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ORIGINAL ARTICLE

Expected returns with leverage constraints and target returns

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Abstract
Classic mean–variance optimization is very sensitive to expected returns. An alternative and more robust approach is to calculate the implied returns given the current portfolio allocation and risk profile. Portfolio managers can then do a reality check on the implied returns and find opportunities for better allocations. The most common implied return calculation assumes normal distribution and unlimited leverage, and use volatility as risk measure and covariance matrix as model input. However, practitioners usually have leverage constraints, often use non-parametric risk models, and care about portfolio tail risk. This paper presents a new approach to calculate expected returns with leverage constraints. This approach is flexible enough to alleviate normal distribution assumption, connect with non-parametric risk models, and use tail risk measures, such as conditional VaR.

Keywords Portfolio optimization · Black-Litterman · Implied return · Expected return · Leverage constraint

Introduction

It is well documented that the classic mean–variance optimization is very sensitive to small changes in expected returns.[1] An alternative and more robust approach is to calculate the implied returns given the allocation and risk profile of the current portfolio or benchmark, with the assumption that the current portfolio is efficient. Portfolio managers can then do a reality check on the implied returns and find opportunities for better allocations.

Numerous papers, such as Idzorek (2007), Guangliang and Litterman (2002) and Satchell and Scowcroft (2000), show the way of calculating implied returns with no leverage constraints. Sharpe (1974), Fisher (1975) and Herold (2005) calculated implied returns in a mean–variance optimization with leverage constraint.

We know that asset returns often have fatter tails than normal distribution. So portfolio optimization models based on downside risk measures, such as Value-at-Risk (VaR) and conditional VaR (CVaR), are researched in Lucas and Klaassen (1998), Fabozzi et al. (2007), Harlow (1991), and Krokhmal et al. (2002).[2] Pang and Karan (2018) in search of a close form of CVaR optimization generalized the implied return calculation using elliptical distribution.

In addition, the models mentioned above use variance/covariance, and thus cannot be easily implemented with commercial risk models using non-parametric approach, such as Monte Carlo simulation, or risk measures emphasizing tail risk, such as VaR and conditional VaR.

This paper proposes an implied return calculation that addresses the above issues.

The rest of this paper first review some existing models on implied returns, and then discusses the intuition on the objectives of portfolio optimization with leverage constraints and target returns. This analysis leads to the new implied return model. Finally, we use an example to illustrate the usage and advantages of this model.

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[1] See Fabozzi, F. J., Kolm, P. N., Pachamanova, D. A., and Focardi, S. M. "Robust Portfolio Optimization and Management". (John Wiley & Sons. 2007) p. 4, Black, Fischer, and Robert Litterman. 1992. Global portfolio optimization. Financial Analysts Journal 48(5): 28–43, and Litterman, Robert B. "Modern Investment Management: An Equilibrium Approach" (Hoboken, N.J.: John Wiley, 2003) p. 76, for detailed discussions.

[2] CVaR is a better risk measure than VaR for optimization, because VaR is not coherent (Krokhmal et al. 2002; Artzner et al. 1999). However, VaR is coherent where all portfolios can be modeled as linear combinations of elliptically distributed risk factors (McNeil et al. (2015), so optimization using VaR is reasonable for portfolios without derivatives or strategies with discrete return distributions.

Existing models on implied returns

The objective of mean–variance optimization is to maximize the expectation of utility:

$$U = r_p - \frac{\lambda}{2}\sigma_p^2 \quad (1)$$

where U is investor's utility, r_p is expected portfolio return, λ is risk aversion coefficient, σ_p is the volatility of portfolio.

If the portfolio has N assets, then

$$r_p = R'W, \quad (2)$$

where R is a N by 1 vector of asset returns and W is a N by 1 vector of asset weights. In the portfolio

$$\sigma_p^2 = W'\Sigma W, \quad (3)$$

where Σ is asset covariance matrix.

