varphi \qquad \text{on } \Gamma^+ \setminus \operatorname{supp}(\mu^*), \tag{1.12}$$

while $G\mu^* \leq \varphi$ on $\mathbb{C}^+$.

The equation (1.11) is the Euler-Lagrange variational equality for the minimization problem. Since σu^* is only defined λ-a.e., we cannot expect to have (1.11) everywhere on $\operatorname{supp}(\mu^*)$ and in fact we have it only μ^*-a.e. The equation (1.11) is accompanied by a variational inequality outside of the support of μ^* (where we may and do assume that $u^* = 0$) which says that $G\mu^* \geq \varphi$ on $\Gamma^+ \setminus \operatorname{supp}(\mu^*)$ up to a possible set of zero capacity. In general we may expect strict inequality outside $\operatorname{supp}(\mu^*)$ and then u^* is definitely not a positive solution of (1.5) on the full Γ^+.

The conditions on φ in item (c) however imply equality outside of the support, and this includes the case $\varphi(z) = \operatorname{Im} z$. Under these conditions, the minimizer μ^* will be called a *weak solution* of the equation (1.5) as (1.5) is valid μ^*-a.e. on $\operatorname{supp}(\mu^*)$ and everywhere on $\Gamma \setminus \operatorname{supp}(\mu^*)$.

At points where $\sigma = 0$, the product σu^* in (1.11) is taken to be zero. See Example 1 for a situation where $\sigma(a) = 0$ and $u^*(a) = +\infty$ at a certain $a \in \Gamma^+$, and one may redefine the value of σu^* at a in order to have the identity (1.11) also at a.

An energy functional similar to J_σ, where u^2 in $\int \sigma u^2 d\lambda$ was replaced with a somewhat more complicated expression, was studied in [24] in the context of the KdV equation. In that case Γ^+ should be replaced by $[0,1]$. Existence of a minimizer (without the non-negative requirement) was proven there under the additional assumption $\sigma > 0$ on $[0,1]$.

The arguments in the proof of Theorem 1.3 lead to an important characterization of $\operatorname{supp}\mu^*$. Let Ω denote the unbounded component of $\mathbb{C}^+ \setminus \Gamma^+$. Then conditions (c), Theorem 1.3 together with some mild requirements on $\varphi(z)$ near infinity imply that $\partial\Omega \cap \Gamma^+ \subset \operatorname{supp}\mu^*$, see Proposition 3.5 for exact formulation. In particular, if Γ^+ is a collection of open arcs (each arc has endpoints) then $\operatorname{supp}\mu^* = \Gamma^+$.

Remark 1.4. In the assumptions of Theorem 1.3 let us fix some Γ^+ and φ but allow σ to vary. Then it is clear that

$$J_0(\mu_0^*) \leq J_0(\mu_\sigma^*) \leq J_\sigma(\mu_\sigma^*), \tag{1.13}$$

where μ_σ^* denotes the minimizer of J_σ for a given σ. Thus, *the condensate* $\sigma \equiv 0$ *corresponds to the minimal energy* for given Γ^+ and φ. The converse statement, in general, is not true, as it follows from Example 2 below. We also observe that the condensate maximizes the value of $\int_{\Gamma^+} \varphi d\mu_\sigma^*$ with given Γ^+ and φ since, by the definition (1.8), (1.10) of J_σ

$$\begin{aligned} J_\sigma(\mu_\sigma^*) &= \int_{\Gamma^+} G\mu_\sigma^* d\mu_\sigma^* - 2\int_{\Gamma^+} \varphi d\mu_\sigma^* + \int_{\Gamma^+} \sigma (u_\sigma^*)^2 d\lambda \\ &= \int_{\Gamma^+} \left(G\mu_\sigma^* + \sigma u_\sigma^* - 2\varphi\right) d\mu_\sigma^* \\ &= -\int_{\Gamma^+} \varphi d\mu_\sigma^*, \end{aligned}$$

where for the last line we used the identity (1.11) that is valid a.e. on the support of μ_σ^*. In particular, $\int_{\Gamma^+} \varphi d\mu_\sigma^*$ is related with the average intensity of the fNLS soliton gas when $\varphi(z) = \operatorname{Im} z$, which is maximized in the case of the condensate.

1.3 Equality in variational condition

In our second main result we give conditions that guarantee that the equation (1.11) is valid everywhere on $\operatorname{supp}(\mu^*)$ instead of being valid just μ^*-a.e. We have two such conditions. The first condition is that $\sigma > 0$. Then it turns out that μ^* has a continuous density as we show in part (a) of Theorem 1.6 and (1.11) is satisfied everywhere on Γ^+ where $\sigma > 0$.

The second condition deals with the case when $\sigma = 0$ on Γ^+ or on part of Γ^+. When $\sigma \equiv 0$ on Γ^+ then it is known from potential theory [21, 22] that the identity (1.11) may fail on a subset E of the support of μ^* of capacity zero. The set Γ^+ is thin at the points in E in the following sense, see [21, Definition 3.8.1].

Definition 1.5. Let S be a subset of $\mathbb{C}$ and let $z_0 \in \mathbb{C}$. Then S is thick (or non-thin) at z_0 if $z_0 \in \overline{S \setminus \{z_0\}}$ and if, for every superharmonic function u defined on a neighborhood of z_0,

$$\liminf_{\substack{z \to z_0 \\ z \in S \setminus \{z_0\}}} u(z) = u(z_0),$$

Otherwise, S is thin at z_0.

Thus, if $\sigma \equiv 0$ on Γ^+ and Γ^+ is thick at all of its points, then (1.10) holds on the full support of μ^*, and under the conditions of Theorem 1.2 (c), the identity $G\mu^* = \varphi$ holds on the full set Γ^+.

A connected set with more than one point (for example a contour) is thick at all of its points. On the other hand, a countable set is thin at every point.

The notion of thickness is related to the solvability of the Dirichlet problem for harmonic functions. If Ω is a bounded open set, and f is a continuous function on $\partial\Omega$, then the Dirichlet problem asks for a continuous function u on $\overline{\Omega}$ that is harmonic in Ω and agrees with f on the boundary. The Dirichlet problem is solvable for every continuous function on $\partial\Omega$ if and only if $\partial\Omega$ is thick at all of its points, see e.g. [1, Theorem 7.5.1] or [22, Appendix A.2, Theorem 2.1].

Theorem 1.6. *Under the general assumptions of Theorem 1.3, let μ^* be the minimizer of J_σ with density u^*.*

(a) *Let $S = \{z \in \Gamma^+ \mid \sigma(z) > 0\}$. Then $G\mu^*$ is continuous on S, the density u^* of μ^* is continuous on S (after modifying it on a set of $\sigma\lambda$-measure 0, if necessary), and*

$$G\mu^* + \sigma u^* = \varphi \qquad \text{on } S.$$

(b) *Let φ be positive, continuous, and superharmonic on $\mathbb{C}^+$. Let σ be continuous on Γ^+, and $S_0 = \{z \in \Gamma^+ \mid \sigma(z) = 0\}$. Suppose S_0 is thick at $z_0 \in S_0$ (see Definition 1.5). Then*

$$G\mu^*(z_0) = \varphi(z_0).$$

Combining parts (a) and (b) of Theorem 1.6 we get the following.

Corollary 1.7. *If the zero set S_0 of σ is thick at each of its points (and φ is positive, continuous, and superharmonic) then*

$$G\mu^* + \sigma u^* = \varphi$$

holds on all of Γ^+. This holds in particular if the zero set is empty, or if it is a connected set with more than one point, or a union of such sets.

In the case where the zero set of σ has an isolated point, one may encounter the situation where $\sigma(a) = 0$ and $u^*(a) = \infty$ and then σu^* is not well defined at this point. This happens in the following example.

Example 1. Suppose Γ^+ is a bounded smooth arc in $\mathbb{C}^+$ with arclength measure λ. Take a point $a \in \Gamma^+$ and consider

$$d\mu^*(z) = c|z-a|^{-1/2}d\lambda(z), \qquad z \in \Gamma^+$$

with $c > 0$. This is a finite measure with a Green potential that is bounded and continuous. For small enough $c > 0$ we have $G\mu^* < \varphi$ on Γ^+. Then

$$\sigma(z) := c^{-1}|z-a|^{1/2}\left(\varphi(z) - G\mu^*(z)\right), \qquad z \in \Gamma^+,$$

is non-negative and continuous on Γ^+.

The measure μ^* is the minimizer of J_σ for this σ, with the density

$$u^*(z) = c|z-a|^{-1/2}.$$

By construction, the equality $G\mu^* + \sigma u^* = \varphi$ holds on $\Gamma^+ \setminus \{a\}$. At $z = a$ we have $\sigma(a) = 0$ and $u^*(a) = +\infty$ and the product σu^* is not well-defined at a.

To have the equality at $z = a$ as well, we need to interpret the product $\sigma(a)u^*(a)$ in this situation as $\varphi(a) - G\mu^*(a) > 0$.

Neither of Theorem 1.3, part (c), Theorem 1.6, part (b) or Corollary 1.7 covers the case $\varphi(z) = -4\,\mathrm{Im}\,z\,\mathrm{Re}\,z = -2\,\mathrm{Im}(z^2)$ corresponding to (1.2) since this $\varphi(z)$, although harmonic, takes both positive and negative values in $\mathbb{C}^+$. Nevertheless, in Theorem 4.3, Section 4.3, we construct the solution to (1.2) by representing the right hand side of (1.2) as a difference of two continuous, positive and superharmonic in $\mathbb{C}^+$ functions, to which we can apply the statements mentioned at the beginning of this paragraph.

1.4 Outline

Here is a brief description of the rest of the paper. In Section 2 we give a concise presentation of the ideas leading to NDR (1.1)-(1.2) for the fNLS soliton gas. In fact, we will obtain there the more general NDR (2.16)-(2.17) for the fNLS breather gas, for which soliton gas is a particular case. We also describe special cases of soliton gases, such as fNLS bound state soliton gas and fNLS soliton condensate. It is worth mentioning here that the methods of potential theory, used in this paper, can be applied to the breather gas as well. The authors have obtained partial results in this direction that they hope to complete at a later time. We also observe there that fNLS bound state soliton gas can be seen as essentially equivalent to the KdV soliton gas, which is the first example of a soliton gas that was obtained in [8]. Thus, correspondingly modified Theorems 1.3, 1.6, 4.3, Corollary 1.7 and Propositions 3.5, 1.4, as well as some results of Section 5, are applicable to the KdV soliton gas.

