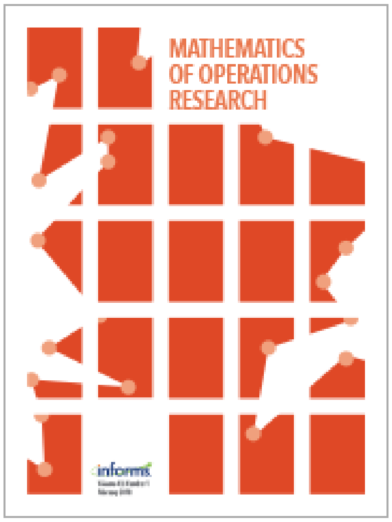

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Finite State Mean Field Games with Wright–Fisher Common Noise as Limits of *N*-Player Weighted Games

Erhan Bayraktar,[a] Alekos Cecchin,[b] Asaf Cohen,[a,*] François Delarue[c]

[a] Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109; [b] Centre de Mathématiques Appliquées, École Polytechnique, 91128 Palaiseau, France; [c] Université Côte d'Azur, CNRS, Laboratoire J.A. Dieudonné, 06108 Nice, France
*Corresponding author

Contact: erhan@umich.edu, https://orcid.org/0000-0002-1926-4570 (EB); alekos.cecchin@polytechnique.edu
https://orcid.org/0000-0003-2396-3638 (AlC); asafc@umich.edu, https://orcid.org/0000-0002-9211-7956 (AsC);
francois.delarue@unice.fr (FD)

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Abstract. Forcing finite state mean field games by a relevant form of common noise is a subtle issue, which has been addressed only recently. Among others, one possible way is to subject the simplex valued dynamics of an equilibrium by a so-called Wright–Fisher noise, very much in the spirit of stochastic models in population genetics. A key feature is that such a random forcing preserves the structure of the simplex, which is nothing but, in this setting, the probability space over the state space of the game. The purpose of this article is, hence, to elucidate the finite-player version and, accordingly, prove that N-player equilibria indeed converge toward the solution of such a kind of Wright–Fisher mean field game. Whereas part of the analysis is made easier by the fact that the corresponding master equation has already been proved to be uniquely solvable under the presence of the common noise, it becomes however more subtle than in the standard setting because the mean field interaction between the players now occurs through a weighted empirical measure. In other words, each player carries its own weight, which, hence, may differ from $1/N$ and which, most of all, evolves with the common noise.

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Keywords: mean field games • diffusion approximation • convergence problem • Wright–Fischer common noise

1. Introduction

In our earlier article (Bayraktar et al. [3]), we introduced a form of mean field games (MFGs) on a finite state space with the peculiarity of being driven by a kind of common noise that is reminiscent of the Wright–Fischer model in population genetics. Noticeably, thanks to the diffusive effect of the common noise on the finite dimensional simplex, we succeeded in proving that the master equation associated with this MFG admitted a unique solution and, in turn, that the MFG itself was uniquely solvable without any monotonicity assumption. The purpose of this new work is to address the asymptotic behavior of the finite-player counterpart of this mean field game. In a word, we here set up a corresponding finite state, many-player game with mean field interactions driven by a kind of Wright–Fisher common noise, and we establish the convergence of its equilibria toward the solution of the MFG constructed in Bayraktar et al. [3]. Whereas this program looks by now quite classic in the field, the originality of our works lies in the fact that two (instead of one) attributes are here assigned to each of the N players entering the finite game: indeed, not only do players have their own state (location), but they are also given a weight (we sometimes say a "mass"), accounting for their influence on the rest of the population. In other words, the empirical distribution that enters the definition of the weak interaction between the players is no longer a uniform distribution, but a more general finite probability weighted by the masses (or the weights) assigned to all of them; we call it the "weighted empirical distribution." Importantly, the weights are subjected to some external common noise, and this makes the main feature of our model. In a nutshell, the state dynamics of the system should be regarded as a system of controlled interacting Markov chains, each of the players controlling the rate of its transition between states. As for the weights, which are the second attribute of each player, they indeed jump simultaneously according to the common noise, the form of the latter being inspired by the Wright–Fisher model from population genetics (Fisher [23], Wright [35]) consistently with the limiting model addressed in Bayraktar et al. [3]. To make it clear, the common noise shuffles the weights of the players at a rate proportional to N according to a multinomial distribution with parameters N and the weighted empirical distribution of the system itself. A key fact, which we make clear in the sequel of the text, is that the mass shared by one player among all the players with the same state is the same after and before the common shuffling. On the top of those dynamics, each of the players aims to minimize an expected weighted cost. Very importantly, the expected weighted cost to each player includes the own mass of the player, which plays the role of a "density." Put differently, each player has its own perception of the uncertainties, depending on its own mass.

Our main result says that the cost per unit weight under the unique Nash equilibrium or, equivalently, the equilibrium value of each player but renormalized by its own mass, converges to the solution of the master equation. Generally speaking, the latter is a parabolic partial differential equation (PDE) describing the equilibrium cost (or the value) of the MFG analyzed in Bayraktar et al. [3]; see Theorem 2.1. In our setting, the master equation reads as a PDE driven by a so-called Kimura-like operator, which is a second order operator acting on functions defined on the finite dimensional simplex. This Kimura operator should be regarded as (a variant of) the generator of a multidimensional Wright–Fischer diffusion process. One of the main underlying obstacles is that this operator degenerates near the boundary of the simplex as a consequence of which corresponding harmonic functions may be singular at the boundary. A major reference on the subject is the monograph by Epstein and Mazzeo [20], which is repeatedly cited in Bayraktar et al. [3]. In order to avoid as much as possible the influence of the boundary, a key step in Bayraktar et al. [3] is to force the dynamics by an extra drift that points inward the simplex at the boundary. Our finite-player model exhibits the same feature, and in the end, the extra drift must be strong enough to ensure that all the results indeed hold true. We eventually use the convergence of the renormalized value of the finite game to show that the weighted empirical distribution, under the Nash equilibrium, converges to the flow of measures described in the mean field game; see Theorem 2.2.

Before we compare these two results with the existing literature, we feel it fair to recall that the theory of MFGs was initiated by the seminal works of Lasry and Lions [30, 31], and Huang et al. [24, 25]. The very first objective of it was precisely to provide an asymptotic formulation for many-player games with weak interaction. Mathematically speaking, the connection between finite and mean field games raises many subtle and challenging questions. Actually, there are two types of convergence results in the MFG literature for justifying the passage from finite to infinite games: one strand is concerned with constructing asymptotic Nash equilibria in the N-player game using solutions of the MFG; the other one is to show that the costs and empirical measure of the N-player game under Nash equilibria converge to a mean field equilibrium, which is, in fact, the more challenging problem between the two. In order to handle the latter problem, Cardaliaguet et al. [8] use the master equation. This method was later utilized in a finite state Markov chain setup (Bayraktar and Cohen [1], Bayraktar and Zhang [2], Cecchin and Pelino [13], Cecchin et al. [14]), and our own approach is obviously inspired from it. For further discussion on MFGs and the convergence problem, the reader is referred to Carmona and Delarue [9, 10], Lacker [27–29], and Fischer [22], the latter two authors having successfully developed other strategies based upon tightness arguments.

In comparison with all the previously cited references, one peculiarity of our model is the role of the common noise. Differently from the aforementioned references, in which it was already allowed to be present, the common noise is fundamental in our work as it guarantees the smoothness of the solution of the master equation and, subsequently, the uniqueness of the asymptotic equilibrium. Accordingly, our need to have a smooth solution to the limiting master equation dictates the form of the common noise in our finite-player game. In this respect, it is fair to say that, independently of the well posedness of the master equation, adding common noise to finite state mean field games is, in fact, a subtle issue. For instance, this difficulty was pointed out by Bertucci et al. [6], who solved it by introducing a common noise in the form of simultaneous jumps governed by a deterministic transformation of the state space. For an alternative model with the simultaneous jumps to the state space, see Belak et al. [4]. Although it shares some similarities, the idea introduced in Bayraktar et al. [3] is slightly different because the simultaneous jumps therein obey a stochastic rule given in the form of a Wright–Fisher noise. Accordingly, the finite-player version that we construct must obey some rules that permit it to recover asymptotically the same Wright–Fisher noise. This is precisely the step at which the extra attribute, namely, the weight, comes in: thanks to it, we are able to inject the Wright–Fisher common noise in the finite game, and this results in a random measure in the limit. In other words, we shift the simultaneous jumps in locations as used in Bertucci et al. [6] to simultaneous jumps in weights.[1] To our mind, allowing the empirical distribution to be nonuniform is an interesting idea in its own, independently of the precise model that we here address. Typically, weakly interacting many-player games are indeed formulated such that the weight of each player is fixed and deterministic in time, that is, $1/N$. More than often, the population size also remains fixed in time. A few exceptions are the models studied by Campi and Fischer [7], Nutz [32], Claisse et al. [16], and Bertucci [5], in which the size of the population evolves in time, yet the weights are still homogeneous with respect to the current population. In contrast, in our model, whereas the size of the population remains fixed, the common noise yields a stochastic evolution of the weights.

We now outline the key steps in establishing the convergence of the many-player game to the MFG. In Section 2, we characterize the unique Nash equilibrium of the N-player game via a system of nonlocal differential equations, which requires special care because of the possible degeneracies of the weights in the empirical distribution of the system. To prove the convergence of the normalized Nash system to the solution of the master equation, which is our first main result, we plug in the solution of the master equation to the normalized Nash system. This results in a remainder; the estimates of this remainder are stated in Section 4.2 and proved later in Section 7. In comparison with similar approaches in MFG theory, we face additional difficulties that arise when the weighted empirical distribution approaches the boundary of the simplex. Indeed, the challenge in estimating the remainder comes from the aforementioned fact that the weights are shuffled in a way that each player receives a fair share with respect to the players in the same state. Because of this shuffling, the inverse of the weighted empirical distribution shows up in the nonlocal term in the differential equation describing the renormalized Nash system. This requires us to analyze the boundary behavior of the empirical distribution process. Moreover, we also need to estimate the moments of the weight process. These preliminary estimates are stated in Section 4.1 and proved in Section 6 by concentration estimates for the multinomial distribution. Even though boundary estimates were also needed in Bayraktar et al. [3] to analyze the master equation, we use here different methods because the stochastic flow of measures is no longer a diffusion but a jump process. In order to guarantee all these estimates are true, we need the aforementioned inward pointing drift to be strong enough; once again, this is consistent with the analysis carried out in Bayraktar et al. [3]. Finally, Section 3 is another preparatory section, in which we provide another interpretation of the normalized Nash system based on an auxiliary game; interestingly, this alternative representation supplies us with some very useful uniform bounds on the equilibrium feedbacks. Our second main result concerns the convergence of the weighted empirical distribution toward the solution of the MFG. In order to prove it, we combine the hence established convergence of the normalized Nash system together with diffusion approximation arguments. We emphasize that the limiting process is a diffusive stochastic Fokker–Planck equation and, as such, has a different structure from the weighted empirical distribution.

The rest of the paper is organized as follows. Section 2 starts with a description of the MFG model and the master equation studied in Bayraktar et al. [3]. Then, it provides the N-player game with common noise as well as the analysis of the Nash system. It ends with the statements of the main results: Theorems 2.1 and 2.2. In the next two sections, we prepare for the proofs of the main results. Section 3 is devoted to the analysis of the normalized system as well as presenting an auxiliary process associated with it. In Section 4.1, we state estimates on the weighted empirical distribution and the moments of the weights, and in Section 4.2, we reformulate the solution of the master equation as an approximate solution of the normalized N-Nash system. Finally, the proofs of the main results are provided in Section 5. Sections 6 and 7 are devoted to the proofs of the auxiliary results stated in Section 4. A future outlook is given in Section 8. In the appendix, we prove some of the results about uniqueness of the Nash equilibria of the N player that are technically demanding but less important for the proof.

In the rest of this section, we list some frequently used notation.

1.1. Notation

Denote $[\![d]\!] := \{1, \dots, d\}$. For an integer $N \geq 1$, a tuple $\boldsymbol{x} = (x^1, \cdots, x^N) \in [\![d]\!]^N$ and another integer $l \in [\![N]\!]$, we write $\boldsymbol{x}^{-l}$ for the tuple $(x^1, \cdots, x^{l-1}, x^{l+1}, \cdots, x^N)$. For some $j \in [\![d]\!]$, the notation $(j, \boldsymbol{x}^{-l})$ is then understood as $(x^1, \cdots, x^{l-1}, j, x^{l+1}, \cdots, x^N)$. A tuple $\boldsymbol{y} = (y^1, \cdots, y^N) \in (\mathbb{R}_+)^N$ is said to belong to $\mathbb{Y}$ if the entries belong to the set of nonnegative rational numbers $\mathbb{Q}_+$ and sum to N. For two tuples $\boldsymbol{x} \in [\![d]\!]^N$ and $\boldsymbol{y} = (y^1, \cdots, y^N) \in (\mathbb{R}_+)^N$, we also introduce the weighted empirical measure

$$\mu^N_{\boldsymbol{x},\boldsymbol{y}} = \frac{1}{N} \sum_{l=1}^{N} y^l \delta_{x^l}.$$

Clearly, $\mu^N_{\boldsymbol{x},\boldsymbol{y}}$ is a measure on $[\![d]\!]$, which we may regard as a d-tuple; for $i \in [\![d]\!]$, we let

$$\mu^N_{\boldsymbol{x},\boldsymbol{y}}[i] := \frac{1}{N}\sum_{l=1}^{N} y^l \mathbb{1}_{\{x^l=i\}}. \tag{1.1}$$

The scalar product between vectors z and w in $\mathbb{R}^d$ is denoted by $z \cdot w$.

We denote by $\mathcal{P}([\![d]\!])$ the space of probability measures on $[\![d]\!]$, which we identify with the simplex $\mathcal{S}_{d-1} := \{(p^1, \cdots, p^d) \in (\mathbb{R}_+)^d : \sum_{i=1}^d p^i = 1\}$. Also, we define $\widehat{\mathcal{S}}_{d-1} := \{(x^1, \cdots, x^{d-1}) \in (\mathbb{R}_+)^d : \sum_{i=1}^{d-1} x^i \le 1\}$. In particular, we sometimes regard the Dirac mass δ_i, for $i \in [\![d]\!]$, as the ith vector of the canonical basis of $\mathbb{R}^d$.

Whenever μ is a probability measure on $[\![d]\!]$ (i.e., $\mu \in \mathcal{P}([\![d]\!])$), we call $\mathcal{M}_{N,\mu}$ the multinomial distribution of parameters N, $(\mu[1], \cdots, \mu[d])$, namely,

$$\mathcal{M}_{N,\mu}(\boldsymbol{k}) = \frac{N!}{k^1! \cdots k^d!}\prod_{i=1}^{d} \mu[i]^{k^i}.$$

This is for $\boldsymbol{k} \in \mathbb{N}^d$ with $k^1 + \cdots + k^d = N$. In order to have another representation of the multinomial distribution, we assume that we are given a probability space, say $(\Xi, \mathcal{G}, \mathbf{P})$, equipped with a collection of random variables $(S_{N,\mu})_{\mu \in \mathcal{P}([\![d]\!])}$, all of them with values in the set of multi-indices $\boldsymbol{k} \in \mathbb{N}^d$ that sum to N such that $\mathbf{P} \circ S^{-1}_{N,\mu} = \mathcal{M}_{N,\mu}$ for all $\mu \in \mathcal{P}([\![d]\!])$. We denote the d entries of $S_{N,\mu}$ in the form $(S_{N,\mu}[i])_{i \in [\![d]\!]}$. Expectation under $\mathbf{P}$ is denoted $\mathbf{E}$. When the value of N is fixed and there is no ambiguity, we just write $(S_\mu)_\mu$ for $(S_{N,\mu})_\mu$.

For a real-valued function v on $[\![d]\!]^N \times (\mathbb{Q}_+)^N$, we define the first order variation (or discrete gradient) at point $(\boldsymbol{x}, \boldsymbol{y})$ and in the direction l as the tuple $\Delta^l v(\boldsymbol{x}, \boldsymbol{y})[\bullet] \in \mathbb{R}^d$ defined by $\Delta^l v(\boldsymbol{x}, \boldsymbol{y})[j] = v((j, \boldsymbol{x}^{-l}), \boldsymbol{y}) - v(\boldsymbol{x}, \boldsymbol{y})$. Quite often in the text, we indeed put a bullet symbol $\bullet$ to emphasize that the related quantity has to be understood as a d-tuple.

Finally, for a real x, the positive part of x is denoted by $x_+ = \max(x, 0)$.

1.2. Derivatives on the Simplex

The formulation of the master equation is given using intrinsic derivatives. Because this is quite common material, we feel it is more convenient to introduce them now as part of our notation. For a real-valued function h defined on $\mathcal{S}_{d-1}$, define the functions $\widehat{h}^i : \widehat{\mathcal{S}}_{d-1} \to \mathbb{R}$, $i \in [\![d]\!]$ as follows:

$$\widehat{h}^i(p^{-i}) := h(p) = h\Big(p_1, \cdots, p_{i-1}, 1 - \sum\nolimits_{k=i} p_k, p_{i+1}, \cdots, p_d\Big),$$
$$\text{with } p^{-i} = (p_1, \cdots, p_{i-1}, p_{i+1}, \cdots, p_d),$$

and we then say that h is differentiable on $\mathcal{S}_{d-1}$ if $\widehat{h}^i$ is differentiable on $\widehat{\mathcal{S}}_{d-1}$ for some (and, hence, for any) $i \in [\![d]\!]$. The corresponding intrinsic gradient reads in the form $\mathfrak{D}h = (\mathfrak{d}_1 h, \ldots, \mathfrak{d}_d h)$ with

$$\mathfrak{d}_i h(p) = -\frac{1}{d}\sum_{j \ne i} \partial_{p_j} \widehat{h}^i(p^{-i}), \quad p \in \mathcal{S}_{d-1},\ i \in [\![d]\!].$$

It follows that $\sum_j \mathfrak{d}_j h = 0$, meaning that the gradient belongs to the tangent space to the simplex. In particular, note that, if h is defined on a neighborhood of $\mathcal{S}_{d-1}$ in $\mathbb{R}^d$, the following holds true:

$$\mathfrak{d}_i h(p) - \mathfrak{d}_j h(p) = \partial_{p_i} h(p) - \partial_{p_j} h(p),$$

for $i, j \in [\![d]\!]$, and $p \in \mathcal{S}_{d-1}$. We may define in a similar manner the second order derivatives $(\mathfrak{d}^2_{i,j} h)_{i,j \in [\![d]\!]}$ on the simplex. We refer to Bayraktar et al. [3, section 3.1.2] for more details.

2. The Model and the Main Results

2.1. Finite Mean Field Games with Wright–Fisher Common Noise

Let us prepare the ground and recall the setup and the main results from the finite state MFG with common noise analyzed in Bayraktar et al. [3].

2.1.1. Formulation of the Mean Field Game and the Master Equation. Consider a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \le t \le T}, \mathbb{P})$, where the filtration $\{\mathcal{F}_t\}_{0 \le t \le T}$ satisfies the usual conditions. The probability space supports a collection of independent standard Brownian motions $((W^{i,j})_{0\le t\le T})_{i,j\in[\![d]\!]:i\neq j}$. To describe the MFG, we consider a *representative player*, whose statistical state (conditional on the noise $((W^{i,j})_{0\le t\le T})_{i,j\in[\![d]\!]:i\neq j}$) is tracked through a process $(Q_t=(Q_t^i : i\in[\![d]\!]))_{0\le t\le T}$ with the specific feature that each Q_t is a density on the product space $\Omega\times[\![d]\!]$. According to Bayraktar et al. [3] and consistent with the finite-player version given in Section 2.2, Q_t describes the statistical distribution of the state X_t of the representative player (which takes values in $[\![d]\!]$) under its own perception of the world, the latter being formalized through a density Y_t on the product space $\Omega\times[\![d]\!]$. The formulation of the MFG then relies on

• The own control of the representative player, which is given in feedback form by a (bounded and measurable) function $\beta:[0,T]\times[\![d]\!]\times\mathcal{S}_{d-1}\to\mathbb{R}^d$ such that, for any $(t,i,p)\in[0,T]\times[\![d]\!]\times\mathcal{S}_{d-1}$,

$$\beta_j(t,i,p)\ge 0,\quad z\in[\![d]\!]\backslash\{i\},\quad \beta_i(t,i,p)=-\sum_{j\neq i}\beta_j(t,i,p);$$

for $j\neq i$, the value $\beta_j(t,i,p)$ is the rate of transition for the representative player at time t from its state i to state j, whereas the environment is in state p.

• The *stochastic environment* $(P_t=(P_t^i : i\in[\![d]\!]))_{0\le t\le T}$, which is a progressively measurable simplex-valued process whose dynamics are given through a fixed point type of argument in the sequel. This is often referred to as the mean field game solution. Essentially, this is the conditional distribution of the system under equilibrium.

• The common noise, which is formed by the antisymmetric Brownian motions $((\overline{W}_t^{i,j}:=(W_t^{i,j}-W_t^{j,i})/\sqrt{2})_{0\le t\le T})_{i,j\in[\![d]\!]:i\neq j}$ and a parameter[2] $\varepsilon\in(0,1)$, referred to as the *intensity* of the common noise.

The dynamics of the representative player are given explicitly by

$$\mathrm{d}Q_t^i=\sum_{j\in[\![d]\!]}(Q_t^j(\varphi(P_t^i)+\beta_i(t,j,P_t))-Q_t^i(\varphi(P_t^j)+\beta_j(t,i,P_t)))\mathrm{d}t+\varepsilon Q_t^i\sum_{j\in[\![d]\!]}\sqrt{\frac{P_t^j}{P_t^i}}\,\mathrm{d}\overline{W}_t^{i,j}, \tag{2.1}$$

for $t\in[0,T]$ with the same deterministic initial condition as P; that is, $Q_0=P_0$. Here, φ is a nonincreasing Lipschitz function from $[0,\infty)$ into itself such that

$$\varphi(r):=\begin{cases}\kappa, & r\le\delta,\\ 0, & r>2\delta,\end{cases} \tag{2.2}$$

where δ is a fixed arbitrary positive parameter. The role of φ is clarified in the statement of Proposition 2.1. For a sufficiently large value of κ, it forces the coordinates of the process $(Q_t)_{0\le t\le T}$ (and in the end of $(P_t)_{0\le t\le T}$ itself, at least whenever $(P_t)_{0\le t\le T}$ is indeed chosen as the fixed point) to stay (strictly) positive and even more to stay away from zero with large probability. In this respect, it is worth recalling from Bayraktar et al. [3] that $(Q_t)_{0\le t\le T}$ does *not* take values in the simplex. To put it clearly, Q_t only defines a density on the product space $\Omega\times[\![d]\!]$: using terminologies from statistical mechanics, it is an annealed but not a quenched density.