Assuming the current portfolio or benchmark allocation is efficient, the implied returns are [see Idzorek (2007)]

$$R_{\text{imp}} = \lambda \Sigma W, \quad (4)$$

Herold (2005) added a condition that the portfolio leverage is equal to 1, so the optimization problem (1) becomes to maximize

$$U = r_p - \frac{\lambda}{2}\sigma_p^2 + \theta(W'1_N - 1), \quad (5)$$

where θ is the Lagrange multiplier. Then,

$$R_{\text{imp}} = \lambda \Sigma W - \theta 1_N \quad (6)$$

Herold's model can be improved on three directions:

- First, the model uses covariance matrix of investment assets. In practice, institutional investment portfolios usually have hundreds or thousands of assets, so the covariance matrix is too large to be directly used in risk calculation.
- Second, the model cannot be easily generalized to tail risk measures, such as VaR or conditional VaR.
- Finally, the economic meaning of θ is unclear, other than the fact that it is the Lagrange multiplier used for optimization. Herold (2005) assumes two of the asset returns are known, so as to calibrate the Lagrange multiplier and risk aversion ratio. This might partially defeat the purpose of finding implied returns.

The new model proposed in this article will improve Herold's model on those directions. Before we dive into technical development, let's first look at the investment case we work with.

Proposed model

In practice, institutional investors often need to reach a target return with constraints on leverage. Active mutual funds, hedge funds, pension, insurance company and endowment funds usually have target returns in their marketing documents, offering memorandum, financial statements or investment policies. From modeling perspective, without target returns, there would be no basis for portfolio managers to choose the best leverage, and thus the constraints on leverage would be irrelevant. Exhibit 1 shows the efficient frontier and the capital market line.

There can be three cases on the leverage constraints:

- Case 1: If there's no leverage limit, the optimal portfolio is on the line of (r_f, b, d). This is the case for Eq. (4).
- Case 2: If the leverage = 1 at all times, the optimal portfolio is on the efficient frontier (a, b, c). This is the case for Eq. (6).
- Case 3: If the portfolio has maximum leverage limit at 1, but can invest in risk free asset, the optimal portfolio is on (r_f, b, c). This is the most realistic case. Obviously, implied return calculation must be a piece wise function.

In Exhibit 1,

Point a is the minimum variance portfolio, Point b is the tangency portfolio, r_f is the return of risk-free asset,r_b is the return of tangency portfolio.

Now, we introduce some propositions. The proofs are given in "Appendix 1".

To easily work with leverage constraint, we define r_p in the following form to separate the returns from investment assets and financing.

$$r_p = \sum_{i=1}^{N} w_i r_i - \left(\sum_{i=1}^{N} w_i - 1\right) r_f = W'R - W'1_N r_f + r_f, \quad (7)$$

where r_f is assumed to be fixed or known.

Proposition 1 By maximizing the utility function (1) with r_p defined in (7), the following equation holds:

$$r_i - r_f = \frac{r_p - r_f}{\sigma_p}\left(\frac{IVol_i}{w_i}\right), \quad (8)$$

for any $i \in [1, 2, \ldots, N]$, where r_i are asset returns, $IVol_i$ are incremental volatilities, defined as $IVol_i = w_i \frac{\partial \sigma_p}{\partial w_i}$.

We prove the proposition in our framework mathematically in "Appendix 1". The intuition of Eq. (8) is that the returns of assets contributing to the portfolio should be proportional with their contributions to risk.

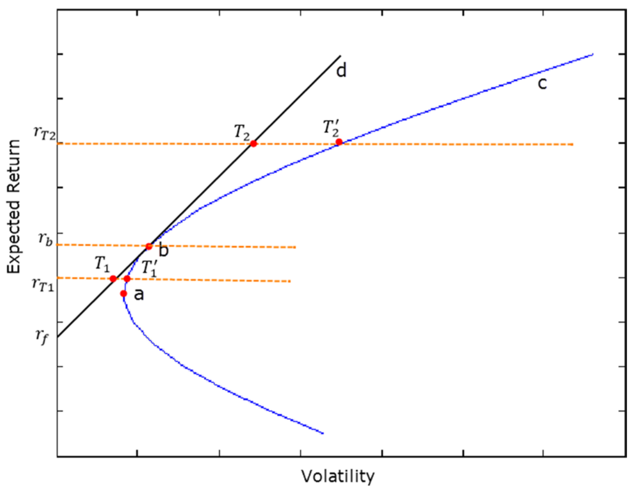

Exhibit 1 Efficient frontier

In fact, Eq. (8) would hold for a general form of risk measures, as long as the derivative of the risk measure with respect to W is well defined. Litterman (2004, pp. 14–15) has detailed proof on that argument.

