RESOLVENT ESTIMATES ON ASYMPTOTICALLY CYLINDRICAL MANIFOLDS AND ON THE HALF LINE

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Abstract. Manifolds with infinite cylindrical ends have continuous spectrum of increasing multiplicity as energy grows, and in general embedded resonances (resonances on the real line, embedded in the continuous spectrum) and embedded eigenvalues can accumulate at infinity. However, we prove that if geodesic trapping is sufficiently mild, then the number of embedded resonances and eigenvalues is finite, and moreover the cutoff resolvent is uniformly bounded at high energies. We obtain as a corollary the existence of resonance free regions near the continuous spectrum.

We also obtain improved estimates when the resolvent is cut off away from part of the trapping, and along the way we prove some resolvent estimates for repulsive potentials on the half line which may be of independent interest.

1. Introduction

1.1. **Resolvent estimates for manifolds with infinite cylindrical ends.** The high energy behavior of the Laplacian on a manifold of infinite volume is, in many situations, well known to be related to the geometry of the *trapped set*; this is the set of bounded maximally extended geodesics. In the best understood cases, such as when the manifold has asymptotically Euclidean or hyperbolic ends (see [Zw2, §3] for a recent survey), the trapped set is compact. Some results have been obtained for more general trapped sets (e.g. manifolds with cusps were studied in [CaVo]) but less detailed information is available.

In this paper we study manifolds with infinite asymptotically cylindrical ends, which have noncompact trapped sets. A motivation for this study comes from waveguides and quantum dots connected to leads. The spectral geometry of these is closely related to that of asymptotically cylindrical manifolds, and they appear in certain models of electron motion in semiconductors and of propagation of electromagnetic and sound waves. We give just a few pointers to the physics and applied math literature here [LoCaMu, Ra, RaBaBaHu, ExKo, BoGaWo]. In [ChDa2], we prove analogues of some of the results below for suitable (star-shaped) waveguides.

The fundamental example of a manifold with cylindrical ends is the Riemannian product $\mathbb{R}\times\mathbb{S}^1$, which has an unbounded trapped set consisting of the circular geodesics. We are interested in the behavior of the resolvent of the Laplacian (and its meromorphic continuation, when this exists) for perturbations of such cylinders and their generalizations. As we discuss below, this behavior can sometimes be very complicated, but we show that if some geometric properties of the manifold are favorable, then the resolvent is uniformly bounded at high energy. In the companion paper [ChDa1], we study the closely related problem of long time wave asymptotics on such manifolds.

We begin with an illustration of a more general theorem to follow, by stating a high energy resolvent estimate for two kinds of *mildly trapping* manifolds (X, g) with infinite cylindrical ends.

Example 1. Let (r, θ) be polar coordinates in $\mathbb{R}^d$ for some $d \geq 2$, and let

$$X = \mathbb{R}^d, \qquad g_0 = dr^2 + F(r)dS,$$

where dS is the usual metric on the unit sphere, $F(r) = r^2$ near $r = 0$, and F' is compactly supported on some interval $[0, R]$ and positive on $(0, R)$; see Figure 1.1.

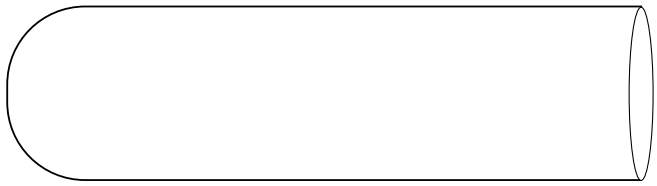

FIGURE 1.1. A cigar-shaped warped product.

Then for $r(t) > 0$ all g_0-geodesics obey

$$\ddot{r}(t) := \frac{d^2}{dt^2} r(t) = 2|\eta|^2 F'(r(t)) F(r(t))^{-2} \geq 0,$$

where $r(t)$ is the r coordinate of the geodesic at time t and η is the angular momentum. Consequently, the only trapped geodesics are the ones with $\dot{r}(t) \equiv F'(r(t)) \equiv 0$, that is the circular ones in the cylindrical end. This is the smallest amount of trapping a manifold with a cylindrical end can have.

Let g be any metric such that $g - g_0$ is supported in $\{(r, \theta) \mid r < R\}$, and such that g and g_0 have the same trapped geodesics. For example we may take $g = g_0 + cg_1$, where g_1 is any symmetric two-tensor with support in $\{(r, \theta) \mid r < R\}$, and $c \in \mathbb{R}$ is chosen sufficiently small depending on g_1. Alternatively, we may take $g = dr^2 + g_S(r)$, where $g_S(r)$ is a smooth family of metrics on the sphere such that $g_S(r) = r^2 dS$ near $r = 0$ and $g_S(r) = F(r)dS$ near $r \geq R$, and such that $\partial_r g_S(r) > 0$ on $(0, R)$. This way we can construct examples where $g - g_0$ is not small.

Example 2. Let (X, g_H) be a convex cocompact hyperbolic surface, such as the symmetric hyperbolic 'pair of pants' surface with three funnels depicted in Figure 1.2.

In particular, there is a compact set $N \subset X$ (the convex core of X) such that

$$X \setminus N = (0, \infty)_r \times Y_y, \qquad g_H|_{X \setminus N} = dr^2 + \cosh^2 r \, dy^2,$$

where Y is a disjoint union of $k \geq 1$ geodesic circles (possibly having different lengths).

We modify the metric in the funnel ends so as to change them into cylindrical ends in the following way. Take g such that

$$g|_N = g_H|_N, \qquad g|_{X \setminus N} = dr^2 + F(r)dy^2,$$

where $F(r) = \cosh^2 r$ near $r = 0$, and F' is compactly supported and positive on the interior of the convex hull of its support.

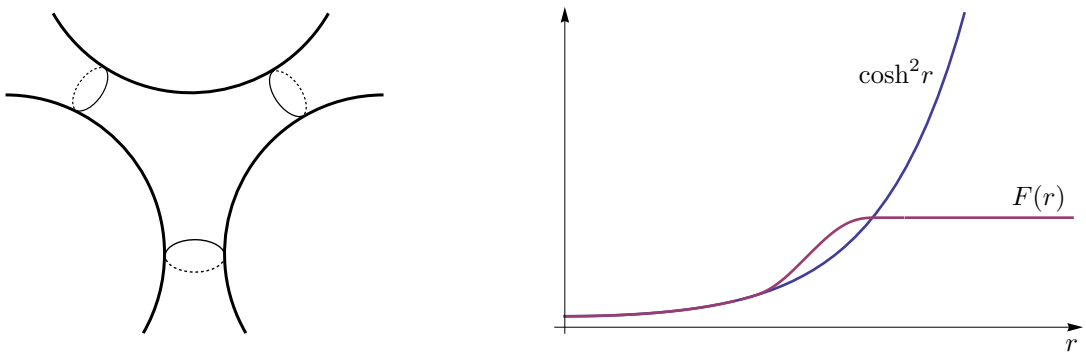

FIGURE 1.2. A hyperbolic surface (X, g_H) with three funnels, and a modification of the metric which changes the funnel ends to cylindrical ends.

To obtain higher dimensional examples, we can take (X, g_H) to be a conformally compact manifold of constant negative curvature, with dimension $d \geq 3$, but in this case we need the additional assumption that the dimension of the limit set is less than $(d-1)/2$. The construction of g now becomes more complicated and we give it in §3.3 below.

Our first result concerns only the above examples.

Theorem 1.1. *Let (X, g) be as in Example 1 or 2 above, and let $\Delta \leq 0$ be its Laplacian. There is $z_0 > 0$ such that for any $\chi \in C_c^\infty(X)$ there is $C > 0$ such that*

$$\|\chi(-\Delta - z)^{-1}\chi\|_{L^2(X)\to L^2(X)} \leq C, \tag{1.1}$$

for all $z \in \mathbb{C}$ with $\operatorname{Re} z \geq z_0$ and $\operatorname{Im} z \neq 0$.

Here $(-\Delta - z)^{-1}$ denotes the standard resolvent which maps $L^2(X) \to L^2(X)$, and not its meromorphic continuation. Below, in Theorem 5.6, we also obtain bounds for the meromorphic continuation, but these are more complicated to state.

The bound (1.1) is optimal in the sense that we cannot replace the right hand side by a function of z which tends to 0 as $\operatorname{Re} z \to \infty$. Indeed, taking the case of Example 1 with $d = 2$ for definiteness, we have $(-\Delta - k^2)v(r)e^{ik\theta} = -v''(r)e^{ik\theta}$ for any $v \in C_c^\infty((R, \infty))$ and $k \in \mathbb{Z}$.

Note also that the resolvent in these examples is better behaved than it is for the (geometrically simpler) Riemannian product $(X, g) = (\mathbb{R} \times Y, g = dr^2 + g_Y)$, where (Y, g_Y) is a compact Riemannian manifold. Indeed, take $\chi \in C_c^\infty(X)$ a function of r such that $\chi \geq 0$ and $\chi \not\equiv 0$, and take $\chi_0 \in C_c^\infty(X)$ such that $\chi_0\chi = \chi$, and let ϕ be an eigenfunction of the Laplacian on (Y, g_Y) with $-\Delta\phi = \sigma^2\phi$. Then, by separation of variables,

$$\|\chi(-\Delta - z)^{-1}\chi\chi_0\phi\|_{L^2(X)} = \|\phi\|_{L^2(Y)}\|\chi(-\partial_r^2 - z + \sigma^2)^{-1}\chi\|_{L^2(\mathbb{R})} \xrightarrow{z\to\sigma^2} +\infty, \tag{1.2}$$

where we take the limit using the explicit formula for the resolvent [DyZw, (2.2.1)]. For our proof of Theorem 1.1 it will be crucial that $F' > 0$ near the cylindrical ends in Examples 1 and 2, and this is what is missing in the Riemannian product just discussed.

We will deduce Theorem 1.1 from Theorem 3.1 below, which gives a stronger result (allowing χ to be replaced by a noncompactly supported weight) and also applies to Schrödinger operators on more general manifolds with asymptotically cylindrical ends. We will further prove in Theorem 3.2 that we can obtain stronger resolvent bounds by suitably refining the cutoffs χ.

An estimate like (1.1) has well-known implications for the spectrum of $-\Delta$. In particular, by [ReSi, Theorem XIII.20], the spectrum is purely absolutely continuous on (z_0, ∞), which rules out any embedded eigenvalues there, and we will see below, in §5, that embedded resonances (resonances on the real line, embedded in the continuous spectrum) are also ruled out.

To our knowledge ours is the first result ruling out the presence of infinitely many embedded eigenvalues or resonances for a large class of examples of manifolds with infinite cylindrical ends.

The situation can be very different for other manifolds with cylindrical ends. For example, let $X = \mathbb{R} \times Y$ and $g = dr^2 + F(r)g_Y$, where (Y, g_Y) is a compact Riemannian manifold and $F \in C^\infty(\mathbb{R}; (0, \infty))$, $1 - F$ is compactly supported, and $\max F > 1$. Then $-\Delta$ has infinitely many embedded eigenvalues converging to $+\infty$ ([ChZw, §3], [Pa2, (3.6)]).

The study of the spectral and scattering theory of the Laplacian on manifolds with cylindrical ends, and their perturbations, goes back to Guillopé [Gu] and Melrose [Me] and is an active and wide-ranging area of research: see for example [IsKuLa, MüSt, RiTi] for some recent results and more references. There is also a large of body of literature on the closely related study of the Laplacian on waveguides: something of a survey can be found in [KrKř], and let us also mention the older result [Go], and that there is a nonexistence result for eigenvalues in [DaPa]. In a slightly different direction, weighted resolvent estimates up to the spectrum and limiting absorption principles have been investigated using Mourre theory [Mo, AmBoGe, DeGé], and this has been applied to geometric situations such as ours in [Ni].

Our results also have implications for the distribution of *resonances*; these are the poles of the meromorphic continuation of the resolvent, and their study in this context also goes back to [Gu, Me]. An existence result for resolvent poles (in the presence of appropriate quasimodes, and which may be embedded in the real line or complex) on waveguides can be found in [Ed], and for more such results see [KrKř]. Upper bounds on the number resonances for manifolds with infinite cylindrical ends are given in [Ch1].

In Theorem 5.6, we will use an identity due to Vodev [Vo] to prove that (1.1) (or a more general resolvent estimate up to the spectrum) implies the existence of a resonance free region near the continuous spectrum. In a companion paper to this one, [ChDa1], we use these results to prove an asymptotic expansion for solutions to the wave equation.

1.2. **Repulsive potentials on the half line.** In this paper we also obtain some resolvent estimates for Schrödinger operators on the half line which we need in the course of the proofs of our main results, and which may be of independent interest. We state them here.

Let V_D be a bounded, nonnegative, nonincreasing potential on the half line, such that

$$V_D'(r) \leq -\delta_V (1 + r)^{-1} V_D(r) \leq 0, \tag{1.3}$$

for some $\delta_V > 0$ and for all $r \geq 0$, where if V_D is not everywhere differentiable then (1.3) is meant in the sense of measures. Note that in particular the potential is repulsive in the sense of classical mechanics, since $V_D'(r) < 0$ except where $V_D(r) = 0$.

For $h > 0$ and $\zeta \in \mathbb{C} \setminus [0, \infty)$ let

$$(-h^2\partial_r^2 + V_D - \zeta)^{-1}$$

denote the Dirichlet resolvent. In this paper we prove the following semiclassical resolvent estimates:

Theorem 1.2. *For all $s,\ s_1,\ s_2 > 1/2$ with $s_1 + s_2 > 2$ there is $C > 0$ such that for all $\zeta \in \mathbb{C} \setminus [0, \infty)$ and $h > 0$ we have*

$$\|(1+r)^{-s}(-h^2\partial_r^2 + V_D(r) - \zeta)^{-1}(1+r)^{-s}\| \leq \frac{C}{h\sqrt{|\zeta|}}, \tag{1.4}$$

$$\|(1+r)^{-s_1}(-h^2\partial_r^2 + V_D(r) - \zeta)^{-1}(1+r)^{-s_2}\| \leq \frac{C}{h^2}, \tag{1.5}$$

and

$$\|V_D(r)^{1/2}(1+r)^{-1/2}(-h^2\partial_r^2 + V_D(r) - \zeta)^{-1}(1+r)^{-s}\| \leq \frac{C}{h}, \tag{1.6}$$

where the norms are $L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$.

Recall that, in the case $V_D \equiv 0$, (1.4) and (1.5) are well known to be sharp as $\operatorname{dist}(\zeta, [0, \infty)) \to 0$; this can be checked from the explicit formula for the resolvent in that case, which we give below in (5.4).

In fact, we will deduce these estimates from some uniform estimates for Schrödinger operators with repulsive potentials, replacing C by an explicit constant. To state them, let

$$P_D := -\partial_r^2 + V_D(r),$$

regarded as a self-adjoint operator on $L^2(\mathbb{R}_+)$ with domain $\{u \in H^2(\mathbb{R}_+) \mid u(0) = 0\}$.

Theorem 1.3. *For all $\delta > 0$, $\theta \in [0, 1]$, and $z \in \mathbb{C} \setminus [0, \infty)$, we have*

$$\|(1+r)^{-\frac{1+\delta}{2}}(P_D - z)^{-1}(1+r)^{-\frac{1+\delta}{2}}\| \leq \frac{1+\sqrt{2}}{\sqrt{|z|}}\left(\frac{1}{\delta} + \frac{1}{\delta_V}\right), \tag{1.7}$$

$$\|(1+r)^{-\frac{1+\delta}{2}-\theta}(P_D - z)^{-1}(1+r)^{-\frac{1+\delta}{2}-(1-\theta)}\| \leq (1+\sqrt{2})\left(\frac{1}{\delta} + \frac{1}{\delta_V}\right), \tag{1.8}$$

and

$$\|V_D(r)^{\frac{\theta}{2}}(1+r)^{-\frac{1+(1-\theta)\delta}{2}}(P_D - z)^{-1}V_D(r)^{\frac{1-\theta}{2}}(1+r)^{-\frac{1+\theta\delta}{2}}\| \leq \frac{2\sqrt{2}}{\delta_V}\sqrt{1 + \frac{\delta_V}{\delta}}, \tag{1.9}$$

where the norms are $L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$.

Note that Theorem 1.3 implies Theorem 1.2.