The representative player aims to minimize the following *cost function*:

$$\mathcal{J}(\boldsymbol{\beta},\boldsymbol{P}):=\sum_{i\in[\![d]\!]}\mathbb{E}\left[Q_T^i g(i,P_T)+\int_0^T Q_t^i\left(f(i,P_t)+\frac{1}{2}\sum_{j\neq i}|\beta_j(t,i,P_t)|^2\right)\mathrm{d}t\right],$$

where $f,g:[\![d]\!]\times\mathcal{S}_{d-1}\to\mathbb{R}$ satisfy suitable assumptions given in Proposition 2.1. Finally, a *solution* of the mean field game (with common noise) is a pair $(\boldsymbol{P},\alpha)$ such that

i. $\boldsymbol{P}=(P_t)_{0\le t\le T}$ is a $\mathcal{S}_{d-1}$-valued process, progressively measurable with respect to $\mathbb{F}^{\boldsymbol{W}}$ with some fixed $p_0=(p_{0,i})_{i\in[\![d]\!]}\in\mathcal{S}_{d-1}$ as the initial condition, and $\alpha:[0,T]\times[\![d]\!]\times\mathcal{S}_{d-1}\times[\![d]\!]\to\mathbb{R}$ is a bounded feedback strategy;

ii. $\boldsymbol{P}$ and α satisfy in the strong sense the equation

$$\mathrm{d}P_t^i=\sum_{j\in[\![d]\!]}\Big(P_t^j(\varphi(P_t^i)+\alpha(t,j,P_t)(i))-P_t^i(\varphi(P_t^j)+\alpha(t,i,P_t)(j))\Big)\mathrm{d}t+\varepsilon\sum_{j\in[\![d]\!]}\sqrt{P_t^iP_t^j}\mathrm{d}\overline{W}_t^{i,j};$$

iii. $\mathcal{J}(\alpha,\boldsymbol{P})\le\mathcal{J}(\boldsymbol{\beta},\boldsymbol{P})$ for any admissible strategy $\boldsymbol{\beta}$.

We say that the solution $(\boldsymbol{P},\alpha)$ is unique if, given another solution $(\widetilde{\boldsymbol{P}},\widetilde{\alpha})$, we have $P_t=\widetilde{P}_t$ for any $t\in[0,T]$, $\mathbb{P}$-almost surely, and $\alpha(t,i,P_t)(j)=\widetilde{\alpha}(t,i,P_t)(j)\,dt\otimes\mathbb{P}$-almost everywhere for each $i,j\in[\![d]\!]$.

The existence of a unique MFG solution is established in Bayraktar et al. [3] by showing that the associated master equation has a unique smooth solution. In this case, the master equation is a system of d parabolic PDEs set on the $(d-1)$-dimensional simplex:

$$\begin{aligned}
&\partial_t U^i(t,p) - \frac{1}{2}\sum_{j\neq i}(U^i(t,p) - U^j(t,p))_+^2 + f^i(p) + \sum_{j\in[\![d]\!]}\varphi(p_j)[U^j(t,p) - U^i(t,p)] \\
&\quad + \sum_{j,k\in[\![d]\!]} p_k[\varphi(p_j) + (U^k(t,p) - U^j(t,p))_+]\Big(\mathfrak{d}_j U^i(t,p) - \mathfrak{d}_k U^i(t,p)\Big) \\
&\quad + \varepsilon^2\sum_{j\neq i} p_j\Big(\mathfrak{d}_i U^i(t,p) - \mathfrak{d}_j U^i(t,p)\Big) + \frac{\varepsilon^2}{2}\sum_{j,k\in[\![d]\!]}(p_j\delta_{j,k} - p_j p_k)\mathfrak{d}^2_{j,k}U^i(t,p) = 0, \qquad (2.3) \\
&U^i(T,p) = g^i(p),
\end{aligned}$$

for $(t,p)\in[0,T]\times \mathrm{Int}(\widehat{\mathcal{S}}_{d-1})$.

The statement of the theorem requires that the cost functions are smooth enough in the sense of so-called Wright–Fisher spaces. The reader is referred to Epstein and Mazzeo [20] (see also Bayraktar et al. [3, section 3.2.2]) for a thorough discussion on these spaces. We provide a short reminder along the lines of Cecchin and Delarue [11]:

1. $\mathcal{C}^{\gamma}_{\mathrm{WF}}(\mathcal{S}_{d-1})$ consists of continuous functions on $\mathcal{S}_{d-1}$ that are γ-Hölder continuous up to the boundary with respect to the metric associated with the Wright–Fisher noise.
2. $\mathcal{C}^{2+\gamma}_{\mathrm{WF}}(\mathcal{S}_{d-1})$ consists of continuous functions on $\mathcal{S}_{d-1}$ that are twice continuously differentiable in the $(d-1)$-dimensional interior of $\mathcal{S}_{d-1}$ with derivatives satisfying a suitable behavior at the boundary and a suitable Hölder regularity that depends on the order of the derivative; in particular, the derivatives of order 1 are Hölder continuous up to the boundary, but the derivative of order 2 may blow up at the boundary and be only locally Hölder continuous in the interior.
3. $\mathcal{C}^{1+\gamma/2,2+\gamma}_{\mathrm{WF}}([0,T]\times\mathcal{S}_{d-1})$ is the parabolic version of $\mathcal{C}^{2+\gamma}_{\mathrm{WF}}(\mathcal{S}_{d-1})$; it consists of continuous functions on $[0,T]\times\mathcal{S}_{d-1}$ that are continuously differentiable in time $t\in[0,T]$ and that are twice continuously differentiable in space in the $(d-1)$-dimensional interior of $\mathcal{S}_{d-1}$ with derivatives satisfying a suitable behavior at the boundary and a suitable Hölder regularity; in particular, the time derivative and the space derivatives of order 1 are Hölder continuous up to the boundary, but the derivative of order 2 may blow up at the boundary.

Proposition 2.1 (Bayraktar et al. [3, Theorems 3.2 and 3.4]). *Assume that, for some $\gamma>0$, each $f(i,\cdot)$ for $i\in[\![d]\!]$ belongs to $\mathcal{C}^{\gamma}_{\mathrm{WF}}(\mathcal{S}_{d-1})$, and each $g(i,\cdot)$ for $i\in[\![d]\!]$ belongs to $\mathcal{C}^{2+\gamma}_{\mathrm{WF}}(\mathcal{S}_{d-1})$. Then, for any $\varepsilon\in(0,1)$, there exists a universal exponent $\eta\in(0,1)$ (hence, independent of ε) and a threshold $\kappa_0>0$ only depending on ε, $\|f\|_\infty$, $\|g\|_\infty$, and T such that, for any $\kappa\geq\kappa_0$ and $\delta\in(0,1/(4\sqrt{d}))$, the master Equation (2.3) has a unique solution in $[\mathcal{C}^{1+\gamma'/2,2+\gamma'}_{\mathrm{WF}}([0,T]\times\mathcal{S}_{d-1})]^d$ for $\gamma'=\min(\gamma,\eta)/2$. Furthermore, for any (deterministic) initial condition $p_0=(p_{0,i})_{i\in[\![d]\!]}\in\mathcal{S}_{d-1}$ with positive entries, the mean field game has a unique solution, and it is given by*

$$dP^i_t = \sum j\in[\![d]\!]\Big[P^j_t(\varphi(P^i_t) + (U^j - U^i)_+(t,P_t)) - P^i_t(\varphi(P^j_t) + (U^i - U^j)_+(t,P_t))\Big]dt + \frac{\sqrt{\varepsilon}}{\sqrt{2}}\sum_{j\in[\![d]\!]}\sqrt{P^i_tP^j_t}\,d[W^{i,j}_t - W^{j,i}_t], \qquad (2.4)$$

with p_0 as initial condition.

Observe that, when $\varepsilon=0$, we reduce to the finite state MFG without common noise analyzed in Bayraktar and Cohen [1] and Cecchin and Pelino [13]. Also, in the sequel, it is implicitly understood that the assumptions of Proposition 2.1 are in force. In particular, f and g have the same regularity as therein for some $\gamma\in(0,1)$. Also, the parameter κ in (2.2) of φ is indeed assumed to be large enough. Depending on our needs, we may even require the value of κ to be larger in some of our statements.

Now, we are ready to present the N-player game and the main results.

2.2. Finite-Player Version

We now introduce the finite-player analogue of the mean field game. Not only should we assign a time-dependent state with each player in the finite game but also a weight. Throughout, we use the following generic notations: N is the number of players in the finite game; for each index $l\in[\![N]\!]:=\{1,\cdots,N\}$ and at each time $t\in[0,T]$, X^l_t denotes the state of player l at time t and Y^l_t denotes its *weight*, which is a nonnegative rational number.

The tuple $(X_t^1, \cdots, X_t^N)$ is written with a boldface letter $\boldsymbol{X}_t$ and similarly for the weights. The weighted empirical measure is denoted by

$$\mu_t^N := \mu^N_{\mathbf{X}_t,\mathbf{Y}_t} = \frac{1}{N}\sum_{l=1}^N Y_t^l \delta_{X_t^l}.$$

2.2.1. Controlled Dynamics. The dynamics of $(\boldsymbol{X}_t,\boldsymbol{Y}_t)_{0\le t\le T}$ are constructed on the same probability space $(\Omega,\mathcal{F},\mathbb{P})$ as before. Consistently with the setup used for the formulation of the mean field game, they are subjected to a tuple of controls, formalized in the form of a collection of transition rates chosen by each player. We may indeed denote by $\mathbb{A}$ the collection of tuples $(\alpha^i[j])_{i,j\in[\![d]\!]:i\neq j}$ with coordinates in $\mathbb{R}_+$. A control is then a function $\boldsymbol{\alpha} : [0,T] \times [\![d]\!]^N \times \mathbb{Q}_+^N \to \mathbb{A}^N$ that maps $(t,\boldsymbol{x},\boldsymbol{y}) \in [0,T]\times[\![d]\!]^N \times \mathbb{Q}_+^N$ onto $\boldsymbol{\alpha}(t,\boldsymbol{x},\boldsymbol{y}) = ((\alpha^l(t,(i,\boldsymbol{x}^{-l}),\boldsymbol{y})[j])_{i,j\in[\![d]\!]:i\neq j})_{l\in[\![N]\!]} \in \mathbb{A}^N$. In other words, $\alpha^l : [0,T]\times[\![d]\!]^{N-1}\times\mathbb{Q}_+^N \ni (t,\boldsymbol{x}^{-l},\boldsymbol{y}) \longmapsto (\alpha^l(t,(i,\boldsymbol{x}^{-l}),\boldsymbol{y})[j])_{i,j\in[\![d]\!]:i\neq j} \in \mathbb{A}$ is the feedback function chosen by player l; for sure, another (but equivalent) way to do it is to consider α^l as the application $\alpha^l : [0,T]\times[\![d]\!]^N \times \mathbb{Q}_+^N \ni (t,\boldsymbol{x},\boldsymbol{y}) \longmapsto (\alpha^l(t,\boldsymbol{x},\boldsymbol{y})[j])_{j\in[\![d]\!]:j\neq x^l}$ that no longer takes its values in $\mathbb{A}$ but in $(\mathbb{R}_+)^{d-1}$. We mostly adopt the latter point of view in the sequel and, hence, reserve the notation α^l for the $(\mathbb{R}_+)^{d-1}$-valued mapping. For simplicity, we assume that the feedback functions are bounded and measurable.

For a given control $\boldsymbol{\alpha}$ and for the same function φ and the same viscosity parameter $\varepsilon\in(0,1]$ as in the previous paragraph, the population evolves according to a continuous time Markov chain with transition rates given by

$$\begin{aligned}\mathbb{P}\Big(X_{t+h}^l = j, \boldsymbol{X}_{t+h}^{-l} = \boldsymbol{x}^{-l}, \boldsymbol{Y}_{t+h} = \boldsymbol{y}\,\Big|\,\boldsymbol{X}_t=\boldsymbol{x},\boldsymbol{Y}_t=\boldsymbol{y}\Big)\\ = (\varphi(\mu^N_{\boldsymbol{x},\boldsymbol{y}}[j]) + \alpha^l(t,\boldsymbol{x},\boldsymbol{y})[j])h + o(h),\end{aligned} \tag{2.5}$$

whenever $j\neq x^l$, and

$$\mathbb{P}\left(\boldsymbol{X}_{t+h}=\boldsymbol{x}, Y_{t+h}^l = y^l\frac{k^{x^l}}{N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^l]}\ \forall l\in[\![N]\!]\,\middle|\,\boldsymbol{X}_t=\boldsymbol{x},\boldsymbol{Y}_t=\boldsymbol{y}\right) = \varepsilon\mathcal{M}_{N,\mu^N_{\boldsymbol{x},\boldsymbol{y}}}(\boldsymbol{k})Nh + o(h), \tag{2.6}$$

where $\boldsymbol{k}$ is a d-tuple of integers $(k^1,\cdots,k^d)$ with $k^1+\cdots+k^d = N$. In fact, this definition requires some care: (i) the ratio $k_{x^l}/(N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^l])$ is treated as one if $\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^l]=0$, so on the left-hand side, we should write the value of Y^l_{t+h} in the form $Y^l_{t+h} = y^l k_{x^l}/(N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^l])\mathbb{1}_{\{\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^l]\neq 0\}}$; (ii) the multinomial distribution on the right-hand side is well-defined if $\mu^N_{\boldsymbol{x},\boldsymbol{y}}$ is a probability measure on $[\![d]\!]$, meaning that $y^1+\cdots+y^N = N$.

In the sequel, we, thus, always assume that the Markov chain is initialized from a point $(\boldsymbol{x},\boldsymbol{y})$ such that $y^1+\cdots+y^N=N$, namely, $\boldsymbol{y}\in\mathbb{Y}$. In order to guarantee that the Markov chain is well-defined, it is then needed to check that the latter condition is preserved by the dynamics, meaning that, at any time $t>0$, $Y_t^1+\cdots+Y_t^N = N$. To do so, it suffices to check that, for any $(\boldsymbol{x},\boldsymbol{y})\in[\![d]\!]^N\times\mathbb{Y}$ and for any $\boldsymbol{k}\in\mathbb{N}^d$ with $k^1+\cdots+k^d=N$, it holds that

$$\sum_{l\in[\![N]\!]} y^l \frac{k^{x^l}}{N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^l]} = N. \tag{2.7}$$

This is well-checked. Indeed, by (1.1),

$$\sum_{l\in[\![N]\!]} y^l\frac{k^{x^l}}{N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^l]} = \sum_{i\in[\![d]\!]}\sum_{l\in[\![N]\!]}\mathbb{1}_{\{x^l=i\}}y^l\frac{k^i}{N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[i]} = \sum_{i\in[\![d]\!]}k^i = N.$$

Another important point related to the dynamics of $\boldsymbol{Y}$ is that zero is an absorption point for each coordinate, meaning that Y^l remains in zero once it has reached it.

Finally, it is implicitly understood that there are no other possible jumps for the whole system than those described in (2.5) and (2.6), meaning that

$$\begin{aligned}\mathbb{P}(\boldsymbol{X}_{t+h}=\boldsymbol{x},\boldsymbol{Y}_{t+h}=\boldsymbol{y}\mid\boldsymbol{X}_t=\boldsymbol{x},\boldsymbol{Y}_t=\boldsymbol{y})\\ = 1-\sum_{l=1}^N\sum_{j\neq x^l}\Big(\varphi(\mu^N_{\boldsymbol{x},\boldsymbol{y}}[j]) + \alpha^l(t,\boldsymbol{x},\boldsymbol{y})[j]\Big)h - \varepsilon Nh + o(h).\end{aligned} \tag{2.8}$$

We make an intense use of the generator of the process $(\boldsymbol{X},\boldsymbol{Y})$. It is given by

$$\mathcal{L}_t^N v(\boldsymbol{x},\boldsymbol{y}) = \sum_{l\in[\![N]\!]}\sum_{j\in[\![d]\!]}\Big(\varphi(\mu^N_{\boldsymbol{x},\boldsymbol{y}}[j]) + \alpha^l(t,\boldsymbol{x},\boldsymbol{y})[j]\Big)[v((j,\boldsymbol{x}^{-l}),\boldsymbol{y}) - v(\boldsymbol{x},\boldsymbol{y})]$$
$$+ \varepsilon N \sum_{\boldsymbol{k}\in[\![N]\!]^d}\mathcal{M}_{N,\mu^N_{\boldsymbol{x},\boldsymbol{y}}}(\boldsymbol{k})\left[v\left(\boldsymbol{x}, y^1\frac{k_{x^1}}{N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^1]},\ldots,y^N\frac{k_{x^N}}{N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^N]}\right) - v(\boldsymbol{x},\boldsymbol{y})\right],$$

with the same convention as before for the ratios in the second line whenever one of the denominators cancels. Using our notations, this can be written as

$$\mathcal{L}_t^N v(\boldsymbol{x},\boldsymbol{y}) = \sum_{l=1}^N\Big(\varphi(\mu^N_{\boldsymbol{x},\boldsymbol{y}}[\bullet]) + \alpha^l(t,\boldsymbol{x},\boldsymbol{y})[\bullet]\Big)\cdot\Delta^l v(\boldsymbol{x},\boldsymbol{y})[\bullet]$$
$$+ \varepsilon N\mathbf{E}\left[v\left(\boldsymbol{x}, y^1\frac{S_{\mu^N_{\boldsymbol{x},\boldsymbol{y}}}[x^1]}{N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^1]},\ldots,y^N\frac{S_{\mu^N_{\boldsymbol{x},\boldsymbol{y}}}[x^N]}{N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^N]}\right) - v(\boldsymbol{x},\boldsymbol{y})\right], \tag{2.9}$$

where the inner product in the first line is in $\mathbb{R}^d$ and we recall that $\Delta^l v(\boldsymbol{x},\boldsymbol{y})[\bullet]\in\mathbb{R}^d$ is defined by $\Delta^l v(\boldsymbol{x},\boldsymbol{y})[j] = v((j,\boldsymbol{x}^{-l}),\boldsymbol{y}) - v(\boldsymbol{x},\boldsymbol{y})$.

2.2.2. Cost Functional. With the same coefficients $f,g:[\![d]\!]\times\mathcal{S}_{d-1}\to\mathbb{R}$ as before, we can now assign a cost with each player. For each $l\in[\![N]\!]$, player l aims at minimizing the cost[3]

$$J^l(\boldsymbol{\alpha}) = \mathbb{E}\left[\int_0^T Y_t^l(L(X_t^l,\alpha^l(t,\boldsymbol{X}_t,\boldsymbol{Y}_t)) + f(X_t^l,\mu_t^N))dt + Y_T^l g(X_T^l,\mu_T^N)\right], \tag{2.10}$$

where $\boldsymbol{\alpha}=(\alpha^1,\ldots,\alpha^N)$ is the tuple of controls chosen by the N players. As usual in game theory, the cost to player l, hence, implicitly depends on the controls chosen by the others. The function L denotes the same Lagrangian as in the previous section, namely, $L(i,\alpha)=\frac{1}{2}\sum_{j\neq i}|\alpha(j)|^2$ for $i\in[\![d]\!]$ and $\alpha=(\alpha(j))_{j\in[\![d]\!]:j\neq i}\in(\mathbb{R}_+)^{d-1}$. Hence, we may rewrite the cost as

$$J^l(\boldsymbol{\alpha}) \;=\sum_{i\in[\![d]\!]}\mathbb{E}\left[\int_0^T Y_t^l\mathbb{1}_{\{X_t^l=i\}}\left(\frac{1}{2}\sum_{j\neq i}|\alpha^l(t,(i,\boldsymbol{X}_t^{-l}),\boldsymbol{Y}_t)[j]|^2 + f(i,\mu_t^N)\right)dt + Y_T^l\mathbb{1}_{\{X_T^l=i\}}g(i,\mu_T^N)\right].$$

Of course, the occurrence of the weights in the definition of the cost functional (which is one of the unusual features of our model) is reminiscent of the formula used in the mean field limit. Obviously, this is our objective to make the connection rigorous. Accordingly, we can introduce the Hamiltonian

$$H(i,u) = -\frac{1}{2}\sum_{j\neq i}(u^i-u^j)_+^2 = \inf_{\alpha\in(\mathbb{R}_+)^d}\left\{\sum_j \alpha[j](u^j-u^i) + \frac{1}{2}\sum_{j\neq i}|\alpha[j]|^2\right\}, \tag{2.11}$$

whose argmin is $a^*(i,u)[j]=(u^i-u^j)_+$. (Notice that, in this infimum, α is d-dimensional, whereas controls have been regarded as being $(d-1)$-dimensional so far. Obviously, this is for notational convenience only.)

2.2.3. More About Our Choice of Common Noise. The reader might object that, despite the terminology used throughout the paper, our model does not coincide with the standard Wright–Fisher one. Although our model is indeed different, the common noise shares some similarities, hence, our choice to call it "Wright–Fischer." To wit, it is easy to see from (2.6) that, at any random time t when the common noise rings (or jumps),

$$\mu_t^N[i] = \frac{1}{N}\sum_{l\in[\![N]\!]}Y_t^l\mathbb{1}_{\{X_{t-}=i\}} = \frac{1}{N}\sum_{l\in[\![N]\!]}Y_{t-}^l\frac{S_{\mu^N_{t-}}[i]}{N\mu^N_{t-}[i]}\mathbb{1}_{\{X_{t-}=i\}} = \frac{S_{\mu^N_{t-}}[i]}{N}, \quad i\in[\![d]\!],$$

where, conditional on the observations up to $t-$, $S_{\mu^N_{t-}}$ is a multinomial distribution of parameters N and $(\mu^N_{t-}[e])_{e\in[\![d]\!]}$. The way the weighted empirical distribution is, hence, updated is, thus, similar to the way the empirical distribution is updated in the standard Wright–Fisher model.

To make the comparison even stronger, it is interesting to observe that there is, in fact, a more direct way to introduce a Wright–Fisher common noise in the game. Indeed, consistently with the very definition of the Wright–Fisher model, we could require that, at any jump time of the common noise, all the players in the game resample their own state (or location in $[\![d]\!]$), independently of the others and of the past, according to the uniform empirical distribution of the system. Here, what we mean by "uniform empirical distribution" of the system is the standard empirical distribution, obtained by assigning the weight $1/N$ to each player. As a result, this model would not feature any additional weight process $(\boldsymbol{Y}^l)_{l\in[\![N]\!]}$. Even though it is very appealing, this approach has, however, a very strong drawback, which makes it useless for our own purpose: because of the resampling (which would occur very often under the same intensity as in (2.6)), the empirical measure would be strongly attractive; in turn, the latter would preclude any interesting deviating phenomenon. This is in contrast to our model: because the common noise acts on the weights, players may really deviate from the empirical measure of the whole population. This is the rationale for assigning two attributes to each player. This is also the main conceptual innovation of our model.

