ON AN OLD THEOREM OF ERDÖS ABOUT AMBIGUOUS LOCUS

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Abstract. Erdös proved in 1946 that if a set $E \subset \mathbb{R}^n$ is closed and non-empty, then the set, called ambiguous locus or medial axis, of points in $\mathbb{R}^n$ with the property that the nearest point in E is not unique, can be covered by countably many surfaces, each of finite $(n-1)$-dimensional measure. We improve the result by obtaining a new regularity result for these surfaces in terms of convexity and C^2 regularity.

Given a closed set $\varnothing \neq E \subset \mathbb{R}^n$, let $\mathrm{Unp}(E)$ be the set of all points $x \in \mathbb{R}^n$ for which there is a unique point $y \in E$ nearest to x. Clearly $E \subset \mathrm{Unp}(E)$. If we denote this nearest point by $\pi(x) := y$, the mapping $\pi : \mathrm{Unp}(E) \to E$ is called the *metric projection*. In order to understand the properties of this mapping it is important to understand the structure of the set $\mathbb{R}^n \setminus \mathrm{Unp}(E) \subset \mathbb{R}^n \setminus E$, where we lack uniqueness of the metric projection. This set is often called the *ambiguous locus* of the metric projection. It is also called the *medial axis* or the *skeleton* of $\mathbb{R}^n \setminus E$. Zamfirescu [23] (see also [16, 4A], [24]) proved that for most compact sets $\varnothing \neq E \subset \mathbb{R}^n$ (in the Baire category sense with respect to the Hausdorff distance on the spaces on compact subsets of $\mathbb{R}^n$), the set $\mathrm{Unp}(E)$ has empty interior, meaning that the set of points in $\mathbb{R}^n$ without a unique nearest point in E is dense. On the other hand, it is known that the Lebesgue measure of the set $\mathbb{R}^n \setminus \mathrm{Unp}(E)$ equals zero, $|\mathbb{R}^n \setminus \mathrm{Unp}(E)| = 0$. This result is due to Erdös [13]. For a simple folklore proof (different from that in [13]), see Lemma 12 and Remark 13 below.

Erdös [12], proved however, a much stronger result: *The set* $\mathbb{R}^n \setminus \mathrm{Unp}(E)$ *is contained in the sum of countably many surfaces of finite* $(n-1)$*-dimensional measure.* His proof is based on Roger's [20] proof of the *contingent theorem* (see also [21, pp. 264-266 and 304-307] and [18, Section 2.1.8]). Fifty years later, Erdös' result was rediscovered by Fremlin [16, Theorem 1G] with a different proof, but the author was not aware of the work of Erdös. For more results about the structure of the set $\mathbb{R}^n \setminus \mathrm{Unp}(E)$, see [1] and references therein. Another interesting and related reference is [9].

In Theorem 1 which is the main result of the paper, we substantially improve Erdös' result by showing convexity and C^2 regularity properties of the surfaces covering the set $\mathbb{R}^n \setminus \mathrm{Unp}(E)$. Our argument is different from that of Erdös. To the best of my knowledge, Theorem 1 is new. While, it can be easily deduced from the results existing in the literature (as we do it here), I believe, the result is of a substantial interest and it deserves a clear, self-contained, and easy to read proof. In addition to Theorem 1, I believe, the proof of Theorem 8 is of independent interest as I explained it in Remark 9.

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We say that the set

$$G = \{x \in \mathbb{R}^n : x_i = f(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)\}, \tag{1}$$

where $1 \le i \le n$ and $f : \mathbb{R}^{n-1} \to \mathbb{R}$ is continuous, is a C^2-*graph* if $f \in C^2$ and it is a $(c-c)$-graph if $f = g - h$ is the difference of convex functions $g, h : \mathbb{R}^{n-1} \to \mathbb{R}$. If the convex functions g, h are of class C^2, we say that G is a $C^2 - (c-c)$-*graph*.