If $V_D \in C^1([0, \infty))$ is compactly supported and has $V_D' < 0$ on the interior of the support of V_D, then (1.3) is satisfied for some $\delta_V > 0$ (because $\log V_D$ and $(\log V_D)'$ tend to $-\infty$ at the boundary of the support). Moreover the class of potentials satisfying (1.3) for a given $\delta_V > 0$ is closed under

nonnegative linear combinations and contains all functions of the form $(1+r)^{-m}$ with $m \geq \delta_V$. The same proof could also handle potentials V_D satisfying (1.3) and such that $V_D(r) \to \infty$ as $r \to 0$, provided $V_D(r)|u(r)|^2 \to 0$ as $r \to 0$ for all u in the domain of P_D.

The bounds (1.4) and (1.7) are best when the spectral parameter is not too close to 0, and (1.5) and (1.8) are best when the spectral parameter is close to 0. We can think of (1.6) and (1.9) as being a kind of Agmon or elliptic estimate in the limit $|z| \to 0$ (see also (4.14) below); they give an improvement when we are looking at the resolvent in the elliptic and classically forbidden range in the interior of the support of V_D. When $V_D(r) \sim (1+r)^{-m}$ as $r \to \infty$ for some $m > 0$, the weights in (1.9) are also to be compared to the weights in [Ya, Na]; see in particular [Na, Theorem 1.3].

If we do not demand explicit constants in the estimates, then Theorem 1.3 is essentially well-known if either $V_D(0)$ (which we can think of as a coupling constant) is not large (see [Ya, Chapter 4] for a more general discussion of scattering on the half line, and [KoTr] for some more recent results and references), or if $V_D(0)$ and $|z|$ are large (this is the semiclassical, nontrapping regime: see [Ya, Chapter 7, Theorem 1.6] for a similar result). The main novelty here is that we cover all values of $V_D(0)$ and $|z|$ uniformly, and for our applications in §3 we will especially need the case where $V_D(0)$ is large compared to $|z|$: this corresponds to a low-energy semiclassical problem.

We prove Theorem 1.3 in §2 below.

1.3. **Notation.** Throughout the paper C is a large constant which can change from line to line, and all estimates are uniform for $h \in (0, h_1]$, where h_1 can change from line to line. It will sometimes be convenient to write derivatives with respect to r using the notation $D_r := -i\partial_r$. We use

$$\|u\|_{H_h^m(X)} := \|(-h^2\Delta + 1)^{m/2} u\|_{L^2(X)},$$

and similarly define $\|u\|_{H_h^m(\mathbb{R})}$ and $\|u\|_{H_h^m(\mathbb{R}_+)}$ (in the latter case we will only be concerned with u vanishing near $r = 0$, so the boundary condition on the Laplacian implicit in the notation in this case is immaterial).

The energy level $E_0 > 0$ is fixed in §3.1, along with the rest of the notation needed for our general abstract setup of a mildly trapping Schrödinger operator on a manifold with asymptotically cylindrical ends. The auxiliary notations E_j and E_* are defined in §4.2 in terms of this setup. The notation E without a subscript is used in §2 and §5 to denote a variable positive energy, not related in any particular way to E_0 or E_j or E_*.

The radial variable r on the cylindrical end has the same meaning in §3.1, in §4, and in §5. The usage in §2 is consistent with this usage, if we separate variables to write Schrödinger operator on an asymptotically cylindrical end as a sum of Schrödinger operators on $\mathbb{R}_+$. For example, if Δ is the Laplacian on $((0,\infty) \times Y, dr^2 + g_Y)$ we write

$$-\Delta = \sum_{j=0}^{\infty} (-\partial_r^2 + \sigma_j^2)\phi_j \otimes \phi_j, \text{ to mean } -\Delta u = \sum_{j=0}^{\infty} \phi_j \int_Y (-\partial_r^2 + \sigma_j^2) u(r,y)\phi_j(y) d\mathrm{vol}(y),$$

where $\{\phi_j\}_{j=0}^\infty$ is a complete set of real-valued orthonormal eigenfunctions of the Laplacian on Y and $-\Delta_Y \phi_j = \sigma_j^2 \phi_j$.

Of course the results of §2 also apply to more general Schrödinger operators on $\mathbb{R}_+$.

The variable r is used a little differently in §1.1, §3.3, and §3.4. To convert the r in one of these sections to the r in the rest of the paper, use the affine map

$$r \mapsto 6(r - R_1)/(R_2 - R_1), \tag{1.10}$$

for suitably chosen R_1 and R_2, and then multiply g by $(R_2 - R_1)^2/36$ to remove the factor that appears in front of dr^2. For Example 1, take R_1 such that $\inf\{r > 0 \mid g(r, y) = g_0(r, y) \text{ for all } y\} < R_1 < R$ and use $R_2 = R$. For Example 2, let $R_2 = \max \operatorname{supp} F'$, and take $R_1 \in (0, R_2)$. For §3.3, let $R_1 = R + 1$ and $R_2 = \max \operatorname{supp} F'$. For §3.4, let $R_1 = R/2$ and $R_2 = R$.

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2. Resolvent estimates on the half line

In this section we prove Theorem 1.3. All function norms and inner products in this section are in $L^2(\mathbb{R}_+)$, and operator norms are $L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$.

Proof of (1.7). Let $E := \operatorname{Re} z$ and $\varepsilon := |\operatorname{Im} z|$. We begin by proving an a priori estimate when $E > 0$ and $\varepsilon > 0$. Roughly speaking, the idea is to exploit the fact that, since $V_D' \le 0$, we have the positive commutator $[P_D, r\partial_r] = -2\partial_r^2 - rV_D'(r) \ge 0$. However, to be able to control the remainder terms in our positive commutator argument, we must replace $r\partial_r$ with $w(r)\partial_r$ where w grows more slowly. Such commutants have been used by many authors (see [ReSi, §XIII.7] and references therein); below we take an approach inspired by [Vo, Da1] and papers cited therein.

Take $w \in C^1([0, \infty); [0, 1])$ such that $w'(r) > 0$ for all $r \ge 0$, and take $u \in H^2(\mathbb{R}_+)$ such that $u(0) = 0$ and $(w')^{-1/2}(P_D - z)u \in L^2$; in particular, $u(r)$ and $u'(r)$ tend to 0 as $r \to \infty$. Adding together the integration by parts identities

$$-\langle (w(V_D - E))' u, u\rangle = 2 \operatorname{Re}\langle w(V_D - E)u, u'\rangle,$$

and

$$\langle w'u', u'\rangle + w(0)|u'(0)|^2 = -2\,\mathrm{Re}\langle wu'', u'\rangle,$$

gives

$$E\|\sqrt{w'}u\|^2 + \|\sqrt{w'}u'\|^2 - \langle (wV_D)'u, u\rangle + w(0)|u'(0)|^2 = 2\,\mathrm{Re}\langle w(P_D - z)u, u'\rangle - 2\,\mathrm{Im}\, z\,\mathrm{Im}\langle wu, u'\rangle.$$

Since $0 \le w \le 1$, this implies

$$E\|\sqrt{w'}u\|^2 + \|\sqrt{w'}u'\|^2 - \langle (wV_D)'u, u\rangle \le 2\left\|\frac{1}{\sqrt{w'}}(P_D - z)u\right\| \|\sqrt{w'}u'\| + 2\varepsilon\|u\|\|u'\|. \tag{2.1}$$

Later we will choose w so that $(wV_D)' \le 0$, but first we estimate the second term on the right, which we think of as a remainder term. Since $V_D \ge 0$, integrating by parts gives

$$\|u'\|^2 \le \mathrm{Re}\langle (P_D - z)u, u\rangle + E\|u\|^2 \le \left\|\frac{1}{\sqrt{w'}}(P_D - z)u\right\| \|\sqrt{w'}u\| + E\|u\|^2,$$

and we also have

$$\varepsilon\|u\|^2 = |\,\mathrm{Im}\langle (P_D - z)u, u\rangle| \le \left\|\frac{1}{\sqrt{w'}}(P_D - z)u\right\| \|\sqrt{w'}u\|.$$

Combining these gives

$$\varepsilon^2\|u\|^2\|u'\|^2 \le (E + \varepsilon)\left\|\frac{1}{\sqrt{w'}}(P_D - z)u\right\|^2 \|\sqrt{w'}u\|^2,$$

and then plugging this into (2.1) gives

$$E\|\sqrt{w'}u\|^2 + \|\sqrt{w'}u'\|^2 - \langle (wV_D)'u, u\rangle \le 2\left\|\frac{1}{\sqrt{w'}}(P_D - z)u\right\| \left(\|\sqrt{w'}u'\| + \sqrt{E+\varepsilon}\|\sqrt{w'}u\|\right).$$

Completing the square gives

$$\begin{aligned}\left(\sqrt{E}\|\sqrt{w'}u\| - \frac{\sqrt{E+\varepsilon}}{\sqrt{E}}\left\|\frac{1}{\sqrt{w'}}(P_D - z)u\right\|\right)^2 + \left(\|\sqrt{w'}u'\| - \left\|\frac{1}{\sqrt{w'}}(P_D - z)u\right\|\right)^2 \\ -\langle (wV_D)'u, u\rangle \le \frac{2E+\varepsilon}{E}\left\|\frac{1}{\sqrt{w'}}(P_D - z)u\right\|^2.\end{aligned} \tag{2.2}$$

We now take

$$w(r) := 1 - \frac{\delta_V}{\delta_V + \delta}(1 + r)^{-\delta}, \tag{2.3}$$

so that, by (1.3), we have

$$(wV_D)'(r) = \frac{\delta\delta_V V_D(r)}{(\delta_V + \delta)(1 + r)^{1+\delta}} + w(r)V_D'(r) \le \frac{\delta_V V_D(r)}{1 + r}\left((1 + r)^{-\delta} - 1\right) \le 0, \tag{2.4}$$

where, as with (1.3), we understand (2.4) in the sense of measures in the case that V_D is not differentiable everywhere. We may now drop the second and third terms from the left hand side of (2.2), giving

$$\sqrt{E}\|\sqrt{w'}u\| \le \frac{\sqrt{E+\varepsilon} + \sqrt{2E+\varepsilon}}{\sqrt{E}}\left\|\frac{1}{\sqrt{w'}}(P_D - z)u\right\|. \tag{2.5}$$

From (2.5) we can deduce a weighted resolvent estimate when $\operatorname{Re} z > 0$, $\operatorname{Im} z \neq 0$. To obtain an estimate for all $z \in \mathbb{C} \setminus [0, \infty)$, we use the Phragmén–Lindelöf principle in the following way. For $u,\ v \in L^2(\mathbb{R}_+)$, put

$$U(z) := \langle (1+r)^{-\frac{1+\delta}{2}} (P_D - z)^{-1} (1+r)^{-\frac{1+\delta}{2}} u, v \rangle \sqrt{z}, \tag{2.6}$$

and for $\alpha > 0$ put

$$\Omega_\alpha := \{ z \in \mathbb{C} \mid \alpha \operatorname{Re} z < |\operatorname{Im} z| \}.$$

Then U is holomorphic in Ω_α, where it obeys

$$|U(z)| \leq \frac{|\sqrt{z}| \|u\| \|v\|}{\operatorname{dist}(z, [0, \infty))} \leq \frac{\sqrt{1+\alpha^{-2}} \|u\| \|v\|}{|\sqrt{z}|}$$

Moreover, by (2.5), for $z \in \partial\Omega_\alpha \setminus \{0\}$, we have

$$|U(z)| \leq \left(\sqrt{1+\alpha} + \sqrt{2+\alpha} \right) \left(\delta^{-1} + \delta_V^{-1} \right) \|u\| \|v\|. \tag{2.7}$$

Then the Phragmén–Lindelöf principle (see e.g. [ReSi, p. 236]) implies (2.7) for all $z \in \Omega_\alpha$. Taking $\alpha \to 0$ gives (1.7). □

Proof of (1.8). We begin by following the proof of (1.7), but we drop the first term, rather than the second, from the left hand side of (2.2), so that in place of (2.5) we have

$$\|\sqrt{w'} u'\| \leq \left(1 + \sqrt{2 + \varepsilon E^{-1}} \right) \left\| \frac{1}{\sqrt{w'}} (P_D - z) u \right\|.$$

We now integrate by parts to obtain a weighted version of the Poincaré inequality:

$$\left\| (1+r)^{\frac{-3-\delta}{2}} u \right\|^2 = \frac{2}{2+\delta} \operatorname{Re} \left\langle (1+r)^{-2-\delta} u', u \right\rangle \leq \left\| (1+r)^{\frac{-1-\delta}{2}} u' \right\| \left\| (1+r)^{\frac{-3-\delta}{2}} u \right\|,$$

giving

$$\left\| (1+r)^{\frac{-3-\delta}{2}} u \right\| \leq \sqrt{\delta_V^{-1} + \delta^{-1}} \left(1 + \sqrt{2 + \varepsilon E^{-1}} \right) \left\| \frac{1}{\sqrt{w'}} (P_D - z) u \right\|. \tag{2.8}$$

We now apply the Phragmén–Lindelöf principle as in the proof of (1.7), with the difference that in place of (2.6) we use

$$U(z) := \langle (1+r)^{-\frac{3+\delta}{2}} (P_D - z)^{-1} (1+r)^{-\frac{1+\delta}{2}} u, v \rangle,$$

to obtain (1.8) when $\theta = 1$. Then taking the adjoint gives the result for $\theta = 0$, and interpolating (that is to say, applying the Phragmén–Lindelöf principle with respect to $\theta \in \mathbb{C}$ such that $\operatorname{Re} \theta \in [0, 1]$) gives the result for $\theta \in (0, 1)$. □

Proof of (1.9). We again proceed as in the proof of (1.7), but this time we replace (2.3) by

$$w(r) := 1 - \frac{\delta_V}{2(\delta_V + \delta)} (1+r)^{-\delta},$$

so that (2.4) is replaced by

$$(w V_D)'(r) \leq -\frac{\delta_V V_D(r)}{2(1+r)}.$$

Now dropping the first two terms on the left hand side of (2.2) gives

$$\left\langle \frac{\delta_V V_D(r)}{2(1+r)} u, u \right\rangle \leq \frac{2E+\varepsilon}{E} \left\| \frac{1}{\sqrt{w'}} (P_D - z) u \right\|^2,$$

or

$$\|V_D(r)^{\frac{1}{2}}(1+r)^{-\frac{1}{2}}(P_D - z)^{-1}(1+r)^{-\frac{1+\delta}{2}}\| \leq \frac{2\sqrt{2+\varepsilon E^{-1}}}{\sqrt{\delta_V}} \sqrt{\delta^{-1} + \delta_V^{-1}}.$$

We now proceed as in the proof of (1.8), applying the Phragmen–Lindelöf principle to obtain (1.9) for $\theta = 1$, and then taking the adjoint and interpolating to obtain (1.9) for $\theta \in [0, 1)$. □

3. Resolvent estimates for mildly trapping manifolds

In §3.1 we state our main resolvent estimates for mildly trapping manifolds with asymptotically cylindrical ends, under suitable abstract assumptions. In the remainder of §3 we give examples which satisfy the assumptions, and then in §4 we prove the estimates.

3.1. Resolvent estimates for asymptotically cylindrical manifolds.

Let (X, g) be a smooth Riemannian manifold of dimension $d \geq 2$, with or without boundary, with the following kind of asymptotically cylindrical ends: we assume there is an open set $X_e \subset X$ such that $\partial X \cap X_e = \varnothing$, $X \setminus X_e$ is compact, and

$$X_e = (0, \infty)_r \times Y, \qquad g|_{X_e} = dr^2 + f(r)^{4/(d-1)} g_Y.$$

Here Y is a compact, not necessarily connected, manifold without boundary of dimension $d-1$, g_Y is a fixed smooth metric on Y and $f \in C^\infty([0, \infty); (0, 1])$. We suppose further that there is $\delta_0 > 0$ such that

$$|(f-1)^{(k)}(r)| \leq C_k (1+r)^{-k-\delta_0} \text{ for all } k \in \mathbb{N}_0 \text{ and } r \geq 0, \tag{3.1}$$

and

$$f'(r) \geq \delta_0 (1+r)^{-1}(1-f) \geq 0 \text{ for all } r \geq 0. \tag{3.2}$$

Suppose finally that $f(r) < 1$ for $r < 6$. Note that if we replace $r < 6$ by $r < r_0$ in this last condition, we can reduce to the case $r_0 = 6$ by multiplying g by a constant and rescaling r (i.e. using (1.10) with $R_1 = 0$ and $R_2 = r_0$).