2.2.4. Application to Models with Heterogeneous Influences. As we already pointed out, one of the main features of the finite game that we have just introduced is that the weights $(Y_t^1,\cdots,Y_t^N)_{t\ge 0}$ of the players in the flow of empirical measures $(\mu_t^N)_{t\ge 0}$ may be different. From the practical point of view, the weight Y_t^l that is carried out by player l at time t may be regarded as the instantaneous influence of player l onto the whole collectivity. In this sense, this type of game may be used in order to describe models with heterogeneous influences. Notice that this interpretation is consistent with the one that we gave earlier in the text. Following Bayraktar et al. [3], we indeed explain that the weight to player l could be regarded as describing the own perception of the world by player l. What we are saying here is that the way player l perceives the world can be determined by the way player l is, in fact, perceived by the others: in other words, the collectivity is acting as a mirror.

Generally speaking, influences here evolve with time according to the transitions (2.6), the very interpretation of which is as follows. Intuitively, influences within the population are updated from time to time when the bell of the common noise rings. Equivalently, common noise defines random times at which polls (or elections) are made in order to determine the current influences of the players. In this regard, (2.6) is reminiscent of models in population genetics: when the common noise rings, every player in the population chooses at random a feature in $[\![d]\!]$ according to the current weighted empirical measure. Obviously, features are sampled independently. The new influence of player l is then recomputed by multiplying the earlier one by the ratio equal to "the number of outcomes for the feature carried by l divided by the expected number of outcomes for this feature" with the later denominator in the ratio being also understood as the total influence of the given feature. In the mean, the influence keeps constant, but obviously, some fluctuations may occur. We then recover the aforementioned notion of "perception" by returning to Equation (2.10) for the cost. Intuitively, an action may not have the same impact whether a player is very or poorly influent. For instance, a strong influencer may gain or lose a lot of followers after a good or bad decision, whence the presence of the weight Y^l as a density in (2.10).

In order to clarify the exposition, we provide two economic and social phenomena that fit this concept. Typically, we may consider a network with N agents, each of them aiming at selling a product, which can be of one of d types. Consistent with the previous description, any player has the player's own influence/importance in the market: the position $X_t^l\in[\![d]\!]$ of player l is, thus, understood as the type the player sells, whereas the weight Y_t^l is seen as the player's influence. Moreover, there are external random shocks that, at random times, resample the influences of players according to (2.6). Any player then chooses a control, which is the rate at which the player changes type in order to maximize a reward given by (the opposite of) (2.10). Therein, a quadratic cost has to be paid in order to trade another type of product; moreover, the rewards $-f$ and $-g$ include the weighted empirical measure $\mu_t^N = N^{-1}\sum_{l=1}^N Y_t^l\delta_{X_t^l}$. The component $\mu_t^N[i] = N^{-1}\sum_{l=1}^N Y_t^l\,\mathbb{1}_{\{X_t^l=i\}}$ is then understood as the total influence of the type i. Hence, f and g can be taken, for example, monotone if it is more advantageous to sell products with a small influence, whereas they can be antimonotone if it is better to sell products with a large influence. The two rewards $-f$ and $-g$ in (2.10) are multiplied by Y^l in order to account for the fact that the reputation of a trademark may have a direct impact on the selling price. Similarly, a highly reputed trademark may pay a higher price for switching from one type of product to another one, which may justify why the quadratic cost in (2.10) is also multiplied by Y^l.

As our second social phenomenon, let us focus on a social voter model in which N people can choose one out of two opposite opinions. Similarly to Cecchin et al. [14], we assume that $X_t^l=\pm 1$. In addition, any player has the player's own influence Y_t^l, which can be thought of, for instance, as the number of followers in a social network. Again, external random shocks might be thought of as elections that shuffle people's influence. People are rational and want to minimize the cost

$$J^l(\boldsymbol{\alpha}) = \mathbb{E}\left[\int_0^T Y_t^l \frac{|\alpha^l(t,\boldsymbol{X}_t,\boldsymbol{Y}_t)[-X_t^l]|^2}{2}dt - Y_T^l X_T^l \mathcal{M}(\mu_T^N)\right],$$

where $\alpha^l(t,\boldsymbol{X}_t,\boldsymbol{Y}_t)[-X_t^l]$ is the switching rate of player l from X_t^l to $-X_t^l$ and $\mathcal{M}(\mu_T^N)$ denotes the mean of the measure, that is, $\mathcal{M}(\mu_T^N)=\mu_T^N[1]-\mu_T^N[-1]=N^{-1}\sum_{l=1}^N Y_t^l\,\mathbb{1}_{\{X_t^l=1\}}-N^{-1}\sum_{l=1}^N Y_t^l\,\mathbb{1}_{\{X_t^l=-1\}}$. Therefore, the terminal cost is antimonotone, and morally, if the total influence of opinion 1 is larger than the total influence of –1, then the mean is positive. In turn, any player wants to be in 1 as the player has to minimize the cost. Still, there is a cost to pay to change one's opinion, which is larger when the influence is large. Thus, any player wishes to imitate the majority, but differently from Cecchin et al. [14], this is here a weighted majority that takes into account the influence of people. This is a typical model with *heterogeneous influences*.

2.3. Existence of Nash Equilibria

We are now able to state the *Nash system*, which is an equation for the equilibrium values of the game; equivalently, the Nash system gives the cost to each player when all of them play the Nash equilibrium (we prove that the latter is indeed unique in a relevant sense). It is a system of N functions indexed by $l\in[\![N]\!]$, $\boldsymbol{x}\in[\![d]\!]^N$, and $\boldsymbol{y}\in\mathbb{Y}$, which formally writes (with $(v^{N,l}(t,\boldsymbol{x},\boldsymbol{y}))_{l\in[\![N]\!]}$ as unknown)

$$\begin{aligned}
&\frac{d}{dt}v^{N,l}+\sum_{m\in[\![N]\!]}\varphi(\mu^N_{\boldsymbol{x},\boldsymbol{y}}[\bullet])\cdot\Delta^m v^{N,l}[\bullet]+\sum_{m\neq l}a^*\left(x^m,\frac{1}{y^m}v^{N,m}_\bullet\right)\cdot\Delta^m v^{N,l}[\bullet]\\
&\quad+y^lH\left(x^l,\frac{1}{y^l}v^{N,l}_\bullet\right)+y^lf(x^l,\mu^N_{\boldsymbol{x},\boldsymbol{y}})\\
&\quad+\varepsilon N\mathbf{E}\left[v^{N,l}\left(t,\boldsymbol{x},y^1\frac{S_{\mu^N_{\boldsymbol{x},\boldsymbol{y}}}[x^1]}{N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^1]},\ldots,y^N\frac{S_{\mu^N_{\boldsymbol{x},\boldsymbol{y}}}[x^N]}{N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^N]}\right)-v^{N,l}(t,\boldsymbol{x},\boldsymbol{y})\right]=0,\\
&v^{N,l}(T,\boldsymbol{x},\boldsymbol{y})=y^lg(x^l,\mu^N_{\boldsymbol{x},\boldsymbol{y}}),
\end{aligned}\tag{2.12}$$

where $v^{N,l}_\bullet$ denotes the vector in $\mathbb{R}^d$ given by $(v^{N,l}(t,\boldsymbol{x},\boldsymbol{y})[j]=v^{N,l}(t,(j,\boldsymbol{x}^{-l}),\boldsymbol{y}))_{j\in[\![d]\!]}$.

In a word, the Nash system here reads as a countable system of ordinary differential equations, which makes it a bit more subtle than the Nash system that arises in the analysis of N-player games over a finite state space (as is the case in the study of the convergence problem for finite state mean field games without common noise; see, for instance, Bayraktar and Cohen [1], Cecchin and Pelino [13]). Also (and this is another difficulty, very specific to our setting), none of these equations makes sense whenever any of the coordinates of $\boldsymbol{y}$ vanishes because of the ratio $v^{N,m}_\bullet/y^m$ in a^* and of the ratio $v^{N,l}_\bullet/y^l$ in the Hamiltonian.

Our first main result in this regard comes as a verification argument, the proof of which is postponed to Appendix A.1:

Proposition 2.2 (Verification Argument). *Consider a collection of measurable functions $(a^l:[0,T]\times[\![d]\!]^N\times\mathbb{Y}\ni(t,\boldsymbol{x},\boldsymbol{y})\longmapsto a^l(t,(i,\boldsymbol{x},\boldsymbol{y})[j])_{j\in[\![d]\!]:j\neq x^l}\in(\mathbb{R}_+)^{d-1})_{l\in[\![N]\!]}$ taking values in a compact domain. Assume that $(v^{N,l})_{l\in[\![N]\!]}$ solves the Nash system (2.12) with the special features that*

1. *$a^*(x^l,v^{N,l}_\bullet(t,\boldsymbol{x},\boldsymbol{y})/y^l)$ is understood as $a^l(t,\boldsymbol{x},\boldsymbol{y})$ whenever $y^l=0$.*
2. *The Hamiltonian $y^lH(x^l,v^{N,l}_\bullet/y^l)$ is understood as zero whenever $y^l=0$.*

Then, the feedback strategy vector $\boldsymbol{\alpha}^*=(\alpha^{*,1},\ldots,\alpha^{*,N})$ given by

$$\begin{aligned}
\alpha^{*,l}(t,\boldsymbol{x},\boldsymbol{y})[j]=a^*\left(x^l,\frac{1}{y^l}v^{N,l}_\bullet(t,\boldsymbol{x},\boldsymbol{y})\right)[j]&:=\frac{1}{y^l}(v^{N,l}(t,\boldsymbol{x},\boldsymbol{y})-v^{N,l}(t,(j,\boldsymbol{x}^{-l}),\boldsymbol{y}))_+\\
&=\frac{1}{y^l}(-\Delta^lv^{N,l}(t,\boldsymbol{x},\boldsymbol{y})[j])_+,
\end{aligned}\tag{2.13}$$

for $l\in[\![N]\!]$ such that $j=x^l$ and $y^l>0$, and by

$$\alpha^{*,l}(t,\boldsymbol{x},\boldsymbol{y})[j]=a^l(t,\boldsymbol{x},\boldsymbol{y})[j],\tag{2.14}$$

for $l\in[\![N]\!]$ such that $j\neq x^l$ and $y^l=0$, defines a Nash equilibrium in Markov feedback form for the N-player game. Moreover, the functions $(v^{N,l})_{l\in[\![N]\!]}$ are the values of the equilibrium, that is,

$$v^{N,l}(t,\boldsymbol{x},\boldsymbol{y})=J^l(t,\boldsymbol{x},\boldsymbol{y},\boldsymbol{\alpha}^*)=\inf_\beta J^l(t,\boldsymbol{x},\boldsymbol{y},\beta,\boldsymbol{\alpha}^{*,-l}),\tag{2.15}$$

where $J^l(t,\boldsymbol{x},\boldsymbol{y},\boldsymbol{\alpha})$ denotes the cost when the process $(\boldsymbol{X},\boldsymbol{Y})$ starts at time t with $(\boldsymbol{X}_t,\boldsymbol{Y}_t)=(\boldsymbol{x},\boldsymbol{y})$.

The indetermination when $y^l=0$ is well-understood: in that case, the coordinate Y^l remains stuck in zero, and the running and terminal costs become zero whatever the choice of the strategy. In the sequel, we circumvent part of the indetermination by restricting uniqueness of a^l in (2.14) to triples $(t,\boldsymbol{x},\boldsymbol{y})$ such that $y^l>0$.

Proposition 2.3 (Existence and Uniqueness of Equilibria). *The Nash system (2.12) has a solution $(v^{N,l})_{l\in[\![N]\!]}$ such that, for each $l\in[\![N]\!]$, the function $(t,\boldsymbol{x},\boldsymbol{y})\longmapsto v^{N,l}(t,\boldsymbol{x},\boldsymbol{y})/y^l$, which is a priori defined on the set $\{(t,\boldsymbol{x},\boldsymbol{y})\in[0,T]\times[\![d]\!]^N\times\mathbb{Y}:y^l>0\}$, extends to $[0,T]\times[\![d]\!]^N\times\mathbb{Y}$ into a function $[0,T]\times[\![d]\!]^N\times\mathbb{Y}\ni(t,\boldsymbol{x},\boldsymbol{y})\longmapsto w^{N,l}(t,\boldsymbol{x},\boldsymbol{y})$ that is bounded by $T\|f\|_\infty+\|g\|_\infty$. It satisfies, for any $l,m\in[\![N]\!]$, any $\boldsymbol{y}\in\mathbb{Y}$ such that $y^l>0$ and $y^n=0$ and any $(t,\boldsymbol{x})\in[0,T]\times[\![d]\!]^N$,*

$$\Delta^n w^l(t,\boldsymbol{x},\boldsymbol{y})[j]=0,\quad j\in[\![d]\!]. \tag{2.16}$$

Accordingly, $\boldsymbol{\alpha}^*$, as given by (2.13), extends to tuples $\boldsymbol{y}$ satisfying $y^l=0$ for some $l\in[\![N]\!]$, replacing therein $v^{N,l}(t,\boldsymbol{x},\boldsymbol{y})/y^l$ by $w^{N,l}(t,\boldsymbol{x},\boldsymbol{y})$, and, hence, defines an equilibrium in Markov feedback form. Any other equilibrium in Markov feedback form $(a^l:[0,T]\times[\![d]\!]^N\times\mathbb{Y}\ni(t,\boldsymbol{x},\boldsymbol{y})\longmapsto a^l(t,(i,\boldsymbol{x},\boldsymbol{y})[j])_{j\in[\![d]\!]:j\neq x^l}\in(\mathbb{R}_+)^{d-1})_{l\in[\![N]\!]}$ satisfies

$$a^l(t,(i,\boldsymbol{x},\boldsymbol{y})[j]=\frac{1}{y^l}\Big(-\Delta^l v^{N,l}(t,\boldsymbol{x},\boldsymbol{y})[j]\Big)_+, \tag{2.17}$$

for $l\in[\![N]\!]$ such that $j\neq x^l$ and $y^l>0$.

Interestingly, Property (2.16) should read as an insensitivity property: it says that, whenever player l has a nonzero mass and implements the equilibrium strategy given by (2.13), it is insensitive to the peculiar state of any player n with a zero mass y^n at the same time. As explained right after, this feature implies a form of uniqueness of the hence constructed Nash equilibrium.

Indeed, although Proposition 2.3 does not give uniqueness of an equilibrium in Markov feedback form (because of the indetermination raised by Proposition 2.2 at points at which the mass of one player vanishes), (2.17) shows that the equilibrium feedback function chosen by player l at any point $a^l(t,\boldsymbol{x},\boldsymbol{y})$ with $y^l>0$ is, in fact, uniquely determined. Accordingly, the equilibrium state process (X^l,Y^l) of player l is uniquely defined up to the first time when its mass Y^l hits zero. Once the mass process Y^l has touched zero, it remains in zero; the position X^l may still vary according to the feedback function a^l, but this has no influence on the remaining expected cost of player l itself (because the running and terminal costs in (2.10) are multiplied by the mass) nor on the remaining expected cost of the other players. The latter feature is a bit subtle and is a consequence of the following two facts:

1. The first one is that, at any time t, the player l has no influence on the running and terminal costs to any player $m\neq l$ if its mass at time t is zero; indeed, its contribution to the empirical measure μ_t^N is then null.
2. The second fact is that, by the insensitivity property (2.16), the player l has no influence on the feedback of any player $m\neq l$ at any time when its mass is zero.

The proofs of Propositions 2.2 and 2.3 are partially postponed to the appendix because only a part of those former two results is needed in the core of our analysis. In fact, what we really need is the existence of the functions $(w^{N,l})_{l\in[\![N]\!]}$ quoted in the statement of Proposition 2.3 and, most of all, the fact that those functions can be bounded independently of N. The latter facts are addressed in Section 3, whereas the remaining claims are just checked in the appendix.

2.4. Convergence Results

Consider the family of solutions $((v^{N,l})_{l\in[\![N]\!]})_{N\geq 1}$ given by Proposition 2.3. Accordingly, set, for any $N\in\mathbb{N}\backslash\{0\}$ and $l\in[\![N]\!]$, the real-valued functions $w^{N,l},z^{N,l}$ on $[0,T]\times[\![d]\!]^N\times\mathbb{Y}$ by

$$w^{N,l}(t,\boldsymbol{x},\boldsymbol{y}):=\frac{1}{y^l}v^{N,l}(t,\boldsymbol{x},\boldsymbol{y}),\qquad z^{N,l}(t,\boldsymbol{x},\boldsymbol{y}):=U^{x^l}(t,\mu^N_{\boldsymbol{x},\boldsymbol{y}}), \tag{2.18}$$

where $U\in[\mathcal{C}_{\rm WF}^{1+\gamma'/2,2+\gamma'}([0,T]\times\mathcal{S}_{d-1})]^d$ for some $\gamma'\in(0,1)$ (depending on γ as mentioned in the statement of Proposition 2.1) is the classic solution to the master Equation (2.3) (recalling that κ is implicitly required to be large enough, which is, in any case, explicitly recalled in our main statements).

Our first main result provides a bound for the difference between $w^{N,l}$ and $z^{N,l}$.

Theorem 2.1 (Distance Between the *N*-Player and Mean Field Value Functions). *Under the assumption of Proposition 2.1, we can find a constant $\overline{\kappa}_0$ only depending on ε, T, $\|f\|_\infty$, and $\|g\|_\infty$ such that, for any $\kappa\geq\overline{\kappa}_0$, there exists an exponent $\chi>0$ only depending on γ, and a constant C, depending on δ, κ, T, $\|f\|_\infty$, $\|g\|_\infty$, and d such that, for any $N\in\mathbb{N}\backslash\{0\}$, $l\in[\![N]\!]$, $\boldsymbol{x}\in[\![d]\!]^N$, $\boldsymbol{y}\in\mathbb{Y}$, one has*

$$|(z^{N,l}-w^{N,l})|(t,\boldsymbol{x},\boldsymbol{y})\leq CN^{-\chi}\left(\prod_{i\in[\![d]\!]}\frac{1}{N^{-\epsilon}+\mu^N_{\boldsymbol{x},\boldsymbol{y}}[i]}\right)^{1/(2d)}\left(\frac{1}{N}\sum_{m\in[\![N]\!]}|y^m|^\ell\right)^{1/2}, \tag{2.19}$$

with $\epsilon=1/8$ and $\ell=3$.

Remark 2.1.

1. We use the bound in (2.19) along a sequence $(\boldsymbol{x}^N, \boldsymbol{y}^N)_{N\geq 1}$ (with each $(\boldsymbol{x}^N, \boldsymbol{y}^N)$ in $[\![d]\!]^N \times \mathbb{Y}_N$, where we put an additional index N in the notation $\mathbb{Y}$ because the latter obviously depends on N) that satisfies the following:

$$\sup_{N\geq 1}\left\{\max_{i\in[\![d]\!]}\frac{1}{N^{-\epsilon}+\mu^N_{\boldsymbol{x}^N,\boldsymbol{y}^N}[i]}+\frac{1}{N}\sum_{m\in[\![N]\!]}|y^{N,m}|^{\ell}\right\}<\infty. \tag{2.20}$$

Importantly, the bound in (2.19) is not uniform in $(\boldsymbol{x},\boldsymbol{y})$. The reason for this is that the proof requires a moment estimate on the weights, but this estimate explodes as $\mu^N_{\boldsymbol{x},\boldsymbol{y}}$ approaches the boundary of the simplex; see (4.15). The fact that singularities may emerge near the boundary can be anticipated from the dynamics (2.6), where $1/\mu^N_{\boldsymbol{x},\boldsymbol{y}}$ appears. Actually, this observation is also consistent with our previous result in Theorem 2.1, which establishes uniqueness of mean field game solutions only for initial distributions in the interior of the simplex and requires the optimal process to be sufficiently far away from the boundary with high probability. Instead, without common noise (i.e., $\varepsilon = 0$), the convergence rate is uniform—if the master equation possesses a smooth solution—because the weights are then constantly one.

2. The value of χ could be made explicit in terms of γ. For sure, this would require an additional effort in the proof to track the dependence of χ upon γ. In fact, we have felt easier not to address this question because the value of γ, as given by Bayraktar et al. [3, theorem 2.10], is itself implicit, so the resulting interest for having an explicit formula for χ in terms of γ seems rather limited.
In the same spirit, it is worth noticing that $\overline{\kappa}_0$ in the statement of Theorem 2.1 may differ from κ_0 in the statement of Proposition 2.1; in other words, we are not able to work with the same[4] κ_0 as in the statement of Proposition 2.1.

3. The reader will notice that, in the following proof, we pay a heavy price for the fact that the model is not elliptic. If the transition rates are assumed to be bounded from below by a positive constant on the whole domain (in our setting, the rates are just lower bounded in the neighborhood of the boundary thanks to the function φ), then the arguments would simplify.
Convergence of the empirical measure is directly proved by means of a diffusion approximation theorem. In particular, it is worth noting that, differently from Cardaliaguet et al. [8], we do not address the distance between the equilibrium particle system $(\boldsymbol{X},\boldsymbol{Y})$ and the auxiliary particle system obtained by replacing the feedback function $\Delta w^{N,l}$ by $\Delta z^{N,l}$. Indeed, if ever we were willing to do so, then we would have to invoke a similar diffusion approximation result but for the convergence of the auxiliary particle system; the proof would be exactly the same. We, hence, feel more straightforward to apply such a diffusion approximation argument to the equilibrium particle system. Also, it must be stressed that a peculiar interest of the auxiliary particle system is to allow for refined convergence results for the fluctuations and the deviations of the finite Nash equilibrium. For instance, this idea is developed in Bayraktar and Cohen [1], Cecchin and Pelino [13], and Delarue et al. [18, 19], but it requires a sufficiently strong rate of convergence for the difference $\Delta w^{N,l} - \Delta z^{N,l}$. Here, the bound obtained in the statement of Theorem 2.1 is rather weak and is below the threshold that would be needed for this approach (even though the value for χ is not explicit, it is clear from the proof that it is small).

In the following statement, we use the notation $(\mu_t := \mu^N_{\boldsymbol{X}^N_t,\boldsymbol{Y}^N_t})_{0\leq t\leq T}$ for the empirical measure of the equilibrium particle system with N players.

Theorem 2.2 (Convergence of the Empirical Measure). *For the same regime of parameters as in the assumption of Theorem 2.1, consider, for each $N\geq 1$, an initial condition $(\boldsymbol{x}^N,\boldsymbol{y}^N) = ((x^{N,l})_{l\in[\![N]\!]},(1,\dots,1)_{l\in[\![N]\!]}) \in [\![d]\!]^N \times \mathbb{Y}$ that satisfies*

$$\lim_{N\to\infty}\mu^N_{\boldsymbol{x}^N,\boldsymbol{y}^N}[i] = p_0^i > 0, \tag{2.21}$$

for all $i\in[\![d]\!]$ and for some $p_0\in\mathcal{P}([\![d]\!])$. Then, the sequence $(\mu^N_t)_{0\leq t\leq T}$ (seen as random elements taking values in the Skorokhod space $\mathcal{D}([0,T],\mathbb{R}^d)$) converges in the weak sense on $\mathcal{D}([0,T];\mathbb{R}^d)$, equipped with the J1 Skorokhod topology, to the solution $(P_t)_{0\leq t\leq T}$ of the SDE (2.4).

