Consensus-Based Distributed Estimation in the Presence of Heterogeneous, Time-Invariant Delays

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***Abstract*—Classical distributed estimation scenarios typically assume timely and reliable exchanges of information over the sensor network. This letter, in contrast, considers single time-scale distributed estimation via a sensor network subject to transmission time-delays. The proposed discrete-time networked estimator consists of two steps: (i) consensus on (delayed) a-priori estimates, and (ii) measurement update. The sensors only share their a-priori estimates with their out-neighbors over (possibly) time-delayed transmission links. The delays are assumed to be fixed over time, heterogeneous, and known. We assume distributed observability instead of local observability, which significantly reduces the communication/sensing loads on sensors. Using the notions of augmented matrices and the Kronecker product, the convergence of the proposed estimator over strongly-connected networks is proved for a specific upper-bound on the time-delay.**

***Index Terms*—Distributed estimation, consensus, Kronecker product, communication time-delays.**

I. Introduction

LATENCY in data transmission networks may significantly affect the performance of decision-making over sensor networks and multi-agent systems [1]. In particular, time-delays may cause instability in networked control systems which are originally stable in the corresponding delay-free case. For example, the consequence of communication delays on the consensus stability are discussed in [2]–[4] among others, and centralized observer design are discussed in [5], [6]. This work extends to *distributed* estimation over a sensor network with random communication time-delays.

The literature on distributed estimation spans from multi time-scale scenarios to single time-scale methods. The former case requires many iterations of averaging/data-sharing (consensus/communication time-scale) between two consecutive system time-steps (system time-scale) [7], [8], where the estimation performance tightly depends on the number of consensus iterations. This is less efficient in terms of computational and communication loads on sensors and, further, requires much faster data sharing/processing rate which might be inaccessible over large networks. In terms of observability, in the multi time-scale method, number of communication/consensus iterations is greater than the network diameter, and therefore, all sensors eventually gain all state information (and system observability) between every two system time-steps. In the single time-scale, however, every sensor performs only one iteration of consensus, and therefore, many works require the system to be *locally* observable in the neighborhood of the sensors [9]–[16]; in this work, we assume *global* observability as in [17]–[20]. Recall that local observability mandates: (i) more network connectivity, and/or (ii) access to more system outputs at each sensor, and may considerably increase the communication/sensing-related costs [21]–[23]. This work, however, considers the least connectivity requirement (strong-connectivity) and least outputs at each sensor (one output), while addressing transmission delays.

The networked estimator in this letter is single time-scale, where sensors perform one consensus iteration on (possibly) *delayed* a-priori estimates in their in-neighborhood, and then, measurement-update using their own outputs. As in [2], we consider arbitrary time-delays at every communication link, but the delays are time-invariant and known. The delays are bounded so that no information is lost over the network and the data would eventually reach the recipient sensor. To avoid considering a trivial case, this work makes no assumption on the stability of the linear system. Further, similar to [10], [12], we assume that the system is full-rank as in structurally-cyclic/self-damped systems [24], [25]. We adopt the notions of augmented representation [2] and the Kronecker network product [26] to simplify the convergence analysis. We show

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that feedback gain design in the absence of delays via the Linear-Matrix-Inequality (LMI) in [17], [27] also results in stable estimation for some upper-bounded delayed cases. Further, we provide a solution to design delay-tolerant networked estimators for a given bound on the delays. Therefore, the gain design requires no information other than the bound on the delays and, the LMI complexity is determined by the original low-order system and not the high-order augmented one. Note that in this work, we assume *no measurement delays*; this is because the sensors take direct measurements, while spatially distributed in large-scale with (possibly) delayed communications. Further, as proved in [6], stability depends *only* on the measurement *packet loss*, not the *packet delays*.

Paper Organization: Section II provides some preliminaries and problem statement. Section III states our main results on delay-tolerant distributed estimation. Section IV provides the simulations, and Section V concludes this letter.

II. The Framework

A. System-Output Model

We consider discrete-time system and measurements as,

$$\mathbf{x}_k = A\mathbf{x}_{k-1} + \nu_k,\ \ \mathbf{y}_k = C\mathbf{x}_k + \zeta_k,\ \ k \geq 0 \tag{1}$$

with $\mathbf{x}_k \in \mathbb{R}^n$ as system states, $\mathbf{y}_k \in \mathbb{R}^N$ as system outputs, $\zeta_k \sim \mathcal{N}(0, R)$ and $\nu_k \sim \mathcal{N}(0, Q)$ as independent noise variables, all at time-step k. It is *not* assumed that $\rho(A) < 1$ (potentially unstable system), while $\det(A) \neq 0$ (full-rank), with $\rho(\cdot)$ and $\det(\cdot)$ as the spectral radius and determinant. Examples of such full-rank systems are structurally cyclic [25] and *self-damped*[1] systems [24], [25] as in, e.g., social opinion dynamics [28] and target tracking [13]. Without loss of generality, we assume N sensors each with one scalar output y^i_k.

Assumption 1: Every sensor knows the system matrix A. The pair (A, C) is observable (similar arguments for detectable case), implying *global* observability. However, in general, the pair (A, C_i) is not necessarily observable at any sensor i.