We briefly discuss the assumptions (3.1) and (3.2). Note that the class of functions f such that (3.1) and (3.2) hold for a given $\delta_0 > 0$ is convex, and contains all functions of the form $f(r) = 1 - (1+r)^{-m}$ whenever $m \geq \delta_0$. Moreover, all functions f, such that f' is compactly supported and positive on the interior of the support of $(1-f)$, obey (3.1) and (3.2) for some $\delta_0 > 0$; indeed, letting $R_f := \max \operatorname{supp}(1-f)$, we have

$$\lim_{r \uparrow R_f} \log(1 - f(r)) = \lim_{r \uparrow R_f} \frac{d}{dr} \log(1 - f(r)) = -\infty.$$

If f' is compactly supported then the ends are cylindrical, rather than just asymptotically cylindrical.

For notational convenience let us extend r to be a continuous function on X with $-1/2 \leq r < 0$ on $X \setminus \overline{X_e}$, and extend f to be constant for $r \leq 0$.

Let $\Delta \leq 0$ be the Laplacian on X. Let

$$P = P_h := -h^2\Delta + V,$$

where $h \in (0, h_0]$ for some $h_0 > 0$, and:

- $V = V_h \in C^\infty(X \times (0, h_0]; \mathbb{R})$ is bounded, together with all derivatives, uniformly in $h \in (0, h_0]$.
- $V|_{X_e}$ is a function of r and h only, and has a decomposition $V|_{X_e} = V_L + hV_S$, where V_L and V_S may also depend on h, and $V_S = 0$ for $r \geq 5$ and $|V_S^{(k)}(r)| + |V_L^{(k)}(r)| \leq C_k(1+r)^{-k-\delta_0}$ for all $k \geq 0$, uniformly in h.
- $V_L'(r) \leq -\delta_0(1+r)^{-1}V_L(r) \leq 0$ for all $r \geq 0$.

Note that the assumptions allow $V \equiv 0$ but not $f \equiv 1$. Such a restriction is necessary to obtain a resolvent bound which is uniform up to the spectrum, in light of the computation in (1.2), which rules out such a bound in the case $(X, g) = (\mathbb{R} \times Y, dr^2 + dS)$ and $P = -h^2\Delta$.

Fix $E_0 > 0$. We suppose that E_0 is a "mildly trapping" energy level for P in the sense that adding a complex absorbing barrier supported on X_e gives a polynomial resolvent bound. More specifically, suppose that for some $W_K \in C^\infty(\mathbb{R}; [0, 1])$ with $W_K = 0$ near $(-\infty, 5]$ and $W_K = 1$ near $[6, \infty)$, there is $N \in \mathbb{R}$ such that

$$\|(P - iW_K(r) - E_0)^{-1}\|_{L^2(X)\to L^2(X)} =: a(h)h^{-1} \leq h^{-N}, \tag{3.3}$$

for all $h \in (0, h_0]$.

We have the following weighted resolvent bound up to the spectrum.

Theorem 3.1. *Let (X, g), P, E_0, and $a(h)$ be as above. Fix s_1, $s_2 > 1/2$ such that $s_1 + s_2 > 2$. There are $C > 0$ and $h_1 > 0$ such that*

$$\|(1+r)^{-s_1}(P - E_0 - i\varepsilon)^{-1}(1+r)^{-s_2}\|_{L^2(X)\to L^2(X)} \leq C(a(h) + h^{-1})h^{-1}, \tag{3.4}$$

for all $\varepsilon \in \mathbb{R} \setminus 0$ and for all $h \in (0, h_1]$.

Note that the condition on s_1 and s_2 is the same as the one in §1.2 above, see in particular (1.5) and (1.8). This is the resolvent weighting needed to have a low energy bound for scattering on the half line (and for more general Euclidean scattering problems).

To deduce Theorem 1.1 from Theorem 3.1, in Examples 1 and 2 we let X_e be the part of X where $r \geq r_1$, for any $r_1 > 0$ such that $F'(r_1) > 0$, and put $V \equiv 0$. Then, after redefining r as in the remark following (3.2), we see that g has the desired form in X_e, and it remains to check that (3.3) holds with $N \leq 2$. Below in §3.2 and §3.3 we will show this for some examples which generalize Examples 1 and 2 above.

We also have an improved bound when we cut off away from the trapping in the end. To state it, let $\chi_\Pi \in C^\infty(\mathbb{R}; [0, 1])$ be 0 near $(-\infty, 0]$ and 1 near $[1, \infty)$. Let $\Delta_Y \leq 0$ be the Laplacian

on (Y, g_Y), and let $\{\phi_j\}_{j=0}^{\infty}$ be a complete real-valued orthonormal set of its eigenfunctions, with $-\Delta_Y \phi_j = \sigma_j^2 \phi_j$, where $0 = \sigma_0 \leq \sigma_1 \leq \cdots$. For any $\mathcal{J} \subset \{0,\ 1,\ \dots\}$, we denote the orthogonal projection onto modes corresponding to $\mathcal{J}$ by $\Pi_{\mathcal{J}} \colon L^2(X_e) \to L^2(X_e)$, so that

$$(\Pi_{\mathcal{J}} u)(r, y) := \sum_{j \in \mathcal{J}} \phi_j(y) \int_Y u(r, y') \phi_j(y') d\mathrm{vol}(y'),$$

where y and y' denote points in Y. Then $\|\Pi_{\mathcal{J}} \chi_\Pi(r)\|_{L^2(X) \to L^2(X)} = 1$, unless $\mathcal{J}$ is empty.

Theorem 3.2. *Fix $s > 1/2$ and $c_{\mathcal{J}} > 0$. Let $\mathcal{J} := \{j \mid E_j := E_0 - h^2 \sigma_j^2 \notin [-c_{\mathcal{J}} h, c_{\mathcal{J}}]\}$. Define a microlocal cutoff $\chi_{\mathcal{J}} \colon L^2(X) \to L^2(X)$ by putting*

$$\chi_{\mathcal{J}} u := \begin{cases} \left(\Pi_{\mathcal{J}} \chi_\Pi(r) + \sqrt{V_L(r) + f(r)^{-4/(d-1)} - 1}\right) u, & u \in L^2(X_e), \\ u, & u \in L^2(X \setminus X_e), \end{cases} \tag{3.5}$$

and then extending to general $u \in L^2(X)$ by linearity. There are $C > 0$ and $h_1 > 0$ such that

$$\|(1+r)^{-s} \chi_{\mathcal{J}} (P - E_0 - i\varepsilon)^{-1} (1+r)^{-s}\|_{L^2(X) \to L^2(X)} \leq C(1 + a(h)) h^{-1}, \tag{3.6}$$

for all $\varepsilon \in \mathbb{R} \setminus 0$ and for all $h \in (0, h_1]$.

By taking the adjoint, we see that (3.6) implies

$$\|(1+r)^{-s} (P - E_0 - i\varepsilon)^{-1} \chi_{\mathcal{J}} (1+r)^{-s}\|_{L^2(X) \to L^2(X)} \leq C(1 + a(h)) h^{-1}. \tag{3.7}$$

Note that the statement is strongest when $c_{\mathcal{J}}$ is chosen very small, much smaller than E_0. We think of $\chi_{\mathcal{J}}$ as cutting off away from (or, almost, projecting away from)

$$T_{\mathcal{J}} = \{u \in L^2(X_e) \mid fu = u,\ V_L u = 0,\ \Pi_{\mathcal{J}} u = 0\} \subset L^2(X).$$

Observe that the condition $E_j \in [-c_{\mathcal{J}} h, c_{\mathcal{J}}]$ corresponds, when $V_L = 0$ and $f = 1$, to the condition that $\rho^2 \in [-c_{\mathcal{J}} h, c_{\mathcal{J}}]$, where ρ is the dual variable to r. In this sense $T_{\mathcal{J}}$ corresponds to a neighborhood of the bicharacteristics in $T^* X_e$ along which r is constant, that is to say bicharacteristics trapped in the cylindrical ends. In this sense $\chi_{\mathcal{J}}$ cuts off away from the trapping in the cylindrical ends. The asymmetry in the interval $[-c_{\mathcal{J}} h, c_{\mathcal{J}}]$ is due to the fact that our estimates are much easier when $E_j \leq -Ch$ for any $C > 0$ (see in particular the sentence following (4.41) below); we do not expect this form of the interval to be optimal.

To simplify matters, in our discussion of the interpretation and context of this result we focus on the special case of the following Corollary, although most of the statements could be adapted to apply to the more general case.

Corollary 3.3. *Let $(X, g) = (\mathbb{R}^d, g)$ be as in Example 1. In the notation of that example, fix $\chi \in C_c^\infty(X)$ with $\operatorname{supp} \chi \subset \{(r, \theta) \in \mathbb{R}^d \mid r < R\}$, and fix $s > 1/2$. Then there are $z_0 > 0$ and $C > 0$ such that*

$$\|(1+r)^{-s} (-\Delta - z)^{-1} \chi\|_{L^2(X) \to L^2(X)} + \|\chi (-\Delta - z)^{-1} (1+r)^{-s}\|_{L^2(X) \to L^2(X)} \leq C/\sqrt{\operatorname{Re} z}, \tag{3.8}$$

for all $z \in \mathbb{C}$ with $\operatorname{Re} z \geq z_0$ and $\operatorname{Im} z \neq 0$.

Note that this χ is local, in contrast to the microlocal $\chi_{\mathcal{J}}$ of Theorem 3.2. Recall that R is the radius at which the cylindrical end begins; hence χ is a cut off away from the trapping on the cylindrical end, and in this example there is no other trapping. The right hand side of (3.8) is the usual nontrapping upper bound, cf. (1.7) and the bound of Ch^{-1} in (1.4). There have been many results in asymptotically Euclidean, conic, and hyperbolic scattering proving that such nontrapping bounds hold when one cuts off away from trapping on *both* sides of the resolvent: these go back to work of Cardoso and Vodev [CaVo], refining an earlier result of Burq [Bu1]. Intriguingly, in (3.8) we get a nontrapping bound by applying a spatial cutoff away from trapping on only *one* side of the resolvent; to our knowledge no such result is known in asymptotically Euclidean, conic, and hyperbolic scattering, although a related weaker bound can be found in [BuZw, Ch2, DaVa2] (and note that the weaker bound is shown to be optimal in a special example in [Dy]). A possible interpretation is the following: unlike in any of the examples studied in [BuZw, DaVa2], in Example 1 the set K of bicharacteristics trapped as $t \to +\infty$ *and* $t \to -\infty$ is the same as the set $\Gamma_\pm$ of bicharacteristics trapped as $t \to +\infty$ *or* $t \to -\infty$, and one expects resolvent estimate losses due to mild trapping to be concentrated on $\Gamma_\pm$.

On the other hand, in [DaDyZw] it is shown that for a "well in an island" semiclassical Schrödinger operator (in which case incidentally K does equal $\Gamma_\pm$), losses due to trapping extend beyond $\Gamma_\pm$ and cutting off on one side only is not enough to give nontrapping bounds; as discussed in that paper, this is closely related to the fact that the trapping in this case is stable (so that tunneling can produce losses away from $\Gamma_\pm$), unlike in Example 1 or in the examples in [DaVa2]. It is then natural to ask: when is cutting off a resolvent away from trapping on one side sufficient to give nontrapping bounds, and when is it necessary to cut off on both sides?

3.2. **Examples with no trapping away from the ends.** Let X have no boundary and let K_{E_0} be the set of bicharacteristics of P at energy E_0 which do not intersect T^*X_e. If $K_{E_0} = \varnothing$, then it is essentially well-known that

$$\|(P - iW_K(r) - E_0)^{-1}\|_{L^2(X)\to L^2(X)} \leq Ch^{-1}; \tag{3.9}$$

the proof of (3.9) follows from the proof of [DyZw, Theorem 6.11] or that of [Da2, Proposition 3.2]. In the case that $|V| \leq Ch$, demanding that $K_{E_0} = \varnothing$ is equivalent to demanding that all maximally extended geodesics on X intersect X_e; specific examples are given in Example 1.

3.3. **Hyperbolic and normally hyperbolic trapped sets.** If $K_{E_0} \neq \varnothing$ we cannot hope to have (3.9), but if K_{E_0} is hyperbolic or normally hyperbolic then we may have

$$\|(P - iW_K(r) - E_0)^{-1}\|_{L^2(X)\to L^2(X)} \leq C\log(h^{-1})h^{-1}. \tag{3.10}$$

In the case of a closed hyperbolic orbit, such bounds are due to Burq [Bu2] and Christianson [Ch2]. For hyperbolic trapped sets satisfying a pressure condition they are due to Nonnenmacher and Zworski [NoZw1], and for normally hyperbolic trapped sets to Wunsch and Zworski [WuZw] and to Nonnenmacher and Zworski [NoZw2] (and see also [Dy]). Some recent surveys of the substantial wider literature concerning estimates like (3.10) can be found in [No, Zw2].

To deduce (3.10) from [NoZw1] or [NoZw2], note that the difference between (3.10) and [NoZw1, (2.7)] or [NoZw2, (1.18)] lies in the assumptions in the region where $W_K = 1$. But in this region $P - iW_K$ is semiclassically elliptic, so the discrepancy can be removed using a parametrix G' analogous to the one in (4.1) below, and rather than having to go through a procedure like that in §4.5 we just have $(P - iW_K(r) - E_0)G' = I + O(h^\infty)$.

Rather than discussing the general dynamical assumptions further, we now specialize to more concrete examples.

Let (X, g_H) be a conformally compact manifold of constant negative curvature. We recall that this means that the metric g_H is asymptotically hyperbolic in the sense of [MaMe] (see also [DyZw, §5.1]), so there is an open set X'_e and $R \in \mathbb{R}$ such that $X \setminus X'_e$ is compact and

$$X'_e = (R, \infty)_r \times Y, \qquad g_H|_{X'_e} = dr^2 + e^{2r} g_Y(e^{-r}),$$

where Y is a compact, not necessarily connected, manifold without boundary and $g_Y(x)$ is a family of metrics on Y depending smoothly on x up to $x = 0$. Such a 'normal form' of the metric was first found in [GrLe], and it is also in [DyZw, §5.1.1].

We modify the metric to obtain a manifold with cylindrical ends in the following way. We first observe that, denoting points in $T^*X'_e$ by (r, y, ρ, η), where $y \in Y$, ρ is dual to r, and η is dual to y, along g_H-geodesics we have

$$\frac{d^2}{dt^2} r =: \ddot{r} = -2\partial_r(e^{-2r}|\eta|^2_{r,y}) = 4e^{-2r}|\eta|^2_{r,y}(1 + O(e^{-r})),$$

where the length $|\eta|_{r,y}$ is taken with respect to the dual metric to $g_Y(e^{-r})$. Hence, after possibly redefining R to be larger, we may suppose that $\ddot{r} \geq 2e^{-2r}|\eta|^2_{r,y}$ for $r \geq R$, and in particular that no bounded g_H-geodesics intersect $\overline{X'_e}$. Indeed, since $E_0 := \rho^2 + e^{-2r}|\eta|^2_{r,y}$ is conserved and $\dot{r} = 2\rho$, in X'_e we have

$$\ddot{r} \geq 2e^{-2r}|\eta|^2_{r,y} = 2E_0 - \dot{r}^2/2,$$

which means r is not bounded for all t.

Fix $\chi_H \in C^\infty(\mathbb{R}; [0, 1])$ such that $\chi_H(r) = 1$ near $(-\infty, R]$ and $\chi_H(r) = 0$ near $[R + 1, \infty)$, and fix $F \in C^\infty([R, \infty), (0, \infty))$ such that F' is compactly supported, positive on the interior of its support, and such that $F'(r) > 0$ for $r \leq R + 2$. Take g such that $g|_{X \setminus X'_e} = g_H|_{X \setminus X'_e}$, and

$$g|_{X'_e} = \chi_H(r) g_H + C_g(1 - \chi_H(r)) \left(dr^2 + F(r) g_Y(0)\right).$$

We claim that if C_g is large enough, then $\ddot{r} \geq 0$ along g-geodesics in $\overline{X_e}$. Indeed,

$$\ddot{r}/2 = -\chi_H(r)\partial_r(e^{-2r}|\eta|^2_{r,y}) + C_g(1 - \chi_H(r))F'(r)|\eta|^2_0 - \chi'_H(r)(e^{-2r}|\eta|^2_{r,y} - C_g F(r)|\eta|^2_0),$$

so it is enough to take C_g large enough that on $T^* \operatorname{supp} \chi'_H(r)$ we have $e^{-2r}|\eta|^2_{r,y} \leq C_g F(r)|\eta|^2_0$.