3. Equilibria with Uniformly Bounded Feedback Functions

One of the purposes of this section is to prove the first part of Proposition 2.3, namely, the fact that we can construct solutions $(v^{N,l})_{l\in[\![N]\!]}$ to the Nash system (2.12) such that $(v^{N,l}(t,\boldsymbol{x},\boldsymbol{y})/y^l)_{l\in[\![N]\!]}$ is bounded independently of N. Actually, we kill here two birds with one stone: not only do we prove the former boundedness property, but we also manage to establish an interpretation of the normalized value functions $(v^{N,l}(t,\boldsymbol{x},\boldsymbol{y})/y^l)_{l\in[\![N]\!]}$ as the value functions of another game. It is precisely this new interpretation that permits proving the former uniform bound in

N. This uniform bound plays a key role in our subsequent analysis of the convergence problem; it is the main ingredient from Proposition 2.3 that is used in the sequel.

3.1. Normalized Nash System

Following (2.18), we are willing to study

$$w^{N,l}(t,\ x,\ y) := \frac{1}{y^l} v^{N,l}(t,\ x,\ y). \tag{3.1}$$

Indeed, dividing (2.12) by y^l, we obtain (pay attention to the fact that the term on the third line is heavily impacted by the change of variable)

$$\begin{aligned}
&\frac{d}{dt} w^{N,l} + \sum_{m\in[\![N]\!]} \varphi(\mu^N_{x,y}[\bullet]) \cdot \Delta^m w^{N,l}[\bullet] + \sum_{m\neq l} a^*(x^m, w^{N,m}_\bullet) \cdot \Delta^m w^{N,l} \\
&+ H(x^l, w^{N,l}_\bullet) + f(x^l, \mu^N_{x,y}) \\
&+ \varepsilon N \mathbf{E}\left[\frac{S_{\mu^N_{x,y}}[x^l]}{N\mu^N_{x,y}[x^l]} w^{N,l}\left(t, x, y^1 \frac{S_{\mu^N_{x,y}}[x^1]}{N\mu^N_{x,y}[x^1]}, \ldots, y^N \frac{S_{\mu^N_{x,y}}[x^N]}{N\mu^N_{x,y}[x^N]}\right) - w^{N,l}(t,\ x,\ y)\right] = 0,
\end{aligned}$$

which, because $\mathbf{E}[S_{\mu^N_{x,y}}[x^l]/(N\mu^N_{x,y}[x^l])] = 1$, can be rewritten as

$$\begin{aligned}
&\frac{d}{dt} w^{N,l} + \sum_{m\in[\![N]\!]} \varphi(\mu^N_{x,y}[\bullet]) \cdot \Delta^m w^{N,l}[\bullet] + \sum_{m\neq l} a^*(x^m, w^{N,m}_\bullet) \cdot \Delta^m w^{N,l} \\
&+ H(x^l, w^{N,l}_\bullet) + f(x^l, \mu^N_{x,y}) \\
&+ \varepsilon N \mathbf{E}\left[\frac{S_{\mu^N_{x,y}}[x^l]}{N\mu^N_{x,y}[x^l]} \left(w^{N,l}\left(t, x, y^1 \frac{S_{\mu^N_{x,y}}[x^1]}{N\mu^N_{x,y}[x^1]}, \ldots, y^N \frac{S_{\mu^N_{x,y}}[x^N]}{N\mu^N_{x,y}[x^N]}\right) - w^{N,l}(t,\ x,\ y)\right)\right] = 0,
\end{aligned} \tag{3.2}$$

with the terminal boundary condition

$$w^{N,l}(T, x, y) = g(x^l, \mu^N_{x,y}). \tag{3.3}$$

Our first result guarantees that (3.2) is well-posed.

Proposition 3.1. *The system of Equations (3.2) has a unique solution among all the N-tuples of bounded functions $(w^{N,l})_{l\in[\![N]\!]}$ of the three variables $t \in [0,\ T]$, $x \in [\![d]\!]^N$, and $y \in \mathbb{Y}$, namely,*

$$\max_{l=1,\cdots,N} \sup_{t,\ x,\ y} |w^{N,l}(t,\ x,\ y)| < \infty,$$

the supremum being taken over $t \in [0,\ T]$, $x \in [\![d]\!]^N$ and $y \in \mathbb{Y}$.

3.1.1. Well Posedness. We argue by using a standard fixed-point argument. Given an input w in the form of an N-tuple of bounded functions $(w^l)_{l\in[\![N]\!]}$ of the three variables t, x, and y as in the statement, we consider the system

$$\begin{aligned}
&\frac{d}{dt} \widetilde{w}^l + \sum_{m\in[\![N]\!]} \varphi(\mu^N_{x,y}[\bullet]) \cdot \Delta^m \widetilde{w}^{N,l}[\bullet] + \sum_{m\neq l} a^*(x^m, \widetilde{w}^m_\bullet) \cdot \Delta^m \widetilde{w}^l + H(x^l, \widetilde{w}^l_\bullet) + f(x^l, \mu^N_{x,y}) \\
&\quad + \varepsilon N \mathbf{E}\left[\frac{S_{\mu^N_{x,y}}[x^l]}{N\mu^N_{x,y}[x^l]} \left(w^l\left(t, x, y^1 \frac{S_{\mu^N_{x,y}}[x^1]}{N\mu^N_{x,y}[x^1]}, \ldots, y^N \frac{S_{\mu^N_{x,y}}[x^N]}{N\mu^N_{x,y}[x^N]}\right) - w^l(t,\ x,\ y)\right)\right] = 0, \\
&\widetilde{w}^l(T, x, y) = g\left(x^l, \mu^N_{x,y}\right).
\end{aligned} \tag{3.4}$$

Following Cecchin and Pelino [13], we know that (3.4) has a unique bounded solution $(\widetilde{w}^l)_{l\in[\![N]\!]}$(notice that the presence of y counts for nothing whenever the input w is frozen: we may then solve the equation y per y). This creates a mapping Φ that sends $w = (w^l)_{l\in[\![N]\!]}$ onto $\widetilde{w} = (\widetilde{w}^l)_{l\in[\![N]\!]}$. System (3.4) can be regarded as the Nash system of a (quite standard) stochastic game on $[\![d]\!]$ with g on the last line as terminal cost and f on the first line plus the whole w^l term on the second line as running cost; see, for instance, Cecchin and Pelino [13, proposition 1]. This allows us to represent $\widetilde{w}^l$, for each l, as the equilibrium cost to player l in this auxiliary game. Bounding each of the cost coefficients therein, we easily deduce that there exists a constant C such that

$$\max_{l\in[\![N]\!]}\sup_{x,y}|\widetilde{w}^l(t,x,y)| \le C\left(1+\int_t^T \max_{l\in[\![N]\!]}\sup_{x,y}|w^l(s,x,y)|\,ds\right).$$

In turn, we get that, whenever the input w satisfies $\max_{l\in[\![N]\!]}\sup_{x,y}|w^l(s,x,y)| \le C\exp(C(T-t))$ for any $t\in[0,T]$, the same holds for the output $\Phi(w)$. We call $\mathcal{E}$ the class of such inputs, and we then prove that, for any two inputs $w^{(1)} = (w^{(1),l})_{l\in[\![N]\!]}$ and $w^{(2)} = (w^{(2),l})_{l\in[\![N]\!]}$ in $\mathcal{E}$,

$$\max_{l\in[\![N]\!]}\sup_{x,y}|\widetilde{w}^{(1),l}(t,x,y)-\widetilde{w}^{(2),l}(t,x,y)| \le C\int_t^T \max_{l\in[\![N]\!]}\sup_{x,y}|w^{(1),l}(t,x,y)-w^{(2),l}(t,x,y)|\,ds,$$

for a possibly new value of the constant C. The end of the proof is standard: $\Phi^{\circ\ell}$ creates a contraction for a large enough integer ℓ, which shows the existence of a solution within the class $\mathcal{E}$. Uniqueness over bounded solutions (that are not a priori assumed to be in $\mathcal{E}$) is proven in the same way. □

3.2. Interpretation of the Renormalized Nash System

The idea is to show that the functions $(w^{N,l})_{l\in[\![N]\!]}$, as given by Proposition 3.1, are the value functions of a new differential game with new dynamics but with an equilibrium whose feedback functions are the same as those given by Proposition 2.3. Although this auxiliary game has no real purpose from the modeling point of view, the resulting formulation of the functions $(w^{N,l})_{l\in[\![N]\!]}$ as the values of this new game gives almost for free some very useful bounds on the $(w^{N,l})_{l\in[\![N]\!]}$'s—see Proposition 3.2—and, in turn, on the feedback strategies of the original game as identified in (2.13). From a mathematical point of view, the intuition behind the construction of such a new game is nothing but recognizing in Equation (3.2) the generator of another process.

However, the definition of this new game requires some care because the state variable of a given player is no longer an element of $[\![d]\!]\times\mathbb{Q}_+$ but of $[\![d]\!]^N\times(\mathbb{Q}_+)^N$. In other words, the state process writes as a tuple of processes $(\widetilde{\boldsymbol{X}}^l,\widetilde{\boldsymbol{Y}}^l)_{l\in[\![N]\!]}$, each $(\widetilde{\boldsymbol{X}}^l,\widetilde{\boldsymbol{Y}}^l)$ writing itself as a tuple $(\widetilde{X}^{l,n},\widetilde{Y}^{l,n})_{n\in[\![N]\!]} = ((\widetilde{X}^{l,n}_t,\widetilde{Y}^{l,n}_t)_{t\in[0,T]})_{n\in[\![N]\!]}$ with values in $[\![d]\!]^N$. For each $l\in[\![N]\!]$, $(\widetilde{\boldsymbol{X}}^l,\widetilde{\boldsymbol{Y}}^l)$ is the state process associated with player l. We must distinguish $(\widetilde{\boldsymbol{X}}^l,\widetilde{\boldsymbol{Y}}^l) = (\widetilde{X}^{l,n},\widetilde{Y}^{l,n})_{n\in[\![N]\!]}$ from $(\widetilde{\boldsymbol{X}}^{\bullet,n},\widetilde{\boldsymbol{Y}}^{\bullet,n}) := (\widetilde{X}^{l,n},\widetilde{Y}^{l,n})_{l\in[\![N]\!]}$. The interpretation of the latter is made clear in the next lines, but say right now that each $\widetilde{\boldsymbol{Y}}^{\bullet,n}$ is required to take values in $\mathbb{Y}$ (which is similar to what we required from $\boldsymbol{Y}$ in the original game).

As before, each player chooses a feedback function. The subtlety is that, even though the state space has been enlarged to $[\![d]\!]^N$, the feedback function to player $l\in[\![N]\!]$ is still regarded as a function $\alpha^l:[0,T]\times[\![d]\!]^N\times\mathbb{Y}\ni(t,x,y)\longmapsto(\alpha^l(t,x,y)[j])_{j\in[\![d]\!]:j\neq x^l}\in(\mathbb{R}_+)^{d-1}$. In particular, the feedback function to player $l\in[\![N]\!]$ does not see the additional index n we use for enlarging the state variable. Now, we postulate that, for each $n\in[\![N]\!]$, the state process $(\widetilde{\boldsymbol{X}}^{\bullet,n},\widetilde{\boldsymbol{Y}}^{\bullet,n})$ is a Markov process with values in $[\![d]\!]^N\times\mathbb{Y}$ with generator

$$\begin{aligned}\widetilde{\mathcal{L}}^{n,N}_t v(\boldsymbol{x},\boldsymbol{y}) &= \sum_{l=1}^N\Big(\varphi(\mu^N_{\boldsymbol{x},\boldsymbol{y}}[\bullet])+\alpha^l(t,\boldsymbol{x},\boldsymbol{y})[\bullet]\Big)\cdot\Delta^l v(\boldsymbol{x},\boldsymbol{y})[\bullet]\\ &\quad+\varepsilon N\mathbf{E}\left[\frac{S_{\mu^N_{\boldsymbol{x},\boldsymbol{y}}}[x^n]}{N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^n]}\left(v\left(\boldsymbol{x},y^1\frac{S_{\mu^N_{\boldsymbol{x},\boldsymbol{y}}}[x^1]}{N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^1]},\ldots,y^N\frac{S_{\mu^N_{\boldsymbol{x},\boldsymbol{y}}}[x^N]}{N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^N]}\right)-v(\boldsymbol{x},\boldsymbol{y})\right)\right],\end{aligned}\tag{3.5}$$

the second line of which differs substantially from the second line of (2.9) (and explicitly depends on the index n appearing on the left-hand side). Equivalently, similar to (2.6), the corresponding jumps of the weight process obey the transitions

$$\mathbb{P}\left(\widetilde{\boldsymbol{X}}^{\bullet,n}_{t+h}=\boldsymbol{x},\widetilde{Y}^{l,n}_{t+h}=y^l\frac{k^{x^l}}{N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^l]}\ \forall l\in[\![N]\!]\,\middle|\,\widetilde{\boldsymbol{X}}^{\bullet,n}_t=\boldsymbol{x},\widetilde{\boldsymbol{Y}}^{\bullet,n}_t=\boldsymbol{y}\right)=\frac{k^{x^n}}{N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^n]}\mathcal{M}_{N,\mu^N_{\boldsymbol{x},\boldsymbol{y}}}(\boldsymbol{k})\varepsilon Nh+o(h).\tag{3.6}$$

As already explained, the ratio $k^{x^n}/N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^n]$ is defined as one if $\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^n]=0$. Also, observe that, by definition of the multinomial distribution $\mathcal{M}_{N,\mu}$, the transition rate $\varepsilon N(k^{x^n}/N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^n])\mathcal{M}_{N,\mu^N_{\boldsymbol{x},\boldsymbol{y}}}(\boldsymbol{k})$ is equal to $\varepsilon N(k^{x^n}/N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^n])$ $\mathcal{M}_{N,\mu^N_{\boldsymbol{x},\boldsymbol{y}}}(\boldsymbol{k})$, which in particular implies that this transition rate is still bounded by εN. Of course, the transitions of the $\boldsymbol{x}$-variable on $[\![d]\!]$ are the same as in (2.5).

For sure, the reader may worry about the correlations between the various Markov processes $((\widetilde{\boldsymbol{X}}^{\bullet,n}, \widetilde{\boldsymbol{Y}}^{\bullet,n}))_{n\in[\![N]\!]}$, but in fact, they do not matter. The reason is that the cost functional to player $l \in [\![N]\!]$ is defined as

$$\widetilde{J}^l(t, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\alpha}) := \mathbb{E}\left[\int_t^T (L(\widetilde{X}_s^{l,l}, \alpha^l(s, \widetilde{\boldsymbol{X}}_s^{\bullet,l}, \widetilde{\boldsymbol{Y}}_s^{\bullet,l})) + f(\widetilde{X}_s^{l,l}, \widetilde{\mu}_s^{\bullet,l}))ds + g(\widetilde{X}_T^l, \widetilde{\mu}_T^{\bullet,l})\right], \tag{3.7}$$

where $(t, \boldsymbol{x}, \boldsymbol{y})$ belongs to $[0, T] \times [\![d]\!]^N \times \mathbb{Y}$ and is understood as the initial condition of all the processes $(\widetilde{\boldsymbol{X}}^{\bullet,n}, \widetilde{\boldsymbol{Y}}^{\bullet,n})_{n\in[\![N]\!]}$ (all of them being, thus, required to start from the same initial condition). In the right-hand side, $\widetilde{\mu}^{\bullet,n}$ is defined as $\widetilde{\mu}_s^{\bullet,n} = \mu^N_{\widetilde{X}_s^{\bullet,n}, \widetilde{Y}_s^{\bullet,n}}$. We must insist once again on the difference between (3.7) and (2.10): in (3.7), the dynamics of the weights of the (other) players are computed with respect to transitions that truly depend on the index l (through the second line in (3.5)), which explains the rather unusual formulation of the game.

Following the proof of the verification argument (see the proof of Proposition 2.2 in the appendix), we can prove that the N-tuple of feedback functions $(\alpha^{*,l} : [0, T] \times [\![d]\!]^N \times \mathbb{Y} \to (\mathbb{R}_+)^{d-1})_{l\in[\![N]\!]}$, defined by $\alpha^{*,l}(t, \boldsymbol{x}, \boldsymbol{y}) = a^*(x^l, w_\bullet^{N,l}(t, \boldsymbol{x}, \boldsymbol{y}))$, is a Nash equilibrium of the new game and that the N functions $(w^{N,l})_{l\in[\![N]\!]}$ are the value functions of this equilibrium.[5] There is, however, a difference with Proposition 2.2 because, for any $l \in [\![N]\!]$, $\alpha^{*,l}(t, \boldsymbol{x}, \boldsymbol{y})$ is well-defined even if $y^l = 0$. In fact, $(\alpha^{*,l})_{l\in[\![N]\!]}$ coincides with the equilibrium given by Proposition 2.3 constructed in the proof of Section 3.3.

Importantly, the interpretation of $(w^{N,l})_{l\in[\![N]\!]}$ as the values of the Nash equilibrium permits to get a bound, independently of N.

Proposition 3.2. *The value functions $(w^{N,l})_{l\in[\![N]\!]}$ are uniformly bounded by the constant $T \| f \|_\infty + \| g \|_\infty$. Accordingly, the feedback functions given by*

$$\alpha^{*,l}(t, \boldsymbol{x}, \boldsymbol{y})[j] = a^*(x^l, w_\bullet^{N,l}(t, \boldsymbol{x}, \boldsymbol{y}))[j] = (w^{N,l}(t, \boldsymbol{x}, \boldsymbol{y}) - w^{N,l}(t, (j, \boldsymbol{x}^{-l}), \boldsymbol{y}))_+,$$

for $l \in [\![N]\!]$, are bounded by $2T \| f \|_\infty + 2 \| g \|_\infty$.

The upper bound for $w^{N,l}$ is found by playing the strategy $\alpha^l \equiv 0$, whereas the lower bound follows by the sign of ℓ. The bound for the feedback functions is obvious. □

3.3. Proof of Proposition 2.3

Here comes now the proof of Proposition 2.3. Existence of a solution follows from Proposition 3.1. It suffices to let $v^{N,l}(t, \boldsymbol{x}, \boldsymbol{y}) = y^l w^{N,l}(t, \boldsymbol{x}, \boldsymbol{y})$. Then, Identity (2.13) becomes

$$\alpha^{*,l}(t, \boldsymbol{x}, \boldsymbol{y})[j] = (w^{N,l}(t, \boldsymbol{x}, \boldsymbol{y}) - w^{N,l}(t, (j, \boldsymbol{x}^{-l}), \boldsymbol{y}))_+,$$

for $l \in [\![N]\!]$ such that $j = x^l$ and $y^l > 0$. The bound for $(w^{N,l})_{l\in[\![N]\!]}$ directly follows from Proposition 3.2.

The uniqueness result (which is not really needed in the rest of the paper) is proven in the appendix.

Remark 3.1. Let us comment on the information that each player uses in case there is or is not common noise. Starting with the latter case, let us compare Cecchin and Pelino [13] and Bayraktar and Cohen [1]. In Cecchin and Pelino [13], admissible strategies for the N-player games are Markov feedback controls with full state information, which is represented by $\boldsymbol{x}$. The characterization of the equilibrium is given by a system of N equations, in which its solution $(\overline{v}^{N,l}(t, \boldsymbol{x}))_{l=1}^N$ stands for the value functions for the players at time t given that, at this time, the states of the players are described by the vector $\boldsymbol{x} \in \mathbb{R}^N$. On the other hand, in Bayraktar and Cohen [1], each player knows the current state and the empirical distribution of the other players. The equilibrium is now characterized by a single ordinary differential equation, the solution of which is denoted by $\overline{V}^N(t, x, \eta)$. This is the value at time t of a representative player, whose state at this time is x, whereas the empirical distribution of the other players is $\eta \in \mathcal{P}^{n-1}([\![d]\!])$. One can show that, for any $l \in [\![N]\!]$, $\overline{v}^{N,l}(t, \boldsymbol{x}) = \overline{V}^N(t, x^l, (1/(N-1))\sum_{n,n\neq l}\delta_{x_n})$. This means that, even if the players have access to the entire configuration $\boldsymbol{x}$, each player may use, instead of the full information, the more concise information: private state and empirical distribution. This should not come as a surprise because the game is symmetric, and the players are anonymous in the sense that the cost for each player depends on its private state and the empirical distribution. Hence, it is reasonable that this information is sufficient to describe the equilibrium.

In contrast, in the model studied here (with common noise), the problem changes if strategies are restricted to time, players' own states, and the weighted empirical measure of the system. Whereas players are anonymous in the sense that the identities of the other players do not matter, players still need to keep track of the weights of the others. A sufficient piece of information for each of the players is the private state and weight and the *distribution* of the weights of the players *within* each state. As before, this observation can be supported by showing that the value function of a representative player in the equilibrium with more concise information coincides with the value functions of the players with the full information $(\boldsymbol{x},\boldsymbol{y})$. Now, recall that, whenever a player moves between states, it carries its weight with it to the new state. Hence, knowing the distribution of the weights of the players within each state is not equivalent to merely knowing the *total* weight of the players in each state, that is, $\mu^N_{\boldsymbol{x},\boldsymbol{y}}$. In other words, $\mu^N_{\boldsymbol{x},\boldsymbol{y}}$ does not provide enough information on the weights of the others. As an example, there is a difference between the case when there is a state that is occupied with a single player or two players, in each case, having the same collective mass.

4. Auxiliary Results for the Proofs of Theorems 2.1 and 2.2

The proofs of Theorems 2.1 and 2.2 rely on several intermediary results, which are stated in this section: in Section 4.1, we collect several estimates on the weight process $\boldsymbol{Y}^N$; Section 4.2 provides a first connection between the solution U of the master equation as defined in (2.3) and the Nash system (2.12), very much in the spirit of Cardaliaguet et al. [8]. Because the proofs of those results are rather lengthy and technical, we feel better to postpone them to Sections 6 and 7, respectively, as otherwise they could distract the reader from the main line of the text.