For a full-rank system A, the output matrix C can be defined via graph-theoretic methods [17], [23], [29], saying that one output from (at least) one state in every irreducible block of A ensures structural (A, C)-observability. The optimal output selection strategies are also of interest as in [25], [30], [31].

B. Preliminaries on Consensus Algorithms

Consider discrete-time consensus algorithms [2], [32] over a network of sensors $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ ($\mathcal{V}$ and $\mathcal{E}$ as the node and link sets) with $\mathbf{z}_k$ as sensors' states which evolve as $\mathbf{z}_k = P\mathbf{z}_{k-1}$. Matrix P (as the consensus weight) represents the communication between the sensors via graph $\mathcal{G}$. The sensor network is in general directed. For notation simplicity denote $P(i, j)$ by p_{ij}, where $0 < p_{ij} < 1$ if $(j, i) \in \mathcal{E}$ and 0 otherwise. P is *row-stochastic*, i.e., $\sum_{j=1}^N p_{ij} = 1$, and $p_{ii} \neq 0$ for all i. Further, $\mathcal{G}$ is *strongly-connected* (SC), i.e., there is a path from every node i to every node j $\forall i, j \in \mathcal{V}$, implying that fusion matrix P is *irreducible*. Such P is stochastic, indecomposable, and aperiodic (SIA), where $\lim_{k\to\infty} P^k \in \text{span}(\mathbf{1}_N)$ [2], with $\mathbf{1}_N$ as all-ones vector of size N, and $\rho(P) = 1$.

[1]A linear system is self-damped if its matrix A has non-zero diagonal entries [24], [25]. Similarly, network $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ is self-damped if for every node $i \in \mathcal{V}$ we have $(i, i) \in \mathcal{E}$, i.e., there is a self-link at every node. Examples of such structurally full-rank CPS models are given in [24].

C. Delay Model

In this work, it is assumed that the data-transmission over the link (j, i) from sensor j to sensor i has a-priori unknown bounded (integer) time-delay, τ_{ij}, where $0 \leq \tau_{ij} \leq \overline{\tau} < \infty$, and $\overline{\tau}$ is an upper bound to the delays in all links. The messages are *time-stamped* (e.g., via a global discrete-time clock over the network $\mathcal{G}$), so the recipient knows the time-step the data was sent. Further, $\tau_{ii} = 0$, i.e., every sensor i knows its own state with no delay. To model the delayed state vectors we adopt the notations in [2]. In a network of N sensors, define an *augmented state* vector $\underline{\mathbf{x}}_k = (\mathbf{x}_k; \mathbf{x}_{k-1}; \ldots; \mathbf{x}_{k-\overline{\tau}})$ with ';' as column concatenation, and $\mathbf{x}_{k-r} = (x^1_{k-r}; \ldots; x^n_{k-r})$ for $0 \leq r \leq \overline{\tau}$. Then, for a given N-by-N matrix P and maximum delay $\overline{\tau}$, define the *augmented matrix* $\overline{P}$ as,

$$\overline{P} = \begin{pmatrix} P_0 & P_1 & P_2 & \ldots & P_{\overline{\tau}-1} & P_{\overline{\tau}} \\ I_N & 0_N & 0_N & \ldots & 0_N & 0_N \\ 0_N & I_N & 0_N & \ldots & 0_N & 0_N \\ 0_N & 0_N & I_N & \ldots & 0_N & 0_N \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_N & 0_N & 0_N & \ldots & I_N & 0_N \end{pmatrix}, \tag{2}$$

with I_N and 0_N as the identity and zero matrix of size N. The non-negative matrices P_r are defined based on the time delay $0 \leq r \leq \overline{\tau}$ on the network links as follows,

$$P_r(i, j) = \begin{cases} p_{ij}, & \text{If } \tau_{ij} = r \\ 0, & \text{Otherwise.} \end{cases} \tag{3}$$

Assuming fixed delays, for every $(j, i) \in \mathcal{E}$, *only one of* $P_0(i, j), P_1(i, j), \ldots, P_{\overline{\tau}}(i, j)$ *is equal to* p_{ij} *and the rest are zero*, and thus, the row-sum of the first N rows of $\overline{P}$ and P are equal, i.e., $\sum_{j=1}^{N(\overline{\tau}+1)} \overline{P}(i, j) = \sum_{j=1}^N P(i, j)$ for $1 \leq i \leq N$ and $P = \sum_{r=0}^{\overline{\tau}} P_r$ for $k \geq 0$. *Therefore, for a row-stochastic matrix* P, *the augmented matrix* $\overline{P}$ *is also row-stochastic.*[2] In the proposed estimator, the matrix $\overline{P}$ is not needed and is only defined to simplify the mathematical analysis.

Assumption 2: For the time-delay τ_{ij} on link (j, i):

i) The delay is known and bounded $\tau_{ij} \leq \overline{\tau}$. The upper-bound $\overline{\tau}$ guarantees no lost information, i.e., the data sent from sensor j at time k would eventually reach the recipient sensor i (at most) at time $k + \overline{\tau}$ (a $\overline{\tau} + 1$-slot transmission buffer).
ii) Delay τ_{ij} is arbitrary, constant, and may/may-not differ for different links (heterogeneous/homogeneous delays).