Now we may take X_e to be the part of X'_e in which $r > R + 1$, and, after redefining r by (1.10), we see that it remains only to check (3.3).

We take $W_K \in C^\infty(\mathbb{R}; [0, 1])$ which is 1 near $[R + 2, \infty)$ and 0 near $(-\infty, R + 1]$, and suppose $|V| \leq Ch$ and $E_0 = 1$. Let K denote the set of trapped unit speed geodesics of (X, g_H), regarded

as a subset of T^*X. We see that K is also the set of the bicharacteristics of P at energy E_0 which do not intersect T^*X_e, and that $g_H = g$ near the projection of K onto X.

Let d_K be the Hausdorff dimension of K. If $d_K < d$, then the assumptions of [NoZw1] are satisfied, and (3.10) holds.

If $d = 2$ and $V \equiv 0$, then we can dispense with the requirement that $d_K < d$ thanks to a recent result of Bourgain and Dyatlov [BoDy, Theorem 2] (this is the case presented in Example 2 above). To do this we use the fact (see [Bu2, Lemma 4.7] or e.g. [DyZw, Proof of (6.3.10)]) that [BoDy, (1.1)] implies

$$\|\chi(-h^2\Delta_0 - E_0 - i0)^{-1}\chi\|_{L^2(X)\to L^2(X)} \leq C\log(h^{-1})h^{-1},$$

for any $\chi \in C_c^\infty(X)$. Then the gluing result of [DaVa1, Theorem 2.1] together with the semiclassically outgoing property of $(-h^2\Delta_0 - E_0 - i0)^{-1}$ (established by Vasy in [Va] and see also [DyZw, Theorem 5.34]) implies (3.10). In the interest of brevity we do not discuss this further here.

3.4. **Warped products with embedded eigenvalues.** Let $X := \mathbb{R} \times Y$ and $g := dr^2 + f(r)^{4/(d-1)}g_Y$ for some $f \in C^\infty(\mathbb{R}; (0, 1])$ which is 1 on $\mathbb{R} \setminus (-R, R)$ for some $R > 0$ and has a nondegenerate minimum as its only critical point in $(-R, R)$: see Figure 3.1.

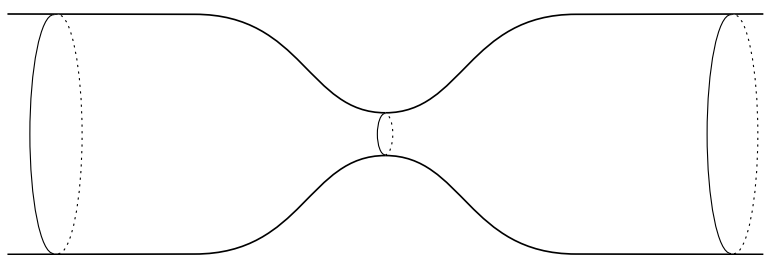

FIGURE 3.1. An hourglass shaped surface of revolution.

Suppose $V = h^2V_W$, with $V_W = V_W(r) \in C_c^\infty((-R, R))$. Then the part of the trapped set away from the cylindrical ends is normally hyperbolic and we have (3.10) (see [DyZw, (6.3.10)], and see also [ChWu, Ch3] for the case of a degenerate minumum where incidentally we also have (3.3)). Consequently, by Theorem 3.1, there is $z_0 > 0$ such that for all $s_1,\ s_2 > 1/2$ such that $s_1 + s_2 > 2$, there is $C > 0$ such that

$$\|(1 + |r|)^{-s_1}(-\Delta + V_W - z)^{-1}(1 + |r|)^{-s_2}\|_{L^2(X)\to L^2(X)} \leq C,$$

for all $z \in \mathbb{C}$ with $\operatorname{Re} z \geq z_0$ and $\operatorname{Im} z \neq 0$. In particular the spectrum of $-\Delta + V_W$ is absolutely continuous on (z_0, ∞).

But if f and V_W are suitably chosen, then $\Delta + V_W$ has an eigenvalue embedded in the spectrum in $[0, z_0]$. Indeed, we have

$$\Delta = f(r)^{-1}\left(\sum_{j=0}^{\infty}\left(\partial_r^2 - f''(r)f(r)^{-1} - \sigma_j^2 f(r)^{-4/(d-1)}\right)\phi \otimes \phi\right) f(r),$$

where $\{\phi_j\}_{j=0}^\infty$ is a complete set of real-valued orthonormal eigenfunctions of the Laplacian on Y and $-\Delta_Y\phi_j = \sigma_j^2\phi_j$. For $J \in \mathbb{N}_0$, consider the effective potential

$$V_J(r) := f''(r)f(r)^{-1} + \sigma_J^2(f(r)^{-4/(d-1)} - 1) + V_W(r).$$

Then $D_r^2 + V_J$ has an eigenvalue as long as $\int V_J(r)dr \leq 0$ by [ReSi, Theorem XIII.110], and this corresponds to an embedded eigenvalue for $-\Delta + V_W$ as long as it is positive, for which it suffices to have $\min V_J(r) > -\sigma_J^2$. For example, we may take f such that $\int(f(r)^{-4/(d-1)} - 1) \leq 1/4$ and $V_W \in C_c^\infty((-R,R);[-\sigma_J^2/2, 0])$ such that $V_W(r) = -\sigma_J^2/2$ on $[-R/2, R/2]$, and then J sufficiently large.

By elaborating the above constuction one can also find examples with any finite number of embedded eigenvalues.

It is not clear whether there are examples of manifolds with cylindrical ends such that $-\Delta$ has a finite but nonzero number of eigenvalues. For all known examples where eigenvalues occur, the existence of infinitely many eigenvalues is either also established [ChZw, Pa2] or at the least it is not ruled out [KrKř]. On the other hand 0 is always a resonance of $-\Delta$ on a manifold with cylindrical ends, with the constant functions as resonant states, unless there is a boundary condition somewhere that eliminates them.

4. Proof of Theorems 3.1 and 3.2

4.1. **Outline of proof.** The idea of the proofs is to define a parametrix for $P - z$ by

$$G := \chi_K(r-1)(P - iW_K(r) - z)^{-1}\chi_K(r) + \chi_e(r+1)(P_e - z)^{-1}\chi_e(r), \tag{4.1}$$

where $\chi_e, \chi_K \in C^\infty(\mathbb{R})$ obey $\chi_e + \chi_K = 1$, $\operatorname{supp}\chi_e \subset (3,\infty)$, and $\operatorname{supp}\chi_K \subset (-\infty, 4)$, and P_e is a suitably chosen differential operator such that $P_e = P$ on the part of X where $r > 2$. Then

$$(P-z)G = I + [h^2D_r^2, \chi_K(r-1)](P - iW_K(r) - z)^{-1}\chi_K(r) + [h^2D_r^2, \chi_e(r+1)](P_e - z)^{-1}\chi_e(r),$$

and we will construct an inverse for $(P - z)$ by removing this remainder using a Neumann series; although the remainder above need not be small, we will see that powers of it are. We call the part of X where $r \in (2,5)$ the *resolvent gluing region*, because the functions in the range of the remainder are supported in that region. To prove that powers of the remainder are small, we will need to know that:

(1) The resolvents of $P - iW_K(r)$ and P_e obey estimates analogous to (3.4) and (3.6). This is the case for $P - iW_K(r)$ thanks to the assumption (3.3), and we will prove it for a suitable choice of P_e in §4.3 and §4.4.
(2) The resolvents of $P - iW_K(r)$ and P_e obey improved estimates when multiplied by cutoffs with suitable support properties in the resolvent gluing region, corresponding to a (special case of a) *semiclassically outgoing condition* so that we are able to remove the remainders. The needed estimates are proved in [DaVa1] for $P - iW_K(r)$ and in §4.3 and §4.4 for P_e.

We combine these estimates to prove Theorems 3.1 and 3.2 in §4.5. There we follow a procedure analogous to that in [DaVa1], but with some finer analysis of remainders to remove the losses due

to trapping in the cylindrical end (see also [Da2, §3] for another, in some ways related, variation on this resolvent gluing procedure).

4.2. **Model operators for X_e.** On X_e, Δ can be written as a direct sum of one-dimensional Schrödinger operators:

$$\Delta|_{X_e} = f(r)^{-1}\left(\sum_{j=0}^{\infty}\left(\partial_r^2 - f''(r)f(r)^{-1} - \sigma_j^2 f(r)^{-4/(d-1)}\right)\phi\otimes\phi\right)f(r),$$

where $\{\phi_j\}_{j=0}^{\infty}$ is a complete set of real-valued orthonormal eigenfunctions of the Laplacian on Y and $-\Delta_Y\phi_j = \sigma_j^2\phi_j$. We will introduce model operators P_j obeying

$$P_j|_{[2,\infty)} = -h^2\partial_r^2 + V_j(r), \qquad V_j(r) := V(r) + h^2 f''(r)f(r)^{-1} + h^2\sigma_j^2(f(r)^{-4/(d-1)} - 1), \quad (4.2)$$

and we will be studying them near the energy levels

$$E_j := E_0 - h^2\sigma_j^2.$$

We will study two ranges of j separately, and the model operators P_j will act on different spaces depending on j. These two ranges correspond to different behavior in the resolvent gluing region, which is the part of X where $r \in (2,5)$ (see §4.1). To define the ranges, fix $E_* \in \mathbb{R}$, independent of h, such that

$$0 < E_* \le c_{\mathcal{J}},$$

where $c_{\mathcal{J}}$ is as in the statement of Theorem 3.2, and

$$E_j \le E_* \Longrightarrow h^2\sigma_j^2 f(5)^{-4/(d-1)} \ge E_0; \quad (4.3)$$

note that the conditions are compatible because $E_j = 0$ when $E_0 = h^2\sigma_j^2$ and $f(5) < 1$.

The first range we consider is $E_j \le E_*$; in this range the set where $r < 5$ is classically forbidden because $V_j > E_j$, and we control remainders in the gluing region using Agmon estimates, taking care to prove that our estimates are uniform as $j \to \infty$ (although the effective potentials V_j become unbounded as $j \to \infty$, they are nonnegative, so the relevant estimates actually get better in this limit). The second range is $E_j \ge E_*$; in this range the set where $r < 5$ is not classically forbidden, but the energy levels E_j are bounded below by a positive constant and the effective potentials V_j are repulsive, so nontrapping propagation of singularities estimates hold, which we can use to control the remainders in the gluing region (once again we take care to prove that the estimates are uniform in j).

For the first range of j we define the operators P_j to act on $L^2(\mathbb{R}_+)$, with a Dirichlet boundary condition at 0, in order to be able to use Theorem 1.3 (the Dirichlet boundary condition makes it easier to analyze the behavior of the resolvent when $|E_j|$ is small). For the second range of j it is more convenient to work over $\mathbb{R}$ than $\mathbb{R}_+$, in order to avoid reflection phenomena when studying propagation of singularities.

4.3. **Analysis when $E_j \le E_*$.** In §4.3 all function norms and inner products are in $L^2(\mathbb{R}_+)$, and operator norms are $L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$, unless otherwise specified.

For this range of j, we put

$$P_j := h^2 D_r^2 + V_j(r), \tag{4.4}$$

regarded as a self-adjoint operator on $L^2(\mathbb{R}_+)$ with a Dirichlet boundary condition at $r = 0$.

We first prove resolvent estimates for P_j analogous to (3.4) and (3.6).

Proposition 4.1. *Fix $s_1,\ s_2,\ s > 1/2$ such that $s_1 + s_2 > 2$. Then*

$$\|(1+r)^{-s_1}(P_j - E_j - i\varepsilon)^{-1}(1+r)^{-s_2}\| \le Ch^{-2}, \tag{4.5}$$

and

$$\|(1+r)^{-s}\chi(r)(P_j - E_j - i\varepsilon)^{-1}(1+r)^{-s}\| + \|(1+r)^{-s}(P_j - E_j - i\varepsilon)^{-1}\chi(r)(1+r)^{-s}\| \le Ch^{-1}, \tag{4.6}$$

for all $\varepsilon \in \mathbb{R} \setminus 0$, $j \in \mathbb{N}$ such that $E_j \le E_*$, where

$$\chi(r) = \sqrt{V_L(r) + f(r)^{-4/(d-1)} - 1}.$$

Proof. The idea of the proof is to apply Theorem 1.3; more precisely (4.5) corresponds to (1.8) (see also (1.5)), and (4.6) corresponds to (1.9) (see also (1.6)).

Before beginning the proof proper, by way of outline let us briefly discuss the terms in V_j, and explain how they each do or do not satisfy (1.3). The term $h^2\sigma_j^2(f(r)^{-4/(d-1)} - 1)$ does satisfy it thanks to (3.2) and (4.3), and moreover those bounds and $f(r) < 1$ for $r < 6$ imply that the term is nontrivial when $r < 6$. The term V_L satisfies it, and we think of it as being harmless. The terms V_S does not satisfy it, but we will show that its effect is compensated by that of the $h^2\sigma_j^2(f(r)^{-4/(d-1)} - 1)$ term. The most difficult term to treat is the $h^2 f''(r)f(r)^{-1}$ term. This term may prevent $h^{-2}V_j$ from satisfying (1.3), but we will show that thanks to (4.3) we can treat it as a small perturbation.

More precisely, let

$$V_M(r) := V_j(r) - h^2 f''(r)f(r)^{-1},$$

and observe that for h sufficiently small V_M obeys (1.3) for some $\delta_V > 0$, since V_L and $f^{-4/(d-1)} - 1$ obey it and $|V_S| + |V_S'| \le C(f^{-4/(d-1)} - 1)$ thanks to (4.3). Indeed, to see that $f^{-4/(d-1)} - 1$ obeys it we write, using $\alpha := 4/(d-1)$ and (3.2),

$$-(f(r)^{-\alpha} - 1)' = \alpha f'(r)f(r)^{-\alpha-1} \ge \alpha\delta_0 \frac{f(r)^{-\alpha-1} - f(r)^{-\alpha}}{1+r} \ge \frac{f(r)^{-\alpha} - 1}{C(1+r)},$$

where we also used the fact that if $a < b$ then

$$C^{-1}(1-f) \le f^a - f^b \le C(1-f). \tag{4.7}$$

Hence by (1.8) with $V_D = h^{-2}V_M$, we have

$$\|(1+r)^{-s_1}(h^2 D_r^2 + V_M - E_j - i\varepsilon)^{-1}(1+r)^{-s_2}\| \le Ch^{-2}. \tag{4.8}$$

Note that by the resolvent identity

$$(1+r)^{-s_1}(P_j-E_j-i\varepsilon)^{-1}(1+r)^{-s_2}=(1+r)^{-s_1}(h^2D_r^2+V_M-E_j-i\varepsilon)^{-1}(1+r)^{-s_2} \\ \times\sum_{k=0}^{\infty}\left[-(1+r)^{s_2}h^2f''(r)f(r)^{-1}(h^2D_r^2+V_M-E_j-i\varepsilon)^{-1}(1+r)^{-s_2}\right]^k, \tag{4.9}$$

the proof of (4.5) is reduced to the proof of

$$\|(1+r)^{s_2}h^2f''(r)f(r)^{-1}(h^2D_r^2+V_M-E_j-i\varepsilon)^{-1}(1+r)^{-s_2}\|\le 1/2. \tag{4.10}$$

But by (1.9), with $\theta=1$ and $V_D=h^{-2}V_M\ge h^{-2}(f^{-4/(d-1)}-1)/C$ (again using (4.3)), we have

$$\|(f(r)^{-4/(d-1)}-1)^{\frac12}(1+r)^{-\frac12}(h^2D_r^2+V_M-E_j-i\varepsilon)^{-1}(1+r)^{-s_2}\|\le Ch^{-1},$$

and interpolating this with (4.8) gives

$$\|(f(r)^{-4/(d-1)}-1)^{\frac14}(1+r)^{-\frac{s_1}{2}-\frac14}(h^2D_r^2+V_M-E_j-i\varepsilon)^{-1}(1+r)^{-s_2}\|\le Ch^{-3/2}.$$

Hence to prove (4.10), and consequently also (4.5), it is enough to show that

$$(1+r)^{s_2}|f''(r)|\le C(f(r)^{-4/(d-1)}-1)^{\frac14}(1+r)^{-\frac{s_1}{2}-\frac14}. \tag{4.11}$$

To prove (4.11) we will use the fact that any bounded $\varphi\in C^2([r,\infty);[0,\infty))$ satisfies

$$|\varphi'(r)|^2\le 2\sup\varphi\sup|\varphi''|, \tag{4.12}$$

where the suprema are taken over $[r,\infty)$. Indeed, by Taylor's theorem, for every $t\ge 0$ there is $t_0\in[r,r+t]$ such that

$$t|\varphi'(r)|=|\varphi(r+t)-\varphi(r)-t^2\varphi''(t_0)/2|\le\sup\varphi+t^2\sup|\varphi''|/2,$$

and taking $t=|\varphi'(r)|/\sup|\varphi''|$ gives (4.12). Applying (4.12) once with $\varphi=f'$ and once with $\varphi=1-f$ gives

$$|f''(r)|^4\le 4\sup|f'|^2\sup|f'''|^2\le 8\sup(1-f)\sup|f''|\sup|f'''|^2=8(1-f(r))\sup|f''|\sup|f'''|^2,$$

where the suprema are still all taken over $[r,\infty)$. Applying (3.1) gives

$$|f''(r)|\le C(1-f(r))^{\frac14}(1+r)^{-2-\frac{3\delta_0}{4}}.$$

By (4.7) this implies (4.11) as long as $s_1+2s_2\le(7+3\delta_0)/2$, which we may suppose without loss of generality. This completes the proof of (4.5).