Theorem 1. *For any closed set $E \subset \mathbb{R}^n$ we have*

(a) *The set $\mathbb{R}^n \setminus \mathrm{Unp}(E)$ can be covered by countably many $(c-c)$-graphs.*
(b) *There are countably many $C^2 - (c-c)$-graphs $\{G_j\}_{j=1}^\infty$ such that*

$$\mathcal{H}^{n-1}\Big((\mathbb{R}^n \setminus \mathrm{Unp}(E)) \setminus \bigcup_{j=1}^\infty G_j\Big) = 0,$$

where $\mathcal{H}^{n-1}$ stands for the Hausdorff measure.

Remark 2. Since convex functions are locally Lipschitz continuous [19, Theorem 41D], compact subsets of $(c-c)$-graphs have finite $(n-1)$-dimensional Hausdorff measure and hence the $(n-1)$-dimensional Hausdorff measure of $\mathbb{R}^n \setminus \mathrm{Unp}(E)$ is σ-finite. Therefore, part (a) implies Erdös' result [12] mentioned earlier.

Remark 3. It follows from (b) that $\mathbb{R}^n \setminus \mathrm{Unp}(E)$ is $(\mathcal{H}^{n-1}, n)$-rectifiable of class C^2 in the sense of [5].

The paper is organized as follows. In Section 1 we collect basic facts about convex functions. In Section 2 we prove a special case of a result of Zajíček [22], and in Section 3 we prove Theorem 1.

1. Convex functions

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. The *subdifferential of f at x*, denoted by $\partial f(x)$, is the set of all $v \in \mathbb{R}^n$ such that

$$f(x+h) \ge f(x) + \langle v, h \rangle \quad \text{for all } h \in \mathbb{R}^n. \tag{2}$$

The geometric interpretation is that each $v \in \partial f(x)$ defines a hyperplane passing through $(x, f(x))$ such that the graph of f is above that hyperplane. Such hyperplanes are called *supporting hyperplanes* of the graph of f at $(x, f(x))$. If $\partial f(x)$ contains more than one vector, it means that there is more than one supporting hyperplane at $(x, f(x))$.

Lemma 4. *If $f : \mathbb{R}^n \to \mathbb{R}$ is convex, then for every $x \in \mathbb{R}^n$, $\partial f(x) \neq \varnothing$. That is at every point, there is at least one supporting hyperplane. If in addition f is differentiable at x, then $\partial f(x) = \{\nabla f(x)\}$ so in that case the tangent hyperplane is a unique supporting hyperplane.*

Proof. It easily follows from the definition of the derivative and convexity, that if f is differentiable at x, then there is a unique supporting hyperplane defined by $\nabla f(x)$ i.e., $\partial f(x) = \{\nabla f(x)\}$. Thus it remains to prove existence in the general case.

For $k = 1, 2, \ldots$, let p_k be the point on the graph of f nearest to $q_k = (x, f(x) - k^{-1})$. Let H_k be the hyperplane orthogonal to the segment $[p_k, q_k]$ and passing through its midpoint. It follows from the convexity of f that the graph of f lies above the hyperplane H_k. Since the unit normal vectors to H_k belong to the compact unit sphere $\nu_k \in \mathbb{S}^{n-1}$, we can select a convergent sequence $\nu_{k_i} \to \nu \in \mathbb{S}^{n-1}$ and it is easy to see that the hyperplane normal to ν and passing through $(x, f(x))$ is a supporting one. $\square$

Lemma 5. *If $f : \mathbb{R}^n \to \mathbb{R}$ is convex and partial derivatives $\partial f/\partial x_i$, $i = 1, 2 \ldots, n$ exist at a point $x \in \mathbb{R}^n$, then f is (Fréchet) differentiable at x.*

For a proof see [19, Theorem 42D]. This lemma follows from Jensen's inequality: any vector $h \in \mathbb{R}^n$ can be expressed as a convex combination of vectors parallel to coordinate axes and this along with the Jensen inequality applied to the convex function $\varphi(h) := f(x+h) - f(x) - \langle \nabla f(x), h\rangle$ allows us to show that $\varphi(h)$ converges to 0 as $o(|h|)$. It might be more rewarding to fill missing details as an exercise rather than to read the proof from [19].

One sided partial derivatives will be denoted by

$$\frac{\partial^\pm f}{\partial x_i}(x) = \lim_{t \to 0^\pm} \frac{f(x + te_i) - f(x)}{t}\, . \tag{3}$$

The next lemma easily follows from the monotonicity of secants of a convex function in one variable.