The above result is obtained when there is no leverage constraint on portfolios. With that insight, when the leverage constraint ($\sum_{i=1}^{N} w_i = 1$) exists for portfolios, Eq. (8) can be reformulated by Proposition 2.

Proposition 2 Under the leverage constraint, by maximizing the utility function (1) with the constraint $\sum_{i=1}^{N} w_i = 1$, we obtain the following equation:

$$r_j - \frac{\sum_i^{i\neq j} w_i * r_i}{1 - w_j} = \varphi\left(\frac{IVol_j}{w_j} - \frac{\sigma_p - IVol_j}{1 - w_j}\right), \tag{9}$$

for any $i \in [1, 2, \ldots, N]$, where φ is a constant.

The proof is given in "Appendix 1". Due to the leverage constraint, when the allocation to asset j increases, the allocation to other assets must decrease proportionally. Thus, the left side of the equation represents the marginal impact to returns for increasing the allocation of asset j, which is the return of asset j, r_j, minus the average return of the rest of the portfolio, $\frac{\sum_i^{i\neq j} w_i * r_i}{1-w_j}$. The part in parenthesis on the right side of the equation is the marginal impact to portfolio volatility. That is, the change of volatility by adding/increasing asset j,$\left(\frac{IVol_j}{w_j}\right)$, minus the average change of volatility by reducing the rest of the portfolio, $\left(\frac{\sigma_p - IVol_j}{1-w_j}\right)$.

Similar to Eq. (8), the part on the left side of the equation must be proportional to the parenthesis term on the right side of equation, and we denote that constant proportion as φ. Intuitively, this ratio φ determines the risk-reward trade-off between asset returns and incremental volatility. That concept is similar to that of the Sharpe ratio.

Equation (9) can be further simplified as

$$r_i - r_p = \varphi\left(\frac{IVol_i}{w_i} - \sigma_p\right), \tag{10}$$

for any $i \in [1, 2, \ldots, N]$. Based on this result, we obtain the formula for implied returns as:

$$R_{imp} = 1_N r_p + \varphi\left(\gamma - 1_N \sigma_p\right), \tag{11}$$

where $\Upsilon = \left(\frac{IVol_1}{w_1}, \frac{IVol_2}{w_2}, \ldots, \frac{IVol_N}{w_N}\right)'$.

Again, like Eqs. (8), (9) (correspondingly (10) and (11)) holds for a general form of risk measures, as long as the derivative of the risk measure with respect to W is well defined.

The ratio $\varphi > 0$, due to the assumption of investor risk aversion. When φ is approaching zero, the asset implied returns are close to identical to each other. The higher the ratio φ, the higher the dispersion among the implied returns of assets. Here, r_p is the target portfolio return defined by portfolio managers.

Equation (11) can be calculated numerically with $IVol_i \underline{\underline{\mathrm{def}}} w_i \frac{\partial \sigma_p}{\partial w_i}$, or parametrically using

$$R_{\mathrm{imp}} = 1_N r_p + \varphi\left(\frac{\Sigma W}{\sigma_p} - 1_N \sigma_p\right) \tag{12}$$

Compare Eq. (12) with Eq. (6) in Herold (2005), they are consistent as one can choose

$$\lambda = \frac{\varphi}{\sigma_p} \quad \text{and} \quad \theta = \varphi\sigma_p - r_p, \tag{13}$$

where λ and θ are parameters used in Herold (2005). This relationship is also shown in our proof of Proposition 2.