The proof of (4.6) proceeds along similar lines. Applying (4.9) with $s_1=s_2=s$ allows us to reduce the proof of the bound on the first term in (4.6) to the proof of

$$\|(1+r)^{-s}\chi(r)(h^2D_r^2+V_M-E_j-i\varepsilon)^{-1}(1+r)^{-s}\|\le Ch^{-1}. \tag{4.13}$$

But (4.13) follows from (1.9) with $\theta=1$ and $V_D=h^{-2}V_M\ge h^{-2}(V_L+f^{-4/(d-1)}-1)/C=h^{-2}\chi^2/C$. The bound on the second term of (4.6) follows from the bound on the first term after taking the adjoint. □

We will also need the following Agmon estimates:

Proposition 4.2. *Let $R \in (0,5]$, $\chi_- \in C_c^\infty((0,R))$, $\chi_+ \in C_c^\infty((R,\infty))$, and $s > 1/2$. Then*

$$\|\chi_-(P_j - E_j - i\varepsilon)^{-1}(1+r)^{-s}\|_{L^2(\mathbb{R}_+)\to H_h^1(\mathbb{R}_+)} + \|(1+r)^{-s}(P_j - E_j - i\varepsilon)^{-1}\chi_-\| \le C, \tag{4.14}$$

$$\|\chi_-(P_j - E_j - i\varepsilon)^{-1}\chi_+\| \le e^{-1/(Ch)}, \tag{4.15}$$

for all $\varepsilon \in \mathbb{R}\setminus 0$, and $j \in \mathbb{N}$ such that $E_j \le E_*$.

Recall that the norms without subscripts are $L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ here, and that χ_- is supported in the classically forbidden region for $P_j - E_j$.

Proof. These are similar to the usual Agmon estimates as in [Zw1, §7.1] but we keep track of the j dependence.

Let $v \in L^2(\mathbb{R}_+)$, and let $u := (P_j - E_j - i\varepsilon)^{-1}(1+r)^{-s}v$. Fix $\varphi_0 \in C_c^\infty((0,R);[0,1])$ which is identically 1 on a neighborhood I of $\operatorname{supp}\chi_-$, and let $\varphi(r) := m\varphi_0(r)$, for a constant m to be chosen later. Then define

$$\begin{aligned} P_\varphi &:= e^{\varphi/h}(P_j - E_j - i\varepsilon)e^{-\varphi/h} \\ &= h^2 D_r^2 + 2i\varphi' h D_r + V_j - \varphi'^2 + h\varphi'' - E_j - i\varepsilon. \end{aligned}$$

Put $w := \chi_0 e^{\varphi/h}u$, where $\chi_0 \in C_c^\infty((0,R))$ is 1 near $\operatorname{supp}\varphi$. Using $\operatorname{Re}\langle 2h\varphi' w', w\rangle = -h\langle \varphi'' w, w\rangle$, write

$$\operatorname{Re}\langle P_\varphi w, w\rangle = \|hw'\|^2 + \langle (V_j - \varphi'^2 - E_j)w, w\rangle.$$

We now observe that, using (4.3) and the fact that $1 - f(r)^{-4/(d-1)} > 1 - f(5)^{-4/(d-1)} > 0$ for $r \in (0,5)$, we can choose $m > 0$ small enough, independent of h and j, such that there is $c_0 > 0$ independent of h and j for which $V_j - \varphi'^2 - E_j > c_0$ on $\operatorname{supp} w$ for h small enough. This implies

$$\|w\| \le C\|P_\varphi w\| \le C\|e^{\varphi/h}\chi_0 v\| + C\|[P,\chi_0]u\|,$$

where we used $\varphi\chi_0' = 0$ to deduce $[P_\varphi, \chi_0]e^{\varphi/h}u = [P,\chi_0]u$. We use an elliptic estimate to bound the commutator term: for $\chi_1 \in C_c^\infty((0,R))$ we have, using $V_j - E_j \ge V_0 - E_0 \ge -C$,

$$\begin{aligned} C\|\chi_1 v\|\|\chi_1 u\| \ge \operatorname{Re}\langle (1+r)^{-s}v, \chi_1^2 u\rangle &= \operatorname{Re}\langle (P_j - E_j)u, \chi_1^2 u\rangle \\ &\ge \|\chi_1 h u'\|^2 - Ch\|\chi_1 h u' u\|_{L^1(\mathbb{R}_+)} - C\|\chi_1 u\|^2, \end{aligned} \tag{4.16}$$

from which it follows that, provided $\chi_2 = 1$ near $\operatorname{supp}\chi_0$,

$$\|[P,\chi_0]u\| \le Ch\|\chi_2 u\| + Ch\|\chi_2 v\| \le Ch^{-1}\|v\|,$$

where we used (4.5). Consequently

$$\int_I |u|^2 = e^{-2m/h}\int_I |w|^2 \le Ce^{-2m/h}\left(\|e^{\varphi/h}\chi_0 v\|^2 + h^{-2}\|v\|^2\right) \le C\|v\|^2, \tag{4.17}$$

where we used $\varphi \le m$.

To estimate u' we apply (4.16) with $\chi_1 \in C_c^\infty(I)$, giving

$$\|\chi_1 h u'\|^2 \le C\left(\int_I |u|^2 dr + \|\chi_1 h v\|^2\right),$$

which implies the bound on the first term of (4.14). The bound on the second term follows from taking the adjoint, and (4.15) follows from the fact that if $\operatorname{supp} v \subset (R, \infty)$, then $\chi_0 v = 0$ and we can improve (4.17) to

$$\int_I |u|^2 = e^{-2m/h} \int_I |w|^2 \leq C e^{-2m/h} h^{-2} \|v\|^2.$$

□

4.4. **Analysis when $E_j > E_*$.** In §4.4 all function norms and inner products are in $L^2(\mathbb{R})$, and operator norms are $L^2(\mathbb{R}) \to L^2(\mathbb{R})$, unless otherwise specified.

For this range of j the Agmon estimate (4.15) must be replaced by a propagation of singularities estimate. It is convenient to introduce a complex absorbing barrier and to work over $\mathbb{R}$: let $W_e \in C^\infty(\mathbb{R}; [0,1])$ be 1 near $(-\infty, 1]$ and 0 near $[2, \infty)$, and let

$$V_{j,0} := \chi_0 V_j,$$

where $\chi_0 \in C^\infty(\mathbb{R}; [0,1])$ is 0 near $(-\infty, 0]$ and 1 near $[1, \infty)$. We now put

$$P_j := h^2 D_r^2 + V_{j,0}(r) - i W_e(r),$$

regarded as an unbounded operator on $L^2(\mathbb{R})$ with domain $H^2(\mathbb{R})$. We will prove

Proposition 4.3. *For any $s > 1/2$ we have*

$$\|(1 + r_+)^{-s} (P_j - E_j - i\varepsilon)^{-1} (1 + r_+)^{-s}\| \leq C h^{-1}, \tag{4.18}$$

where $r_+ := \max\{0, r\}$. For any $\chi_- \in C_c^\infty((0,3))$, $\chi_+ \in C_c^\infty((3,\infty))$, $\psi \in C_c^\infty((0,\infty))$, we have

$$\|\chi_-(r)(P_j - E_j - i\varepsilon)^{-1} \chi_+(r) \psi(hD_r)\| = O(h^\infty). \tag{4.19}$$

Both (4.18) *and* (4.19) *hold uniformly for all $\varepsilon > 0$, and for all $j \in \mathbb{N}_0$ such that $E_j > E_*$.*

Note that since E_j is bounded from below away from 0, we can think of (4.18) as the analogue of (1.7) in this setting; we do not need a weight for $r < 0$ because the $-iW_e$ term makes the operator $P_j - E_j - i\varepsilon$ semiclassically elliptic there. It is also similar to the usual nontrapping resolvent estimate as in [VaZw] and in other papers cited therein, but we need an estimate which is uniform in j.

The propagation of singularities estimate (4.19) is a microlocalized version of (4.18). The improved bound is due to the fact that solutions to the classical equations of motion $\dot{r} = 2\rho$, $\dot{\rho} = -V_j'(r)$ with $r(0) > 3$ and $\rho(0) > 0$ cannot have $r(t) < 3$ for any $t > 0$.

Proof of (4.18). We prove (4.18) using a microlocal positive commutator argument, rather than (as is probably possible) integration by parts arguments as in the proof of (1.7). We do this because the proof of (4.19) follows along very similar lines, and the latter estimate does not seem to be provable by integration by parts arguments. The idea is to construct a microlocal commutant, based on the $w(r)\partial_r$ of the proof of (1.7), but which is nonnegative. This will be obtained as the quantization of an escape function, defined in (4.26) below.

As in [VaZw] we will use the semiclassical scattering calculus, and we begin by recalling its relevant properties. We use (r,ρ) to denote points in $T^*\mathbb{R}$, and for $l,\ m\in\mathbb{R}$ we define the symbol class S^m_l to be the set of $a\in C^\infty(T^*\mathbb{R})$ such that, for any $n_1,\ n_2\in\mathbb{N}_0$ there is C_{n_1,n_2} such that

$$|\partial_r^{n_1}\partial_\rho^{n_2}a(r,\rho)|\le C_{n_1,n_2}(1+|r|)^{l-n_1}(1+|\rho|)^{m-n_2}, \tag{4.20}$$

for all $(r,\rho)\in T^*\mathbb{R}$. We also write $S^\infty_l:=\bigcup_m S^m_l$, $S^{-\infty}_l:=\bigcap_m S^m_l$, and similarly for S^m_∞ and $S^m_{-\infty}$. Below we will consider symbols depending on h and j, and the constants C_{n_1,n_2} in (4.20) will always be uniform with respect to those parameters. For such a, we denote the semiclassical quantization by $\mathrm{Op}_h(a)$, which we define by

$$\mathrm{Op}_h(a)u:=\frac{1}{2\pi h}\iint e^{i(r-r')\rho/h}a(r,\rho)u(r')dr'd\rho. \tag{4.21}$$

When a symbol is denoted by a lowercase letter (with possible subscripts and superscripts), we will denote its quantization by the corresponding uppercase letter (with the same subscripts and superscripts, if any).

We recall the composition and adjoint formulas. If $a\in S^{m_1}_{l_1}$ and $b\in S^{m_2}_{l_2}$, then there is $a\#b\in S^{m_1+m_2}_{l_1+l_2}$ such that

$$AB=\mathrm{Op}_h(a\#b),$$

and, for any $N\in\mathbb{N}$,

$$\begin{aligned} a\#b(r,\rho)&=e^{-ih\partial_{r'}\partial_{\rho'}}\left(a(r,\rho')b(r',\rho)\right)\Big|_{(r,\rho)=(r',\rho')}\\ &=\sum_{k=0}^{N-1}\frac{(-ih)^k}{k!}\partial_\rho^k a(r,\rho)\partial_r^k b(r,\rho)+h^N z_N(r,\rho), \end{aligned} \tag{4.22}$$

where $z_N\in S^{m_1+m_2-N}_{l_1+l_2-N}$ is given by

$$z_N(r,\rho):=\frac{(-i)^N}{(N-1)!}\int_0^1(1-t)^{N-1}e^{-ith\partial_{r'}\partial_{\rho'}}\left(\partial_{\rho'}^N a(r,\rho')\partial_{r'}^N b(r',\rho)\right)\Big|_{(r,\rho)=(r',\rho')}dt. \tag{4.23}$$

Indeed, [Zw1, Theorem 4.14] gives the formula for Schwartz symbols, and [Zw1, Theorems 4.13 and 4.18] give it for a larger class of symbols than the ones we consider, but with weaker bounds on z_N. The statement that $z_N\in S^{m_1+m_2-N}_{l_1+l_2-N}$ follows from applying [Zw1, Theorem 4.17] to (4.23). See also [DyZw, Proposition E.8], [Pa1, (3) and (9)], [Sh], and [Hö, §18.5] for similar expansions, and [HeSj] for a much more general version.

Similarly, if $a\in S^m_l$ there is $a^*\in S^m_l$ such that the formal adjoint of A is given by

$$A^*=\mathrm{Op}_h(a^*),$$

and, for any $N\in\mathbb{N}$,

$$a^*(r,\rho)=e^{-ih\partial_r\partial_\rho}\bar a(r,\rho)=\sum_{k=0}^{N-1}\frac{(-ih)^k}{k!}\partial_r^k\partial_\rho^k\bar a(r,\rho)+h^N z_N(r,\rho), \tag{4.24}$$

where this time $z_N\in S^{m-N}_{l-N}$ is given by

$$z_N(r,\rho):=\frac{(-i)^N}{(N-1)!}\int_0^1(1-t)^{N-1}e^{-ith\partial_r\partial_\rho}\partial_r^N\partial_\rho^N\bar a(r,\rho)dt.$$

Let

$$p_j := \rho^2 + V_{j,0}(r) - iW_e(r)$$

be the semiclassical symbol of P_j (in the sense that $p_j \in S_0^2$ and $P_j = \mathrm{Op}_h(p_j)$), let

$$R_j := \inf\{r > 0 \mid \text{both } V_j(r) = V_{j,0}(r) \text{ and } V_{j,0}(r) \le E_*/2\},$$

so that $R_0 \le R_1 \le \cdots$, and let

$$F_j := \{(r,\rho) \mid r \ge 1 \text{ and } \rho^2 \le 2E_0\} \setminus \{(r,\rho) \mid R_j < r \text{ and } \rho^2 < E_*/3\}.$$

Note that each F_j is a closed neighborhood of the energy surface $p_j = E_j$, and they have been chosen such that they form a nested sequence $F_0 \subset F_1 \subset \cdots$. Moreover, since we only consider j such that $E_j > E_*$, all of the F_j agree outside of a compact set: see Figure 4.1.

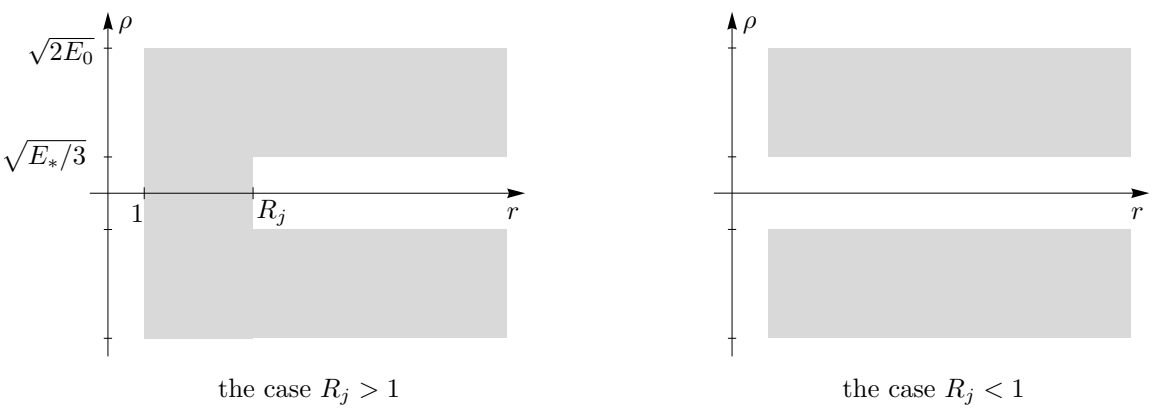

FIGURE 4.1. The shaded regions are the sets F_j. They are closed nested neighborhoods of the energy surfaces $p_j = E_j$ which agree outside of a compact set.