Suppose that we know either an upper-bound on the delays we may have or their probability (which is of finite support), e.g., $r \leq \overline{\tau}$ steps delay with probability $\mathbb{P}(r)$ and zero probability for $\overline{\tau} + 1$ and above. Then, even though either the *exact distribution* or the actual delays are unknown and might be time-varying, agents know the maximum possible delay, say $\overline{\tau}$, and may choose to update and communicate every $\overline{\tau} + 1$ steps (of system dynamics) after all delayed information is received; see more on this in [2] and Section III-D. On the other hand, for static sensors where the communication distances over the wireless sensor network are fixed, it is likely that we have constant propagation delays (which are proportional to the constant distances), see more in [33], [34].

[2]Note the subtle difference between our notation vs. [2]. In [2] *column* augmented matrix is introduced, while we consider *row* augmented matrices.

D. Problem Statement

The problem in this work is to design a networked estimator for the system-output model (1) satisfying Assumption 1, where every sensor relies only on its partial system output (partial observability) and the received (possibly delayed satisfying Assumption 2) information from its in-neighbors. This work particularly differs from [9]–[16] via the following remark.

Remark 1: Let $\mathcal{N}_i = \{j|(j,i) \in \mathcal{E}\}$ denote the set of in-neighborhood of sensor i over the network $\mathcal{G}$. The pair $(A, \sum_{j\in\mathcal{N}_i} C_j)$ is not necessarily observable at any sensor i, implying no *local* observability assumption.

III. Distributed Estimation in Presence of Delays

Every sensor i performs the following two steps for distributed state estimation in the presence of time-delays,

$$\widehat{\mathbf{x}}^i_{k|k-1} = p_{ii}A\widehat{\mathbf{x}}^i_{k-1|k-1} + \sum_{j\in\mathcal{N}_i}\sum_{r=0}^{\overline{\tau}} p_{ij}A^{r+1}\widehat{\mathbf{x}}^j_{k-r|k-r}\mathbf{I}_{k-r,ij}(r), \quad (4)$$

$$\widehat{\mathbf{x}}^i_{k|k} = \widehat{\mathbf{x}}^i_{k|k-1} + K_iC_i^\top\Big(y^i_k - C_i\widehat{\mathbf{x}}^i_{k|k-1}\Big), \quad (5)$$

where $\mathbf{I}_{k,ij}(r)$ is the indicator function defined as [2],

$$\mathbf{I}_{k,ij}(r) = \begin{cases} 1, & \text{if } \tau_{ij} = r \\ 0, & \text{otherwise.} \end{cases} \quad (6)$$

In (4), $\widehat{\mathbf{x}}^i_{k|k-1}$ denotes the sensor i's *a-priori* state estimate at time k given all the (possibly delayed) information up to time $k-1$ from its in-neighbors $\mathcal{N}_i$, with $\widehat{\mathbf{x}}_{k|k-1}$ denoting the column-concatenation of $\widehat{\mathbf{x}}^i_{k|k-1}$'s (similarly for $\widehat{\mathbf{x}}_{k|k}$). Eq. (4) represents one iteration of information-consensus, where sensor i sums the weighted estimates of sensors $j \in \mathcal{N}_i$ as they arrive *knowing their delays*. Recall that, for every link (i,j) the indicator $\mathbf{I}_{k-r,ij}(r)$ is only non-zero for one r between 0 and $\overline{\tau}$ (due to fixed delay assumption). The second step (5) is a measurement-update (or *innovation*) to modify the a-priori estimate using the new sensor measurement y^k_i. Clearly, the protocol (4)-(5) is single time-scale with one step of information-sharing/consensus-update between $k-1$ and k. Using the notion of augmented vector, define $\underline{\widehat{\mathbf{x}}}_{k|k-1} = (\widehat{\mathbf{x}}_{k|k-1}; \widehat{\mathbf{x}}_{k-1|k-2}; \ldots; \widehat{\mathbf{x}}_{k-\overline{\tau}|k-\overline{\tau}-1})$ and similarly $\underline{\widehat{\mathbf{x}}}_{k|k}$. Then, the augmented version of (4)-(5) is,

$$\underline{\widehat{\mathbf{x}}}_{k|k-1} = \overline{PA}\,\underline{\widehat{\mathbf{x}}}_{k-1|k-1}, \quad (7)$$

$$\underline{\widehat{\mathbf{x}}}_{k|k} = \underline{\widehat{\mathbf{x}}}_{k|k-1} + \mathbf{b}_1^{\overline{\tau}+1} \otimes KD_C^\top\big(\mathbf{y}_k - D_C\Xi^{Nn}_{1,\overline{\tau}}\underline{\widehat{\mathbf{x}}}_{k|k-1}\big), \quad (8)$$