Sections 3-4 are devoted to the proofs of Theorems 1.3 and 1.6, respectively. These theorems are also used to prove the existence of a solution to equation (1.2) in the Subsection 4.3. Properties of the minimizer μ^* of of J_σ, such as its support and smoothness under various additional assumptions on Γ^+ and σ are discussed in Section 5. For example, we show there that if $G\mu^* = \varphi$ on the boundary $\partial\Omega \subset \Gamma^+$ of some bounded region Ω and φ is harmonic on the closure $\bar{\Omega}$ of Ω then $\operatorname{supp}\mu^* \cap \Omega = \emptyset$, see Lemma 5.1. In particular, in the case of the soliton condensate ($\sigma \equiv 0$ on Γ^+), the support $\operatorname{supp}\mu^* \subset \partial\Gamma^+$ if Γ^+ is a 2D compact region. Assume now that Γ^+ is a finite collection of piecewise C^∞ smooth curves for soliton condensate (1.1) and $\varphi \in C^\infty(\Gamma^+)$. Then, according to Theorem 5.2, the solution u is C^∞ smooth on Γ^+ except for small neighborhoods of the points of non smoothness of Γ^+ (which include all the endpoints of Γ^+). In Lemmas 5.7 and 5.9 we describe smoothness of u in the general soliton gas with $\sigma \geq 0$. Finally, in Section 6 we prove that in the case of a bound state condensate the solution $u(z)$ to (1.1) is proportional to the density of the quasimomentum meromorphic differential for the hyperelliptic Riemann surface $\mathcal{R}$ defined by $\Gamma^+ \cup \overline{\Gamma^+}$ respectively, see Theorem 6.1. We then discuss extension of these results to the KdV soliton gas.

2 Background

As it is well known, solitons and breathers are localized solutions of integrable systems. At the same time, they can also be viewed as particles of complex statistical objects called soliton and breather gases. The nontrivial relation between the integrability and randomness in these gases falls within the framework of "integrable turbulence", introduced by V. Zakharov in [27]. The latter was motivated by the complexity of many nonlinear wave phenomena in physical systems that can be modeled by integrable equations. In view of the growing evidence of wide spread presence of the integrable gases in fluids and nonlinear optical media, see [11] and references therein, they present a fundamental interest for nonlinear science.

In this section we very briefly describe the relation between the spectral theory for fNLS soliton gas and equations (NDR) (1.1)-(1.2). More details can be found in [11]. We then show that the NDR for the Kortweg - de Vries (KdV) soliton gas is closely related with (1.1)-(1.2) when $\Gamma^+ \subset i\mathbb{R}^+$.

2.1 fNLS soliton gas

The fNLS has the form

$$i\psi_t + \psi_{xx} + 2|\psi|^2\psi = 0, \tag{2.1}$$

where $x, t \in \mathbb{R}$ are the space-time variables and $\psi : \mathbb{R}^2 \to \mathbb{C}$ is the unknown function. The simplest solution of equation (2.1) is a plane wave

$$\psi = qe^{2iq^2t}, \tag{2.2}$$

where $q > 0$ is the amplitude of the wave.

It is well known that the fNLS is an integrable equation [28]; the Cauchy (initial value) problem for (2.1) can be solved using the inverse scattering transform (IST) method for different classes of initial data, also known as potentials.

The scattering transform connects a given potential with scattering data expressed in terms of the spectral variable $z \in \mathbb{C}$. In particular, the scattering data consisting of one pair of spectral points $z = a \pm ib$, where $b > 0$, and a (norming) constant $c \in \mathbb{C}$, defines the famous soliton solution

$$\psi_S(x,t) = 2ib \operatorname{sech}[2b(x + 4at - x_0)] e^{-2i(ax+2(a^2-b^2)t)+i\phi_0}, \tag{2.3}$$

to the fNLS, where c defines its initial position x_0 and the initial phase ϕ_0. The soliton (2.3) represents a spatially localized traveling wave (pulse) on a zero background. It is characterized by two independent papameters: $b = \operatorname{Im} z$ determines the soliton amplitude $2b$ and $a = \operatorname{Re} z$ determines its velocity $s = -4a$. Scattering data that consists of several points $z_j \in \mathbb{C}^+$ (and their complex conjugates), $j \in \mathbb{N}$, together with their norming constants corresponds to the multi-soliton solutions. Assuming that originally (at $t = 0$) the centers of individual solitons are far from each other, we can represent the fNLS time evolution of a multi-soliton solution as propagation and interaction of the individual solitons. It is well known that the interaction of solitons in multi-soliton fNLS solutions reduces to only two-soliton elastic collisions, where the faster soliton (corresponding to z_m) gets a forward shift [28]

$$\Delta_{mj} = \frac{1}{\operatorname{Im}(z_m)} \log \left| \frac{z_m - \overline{z}_j}{z_m - z_j} \right|, \quad \operatorname{Re}(z_m) > \operatorname{Re}(z_j),$$

and the slower "z_j-soliton" is shifted backwards by $-\Delta_{mj}$.

Suppose now we have a "gas" of solitons (2.3) whose spectral characteristics z are distributed over a compact set $\Gamma^+ \subset \mathbb{C}^+$ according to some non negative measure μ. Assume also that the centers of these solitons are distributed uniformly on $\mathbb{R}$ and that $\mu(\Gamma^+)$ is small, i.e the gas is dilute. Let us consider the speed of the trial z - soliton in the gas. Since it undergoes rare but sustained collisions with other solitons, the speed $s_0(z) = -4 \operatorname{Re} z$ of a free solution must be modified as

$$s(z) = s_0(z) + \frac{1}{\operatorname{Im} z} \int_{\Gamma^+} \log \left| \frac{w - \bar{z}}{w - z} \right| [s_0(z) - s_0(w)] d\mu(w). \tag{2.4}$$

Similar modified speed formula was first obtain by V. Zakharov [26] in the context of the KdV equation. How can one find $s(z)$ without the assumption of the diluted gas, that is, when $\mu(\Gamma^+) = O(1)$? The answer is given by the integral equation

$$s(z) = s_0(z) + \frac{1}{\operatorname{Im} z} \int_{\Gamma^+} \log \left| \frac{w - \bar{z}}{w - z} \right| [s(z) - s(w)] d\mu(w) \tag{2.5}$$

for $s(z)$, known as the equation of state for the soliton gas, which was first obtained in [9] using purely physical reasoning. A similar equation in the KdV context was obtained earlier in [8]. In this equation $s(z)$ has the meaning of the speed of the "element of the gas" associated with the spectral parameter z (note that when $\mu(\Gamma^+) = O(1)$ we cannot distinguish individual solitons).

If we now assume some dependence of s and u on space time parameters x, t (here $d\mu = ud\lambda$ with λ being the Lebesgue measure) that occurs on very large spatiotemporal scales, then we complement the equation of state (2.5) by the continuity equation for the density of states

$$\partial_t u + \partial_x(su) = 0, \tag{2.6}$$

which was first suggested in [9] and derived in [11]. Equations (2.5), (2.6) form the kinetic equation for a dynamic (non-equilibrium) fNLS soliton gas. The kinetic equation for the KdV soliton gas was derived in [8]. It is remarkable that recently the kinetic equation having similar structure was derived in the framework of the generalized hydrodynamics for quantum many-body integrable systems, see, for example, [4, 5, 25].

It is interesting to observe that (2.5) is a direct consequence of (1.1)-(1.2), where $s(z) = \frac{v(z)}{u(z)}$. Indeed, multiplying (1.1) by $s(z)$, substituting $v(z) = s(z)u(z)$ into (1.2), subtracting the second equation from the first one and dividing both parts by $\operatorname{Im} z$ we obtain exactly (2.5). In this paper we consider the NDR (1.1)-(1.2) for equilibrium soliton gases, that is, we do not assume any dependence of u, v on the space-time variables x, t.

A mathematical albeit formal (i.e., without error estimates) derivation of the equation of state (2.5) was presented in the recent paper [11]. The first step in this process is derivation of equations (1.1)-(1.2), which describe the density of states u and its temporal analog v. The derivation is based on the idea of thermodynamic limit for a family of finite gap solutions of the fNLS, which was originally developed for the KdV equation in [8]. Finite-gap solutions are quasi-periodic functions in x, t that spectrally can be represented by a finite number of symmetrical with respect to $\mathbb{R}$ (Schwarz symmetrical) arcs (bands) on the complex z plane. Here Schwarz symmetry means that either a band γ coincides with its Schwarz symmetrical image $\bar\gamma$ or if γ is a band then $\bar\gamma$ is another band. Assume additionally that there is a complex constant (initial phase) associated with each band that also respects the Schwarz symmetry, i.e., Schwarz symmetrical bands have Schwarz symmetrical phases. Given a finite set of Schwarz symmetrical bands with the corresponding phases, a finite-gap solution to the fNLS can be written explicitly in terms of the Riemann Theta functions on the hyperelliptic Riemann surface $\mathfrak{R}$, where the bands are the branchcuts of $\mathfrak{R}$, see, for example, [2].

For convenience of the further exposition, we will consider $\mathfrak{R}$ of the genus $2N$, where the genus of $\mathfrak{R}$ is the number of bands minus one. The one exceptional band γ_0 will be crossing $\mathbb{R}$, whereas the remaining N bands $\gamma_j \subset \mathbb{C}^+$, $j = 1, \dots, N$, and their Schwarz symmetrical $\gamma_{-j} := \bar\gamma_j \subset \mathbb{C}^-$. It was shown in [11] that the wavenumbers k_j, $\tilde k_j$ and the frequencies ω_j, $\tilde\omega_j$ of a quasi-periodic finite gap solution ψ_{2N} determined by $\mathfrak{R}$ can be expressed as

$$k_j = -\oint_{\mathrm{A}_j} dp, \quad \omega_j = -\oint_{\mathrm{A}_j} dq, \quad j = 1, \dots, N, \tag{2.7}$$

$$\tilde k_j = \oint_{\mathrm{B}_j} dp, \quad \tilde\omega_j = \oint_{\mathrm{B}_j} dq, \quad j = 1, \dots, N, \tag{2.8}$$

where the cycles A_j, B_j are shown on Figure 1. Here $dp(z)$ and $dq(z)$, known as the quasimomentum and quasienergy differentials, are meromorphic differentials on $\mathfrak{R}$ with the only poles at $z=\infty$ on both sheets. These differentials are real normalized (all the periods of dp, dq are real) and are (uniquely) defined (see e.g. [12], [3]) by local expansions

$$dp \sim \pm 1 + \mathcal{O}(z^{-2}), \qquad dq \sim \pm 4z + \mathcal{O}(z^{-2}) \tag{2.9}$$

near $z=\infty$ on the main and second sheet respectively.