Observe that we have $|p_j - E_j - i\varepsilon| \ge c(1+\rho^2)$ on $T^*\mathbb{R} \setminus F_j$, for some $c > 0$, which implies the following elliptic estimate: for any $a \in S_l^m$, $a' \in S_l^{m-2}$ satisfying $\operatorname{supp} a \cap F_j = \varnothing$ and $|a'(r,\rho)| \ge (1+|r|)^l(1+|\rho|)^{m-2}$ for $(r,\rho) \in \operatorname{supp} a$, and for any $N \in \mathbb{R}$, we have

$$\|Au\| \le C\|A'(P_j - E_j - i\varepsilon)u\| + h^N\|Z_N u\|, \tag{4.25}$$

for some $z_N \in S_{l-N}^{m-N}$. This follows from (4.22) by the usual iterative elliptic parametrix construction as in [DyZw, Theorem E.32].

To handle F_j, we define an escape function (based on the usual $-r\rho$ but modified to be nonnegative near F_j and more slowly growing) as follows. For $\delta \in (0, 1/4)$, take $\tilde{q}_\delta \in C^\infty(\mathbb{R})$ with $\tilde{q}_\delta(x) = x^\delta$ for $x \ge 2$, $\tilde{q}_\delta(x) = |x|^{-\delta}$ for $x \le -2$, and $\tilde{q}_\delta'(x) > 0$ for $|x| < 2$, and put

$$q(r,\rho) := \tilde{q}_\delta(-r\rho)\chi_q(r,\rho), \tag{4.26}$$

where $\chi_q \in S_0^{-\infty}$ is real valued, is 1 near all of the F_j, and vanishes in a neighborhood of

$$\{(r,\rho) \mid r \notin (-1, 1 + \max_j R_j) \text{ and } \rho = 0\}$$

whose boundary consists of two line segments and four half lines as in Figure 4.2.

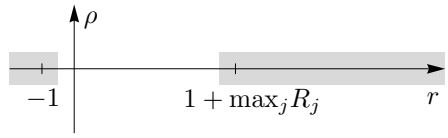

FIGURE 4.2. The kind of neighborhood where χ_q must vanish.

Then $q \in S^{-\infty}_{\delta}$, and near F_j we have

$$\{\operatorname{Re} p_j, q^2\} = 2(-2\rho^2 + rV_j'(r))\tilde{q}_\delta'(-r\rho)\tilde{q}_\delta(-r\rho) \le -cr^{-1-2\delta}, \tag{4.27}$$

for some $c > 0$ (here we used $V_j \ge E_*/2 \Longrightarrow rV_j' \le -1/C$).

Consequently, there are real valued symbols $b \in S^{-\infty}_{-\frac{1}{2}+\delta}$ and $a_0 \in S^{-\infty}_{-1+2\delta}$ such that

$$b^2 = \{q^2, \operatorname{Re} p_j\} + a_0, \tag{4.28}$$

and such that $\operatorname{supp} a_0 \cap F_j = \varnothing$ and $b \ge cr^{-\frac{1}{2}-\delta} > 0$ near F_j; for example we can take $b := \{q^2, \operatorname{Re} p_j\}^{1/2}\chi_b$ for some $\chi_b \in S^{-\infty}_0$ with $\chi_b = 1$ near F_j and supported in the set where (4.27) holds. Note that q depends on δ, and b and a_0 depend on δ and j, although our notation does not reflect this.

Using (4.28), (4.22), and (4.24), we can write

$$B^*B = \frac{i}{h}[Q^*Q, \operatorname{Re} P_j] + A_0 + hA_1,$$

for some $a_1 \in S^{-\infty}_{-2+2\delta}$, giving

$$\|Bu\|^2 = \frac{i}{h}\langle [Q^*Q, \operatorname{Re} P_j]u, u\rangle + \langle A_0 u, u\rangle + h\langle A_1 u, u\rangle,$$

Combining this with (4.25) and the similar elliptic estimate

$$\|B'u\| \le C\|Bu\| + h^N\|Z_N u\|, \tag{4.29}$$

which holds for all $b' \in S^{-\infty}_{-\frac{1}{2}-\delta}$ which is supported in a small enough neighborhood of F_j and for suitable $z_N \in S^{-\infty}_{-\frac{1}{2}-\delta-N}$, we have (since $\delta < 1/4$),

$$\|(1+r_+)^{-\frac{1}{2}-\delta}u\|^2 \le C\frac{i}{h}\langle [Q^*Q, \operatorname{Re} P_j]u, u\rangle + C\|(P_j - E_j - i\varepsilon)u\|^2.$$

Next

$$i\langle [Q^*Q, \operatorname{Re} P_j]u, u\rangle = -2\operatorname{Im}\langle Q(P_j - E_j - i\varepsilon)u, Qu\rangle - 2\operatorname{Re}\langle Q(W_e(r)+\varepsilon)u, Qu\rangle,$$

giving

$$\|(1+r_+)^{-\frac{1}{2}-\delta}u\|^2 \le \frac{C}{h^2}\|(1+r_+)^{\frac{1}{2}+3\delta}(P_j - E_j - i\varepsilon)u\|^2 - \frac{C}{h}\operatorname{Re}\langle Q(W_e(r)+\varepsilon)u, Qu\rangle.$$

But

$$-\operatorname{Re}\langle Q(W_e(r)+\varepsilon)u, Qu\rangle \le |\operatorname{Re}\langle Q^*[Q, W_e(r)]u, u\rangle|,$$

thanks to $W_e + \varepsilon \geq 0$, and by (4.22) and (4.24) we have $\operatorname{Re} Q^*[Q, W_e(r)] = h^2 a_2$ for some $a_2 \in S^{-\infty}_{-\infty}$, giving

$$|\operatorname{Re}\langle Q^*[Q, W_e(r)]u, u\rangle| = h^2\langle A_2 u, u\rangle.$$

This proves (4.18) with $s = \frac{1}{2} + 3\delta$, and taking $\delta > 0$ small enough proves it for all $s > 1/2$. □

Proof of (4.19). Let

$$u := (P_j - E_j - i\varepsilon)^{-1}\chi_+(r)\psi(hD_r)v,$$

with $\|v\| = 1$, and fix $\delta \in (0, 1/4)$. We will use the following argument by induction to prove (4.19).

The inductive hypothesis is that for a given $k \in \mathbb{R}$ there is a neighborhood U of $F_j \setminus (3, \infty) \times (0, \infty)$ such that $\|Au\| \leq Ch^k$ for any $a \in S^{-\infty}_{k+\frac{1}{2}-\delta}$ which is supported in U.

The inductive step is that there is a (smaller) neighborhood U' of $F_j \setminus (3, \infty) \times (0, \infty)$ such that

$$\|A'u\| \leq Ch^{k+1/2}, \tag{4.30}$$

for any $a' \in S^{-\infty}_{k+1+\delta}$ which is supported in U'.

Let us see first that (4.30) for arbitrary k implies (4.19). Indeed, by the elliptic estimate (4.25), the composition formula (4.22), and the resolvent estimate (4.18), we see that

$$\|A''u\| \leq C_N h^N \tag{4.31}$$

for any $N \in \mathbb{R}$ and $a'' \in S^{\infty}_{-\infty}$ such that $\operatorname{supp} a'' \subset (0, 3) \times \mathbb{R}$ and $\operatorname{supp} a'' \cap F_j = \varnothing$. Then we can write

$$\chi_-(r)u = \chi_-(r)\varphi_F(hD_r)u + \chi_-(r)(1 - \varphi_F(hD_r))u$$

for $\varphi_F \in C_c^\infty(\mathbb{R})$ chosen such that (4.30) applies to the first term on the right and (4.31) applies to the second.

We remark in passing that elaborating this argument we can actually show that u is semiclassically trivial everywhere away from the union of two sets (including uniformly as $|r| \to \infty$ and $|\rho| \to \infty$): the first is $\operatorname{supp}\chi_+ \times \operatorname{supp}\psi$, and the second is $F_j \cap (3, \infty) \times (0, \infty)$ which we can think of as a neighborhood of the forward bicharacteristic flowout of the first. Here we are focusing on a more concrete and narrower version of this conclusion which is sufficient for our purposes.

Next observe that the base case (the inductive hypothesis with $k = -1$ and $U = T^*\mathbb{R}$) follows from the resolvent estimate (4.18).

It remains to prove (4.30) under the inductive hypothesis. Roughly speaking, we use an escape function which on $F_j \setminus (3, \infty) \times (0, \infty)$ agrees with the one used in the proof of (4.18) above, but is adapted to vanish near $\operatorname{supp}\chi_+ \times \operatorname{supp}\psi$ and $F_j \setminus U$. (Note that $F_j \setminus U = \varnothing$ when $k = -1$ but that for $k > -1$ we expect $F_j \setminus U \neq \varnothing$ in general).

More specifically, to define the escape function, fix $\chi_k, \psi_k \in C^\infty(\mathbb{R})$ nondecreasing, and satisfying $\chi_k = 0$ near $(-\infty, 3]$, $\psi_k = 0$ near $(-\infty, 0]$, $\psi_k = 1$ near $[\sqrt{E_*/3}, \infty)$, and $\chi_k(r)\psi_k(\rho) = 1$ near $F_j \setminus U$. Then let

$$q_k(r, \rho) := \tilde{q}_{k+\frac{3}{2}-\delta}(-r\rho)\chi_q(r, \rho)(1 - \chi_k(r)\psi_k(\rho)),$$

where $\tilde{q}_{k+\frac{3}{2}-\delta}$ and χ_q are as in (4.26), so that $q_k \in S^{-\infty}_{k+\frac{3}{2}-\delta}$. Calculating as in (4.27), we see that near F_j we have

$$\{\operatorname{Re} p_j, q_k^2\} \leq 0,$$

and near $F_j \setminus (3,\infty)\times(0,\infty)$ we have $\chi_k(r)\psi_k(\rho) = 0$ and hence $\{\operatorname{Re} p_j, q_k^2\} \leq -cr^{2k+2-2\delta} < 0$ (this is slightly better than (4.27) because outside of a compact set we have $\rho < 0$ on $F_j \setminus (3,\infty)\times(0,\infty)$ and in particular we are staying away from the outgoing part of the energy surface).

Consequently, as before, we can write

$$b_k^2 = \{q_k^2, \operatorname{Re} p_j\} + a_{0,k},$$

where $b_k \in S^{-\infty}_{k+1-\delta}$, $a_{0,k} \in S^{-\infty}_{2k+2-2\delta}$, $\operatorname{supp} b_k \subset \operatorname{supp} q_k$, $\operatorname{supp} a_{0,k} \cap (F_j \cup \operatorname{supp}\chi_+ \times \operatorname{supp}\psi) = \varnothing$, and $b_k \geq cr^{k+1-\delta} > 0$ near $F_j \setminus (3,\infty)\times(0,\infty)$. Hence

$$B_k^* B_k = \frac{i}{h}[Q_k^* Q_k, \operatorname{Re} P_j] + A_{0,k} + hA_{1,k},$$

for some $a_{1,k} \in S^{-\infty}_{2k+1-2\delta}$. We refine this by using (4.22) and (4.24) to expand $a_{1,k}$ in powers of h up to h^N in terms of b_k, q_k, p_j, $a_{0,k}$, and their derivatives, which gives

$$B_k^* B_k = \frac{i}{h}[Q_k^* Q_k, \operatorname{Re} P_j] + A_{0,k} + hA'_{1,k} + h^N Z_N,$$

where $a'_{1,k} \in S^{-\infty}_{2k+1-2\delta}$ has $\operatorname{supp} a'_{1,k} \subset \operatorname{supp} q_k$ and $z_N \in S^{-\infty}_{2k+2-2\delta-N}$. Consequently

$$\|B_k u\|^2 = \frac{i}{h}\langle [Q_k^* Q_k, \operatorname{Re} P_j]u, u\rangle + \langle A_{0,k}u, u\rangle + h\langle A'_{1,k}u, u\rangle + h^N\langle Z_N u, u\rangle.$$

By the elliptic estimate (4.29) with b_k in place of b we see that to deduce (4.30) it is enough to show

$$\|B_k u\|^2 \leq Ch^{2k+1}. \tag{4.32}$$

Now $\langle A_{0,k}u, u\rangle = O(h^\infty)$ by (4.25). Also, since q_k vanishes near $F_j \setminus U$, it follows that $a'_{1,k}$ vanishes near $F_j \setminus U$, so by (4.25), (4.22), and the inductive hypothesis, we have

$$|\langle A'_{1,k}u, u\rangle| \leq Ch^{2k}.$$

Hence to show (4.32) it suffices to show that

$$i\langle [Q_k^* Q_k, \operatorname{Re} P_j]u, u\rangle \leq Ch^{2k+2}. \tag{4.33}$$

As before we write, for any $N \in \mathbb{R}$,

$$\begin{aligned} i\langle [Q_k^* Q_k, \operatorname{Re} P_j]u, u\rangle &= -2\operatorname{Im}\langle Q_k(P_j - E_j - i\varepsilon)u, Q_k u\rangle - 2\operatorname{Re}\langle Q_k(W_e(r) + \varepsilon)u, Q_k u\rangle \\ &\leq 2|\operatorname{Re}\langle Q_k^*[Q_k, W_e(r)]u, u\rangle| + O(h^\infty), \end{aligned}$$

where we used $\operatorname{supp} q_k \cap \operatorname{supp}\chi_+ \times \operatorname{supp}\psi = \varnothing$. Now (4.33) follows from the inductive hypothesis together with the fact that (arguing as in the construction of $a'_{1,k}$ above) $\operatorname{Re} Q_k^*[Q_k, W_e(r)] = h^2 A_{2.k} + h^N Z_N$, with $a_{2,k}$, $z_N \in S^{-\infty}_{-\infty}$, and $\operatorname{supp} a_{2,k} \cap F_j \subset U$. □

4.5. **Proof of Theorems 3.1 and 3.2.** In this section all operator norms are $L^2(X) \to L^2(X)$. We implement the outline discussed in §4.1. We assume without loss of generality that $\varepsilon \in (0,1]$, as the statements with $\varepsilon > 1$ follow from self-adjointness and the statements with $\varepsilon < 0$ then follow by taking the adjoint.

We first explain the key dynamical property of the bicharacteristic flow in X_e which allows us to remove the remainders in the parametrix construction.

Let us denote points in T^*X_e by (r, y, ρ, η), where $y \in Y$, ρ is dual to r, and η is dual to y. The *energy surface* for P in T^*X_e at energy E_0 is the subset of T^*X_e defined by

$$p(r, y, \rho, \eta) := \rho^2 + |\eta|^2 f(r)^{-4/(d-1)} + V_L(r) = E_0,$$

and *bicharacteristics* in T^*X_e of this energy surface are solutions $\gamma(t) := ((r(t), y(t), \rho(t), \eta(t))$ to the Hamiltonian equation of motion $\dot\gamma(t) := \frac{d}{dt}\gamma(t) = \{p, \gamma(t)\}$. The *backward bicharacteristic flowout* in T^*X_e of a point $\gamma_0 \in T^*X_e$ is the set of points $\gamma' \in T^*X_e$ such that if $\gamma(t)$ is the bicharacteristic in T^*X_e with $\gamma(0) = \gamma_0$, then $\gamma(t) = \gamma'$ for some $t \le 0$; note that some bicharacteristics enter $T^*(X \setminus X_e)$ in finite time, and our definition only counts them while they stay in T^*X_e.

If $\gamma(t) := ((r(t), y(t), \rho(t), \eta(t))$ is a bicharacteristic, then

$$\dot r(t) = 2\rho(t), \qquad \dot\rho(t) = \frac{4}{d-1}|\eta|^2 f'(r(t)) f(r(t))^{-(d+3)/(d-1)} - V_L'(r(t)) \ge 0, \tag{4.34}$$

and hence $\ddot r = 2\dot\rho \ge 0$. Consequently no bicharacteristic can visit the sets $T^*((0,4))$, $T^*((4,5))$, and $T^*((2,3))$ in that order (here and below $T^*((a,b))$ denotes the subset of T^*X_e on which $a < r < b$), and this fact is exploited to prove the crucial remainder estimate in (4.38) below.