where $\overline{PA}$ is the *modified* augmented version of $P \otimes A$ as,

$$\overline{PA} = \begin{pmatrix} P_0\otimes A & P_1\otimes A^2 & \ldots & P_{\overline{\tau}-1}\otimes A^{\overline{\tau}} & P_{\overline{\tau}}\otimes A^{\overline{\tau}+1} \\ I_{Nn} & 0_{Nn} & \ldots & 0_{Nn} & 0_{Nn} \\ 0_{Nn} & I_{Nn} & \ldots & 0_{Nn} & 0_{Nn} \\ 0_{Nn} & 0_{Nn} & \ddots & 0_{Nn} & 0_{Nn} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0_{Nn} & 0_{Nn} & \ldots & I_{Nn} & 0_{Nn} \end{pmatrix}, \quad (9)$$

and $D_C = \text{blockdiag}(C_i)$, $K = \text{blockdiag}(K_i)$, and the auxiliary matrix $\Xi^m_{i,\overline{\tau}}$ is an $m \times (\overline{\tau}+1)m$ matrix defined as $\Xi^m_{i,\overline{\tau}} = (\mathbf{b}_i^{\overline{\tau}+1} \otimes I_m)^\top$ with $\mathbf{b}_i^{\overline{\tau}+1}$ as the unit column-vector of the i'th coordinate ($1 \le i \le \overline{\tau}+1$). Define the augmented state $\underline{\mathbf{x}}_k = (1_N \otimes \mathbf{x}_k; 1_N \otimes \mathbf{x}_{k-1}; \ldots; 1_N \otimes \mathbf{x}_{k-\overline{\tau}})$ and the augmented error $\underline{\mathbf{e}}_k$ at time k as $\underline{\mathbf{e}}_k = \underline{\mathbf{x}}_k - \underline{\widehat{\mathbf{x}}}_{k|k}$. Define $\widetilde{\nu}_k = 1_N \otimes \nu_k$ and $\underline{\nu}_k = \mathbf{b}_i^{\overline{\tau}+1} \otimes \widetilde{\nu}_k$. We have $D_C^\top\mathbf{y} = D_C^\top D_C(1_N \otimes \mathbf{x}_k) + D_C^\top\zeta$. Recall that the row-stochasticity of P and $\overline{P}$ along with the system dynamics (1) implies that $\underline{\mathbf{x}}_k = \overline{PA}\,\underline{\mathbf{x}}_{k-1} + \underline{\nu}_k$. From with (1) and some calculations (skipped due to space limitation) we get,

$$\begin{aligned} \underline{\mathbf{e}}_k &= \underline{\mathbf{x}}_k - \Big(\underline{\widehat{\mathbf{x}}}_{k|k-1} + \mathbf{b}_1^{\overline{\tau}+1} \otimes KD_C^\top\mathbf{y}_k - D_C\Xi^{Nn}_{1,\overline{\tau}}\underline{\widehat{\mathbf{x}}}_{k|k-1}\Big) \\ &= \underline{\mathbf{x}}_k - \overline{PA}\underline{\widehat{\mathbf{x}}}_{k-1|k-1} - \mathbf{b}_1^{\overline{\tau}+1} \otimes KD_C^\top\big(\mathbf{y}_k - D_C\Xi^{Nn}_{1,\overline{\tau}}\overline{PA}\underline{\widehat{\mathbf{x}}}_{k-1|k-1}\big) \\ &= \overline{PA}\underline{\mathbf{e}}_{k-1} - \mathbf{b}_1^{\overline{\tau}+1} \otimes K\overline{D}_C\Xi^{Nn}_{1,\overline{\tau}}\overline{PA}\underline{\mathbf{e}}_{k-1} + \underline{\eta}_k \\ &= \widehat{A}\underline{\mathbf{e}}_{k-1} + \underline{\eta}_k, \end{aligned} \quad (10)$$

where $\widehat{A} = \overline{PA} - \mathbf{b}_1^{\overline{\tau}+1} \otimes K\overline{D}_C\Xi^{Nn}_{1,\overline{\tau}}\overline{PA}$ is the *closed-loop matrix*, $\overline{D}_C = D_C^\top D_C$, and $\underline{\eta}_k$ collects the noise terms as,

$$\begin{aligned} \underline{\eta}_k &= \underline{\nu}_k - \mathbf{b}_1^{\overline{\tau}+1} \otimes K\overline{D}_C\widetilde{\nu}_k - \mathbf{b}_1^{\overline{\tau}+1} \otimes KD_C^\top\zeta_k \\ &= \mathbf{b}_1^{\overline{\tau}+1} \otimes (\widetilde{\nu}_k - K\overline{D}_C\widetilde{\nu}_k - KD_C^\top\zeta_k). \end{aligned} \quad (11)$$

The error dynamics in the absence of any delay is as follows,

$$\mathbf{e}_k = (P\otimes A - K\overline{D}_C(P\otimes A))\mathbf{e}_{k-1} + \eta_k = \widehat{A}\mathbf{e}_{k-1} + \eta_k \quad (12)$$

where η_k follows the formulation (11) with $\overline{\tau} = 0$ and $\widehat{A} = P\otimes A - K\overline{D}_C(P\otimes A)$ is the *delay-free closed-loop matrix*. For the Schur stability of the error dynamics (10) and (12), we need $\rho(\widehat{\underline{A}}) < 1$ and $\rho(\widehat{A}) < 1$, respectively. We first discuss the condition for Schur stability of $\widehat{A}$ and then extend the results to Schur stability of $\widehat{\underline{A}}$. Following Kalman theorem and justification in [17], [27], for stability of (12) the pair $(P\otimes A, \overline{D}_C)$ needs to be observable (or detectable); this is known as *distributed observability* [17], discussed next.