In Herold's approach, the parameter θ is a Lagrange multiplier and its economic meaning is unclear. We replaced λ and θ with a function of portfolio target return r_p and a ratio φ. Those two new inputs are intuitive to investors. For many institutional investors, target returns are either explicitly or implicitly defined. For example, pension funds, insurance companies, endowments, foundation and family offices usually clearly define target returns in their investment and spending policies or financial statements; financial advisors often use target returns for financial planning; and hedge funds often describe their target return and risk profile in presentations to clients. The ratio φ determines the level of risk premium, such as equity premium and small cap premium, depending on the investment portfolio. The higher the ratio φ the higher the risk premium. Investors can adjust φ to arrive at a reasonable level of risk premium.

When there is no leverage constraint, from (8), $r_i = r_f + \frac{r_p - r_f}{\sigma_p}\left(\frac{IVol_i}{w_i}\right)$. Thus, the implied returns can be formulated as

$$R_{\text{imp}} = 1_N r_f + \varphi Y = 1_N r_p + \varphi\left(Y - 1_N \sigma_p\right),$$

where $\varphi = \frac{r_p - r_f}{\sigma_p}$. This equation is a special case of (11) with a fixed ratio $\varphi = \frac{r_p - r_f}{\sigma_p}$. Thus, the formulation of (11) is general. Note that under the definition of (7), Eq. (4) is slightly modified as $R_{imp} = 1_N r_f + \lambda \Sigma W$. Now, we see that this equation is also a special case of Eq. (11), where $\varphi = \frac{r_p - r_f}{\sigma_p}$ and $\lambda = \frac{\varphi}{\sigma_p}$. In fact, the relationship $\lambda = \frac{\varphi}{\sigma_p}$ is also established in our proof of Proposition 1. By choosing $\varphi = \frac{r_p - r_f}{\sigma_p}$, we can generalize the modified Eq. (4) with Eq. (11) to obtain

$$R_{imp} = 1_N r_f + \frac{r_p - r_f}{\sigma_p} Y. \tag{14}$$

Besides the intuitive inputs, this new model has two additional advantages:

First, because Eq. (11) holds for any general form of risk measures as long as the derivative of that risk measure to w_i is defined, we can replace *IVol* with incremental VaR[3] or incremental conditional VaR, if the portfolio manager focuses more on tail risk management. For example,

$$R_{\text{imp}} = 1_N r_p + \varphi_{\text{VaR}}\left(\gamma_{\text{VaR}} - 1_N VaR_p\right), \tag{15}$$

where $\gamma_{\text{VaR}} = \left(\frac{IVaR_1}{w_1}, \frac{IVaR_2}{w_2}, \ldots, \frac{IVaR_N}{w_N}\right)^T$.

In Eq. (15), we may choose φ_{VaR} directly. In order to make φ_{VaR} consistent with the level of φ when using volatility, we may calculate $\varphi_{VaR} = \frac{\varphi}{Z_p}\sqrt{n}$, where p is the confidence level of VaR, such as 0.95 and 0.99, Z_p is the Z-score according to the confidence level p, and n is the annualize factor of VaR, such as 252 for daily VaR or 52 for weekly VaR.

Second, in practice, institutional investment portfolios usually have so many assets that the covariance matrix of the assets is too large to be directly used in calculation. Equation (11) does not use covariance matrix as input, so σ_p can be calculated with Monte Carlo simulation or factor models, and $IVol_i$ can be calculated numerically with its definition $w_i \frac{\partial \sigma_p}{\partial w_i}$. With Eq. (11), risk managers can use outputs from commercial risk systems as inputs to calculate implied returns. See Mina and Xiao (2001), RiskMetrics (2009) and Barra (1998) for examples of how commercial risk systems calculate risk measures with simulations and factor models. See RiskMetrics (2009) for an example of calculating incremental VaR with numerical approach.

So far we have been assuming that there is no risk free asset in the portfolio. As mentioned earlier, when the target return is lower than the tangent portfolio return, a combination of risk-free asset and tangent portfolio is more efficient than efficient frontier. If the portfolio allocation has risk-free asset for investment, rather than liquidity purpose, then the part of the portfolio excluding risk-free asset is the tangent portfolio. We should then use Eq. (14) to calculate implied returns.