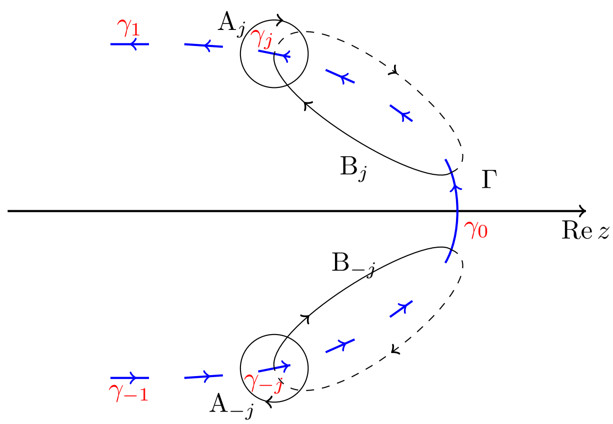

Figure 1: The spectral bands $\gamma_{\pm j}$ and the cycles $\mathrm{A}_{\pm j}, \mathrm{B}_{\pm j}$. The 1D Schwarz symmetrical curve Γ consists of the bands $\gamma_{\pm j}$, $j=0,\ldots,N$, and gaps between the bands (the gaps are not shown on this figure).

We shall call the special set of wavenumbers and frequencies defined by (2.7), (2.8) *the fundamental wavenumber-frequency set*. We note that the wavenumbers and frequencies defined by (2.7) and those defined by (2.8) are of essentially different nature: in the limit of γ_j shrinking to a point, we have

$$k_j, \omega_j \to 0, \quad \tilde{k}_j, \tilde{\omega}_j = \mathcal{O}(1), \qquad j=1,\ldots,N, \tag{2.10}$$

see [11]. Motivated by these properties, k_j, ω_j are called *solitonic wavenumbers and frequencies* whereas the remaining $\tilde{k}_j$, $\tilde{\omega}_j$ are called *carrier wavenumbers and frequencies.*

The standard normalized holomorphic differentials w_j of $\mathfrak{R}$ are defined by

$$w_j = [P_j(z)/R(z)]dz, \quad \oint_{\mathrm{A}_i} w_j = \delta_{ij}, \quad i,j = \pm 1, \ldots, \pm N, \tag{2.11}$$

where the polynomials

$$P_j(z) = \varkappa_{j,1} z^{2N-1} + \varkappa_{j,2} z^{2N-2} + \cdots + \varkappa_{j,2N} \tag{2.12}$$

have complex coefficients and the radical

$$R(z) = \prod_{k=1}^{2N+1} (z-\alpha_k)^{\frac{1}{2}} (z-\bar{\alpha}_k)^{\frac{1}{2}}. \tag{2.13}$$

defines the hyperelliptic surface $\mathfrak{R}$, i.e, the product runs over all the endpoints (of the bands) $\alpha_k \in \mathbb{C}^+$. It was shown in [11] that the solitonic wavenumbers and frequencies satisfy the systems

$$\begin{aligned} &\sum_{|m|=1}^{N} k_m \operatorname{Im} \oint_{\mathrm{B}_m} \frac{P_j(\zeta)d\zeta}{R(\zeta)} = 4\pi \operatorname{Re} \varkappa_{j,1}, \\ &\sum_{|m|=1}^{N} \omega_m \operatorname{Im} \oint_{\mathrm{B}_m} \frac{P_j(\zeta)d\zeta}{R(\zeta)} = 8\pi \operatorname{Re} \left(\varkappa_{j,1} \sum_{k=1}^{2N+1} \operatorname{Re} \alpha_k + \varkappa_{j,2} \right), \\ &\qquad\qquad |j| = 1, \ldots, N, \end{aligned} \tag{2.14}$$

where the latter summation is taken over all the endpoints in $\mathbb{C}^+$. We call (2.14) the solitonic nonlinear dispersion relations (NDR). Indeed, the NDR indirectly connect (through the Riemann surface $\mathfrak{R}$) the solitonic wavenumbers and frequencies of the finite gap solution ψ_{2N}, i.e., (2.14) represents nonlinear dispersion relations.

Equations (2.14) together with (2.10) are our starting point for deriving equations (1.1)-(1.2). Before describing the derivation, we want to point out that the matrix of the systems (2.14) is negative-definite and, therefore, each of the systems (1.1)-(1.2) has a unique solution. The negative-definiteness of the matrix of the systems (1.1)-(1.2) follows from the properties of the the Riemann period matrix τ of the Riemann surface $\mathfrak{R}$ ($\operatorname{Im} \tau$ is positive definite).

Suppose now that we start shrinking each band to a point. Then we will be taking the finite gap solution to its multi-soliton solution limit, where the phases should be transformed into the corresponding norming constants. The idea of thermodynamical limit consists of increasing the number $2N+1$ of bands simultaneously with shrinking the size $2\delta_j$ of each band γ_j (except, possibly, γ_0) at some exponential rate with respect to N, so that the centers z_j of the bands located in $\mathbb{C}^+$ will be filling a certain compact set $\Gamma^+ \subset \mathbb{C}^+$ with some limiting density $\phi(z)$. Moreover, we assume the distance between any bands to be much larger than the size of the bands. Under these assumptions one can show that the leading order behavior of the coefficients of the linear system (2.14) is given by

$$\oint_{\tilde{\mathbf{B}}_m} \frac{P_j(\zeta)d\zeta}{R(\zeta)} = \frac{1}{i\pi} \left[\log \frac{R_0(z_j)R_0(z_m) + z_j z_m + \delta_0^2}{R_0(z_j)R_0(\bar{z}_m) + z_j \bar{z}_m + \delta_0^2} - \log \frac{z_m - z_j}{z_m - \bar{z}_j} \right] \tag{2.15}$$

when $m \neq j$ and

$$\oint_{\tilde{\mathrm{B}}_j} \frac{P_j(\zeta)d\zeta}{R(\zeta)} = i\frac{2\log \delta_j}{\pi},$$

where $\tilde{\mathrm{B}}_m = \mathrm{B}_m + \mathrm{B}_{-m}$, $2\delta_0$ is the distance between the endpoints of the exceptional arc γ_0 and $R_0(z) := \sqrt{z^2+\delta_0^2} \sim z$ as $z \to \infty$ (here WLOG we assume $z_0 = 0$). The imaginary part of (2.15) provides the expression for the kernel of the integral equations for $u(z)$ in the case of the fNLS breather gas

$$\frac{1}{\pi}\int_{\Gamma^+}\left[\log\left|\frac{w-\bar{z}}{w-z}\right| + \log\left|\frac{R_0(z)R_0(w)+zw+\delta_0^2}{R_0(\bar{z})R_0(w)+\bar{z}w+\delta_0^2}\right|\right] u(w)d\lambda(w) + \sigma(z)u(z) \\ = \operatorname{Im} R_0(z), \quad (2.16)$$

whereas the second formula gives rise to the secular term $\sigma(z)u(z)$ in (2.16), where $\sigma(z)$ is defined by $\phi(z)$ and by the rate the bands shrink near z. The same holds for the integral equation for $v(z)$:

$$\frac{1}{\pi}\int_{\Gamma^+}\left[\log\left|\frac{w-\bar{z}}{w-z}\right| + \log\left|\frac{R_0(z)R_0(w)+zw+\delta_0^2}{R_0(\bar{z})R_0(w)+\bar{z}w+\delta_0^2}\right|\right] v(w)d\lambda(w) + \sigma(z)v(z) \\ = -2\operatorname{Im}[zR_0(z)]. \quad (2.17)$$

The breather gas is obtained when in the thermodynamic limit all the bands except γ_0 are shrinking to points while the exceptional band γ_0 approaches some limiting position as $N \to \infty$, where the endpoints of γ_0 approach $\pm i\delta_0$ respectively. Being considered alone, the limiting spectral band γ_0 corresponds to the plane wave solution (2.2) with $q = \delta_0$. The band γ_0 together with Schwarz symmetrical points of discrete spectrum $z, \bar{z}$ correspond to a soliton on the plane wave (carrier) background, also known as a breather. It is remarkable that the kernel in the integral equations (2.16)-(2.17), being divided by $\operatorname{Im} R_0(z)$, provides an elegant expression for the "position shift" of two interacting breathers; some considerably more involved expressions for this phase shift were recently obtained in [17, 16, 13]. Therefore, equations (2.16)-(2.17) represent nonlinear dispersive relations for the breather gas. It is easy to check that equations (2.16)-(2.17) coincide with (1.1)-(1.2) in the limit $\delta_0 \to 0$. Thus, soliton gas can be considered as a particular case of the breather gas, see [11] for details.

In the case of subexponential rate of shrinking of bands γ_j in the thermodynamic limit, the function $\sigma(z)$ turns to be zero and we obtain a breather (or soliton, if $\delta_0 \to 0$) condensate ([11]). As it was mentioned in Remark 1.4, the term "condensate" reflects the fact that for a given Γ^+ and $\varphi(z) = \operatorname{Im} z$ the energy $J_\sigma(\mu_\sigma^*)$ is minimized when $\sigma \equiv 0$ on Γ^+.

Consider a sequence of atomic, possibly signed measures μ_N with weights

$$u_j = \frac{\phi(z_j)k_j}{2\pi} \quad (2.18)$$

at each z_j, $j = 1, \ldots, N$. Assuming that the sequence $\{\mu_N\}_1^\infty$ weakly converges to some measure $d\mu = ud\lambda$ on $\mathbb{C}^+$, we obtain integral equation (1.1) as the thermodynamic limit of the first equation (2.14) (here γ_0 also shrinks to a point $z_0 = 0$). Equation (1.2) can be obtained from the second (2.14) equation similarly.

We want to emphasize that the existence and uniqueness of solution of (2.14) do not imply the existence and uniqueness of solutions (1.1)-(1.2) and, what is especially important, provides no information related to the requirement $u(z) \geq 0$ on Γ^+. The aim of the present paper is to address these questions.

2.2 Bound state fNLS and KdV gases

Soliton gas is called a bound state gas if Γ^+ is a subset of a vertical line $\operatorname{Re} z = c$, where c is a constant. The terminology comes from the fact that all the solitons in a multisoliton fNLS solution with the (discrete) spectrum on $\operatorname{Re} z = c$ have the same speed and therefore such a solution does not decompose into a collection of individual solitons in the process of evolution. In the context of soliton gases, one can note that solutions to the NDR (1.1)-(1.2) for bound state gases are proportional since these equations have proportional right hand sides. Thus, all components of a bound state gas have the same speed $-4c$.

According to [11], equations (1.1)-(1.2) with $\Gamma^+ = [0, iq]$ and $\sigma \equiv 0$ have solutions

$$u(z) = \frac{-iz}{\pi\sqrt{z^2+q^2}}, \quad v \equiv 0, \qquad \text{on } [0, iq], \tag{2.19}$$

which, according to Theorem 5.2 in Section 5 are C^∞ smooth (in fact, analytic) on any proper subarc of Γ^+. The only singularity $z_* = iq$ is at the upper endpoint of Γ^+, which is a point of non-smoothness of Γ^+, see Remark 5.5, Section 5. We note that the local behavior of u near z_* is in full agreement with Remark 5.5.

Remark 2.1. The reader may notice that in the example above $\Gamma^+ \cap \mathbb{R} = \{0\} \neq \emptyset$, so, as stated, Theorems 1.3 and 1.6 are not applicable to this Γ^+. However, our results, not included in this paper, show that these theorems are still applicable to the case when a 1D curve Γ^+ intersects $\mathbb{R}$ transversally.