Lemma 1: Given a full-rank matrix A and output matrix C, following Assumption 1 and Remark 1, the pair $(P\otimes A, \overline{D}_C)$ is structurally observable[3] if the matrix P is irreducible.

Proof: The proof follows the results in [26] on the (structural) observability of composite Kronecker-product networks. Given a system digraph $\mathcal{G}_1$ associated with full-rank system A and measurement matrix C satisfying (A,C)-observability, the observability of the Kronecker-product network (denoted by $\mathcal{G}\times\mathcal{G}_1$) is determined by $\mathcal{G}$ being strongly-connected and self-damped (see [26, Th. 4]). Following the definition of consensus matrix P, we have $p_{ii} \neq 0$ (satisfying the self-damped condition). The strong-connectivity of $\mathcal{G}$, equivalent with irreducibility of P, then ensures structural $(P\otimes A, \overline{D}_C)$-observability. This completes the proof. ■

Note that, in contrast to *Cartesian-product* network observability [35], [36], Lemma 1 recalls on *Kronecker-product* network observability, see the differences in [26, Fig. 1].

Corollary 1: For observable $(P\otimes A, \overline{D}_C)$, the gain matrix K can be designed such that $\rho(\widehat{A}) < 1$.

A. Constrained Feedback Gain Design

It is known that for an observable pair $(P\otimes A, \overline{D}_C)$, the feedback gain K can be designed to ensure Schur stability of the error dynamics (12) (Corollary 1), i.e., $\rho(\widehat{A}) < 1$. Typically, such K is designed via solving the following LMI,

$$X - \widehat{A}^\top X\widehat{A} \succ 0, \quad (13)$$

[3]In this letter the results are based on structured systems theory, and in the rest of this letter we consider *structural* observability, which holds for almost all random entries of system, output, and consensus matrix.

for some $X \succ 0$ with $\succ$ implying positive-definiteness. The solution of (13) is, in general, a *full* matrix. However, for distributed estimation, we need the state feedback to be further *localized*, i.e., the gain matrix K needs to be *block-diagonal* so every sensor uses its own state-feedback. Such a constrained feedback gain design is proposed in [17], [27] based on cone-complementarity LMI algorithms, which are known to be of polynomial-order complexity for application in large scale.

B. Stability of the Delayed Estimator Dynamics

Following the Schur stability of the delay-free error dynamics (12) (via LMI design of K), we extend the results to stability of the delayed dynamics (10), i.e., to get $\rho(\widehat{\underline{A}}) < 1$ in the presence of delays.

Theorem 1: Let conditions in Lemma 1 hold and the feedback gain K is designed such that $\rho(\widehat{A}) < 1$ from Corollary 1. The networked estimator (4)-(5) successfully tracks the system (1) (subject to delays satisfying Assumption 2 and possibly with $\rho(A) > 1$) with stable error for any $\overline{\tau} \leq \overline{\tau}^*$, where $\overline{\tau}^*$ is the largest value of $\overline{\tau}$ satisfying,

$$\rho(P \otimes A^{\overline{\tau}+1} - K\overline{D}_C(P \otimes A^{\overline{\tau}+1})) < 1.$$

See the Appendix for the proof of this theorem. This theorem gives a sufficient condition for stable tracking in the presence of heterogeneous, time-invariant delays $\tau_{ij} = \overline{\tau}$.

C. Convergence Rate

Note that, in general, the exact characterization of the convergence rate/time of the linear systems is difficult. The following lemma gives the order of convergence time.

Lemma 2: The convergence time of the error dynamics (10) and (12) are of order[4] $\Theta(\frac{1}{1-\rho(\widehat{\underline{A}})})$ and $\Theta(\frac{1}{1-\rho(\widehat{A})})$, respectively.

Proof: The proof follows from [28, Lemma 3]. ■

Following Lemma 2 and 5, the geometric decay rate of (10) (for $\tau_{ij} = \overline{\tau}$) is proportional to,

$$1 - \rho(P \otimes A^{\overline{\tau}+1} - K\overline{D}_C(P \otimes A^{\overline{\tau}+1}))^{\frac{1}{\overline{\tau}+1}}, \tag{14}$$

which is positive from Theorem 1. For longer delays, the consensus rate in (4) (and decay rate of (10)) is slower, e.g., for any two $\overline{\tau}_1 < \overline{\tau}_2 \leq \overline{\tau}^*$, Eq. (14) gives a larger value close to 1 and, thus, lower convergence rate for $\overline{\tau}_2$. The convergence rate is lower-bounded by (14) (for time-invariant delays). For linear feedback systems, one can adjust the closed-loop eigenvalues and the convergence rate by desired feedback gain design, where, in the decentralized case, gain K is further constrained to be block-diagonal. Further, additional bound-constraint on the closed-loop eigenvalues makes the LMI-design more complex as it reduces the set of possible solutions; however, it is solvable, for example, using `MATLAB cvx`.