4.1. Analysis of the Weight Process

As we already alluded to several times, the main new point in our model is the weight process $\boldsymbol{Y}^N$. In this regard, we need to establish first some preliminary estimates of $\boldsymbol{Y}^N$ before we address the convergence problem itself. A simple look at (2.6) shows that this might be rather involved: the transitions of the weight process are determined by the empirical measure μ^N_t, but in turn, the latter is itself defined in terms of the weights, and most of all, the weights show up in (2.6) through the inverse quantities $(1/\mu^N_t[i])_{i\in[\![d]\!]}$. For sure, it is worth recalling that, whenever $\mu^N_t[i]$ is zero, the ratio $k^i/(N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[i])$ in the definition of the transition probability is understood as one and is, thus, well-defined, but this conventional rule cannot prevent us from a careful analysis of the boundary behavior of the empirical measure and in particular of the reachability of the boundary of the simplex $\mathcal{P}([\![d]\!])$. Note in this regard that, even though $\mu^N_t[i]$ is positive for some time $t\in[0,T)$, it may jump to zero after an infinitesimal time. The good point is that, whenever $\mu^N_t[i]$ is sufficiently far away from zero, this may happen with a small probability only. As a result, we manage to prove that the coordinates of the empirical measure can hardly touch zero provided the latter start sufficiently far away from it and the constant κ in (2.2) is large enough. In fact, this result is fully consistent with the analysis performed in Bayraktar et al. [3, section 2.2.1], in which we prove that the equilibria of the limiting mean field game cannot touch the boundary of the simplex.

Throughout the section, the number of players N is fixed. Moreover, all the results stated in the section are proved in Section 6.

4.1.1. General Setting. Actually, not only do we need to prove that the empirical measure associated with $(\boldsymbol{X},\boldsymbol{Y})$ remains away from the boundary of the simplex with high probability, but we also need to prove it for the process $(\widetilde{\boldsymbol{X}}^{\bullet,n},\widetilde{\boldsymbol{Y}}^{\bullet,n})_{n\in[\![N]\!]}$ introduced in the previous section; see Section 3.2.

In order to make the statement as general as possible, we, thus, assume that we are given an $[\![d]\!]^N\times\mathbb{Y}$-valued process $(\overline{\boldsymbol{X}},\overline{\boldsymbol{Y}})=(\overline{X}^l,\overline{Y}^l)$ (which must be thought of as $(\boldsymbol{X},\boldsymbol{Y})$ itself or as one of the $(\widetilde{\boldsymbol{X}}^{\bullet,n},\widetilde{\boldsymbol{Y}}^{\bullet,n})$ s for some $n\in[\![N]\!]$) satisfying the analogue of (2.5) (with $(\boldsymbol{X},\boldsymbol{Y})$ replaced by $(\overline{\boldsymbol{X}},\overline{\boldsymbol{Y}})$) for some feedback function $\boldsymbol{\alpha}=(\alpha^l)_{l\in[\![N]\!]}$, bounded by $2(T\|f\|_\infty+\|g\|_\infty)$, together with the analogue of either (2.6) or (3.6).

Whereas the meaning for the analogue of (2.5) should be clear, we feel it is useful to write down explicitly the analogue of (2.6) or (3.6) in the following form:

$$\begin{aligned}&\mathbb{P}\left(\overline{\boldsymbol{X}}_{t+h}=\boldsymbol{x},\overline{Y}^l_{t+h}=y^l\frac{k^{x^l}}{N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^l]}\ \forall l\in[\![N]\!]\ \middle|\ \overline{\boldsymbol{X}}_t=\boldsymbol{x},\overline{\boldsymbol{Y}}_t=\boldsymbol{y}\right)\\&\qquad=\varepsilon N\left(\frac{k^{x^n}}{N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^n]}\right)^{\iota}\mathcal{M}_{N,\mu^N_{\boldsymbol{x},\boldsymbol{y}}}(\boldsymbol{k})h+o(h),\end{aligned}\tag{4.1}$$

where $\iota\in\{0,1\}$ and $n\in[\![N]\!]$ are fixed once for all in the dynamics.

In fact, following Cecchin and Fischer [12], it is convenient to represent the dynamics of $(\overline{\boldsymbol{X}},\overline{\boldsymbol{Y}})$ as the solutions of stochastic differential eqations (SDEs) driven by Poisson random measures. To do so, we let $\mathcal{N}^0,\mathcal{N}^1,\dots,\mathcal{N}^N$ be independent Poisson random measures with respective intensity measures ν^0 on $[0,\varepsilon N]^{[\![N]\!]^d}$ and ν^l on $[0,M]^d$ for $l\in[\![N]\!]$ with

$$M:=\kappa+2(T\|f\|_\infty+\|g\|_\infty).\tag{4.2}$$

Intuitively, the $(\mathcal{N}^l)_{l\in[\![N]\!]}$'s can be thought of as the idiosyncratic noises and $\mathcal{N}^0$ as the common noise. Also,

$$\forall l \in [\![N]\!], \quad d\nu^l(\theta) = \sum_{j\in[\![d]\!]} \mathbb{1}_{[0,M]^d}(\theta) d\theta^j d\delta_{0_{\mathbb{R}^{d-1}}}(\theta^{-j}),$$
$$d\nu^0(\theta) = \sum_{\boldsymbol{k}\in[\![N]\!]^d} \mathbb{1}_{[0,\varepsilon N]^{[\![N]\!]^d}}(\theta) d\theta^{\boldsymbol{k}} d\delta_{0_{\mathbb{R}^{N^d-1}}}(\theta^{-\boldsymbol{k}}), \tag{4.3}$$

where θ^{-j} in the first line is the $(d-1)$-tuple $(\theta^1, \cdots, \theta^{j-1}, \theta^{j+1}, \cdots)$ and similarly for $\theta^{-\boldsymbol{k}}$ on the second line. In particular, all the N measures $\nu^1, \cdots, \nu^N$ are, in fact, the same and can be just denoted by ν. Notice also that $\mathcal{N}^0$ depends on N. Then, the dynamics of $(\overline{X}^l, \overline{Y}^l)$ can be written, for any $l \in [\![N]\!]$, as

$$d\overline{X}_t^l = \int_{[0,M]^d} \sum_{j\in[\![d]\!]} (j - \overline{X}_{t-}^l) \mathbb{1}_{(0,\beta_{t-}^l(j)]}(\theta^j) \mathcal{N}^l(d\theta, dt),$$
$$d\overline{Y}_t^l = \int_{[0,\varepsilon N]^{[\![N]\!]^d}} \sum_{\boldsymbol{k}\in[\![N]\!]^d} \left(\overline{Y}_{t-}^l \frac{k^{\overline{X}_{t-}^l}}{N\overline{\mu}_{t-}[\overline{X}_{t-}^l]} - \overline{Y}_{t-}^l \right) \mathbb{1}_{\{\overline{\mu}_{t-}[\overline{X}_{t-}^l]\neq 0\}} \mathbb{1}_{(0,\beta_{t-}^0(\boldsymbol{k})]}(\theta^{\boldsymbol{k}}) \mathcal{N}^0(d\theta, dt), \tag{4.4}$$

where

$$\overline{\mu}_t := \mu^N_{\overline{\boldsymbol{X}}_t, \overline{\boldsymbol{Y}}_t}$$

and

$$\beta_t^l(j) := \beta^l(t, \overline{\boldsymbol{X}}_t, \overline{\boldsymbol{Y}}_t)[j]; \quad \beta^l(t, \boldsymbol{x}, \boldsymbol{y})[j] := \varphi(\mu^N_{\boldsymbol{x},\boldsymbol{y}}[j]) + \alpha^l(t, \boldsymbol{x}, \boldsymbol{y})[j], \quad l \in [\![N]\!],$$
$$\beta_t^0(\boldsymbol{k}) := \beta^0(t, \overline{\boldsymbol{X}}_t, \overline{\boldsymbol{Y}}_t)[\boldsymbol{k}]; \quad \beta^0(t, \boldsymbol{x}, \boldsymbol{y})[\boldsymbol{k}] := \varepsilon N \left(\frac{k^{x^n}}{N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^n]} \right)^{\iota} \mathcal{M}_{N,\mu^N_{\boldsymbol{x},\boldsymbol{y}}}(\boldsymbol{k}). \tag{4.5}$$

Formulation (4.4) prompts us to let

$$\overline{f}^l(t, \boldsymbol{x}, \boldsymbol{y}, \theta) = \sum_{j\in[\![d]\!]} (j - x^l) \mathbb{1}_{(0,\beta^l(t,\boldsymbol{x},\boldsymbol{y})[j]]}(\theta^j),$$
$$\overline{g}^l(t, \boldsymbol{x}, \boldsymbol{y}, \theta) = \sum_{\boldsymbol{k}\in[\![N]\!]^d} \left(y^l \frac{k^{x^l}}{N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^l]} - y^l \right) \mathbb{1}_{\{\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^l]\neq 0\}} \mathbb{1}_{(0,\beta^0(t,\boldsymbol{x},\boldsymbol{y})[\boldsymbol{k}]]}(\theta^{\boldsymbol{k}}). \tag{4.6}$$

We then have, for any test function v of the two variables $\boldsymbol{x}$ and $\boldsymbol{y}$,

$$\begin{aligned}
&\int_{[0,M]^d} [v((x^l + \overline{f}^l(t, \boldsymbol{x}, \boldsymbol{y}, \theta), \boldsymbol{x}^{-l}), \boldsymbol{y}) - v(\boldsymbol{x}, \boldsymbol{y})] \nu(d\theta) \\
&= \sum_{j\in[\![d]\!]} \left(\varphi(\mu^N_{\boldsymbol{x},\boldsymbol{y}}[j]) + \alpha^l(t, \boldsymbol{x}, \boldsymbol{y})[j] \right) [v((j, \boldsymbol{x}^{-l}), \boldsymbol{y}) - v(\boldsymbol{x}, \boldsymbol{y})], \\
&\int_{[0,\varepsilon N]^{[\![N]\!]^d}} [v(\boldsymbol{x}, y^1 + \overline{g}^1(t, \boldsymbol{x}, \boldsymbol{y}, \theta), \ldots, y^N + \overline{g}^N(t, \boldsymbol{x}, \boldsymbol{y}, \theta)) - v(\boldsymbol{x}, \boldsymbol{y})] \nu^0(d\theta) \\
&= \varepsilon N \mathbf{E}\left[\left(\frac{S_{\mu^N_{\boldsymbol{x},\boldsymbol{y}}}[x^n]}{N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^n]} \right)^{\iota} \left(v\left(\boldsymbol{x}, y^1 \frac{S_{\mu^N_{\boldsymbol{x},\boldsymbol{y}}}[x^1]}{N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^1]}, \ldots, y^N \frac{S_{\mu^N_{\boldsymbol{x},\boldsymbol{y}}}[x^N]}{N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^N]} \right) - v(\boldsymbol{x}, \boldsymbol{y}) \right) \right].
\end{aligned} \tag{4.7}$$

In the rest of the paper, we then make an intense use of Itô's formula for the process $(\overline{\boldsymbol{X}}, \overline{\boldsymbol{Y}})$. It is written, for a general test function v, as

$$\begin{aligned}
dv(\overline{\boldsymbol{X}}_t, \overline{\boldsymbol{Y}}_t) = &\sum_{l\in[\![N]\!]} \int_{[0,M]^d} (v((\overline{X}_{t-}^l + \overline{f}_{t-}^l(\theta), \overline{\boldsymbol{X}}_{t-}^{-l}), \overline{\boldsymbol{Y}}_{t-}) - v(\overline{\boldsymbol{X}}_{t-}, \overline{\boldsymbol{Y}}_{t-})) \mathcal{N}^l(d\theta, dt) \\
&+ \int_{[0,\varepsilon N]^{[\![N]\!]^d}} (v(\overline{\boldsymbol{X}}_{t-}, \overline{Y}_{t-}^1 + \overline{g}_{t-}^1(\theta), \ldots, \overline{Y}_{t-}^N + \overline{g}_{t-}^N(\theta)) - v(\overline{\boldsymbol{X}}_{t-}, \overline{\boldsymbol{Y}}_{t-})) \mathcal{N}^0(d\theta, dt),
\end{aligned} \tag{4.8}$$

with the notations

$$\overline{f}_t^l(\theta) = \overline{f}^l(t, \overline{\boldsymbol{X}}_t, \overline{\boldsymbol{Y}}_t, \theta), \quad \overline{g}_t^l(\theta) = \overline{g}^l(t, \overline{\boldsymbol{X}}_t, \overline{\boldsymbol{Y}}_t, \theta). \tag{4.9}$$

4.1.2. Integrability of the Inverse of the Empirical Measure. Recall now the parameter κ provided in (2.2). Together with the preceding notation, we have the following theorem, which plays a crucial role in our subsequent analysis. Its proof is given in Section 6.1.

Theorem 4.1. *Under the preceding setting (which comprises in particular some feedback function $\boldsymbol{\alpha} = (\alpha^l)_{l\in[\![N]\!]}$ that is bounded by $2(T\|f\|_\infty + \|g\|_\infty)$), assuming in addition that the initial condition $(\overline{\boldsymbol{X}}_0, \overline{\boldsymbol{Y}}_0)$ is deterministic, let, for any $\epsilon \in (0, 1/4)$,*

$$\overline{\tau}_N := \inf\left\{t \geq 0 : \min_{i\in[\![d]\!]} \overline{\mu}_t[i] < N^{-\epsilon} \text{ or } \max_{l\in[\![N]\!]} \overline{Y}_t^l > \frac{1}{2}N^{1-\epsilon}\right\} \wedge T. \tag{4.10}$$

Then, for any $\lambda \geq 1$, there exists a constant $\overline{\kappa}_0$, depending only on λ such that, for any $\kappa \geq \overline{\kappa}_0$ and any $i \in [\![d]\!]$,

$$\mathbb{E}\left[\exp\left\{\int_0^{\overline{\tau}_N} \frac{\lambda}{\overline{\mu}_t[i]} dt\right\}\right] \leq \frac{C}{N^{-\epsilon} + \overline{\mu}_0[i]}, \tag{4.11}$$

for a constant C depending on δ, κ, T, d, M but independent of N and of the initial condition. Even more, for any $t \in [0, T]$,

$$\mathbb{E}\left[\frac{1}{N^{-\epsilon} + \overline{\mu}_{t\wedge\overline{\tau}_N}[i]} \exp\left(\int_0^{t\wedge\overline{\tau}_N} \frac{\lambda}{\overline{\mu}_s[i]} ds\right)\right] \leq \frac{C}{N^{-\epsilon} + \overline{\mu}_0[i]}. \tag{4.12}$$

Remark 4.1. Combining (4.11) and Hölder's inequality, we deduce that, for any $\lambda \geq 1$, there exists a constant $\overline{\kappa}_0$, depending only on λ such that, for any $\kappa \geq \overline{\kappa}_0$ and any $i \in [\![d]\!]$,

$$\mathbb{E}\left[\exp\left\{\sum_{i\in[\![d]\!]} \int_0^{\overline{\tau}_N} \frac{\lambda}{\overline{\mu}_t[i]} dt\right\}\right] \leq \prod_{i\in[\![d]\!]} \left(\frac{C}{N^{-\epsilon} + \overline{\mu}_0[i]}\right)^{1/d} = C \prod_{i\in[\![d]\!]} (N^{-\epsilon} + \overline{\mu}_0[i])^{-1/d}, \tag{4.13}$$

for a constant C depending on δ, κ, T, d, M but independent of N.

Similarly, for any $t \in [0, T]$ (and by enlarging $\overline{\kappa}_0$ if necessary),

$$\mathbb{E}\left[\prod_{i\in[\![d]\!]} (N^{-\epsilon} + \overline{\mu}_{t\wedge\overline{\tau}_N}[i])^{-1/d} \exp\left(\int_0^{t\wedge\overline{\tau}_N} \frac{\lambda}{\overline{\mu}_s[i]} ds\right)\right] \leq C \prod_{i\in[\![d]\!]} (N^{-\epsilon} + \overline{\mu}_0[i])^{-1/d}. \tag{4.14}$$

Notice that all these estimates are true if $\overline{\tau}_N = 0$.

4.1.3. Moment Bounds. We can now state several bounds on the moments of $(\overline{Y}^l)_{l\in[\![N]\!]}$ uniformly in N, recalling that the index n and the parameter $\iota \in \{0, 1\}$ are fixed as in (4.1). We also work with the same general feedback function $\boldsymbol{\alpha} = (\alpha^l)_{l\in[\![N]\!]}$ as in (4.1), keeping in mind that it is bounded by $2(T\|f\|_\infty + \|g\|_\infty)$.

Proposition 4.1. *Let $\overline{\tau}_N$ be as in Theorem 4.1. Then, for any integer $\ell \geq 1$, there exists a constant $\overline{\kappa}_0$, depending only on ℓ such that, for any $\kappa \geq \overline{\kappa}_0$ and any $\epsilon < 1/4$ (recalling that ϵ shows up in (4.10)),*

$$\sup_{0\leq t\leq T} \mathbb{E}\left[\frac{1}{N}\sum_{l\in[\![N]\!]} |\overline{Y}^l_{t\wedge\overline{\tau}_N}|^\ell\right] \leq C\left(\frac{1}{N}\sum_{l\in[\![N]\!]} |y_0^l|^{2\ell}\right)^{1/2} \prod_{i\in[\![d]\!]} (N^{-\epsilon} + \overline{\mu}_0[i])^{-1/(2d)}, \tag{4.15}$$

for a constant C depending on δ, κ, T, d, M but independent of N and of the initial condition $\overline{\mu}_0$, the latter being assumed to be deterministic. Moreover, if $\ell \geq 3$,

$$\mathbb{P}(\overline{\tau}_N < T) \leq \frac{C}{N^{\epsilon/d}}\left\{\prod_{i\in[\![d]\!]} (N^{-\epsilon} + \overline{\mu}_0[i])^{-1/d} + \left(\frac{1}{N}\sum_{l\in[\![N]\!]} |y_0^l|^{2\ell}\right)^{1/2} \prod_{i\in[\![d]\!]} (N^{-\epsilon} + \overline{\mu}_0[i])^{-1/(2d)}\right\}. \tag{4.16}$$

As for the inverse, we have the following.

Proposition 4.2. *Let $\overline{\tau}_N$ be as in Theorem 4.1. Then, for any $\ell \geq 1$, there exists a constant $\overline{\kappa}_0$, depending only on ℓ such that, for any $\kappa \geq \overline{\kappa}_0$ and any $\epsilon < 1/4$ (with ϵ showing up in (4.10)),*

$$\sup_{0\leq t\leq T} \mathbb{E}\left[\left(\frac{1}{N}\sum_{l\in[\![N]\!]} |\overline{Y}^l_{t\wedge\overline{\tau}_N}|^\ell\right)^{-1}\right] \leq C\left[\left(\frac{1}{N}\sum_{l\in[\![N]\!]} |y^l_0|^\ell\right)^{-1/2} + \exp(-cN^{1-2\epsilon})\right]\prod_{i\in[\![d]\!]}(N^{-\epsilon}+\overline{\mu}_0[i])^{-1/(2d)}, \tag{4.17}$$

for a constant C depending on δ, κ, T, d, M but independent of N and of the initial condition $\overline{\mu}_0$, the latter being assumed to be deterministic.

Notice that, here as well, all the preceding estimates are true if $\overline{\tau}_N = 0$. The proofs of Propositions 4.1 and 4.2 are given in Sections 6.3 and 6.4, respectively. Both rely on the following lemma, which is also used in Section 5 and the proof of which is given in Section 6.2.

Lemma 4.1. *For any integer $\ell \geq 1$, we can find a constant C such that, for any probability measure $\mu \in \mathcal{P}([\![d]\!])$ with $\mu[i] > 0$ for all $i \in [\![d]\!]$, the following two inequalities hold true for all $i, j \in [\![d]\!]$ and $N \geq 1$:*

$$\begin{aligned} \left|\mathbf{E}\left[\left(\frac{S_\mu[i]}{N\mu[i]}\right)^\ell - 1\right]\right| &\leq \frac{\ell(\ell-1)}{2N\mu[i]} + C\sum_{k=3}^{\ell}\frac{1}{N^{k/2}\mu[i]^{k-1}}, \\ \left|\mathbf{E}\left[\frac{S_\mu[j]}{N\mu[j]}\left\{\left(\frac{S_\mu[i]}{N\mu[i]}\right)^\ell - 1\right\}\right]\right| &\leq \frac{\ell(\ell+1)}{2N\min_{e\in[\![d]\!]}\mu[e]} + C\sum_{k=3}^{\ell+1}\frac{1}{N^{k/2}\min_{e\in[\![d]\!]}\mu[e]^{k-1}}, \end{aligned} \tag{4.18}$$

where we use the convention that $\sum_{k=3}^2 = 0$. Moreover, for any $p \geq 2$, we can find another constant C such that, for any $i \in [\![d]\!]$ and $N \geq 1$,

$$\mathbf{E}\left[\left|\left(\frac{S_\mu[i]}{N\mu[i]}\right)^\ell - 1\right|^p\right]^{1/p} \leq C\sum_{k=1}^{\ell}\left(N\mu[i]\right)^{-k/2}. \tag{4.19}$$

Finally, for any $\eta > 0$, there exists a constant $c > 0$ such that, for all $i \in [\![d]\!]$,

$$\mathbf{P}\left(\left|\left(\frac{S_\mu[i]}{N\mu[i]}\right)^\ell - 1\right| \geq \eta\right) \leq C\exp(-cN(\mu[i])^2). \tag{4.20}$$

4.2. The Master Equation as an Approximation of the Normalized Nash System

The purpose of this section is to formulate the analogue of Cardaliaguet et al. [8, proposition 6.1.3] (for the continuous state case, and Cecchin and Pelino [13, proposition 4] for the finite state space), namely, to regard finite-dimensional projections of classical solutions to the master Equation (2.3) as almost solutions of the Nash system (2.12).

To make it clear, recall that we define in (2.18) $z^{N,l}(t,x,y) = U^{x^l}(t,\mu^N_{x,y})$, where $U \in [\mathcal{C}^{1+\gamma'/2,2+\gamma'}_{\mathrm{WF}}([0,T]\times\mathcal{S}_{d-1})]^d$ for the same $\gamma' \in (0,1)$ as in (2.1) is the classic solution to the master equation. Then, we want to show that

$$u^{N,l}(t,x,y) := y^l z^{N,l}(t,x,y) \tag{4.21}$$

almost solves the Nash system (2.12), at least when

$$(x,y) \in \mathcal{T}_N := \left\{(x,y) \in [\![d]\!]^N \times \mathbb{Y} : \min_{i\in[\![d]\!]}\overline{\mu}^N_{x,y}[i] \geq N^{-\epsilon},\ \max_{l\in[\![N]\!]} y^l \leq \frac{1}{2}N^{1-\epsilon}\right\}, \tag{4.22}$$

for some $\epsilon \in (0,1/4)$, which has to be understood as the same ϵ as in (4.10). In comparison with the proof performed in Cardaliaguet et al. [8, proposition 6.1.3], one difficulty comes from the definition of $\mathcal{C}^{1+\gamma'/2,2+\gamma'}_{\mathrm{WF}}$ as the second order derivatives of elements of the latter space may be singular at the boundary of the simplex.

Our result takes the following form.