Lemma 6. *If $f : \mathbb{R}^n \to \mathbb{R}$ is convex, then one-sided partial derivatives (3) exist at every point $x \in \mathbb{R}^n$ and $\partial^- f(x)/\partial x_i \leq \partial^+ f(x)/\partial x_i$. Moreover, for any $x \in \mathbb{R}^n$, and $1 \leq i \leq n$*

$$f(x + te_i) \geq f(x) + st \text{ for all } t \in \mathbb{R} \text{ and all } s \text{ satisfying } \frac{\partial^- f}{\partial x_i}(x) \leq s \leq \frac{\partial^+ f}{\partial x_i}(x). \tag{4}$$

Thus a convex function f is *not* differentiable at x if and only if there is $i \in \{1, 2, \ldots, n\}$ such that

$$\frac{\partial^- f}{\partial x_i}(x) < \frac{\partial^+ f}{\partial x_i}(x). \tag{5}$$

Indeed, according to Lemma 5, f is differentiable at x if and only if partial derivatives exist and that is equivalent to equality of all one-sided partial derivatives.

Therefore, if $A \subset \mathbb{R}^n$ is the set of points where a convex function f is not differentiable, then

$$A = \bigcup_{i=1}^n \bigcup_{\substack{\alpha<\beta \\ \alpha,\beta\in\mathbb{Q}}} \underbrace{\left\{x : \frac{\partial^- f}{\partial x_i}(x) \leq \alpha < \beta \leq \frac{\partial^+ f}{\partial x_i}(x)\right\}}_{A^i_{\alpha,\beta}}. \tag{6}$$

We say that a convex function $f : \mathbb{R}^n \to \mathbb{R}$ *is coercive* if $f(x) \to \infty$ as $|x| \to \infty$, and f is *strongly convex* if $f(x) - \mu|x|^2$ is convex for some $\mu > 0$. Clearly, if f is convex, then $f(x) + |x|^2$ is strongly convex.

If f is strongly convex, then it is coercive in the following stronger sense:

$$\lim_{|x|\to\infty} (f(x) - \ell(x)) = \infty \quad \text{for any linear function } \ell : \mathbb{R}^n \to \mathbb{R}. \tag{7}$$

Indeed, $f(x)-\mu|x|^2$ is convex and hence an affine function (supporting hyperplane) bounds it from below, $f(x)-\mu|x|^2 \geq A(x)$ so $f(x)-\ell(x) \geq A(x)+\mu|x|^2-\ell(x) \to \infty$ as $|x| \to \infty$.

Lemma 7. *If $f : \mathbb{R}^k \times \mathbb{R}^\ell \to \mathbb{R}$ is convex and coercive, then $F(x) := \inf_{y\in\mathbb{R}^\ell} f(x,y)$ defines a convex function $F : \mathbb{R}^k \to \mathbb{R}$.*

Proof. Let $x_1, x_2 \in \mathbb{R}^k$ and $\lambda \in [0,1]$. Then for any $y_1, y_2 \in \mathbb{R}^\ell$ we have

$$
\begin{aligned}
&F(\lambda x_1 + (1-\lambda)x_2) \leq f(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda) y_2) \\
&= f(\lambda(x_1,y_1) + (1-\lambda)(x_2,y_2)) \leq \lambda f(x_1,y_1) + (1-\lambda) f(x_2,y_2)
\end{aligned}
$$

and the result follows upon taking the infima over $y_1 \in \mathbb{R}^\ell$ and $y_2 \in \mathbb{R}^\ell$. □

2. A theorem of Luděk Zajíček

The next result is a special case of a theorem of Zajíček [22, Theorem 1].

Theorem 8. *If $f : \mathbb{R}^n \to \mathbb{R}$ is convex, then the set of points where f is not differentiable is contained in a countable union of $(c-c)$-graphs.*

Remark 9. While our proof is almost the same as the original one, the result is a slight improvement of that of Zajíček, as Proposition 10 and its proof provide a more detailed description of surfaces covering the non-differentiability sets $A^i_{\alpha,\beta}$ in the case of strongly convex functions. Other reasons to include a proof are: (1) to make the paper self-contained; (2) the result [22, Theorem 1] whose special case we prove here is more difficult to read due to its generality; (3) while the theorem of Zajíček has been cited many times, it is a good idea to provide an easy to read proof as it might contribute to popularization of this beautiful result.