D. Discussion

1) Each sensor processes the a-priori estimates of its in-neighbors as they arrive. The messages are time-stamped and the sensor knows the time-step (and hence the delay) of the received information. The proposed solution works *for both heterogeneous and homogeneous delays*. In case of homogeneous *known* delays ($\tau_{ij} = \overline{\tau}, \forall i, j$) only the term $\sum_{j\in\mathcal{N}_i} p_{ij}A^{\overline{\tau}+1}\widehat{\mathbf{x}}^j_{\overline{k}-\overline{\tau}|\overline{k}-\overline{\tau}}$ appears in (4). For *unknown* delays bounded by $\overline{\tau}$, following the strategy in [2], state-updates and communications at all sensors take place every $\overline{\tau}+1$ steps of system dynamics, i.e., over a longer time-scale $\overline{k} = \lfloor \frac{k}{\overline{\tau}+1} \rfloor$. In this scenario, all sensors wait to receive all the possibly delayed information from the neighbors and then update their state as,

$$\widehat{\mathbf{x}}^i_{\overline{k}|\overline{k}-1} = \sum_{j\in\{\mathcal{N}_i\cup\{i\}\}} p_{ij}A^{\overline{\tau}+1}\widehat{\mathbf{x}}^j_{\overline{k}-1|\overline{k}-1}, \tag{15}$$

$$\widehat{\mathbf{x}}^i_{\overline{k}|\overline{k}} = \widehat{\mathbf{x}}^i_{\overline{k}|\overline{k}-1} + K_i C_i^\top \left(y^i_{\overline{k}} - C_i\widehat{\mathbf{x}}^i_{\overline{k}|\overline{k}-1}\right), \tag{16}$$

The above models a non-delayed LTI distributed estimator as in [17] with system matrix $A^{\overline{\tau}+1}$, where the stability analysis follows similar to Theorem 1.

2) The structural observability results are, in general, hold for almost all choices of numerical entries of system parameters as long as its structure (the sensor network) is fixed/time-invariant [37], [38]. The observability can be *checked* for different random consensus weights for *stochastic* P. For constrained cases, e.g., lazy Metropolis [32] or simply $\frac{1}{|\mathcal{N}_i|+1}$ [2], it may not necessarily hold. The structure of P, however, may affect the LMI-based gain design, the bound in Eq. (26), $\rho(\widehat{\underline{A}})$, and the convergence rate (14). Note from Lemma 3 that $\rho(P \otimes A) = \rho(A)$ since $\rho(P) = 1$ for any row-stochastic P.
3) The cost-optimal network design and sensor placement [21], [22] can be considered to reduce the communication-related and/or sensing-related costs.
4) In case of sensor failure, the concept of *observational equivalence* in both *centralized* [37] and *distributed* [38] scenarios can recover the loss of observability.
5) To design a distributed estimator to tolerate time-delays bounded by $\overline{\tau}_1$, one can redesign the LMI gain matrix K by replacing $\widehat{A} = P \otimes A^{\overline{\tau}_1+1} - K\overline{D}_C(P \otimes A^{\overline{\tau}_1+1})$ in (13). Clearly, from Theorem 1 and (26), such K results in $\rho(\widehat{\underline{A}}) < 1$ for $\overline{\tau} \leq \overline{\tau}_1$ (simply replace $\overline{\tau}^* = \overline{\tau}_1$). Such LMI gain design for the delay-free closed-loop matrix $\widehat{A}$ of size nN instead of the delayed matrix $\widehat{\underline{A}}$ of size $nN(\overline{\tau}+1)$, significantly reduces the complexity order with no need of using the augmented matrix $\overline{P}$.
6) On a different setup, [1, Th. 1] claims that for *weakly diagonally dominant* matrix P, the off-diagonal delays are harmless for stability. This is not applicable to error dynamics (10), as the entries of $\widehat{\underline{A}}$ (and weak/strong diagonal dominance of the *closed-loop* system) depend on the feedback gain K and cannot be evaluated only based on the open-loop matrices A and P.

IV. Simulation

For `MATLAB` simulation, we consider a linear structurally-cyclic system of $n = 6$ states with 4 irreducible sub-systems and $N = 4$ sensors each taking one system output (from each irreducible block) satisfying Assumption 1. The system is full-rank and unstable with $\rho(A) = 1.04$. System and output noise are considered as $\mathcal{N}(0, 0.004)$. The network of sensors $\mathcal{G}$ is a simple directed self-damped cycle $1 \to 2 \to 3 \to 4 \to 1$, over which the system is not observable in the neighborhood of any sensor (Remark 1). Consensus weights are considered random while satisfying row-stochasticity of

[4]Given functions $f(\cdot)$ and $g(\cdot)$, say $f = O(g)$ if $\sup_n |\frac{f(n)}{g(n)}| < \infty$ and $f = \Omega(g)$ if $g = O(f)$. If both $f = O(g)$ and $f = \Omega(g)$ holds, then $f = \Theta(g)$ [28].