Proposition 4.3. *Let $\epsilon < 1/4$ be as in the definition of (4.22). Then, the function $u^{N,l}$ defined in (4.21) solves*

$$
\begin{aligned}
&\frac{d}{dt}u^{N,l} + \sum_{m\in[\![N]\!]} \varphi(\mu^N_{\boldsymbol{x},\boldsymbol{y}}[\bullet]) \cdot \Delta^m u^{N,l}[\bullet] + \sum_{m\neq l} a^*\Big(x^m, \frac{1}{y^m}u^{N,m}_{\bullet}\Big) \cdot \Delta^m u^{N,l}[\bullet] \\
&\quad + y^l H\Big(x^l, \frac{1}{y^l}u^{N,l}_{\bullet}\Big) + y^l f(x^l, \mu^N_{\boldsymbol{x},\boldsymbol{y}}) \\
&\quad + \varepsilon N \mathbf{E}\Bigg[u^{N,l}\Bigg(t, \boldsymbol{x}, y^1\frac{S_{\mu^N_{\boldsymbol{x},\boldsymbol{y}}}[x^1]}{N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^1]}, \dots, y^N\frac{S_{\mu^N_{\boldsymbol{x},\boldsymbol{y}}}[x^N]}{N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^N]}\Bigg) - u^{N,l}(t, \boldsymbol{x}, \boldsymbol{y})\Bigg] \\
&= y^l r^{N,l}(t, \boldsymbol{x}, \boldsymbol{y}), \\
&u^{N,l}(T, \boldsymbol{x}, \boldsymbol{y}) = y^l g(x^l, \mu^N_{\boldsymbol{x},\boldsymbol{y}}),
\end{aligned}
\tag{4.23}
$$

and there exists an exponent η only depending on γ' and ϵ and a constant C only depending on d and on the norm of U in the space $[\mathcal{C}^{1+\gamma'/2,2+\gamma'}_{\mathrm{WF}}([0,T]\times\mathcal{S}_{d-1})]^d$ such that the rest $r^{N,l}(t, \boldsymbol{x}, \boldsymbol{y})$ is bounded as follows:

$$
|r^{N,l}(t, \boldsymbol{x}, \boldsymbol{y})| \le \frac{C}{N^\eta},
\tag{4.24}
$$

for $t \in [0, T]$ and $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{T}_N$.

Remark 4.2. Importantly, Proposition 4.3 may be reformulated in a similar result for the function $z^{N,l}$ defined in (2.18). In short, $z^{N,l}$ solves (3.2) plus the same rest $r^{N,l}$ as in (4.23) (but without the leading factor y^l). We felt better to formulate Proposition 4.3 as it makes a direct connection with the Nash system (2.12), but we mostly use the version of (4.23) for $z^{N,l}$ in the core of the proof of Theorem 2.1. Anyway, the reader must be convinced that there is no difficulty in passing from one version to the other.

4.2.1. Recovering the Common Noise.

The strategy of proof of Proposition 4.3 consists in identifying the various terms of the master equation with the terms of (2.12). In this respect, the most subtle term to deal with is certainly the term associated with the common noise in (4.23). The following lemma makes the connection between the latter and the second order term in the master equation, and the proof of this connection is given in Section 7.1.

Lemma 4.2. *Under the assumption of Proposition 4.3, we have, for $l \in [\![N]\!]$,*

$$
\begin{aligned}
&\frac{1}{y^l} N \mathbf{E}\Bigg[u^{N,l}\Bigg(\boldsymbol{x}, y^1\frac{S_{\mu^N_{\boldsymbol{x},\boldsymbol{y}}}[x^1]}{N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^1]}, \dots, y^N\frac{S_{\mu^N_{\boldsymbol{x},\boldsymbol{y}}}[x^N]}{N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^N]}\Bigg) - u^{N,l}(t, \boldsymbol{x}, \boldsymbol{y})\Bigg] \\
&= \sum_{j\in[\![d]\!]} (\delta_{j,x_l} - \mu^N_{\boldsymbol{x},\boldsymbol{y}}[j])\mathfrak{d}_j U(t, x^l, \mu^N_{\boldsymbol{x},\boldsymbol{y}}) \\
&\quad + \frac{1}{2}\sum_{j,k\in[\![d]\!]} (\mu^N_{\boldsymbol{x},\boldsymbol{y}}[j]\delta_{j,k} - \mu^N_{\boldsymbol{x},\boldsymbol{y}}[j]\mu^N_{\boldsymbol{x},\boldsymbol{y}}[k])\mathfrak{d}^2_{j,k} U(t, x^l, \mu^N_{\boldsymbol{x},\boldsymbol{y}}) + r^{N,l}_1(t, \boldsymbol{x}, \boldsymbol{y}),
\end{aligned}
$$

where the rest $r^{N,l}_1$ is such that, for $t \in [0, T]$ and $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{T}_N$,

$$
|r^{N,l}_1(t, \boldsymbol{x}, \boldsymbol{y})| \le \frac{C}{N^\eta},
\tag{4.26}
$$

for a constant C only depending on d, the norm of U in the space $[\mathcal{C}^{1+\gamma'/2,2+\gamma'}_{\mathrm{WF}}([0,T]\times\mathcal{S}_{d-1})]^d$, and an exponent η, which, in turn, only depend on γ' and ϵ.

4.2.2. Other Terms.

Back to the statement of Proposition 4.3, we now address the terms of the master equation that are not associated with the common noise. In this respect, we have the following sequence of lemmas, the first of which is of independent interest as it allows us to connect first order variations and first order derivatives on the simplex. All three lemmas are proved in Section 7.2.

Lemma 4.3. *Under the assumption of Proposition 4.3, we have*

$$
\frac{1}{y^l}\Delta^m u^{N,l}(t, \boldsymbol{x}, \boldsymbol{y})[j] = \frac{y^m}{N}(\mathfrak{d}_j U^{x^l}(t, \mu^N_{\boldsymbol{x},\boldsymbol{y}}) - \mathfrak{d}_{x^m} U^{x^l}(t, \mu^N_{\boldsymbol{x},\boldsymbol{y}})) + \varrho^{N,l,m}(t, \boldsymbol{x}, \boldsymbol{y})[j], \quad m \neq l,
\tag{4.27}
$$

and

$$\frac{1}{y^l}\Delta^l u^{N,l}(t,\boldsymbol{x},\boldsymbol{y})[j] = U^j(t,\mu^N_{\boldsymbol{x},\boldsymbol{y}}) - U^{x^l}(t,\mu^N_{\boldsymbol{x},\boldsymbol{y}}) + \frac{y^l}{N}(\mathfrak{d}_j U^{x^l}(t,\mu^N_{\boldsymbol{x},\boldsymbol{y}}) - \mathfrak{d}_{x^l}U^{x^l}(t,\mu^N_{\boldsymbol{x},\boldsymbol{y}})) + \varrho^{N,l,l}(t,\boldsymbol{x},\boldsymbol{y})[j], \tag{4.28}$$

where, for $t\in[0,T]$, $(\boldsymbol{x},\boldsymbol{y})\in[\![d]\!]^N\times\mathbb{Y}$ *and* $m,l\in[\![N]\!]$,

$$\sup_{j\in[\![d]\!]}|\varrho^{N,l,m}(t,\boldsymbol{x},\boldsymbol{y})[j]| \le C\frac{(y^m)^{1+\gamma'/2}}{N^{1+\gamma'/2}}, \tag{4.29}$$

for a constant C only depending on d and on the norm of U in the space $[\mathcal{C}^{1+\gamma'/2,2+\gamma'}_{\mathrm{WF}}([0,T]\times\mathcal{S}_{d-1})]^d$.

The following lemma permits us to handle the first order terms in the Nash system and in the master equation.

Lemma 4.4. *Under the assumption of Proposition 4.3, the analogue of the drift term in the Nash system (2.12), but for* $u^{N,l}$*, has the following expansion*

$$\begin{aligned}
&\frac{1}{y^l}\sum_{m\in[\![N]\!]}\varphi(\mu^N_{\boldsymbol{x},\boldsymbol{y}}[\bullet])\cdot\Delta^m u^{N,l}[\bullet] + \frac{1}{y^l}\sum_{m\ne l}a^*\Big(x^m,\frac{1}{y^m}u^{N,m}_\bullet\Big)\cdot\Delta^m u^{N,l}[\bullet]\\
&= \sum_{k,j\in[\![d]\!]}\mu^N_{\boldsymbol{x},\boldsymbol{y}}[k]\varphi(\mu^N_{\boldsymbol{x},\boldsymbol{y}}[j])(\mathfrak{d}_j U^{x^l}(t,\mu^N_{\boldsymbol{x},\boldsymbol{y}}) - \mathfrak{d}_k U^{x^l}(t,\mu^N_{\boldsymbol{x},\boldsymbol{y}}))\\
&\quad+\sum_{k,j\in[\![d]\!]}\mu^N_{\boldsymbol{x},\boldsymbol{y}}[k](U^k(t,\mu^N_{\boldsymbol{x},\boldsymbol{y}}) - U^j(t,\mu^N_{\boldsymbol{x},\boldsymbol{y}}))_+(\mathfrak{d}_j U^{x^l}(t,\mu^N_{\boldsymbol{x},\boldsymbol{y}}) - \mathfrak{d}_k U^{x^l}(t,\mu^N_{\boldsymbol{x},\boldsymbol{y}}))\\
&\quad+\sum_{j\in[\![d]\!]}\varphi(\mu^N_{\boldsymbol{x},\boldsymbol{y}}[j])(U^j(t,\mu^N_{\boldsymbol{x},\boldsymbol{y}}) - U^{x^l}(t,\mu^N_{\boldsymbol{x},\boldsymbol{y}})) + r_2^{N,l}(t,\boldsymbol{x},\boldsymbol{y}),
\end{aligned} \tag{4.30}$$

where the functions $u^{N,m}$ *and* $\Delta^m u^{N,l}$ *are evaluated at point* $(t,\boldsymbol{x},\boldsymbol{y})\in[0,T]\times\mathcal{T}_N$*, and where*

$$|r_2^{N,l}(t,\boldsymbol{x},\boldsymbol{y})| \le \frac{C}{N^\eta}, \tag{4.31}$$

for an exponent η *only depending on* γ *and* ϵ *and a constant C only depending on d and on the norm of U in the space* $[\mathcal{C}^{1+\gamma'/2,2+\gamma'}_{\mathrm{WF}}([0,T]\times\mathcal{S}_{d-1})]^d$.

It now remains to deal with the Hamiltonian part of the Nash system.

Lemma 4.5. *We fix* $t\in[0,T]$, $(\boldsymbol{x},\boldsymbol{y})\in\mathcal{T}_N$ *and* $l\in[\![N]\!]$*. Under the assumption of Proposition 4.3, we have*

$$H\Big(x^l,\frac{1}{y^l}u^{N,l}_\bullet\Big) = H(x^l,U(t,\mu^N_{\boldsymbol{x},\boldsymbol{y}})) + r_3^{N,l}(t,\boldsymbol{x},\boldsymbol{y}), \tag{4.32}$$

where

$$|r_3^{N,l}(t,\boldsymbol{x},\boldsymbol{y})| \le C\frac{y^l}{N},$$

for a constant C that only depends on d and on the norm of U in the space $[\mathcal{C}^{1+\gamma'/2,2+\gamma'}_{\mathrm{WF}}([0,T]\times\mathcal{S}_{d-1})]^d$.

4.2.3. Conclusion.

We now complete the proof of Proposition 4.3 by replacing the various terms on the left-hand side of (4.23) by the expansions obtained in Lemmas 4.2, 4.4, and 4.5. Using the fact that $y^l\le N^{1-\epsilon}$, we easily complete the proof thanks to the fact that U satisfies the master Equation (2.3).

5. Proofs of Theorems 2.1 and 2.2

We now prove the main results of the paper.

5.1. Proof of Theorem 2.1

We elaborate on the idea developed in Cardaliaguet et al. [8] on a continuous state space and then employed in Bayraktar and Cohen [1] and Cecchin and Pelino [13] for finite state spaces, paying attention to the fact that our setting here requires some care. The main noticeable difference with these references—as explained in Remark

2.1(1)—is that we cannot provide a direct estimate for[6]

$$\sup_{x,y} |(z^{N,n} - w^{N,n})(t, x, y)|, \quad t \in [0, T],\ n \in [\![N]\!],$$

the supremum being taken over $x \in [\![d]\!]^N$ and $y \in \mathbb{Y}$.

Here, instead, we must introduce a suitable weight and focus on the normalized quantity:

$$\theta_t^n := \sup_{x,y} \left[\prod_{i \in [\![d]\!]} (N^{-\epsilon} + \mu_{x,y}^N[i])^{1/d} \Phi^N(y)(z^{N,n} - w^{N,n})^2(t, x, y) \right], \quad t \in [0, T],\ n \in [\![N]\!], \tag{5.1}$$

with $\epsilon = 1/8$ and $\ell = 3$, where

$$\Phi^N(y) := \left(N^{-1} \sum_{m \in [\![N]\!]} |y^m|^\ell \right)^{-1}.$$

Observe in particular that the leading factor inside the supremum in the definition of (5.1) decays as the m-moment of y increases or as the empirical distribution of one of the states decreases. In other words, the accuracy of our estimate for $|(z_t^{N,n} - w_t^{N,n})(x, y)|$ becomes rather bad as $\mu_{x,y}^N$ gets closer to the boundary of the simplex or as the ℓ-moment of y tends to ∞.

A key fact in the proof is that $\Phi^N(y) \le 1$ for any $y \in \mathbb{Y}$ because $\ell \ge 1$ and $N^{-1}\sum_{m \in [\![N]\!]} y^m = 1$.

5.1.1. First Step. For ℓ as before and for a fixed index $n \in [\![N]\!]$, we let

$$\Psi(t, x, y) := \Phi^N(y)[z^{N,l}(t, x, y) - w^{N,l}(t, x, y)]^2.$$

We then remind the reader of the definition of $(\widetilde{X}^{\bullet,l}, \widetilde{Y}^{\bullet,l})$ in Section 3.2 (see (3.5) and (3.6)) with α in (3.5) being given by α^* as in the statement of Proposition 3.2. Also, we denote by $(e_t)_{0 \le t \le T}$ some real-valued (adapted) absolutely continuous nondecreasing process, whose precise form is specified later on in the proof. Following (4.8), Itô's lemma implies that, for any $t \in [0, T]$,

$$\begin{aligned} d[e_t \Psi(t, \widetilde{X}_t^{\bullet,n}, \widetilde{Y}_t^{\bullet,n})] &= [e_t \partial_t \Psi(t, \widetilde{X}_t^{\bullet,n}, \widetilde{Y}_t^{\bullet,n}) + \dot{e}_t \Psi(t, \widetilde{X}_t^{\bullet,n}, \widetilde{Y}_t^{\bullet,n})]dt \\ &\quad + \sum_{l \in [\![N]\!]} \int_{[0,M]^d} e_t(\Psi(t, (\widetilde{X}_{t-}^{l,n} + \widetilde{f}^l_{t-}(\theta), \widetilde{X}_{t-}^{-l,n}), \widetilde{Y}_{t-}^{\bullet,n}) - \Psi(t, \widetilde{X}_{t-}^{\bullet,n}, \widetilde{Y}_{t-}^{\bullet,n}))\mathcal{N}^l(d\theta, dt) \\ &\quad + \int_{[0,\varepsilon N]^{[\![N]\!]^d}} e_t\Big(\Psi\Big(t, \widetilde{X}_{t-}^{\bullet,n}, \widetilde{Y}_{t-}^{1,n} + \widetilde{g}^1_{t-}(\theta), \ldots, \widetilde{Y}_{t-}^{N,n} + \widetilde{g}^N_{t-}(\theta)\Big) - \Psi(t, \widetilde{X}_{t-}^{\bullet,n}, \widetilde{Y}_{t-}^{\bullet,n})\Big)\mathcal{N}^0(d\theta, dt), \end{aligned} \tag{5.2}$$

where we use the same representation as in (4.4) with $(\overline{X}, \overline{Y})$ therein being understood as $(\widetilde{X}^{\bullet,n}, \widetilde{Y}^{\bullet,n})$. Equivalently, $(\boldsymbol{\beta}^l)_{l \in [\![N]\!]}$ and $\boldsymbol{\beta}^0$, as originally defined in (4.5), now read

$$\begin{aligned} \beta_t^l(j) &:= \beta^l(t, \widetilde{X}_t^{\bullet,n}, \widetilde{Y}_t^{\bullet,n})[j]; \quad \beta^l(t, x, y)[j] := \varphi(\mu_{x,y}^N[j]) + a^*(x^l, w_\bullet^{N,l}(t, x, y)), \\ \beta_t^0(\boldsymbol{k}) &:= \beta^0(t, \widetilde{X}_t^{\bullet,n}, \widetilde{Y}_t^{\bullet,n})[\boldsymbol{k}]; \quad \beta^0(t, x, y)[\boldsymbol{k}] := \varepsilon N\left(\frac{k^{x^n}}{N\mu_{x,y}^N[x^n]}\right)\mathcal{M}_{N,\mu_{x,y}^N}(\boldsymbol{k}), \end{aligned}$$

ι being here equal to one. Following (4.6), we let

$$\begin{aligned} \widetilde{f}^l(t, x, y, \theta) &= \sum_{j \in [\![d]\!]} (j - x^l) \mathbb{1}_{(0, \beta^l(t,x,y)[j]]}(\theta^j), \\ \widetilde{g}^l(t, x, y, \theta) &= \sum_{\boldsymbol{k} \in [\![N]\!]^d} \left(y^l \frac{k^{x^l}}{N\mu_{x,y}^N[x^l]} - y^l \right) \mathbb{1}_{\{\mu_{x,y}^N[x^l] \ne 0\}} \mathbb{1}_{(0, \beta^0(t,x,y)[\boldsymbol{k}]]}(\theta^{\boldsymbol{k}}). \end{aligned}$$

Then, the processes $(\widetilde{f}^l_t(\theta))_{l\in[\![N]\!]}$ and $(\widetilde{g}^l_t(\theta))_{l\in[\![N]\!]}$ in (5.2) are defined as (compare if needed with (4.9))

$$\widetilde{f}_t(\theta) = f(t, \widetilde{\boldsymbol{X}}^{\bullet,n}_t, \widetilde{\boldsymbol{Y}}^{\bullet,n}_t, \theta), \quad \widetilde{g}_t(\theta) = g(t, \widetilde{\boldsymbol{X}}^{\bullet,n}_t, \widetilde{\boldsymbol{Y}}^{\bullet,n}_t, \theta).$$

Now, for any $s \in [0, T]$, denote

$$\begin{aligned}
&\Phi^n_s := \Phi^N(\widetilde{\boldsymbol{Y}}^{\bullet,n}_s),\\
&z^n_s := z^{N,n}(s, \widetilde{\boldsymbol{X}}^{\bullet,n}_s, \widetilde{\boldsymbol{Y}}^{\bullet,n}_s), \qquad \partial_t z^n_s := \partial_t z^{N,n}(s, \widetilde{\boldsymbol{X}}^{\bullet,n}_s, \widetilde{\boldsymbol{Y}}^{\bullet,n}_s),\\
&w^n_s := w^{N,n}(s, \widetilde{\boldsymbol{X}}^{\bullet,n}_s, \widetilde{\boldsymbol{Y}}^{\bullet,n}_s), \qquad \partial_t w^n_s := \partial_t w^{N,n}(s, \widetilde{\boldsymbol{X}}^{\bullet,n}_s, \widetilde{\boldsymbol{Y}}^{\bullet,n}_s).
\end{aligned}$$

Let $t_0 \in [0, T]$ be the initial time of the process $(\widetilde{\boldsymbol{X}}^{\bullet,n}, \widetilde{\boldsymbol{Y}}^{\bullet,n})$ for some initial condition $(\boldsymbol{x}, \boldsymbol{y}) \in [\![d]\!]^N \times \mathbb{Y}$. Letting $(\widetilde{\mu}^{\bullet,n}_t[i] := \mu^N_{\widetilde{\boldsymbol{X}}^{\bullet,n}_t, \widetilde{\boldsymbol{Y}}^{\bullet,n}_t}[i])_{t_0\le t\le T}$, following (4.10) with $\iota = 1$ therein, we introduce the stopping time

$$\widetilde{\tau}_N := \inf\left\{t \ge t_0 : \min_{i\in[\![d]\!]} \widetilde{\mu}_t[i] < N^{-\epsilon} \text{ or } \max_{l\in[\![N]\!]} \widetilde{Y}^{l,n}_t > \frac{1}{2}N^{1-\epsilon}\right\} \wedge T. \tag{5.3}$$

Letting $t \in [t_0, T]$, by integrating both sides of (5.2) on the interval $[t\wedge\widetilde{\tau}_N, \widetilde{\tau}_N]$ and recalling that $z^n_T = w^n_T$, we get

$$\begin{aligned}
&e_{\widetilde{\tau}_N}\Phi^n_{\widetilde{\tau}_N}\left(z^n_{\widetilde{\tau}_N} - w^n_{\widetilde{\tau}_N}\right)^2 = e_t\Phi^n_t\left(z^n_t - w^n_t\right)^2 + \int_{t\wedge\widetilde{\tau}_N}^{\widetilde{\tau}_N} \Phi^n_s\Big[2e_s(z^n_s - w^n_s)(\partial_t z^n_s - \partial_t w^n_s) + \dot{e}_s(z^n_s - w^n_s)^2\Big]ds\\
&+ \sum_{l\in[\![N]\!]}\int_{t\wedge\widetilde{\tau}_N}^{\widetilde{\tau}_N}\int_{[0,M]^d} e_s\Phi^n_{s-}\Big[(z^{N,n} - v^{N,n})^2\Big(s, (\widetilde{X}^{l,n}_{s-} + \widetilde{f}^l_{s-}(\theta), \widetilde{\boldsymbol{X}}^{-l,n}_{s-}),\ \widetilde{\boldsymbol{Y}}^{\bullet,n}_{s-}\Big) - (z^n_{s-} - w^n_{s-})^2\Big]\mathcal{N}^l(d\theta, ds)\\
&+ \int_{t\wedge\widetilde{\tau}_N}^{\widetilde{\tau}_N}\int_{[0,\epsilon N]^{[\![N]\!]^d}} e_s\Big[-\Phi^n_{s-}(z^n_{s-} - w^n_{s-})^2 + \Phi^N(\widetilde{Y}^{1,n}_{s-} + \widetilde{g}^1_{s-}(\theta), \dots)(z^{N,n} - v^{N,n})^2(s, \widetilde{\boldsymbol{X}}^{\bullet,n}_{s-}, \widetilde{Y}^{1,n}_{s-} + \widetilde{g}^1_{s-}(\theta), \dots)\Big]\mathcal{N}^0(d\theta, ds).
\end{aligned}$$