In practice, if the portfolio or benchmark has allocation to cash, we need to decide the purpose of the cash allocation. Suppose that the cash allocation in the portfolio is w_f.

1. If the cash allocation is for liquidity or consumption purpose, then that means there is a leverage constraint. In this case, we need to exclude the cash from the investment portfolio, and adjust the target return and risky asset weights using

$$\tilde{r}_p = \frac{r_p - r_f w_f}{1 - w_f} \quad \text{and} \quad \tilde{W} = \frac{W}{1 - w_f}, \tag{16}$$

where w_f is the weight for risk-free asset.

Then use $\tilde{r}_p$ and $\tilde{W}$ and Eqs. (11) or (12) to calculate implied returns.

2. If the cash allocation is for investment purpose, then that means the target return of the investor is so low that it can be achieved with some asset allocated to cash. In that case, leverage is not needed, so it's not a constraint anymore. Therefore, we should use the model without constraint, i.e., Eq. (14). Since the optimized portfolio is a combination of cash and tangency portfolio, we should use Eq. (16) to adjust r_p and W, and then use Eq. (14) to calculate implied returns.

That two-step process is consistent with the piecewise optimization function discussed earlier in Case 3 of the efficient frontier analysis.

Example analysis

Now let's look at an example. Suppose a portfolio manager starts from the initial portfolio or benchmark in Exhibit 2.

We now compare three approaches in calculating implied returns:

1. The original model, Eq. (4).
2. The new model with parametric approach, Eq. (12).

[3] Incremental VaR does not always exist, as VaR is non-convex and non-smooth as a function of positions in some special cases, but it exists for most investment portfolios or strategies, so is calculated by many commercial risk systems. See Rockafellar et al. (2000), Artzner et al. (1999) and RiskMetrics (2009).

3. The new model with historical simulation VaR, Eq. (15).

Let's assume

- Target return $r_p = 7\%$
- Risk aversion ratio used for Eq. (4) $\lambda = \frac{r_p}{\sigma_p^2} = 10$. Again, please note that $\lambda = \frac{r_p}{\sigma_p^2}$ does not necessarily hold for Eqs. (12) and (13).

  $\varphi = 0.4$

- Risk-free return $r_f = 0$
- Portfolio weight $\mathbf{W}$ is in Exhibit 2

We use the daily returns of the indices in Exhibit 2 from January 1st, 2011 to Sept. 30th, 2016, to obtain the rest of inputs to Eqs. (4), (12) and (15) as following:

- Obtain the annualized sample covariance matrix as an estimate of Σ. The annualized volatility and the corresponding sample correlation matrix based on the sample covariance matrix are given in "Appendix 2".
- Use Eq. (3) to calculate portfolio volatility $\sigma_p = 8.36\%$.
- Portfolio historical VaR, $\text{VaR}_\text{p} = 0.858\%$, is calculated as the 95th percentile of portfolio return from Eq. (2).
- Historical incremental VaR, $IVaR_i$, for asset $i \in [1, \ldots, N]$ is calculated with the simulation method detailed in RiskMetrics (2009). The results are in "Appendix 2".
- To make the results from VaR comparable to other models, use $\varphi_{\text{VaR}} = \frac{\varphi}{Z_\text{p}}\sqrt{n} = \frac{0.4}{1.645}\sqrt{252} = 3.86$.

The results are in Exhibit 3.

Here, we have some observations:

First, the implied returns from the new models have much lower equity risk premium than those from the original model, and the lower equity risk premiums are closer to reality. The implied returns from the original model are negative or close to zero for bonds and double digits for equities, and thus imply a double digit equity risk premium. According to Appendix 6 in Damodaran, et al. (2017), equity risk premium in the US market was never higher than 6.5% from 1961 to 2016. Additionally, projected equity risk premiums in the next 10–15 years by JP Morgan (2020) are in low single digit as well, closer to the results from the new models.