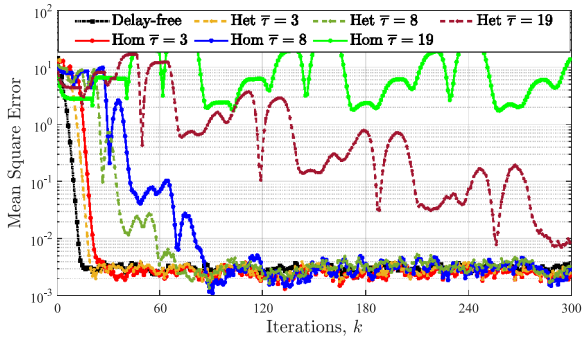

Fig. 1. MSEs are bounded steady-state stable for $\overline{\tau}=0,3,8$. For $\overline{\tau}=19>\overline{\tau}^*=10$ the MSE is marginally unstable for the homogeneous case, while it is stable for the heterogeneous case. Note that Theorem 1 gives the *a sufficient (and not a necessary) condition on* $\overline{\tau}$ *for convergence.*

irreducible P matrix (satisfying Lemma 1). Therefore, solving the LMI in [17], [27] via MATLAB CVX, the block-diagonal gain matrix K is designed such that $\rho(\widehat{A})=0.64<1$ and $\overline{\tau}^*=10$, implying stable error dynamics for any $\overline{\tau}\leq 10$ (sufficiency from Theorem 1). Next, following Assumption 2, we consider both heterogeneous delays (*uniformly distributed* between 0 and $\overline{\tau}$ for different links [2]) and homogeneous delays (equal to $\overline{\tau}$ at all links). Fig. 1 shows the Monte-Carlo simulation (100 trials) of mean-squared-error (MSE) over the network $\mathcal{G}$ for (i) no time-delay and (ii) homogeneous/heterogeneous delays with $\overline{\tau}=3,8,19$ and bounds in (26) as $\rho(\widehat{A})\leq 0.74, 0.94, 1.45$. For fixed homogeneous delays $\overline{\tau}=19$, we have $\rho(\widehat{A})=1$. From Fig. 1 and Eq. (14), longer delays decrease the MSE decay rate for $\overline{\tau}\leq\overline{\tau}^*$ (with $\overline{\tau}^*=10$ for this example), while for $\overline{\tau}>\overline{\tau}^*$ the error *may not necessarily* converge.

V. Conclusion and Future Directions

This letter extends the recent literature on distributed estimation over linear networks to time-delayed ones. For a given bound on the delays, we design distributed estimators over delayed communication networks. Our ongoing research is focused on (i) rank-deficient systems (with more complexity in terms of system outputs, network connectivity, and data-sharing [20], [29]), (ii) detecting faults/attacks [39], [40] on the distributed estimation networks with latency, (iii) time-varying delays, and (iv) network pruning to improve observability and convergence [23], [41].

Appendix

Some of the following lemmas can be found in standard matrix theory books, e.g., in [42].

Lemma 3: Consider two square matrices P and A of size N and n, respectively, with the set of eigenvalues $\{\lambda_1,\ldots,\lambda_N\}$ and $\{\mu_1,\ldots,\mu_n\}$. Then, the set of eigenvalues of $P\otimes A$ is $\{\lambda_i\mu_j | i=1,\ldots,N,\ j=1,\ldots,n\}$.

Lemma 4: Define the following nN-by-nN block matrix,

$$\overline{A}_{n,i}=\begin{pmatrix} 0_N & \ldots & A_i & \ldots & 0_N\\ I_N & 0_N & \ldots & \ldots & 0_N\\ 0_N & I_N & \ddots & \ldots & 0_N\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ 0_N & 0_N & \ldots & I_N & 0_N\end{pmatrix}, \tag{17}$$

where N-by-N matrix A_i is located at the ith block (and the only non-zero block) in the first block-row of $\overline{A}_{n,i}$. Let $p(\lambda)$ and $q(\overline{\lambda})$ represent the characteristic polynomials of A_i and $\overline{A}_{n,i}$, respectively. Then, $q(\overline{\lambda})=\overline{\lambda}^{N(n-i)}p(\overline{\lambda}^i)$.

Proof: Consider,

$$\overline{\lambda}I_{nN}-\overline{A}_{n,i}=\begin{pmatrix} E & F\\ G & H\end{pmatrix}, \tag{18}$$

where block-matrix E is $N(i-1)$-by-$N(i-1)$, F is $N(i-1)$-by-$N(n-i+1)$, G is $N(n-i+1)$-by-$N(i-1)$, and H is $N(n-i+1)$-by-$N(n-i+1)$ defined as,

$$E=\begin{pmatrix} \overline{\lambda}I_N & 0_N & \ldots & \ldots & 0_N\\ -I_N & \overline{\lambda}I_N & \ldots & \ldots & 0_N\\ 0_N & -I_N & \ddots & \ldots & 0_N\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ 0_N & 0_N & \ldots & -I_N & \overline{\lambda}I_N\end{pmatrix} \tag{19}$$

$$F=\begin{pmatrix} -A_i & 0_N & \ldots & 0_N\\ 0_N & 0_N & \ldots & 0_N\\ \vdots & \vdots & \vdots & \vdots\\ 0_N & 0_N & \ldots & 0_N\end{pmatrix} G=\begin{pmatrix} 0_N & \ldots & 0_N & -I_N\\ 0_N & \ldots & 0_N & 0_N\\ \vdots & \vdots & \vdots & \vdots\\ 0_N & 0_N & \ldots & 0_N\end{pmatrix} \tag{20}$$

$$H=\begin{pmatrix} \overline{\lambda}I_N & 0_N & \ldots & \ldots & 0_N\\ -I_N & \overline{\lambda}I_N & \ldots & \ldots & 0_N\\ 0_N & -I_N & \ddots & \ldots & 0_N\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ 0_N & 0_N & \ldots & -I_N & \overline{\lambda}I_N\end{pmatrix}. \tag{21}$$