Solution (2.19) of $Gu(z) = \operatorname{Im} z$ was obtained by first extending (1.1) symmetrically to $\mathbb{C}^-$ (see equation (5.1), Section 5), then differentiating both sides in $s = \operatorname{Im} z$ and, finally, inverting the obtained Finite Hilbert Transform (FHT) on $[-iq, iq]$. We will use this approach in Section 6 below to solve (1.1) for any bound state condensate.

It is interesting to observe that the NDR for the KdV soliton gas

$$\frac{1}{\pi}\int_{\Gamma^+} \log\left|\frac{\omega+\zeta}{\omega-\zeta}\right| u(\omega)d\omega + \sigma(\zeta)u(\zeta) = \frac{\zeta}{2}, \tag{2.20}$$

$$\frac{1}{\pi}\int_{\Gamma^+} \log\left|\frac{\omega+\zeta}{\omega-\zeta}\right| v(\omega)d\omega + \sigma(\zeta)v(\zeta) = -2\zeta^3, \tag{2.21}$$

first obtained in [8], are closely related with the fNLS bound state NDR with $\Gamma^+ \subset i\mathbb{R}^+$. Indeed, substituting $z = i\zeta$ and $w = i\omega$, we convert the left hand sides of (1.1)-(1.2) into the left hand sides of (2.20)-(2.21). In fact, solutions of the corresponding first NDRs coincide up to the factor 2. Therefore, all the results obtained in this paper applicable to equation (1.1) and most applicable to equation (1.2) with $\Gamma^+ \subset i\mathbb{R}^+$ are automatically applicable to the KdV soliton gas from [8], see Section 6 for details.

Remark 2.2. Realizations of an fNLS soliton gas can be related with the semiclassical limit of the fNLS equation with rapidly decaying real one hump potential that typically has $O(1/\epsilon)$ points of discrete spectrum (solitons) located on $i\mathbb{R}^+$, where $\epsilon > 0$ is a small semiclassical parameter. Such potentials include, for example, $\operatorname{sech} x$, the barrier (box) potential and many others. The fNLS time evolution of such potentials is known to typically lead to the appearance of coherent structures of increasing complexity that can be locally approximated by genus n finite-gap solutions with n increasing in time [10]. There are strong indications that for sufficiently large time t (and consequently large n), the semiclassical spectrum of these solutions fits into one of the thermodynamic scaling requirements described above. Taking into account the effective randomization of phases, the large t evolution of semiclassical solutions is expected to provide the dynamical realization of a bound state soliton gas studied in this paper. It is interesting that the first rigorous study of the large n limit of a special n-soliton solution to the KdV was recently conducted in [14]. It is based on the idea of the primitive potential from [6].

3 Proof of Theorem 1.3

3.1 Proof of part (a): the variational problem

We are going to show that the energy functional J_σ defined in (1.2) has a unique minimizer on the set of Borel measures on Γ^+. As a first step we show that J_σ is lower semicontinuous.

Lemma 3.1. *The functional J_σ is lower semicontinuous on the set of positive Borel measures on Γ^+ with the weak* topology.*

Proof. Let $(\mu_k)_k$ be a sequence of positive Borel measures on Γ^+ with μ as the weak* limit. We have to show that

$$J_\sigma(\mu) \leq \liminf_k J_\sigma(\mu_k). \tag{3.1}$$

To do so, we may assume (by passing to a subsequence if necessary) that $J_\sigma(\mu_k) < +\infty$ for every k, and that the limit

$$J^* := \lim_{k\to\infty} J_\sigma(\mu_k) \tag{3.2}$$

exists with $J^* < +\infty$. (If it would be infinite, there would be nothing to prove).

It is known that the quadratic term $\mu \mapsto \int G\mu d\mu$ in (1.10) is lower semicontinuous [22], while the linear term $\mu \mapsto \int \varphi d\mu$ is continuous with respect to the weak* topology. Thus to prove (3.1) it suffices to show that $\sigma\mu$ is absolutely continuous with respect to λ, say $\sigma\mu = \sigma u\lambda$, and

$$\int \sigma u^2 d\lambda \leq \liminf_{k\to\infty} \int \sigma u_k^2 d\lambda, \tag{3.3}$$

and this is what we are going to do.

For each k we have that $J_\sigma(\mu_k)$ is finite and thus by the definition (1.10) there exists a non-negative measurable function u_k on Γ^+ such that $\sigma\mu_k = \sigma u_k\lambda$. Since $\int G\mu_k d\mu_k \geq 0$, we have

$$\begin{aligned}\int \sigma u_k^2 d\lambda &= J_\sigma(\mu_k) - J_0(\mu_k)\\ &\leq J_\sigma(\mu_k) + 2\int \varphi d\mu_k \to J^* + 2\int \varphi d\mu \quad \text{as } k\to\infty,\end{aligned}$$

by weak* convergence. Hence the integrals $\int \sigma u_k^2 d\lambda$ remain bounded as $k\to\infty$, and passing to a further subsequence, if necessary, we may assume that

$$\lim_{k\to\infty} \int \sigma u_k^2 d\lambda = R^2 \tag{3.4}$$

exists and R is finite.

Let ψ be a continuous function on Γ^+. Then by the Cauchy-Schwarz inequality

$$\int_{\Gamma^+} |\psi|\sigma^{1/2} d\mu_k = \int_{\Gamma^+} |\psi|\sigma^{1/2} u_k d\lambda \leq \left(\int_{\Gamma^+} |\psi|^2 d\lambda\right)^{1/2} \left(\int_{\Gamma^+} \sigma u_k^2 d\lambda\right)^{1/2}.$$

Taking the limit $k\to\infty$ we obtain by weak* convergence $\mu_k \to \mu$ and (3.4) that

$$\int |\psi|\sigma^{1/2} d\mu \leq R\left(\int_{\Gamma^+} |\psi|^2 d\lambda\right)^{1/2} \tag{3.5}$$

for any continuous function ψ on Γ^+. Since continuous functions are dense in $L^2(\Gamma^+,\lambda)$ the inequality (3.5) continues to hold for every $\psi \in L^2(\Gamma^+,\lambda)$.

We take $\psi = 1_A$, where A is a Borel subset of Γ^+ and 1_A denotes the characteristic function of A. Then (3.5) gives

$$\int_A \sigma^{1/2} d\mu \leq R\sqrt{\lambda(A)},$$

which implies that $\sigma^{1/2}\mu$ is absolutely continuous with respect to λ. Hence there is a non-negative density u such that $\sigma^{1/2}\mu = \sigma^{1/2}u\lambda$ and then also

$$\sigma\mu = \sigma u\lambda. \tag{3.6}$$

Next take $\psi_M = \min(\sigma^{1/2}u, M)$ for some $M > 0$. Then ψ_M is bounded and thus certainly in $L^2(\Gamma^+,\lambda)$. Hence by (3.5) and (3.6)

$$\int_{\Gamma^+} \psi_M \sigma^{1/2} u d\lambda \leq R\left(\int_{\Gamma^+} \psi_M^2 d\lambda\right)^{1/2} \leq R\left(\int_{\Gamma^+} \psi_M \sigma^{1/2} u d\lambda\right)^{1/2}$$

since $\psi_M \le \sigma^{1/2}u$. Since the integral is finite we deduce

$$\int_{\Gamma^+} \psi_M \sigma^{1/2} u d\lambda \le R^2.$$

Lettting $M \to +\infty$ and noting that $\psi_M \nearrow \sigma^{1/2}u$, we obtain by monotone convergence and (3.4)

$$\int \sigma u^2 d\lambda \le R^2 = \lim_{k\to\infty} \int \sigma u_k^2 d\lambda. \tag{3.7}$$

This implies (3.3) and the lemma is proven. □

Lemma 3.2. *Suppose $\Gamma^+ \cap \mathbb{R} = \emptyset$. Then J_σ has compact sub-level sets, i.e., for every $c \in \mathbb{R}$ the set*

$$\{\mu \ge 0 \mid J_\sigma(\mu) \le c\} \tag{3.8}$$

is compact in the weak topology.*

Proof. The functional $\mu \mapsto \int G\mu d\mu$ has a unique minimizer among probability measures on Γ^+, and the minimum is positive, say $c_0 > 0$, since $\Gamma^+ \cap \mathbb{R} = \emptyset$. Then, as it is a quadratic functional,

$$\int G\mu d\mu \ge c_0 \left(\int d\mu\right)^2$$

By Definition 1.2, since $\sigma \ge 0$,

$$J_\sigma(\mu) \ge J_0(\mu) \ge c_0 \left(\int d\mu\right)^2 - 2 \max_{\Gamma^+} \varphi \int d\mu. \tag{3.9}$$

This immediately implies that for every $c \in \mathbb{R}$ there is $M > 0$ such that for every $\mu \ge 0$ on Γ^+ we have

$$J_\sigma(\mu) \le c \implies \int d\mu \le M.$$

In other words, the sub-level set (3.8) is contained in the set

$$\{\mu \ge 0 \mid \int d\mu \le M\} \tag{3.10}$$

which is compact in the weak* topology since Γ^+ is compact set. Because of the lower semi-contuinity of J_σ, see Lemma 3.1, the set (3.8) is a closed subset of (3.10) and the lemma follows. □

Now we can prove part (a).

Proof of Theorem 1.3 (a). It follows from (3.9) that J_σ is bounded away from $-\infty$. Let $(\mu_k)_k$ be a sequence of non-negative measures on Γ^+ such that

$$\lim_{k\to\infty} J_\sigma(\mu_k) = \inf_{\mu\geq 0} J_\sigma(\mu). \tag{3.11}$$

Because of Lemma 3.2 the sequence $(\mu_k)_k$ is in a weak* compact set, and therefore it has a subsequence with a weak* limit, say μ^*. Because of Lemma 3.1 and (3.11) we then have

$$J_\sigma(\mu^*) \leq \inf_{\mu\geq 0} J_\sigma(\mu)$$

which implies that μ^* is a minimizer.

The minimizer is unique, since the functional J_σ is strictly convex, which follows from Definition 1.2 since J_0 is known to be strictly convex. From Definition 1.2 it is also clear that $\sigma\mu^*$ is continuous with respect to λ. This completes the proof of part (a). □

3.2 Proof of part (b): variational condition on $\operatorname{supp}(\mu^*)$

The proof relies on standard arguments in variational calculus. Since we use these arguments later on as well, we state them in a separate lemma.