Second, the implied returns calculated with parametric approach are slightly different from those calculated with historical simulation VaR. For example, the implied returns for US Long-Term Bonds calculated by parametric approach is 2.6% more than those by VaR approach, while for US Large Cap Equity the difference is the other way around. From the following QQ plot (Exhibit 4), we can see that US Large Cap Equities not only have higher volatility, but also have larger left tail than US Long-Term Bonds. If the portfolio manager focuses more on the left tail and uses VaR as the risk measure, the implied or required return for US Long-Term Bonds should be even smaller.

Finally, the implied returns from the new model can be very useful for portfolio managers. For example, using the returns calculated with VaR, a portfolio manager may consider the 3.5% implied return for emerging market debt is low compared with Global ex-US Government Bonds. With that observation, the portfolio manager may increase allocation to Emerging Market Bonds and reduce allocation to ex-US Government Bonds until the implied returns are consistent with expectation.

One may have noticed that the sample portfolio is very similar to Morningstar Asset Allocation Index[4] and just doesn't have allocation to cash. If we use Morningstar Asset Allocation Index as the benchmark and think the allocation to cash from the index is for liquidity or consumption purpose, the above analysis is exactly what we should do. Of course, before calculating implied returns, we should first adjust target returns and risky asset weight using Eq. (16).[5]

If we believe that the 4% allocation to cash in the index is purely for investment purpose, we should use the implied returns from the original model. As discussed before, the large divergence of the equity and fixed income asset returns look too big to be realistic. That means the 40/60 asset allocation is not optimal, and the allocation to fixed income assets are too low. If we increase the allocation to fixed income assets, their implied returns will increase and implied returns for equities will decrease to more realistic levels. Intuitively, that makes a lot of sense. If the portfolio manager's target return is so low that he/she didn't even deploy 100% of assets, he/she probably should allocate more assets into fixed income than the typical 40/60 asset allocation.

Conclusion

To calculate implied returns, we introduced a new model that provide improvements for practical usage, and yet interconnected with existing models. The improvements include:

[4] Morningstar, "Morningstar Global Allocation Index Fact Sheet," accessed October 20, 2016, https://corporate.morningstar.com/US/documents/Indexes/MorningstarGlobalAllocationIndexFactSheet.pdf I replaced the Morningstar Emerging Market Index with MSCI Emerging Market index, as the history of the Morningstar index is incomplete on Bloomberg.

[5] To avoid too many decimal points and for illustration purpose, I rounded the weights calculated by Eq. (16) in sample portfolio earlier.

Exhibit 2 Sample Portfolio

Asset class	Index	Allocation %
Equities		*60*
U.S. large cap equities	Morningstar US Large Cap Index	13.5
U.S. mid cap equities	Morningstar US Mid Cap Index	11.5
U.S. small cap equities	Morningstar Small Cap Index	3
Non-U.S. developed equities	Morningstar Developed Markets ex-US Index	28
Emerging markets equities	MSCI Emerging Market Index	6
Fixed income		*40*
Long-term U.S. bonds	Morningstar Long-Term Core Bond Index	4
Intermediate-term U.S. bonds	Morningstar Intermediate Core Bond Index	11
Short-term U.S. bonds	Morningstar Short-Term Core Bond Index	3
Non-U.S. developed bonds	Morningstar Global ex-US Government Bond Index	15
Emerging markets bonds	Morningstar Emerging Markets Composite Bond Index	5

Exhibit 3 Implied returns of sample portfolio

Name	Implied returns		
	Original model (%)	New model with parametric approach (%)	New model with historical VaR (%)
Morningstar US large cap index	10.2	8.5	10.2
Morningstar US mid cap index	11.9	9.4	10.3
Morningstar small cap index	13.2	10.0	10.6
Morningstar developed markets ex-US index	11.7	9.3	9.5
MSCI emerging markets index	10.3	8.6	5.5
Morningstar long-term core bond index	−2.0	2.7	0.1
Morningstar intermediate core bond index	−0.3	3.5	2.6
Morningstar short-term core bond index	0.1	3.7	3.2
Morningstar global ex-US government bond index	−0.1	3.6	3.8
Morningstar emerging markets composite bond index	1.6	4.4	3.5

- First, this model is applicable for general form of risk measures so that investors who care about tail risk can use VaR or CVaR to calculate implied returns.
- Second, the model does not directly use the covariance matrix of assets in calculation, so that for many institutional portfolios where covariance matrix of assets is very large, risk measures can be calculated with simulation or factor models.
- Finally, the assumption parameters, including the target return r_p and ratio φ, are more intuitive than parameters in existing models as they have economic meanings.