Recall that $p(\lambda)=|\lambda I_N-A_i|$, and

$$q(\overline{\lambda})=|\overline{\lambda}I_{nN}-\overline{A}_{n,i}|=|E||H-GE^{-1}F|. \tag{22}$$

We have,

$$E^{-1}=\begin{pmatrix} \frac{I_N}{\overline{\lambda}} & 0_N & \ldots & \ldots & 0_N\\ \frac{I_N}{\overline{\lambda}^2} & \frac{I_N}{\overline{\lambda}} & \ldots & \ldots & 0_N\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ \frac{I_N}{\overline{\lambda}^{i-1}} & \frac{I_N}{\overline{\lambda}^{i-2}} & \ldots & \frac{I_N}{\overline{\lambda}^2} & \frac{I_N}{\overline{\lambda}}\end{pmatrix}, \tag{23}$$

and $H-GE^{-1}F$ is equal to,

$$\begin{pmatrix} \overline{\lambda}I_N-\frac{A_i}{\overline{\lambda}^{i-1}} & 0_N & \ldots & \ldots & 0_N\\ -I_N & \overline{\lambda}I_N & \ldots & \ldots & 0_N\\ 0_N & -I_N & \ddots & \ldots & 0_N\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ 0_N & 0_N & \ldots & -I_N & \overline{\lambda}I_N\end{pmatrix}. \tag{24}$$

Then, $|E|=\overline{\lambda}^{N(i-1)}$, $|H-GE^{-1}F|=\overline{\lambda}^{N(n-i)}|\overline{\lambda}I_N-\frac{A_i}{\overline{\lambda}^{i-1}}|$, and substituting these in (22),

$$q(\overline{\lambda})=\overline{\lambda}^{N(n-i)}|\overline{\lambda}^i I_N-A_i|=\overline{\lambda}^{N(n-i)}p(\overline{\lambda}^i). \tag{25}$$

The proof is complete. ■

Lemma 5: Given matrix A with $\rho(A)<1$, we have $\rho(\overline{A})\leq\rho(A)^{\frac{1}{\overline{\tau}+1}}<1$ with $\overline{A}$ as the augmented form of A via Eq. (2).

Proof: The characteristic polynomial of $\overline{A}$ can be defined based on Lemma 4. Let $p(\lambda)$ and $q(\bar{\lambda})$ respectively represent the characteristic polynomial of A and $\overline{A}$. For $\tau_{ij} = \overline{\tau}$ for all i, j and $A_{\overline{\tau}} = A$. Therefore, $q(\bar{\lambda}) = p(\bar{\lambda}^{\overline{\tau}+1})$ and $\rho(\overline{A}) = \rho(A)^{\frac{1}{\overline{\tau}+1}} < 1$. We know that the function $\rho(A)^{\frac{1}{\tau+1}}$ is an increasing function of τ (given $\rho(A) < 1$), then, for $\tau_{ij} = r < \overline{\tau}$ for all i, j and $A_r = A$, we have $\rho(\overline{A}) = \rho(A)^{\frac{1}{r+1}} < \rho(A)^{\frac{1}{\overline{\tau}+1}} < 1$. This can be generalized for any choice of bounded time-delay and associated augmented matrix in the form of (2). This completes the proof. ■

Proof of Theorem 1: For the proof, from Lemma 5, we show that $\rho(\widehat{\underline{A}}) < 1$ for $\overline{\tau} \leq \overline{\tau}^*$, implying Schur stable error dynamics (10). Recall that $\rho(\widehat{A}) < 1$ implies that the networked estimator (4)-(5) successfully tracks system (1) for the delay-free case. From (9) and Lemma 5, for the closed-loop matrix $\widehat{\underline{A}}$ (as modified augmented version of $\widehat{A}$) and $\rho(A) > 1$,

$$\rho(\widehat{\underline{A}}) \leq \rho(P \otimes A^{\overline{\tau}+1} - K\overline{D}_C(P \otimes A^{\overline{\tau}+1}))^{\frac{1}{\overline{\tau}+1}}, \qquad (26)$$

which implies that $\rho(\widehat{\underline{A}}) < 1$ for any $\overline{\tau} \leq \overline{\tau}^*$. In case $\rho(A) < 1$, since $\rho(A)^{\overline{\tau}+1} < \rho(A)$, Schur stability of $\widehat{A}$ also ensures the stability of $\widehat{\underline{A}}$ (for all $\overline{\tau}$). This completes the proof.

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