Lemma 3.3. *Let ν be a measure on Γ^+.*

(a) *If $J_\sigma(\nu) < \infty$, then*

$$\int (G\mu^* + \sigma u^* - \varphi) d\nu \geq 0. \tag{3.12}$$

(b) *If $\nu \leq \mu^*$, then*

$$\int (G\mu^* + \sigma u^* - \varphi) d\nu = 0. \tag{3.13}$$

Proof. (a) Suppose $J_\sigma(\nu) < \infty$. Then $\sigma\nu = \sigma v \lambda$ for some v. Note that

$$\begin{aligned}\int \sigma(u^* + \varepsilon v)^2 d\lambda - \int \sigma(u^*)^2 d\lambda &= 2\varepsilon \int \sigma u^* v d\lambda + O(\varepsilon^2)\\ &= 2\varepsilon \int \sigma u^* d\nu + O(\varepsilon^2)\end{aligned}$$

as $\varepsilon \to 0$, and it follows that

$$J_\sigma(\mu^* + \varepsilon\nu) - J_\sigma(\mu^*) = 2\varepsilon \int (G\mu^* + \sigma u^* - \varphi)\, d\nu + O(\varepsilon^2). \tag{3.14}$$

Since μ^* is the minimizer of J_σ, the left-hand side of (3.14) is non-negative for every $\varepsilon > 0$ and (3.12) follows by letting $\varepsilon \to 0+$.

(b) If $\nu \leq \mu^*$, then also $J_\sigma(\nu) < \infty$, and then the left-hand side of (3.14) is non-negative for every $\varepsilon \in (-1, \infty)$. Then the equality (3.13) follows by letting $\varepsilon \to 0$ through negative values as well. □

Proof of part (b). Let $E = \{z \in \Gamma^+ \mid (G\mu^* + \sigma u^* - \varphi)(z) < 0\}$ and suppose $\mu^*(E) > 0$. Then there is $\delta > 0$ such that

$$E_\delta = \{z \in \Gamma^+ \mid (G\mu^* + \sigma u^* - \varphi)(z) < -\delta\}$$

has $\mu^*(E_\delta) > 0$. Let ν be the restriction of μ^* to E_δ. We get

$$\int (G\mu^* + \sigma u^* - \varphi) d\nu < -\delta\nu(E_\delta) < 0$$

which is in contradiction with (3.13), since clearly $\nu \le \mu^*$. We obtain a similar contradiction in case $\mu^*(\{z \in \Gamma^+ \mid (G\mu^* + \sigma u^* - \varphi)(z) > 0\}) > 0$ and (1.11) follows. □

3.3 Proof of part (c): variational condition outside $\mathrm{supp}(\mu^*)$

To obtain the equality (1.12) on $\Gamma^+ \setminus \mathrm{supp}(\mu^+)$ we need the conditions of part (c) Theorem 1.3. That is, we require that φ is positive, continuous and superharmonic on $\mathbb{C}^+$. We also use Assumption 1.1 on λ. It follows from this assumption that the Green potential of the restriction of λ to any open subset of Γ^+ is continuous as well, as this is a consequence of the following simple general fact.

Lemma 3.4. *Suppose μ, ν are measures with $\nu \le \mu$. If $G\mu$ is continuous then $G\nu$ is continuous as well.*

Proof. Being a Green potential, $G\nu$ is lower semicontinuous. Since $\mu - \nu \ge 0$, also $G(\mu - \nu)$ is lower semicontinuous. Hence if $G\mu$ is continuous, then $G\nu = G\mu - G(\mu - \nu)$ is upper semicontinuous as well, and therefore continuous. □

Proof of part (c) of Theorem 1.3. Suppose $z_0 \in \mathrm{supp}(\mu^*)$ is such that $G\mu^*(z_0) > \varphi(z_0)$. Since $G\mu^* - \varphi$ is lower semicontinuous, we can then find $\varepsilon > 0$ and a disk $D_0 = D(z_0, r_0)$ around z_0 such that $G\mu^* - \varphi \ge \varepsilon$ on D_0. The restriction of μ^* to the complement of D_0 is a measure that is obviously bounded by μ^*, and thus by (3.13) we have

$$\int_{\mathbb{C}\setminus D_0} (G\mu^* + \sigma u^* - \varphi) d\mu^* = 0.$$

We conclude, since $G\mu^* + \sigma u^* - \varphi \ge G\mu^* - \varphi \ge \varepsilon$ on D_0,

$$\begin{aligned}\int (G\mu^* + \sigma u^* - \varphi) d\mu^* &= \int_{D_0} (G\mu^* + \sigma u^* - \varphi) d\mu^* \\ &\ge \varepsilon\mu^*(D_0) > 0.\end{aligned}$$

The last inequality holds since D_0 is a disk around z_0 and $z_0 \in \mathrm{supp}(\mu^*)$. On the other hand, (3.13) also holds for $\nu = \mu^*$ itself, and we find a contradiction.

Thus $G\mu^* \leq \varphi$ on $\mathrm{supp}(\mu^*)$ and we first extend the inequality to all of $\mathbb{C}^+$. This will follow from the maximum principle for subharmonic functions as we now show. Note that

$$\limsup_{z \to x \in \mathbb{R}} (G\mu^* - \varphi)(z) \leq 0,$$

since φ is non-negative and $G\mu^*$ is zero on the real line. Similarly

$$\limsup_{z \to \infty} (G\mu^* - \varphi)(z) \leq 0. \tag{3.15}$$

Due to the assumption that φ is superharmonic, and due to the fact that $G\mu^*$ is harmonic away from the support of μ^*, we also have that $G\mu^* - \varphi$ is subharmonic on $\mathbb{C}^+ \setminus \mathrm{supp}(\mu^*)$. Then by the maximum principle for subharmonic functions (which says that the maximum is attained on the boundary) we indeed have that

$$G\mu^* \leq \varphi \text{ on } \mathbb{C}^+ \tag{3.16}$$

To prove (1.12) we take $z_0 \in \Gamma^+ \setminus \mathrm{supp}(\mu^*)$ and in order to get a contradiction we suppose $(G\mu^* - \varphi)(z_0) \neq 0$. Because of (3.16) we then have $(G\mu^* - \varphi)(z_0) < 0$.

Since $G\mu^* - \varphi$ is continuous away from the support of μ^*, there is $r_0 > 0$ and $\varepsilon > 0$ such that $D_0 = D(z_0, r_0) \subset \Gamma^+ \setminus \mathrm{supp}(\mu^*)$ and $G\mu^* - \varphi < -\varepsilon$ on D_0. Let ν denote the restriction of λ to D_0. Then $\nu(D_0) > 0$ by Assumption 1.1 (since $\mathrm{supp}(\lambda) = \Gamma^+$), and

$$\int (G\mu^* - \varphi)\, d\nu < -\varepsilon \nu(D_0) < 0. \tag{3.17}$$

Because of Assumption 1.1 and its consequence that is noted before the statement of Assumption 1.1, we have that $G\nu$ is continuous and bounded. Hence $\int G\nu d\nu$ is finite. Also ν has a density v with respect to λ (which is just the characteristic function of $D_0 \cap \Gamma^+$) and $\int \sigma v^2 d\lambda$ is finite as well. Hence by (1.10) we have $J_\sigma(\nu) < \infty$, and then by (3.12) we get

$$\int (G\mu^* + \sigma u^* - \varphi) d\nu \geq 0.$$

However $\sigma u^* = 0$ on $\mathrm{supp}(\nu)$, and we find a contradiction with (3.17). $\square$

A slight extension of the proof also yields the following fact about $\mathrm{supp}\, \mu^*$. Let Ω denote the unbounded connected component of $\mathbb{C}^+ \setminus \Gamma^+$, so that $\partial\Omega \subset \mathbb{R} \cup \{\infty\} \cup \Gamma^+$.

Proposition 3.5. *In the conditions of Theorem 1.3, part (c), assume that*

$$\liminf_{z \to \infty} \varphi(z) > 0. \tag{3.18}$$

Then $\partial\Omega \cap \Gamma^+$, i.e., the outer boundary of Γ^+, is contained in $\mathrm{supp}(\mu^*)$.

In particular, if Γ^+ has empty interior and a connected complement then $\mathrm{supp}(\mu^*) = \Gamma^+$.

Proof. As in the proof of Theorem 1.3, part (c), we have (3.16) but now

$$\limsup_{z\to\infty} \left(G\mu^* - \varphi\right)(z) < 0.$$

Then by the maximum principle $G\mu^* - \varphi < 0$ on the unbounded connected component of $\mathbb{C}^+ \setminus \operatorname{supp}(\mu^*)$. But if $z_0 \in \Gamma^+ \setminus \operatorname{supp}\mu^*$, then $(G\mu^* - \varphi)(z_0) = 0$ according to (1.12). Thus, $\partial\Omega \cap \Gamma^+ \subset \operatorname{supp}\mu^*$. □

4 Proof of Theorem 1.6 and the second NDR equation

4.1 Proof of part (a)

Proof. Take $z_0 \in S$ with $\sigma(z_0) > 0$. Since σ is continuous there exist a disk $D = D_0(z_0, r_0)$ around z_0 and a number $C_0 > 0$ such that $\sigma(z) \geq C_0 > 0$ for all $z \in D \cap \Gamma^+$. Let ν be the restriction of λ to $D \cap \Gamma^+$. Then $G\nu$ is continuous by Assumption 1.1. Let μ_0^* be the restriction of μ^* to $D \cap \Gamma^+$. Then μ_0^* has the density u^* on $D \cap \Gamma^+$ and

$$\sigma u^* \leq \varphi, \quad \mu_0^*\text{-a.e.}$$

which is a consequence of (1.11).

Since $\sigma \geq C_0$ on $\operatorname{supp}(\mu_0^*)$, we find

$$u^* \leq \frac{1}{C_0} \max_{z\in\Gamma^+} \varphi(z), \quad \mu_0^*\text{-a.e..}$$

This means that there is a constant $C > 0$ such that $\mu_0^* \leq C\nu$. Since $G\nu$ is continuous, it follows from Lemma 3.4 that $G\mu_0^*$ is continuous.

Then $G\mu^*$ is continuous on $D \cap \Gamma^+$, and in particular at z_0, since $G\mu$ is the sum of $G\mu_0^*$ and $G(\mu^* - \mu_0^*)$, and the latter is continuous on $D \cap \Gamma^+$ as $\mu^* - \mu_0^*$ is supported away from this open set. Since $z_0 \in S$ is arbitrary, we find that $G\mu^*$ is continuous on S.

Because of (1.11) we have

$$u^* = \frac{\varphi - G\mu^*}{\sigma} \quad \mu^*\text{-a.e. on } S.$$

The right-hand side is continuous on S, and if we just redefine u^* by the right-hand side, then the new u^* is still a valid density for μ^* and it is continuous. We also find that the identity $G\mu^* + \sigma u^* = \varphi$ holds on S, which concludes the proof of part (a). □

4.2 Proof of part (b)

We need some notions from potential theory that we briefly summarize, see [1, 15, 21, 22] for fuller accounts. A set where a superharmonic function is

$+\infty$ is called a polar set, otherwise it is non-polar. A compact set $A \subset \mathbb{C}^+$ is non-polar if and only if its capacity is positive, which means that there is a probability measure ν with $\mathrm{supp}(\nu) \subset A$ and $\int G\nu d\nu < \infty$.