We now take expectations (recalling that $(\widetilde{\boldsymbol{X}}^{\bullet,n}_{t_0}, \widetilde{\boldsymbol{Y}}^{\bullet,n}_{t_0}) = (\boldsymbol{x}, \boldsymbol{y})$). We get

$$\begin{aligned}
&\mathbb{E}[e_{\widetilde{\tau}_N}\Phi^n_{\widetilde{\tau}_N}(z^n_{\widetilde{\tau}_N} - w^n_{\widetilde{\tau}_N})^2] = \mathbb{E}[e_t\Phi^n_t(z^n_t - w^n_t)^2] + \mathbb{E}\int_{t\wedge\widetilde{\tau}_N}^{\widetilde{\tau}_N} \Phi^n_s\Big[2e_s(z^n_s - w^n_s)(\partial_t z^n_s - \partial_t w^n_s) + \dot{e}_s(z^n_s - w^n_s)^2\Big]ds\\
&+ \sum_{l\in[\![N]\!]}\mathbb{E}\int_{t\wedge\widetilde{\tau}_N}^{\widetilde{\tau}_N} e_s\Phi^n_s[(\widetilde{\varphi}^{\bullet,n}_s[\bullet] + a^*(\widetilde{X}^{l,n}_s, w^l_s)[\bullet]) \cdot \Delta^l\{(z^n_s - w^n_s)^2\}[\bullet]]ds\\
&+ \epsilon N\mathbb{E}\int_{t\wedge\widetilde{\tau}_N}^{\widetilde{\tau}_N} e_s\mathbf{E}\left\{\frac{S_{\widetilde{\mu}^{\bullet,n}_s}[\widetilde{X}^{n,n}_s]}{N\widetilde{\mu}^{\bullet,n}_s[\widetilde{X}^{n,n}_s]}\Big[-\Phi^n_s(z^n_s - w^n_s)^2 + \Phi^N\Big(h\Big(S_{\widetilde{\mu}^{\bullet,n}_s}, \widetilde{\boldsymbol{X}}^{\bullet,n}_s, \widetilde{\boldsymbol{Y}}^{\bullet,n}_s\Big)\Big)(z^{N,n} - w^{N,n})^2\Big(s, \widetilde{\boldsymbol{X}}^{\bullet,n}_s, h\Big(S_{\widetilde{\mu}^{\bullet,n}_s}, \widetilde{\boldsymbol{X}}^{\bullet,n}, \widetilde{\boldsymbol{Y}}^{\bullet,n}\Big)\Big)\Big]\right\}ds,
\end{aligned} \tag{5.4}$$

where we use the notation $\widetilde{\varphi}^{\bullet,n}_s = (\widetilde{\varphi}^{\bullet,n}_s[j] : j \in [\![d]\!])$, $\widetilde{\mu}^{\bullet,n}_s = (\widetilde{\mu}^{\bullet,n}_s[j] : j \in [\![d]\!])$ with

$$\widetilde{\mu}^{\bullet,n}_s[j] = \mu^N_{\widetilde{X}^{\bullet,n}_s, \widetilde{Y}^{\bullet,n}_s}[j], \qquad \text{and} \qquad \widetilde{\varphi}^{\bullet,n}_s[j] := \varphi(\widetilde{\mu}^{\bullet,n}_s[j]).$$

And

$$h(k, \boldsymbol{x}, \boldsymbol{y}) := \left(\boldsymbol{y}^1\frac{k[x^1]}{N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^1]}, \dots, \boldsymbol{y}^N\frac{k[x^N]}{N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^N]}\right).$$

Recalling the notation from (4.23), set $r^n_s := r^{N,n}(s, \widetilde{\boldsymbol{X}}^n_s, \widetilde{\boldsymbol{Y}}^n_s)$ and

$$\begin{aligned}
\alpha^{w,l,n}_s[\bullet] &:= \widetilde{\varphi}^{\bullet,n}_s[\bullet] + a^*(\widetilde{X}^{l,n}_s, w^l_s)[\bullet], \qquad H^{w,n}_s := H(\widetilde{X}^{n,n}_s, w^{N,n}_\bullet(s, \widetilde{\boldsymbol{X}}^{\bullet,n}_s, \widetilde{\boldsymbol{Y}}^{\bullet,n}_s)),\\
\alpha^{z,l,n}_s[\bullet] &:= \widetilde{\varphi}^{\bullet,n}_s[\bullet] + a^*(\widetilde{X}^{l,n}_s, z^l_s)[\bullet], \qquad H^{z,n}_s := H(\widetilde{X}^{n,n}_s, z^{N,n}_\bullet(s, \widetilde{\boldsymbol{X}}^{\bullet,n}_s, \widetilde{\boldsymbol{Y}}^{\bullet,n}_s)),
\end{aligned}$$

and using Equations (2.12) and (4.23), we obtain

$$
\begin{aligned}
&\mathbb{E}[e_{\tilde\tau_N}\Phi^n_{\tilde\tau_N}(z^n_{\tilde\tau_N}-w^n_{\tilde\tau_N})^2]=\mathbb{E}[e_t\Phi^n_t(z^n_t-w^n_t)^2]+\mathbb{E}\int_{t\wedge\tilde\tau_N}^{\tilde\tau_N}\dot e_s\Phi^n_s(z^n_s-w^n_s)^2ds\\
&\quad+2\mathbb{E}\Bigg[\int_{t\wedge\tilde\tau_N}^{\tilde\tau_N}e_s\Phi^n_s(z^n_s-w^n_s)\Bigg\{\sum_{l\neq n}[\alpha^{w,l,n}_s[\bullet]\cdot\Delta^l w^n_s[\bullet]-\alpha^{z,l,n}_s[\bullet]\cdot\Delta^l z^n_s[\bullet]]\\
&\quad+\widetilde\varphi^{\bullet,n}_s[\bullet]\cdot\Delta^n(w^n_s-z^n_s)[\bullet]+H^{w,n}_s-H^{z,n}_s-r^n_s\Bigg\}ds\Bigg]+\sum_{l\in[\![N]\!]}\mathbb{E}\Bigg[\int_{t\wedge\tilde\tau_N}^{\tilde\tau_N}e_s\Phi^n_s\{\alpha^{w,l,n}_s[\bullet]\cdot\Delta^l(z^n_s-w^n_s)^2[\bullet]\}ds\Bigg]\\
&\quad+2\varepsilon N\mathbb{E}\Bigg[\int_{t\wedge\tilde\tau_N}^{\tilde\tau_N}e_s\Phi^n_s\mathbf{E}\Bigg\{\frac{S_{\tilde\mu^{\bullet,n}_s}[\widetilde X^{n,n}_s]}{N\tilde\mu^{\bullet,n}_s[\widetilde X^{n,n}_s]}(z^n_s-w^n_s)\times((w^{N,n}-z^{N,n})(s,\widetilde{\boldsymbol X}^{\bullet,n}_s,h(S_{\tilde\mu^{\bullet,n}_s},\widetilde{\boldsymbol X}^{\bullet,n}_s,\widetilde{\boldsymbol Y}^{\bullet,n}_s))-(w^n_s-z^n_s))\Bigg\}ds\Bigg]\\
&\quad+\varepsilon N\mathbb{E}\Bigg[\int_{t\wedge\tilde\tau_N}^{\tilde\tau_N}e_s\mathbf{E}\Bigg\{\frac{S_{\tilde\mu^{\bullet,n}_s}[\widetilde X^{n,n}_s]}{N\tilde\mu^{\bullet,n}_s[\widetilde X^{n,n}_s]}\Big[-\Phi^n_s(z^n_s-w^n_s)^2+\Phi^N(h(S_{\tilde\mu^{\bullet,n}_s},\widetilde{\boldsymbol X}^{\bullet,n}_s,\widetilde{\boldsymbol Y}^{\bullet,n}_s))(z^{N,n}-w^{N,n})^2(s,\widetilde{\boldsymbol X}^{\bullet,n}_s,h(S_{\tilde\mu^{\bullet,n}_s},\widetilde{\boldsymbol X}^{\bullet,n},\widetilde{\boldsymbol Y}^{\bullet,n}))\Big]\Bigg\}ds\Bigg]\\
&=:T_1+T_2+T_3+T_4+T_5+T_6.
\end{aligned}
\tag{5.5}
$$

5.1.2. Second Step. We first treat the sum of the last two terms T_5 and T_6:

$$
\begin{aligned}
T_5+T_6&=\varepsilon N\mathbb{E}\Bigg[\int_{t\wedge\tilde\tau_N}^{\tilde\tau_N}e_s\Phi^n_s\mathbf{E}\Bigg\{\frac{S_{\tilde\mu^{\bullet,n}_s}[\widetilde X^{n,n}_s]}{N\tilde\mu^{\bullet,n}_s[\widetilde X^{n,n}_s]}\Big[(z^{N,n}-w^{N,n})^2(s,\widetilde{\boldsymbol X}^{\bullet,n}_s,h(S_{\tilde\mu^{\bullet,n}_s},\widetilde{\boldsymbol X}^{\bullet,n}_s,\widetilde{\boldsymbol Y}^{\bullet,n}_s))\\
&\qquad-2(z^n_s-w^n_s)(z^{N,n}-w^{N,n})(s,\widetilde{\boldsymbol X}^{\bullet,n}_s,h(S_{\tilde\mu^{\bullet,n}_s},\widetilde{\boldsymbol X}^{\bullet,n}_s,\widetilde{\boldsymbol Y}^{\bullet,n}_s))+(z^n_s-w^n_s)^2\Big]\Bigg\}ds\Bigg]\\
&\qquad+\varepsilon N\mathbb{E}\Bigg[\int_{t\wedge\tilde\tau_N}^{\tilde\tau_N}e_s\mathbf{E}\Bigg\{\frac{S_{\tilde\mu^{\bullet,n}_s}[\widetilde X^{n,n}_s]}{N\tilde\mu^{\bullet,n}_s[\widetilde X^{n,n}_s]}\Big[\Phi^N(h(S_{\tilde\mu^{\bullet,n}_s},\widetilde{\boldsymbol X}^{\bullet,n}_s,\widetilde{\boldsymbol Y}^{\bullet,n}_s))-\Phi^n_s\Big]\times(z^{N,n}-w^{N,n})^2(s,\widetilde{\boldsymbol X}^{\bullet,n}_s,h(S_{\tilde\mu^{\bullet,n}_s},\widetilde{\boldsymbol X}^{\bullet,n}_s,\widetilde{\boldsymbol Y}^{\bullet,n}_s))\Bigg\}ds\Bigg]\\
&\quad:=T^{5,6}_0+T^{5,6}_1.
\end{aligned}
\tag{5.6}
$$

We start with the analysis of $T^{5,6}_0$.

We get

$$
T^{5,6}_0=\mathbb{E}\Bigg[\int_{t\wedge\tilde\tau_N}^{\tilde\tau_N}e_sJ^n_sds\Bigg]\quad\text{with}\quad J^n_s:=\varepsilon N\Phi^n_s\mathbf{E}\Bigg[\frac{S_{\tilde\mu^{\bullet,n}_s}[\widetilde X^{n,n}_s]}{N\tilde\mu^{\bullet,n}_s[\widetilde X^{n,n}_s]}\times((z^{N,n}-w^{N,n})(s,\widetilde{\boldsymbol X}^{\bullet,n}_s,h(S_{\tilde\mu^{\bullet,n}_s},\widetilde{\boldsymbol X}^{\bullet,n}_s,\widetilde{\boldsymbol Y}^{\bullet,n}_s))-(z^n_s-w^n_s))^2\Bigg].
\tag{5.7}
$$

Back to (5.6), we now address the term $T^{5,6}_1$. We write

$$
T^{5,6}_1=\mathbb{E}\int_{t\wedge\tilde\tau_N}^{\tilde\tau_N}e_sT^{5,6}_2(s)ds,\ \text{with}\quad T^{5,6}_2(s):=\varepsilon N\mathbf{E}\Bigg[\frac{S_{\tilde\mu^{\bullet,n}_s}[\widetilde X^{n,n}_s]}{N\tilde\mu^{\bullet,n}_s[\widetilde X^{n,n}_s]}\Big[\Phi^N(h(S_{\tilde\mu^{\bullet,n}_s},\widetilde{\boldsymbol X}^{\bullet,n}_s,\widetilde{\boldsymbol Y}^{\bullet,n}_s))-\Phi^n_s\Big]\times(z^{N,n}-w^{N,n})^2(s,\widetilde{\boldsymbol X}^{\bullet,n}_s,h(S_{\tilde\mu^{\bullet,n}_s},\widetilde{\boldsymbol X}^{\bullet,n}_s,\widetilde{\boldsymbol Y}^{\bullet,n}_s))\Bigg].
$$

We now recall that $\Phi(\boldsymbol y)=\Big(N^{-1}\sum_{l\in[\![N]\!]}|y^l|^\ell\Big)^{-1}$. By the convexity of the mapping $x\longmapsto x^{-1}$, it is well-checked that

$$
\Phi(h(S_{\tilde\mu^{\bullet,n}_s},\widetilde{\boldsymbol X}^{\bullet,n},\widetilde{\boldsymbol Y}^{\bullet,n}))-\Phi(\widetilde{\boldsymbol Y}^{\bullet,n}_s)\geq-\Phi^2(\widetilde{\boldsymbol Y}^{\bullet,n}_s)D\,\widetilde{\boldsymbol Y}^{\bullet,n}_s,
$$

where

$$
D\,\widetilde{\boldsymbol Y}^{\bullet,n}_s:=\frac{1}{N}\sum_{l\in[\![N]\!]}|\widetilde{\boldsymbol Y}^{l,n}_s|^\ell\Bigg[\Bigg(\frac{S_{\tilde\mu^{\bullet,n}_s}[\widetilde X^{l,n}_s]}{N\tilde\mu^{\bullet,n}_s[\widetilde X^{l,n}_s]}\Bigg)^\ell-1\Bigg].
\tag{5.8}
$$

From this, we deduce that

$$
\begin{aligned}
T^{5,6}_2(s)&\geq-\varepsilon N\mathbf{E}\Bigg[\frac{S_{\tilde\mu^{\bullet,n}_s}[\widetilde X^{n,n}_s]}{N\tilde\mu^{\bullet,n}_s[\widetilde X^{n,n}_s]}(z^{N,n}-w^{N,n})^2(s,\widetilde{\boldsymbol X}^{\bullet,n}_s,h(S_{\tilde\mu^{\bullet,n}_s},\widetilde{\boldsymbol X}^{\bullet,n}_s,\widetilde{\boldsymbol Y}^{\bullet,n}_s))(\Phi^n_s)^2D\,\widetilde{\boldsymbol Y}^{\bullet,n}_s\Bigg]\\
&=-T^{5,6}_{2,1}(s)-T^{5,6}_{2,2}(s)+T^{5,6}_{2,3}(s),
\end{aligned}
$$

with

$$T^{5,6}_{2,1}(s) := \varepsilon N \mathbf{E}\left[\frac{S_{\widetilde{\mu}^{\bullet,n}_s}[\widetilde{X}^{n,n}_s]}{N\widetilde{\mu}^{\bullet,n}_s[\widetilde{X}^{n,n}_s]}((z^{N,n}-w^{N,n})(s,\widetilde{\boldsymbol{X}}^{\bullet,n}_s, h(S_{\widetilde{\mu}^{\bullet,n}_s},\widetilde{\boldsymbol{X}}^{\bullet,n}_s,\widetilde{\boldsymbol{Y}}^{\bullet,n}_s)) - (z^n_s - w^n_s))^2(\Phi^n_s)^2 D\,\widetilde{\boldsymbol{Y}}^{\bullet,n}_s\right],$$

$$T^{5,6}_{2,2}(s) := 2\varepsilon N \mathbf{E}\left[\frac{S_{\widetilde{\mu}^{\bullet,n}_s}[\widetilde{X}^{n,n}_s]}{N\widetilde{\mu}^{\bullet,n}_s[\widetilde{X}^{n,n}_s]}((z^{N,n}-w^{N,n})(s,\widetilde{\boldsymbol{X}}^{\bullet,n}_s, h(S_{\widetilde{\mu}^{\bullet,n}_s},\widetilde{\boldsymbol{X}}^{\bullet,n}_s,\widetilde{\boldsymbol{Y}}^{\bullet,n}_s)) - (z^n_s - w^n_s))(z^n_s-w^n_s)(\Phi^n_s)^2 D\,\widetilde{\boldsymbol{Y}}^{\bullet,n}_s\right]$$

$$T^{5,6}_{2,3}(s) := 3\varepsilon N \mathbf{E}\left[\frac{S_{\widetilde{\mu}^{\bullet,n}_s}[\widetilde{X}^{n,n}_s]}{N\widetilde{\mu}^{\bullet,n}_s[\widetilde{X}^{n,n}_s]}(z^n_s - w^n_s)^2(\Phi^n_s)^2 D\,\widetilde{\boldsymbol{Y}}^{\bullet,n}_s\right].$$

We start with the analysis of $T^{5,6}_{2,3}(s)$.

Denote $\widetilde{\mu}^{\bullet,\min}_s := \min_{e\in[\![d]\!]}\widetilde{\mu}^{\bullet,n}_s[e]$. By the definition of $D\,\widetilde{\boldsymbol{Y}}^{\bullet,n}$, the uniform bounds of z^n_s and w^n_s, and the second part of (4.18), we have

$$\begin{aligned}
|T^{5,6}_{2,3}(s)| &\le 3\varepsilon(z^n_s-w^n_s)^2(\Phi^n_s)^2\sum_{l\in[\![N]\!]}|\widetilde{\boldsymbol{Y}}^{l,n}_s|^\ell\left|\mathbf{E}\left[\frac{S_{\widetilde{\mu}^{\bullet,n}_s}[\widetilde{X}^{n,n}_s]}{N\widetilde{\mu}^{\bullet,n}_s[\widetilde{X}^{n,n}_s]}\left(\left(\frac{S_{\widetilde{\mu}^{\bullet,n}_s}[\widetilde{X}^{l,n}_s]}{N\widetilde{\mu}^{\bullet,n}_s[\widetilde{X}^{l,n}_s]}\right)^\ell - 1\right)\right]\right|\\
&\le cN(z^n_s-w^n_s)^2\Phi^n_s\left(\frac{1}{N\widetilde{\mu}^{\bullet,\min}_s}+\frac{1}{N^{\frac{3}{2}-2\epsilon}}\right)\\
&\le \frac{c}{\widetilde{\mu}^{\bullet,\min}_s}(z^n_s-w^n_s)^2\Phi^n_s + \frac{1}{N^{\frac{1}{2}-2\epsilon}},
\end{aligned} \tag{5.9}$$

for a c whose value is allowed to vary from line to line as long as it only depends on T, $\|f\|_\infty$, and $\|g\|_\infty$.

We now turn to the analysis of $T^{5,6}_{2,1}(s)$. Generally speaking, the strategy is to prove that it is smaller (up to a small remainder) than a small fraction of J^n_s in (5.7) (because the latter is positive, it, hence, permit us to absorb $-T^{5,6}_{2,1}$ in the expansion of $T^{5,6}_2$). The proof is as follows: On the event

$$F := \left\{\left|\left(\frac{S_{\widetilde{\mu}^{\bullet,n}_s}[\widetilde{X}^{l,n}_s]}{N\widetilde{\mu}^{\bullet,n}_s[\widetilde{X}^{l,n}_s]}\right)^\ell - 1\right| \le \frac{1}{2} : l\in\{1,2,\cdots,N\}\right\},$$

we have $|D\widetilde{\boldsymbol{Y}}^{\bullet,n}_s| \le (\Phi^n_s)^{-1}/2$ so that (recalling the definition of J^n_s in (5.7))

$$T^{5,6}_{2,1}(s) \le \frac{1}{2}J^n_s + c\mathbf{E}\left[\mathbb{1}_{F^\complement}\frac{S_{\widetilde{\mu}^{\bullet,n}_s}[\widetilde{X}^{n,n}_s]}{N\widetilde{\mu}^{\bullet,n}_s[\widetilde{X}^{n,n}_s]}(\Phi^n_s)^2|D\widetilde{\boldsymbol{Y}}^{\bullet,n}_s|\right].$$

Using (4.19) and recalling from (4.20) that $\mathbb{P}(F^\complement) \le CN\exp(-cN^{1-2\epsilon})$, we deduce that

$$T^{5,6}_{2,1}(s) \le \frac{1}{2}J^n_s + \frac{c}{N^{1/2-2\epsilon}}. \tag{5.10}$$

It then remains to tackle $T^{5,6}_{2,2}(s)$. We use Young's inequality, Jensen's inequality, the Cauchy–Schwartz inequality, and finally (4.19):

$$\begin{aligned}
T^{5,6}_{2,2}(s) &\le \frac{1}{4}J^n(s) + cN\mathbf{E}\left[\frac{S_{\widetilde{\mu}^{\bullet,n}_s}[\widetilde{X}^{n,n}_s]}{N\widetilde{\mu}^{\bullet,n}_s[\widetilde{X}^{n,n}_s]}(z^n_s-w^n_s)^2(\Phi^n_s)^3(D\widetilde{\boldsymbol{Y}}^{\bullet,n}_s)^2\right]\\
&= \frac{1}{4}J^n(s) + cN(z^n_s-w^n_s)^2\Phi^n_s \times \mathbf{E}\left[\frac{S_{\widetilde{\mu}^{\bullet,n}_s}[\widetilde{X}^{n,n}_s]}{N\widetilde{\mu}^{\bullet,n}_s[\widetilde{X}^{n,n}_s]}\left\{\sum_{l\in[\![N]\!]}\frac{\frac{1}{N}|\widetilde{\boldsymbol{Y}}^{l,n}_s|^\ell}{(\Phi^n_s)^{-1}}\sum_{i\in[\![d]\!]}\mathbb{1}_{\{\widetilde{X}^{l,n}_s=i\}}\left(\left(\frac{S_{\widetilde{\mu}^{\bullet,n}_s}[i]}{N\widetilde{\mu}^{\bullet,n}_s[i]}\right)^\ell - 1\right)\right\}^2\right]\\
&\le \frac{1}{4}J^n(s) + cN(z^n_s-w^n_s)^2\Phi^n_s \times \sum_{l\in[\![N]\!]}\frac{\frac{1}{N}|\widetilde{\boldsymbol{Y}}^{l,n}_s|^\ell}{(\Phi^n_s)^{-1}}\sum_{i\in[\![d]\!]}\mathbf{E}\left[\frac{S_{\widetilde{\mu}^{\bullet,n}_s}[\widetilde{X}^{n,n}_s]}{N\widetilde{\mu}^{\bullet,n}_s[\widetilde{X}^{n,n}_s]}\left(\left(\frac{S_{\widetilde{\mu}^{\bullet,n}_s}[i]}{N\widetilde{\mu}^{\bullet,n}_s[i]}\right)^\ell - 1\right)^2\right] \le \frac{1}{4}J^n(s) + cN(z^n_s-w^n_s)^2\Phi^n_s\\
&\quad\times\sum_{i\in[\![d]\!]}\mathbf{E}\left[\left(\frac{S_{\widetilde{\mu}^{\bullet,n}_s}[\widetilde{X}^{n,n}_s]}{N\widetilde{\mu}^{\bullet,n}_s[\widetilde{X}^{n,n}_s]}\right)^2\right]^{1/2}\mathbf{E}\left[\left(\left(\frac{S_{\widetilde{\mu}^{\bullet,n}_s}[i]}{N\widetilde{\mu}^{\bullet,n}_s[i]}\right)^\ell - 1\right)^4\right]^{1/2} \le \frac{1}{4}J^n(s) + \frac{c}{\widetilde{\mu}^{\bullet,\min}_s}\Phi^n_s(z^n_s-w^n_s)^2,
\end{aligned} \tag{5.11}$$

where we indeed apply (4.19) with $p = 4$ in the penultimate line (noticing that the leading exponent of the last term therein is $1/2$). Moreover, the last inequality follows because $N\widetilde{\mu}^{\bullet,\min} \ge 1$ on the event $\{t \le \widetilde{\tau}_N\}$. As a consequence, invoking once again (4.19), we have a universal bound for

$$\mathbf{E}\left[\left(\frac{S_{\widetilde{\mu}_s^{\bullet,n}}[\widetilde{X}_s^{n,n}]}{N\tilde{\mu}_s^{\bullet,n}[\widetilde{X}_s^{n,n}]}\right)^2\right]^{1/2}.$$