Since f is non-differentiable at x if and only if the strongly convex function $f(x)+|x|^2$ is non-differentiable at x, Theorem 8 follows from the next result whose proof provides a more detailed description of the structure of the discontinuity sets $A^i_{\alpha,\beta}$.

We say that graphs of the form (1) are graphs *in the direction of x_i*.

Proposition 10. *If $f : \mathbb{R}^n \to \mathbb{R}$ is strongly convex, then each of the sets $A^i_{\alpha,\beta}$ is contained in a $(c-c)$-graph in the direction of x_i. Therefore, the set A is contained in a countable union of $(c-c)$-graphs.*

Proof. Without loss of generality we may assume that $i=1$. For any $s \in \mathbb{R}$, the function $f_s(x) = f(x) - sx_1$ is convex and coercive by (7). Therefore, Lemma 7 implies that the function

$$
g_s : \mathbb{R}^{n-1} \to \mathbb{R}, \quad g_s(x_2,\ldots,x_n) := \inf_{x_1\in\mathbb{R}} f_s(x_1,x_2,\ldots,x_n)
$$

is convex. If $a \in A^1_{\alpha,\beta}$, then (4) yields

$$
f(a+te_1) \geq f(a) + \alpha t \quad \text{for all } t \in \mathbb{R}
$$

or equivalently,

$$
f_\alpha(a+te_1) \geq f_\alpha(a) \quad \text{for all } t \in \mathbb{R}.
$$

The last condition however, means that the function $x_1 \mapsto f_\alpha(x_1, a_2, \dots, a_n)$ attains minimum at $x_1 = a_1$ so

$$g_\alpha(a_2, \dots, a_n) = f_\alpha(a_1, a_2, \dots, a_n) = f(a_1, a_2, \dots, a_n) - \alpha a_1. \tag{8}$$

Similarly, inequality

$$f(a + te_1) \geq f(a) + \beta t \quad \text{for all } t \in \mathbb{R}$$

(also guaranteed by (4)) implies that

$$g_\beta(a_2, \dots, a_n) = f(a_1, a_2, \dots, a_n) - \beta a_1. \tag{9}$$

Now (8) and (9) yield

$$a_1 = \frac{1}{\beta - \alpha}(g_\alpha(a_2, \dots, a_n) - g_\beta(a_2, \dots, a_n))$$

and $A^1_{\alpha,\beta}$ is contained in the graph of the $(c-c)$-function

$$g(x_2, \dots, x_n) = \frac{1}{\beta - \alpha}(g_\alpha(x_2, \dots, x_n) - g_\beta(x_2, \dots, x_n)). \tag{10}$$

The proof is complete. □

Remark 11. Note that in general not all points of the graph of (10) belong to the set $A^1_{\alpha,\beta}$ as otherwise we would obtain a whole surface of points of discontinuity of the derivative and $f(x) = |x|$ has only one such point.

3. Proof of Theorem 1

Proof of (a). In the proof we will need the following two lemmata. Erdös [13], proved that $|\mathbb{R}^n \setminus \mathrm{Unp}(E)| = 0$, but Lemma 12 provides a different proof of this fact, see [11, Proposition 2.4], [10, Lemma 2.1], [15, Theorem 3.3], [8, Lemma 2.21].

Lemma 12. *If a set $\varnothing \neq E \subset \mathbb{R}^n$ is closed, and the distance function $d(x) = \mathrm{dist}(x, E)$ is differentiable at $x \in \mathbb{R}^n \setminus E$, then $x \in \mathrm{Unp}(E)$.*

Remark 13. Since d is Lipschitz continuous, it is differentiable a.e. by the Rademacher theorem, and hence Lemma 12 yields $|\mathbb{R}^n \setminus \mathrm{Unp}(E)| = 0$.