The fine topology on $\mathbb{C}^+$ is the coarsest topology for which all Green potentials $G\mu$, $\mu \geq 0$ are continuous. The fine topology is finer than the usual Euclidean topology, since there exist non-continuous Green potentials. A fine neighborhood of z_0 is a neighborhood in the fine topology. Then a set S is thick at z_0 if and only if every fine neighborhood of z_0 has a non-empty intersection with $S \setminus \{z_0\}$, see e.g. [1, Theorem 7.2.3].

Lemma 4.1. *Suppose S is thick at z_0. Let U be a fine neighborhood of z_0. Then $S \cap U$ is non-polar.*

Proof. Suppose $S \cap U$ is polar. By [1, Theorem 7.2.2] a polar set is thin everywhere, so in particular $S \cap U$ is thin at z_0. Thus there is a fine neighborhood V of z_0 that does not intersect $(S \cap U) \setminus \{z_0\}$.

Then $U \cap V$ is a fine neighborhood of z_0 that does not intersect $S \setminus \{z_0\}$ which is a contradiction, since S is thick at z_0. □

Now we turn to the proof of part (b) of Theorem 1.6.

Proof of part (b). Suppose S_0 is thick at z_0, and assume $G\mu^*(z_0) \neq \varphi(z_0)$. From Theorem 1.3 (c) we know that $G\mu^*(z_0) \leq \varphi(z_0)$, and therefore there is $\delta > 0$ such that

$$G\mu^*(z_0) < \varphi(z_0) - \delta.$$

Then $A = \{z \in \mathbb{C}^+ \mid G\mu^*(z) \leq \varphi(z) - \delta\}$ is a fine neighborhood of z_0, and it is also closed in the usual topology, since $G\mu^*$ is lower semicontinuous and φ is continuous.

Since S_0 is thick at z_0, we conclude that $A \cap S_0$ has positive capacity. Thus there is a probability measure ν on $A \cap S_0$ with $\int G\nu d\nu < \infty$. Since $\sigma = 0$ on $\mathrm{supp}(\nu) \subset S_0$ we then also have $J_\sigma(\nu) < \infty$. Since $\sigma = 0$ and $G\mu^* \leq \varphi - \delta$ on the support of ν, we find

$$\int (G\mu^* + \sigma u^* - \varphi) d\nu \leq -\delta < 0$$

which is in contradiction with Lemma 3.3 (a). This proves part (b) of Theorem 1.6. □

4.3 Solution of equation (1.2)

In this section we use Theorems 1.3 and 1.6 to prove the existence of a solution to (1.5) with $\varphi(z)$ that is sufficiently smooth on some neighborhood of Γ^+, but is not necessarily positive or superharmonic in $\mathbb{C}^+$. As an example, $\varphi(z) = -4 \operatorname{Im} z \operatorname{Re} z$ corresponds to equation (1.2). We start with the following lemma.

Lemma 4.2. *The energy functional $J_0(\mu)$ with $\varphi \equiv 0$, see (1.8), is non negative on the set of all signed compactly supported Borel measures on $\mathbb{C}^+$. Moreover, it is zero if and only if $\mu = 0$.*

Proof. Let $\Gamma^+ \subset \mathbb{C}^+$ be a compact set containing $\operatorname{supp}\mu$ Denote $\Gamma := \Gamma^+ \cup \Gamma^-$, where the compact set $\Gamma^- \subset \mathbb{C}^-$ is Schwarz symmetrical to Γ^+. Any (signed) Borel measure μ with $\operatorname{supp}\mu \subset \Gamma^+$ we anti-symmetrically extend to the signed measure $\tilde{\mu}$ on Γ by setting $\tilde{\mu}(S) = \mu(S)$ if $S \subset \Gamma^+$ and $\tilde{\mu}(S) = -\mu(S)$ if $S \subset \Gamma^-$. Then $\tilde{\mu}(\Gamma) = 0$ and it is easy to check that

$$G\mu(z) = -\frac{1}{\pi}\int \log|z-w|\,d\tilde{\mu}(w) =: U\tilde{\mu}(z), \tag{4.1}$$

where $U\tilde{\mu}$ is the standard logarithmic potential of $\tilde{\mu}$. Now the statement follows from [22], Lemma I.1.8. $\square$

Lemma 4.2 implies that the operator $v \mapsto Gv$ defined by (1.6) (in some suitable function space) is positive definite. Therefore, if $\sigma > 0$ on Γ^+ then the operator $v \mapsto Gv + \sigma v$ is also positive definite, with the spectrum bounded away from 0; then its inverse exists and $v = (G+\sigma)^{-1}\varphi$ is the solution. If σ has zeros on Γ^+ then this argument does not work. However, Theorem 4.3 stated below covers the latter case.

Theorem 4.3. *If: a) φ is a C^2 function in a neighborhood of Γ^+, and; b) the zero set $S_0 = \{z \in \Gamma^+ \mid \sigma(z) = 0\}$ of σ is thick at each of its points then there exists a unique signed measure μ with a density u with respect to λ such that the equation* (1.5) *is valid everywhere on Γ^+.*

Proof. Let h be a compactly supported C^2 function in $\mathbb{C}$ with $h = \varphi$ on Γ^+ and $h = 0$ on $\mathbb{R}$. Let $K \subset \mathbb{C}^+$ be a compact set containing Γ^+ as well as the support of h. Since h is C^2 and has compact support there is $c > 0$ such that $\Delta h < 2c$ in $\mathbb{C}$, where Δ is the Laplace operator.

Let w be a non-negative C^2 function in $\mathbb{C}^+$ with compact support such that $w = c$ on K. Then the function

$$\varphi_1(z) = \frac{1}{\pi}\int_{\mathbb{C}^+} \log\left|\frac{z-\bar{s}}{z-s}\right| w(s)\,dA(s)$$

where dA is planar Lebesgue measure, is superharmonic and $\Delta\varphi_1 = -2w$. So,

$$\Delta(h+\varphi_1) = \Delta h + \Delta\varphi_1 < 2c - 2w = 0 \qquad \text{on } K,$$

while $\Delta(h+\varphi_1) = \Delta\varphi_1 = -2w \le 0$ on $\mathbb{C}\setminus K$.

Then we split

$$\varphi = h + \varphi_1 - \varphi_1 \qquad \text{on } \Gamma^+$$

where both φ_1 and $\varphi_2 = h+\varphi_1$ are continuous and superharmonic on $\mathbb{C}^+$. Both functions are non-negative, and since they are not-identically zero, they must be positive on $\mathbb{C}^+$ by the minimum principle for superharmonic functions.

Thus Theorem 1.3 part (c) applies, and there are positive measures μ_1^* and μ_2^* on Γ^+ with corresponding densities u_1^* and u_2^* with respect to λ, such that for $j = 1, 2$,

$$G\mu_j^* + \sigma u_j^* = \varphi_j, \tag{4.2}$$

in the weak sense (i.e., μ_j^*-a.e., and equality outside the support of μ_j^*) Hence

$$G(\mu_2^* - \mu_1^*) + \sigma(u_2^* - u_1^*) = \varphi_2 - \varphi_1 = \varphi \quad \text{on } \Gamma^+$$

in the weak sense. Thus, the signed measure $\mu_2^* - \mu_1^*$ solves (1.5), where σu term is understood in the same way as in Theorem 1.3.

Since the zero set S_0 of σ is thick at each of its points then Corollary 1.7 applies and the equations (4.2) are satisfied everywhere on Γ^+. Then $\mu = \mu_2^* - \mu_1^*$ solves (1.5) on Γ^+. Finally, the uniqueness of solution follows from Lemma 4.2. $\square$

Remark 4.4. If in Theorem 4.3 we keep only the assumption a) then, according to Theorem 1.6, the equation (1.5) is satisfied everywhere on Γ^+ except at possibly non-thick (thin) points of the set S_0.

5 Properties of the minimizer μ^* under additional assumptions on Γ^+ and σ

In this section we study the support of the minimizer μ^* and its smoothness under some additional assumptions. Everywhere in this section we assume that Γ^+ is a finite union of compact 1D arcs and 2D closed regions equipped with the standard Lebesgue measure λ each. Moreover, unless specified otherwise, we assume that φ satisfies conditions of Theorem 1.3, part (c), that is, the equation (1.5) has a weak solution.

5.1 Geometry of $\operatorname{supp} \mu^*$

We start with the following lemma.

Lemma 5.1. *In the assumptions of Theorem 1.3 (c), let Ω be a bounded open set such that: a) $\partial\Omega \subset \Gamma^+$, and; b) $\partial\Omega$ is thick at all of its points. Assume that $\sigma = 0$ on $\partial\Omega$ and φ is harmonic on Ω. Then $\operatorname{supp} \mu^* \cap \Omega = \emptyset$, i.e., there is no support of μ^* inside Ω, where μ^* is the minimizer of J_σ.*

Proof. According to Theorem 1.6, part (b), the equation (1.5) holds everywhere on $\partial\Omega$. Since $G\mu^*$ is superharmonic with $G\mu^* = \varphi$ on $\partial\Omega$ and φ harmonic in Ω, the minimum principle tells us that $G\mu^* \geq \varphi$ on Ω. Since $G\mu^* \leq G\mu^* + \sigma u^* = \varphi$ on Γ^+, it follows that $G\mu^* = \varphi$ on $\Omega \cap \Gamma^+$.

Then $G\mu^*$ is harmonic on $\Omega \setminus \Gamma^+$ (since μ^* is supported on Γ^+) and it agrees with φ on its boundary. Then by the maximum/minimum principle for harmonic functions we get $G\mu^* = \varphi$ on Ω and $G\mu^*$ is harmonic on Ω. Since (in distributional sense) $\Delta G\mu^* = -2\mu^*$, we then conclude that $\mu^* = 0$ on Ω. $\square$

Lemma 5.1 leads to some interesting consequences for equation (1.1), where $\varphi(z) = \operatorname{Im} z$ is harmonic in $\mathbb{C}$. The most obvious is that in the case $\sigma \equiv 0$ on a compact connected region Γ^+ then $\operatorname{supp} \mu^* \subset \partial\Gamma^+$. This is a well known fact in potential theory. Moreover, using Proposition 3.5, we obtain that $\operatorname{supp}\mu^*$ coincides with the outer boundary of Γ^+. In fact, the latter result holds even if $\sigma = 0$ only on the outer boundary of Ω.

Consider another case when equation (1.1) has a solution u on the compact domain Γ^+ and $\sigma = 0$ on a simple piece-wise smooth closed curve $\gamma \subset \Gamma^+$. Then, according to Lemma 5.1, if u is bounded on γ then $u \equiv 0$ inside the region bounded by γ.