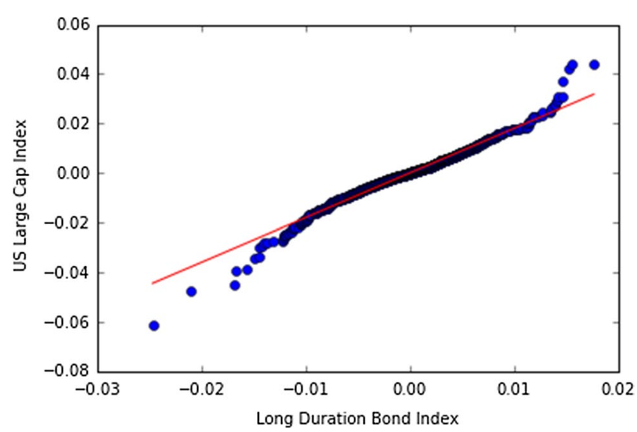

Exhibit 4 QQ plot of index returns

Appendix 1: Proofs of Propositions

Proof of Proposition 1 Under the definition (7), the utility function (1) becomes

Exhibit 5 Statistics on sample portfolio

Name	Weight (%)	CVaRAsset (%)	ICVaR (%)	ICVaRPct (%)	IVaR (%)	IVaRPct (%)	VaRAsset (%)	VolAsset (%)
US large cap	13.5	2.182	0.2382	18.6	0.225	26.80	1.469	14.43
US mid cap	11.5	2.508	0.2789	21.8	0.195	23.24	1.653	16.58
Small cap	3.0	2.809	0.3068	24.0	0.053	6.29	1.846	19.05
Developed markets ex-US	28.0	2.346	0.2697	21.1	0.416	49.54	1.569	15.37
Emerging markets	6.0	2.353	0.2400	18.8	0.028	3.31	1.666	16.00
Long-term core bond	4.0	1.099	−0.0641	−5.0	−0.038	−4.47	0.845	7.92
Intermediate core bond	11.0	0.363	−0.0149	−1.2	−0.032	−3.85	0.254	2.57
Short-term core bond	3.0	0.154	−0.0001	0.0	−0.005	−0.54	0.101	1.10
Global ex-US government bond	15.0	0.392	−0.0091	−0.7	0.000	0.01	0.246	2.73
Emerging markets composite bond	5.0	0.852	0.0338	2.6	−0.003	−0.34	0.359	6.58
Portfolio Vol	8.357							
Portfolio VaR	0.858							
Portfolio CVaR	1.301							

$$U = r_p - \frac{\lambda}{2}\sigma_p^2 = W'R - W'1_N r_f + r_f - \frac{\lambda}{2}W'\Sigma W \tag{17}$$

To maximize (17), consider

$$\frac{\partial U}{\partial W} = R - 1_N r_f - \lambda \Sigma W = 0.$$

Then, we have

$$R = 1_N r_f + \lambda \Sigma W. \tag{18}$$

Since $\frac{\partial \sigma_p}{\partial W} = \frac{\Sigma W}{\varsigma_p}$, then $\Sigma W = \sigma_p\left(\frac{IVol_1}{w_1}, \frac{IVol_2}{w_2}, \ldots, \frac{IVol_N}{w_N}\right)'$, and thus $\left(r_1 - r_f, \ldots, r_N - r_f\right)' = \lambda\sigma_p\left(\frac{IVol_1}{w_1}, \ldots, \frac{IVol_N}{w_N}\right)'$. Correspondingly,

$$r_i - r_f = \lambda\sigma_p(IVol_i/w_i), \tag{19}$$

for any $i \in [1, 2, \ldots, N]$. Now from (18), it is easy to see that $W'R - W'1_N r_f = r_p - r_f = \lambda\sigma_p^2$ and thus $\frac{r_p - r_f}{\sigma_p} = \lambda\sigma_p$. Consequently, $r_i - r_f = \frac{r_p - r_f}{\sigma_p}\left(IVol_i/w_i\right)$, for any $i \in [1, 2, \ldots, N]$. This completes the proof. From the proof, we see that $\lambda = \frac{r_p - r_f}{\sigma_p^2}$.