Fix $\chi_e, \chi_K \in C^\infty(\mathbb{R})$ such that $\chi_e + \chi_K = 1$, $\operatorname{supp}\chi_e \subset (3, \infty)$, and $\operatorname{supp}\chi_K \subset (-\infty, 4)$. Define a parametrix for $P - E - i\varepsilon$ by

$$G := \chi_K(r-1) R_K \chi_K(r) + \chi_e(r+1) R_e \chi_e(r).$$

Here

$$R_K = R_K(E_0 + i\varepsilon) := (-h^2\Delta - iW_K(r) - E_0 - i\varepsilon)^{-1},$$

and

$$R_e = R_e(E_0 + i\varepsilon) := f(r) \sum_{j=0}^{\infty} \left((P_j - i\varepsilon)^{-1} \phi_j \otimes \phi_j\right) f(r)^{-1},$$

and

$$\|R_K\| \le Ca(h)h^{-1}, \qquad \|(1+r)^{-s_1}\chi_e(r+1) R_e \chi_e(r)(1+r)^{-s_2}\| \le Ch^{-2}. \tag{4.35}$$

Indeed, R_K is well defined and obeys (4.35) thanks to (3.3); this follows from the resolvent identity for $\varepsilon > 0$ small enough and then from the bound $\operatorname{Im}(-h^2\Delta - iW_K(r) - E_0 - i\varepsilon) \le -\varepsilon$ for all $\varepsilon > 0$. Meanwhile $\chi_e(r+1)R_e\chi_e(r)$ acts on $L^2(X)$ thanks to (4.2) and the support property of χ_e, even though R_e acts on a funny space due to the way we defined the operators P_j differently depending on j; moreover R_e obeys (4.35) by (4.5) and (4.18).

Define operators A_K and A_e by

$$(P - E_0 - i\varepsilon)G = I + [h^2 D_r^2, \chi_K(r-1)] R_K \chi_K(r) + [h^2 D_r^2, \chi_e(r+1)] R_e \chi_e(r) =: I + A_K + A_e.$$

Our next step is to remove the remainders A_K and A_e. The idea of [DaVa1] is to do this using a *semiclassically outgiong* property of the resolvents R_K and R_e.

To explain this property, we use the following notation: if $U \subset T^*X_e$, then $\Gamma_+ U$ is the set of points in T^*X_e whose backward bicharacteristic flowout intersects U. Now in the case of R_K, the needed semiclassically outgoing property says (in the notation of (4.20) and (4.21)) that if $\tilde{\chi} \in C_c^\infty((0,\infty))$ and $a \in S_l^0$, then

$$\|\tilde{\chi}(r)\operatorname{Op}_h(a)A_K\| = O(h^\infty), \tag{4.36}$$

provided $|\partial_r^{n_1}\partial_\rho^{n_2} a(r,\rho)| = O(h^\infty)$ for every $n_1,\ n_2 \in \mathbb{N}_0$ and for every $(r,\rho) \in T^*((0,4)) \cup \Gamma_+ T^*((0,4))$. This property follows from [DaVa1, Lemma 5.1].

On the other hand, the resolvent R_e is only semiclassically outgoing for j such that $E_j \geq c > 0$ (the relevant statement for us is (4.19)); as $E_j \to 0$ this property fails, but then the gluing region (the part of X such that $r \in (2,5)$) becomes classically forbidden, and so we will be able to estimate and remove remainders using the Agmon estimates of §4.3.

More specifically, we observe that

$$\|A_K\| \leq C(1+a(h)), \qquad \|A_e(1+r)^{-s_2}\| \leq C. \tag{4.37}$$

Indeed, A_K obeys the bound thanks to the corresponding bound on R_K in (4.35); note that $\|R_K\|_{L^2\to H_h^2(X)} \leq C\|R_K\|$ since V, W, and ε are bounded, and E_0 is fixed. Meanwhile A_e obeys the bound by (4.14) and (4.18).

We refine the parametrix with some correction terms, observing that $A_K^2 = A_e^2 = 0$:

$$(P - E_0 - i\varepsilon)G(I - A_K - A_e + A_K A_e) = I - A_e A_K + A_e A_K A_e.$$

We will show that

$$\|A_e A_K\| = O(h^\infty). \tag{4.38}$$

Assuming (4.38) for the moment, we may write (using $R_e\chi_e(r)A_e = R_K\chi_K(r)A_K = 0$)

$$\begin{aligned}(P - E_0 - i\varepsilon)^{-1} &= G(I - A_K - A_e + A_K A_e)(I - A_e A_K + A_e A_K A_e)^{-1}\\ &= \chi_e(r+1)R_e\chi_e(r) + \chi_K(r-1)R_K\chi_K(r) - \chi_e(r+1)R_e A_K\\ &\quad - \chi_K(r-1)R_K A_e + \chi_e(r+1)R_e A_K A_e + O(h^\infty).\end{aligned} \tag{4.39}$$

Note that by by (4.14), (4.18), and the bound on $\|R_K\|$ in (4.35), we have

$$\|(1+r)^{-s_1}\chi_e(r+1)R_e A_K\| \leq Ca(h)h^{-1}. \tag{4.40}$$

Now multiplying (4.39) on the left by $(1+r)^{-s_1}$ and on the right by $(1+r)^{-s_2}$ and estimating the norm on the right term by term, we see that by (4.35) the first term on the right has norm bounded by Ch^{-2}, while by (4.35), (4.37), and (4.40), the next four terms have norm bounded by $Ca(h)h^{-1}$. This implies (3.4).

We similarly deduce (3.6) from (4.39), but rather than using the bound on R_e in (4.35), we use

$$\|(1+r)^{-s}\chi_e(r+1)\chi_{\mathcal{J}} R_e \chi_e(r)(1+r)^{-s}\| \leq Ch^{-1}. \tag{4.41}$$

To prove (4.41), we use (4.6) when $E_j \in [-c_{\mathcal{J}}h, c_{\mathcal{J}}]$, we use (4.18) when $E_j \geq c_{\mathcal{J}}$, and we use the fact that P_j is almost nonnegative (more precisely, $P_j \geq -Ch^2$ by (4.2) and (4.4)) when $E_j \leq -c_{\mathcal{J}}h$.

To complete the proofs of Theorems 3.1 and 3.2, it remains to show (4.38). We have

$$A_e A_K = [\chi_e(r+1), h^2 D_r^2] R_e [\chi_K(r-1), h^2 D_r^2] R_K \chi_K(r).$$

Fix $\tilde{\chi} \in C_c^\infty((3,6))$ which is 1 on $[4,5]$, so that

$$A_e A_K = [\chi_e(r+1), h^2 D_r^2] R_e \tilde{\chi}(r) [\chi_K(r-1), h^2 D_r^2] R_K \chi_K(r).$$

For any $\psi \in C_c^\infty((0,\infty))$ we have

$$\|[\chi_e(r+1), h^2 D_r^2] R_e \tilde{\chi}(r) \psi(hD_r) [\chi_K(r-1), h^2 D_r^2]\| = O(h^\infty),$$

by (4.15) and (4.19), so it remains to show that there is $\psi \in C_c^\infty((0,\infty))$ such that

$$\|\tilde{\chi}(r)(I - \psi(hD_r))[\chi_K(r-1), h^2 D_r^2] R_K \chi_K(r)\| = O(h^\infty).$$

We will deduce this from (4.36). Indeed, it is enough to check that there is $\rho_0 > 0$ such that if $\gamma(t)$ is a bicharacteristic at energy E_0 with $\gamma(0) \in T^* \operatorname{supp} \chi_K(r)$ and with $\gamma(T) \in T^* \operatorname{supp} \chi_K'(r-1)$ for some $T > 0$, then $\rho(T) \geq \rho_0$ (we already know that $\rho(T)^2 \leq E_0$, so we may then take ψ to be 1 near $[\rho_0, \sqrt{E_0}]$).

Thanks to (4.34) we know that $\rho(t)$ is nondecreasing, so we may assume that $\max \operatorname{supp} \chi_K(r) < r(t) < \min \operatorname{supp} \chi_K'(r-1)$ when $t \in (0,T)$, which implies in particular $\rho(0) \geq 0$. Then, for $t \in (0,T)$, we have $f(r(t)) \leq C f'(r(t))$ and $V_L(r(t)) \leq -C V_L'(r(t))$, so that

$$\dot{\rho}(t) \geq (|\eta|^2 f(r)^{-4/(d-1)} + V_L(r))/C_0 = (E_0 - \rho(t)^2)/C_0.$$

If $\rho(0) = \sqrt{E_0}$, then $\rho(T) = \sqrt{E_0}$ and we are done; otherwise we can integrate and use $\rho(0) \geq 0$ to obtain

$$\frac{C_0}{\sqrt{E_0}} \tanh^{-1}\left(\frac{\rho(T)}{\sqrt{E_0}}\right) \geq T = \frac{r(T) - r(0)}{2\bar{\rho}} \geq \frac{r(T) - r(0)}{2\rho(T)},$$

where we used $\bar{\rho} := T^{-1} \int_0^T \rho(t) dt \leq \rho(T)$. This implies $\rho(T) \geq \rho_0$, for some $\rho_0 > 0$ depending on C_0, E_0, and χ_K.

5. Continuation of the resolvent

In this section we keep all of the assumptions of §3.1, and add the assumption that

$$r \geq 6 \Longrightarrow V_L(r) = f(r) - 1 = 0.$$

In §5.1 we briefly review how meromorphic continuation works in this setting, following [Gu] and [Me, §6.7], and introduce the relevant notation. In §5.2 we prove some useful estimates for a model problem on the cylindrical end. In §5.3 we use an identity of Vodev from [Vo] to deduce the existence of a resonance free region.

Roughly speaking, writing $R(z)$ for the resolvent $(P-z)^{-1}$ and for its meromorphic continuation, we deduce from (3.4) that

$$\|\chi R(E_0 \pm i0)\chi\| \lesssim 1/\mu(h),$$

where $\chi \in C_c^\infty(X)$ and $0 < \mu(h) \leq h^2$. Then we use Vodev's identity to show that this implies

$$\|\chi R(z)\chi\| \lesssim 1/\mu(h),$$

as long as the distance from z to $E_0 \pm i0$ is small compared to $\mu(h)$. However some care is needed due to the complicated nature of the Riemann surface to which $R(z)$ continues (see §5.1), and due to the fact that our model resolvent obeys somewhat weaker bounds than the one used in [Vo] (see §5.2). The precise statement and proof are in §5.3.

Although we keep all of the assumptions of §3.1 in this section, strictly speaking they are not all needed once we have (3.4). Instead, as long as we had (3.4), we could allow X to be a more general manifold with cylindrical ends, or allow P to be a black-box perturbation of the Laplacian e.g. in the sense of [ChDa1, §2]. The proof could also be adapted to include the case of waveguides. We omit these generalizations here, to simplify the presentation and because all of our interesting examples satisfy the assumptions of §3.1.

5.1. **Meromorphic continuation of the resolvent.** In §5.1 we think of $h > 0$ as being fixed, until Lemma 5.2, in which we prove an estimate which is uniform as $h \to 0$.

The spectrum of P is given by $[0, \infty)$ together with a finite (possibly empty) set of negative eigenvalues. For z not in the spectrum we define the resolvent

$$R(z) := (P-z)^{-1} \colon L^2(X) \to L^2(X).$$

To define the Riemann surface onto which $R(z)$ meromorphically continues, for each $j \in \mathbb{N}_0$, and $z \in \mathbb{C} \setminus [h^2\sigma_j^2, \infty)$, we introduce the notation

$$\rho_j(z) := \sqrt{z - h^2\sigma_j^2},$$

with the branch of the square root chosen such that $\operatorname{Im} \rho_j(z) > 0$ for this range of z (recall that $0 = \sigma_0 \leq \sigma_1 \leq \cdots$ are the square roots of the eigenvalues of the nonnegative Laplacian on (Y, g_Y) included according to multiplicity).

For each $j \in \mathbb{N}_0$, there is a minimal Riemann surface $\hat{Z}_{h,j}$ onto which ρ_j continues analytically from $\mathbb{C} \setminus [h^2\sigma_j^2, \infty)$; this is a double cover of $\mathbb{C}$ ramified at the singular point $z = h^2\sigma_j^2$. By elaborating the construction of $\hat{Z}_{h,j}$, we see that there is a minimal Riemann surface $\hat{Z}_h$ onto which all the ρ_j extend simultaneously from $\mathbb{C} \setminus [0, \infty)$. This is a countable cover of $\mathbb{C}$, ramified at $z = h^2\sigma_j^2$ for each j, and for each $z \in \hat{Z}_h$ we have $\operatorname{Im} \rho_j(z) > 0$ for all but finitely many j. For more details, see [Gu] and [Me, §6.7].

We use p to denote the projection $\hat{Z}_h \to \mathbb{C}$, we use the term *physical region* to refer to the sheet over $\mathbb{C} \setminus [0, \infty)$ on which $\operatorname{Im} \rho_j > 0$ for all j, and for notational convenience we identify the physical region with $\mathbb{C} \setminus [0, \infty)$. Then $R(z)$ continues meromorphically from the resolvent set

in $\mathbb{C} \setminus [0, \infty)$ to all of $\hat{Z}_h$, as an operator from compactly supported L^2 functions to locally L^2 functions, and we have $(P - p(z))R(z) = I$. We refer to the poles of $R(z)$ as *resonances*.

For $E \geq 0$, we denote by $E \pm i0$ the points in $\hat{Z}_h$ on the boundary of the physical region which are obtained as limits $\lim_{\pm\delta\downarrow 0} E + i\delta$. Note that $\rho_j(E \pm i0) \in i\mathbb{R}_+$ if $E < h^2\sigma_j^2$, and $\pm\rho_j(E \pm i0) > 0$ if $h^2\sigma_j^2 < E$. Below we will only be concerned with points on $\hat{Z}_h$ which are quite close to the boundary of the physical region. To measure how far apart two points on $\hat{Z}_h$ are we use the following

Lemma 5.1. *The function $d_h : \hat{Z}_h \times \hat{Z}_h \to [0, \infty]$ given by*

$$d_h(z, z') := \sup_j |\rho_j(z) - \rho_j(z')| \tag{5.1}$$

takes only finite values and is a metric on $\hat{Z}_h$.

Proof. To see that $|\rho_j(z) - \rho_j(z')|$ is bounded in j, note that

$$p(z) - p(z') = \rho_j^2(z) - \rho_j^2(z') = (\rho_j(z) - \rho_j(z'))(\rho_j(z) + \rho_j(z')). \tag{5.2}$$

Using that $\rho_j^2(z) = p(z) - h^2\sigma_j^2$, we find $\operatorname{Re}\rho_j^2(z) \to -\infty$ as $j \to \infty$. Since $\operatorname{Im}\rho_j(z) > 0$ if j is sufficiently large, $\operatorname{Im}\rho_j(z) \to \infty$ as $j \to \infty$ and we find, since the same is true for z', that for j large enough $|\rho_j(z) - \rho_j(z')| < |\rho_j(z) + \rho_j(z')|$. Since by (5.2), we have

$$\min\{|\rho_j(z) - \rho_j(z')|, |\rho_j(z) + \rho_j(z')|\} \leq |p(z) - p(z')|^{1/2},$$

we have for j sufficiently large, $|\rho_j(z) - \rho_j(z')| \leq |p(z) - p(z')|^{1/2}$.

That d_h is a metric is fairly straightforward; for completeness we check the triangle inequality. Let $z,\ z', w \in \hat{Z}_h$. Then

$$|\rho_j(z) - \rho_j(z')| \leq |\rho_j(z) - \rho_j(w)| + |\rho_j(w) - \rho_j(z')|.$$

But then

$$\begin{aligned} d_h(z, z') = \sup_j |\rho_j(z) - \rho_j(z')| &\leq \sup_j (|\rho_j(z) - \rho_j(w)| + |\rho_j(w) - \rho_j(z')|) \\ &\leq \sup_j |\rho_j(z) - \rho_j(w)| + \sup_j |\rho_j(w) - \rho_j(z')| = d_h(z, w) + d_h(w, z'). \end{aligned}$$

□

Later we will want to use $d_h(z, z')$ in a resolvent identity, and now we show that $d_h(z, z')$ controls $|p(z) - p(z')|$, at least when z' is on the boundary of the physical region:

Lemma 5.2. *Let $E > 0$, and let $E \pm i0$ denote one of the points on the boundary of the physical space in $\hat{Z}_h$ as described above. Then for any $\delta > 0$, if $h > 0$ is sufficiently small,*

$$|p(z) - E| \leq d_h(z, E \pm i0)[d_h(z, E \pm i0) + O(h^{1/2-\delta})]$$

for $z \in \hat{Z}_h$.