Example 2. Consider the example of circular condensate from [11], where Γ^+ consists of the upper semicircle $|z| = \rho$, $\operatorname{Im} z \geq 0$, with some $\rho > 0$ and $\sigma \equiv 0$ on Γ^+. Then $u = \frac{\operatorname{Im} z}{\pi\rho}$ and $v = \frac{-4 \operatorname{Im} z^2}{\pi\rho}$ are solutions of (1.1)-(1.2) respectively. If we replace Γ^+ by the upper semi disk $|z| \leq \rho$, $\operatorname{Im} z \geq 0$ and let σ to be any positive continuous function on Γ^+ such that $\sigma(z) = 0$ when $|z| = \rho$, then the same u, v on the upper semi circle $|z| = \rho$, $\operatorname{Im} z \geq 0$ with trivial continuation $u = v \equiv 0$ for $|z| < \rho$ solve (1.1)-(1.2) respectively. This example, strictly speaking, does not satisfy conditions of Lemma 5.1 since $\Gamma^+ \cap \mathbb{R} \neq \emptyset$ but, nevertheless, it illustrates the idea.

5.2 Smoothness in 1D case with $\sigma \equiv 0$

In the rest of this section we consider the smoothness of $d\mu^*$ in the 1D case, i.e., when Γ^+ is a finite collection of piece-wise smooth curves that can be closed or opened. Transversal intersections of different curves are allowed, but we consider intersection points, as well as the end points of open arcs, as points of non smoothness. In this case the reference measure λ is simply the arclength measure on the curves. We also assume that φ is a C^∞ function in some neighborhood containing Γ^+.

In this subsection we consider the case of a soliton condensate, i.e., the case of $\sigma \equiv 0$ (on Γ^+). There are certain results about the smoothness of the minimizing measure μ on a 1D compact set Γ^+ in terms of its logarithmic potential $U\mu = -\frac{1}{\pi} \int_{\Gamma^+} \log|z-w||dw|$ in the literature, see, for example, [22]. Many of these results can also be applied to signed measures μ. It turns out that these results can be applied to the Green potential $G\mu^*$ of the minimizer μ^* of J_0.

Indeed, denote $\Gamma := \Gamma^+ \cup \Gamma^-$, where the compact set $\Gamma^- \subset \mathbb{C}^-$ is Schwarz symmetrical to Γ^+. Any Borel measure μ with $\operatorname{supp}\mu \subset \Gamma^+$ we anti-symmetrically extend to the signed measure $\tilde{\mu}$ on Γ by setting $\tilde{\mu}(S) = \mu(S)$ if $S \subset \Gamma^+$ and $\tilde{\mu}(S) = -\mu(S)$ if $S \subset \Gamma^-$. The function σ is extended to Γ Schwarz symmetrically. Then, see (4.1), $G\mu = U\tilde{\mu}$, so that $G\mu + \sigma u = \varphi$ on Γ^+ if and only if

$$-\frac{1}{\pi} \int_\Gamma \log|z-w| d\tilde{\mu}(w) + \sigma(z)\tilde{u}(z) = \tilde{\varphi}(z) \qquad \text{on } \Gamma, \tag{5.1}$$

where $\tilde{\varphi}$ denotes anti Schwarz symmetrical (odd) continuation of the function φ from Γ^+ to Γ and $\sigma\tilde{u}$ is the density of $\sigma\tilde{\mu}$.

Take a smooth subarc $\gamma \subset \Gamma^+$ - it is enough to assume that γ is $C^{1+\delta}$ smooth with some $\delta > 0$ - and a measure μ such that $\gamma \in \operatorname{supp}\mu$. Then, according to

Theorem II.1.5 of [22], if the logarithmic potential $U\mu$ is Lip 1 in a neighborhood of γ, then μ is absolutely continuous on γ and its density u is given by

$$u(s) = \frac{d\mu}{ds}(s) = -\frac{1}{2}\left(\frac{\partial U\mu}{\partial n_+}(s) + \frac{\partial U\mu}{\partial n_-}(s)\right), \tag{5.2}$$

where $\frac{\partial}{\partial n_\pm}$ denote two (opposite) normals to γ, and s is the arclength parameter on γ. It is clear that, because of (5.1), we can replace $U\mu$ with the Green potential $G\mu$ in the above statement.

If φ satisfies the (mild) condition (3.18) then, by Proposition 3.5, the requirement $\gamma \in \operatorname{supp}\mu^*$ is guaranteed when Γ^+ belongs to the boundary of the unbounded component Ω of $\mathbb{C}^+ \setminus \Gamma^+$, i.e., when $\Gamma^+ \subset \partial\Omega$. Here μ^* is the minimizer of J_0.

Equations (5.1) and (5.2) show that the smoothness of u can be derived from the smoothness of $G\mu$ at the boundary Γ^+ of $\mathbb{C}^+ \setminus \Gamma^+$. Since the Green potential $\nu(z) = Gu(z)$ satisfies the Dirichlet boundary value problem

$$\begin{cases} \nu \text{ is harmonic on } \mathbb{C}^+ \setminus \Gamma^+, \\ \nu = \varphi \text{ on } \Gamma^+, \\ \nu = 0 \text{ on } \mathbb{R} \text{ and at infinity,} \end{cases} \tag{5.3}$$

we can use regularity theorems for boundary value problems from classical PDEs to estimate the smoothness of ν. Under our assumptions we can represent

$$\mathbb{C}^+ \setminus \Gamma^+ = \Omega \cup \left(\bigcup_{j=1}^{k} D_j\right), \tag{5.4}$$

where Ω denotes the unbounded and D_j's denote bounded, connected and mutually disjoint components. For simplicity, let us for now assume that all D_j's are simply connected.

Let $\Gamma_1 \subset \Gamma^+$ be a C^∞ closed curve that is the boundary of D_1. Then ν is harmonic in D_1 and satisfies the Dirichlet condition $\nu = \varphi$ on $\partial D_1 = \Gamma_1$. Assume that Γ_1 is positively oriented. Then, using regularity theorems, in particular Schauder Estimates in Hölder spaces, see for example [7], Section II.2.14, or [23], we obtain that $\nu(z)$ exists and is a C^∞ function on the closure of D_1, i.e., on $D_1 \cup \Gamma_1$. Thus, partial derivatives of $Gu(z)$ are C^∞ functions on $D_1 \cup \Gamma_1$. In particular, so is $\frac{\partial Gu}{\partial n_+}(z)$.

In the case when $\Gamma_1 = \partial D_1$ is only a piece-wise C^∞ curve, ν is C^∞ smooth everywhere on Γ_1 except the points of non smoothness (since smoothness of ν is a local property). Similar arguments show that ν is C^∞ smooth everywhere on the closure of Ω except points of non smoothness of Γ^+. Thus, $\frac{\partial Gu}{\partial n_-}(z)$ is also a C^∞ function on Γ^+ away from the points of non smoothness and, according to (5.2), so is $u(s)$. This result is summarized in the following theorem.

Theorem 5.2. *Let $\Gamma^+ \subset \mathbb{C}^+$ be a piece-wise C^∞ smooth curve such that $\Gamma^+ \subset \partial\Omega$, where Ω is the unbounded component of $\mathbb{C}^+ \setminus \Gamma^+$. Let φ be a C^∞*

function in some domain B containing Γ^+ that also satisfies condition (3.18) *from Proposition 3.5. Then the solution u^* of the integral equation* (1.5) *is a C^∞ function on any compact subarc of a smooth arc of Γ^+.*

Proof. As it was shown above, both normal derivatives $\partial\nu/\partial n$ are C^∞ functions on any compact subarc γ of a smooth arc of Γ^+ together with the adjacent regions of $\mathbb{C}^+ \setminus \Gamma^+$. Then, by formula (5.2), $u^* = \frac{d\mu^*}{ds}$, where μ^* is the minimizer of the energy functional J_0, is a C^∞ function on γ. $\square$

Remark 5.3. According to Lemma 5.1, the requirement $\Gamma^+ \subset \partial\Omega$ in Theorem 5.2 can be dropped if φ is harmonic in $\mathbb{C}^+$, for example, if $\varphi = \operatorname{Im} z$ as in (1.1). In fact, it is sufficient to require that φ is harmonic only in every D_j, see (5.4), such that $\partial D_j \not\subset \partial\Omega$.

Theorem 5.2 obviously applies to equation (1.1) but not to (1.2) since $\varphi(z) = -2\operatorname{Im} z^2$ is not necessarily non negative on Γ^+. However, as in the proof of Theorem 4.3, we can represent $\varphi = \varphi_2 - \varphi_1$, where both $\varphi_{1,2}$ are positive and superharmonic in $\mathbb{C}^+$. By construction, $\varphi_{1,2}$ do not satisfy condition (3.18), but that can be corrected by replacing $\varphi_{1,2}$ with $\varphi_{1,2} + 1$ respectively. Moreover, the C^∞ smoothness of φ implies that both $\varphi_{1,2}$ are also C^∞ functions. We can now apply Theorem 5.2 to the equations (1.5) with the right hand side $\varphi_{1,2}$ to obtain the following corollary.

Corollary 5.4. *Theorem 5.2 remains true without the requirement that φ is nonnegative in $\mathbb{C}^+$.*

Remark 5.5. The case when Γ^+ is a piecewise C^∞ collection of contours includes the case when Γ^+ contains arcs with endpoints (not closed curves). In that case behavior near the endpoints is given by Theorem IV.2.6 of [22], where $\alpha = 2\pi$. In particular, we obtain that $u(z)|z - z_0|^\epsilon \in L^2_{loc}$ for any $\epsilon > 0$ on a piece of Γ^+ that includes an endpoint z_0.

Remark 5.6. Let Γ^+ be a piece-wise finitely smooth curve (with sufficient smoothness). Then one can use similar arguments to show that $u^* = \frac{d\mu^*}{ds}$ is also piece-wise finitely smooth.

5.3 Smoothness in 1D case with $\sigma \geq 0$

Consider now the case when $\sigma \geq 0$ on Γ^+. Take some smooth closed subarc γ of Γ^+ (it contains its endpoints or encircles a region). We assume σ and φ to be sufficiently smooth on γ and also that $\sigma > 0$ on γ According to Theorem 1.6, the density $u^* = d\mu^*/ds$ of the minimizer is continuous on γ. Let us write the equation (1.5) as

$$G_1 u^* + \sigma u^* = \varphi - G_2 u^* \qquad \text{on } \gamma, \tag{5.5}$$

where the integration in G_1 is over γ and $G_2 = G - G_1$ We can perceive u^* on $\Gamma^+ \setminus \gamma$ as to be given and consider (5.5) as a second kind Fredholm integral equation for u^* on γ. The right hand side of (5.5) is harmonic at the interior points of γ. It can also be shown to be Hölder continuous on γ with any

Hölder exponent $\nu \in (0,1)$. Under these conditions it follows, see, for example, [18], Section 51.1, that u^* is Hölder continuous on γ with any Hölder exponent $\nu \in (0,1)$.