Proof of Proposition 2 When the leverage constraint ($\sum_{i=1}^N w_i = 1$) exists for portfolios, we have $r_p = \sum_i w_i r_i = W'R$ with $\sum_{i=1}^N w_i = 1$, according to (7). To maximize the utility function (1) under the leverage constraint, we consider the utility function incorporating the constraint as:

$$U_1 = r_p - \frac{\lambda}{2}\sigma_p^2 + \theta(W'1_N - 1) = W'R - \frac{\lambda}{2}W'\Sigma W + \theta W'1_N - \theta. \tag{20}$$

By setting $\frac{\partial U_1}{\partial W} = R + 1_N\theta - \lambda\Sigma W = 0$, we have $R = -1_N\theta + \lambda\Sigma W$. Then, following the proof of Proposition 1 with r_f replaced by $-\theta$, we can obtain

$$r_j + \theta = \frac{r_p + \theta}{\sigma_p}\left(IVol_j/w_j\right), \tag{21}$$

for any $j \in [1, 2, \ldots, N]$.

Let $\varphi = \frac{r_p + \theta}{\sigma_p}$. Then, we have $(r_j + \theta)w_j = \varphi IVol_j$, for $j \in [1, 2, \ldots, N]$, and $r_p + \theta = \varphi\sigma_p$. Therefore, $r_p + \theta - (r_j + \theta)w_j = \varphi(\sigma_p - IVol_j)$. Correspondingly, $r_p - w_j r_j + \theta\left(1 - w_j\right) = \varphi(\sigma_p - IVol_j)$. Under the leverage constraint $\sum_{i=1}^N w_i = 1$, it follows that $\sum_i^{i\neq j} w_i r_i + \theta\left(1 - w_j\right) = \varphi(\sigma_p - IVol_j)$, and thus

$$\frac{\sum_i^{i\neq j} w_i r_i}{1 - w_j} + \theta = \varphi\left(\frac{\sigma_p - IVol_j}{1 - w_j}\right). \tag{22}$$

Subtracting (22) from (21), we have $r_j - \frac{\sum_i^{i\neq j} w_i * r_i}{1 - w_j} = \varphi\left(\frac{IVol_j}{w_j} - \frac{\sigma_p - IVol_j}{1 - w_j}\right)$, for any $j \in [1, 2, \ldots, N]$. This completes the proof. From the proof, we see the relationship $\theta = \varphi\sigma_p - r_p$. With this new ratio φ, the proposed model has more economic meanings and is more interpretable.

Exhibit 6 Correlation matrix of sample portfolio

	Large cap (%)	Mid cap (%)	Small cap (%)	Non-US equities (%)	EM equities (%)	Long-term (%)	Intermediate-term (%)	Short-term (%)	Non-US bonds (%)	EM bonds (%)
Large cap	100	95	91	60	47	−43	−28	−12	−14	14
Mid cap	95	100	97	61	49	−42	−26	−10	−12	14
Small cap	91	97	100	58	47	−42	−27	−11	−12	12
Non-US equities	60	61	58	100	78	−28	−15	6	−13	25
EM equities	47	49	47	78	100	−21	−11	8	−7	32
Long-term	−43	−42	−42	−28	−21	100	84	64	43	10
Intermediate-term	−28	−26	−27	−15	−11	84	100	80	41	11
Short-term	−12	−10	−11	6	8	64	80	100	29	19
Non-US bonds	−14	−12	−12	−13	−7	43	41	29	100	14
EM bonds	14	14	12	25	32	10	11	19	14	100

Appendix 2: Statistics for the sample portfolio

For the risk calculation of the sample portfolio, we use the daily returns from January 1st, 2011 to September 30th, 2016. Following are some statistics of the sample portfolio (Exhibits 5, 6).

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