Proof. Assume that d is differentiable at $x \in \mathbb{R}^n \setminus E$ and $p \in E$ is such that $|x - p| = d(x)$. We will prove that $p = x - d(x)\nabla d(x)$ and this will imply uniqueness of p.

It follows from the triangle inequality that if y belongs to the interval with endpoints x and p, $y \in [x, p]$, then $d(y) = |y - p|$. Therefore, the function d decreases linearly with the slope 1 along the segment $[x, p]$. Since d is differentiable at x, the directional derivative at x in the direction of the unit vector $v = (p - x)/|p - x|$ satisfies

$$\langle \nabla d(x), v \rangle = D_v d(x) = -1. \tag{11}$$

On the other hand, d is 1-Lipschitz so $|\nabla d(x)| \leq 1$ and hence equality (11) implies that

$$\nabla d(x) = -v = \frac{x - p}{d(x)} \quad \text{so} \quad p = x - d(x)\nabla d(x).$$

□

The next lemma was observed by Asplund [6, p. 235].

Lemma 14. *If* $\varnothing \neq E \subset \mathbb{R}^n$ *is closed, and* $d(x) = \operatorname{dist}(x, E)$, *then the function* $f : \mathbb{R}^n \to \mathbb{R}$ *defined by* $f(x) = |x|^2 - d(x)^2$ *is convex.*

Proof. We have

$$f(x) = |x|^2 - \inf_{y \in E} |x-y|^2 = |x|^2 + \sup_{y\in E}(-|x-y|^2) = \sup_{y \in E} \big(2\langle x, y\rangle - |y|^2\big).$$

Therefore, f is a supremum of a family of affine functions, and hence it is convex. □

By Lemma 12 the set $\mathbb{R}^n \setminus \operatorname{Unp}(E)$ is contained in the set A_+ of points where d is strictly positive and non-differentiable. Since $d > 0$ in A_+, the set A_+ is contained in the set where the convex function $f(x) = |x|^2 - d(x)^2$ is non-differentiable and the result follows from Theorem 8. □

Proof of (b). It is well known that if $f : \mathbb{R}^{n-1} \to \mathbb{R}$ is convex, then for any $\varepsilon > 0$ there is a function $f_\varepsilon \in C^2(\mathbb{R}^{n-1})$ such that the Lebesgue measure of the set where the two functions differ satisfies $|\{x : \ f(x) \neq f_\varepsilon(x)\}| < \varepsilon$. (In general, the function $f_\varepsilon \in C^2$ cannot be convex, see Example 1.9 and Proposition 1.10 in [7].) This is a consequence of the Aleksandrov theorem [4] about second order differentiability of convex functions which implies that a convex function satisfies assumptions of the C^2-Whitney extension theorem outside a set of measure less than ε. While the idea is simple, the details are rather difficult, see [2], [7, Corollary 1.5], [14, Proposition A1], [17]. This and part (a) easily imply that the set $\mathbb{R}^n \setminus \operatorname{Unp}(E)$ can be covered up to a set of $\mathcal{H}^{n-1}$-measure zero by a sequence of graphs of C^2-functions. One only needs to note that any $(c-c)$-function, $g : \mathbb{R}^{n-1} \to \mathbb{R}$ is locally Lipschitz and hence for a set $A \subset \mathbb{R}^{n-1}$ of measure zero, the corresponding set on the graph of g has vanishing Hausdorff measure $\mathcal{H}^{n-1}$.

Therefore, it remains to show that if $f \in C^2$, then on every bounded set, f coincides with the difference of two convex functions of class C^2.

Given $R > 0$, let φ be a compactly supported smooth function that equals 1 for $|x| \leq R$. Then $\varphi f \in C^2$ has bounded second order derivatives so there is $C_R > 0$ such that the matrix $D^2(\varphi(x)f(x) + C_R|x|^2) = D^2(\varphi f) + 2C_R I$ is positive definite and hence the function $\varphi(x)f(x) + C_R|x|^2$ is convex and of class C^2. Now, $f(x) = (\varphi(x)f(x) + C_R|x|^2) - C_R|x|^2$, $|x| \leq R$, represents f for $|x| \leq R$ as a difference of convex functions of class $C^2(\mathbb{R}^n)$. This completes the proof. □

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