Proof. We have, for any $j \in \mathbb{N}$,

$$\begin{aligned} |p(z) - E| &= |\rho_j^2(z) - \rho_j^2(E \pm i0)| \\ &= |\rho_j(z) - \rho_j(E \pm i0))||\rho_j(z) - \rho_j(E \pm i0) + 2\rho_j(E \pm i0)| \\ &\leq |\rho_j(z) - \rho_j(E \pm i0)| \left(|\rho_j(z) - \rho_j(E \pm i0)| + 2|\rho_j(E \pm i0)|\right). \end{aligned} \tag{5.3}$$

By the Weyl law, for any $\delta' > 0$ there is an $h_0 = h_0(\delta') > 0$ so that if $0 < h < h_0$, the interval $[Eh^{-2} - h^{-1-\delta'}, Eh^{-2} + h^{-1-\delta'}]$ contains an element of the spectrum of $-\Delta_Y$; call this $\sigma_{j_0}^2$. We note that j_0 depends on E and on h, but our notation does not reflect that dependence. Then

$$|\rho_{j_0}(E \pm i0)|^2 = |E - h^2\sigma_{j_0}^2| \leq h^{1-\delta'}.$$

Using this in (5.3) with $j = j_0$ proves the lemma, since $|\rho_{j_0}(z) - \rho_{j_0}(E \pm i0)| \leq d_h(z, E \pm i0)$. □

5.2. **Resolvent estimates for the model problem on the cylindrical end.** Let $X_0 = [0, \infty) \times Y$, let $\Delta_0 \leq 0$ be the Laplacian on $(X_0, dr^2 + g_Y)$, and for $h > 0$ and $z \in \mathbb{C} \setminus [0, \infty)$, let

$$R_0(z) := (-h^2\Delta_0 - z)^{-1}$$

denote the semiclassical Dirichlet resolvent.

For $\operatorname{Im}\xi > 0$, let $R_D(\xi)$ be the resolvent for the Dirichlet Laplacian on the half-line with spectral parameter ξ^2 and Schwartz kernel given by

$$R_D(\xi, r, r') = \frac{i}{2h\xi}(e^{i\xi|r-r'|/h} - e^{i\xi(r+r')/h}). \tag{5.4}$$

Then, for z in the physical region of $\hat{Z}_h$ (see §5.1), we have

$$R_0(z) = \sum_{j=0}^{\infty} R_D(\rho_j(z))\phi_j \otimes \phi_j, \tag{5.5}$$

where $\{\phi_j\}_{j=0}^{\infty}$ is a complete set of real-valued orthonormal eigenfunctions of the Laplacian on Y and $-\Delta_Y\phi_j = \sigma_j^2\phi_j$.

Moreover, $R_0(z)$ continues holomorphically to $\hat{Z}_h$ as an operator from compactly supported L^2 functions to locally L^2 functions. In this section we prove some estimates for $R_0(z)$ which will be needed when we use a resolvent identity to find a neighborhood of the boundary of the physical region in which $R(z)$ has no poles.

Proposition 5.3. *Let $\chi \in C_c^\infty([0, \infty))$ and fix $N > 0$. If $\operatorname{Im}\xi,\ \operatorname{Im}\xi' > -Nh$, then*

$$\|\chi R_D(\xi)\chi - \chi R_D(\xi')\chi\| \leq Ch^{-3}|\xi - \xi'|. \tag{5.6}$$

If $\operatorname{Im}\xi,\ \operatorname{Im}\xi' > -Nh$ and $\alpha_1 + \alpha_2 = 1, 2$, then

$$\|\chi h^{\alpha_1}D_r^{\alpha_1}R_D(\xi)h^{\alpha_2}D_r^{\alpha_2}\chi - \chi h^{\alpha_1}D_r^{\alpha_1}R_D(\xi')h^{\alpha_2}D_r^{\alpha_2}\chi\| \leq Ch^{-2}|\xi - \xi'|(|\xi| + |\xi'| + 1)^{\alpha_1+\alpha_2-1}. \tag{5.7}$$

Fix $\delta > 0$ and suppose $\delta < \arg\xi, \arg\xi' < \pi - \delta$ and $|\xi|, |\xi'| \geq 1$. Then if $\alpha_1 + \alpha_2 \leq 2$,

$$\|h^{\alpha_1}D_r^{\alpha_1}R_D(\xi)h^{\alpha_2}D_r^{\alpha_2}\chi - \chi h^{\alpha_1}D_r^{\alpha_1}R_D(\xi')h^{\alpha_2}D_r^{\alpha_2}\| \leq C|\xi - \xi'|. \tag{5.8}$$

All the norms above are $L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$, and the constants depend on χ, N, and δ.

Proof. We begin with (5.6). Note that $\chi \frac{d}{d\xi} R_D(\xi)\chi$ has Schwartz kernel

$$\frac{i\chi(r)}{2h^3(\xi/h)^2}\left((-1+i|r-r'|\frac{\xi}{h})e^{i\xi|r-r'|/h}-(-1+i(r+r')\frac{\xi}{h})e^{i\xi(r+r')/h}\right)\chi(r').$$

With $\operatorname{Im}\xi > -Nh$, this can be pointwise bounded by C/h^3, even when $\xi \to 0$, and hence since χ is compactly supported we have $\|\chi \frac{d}{d\tau} R_D(\tau)\chi\| \le \frac{C}{h^3}$. Integrating from ξ to ξ' gives (5.6). We note for future reference that if $|\xi| \ge h$, then we can improve the estimate to

$$\begin{aligned}\left|\frac{i\chi(r)}{2h^3(\xi/h)^2}\left(\left(-1+i|r-r'|\frac{\xi}{h}\right)e^{i\xi|r-r'|/h}-\left(-1+i(r+r')\frac{\xi}{h}\right)e^{i\xi(r+r')/h}\right)\chi(r')\right| \\ \le C/(h^2|\xi|), \text{ when } |\xi| \ge h.\end{aligned} \tag{5.9}$$

Next consider the operator $h\frac{\partial}{\partial r}R_D(\xi)$. It has Schwartz kernel

$$\frac{-1}{2h}\left(\operatorname{sgn}(r-r')e^{i\xi|r-r'|/h}-e^{i\xi(r+r')/h}\right).$$

Differentiating this with respect to ξ and proceeding as above gives $\left\|\chi \frac{d}{d\tau} h\frac{\partial}{\partial r}R_D(\tau)\chi\right\| \le \frac{C}{h^2}$. Integrating in τ from ξ to ξ' gives (5.7) for $\alpha_1 = 1$, $\alpha_2 = 0$. To prove (5.7) for $\alpha_1 = 2$, $\alpha_2 = 0$, we can argue as before using the Schwartz kernel. Alternately, we can note that $h^2\frac{\partial^2}{\partial r^2}R_D(\xi) = I + \xi^2 R_D(\xi)$ and proceed as in the proof of the first inequality, using the improvement (5.9). Similar techniques give (5.7) when $\alpha_2 \ne 0$, if we consider the Schwartz kernel of $R_D(\xi)\frac{\partial}{\partial_r}$.

When ξ, ξ' satisfy $\delta < \arg\xi, \arg\xi' < \pi - \delta$ they are both in the physical region and we can use the resolvent equation $R_D(\xi) - R_D(\xi') = (\xi^2 - \xi'^2)R_D(\xi)R_D(\xi')$. If $|\xi| \ge 1$, using the bound on $\arg\xi$ we have $\|h^{\alpha_1}D_r^{\alpha_1}R_D(\xi)h^{\alpha_2}D_r^{\alpha_2}\| \le C|\xi|^{\alpha_1+\alpha_2-2}$, where the constant depends on δ. The same inequality holds if ξ is replaced by ξ' everywhere. Using this in the resolvent equation proves (5.8). □

Proposition 5.4. *Let $E > 0$ and consider one of the points $E \pm i0 \in \hat{Z}_h$ which lies on the boundary of the physical region. Fix $N > 0$ and $\chi \in C_c^\infty(X_0)$. Then*

$$\|\chi R_0(z)\chi - \chi R_0(E \pm i0)\chi\| \le Ch^{-3}d_h(z, E \pm i0), \tag{5.10}$$

for all $z \in \hat{Z}_h$ such that $d_h(z, E \pm i0) < Nh$. If $\alpha_1 + \alpha_2 = 1, 2$, then instead

$$\|\chi h^{\alpha_1}D_r^{\alpha_1}R_0(z)h^{\alpha_2}D_r^{\alpha_2}\chi - \chi h^{\alpha_1}D_r^{\alpha_1}R_0(E \pm i0)h^{\alpha_2}D_r^{\alpha_2}\chi\| \le Ch^{-2}d_h(z, E \pm i0), \tag{5.11}$$

for all $z \in \hat{Z}_h$ such that $d_h(z, E \pm i0) < Nh$.

Proof. We begin by noting that for any $j \in \mathbb{N}$, $\operatorname{Im}\rho_j(E \pm i0) \ge 0$, and for $h^2\sigma_j^2 > E$ we have $\rho_j(E \pm i0) \in i\mathbb{R}_+$. Hence if $d_h(z, E \pm i0) < Nh$, then $\operatorname{Im}\rho_j(z) \ge -Nh$ and $\operatorname{Im}\rho_j(z) \to \infty$ as $j \to \infty$.

Without loss of generality, we may assume χ is a function of r only, so that we may consider χ as a function defined on $[0, \infty)$. Using the expression (5.5), we find that

$$\|\chi R_0(z)\chi - \chi R_0(E \pm i0)\chi\|_{L^2(X_0)\to L^2(X_0)} = \sup_j \|\chi R_D(\rho_j(z))\chi - \chi R_D(\rho_j(E \pm i0))\chi\|_{L^2(\mathbb{R}_+)\to L^2(\mathbb{R}_+)}.$$

Now (5.10) follows directly from (5.6) and the definition (5.1) of $d_h(z, E \pm i0)$.

To prove (5.11), we note that for j sufficiently large we have $h^2\sigma_j^2 > E+5$, and $\pi/4 < \arg\rho_j(z), \arg\rho_j(E\pm i0) < 3\pi/4$. Using (5.7) when $h^2\sigma_j^2 \le E+5$ and (5.8) when $h^2\sigma_j^2 > E+5$, along with the definition of $d_h(z, E\pm i0)$ proves (5.11). □

5.3. **The resonance free region.** Throughout §5.3, we keep all of the assumptions of §3.1, as well as the assumption that

$$r \ge 6 \Longrightarrow V_L(r) = f(r) - 1 = 0.$$

To show the existence of a resonance free region, we use an identity due to Vodev [Vo, (5.4)]. In [Vo] the identity is stated only for operators which are potential perturbations of the Laplacian on $\mathbb{R}^d$. However, it in fact holds in far greater generality for operators which are, in an appropriate sense, compactly supported perturbations of each other. Here we state a version adapted to our circumstance.

Lemma 5.5. *([Vo, (5.4)]) Let $\chi_1 \in C_c^\infty(X;[0,1])$ be such that $r \ge 6$ near $\operatorname{supp} 1-\chi_1$. Choose $\chi \in C_c^\infty(X;[0,1])$ so that $\chi\chi_1 = \chi_1$. Then for $z, z_0 \in \hat{Z}_h$,*

$$\begin{aligned}\chi R(z)\chi - \chi R(z_0)\chi &= (p(z)-p(z_0))\chi R(z)\chi\chi_1(2-\chi_1)\chi R(z_0)\chi \\ &\quad + (1-\chi_1-\chi R(z)\chi[h^2\Delta,\chi_1])\,(\chi R_0(z)\chi - \chi R_0(z_0)\chi)\,(1-\chi_1+[h^2\Delta,\chi_1]\chi R(z_0)\chi).\end{aligned}$$

It is important to note in the identity above that $\chi R_0\chi$ only appears where it is multiplied both on the left and right by an operator (either $1-\chi_1$ or $[h^2\Delta,\chi_1]$) supported in the set where $r \ge 6$. If we think of this set as a subset of $X_0 = [0,\infty)\times Y$, then the appearance of $\chi R_0\chi$ makes sense.

We omit the proof of Lemma 5.5 because it is essentially the same as that of [Vo, (5.4)] (see also [DyZw, Lemma 6.26] and, for another version in the setting of cylindrical ends, [ChDa1, Lemma 2.1]).

The proof we give of the following theorem follows the proof of [Vo, Theorem 1.5], but we write it out in detail because it is short and to highlight the role of the estimates we proved in §5.2.

Theorem 5.6. *With χ as in Lemma 5.5, using (3.4) take constants C and $\mu(h)$ such that*

$$\|\chi R(E\pm i0)\chi\|_{L^2(X)\to L^2(X)} \le \frac{C}{\mu(h)},$$

where $E = E_0$ and $0 < \mu(h) \le h^2$. Then there are constants C', $\tilde{C}$ so that for $h > 0$ sufficiently small, $\chi R(z)\chi$ is analytic in $\{z \in \hat{Z}_h : d_h(z, E\pm i0) < C'\mu(h)\}$. Moreover, in this region the cutoff resolvent satisfies the estimate

$$\|\chi R(z)\chi\|_{L^2(X)\to L^2(X)} \le \frac{\tilde{C}}{\mu(h)},$$

with $\tilde{C}$ depending on χ.

Proof. We use the identity from Lemma 5.5, with $z_0 = E \pm i0$. Rearranging, we find (all norms here are $L^2(X) \to L^2(X)$)

$$\begin{aligned}\|\chi R(z)\chi\| \leq \|\chi R(E \pm i0)\chi\| &+ 2|p(z) - E|\|\chi R(z)\chi\|\|\chi R(E \pm i0)\chi\| \\ &+ \|(1-\chi_1)(\chi R_0(z)\chi - \chi R_0(E \pm i0)\chi)(1-\chi_1)\| \\ &+ \|\chi R(z)\chi\|\|[h^2\Delta, \chi_1])(\chi R_0(z)\chi - \chi R_0(E \pm i0)\chi)(1-\chi_1)\| \\ &+ \|(1-\chi_1)\left(\chi(R_0(z)\chi - \chi R_0(E \pm i0)\chi\right)[h^2\Delta, \chi_1]\|\|\chi R(E \pm i0)\chi\| \\ &+ \|\chi R(z)\chi\|\|\chi R(E \pm i0)\chi\|\|[h^2\Delta, \chi_1]\left(\chi R_0(z)\chi - \chi R_0(E \pm i0)\chi\right)[h^2\Delta, \chi_1]\|.\end{aligned}$$

By writing this bound in this detailed fashion we hope to indicate the importance of the improved estimate (5.11) as compared to (5.10), so that, for example,

$$\begin{aligned}&\|[h^2\Delta, \chi_1](\chi R_0(z)\chi - \chi R_0(E \pm i0)\chi)(1-\chi_1)\| \\ &\qquad = \|[h^2\Delta, \chi_1](R_0(z)\chi - R_0(E \pm i0)\chi)(1-\chi_1)\| \leq Cd_h(z, E \pm i0)/h. \quad (5.12)\end{aligned}$$

Using the bound on $\|\chi R(E \pm i0)\chi\|$ from the assumptions along with bounds of Proposition 5.4, we find

$$\begin{aligned}\|\chi R(z)\chi\| \leq \frac{C}{\mu(h)} + \frac{Cd_h(z, E \pm i0)}{\mu(h)}\|\chi R(z)\chi\| + \frac{Cd_h(z, E \pm i0)}{h\mu(h)} \\ + Cd_h(z, E \pm i0)\left(\frac{1}{h} + \frac{1}{\mu(h)}\right)\|\chi R(z)\chi\|.\end{aligned}$$

Here we have also bounded $|p(z) - E| \leq d_h(z, E \pm i0)$, which is weaker than the estimate from Lemma 5.2 since we will have $d_h(z, E \pm i0) = O(\mu(h))$. If we choose C' sufficiently small, the coefficients of $\|\chi R(z)\chi\|$ on the right hand side above will be small enough that the terms with $\|\chi R(z)\chi\|$ can be absorbed in the left hand side, proving the result. □

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