Let us differentiate (5.5) with respect to the arclength s. We obtain

$$\sigma(u^*)' = -\sigma' u^* - (G_1 u^*)' + \varphi' - (G_2 u^*)' =: RH. \tag{5.6}$$

We now prove that the right hand side of (5.6) is Hölder continuous on a proper compact subarc γ_0 of γ. It is obvious that the first and the third terms of RH are Hölder continuous on γ. The second term $(G_1 u^*)'$ is a singular integral of a Hölder continuous function and therefore must be Hölder continuous on γ with the same ν. Finally, since u^* is Hölder continuous on γ, $(G_2 u^*)'$ is harmonic on γ_0. So, we proved the following lemma.

Lemma 5.7. *Let $\gamma \subset \Gamma^+$ be a smooth closed arc such that $\sigma > 0$ and is smooth on γ. Then $\frac{du^*}{ds}$ is Hölder continuous on any compact subarc $\gamma_0 \subset \gamma$ with any Hölder exponent $\nu \in (0,1)$. Here s is the arclength parameter on γ.*

Remark 5.8. Lemma 5.7 is valid for equation (1.1). It can be also applied to equation (1.2) after we represent $\varphi = \varphi_2 - \varphi_1$ as in Theorem 4.3.

Lemma 5.9. *Let $\sigma = 0$ on some open C^∞ smooth arc $\Gamma_1 \subset \Gamma^+$ such that $\Gamma_1 \subset \operatorname{supp} \mu^*$. Then $u^* = d\mu^*/ds$ is C^∞ smooth of Γ_1.*

Proof. We write $Gu^* = G_1 u^* + G_2 u^*$, where in $G_j u^*$ we integrate over Γ_j, $j = 1,2$ and $\Gamma_2 = \Gamma^+ \setminus \Gamma_1$. Then

$$G_1 u^*(z) = \varphi(z) - G_2 u^*(z) \tag{5.7}$$

on Γ_1. Let B be a region containing Γ_1 and separated from $\mathbb{R}$. Take a function ϕ, harmonic in $B \setminus \Gamma_1$, continuous in the closure of B and satisfying $\phi = \varphi(z) - G_2 u^*(z)$ on Γ_1. Denote by f a C_0^∞ extension of ϕ to the closure of $\mathbb{C}^+$ satisfying $f_\mathbb{R} = 0$. Denote $w := G_1 u^*(x)$. Then w satisfies the Dirichlet problem for the Laplace equation in $\mathbb{C}^+ \setminus \Gamma_1$, with the boundary values $w = 0$ on $\mathbb{R}$ and $w = f$ on Γ_1. We can now apply Theorem 5.2 to prove that u^* is C^∞ smooth on any compact subarc of Γ_1. $\square$

6 Bound state fNLS and KdV condensates

The connection between a bound state fNLS soliton gas and the corresponding KdV soliton gas was described in Section 2.2. That connection implies that all the obtained above results about fNLS soliton gases, applicable to bound state gases, can be reformulated for KdV soliton gases. That includes existence and uniqueness of solutions, non negativity of the density of states u, smoothness and geometry of $\operatorname{supp} u$. In particular, the main Theorem 1.3 and Theorem 1.6, are applicable to the KdV soliton gas NDR (2.20)- (2.21) with a given continuous and non negative on Γ^+ function $\sigma(z)$.

Moreover, it turns out that the solution u of (1.1) in the case of the bound state fNLS condensate ($\sigma \equiv 0$) is proportional to the density of the quasimomentum differential dp, see (2.9), on the hyperelliptic Riemann surface $\mathcal{R}$ associated with Γ. This result is formulated (Theorem 6.1) and proven in this section. Its extension to the KdV condensate is also addressed below.

Consider $\sigma \equiv 0$ and $\Gamma = \Gamma^+ \cup \Gamma^- \subset i\mathbb{R}$, which is Schwarz symmetrical and consists of $2N+1$ segments with endpoints ib_j, $j = 0, 1 \ldots, N$, and beginning points ia_j, $j = 1 \ldots, N$, in $\mathbb{C}^+$, where $0 < b_0 < a_1 < b_1 < \cdots < b_N$, and their complex conjugates in $\mathbb{C}^-$. This is the case of a general (even genus) bound state fNLS condensate mentioned in Section 2.2, which has $v(z) \equiv 0$ solution to (1.2). Our goal is the following theorem.

Theorem 6.1. *Denote by dp the real normalized quasimomentum differential on the Riemann surface $\mathcal{R}$ (see Section 2). Then: i) dp has zero* $\mathbf{B}$ *- periods; ii) $\frac{dp}{dz}$ is Schwarz symmetrical (odd on $i\mathbb{R}$); iii) $u(z) = \frac{idp}{\pi dz} > 0$ on $\Gamma^+ \setminus \{0\}$ and it satisfies* (1.1) *with $\sigma \equiv 0$ on Γ^+.*

Proof. Switching from the condensate equation (1.1) to equation (5.1) (with anti Schwarz symmetrical u) and differentiating the latter in z, we obtain $\pi H u = 1$ on Γ, where H denotes the Finite Hilbert Transform (FHT) on Γ (which is oriented upwards). The inversion formula for FHT H (see, for example, [19] when $N = 0$) yields

$$u(z) = \frac{-1}{\pi^2 R(z)} \int_\Gamma \frac{R_+(w)dw}{w-z} = \frac{-1}{2\pi^2 R(z)} \oint_{\hat{\gamma}} \frac{R(w)dw}{w-z} = \frac{i}{\pi R(z)} \left(\operatorname{Res} \frac{R(w)}{w-z}\bigg|_{w=\infty} - \varkappa \operatorname{Res} \frac{R(w)}{w-z}\bigg|_{w=z} \right) \tag{6.1}$$

where $\hat{\gamma}$ is a negatively oriented circle containing Γ but not containing z if $z \notin \Gamma$ and $\varkappa = 0$ if $z \in \Gamma$ with $\varkappa = 1$ otherwise. Calculating the residue at $w = \infty$ we obtain

$$u(z) = \frac{iP(z)}{\pi R(z)} \qquad \text{on } \Gamma, \tag{6.2}$$

where $P(z)$ is a monic odd polynomial of degree $2N+1$ with real coefficients. The exact values of these coefficients, which can be obtained from $\operatorname{Res} \frac{R(w)}{w-z}\Big|_{w=\infty}$, are not essential because the null space of H is spanned by $\frac{z^k}{R(z)}$, where $k = 0, \ldots, 2N-1$. Since $u(z)$ must be odd (anti-Schwarz symmetrical), we consider only the odd powers of k. It is clear that $u(z) \in \mathbb{R}$ when $z \in \Gamma$. Note that $R(z)$ has opposite signs on the neighboring segments of Γ, where by convention we evaluate R on the positive (left) side of Γ. It is clear that in order to have $u > 0$ on Γ^+ and $u < 0$ on Γ^- the polynomial $P(z)$ must have a zero in each of the $2N$ gaps between consecutive segments of Γ (the remaining zero is $z = 0$). To determine these zeros of $P(z)$ we use the fact that the logarithmic potential

$$Gu(z) = -\frac{1}{\pi} \int_\Gamma \log|w-z| \, \frac{iP(w)(-idw)}{R(w)} \tag{6.3}$$

must be continuous, see Theorem I.5.1, assertion 4, [22]. Differentiating (6.3) in z and using the residues we obtain

$$[Gu(z)]' = \frac{1}{\pi}\int_{\Gamma} \frac{P(w)dw}{(w-z)R(w)} = -i\left(1 - \varkappa \frac{P(z)}{R(z)}\right), \tag{6.4}$$

so that

$$Gu(z) = -iz + i\varkappa \int_{m}^{z} \frac{P(w)dw}{R(w)}, \tag{6.5}$$

where $\varkappa$ is the same as in (6.1), $m = ib_{j-1}$ if z is on the jth gap (ib_{j-1}, ia_j), $j = 1, \dots, N$ and $m = -ib_{j-1}$ if z is on the complex conjugate gap $-j$ in $\mathbb{C}^-$. Now the continuity of Gu requirement is translated into the system of linear equations

$$\int_{b_{j-1}}^{a_j} \frac{P(w)dw}{R(w)} = 0, \quad j = 1, \dots, N, \tag{6.6}$$

for the coefficients of the odd monic polynomial P. By the symmetry, the corresponding equations hold on the gaps in $\mathbb{C}^-$.

The system (6.6) has a unique real solution. That follows from that fact that $\operatorname{Im}\tau$, where τ is the Riemann period matrix for $\mathcal{R}$, is positive definite. Thus, $iudz$ is a real normalized meromorphic differential with the poles at infinity of both sheets, and, according to (2.9), $dp = -i\pi u(z)dz$ is the quasi momentum differential on $\mathcal{R}$. Moreover, all the $\mathbf{B}$ periods of dp are zeros.

It remains only to prove that $u(z) > 0$ on Γ^+. In fact, since the system (6.6) requires that there must be just one zero of an odd polynomial $P(z)$ in every gap, it is sufficient to prove that $u > 0$ on the last segment (ia_N, ib_N).

Indeed, since all the zeros of $P(z)$ are on $(-ib_N, ib_N)$, $\arg u(z) = \frac{\pi}{2}$ on $(ib_N, +i\infty)$. When z crosses ib_N and stays on the positive (left) side of (ia_N, ib_N), the argument $of R(z)$ gains $\frac{\pi}{2}$ whereas the argument of $P(z)$ does not change. Hence, the lemma is proved. $\square$

Remark 6.2. The above arguments can be repeated for Schwarz symmetrical $\Gamma \subset i\mathbb{R}$ that consists of $2N$ segments. The corresponding $\mathcal{R}$ has genus $2N-1$ and $u(z) = \frac{iP(z)}{\pi R(z)}$ on Γ, where $R(z)$ is odd and $P(z)$ is even. $P(z)$ must have exactly one zero in each of the $2N-2$ gaps lying entirely in $\mathbb{C}^+$ or $\mathbb{C}^-$ and exactly two symmetrical zeros in the central gap $[-ia_1, ia_1]$.

Theorem 6.1 and Remark 6.2 imply that solutions of the NDR (2.20)- (2.21) for the KdV soliton condensate ($\sigma \equiv 0$) is always represented by the density of the corresponding meromorphic differentials on $\mathcal{R}$. In particular, $u_{KdV}(z) = \frac{1}{2}u_{fNLS}(iz)$, $z \in \Gamma^+$, where $\Gamma^+ \subset \mathbb{R}$ and u_{KdV} are defined for (2.20) and u_{fNLS} is defined by the corresponding (1.1). Equation (2.21) can be solved similarly to (2.20). Its solution is given by the density of the corresponding real normalized meromorphic differential that has $O(z^2)$ behavior as $z \to \infty$ (on both sheets).

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