Collecting all the terms (5.9)–(5.11), we finally have

$$T_1^{5,6} \geq -\frac{3}{4}\mathbb{E}\int_{t\wedge\tilde{\tau}_N}^{\tilde{\tau}_N} e_s J_s^n ds - c\mathbb{E}\left[\int_{t\wedge\tilde{\tau}_N}^{\tilde{\tau}_N} \frac{e_s\Phi_s^n}{\widetilde{\mu}_s^{\bullet,\min}}(z_s^n - w_s^n)^2 ds\right] - \frac{c}{N^{1/2-2\epsilon}}\mathbb{E}\left[\int_{t\wedge\tilde{\tau}_N}^{\tilde{\tau}N} e_s ds\right].$$

Combining with (5.7), we get

$$T_5 + T_6 \geq -c\mathbb{E}\left[\int_{t\wedge\tilde{\tau}_N}^{\tilde{\tau}_N} \frac{e_s\Phi_s^n}{\widetilde{\mu}_s^{\bullet,\min}}(z_s^n - w_s^n)^2 ds\right] - \frac{c}{N^{1/2-2\epsilon}}\mathbb{E}\left[\int_{t\wedge\tilde{\tau}_N}^{\tilde{\tau}_N} e_s ds\right].$$

5.1.3. Third Step. We now simplify the sum of T_3 and T_4 in (5.5). Simple algebraic manipulations imply that, for any $l \neq n$,

$$\begin{aligned}
&2(z_s^n - w_s^n)\{\alpha_s^{w,l,n}[\bullet]\cdot\Delta^l w_s^n[\bullet] - \alpha_s^{z,l,n}[\bullet]\cdot\Delta^l z_s^n[\bullet]\} + \alpha_s^{w,l,n}[\bullet]\cdot\Delta^l(z_s^n - w_s^n)^2[\bullet]\\
&\quad= 2(z_s^n - w_s^n)\{\alpha_s^{w,l,n}[\bullet]\cdot\Delta^l(w_s^n - z_s^n)[\bullet] + (\alpha_s^{w,l,n} - \alpha_s^{z,l,n})[\bullet]\cdot\Delta^l z_s^n[\bullet]\}\\
&\qquad+ \alpha_s^{w,l,n}[\bullet]\cdot\Delta^l(z_s^n - w_s^n)^2[\bullet]\\
&\quad= 2(z_s^n - w_s^n)(\alpha_s^{w,l,n}[\bullet] - \alpha_s^{z,l,n}[\bullet])\cdot\Delta^l z_s^n[\bullet]\\
&\qquad+ \alpha_s^{w,l,n}[\bullet]\cdot[\Delta^l(z_s^n - w_s^n)[\bullet]\odot\Delta^l(z_s^n - w_s^n)[\bullet]],
\end{aligned}$$

where $a\odot b = (a_i b_i : i\in[[d]])$ is the element-by-element product between vectors. Similarly,

$$\begin{aligned}
\alpha_s^{w,n,n}[\bullet]\cdot\Delta^n(z_s^n - w_s^n)^2[\bullet] &= \alpha_s^{w,n,n}\cdot[\Delta^n(z_s^n - w_s^n)[\bullet]\odot\Delta^n(z_s^n - w_s^n)[\bullet]]\\
&\quad+ 2(z_s^n - w_s^n)\alpha_s^{w,n,n}[\bullet]\cdot[\Delta^n(z_s^n - w_s^n)[\bullet]]\\
&\geq 2(z_s^n - w_s^n)\alpha_s^{w,n,n}[\bullet]\cdot[\Delta^n(z_s^n - w_s^n)[\bullet]],
\end{aligned}$$

where, in the last line, we use the fact that the off-diagonal components of $\alpha_s^{w,n}$ are nonnegative.

Back to (5.5), we obtain

$$\begin{aligned}
&\mathbb{E}[e_{\tilde{\tau}_N}\Phi_{\tilde{\tau}_N}^n(z_{\tilde{\tau}_N}^n - w_{\tilde{\tau}_N}^n)^2] \geq \mathbb{E}[e_t\Phi_t^n(z_t^n - w_t^n)^2] + \mathbb{E}\int_{t\wedge\tilde{\tau}_N}^{\tilde{\tau}_N}\dot{e}_s\Phi_s^n(z_s^n - w_s^n)^2 ds\\
&\quad+ 2\mathbb{E}\left[\int_{t\wedge\tilde{\tau}_N}^{\tilde{\tau}_N} e_s\Phi_s^n(z_s^n - w_s^n)\left\{\sum_{l\neq n}[(\alpha_s^{w,l,n}[\bullet] - \alpha_s^{z,l,n}[\bullet])\cdot\Delta^l z_s^n[\bullet]]\right.\right.\\
&\quad\left.\left.+ (\alpha_s^{w,n,n}[\bullet] - \widetilde{\varphi}_s^{\bullet,n}[\bullet])\cdot\Delta^n(z_s^n - w_s^n)[\bullet] + H_s^{w,n} - H_s^{z,n} - r_s^n\right\}ds\right]\\
&\quad+ \sum_{l\neq n}\mathbb{E}\left[\int_{t\wedge\tilde{\tau}_N}^{\tilde{\tau}_N} e_s\Phi_s^n\alpha_s^{w,l,n}[\bullet]\cdot[\Delta^l(z_s^n - w_s^n)[\bullet]\odot\Delta^l(z_s^n - w_s^n)[\bullet]]ds\right]\\
&\quad- c\mathbb{E}\left[\int_{t\wedge\tilde{\tau}_N}^{\tilde{\tau}_N}\frac{e_s\Phi_s^n}{\tilde{-}->\mu_s^{\bullet,\min}}(z_s^n - w_s^n)^2 ds\right] - c\frac{1}{N^{1/2-2\epsilon}}\mathbb{E}\left[\int_{t\wedge\tilde{\tau}_N}^{\tilde{\tau}_N} e_s ds\right].
\end{aligned}\tag{5.12}$$

5.1.4. Fourth Step. We now handle the third expectation on the right-hand side of (5.12). We start with the term on the second line of (5.12). From (4.27),

$$\begin{aligned}
\Delta^l z_s^n[j] &= \Delta^l z^{N,n}(s, \widetilde{\boldsymbol{X}}_s^{\bullet,n}, \widetilde{\boldsymbol{Y}}_s^{\bullet,n})[j]\\
&= \frac{\widetilde{Y}_s^{l,n}}{N}\left[\mathfrak{d}_j U\Big(t, X_s^{n,n}, \mu^N_{\widetilde{\boldsymbol{X}}_s^{\bullet,n},\widetilde{\boldsymbol{Y}}_s^{\bullet,n}}\Big) - \mathfrak{d}_{\widetilde{X}_s^{l,n}} U\Big(s, X_s^{n,n}, \mu^N_{\widetilde{\boldsymbol{X}}_s^{\bullet,n},\widetilde{\boldsymbol{Y}}_s^{\bullet,n}}\Big)\right]\\
&\quad+ \varrho^{N,n,l}(s, \widetilde{\boldsymbol{X}}_s^{\bullet,n}, \widetilde{\boldsymbol{Y}}_s^{\bullet,n})[j],
\end{aligned}$$

with $|\varrho^{N,n,l}(s,\widetilde{\boldsymbol{X}}_s^{\bullet,n},\widetilde{\boldsymbol{Y}}_s^{\bullet,n})| \le C(\widetilde{Y}_s^{l,n})^{1+\gamma}/N^{1+\gamma} \le C\,\widetilde{Y}_s^{l,n}/N$, the last inequality following from the fact that $\widetilde{Y}_s^{l,n} \le N$ and the constant C depending only on the various parameters in the assumption (including κ).

Moreover, by the regularity of U, its gradient is uniformly bounded. Together with the Lipschitz property of α^*, we obtain that, for $l \ne n$,

$$\left|(\alpha_s^{w,l,n}[\bullet] - \alpha_s^{z,l,n}[\bullet]) \cdot \Delta^l z_s^n[\bullet]\right| \le C\frac{\widetilde{Y}_s^{l,n}}{N}\left|(-\Delta^l w_s^n[\bullet])_+ - (-\Delta^l z_s^n[\bullet])_+\right|,$$

and then, recalling that $\sum_{l\ne n}(\widetilde{Y}_s^{l,n}/N) \le 1$, we get (by Jensen's inequality)

$$\begin{aligned}
&\left(\sum_{l\ne n}|(\alpha_s^{w,l,n}[\bullet] - \alpha_s^{z,l,n}[\bullet]) \cdot \Delta^l z_s^n[\bullet]|\right)^2 \\
&\le C\sum_{l\ne n}\frac{\widetilde{Y}_s^{l,n}}{N}|(-\Delta^l w_s^n[\bullet])_+ - (-\Delta^l z_s^n[\bullet])_+|^2 \\
&= C\sum_{l\ne n}\sum_{i\in[\![d]\!]}\left[\frac{\widetilde{Y}_s^{l,n}}{N}\left|(-\Delta^l w_s^n[i])_+ - (-\Delta^l z_s^n[i])_+\right|^2 \mathbb{1}_{\{(-\Delta^l w_s^n[i])_+ \le \widetilde{Y}_s^{l,n}/N\}}\right] \\
&\quad + C\sum_{l\ne n}\sum_{i\in[\![d]\!]}\left[\frac{\widetilde{Y}_s^{l,n}}{N}\left|(-\Delta^l w_s^n[i])_+ - (-\Delta^l z_s^n[i])_+\right|^2 \mathbb{1}_{\{(-\Delta^l w_s^n[i])_+ > \widetilde{Y}_s^{l,n}/N\}}\right] \\
&\le C\sum_{l\ne n}\frac{(\widetilde{Y}_s^{l,n})^3}{N^3} + C\sum_{l=n}\sum_{i\in[\![d]\!]}[(-\Delta^l w_s^n[i])_+\,|(-\Delta^l w_s^n[i])_+ - (-\Delta^l z_s^n[i])_+|^2],
\end{aligned}$$

where, in order to pass from the third to the last line, we use once again the fact that $(-\Delta^l z_s^n[i])_+ \le C\,\widetilde{Y}_s^{l,n}/N$. Recalling that $s < \widetilde{\tau}_N$, we have

$$\sum_{l\ne n}\frac{(\widetilde{Y}_s^{l,n})^3}{N^3} \le \left(\frac{N^{1-\epsilon}}{N}\right)^2\sum_{l\ne n}\frac{\widetilde{Y}_s^{l,n}}{N} = N^{-2\epsilon}.$$

Using in addition the upper bound $(-\Delta^l w_s^n[i])_+ \le \alpha_s^{w,l,n}[i]$ if $i \ne \widetilde{X}_s^{l,n}$, we deduce that

$$\left(\sum_{l\ne n}|(\alpha_s^{w,l,n}[\bullet] - \alpha_s^{z,l,n}[\bullet]) \cdot \Delta^l z_s^n[\bullet]|\right)^2 \le CN^{-2\epsilon} + C\sum_{l\ne n}\alpha_s^{w,l,n}[\bullet] \cdot [\Delta^l(z_s^n - w_s^n)[\bullet] \odot \Delta^l(z_s^n - w_s^n)[\bullet]]. \tag{5.13}$$

Hence, by Young's inequality and because $\epsilon = 1/8$, we deduce from (5.12) that

$$\begin{aligned}
\mathbb{E}[e_{\widetilde{\tau}_N}\Phi^n_{\widetilde{\tau}_N}(z^n_{\widetilde{\tau}_N} - w^n_{\widetilde{\tau}_N})^2] &\ge \mathbb{E}[e_t\Phi^n_t(z^n_t - w^n_t)^2] + \mathbb{E}\int_{t\wedge\widetilde{\tau}_N}^{\widetilde{\tau}_N}\dot{e}_s\Phi^n_s(z^n_s - w^n_s)^2 ds \\
&\quad + 2\mathbb{E}\bigg[\int_{t\wedge\widetilde{\tau}_N}^{\widetilde{\tau}_N} e_s\Phi^n_s(z^n_s - w^n_s)\{-C(z^n_s - w^n_s) \\
&\quad + (\alpha_s^{w,n,n}[\bullet] - \widetilde{\varphi}_s^{\bullet,n}[\bullet]) \cdot \Delta^n(z_s^n - w_s^n)[\bullet] + H_s^{w,n} - H_s^{z,n} - r_s^n\}ds\bigg] \\
&\quad - c\mathbb{E}\left[\int_{t\wedge\widetilde{\tau}_N}^{\widetilde{\tau}_N}\frac{e_s\Phi^n_s}{\widetilde{\mu}_s^{\bullet,\min}}(z^n_s - w^n_s)^2 ds\right] - C\frac{1}{N^{2\epsilon}}\mathbb{E}\left[\int_{t\wedge\widetilde{\tau}_N}^{\widetilde{\tau}_N} e_s ds\right].
\end{aligned} \tag{5.14}$$

We now turn to the analysis of the third line on the right-hand side of (5.14).

Recalling that $\widetilde{\varphi}^{\bullet,n}$, $\alpha^{w,l,n}$, $z^{N,n}$, and $w^{N,n}$ are bounded, we get, by the definition of the Hamiltonian,

$$|H_t^{w,n} - H_t^{z,n}| + |\widetilde{\varphi}_s^{\bullet,n} - \alpha_s^{w,n,n}||\Delta^n(z_s^n - w_s^n)| \le C\,|\Delta^n(z_s^n - w_s^n)| \le C\Theta_s^n,$$

where

$$\Theta_s^n := \sum_{j\in[\![d]\!]}|(z^{N,n} - w^{N,n})(s,(j,\widetilde{\boldsymbol{X}}_s^{\bullet,-n}),\widetilde{\boldsymbol{Y}}_s^n)|\,, \tag{5.15}$$

where $(j, \widetilde{\boldsymbol{X}}_s^{\bullet,-n}) = (\widetilde{X}_s^{1,n}, \ldots, \widetilde{X}_s^{n-1,n}, j, \widetilde{X}_s^{n+1,n}, \ldots)$. Finally, from Proposition 4.3 (with $\epsilon = 1/8$ therein),

$$|r^{N,n}(s, \widetilde{\boldsymbol{X}}_s^n, \widetilde{\boldsymbol{Y}}_s^n)| \le \frac{C}{N^\eta}.$$

Using Young's inequality $ab \le (a^2 + b^2)/2$ in (5.14) and choosing

$$e_s = \exp\left(\int_{t_0}^t \frac{c}{\widetilde{\mu}_s^{\bullet,\min}} ds\right), \tag{5.16}$$

we obtain

$$\begin{aligned}
&\mathbb{E}[e_{\widetilde{\tau}_N} \Phi^n_{\widetilde{\tau}_N} (z^n_{\widetilde{\tau}_N} - w^n_{\widetilde{\tau}_N})^2] \ge \mathbb{E}[e_t \Phi^n_t (z^n_t - w^n_t)^2] \\
&-C\mathbb{E}\left[\int_{t\wedge\widetilde{\tau}_N}^{\widetilde{\tau}_N} e_s \Phi^n_s (\Theta^n_s)^2 ds\right] - \frac{C}{N^{2\epsilon}} \mathbb{E}\left[\int_{t\wedge\widetilde{\tau}_N}^{\widetilde{\tau}_N} e_s ds\right] - \frac{C}{N^{2\eta}}.
\end{aligned} \tag{5.17}$$

Here, it is worth recalling that c in (5.16) only depends on T, $\|f\|_\infty$, and $\|g\|_\infty$.

5.1.5. Fifth Step. We now proceed with the analysis of the various terms on the right-hand side of (5.17). We start with the first term. By (4.13), (4.16), and (4.17) and for κ large enough (we comment more on this requirement right after the inequality), we observe that[7] (using the fact that $\Phi^n_{\widetilde{\tau}_N} \le 1$)

$$\begin{aligned}
&\mathbb{E}[e_{\widetilde{\tau}_N} \Phi^n_{\widetilde{\tau}_N} (z^n_{\widetilde{\tau}_N} - w^n_{\widetilde{\tau}_N})^2] \\
&\le C\mathbb{E}[(e_{\widetilde{\tau}_N})^4]^{1/4} \mathbb{E}[(\Phi^n_{\widetilde{\tau}_N})^2]^{1/2} \mathbb{P}(\widetilde{\tau}_N < T)^{1/4} \\
&\le C N^{-\epsilon/(4d)} \prod_{i\in[\![d]\!]} (N^{-\epsilon} + \widetilde{\mu}_{t_0}[i])^{-5/(8d)} \left[\left(\frac{1}{N}\sum_{l\in[\![N]\!]} |y_0^l|^\ell\right)^{-1/4} + \exp(-cN^{1-2\epsilon})\right] \\
&\quad \times \left[\prod_{i\in[\![d]\!]} (N^{-\epsilon} + \widetilde{\mu}_{t_0}[i])^{-1/(8d)} + \left(\frac{1}{N}\sum_{l\in[\![N]\!]} |y_0^l|^\ell\right)^{1/8}\right].
\end{aligned} \tag{5.18}$$

Notice that, in order to get an estimate for $\mathbb{E}[(e_{\widetilde{\tau}_N})^4]^{1/4}$ (using (4.13)), we must assume that κ is large enough with respect to c, but this makes sense because the constant c in the definition of e_s (see (5.16)) depends only on $\|f\|_\infty$, $\|g\|_\infty$, and T and, hence, does not depend on κ. It also interesting to observe that the polynomial decay in $\boldsymbol{y}$ in the penultimate line permits to balance the polynomial growth in the last line.

If $(N^{-1}\sum_{l\in[\![N]\!]} |y_0^l|^\ell)^{-1/4} \ge \exp(-cN^{1-2\epsilon})$, then using the fact that $(N^{-1}\sum_{l\in[\![N]\!]} |y_0^l|^\ell)^{-1/4}$ is always less than one, we get that the right-hand side in (5.18) can be bounded by $CN^{-\epsilon/(4d)} \prod_{i\in[\![d]\!]} (N^{-\epsilon} + \widetilde{\mu}_{t_0}[i])^{-3/(4d)}$, that is,

$$\mathbb{E}[e_{\widetilde{\tau}_N} \Phi^n_{\widetilde{\tau}_N} (z_{\widetilde{\tau}_N}{}^n - w^n_{\widetilde{\tau}_N})^2] \le CN^{-\epsilon/(4d)} \prod_{i\in[\![d]\!]} (N^{-\epsilon} + \widetilde{\mu}_{t_0}[i])^{-3/(4d)}. \tag{5.19}$$

If $(N^{-1}\sum_{l\in[\![N]\!]} |y_0^l|^\ell)^{-1/4} \le \exp(-cN^{1-2\epsilon})$, that is, $N^{-1}\sum_{l\in[\![N]\!]} |y_0^l|^\ell \ge \exp(4cN^{1-2\epsilon})$, then $\widetilde{\tau}_N = t_0$ with probability one. But, in the latter case,

$$\mathbb{E}\left[e_{\widetilde{\tau}_N} \Phi^n_{\widetilde{\tau}_N} (z^n_{\widetilde{\tau}_N} - w^n_{\widetilde{\tau}_N})^2\right] \le C\Phi^n_{t_0} = C\left(\frac{1}{N}\sum_{l\in[\![N]\!]} |y_0^l|^\ell\right)^{-1} \le C\exp(-4cN^{1-2\epsilon}),$$

and obviously (5.19) also holds true. So the latter holds true in any case. Similarly, by (4.13) again and also for κ large enough as before,

$$\begin{aligned}
N^{-2\epsilon}\mathbb{E}\left[\int_{t\wedge\widetilde{\tau}_N}^{\widetilde{\tau}_N} e_s ds\right] &\le CN^{-2\epsilon} \prod_{i\in[\![d]\!]} (N^{-\epsilon} + \widetilde{\mu}_{t_0}[i])^{-1/d} \\
&\le CN^{-\epsilon/(4d)} \prod_{i\in[\![d]\!]} (N^{-\epsilon} + \widetilde{\mu}_{t_0}[i])^{-3/(4d)}.
\end{aligned} \tag{5.20}$$

To finish, we recall from (5.1) that

$$\theta^n_t = \sup_{x,y} \left[\prod_{i\in[\![d]\!]} (N^{-\epsilon} + \mu^N_{x,y}[i])^{1/d} \Phi^N(\boldsymbol{y})(z^{N,n} - w^{N,n})^2(t, \boldsymbol{x}, \boldsymbol{y})\right], \quad t \in [0, T], \tag{5.21}$$

This allows us to write, using the notation from (5.15),

$$\Phi^n_s(\Theta^n_s)^2 \le C \sum_{j\in[\![d]\!]} \prod_{i\in[\![d]\!]} (N^{-\epsilon} + \mu^N_{(j,\tilde{X}^{\bullet,-n}_s),\tilde{Y}^{\bullet,n}_s}[i])^{-1/d} \theta^n_s.$$

Now, because $s < \widetilde{\tau}_N$, we have, for all $i \in [\![d]\!]$,

$$N^{-\epsilon} + \mu^N_{(j,\tilde{X}^{\bullet,-n}_s),\tilde{Y}^{\bullet,n}_s}[i] \ge N^{-\epsilon} + \mu^N_{\tilde{X}^{\bullet,n}_s,\tilde{Y}^{\bullet,n}_s}[i] - \frac{Y^{n,n}_t}{N} \ge \frac{1}{2}(N^{-\epsilon} + \widetilde{\mu}^{\bullet,n}_s[i]),$$

from which we deduce that

$$\mathbb{E}\left[\int_{t\wedge\tilde{\tau}_N}^{\tilde{\tau}_N} e_s \Phi^n_s(\Theta^n_s)^2 ds\right] \le C\,\mathbb{E}\left[\int_{t\wedge\tilde{\tau}_N}^{\tilde{\tau}_N} \theta^n_s e_s \prod_{i\in[\![d]\!]} (N^{-\epsilon} + \widetilde{\mu}^{\bullet,N}_s[i])^{-1/d} ds\right].$$

By (4.14), the preceding is less than (again, provided that κ is chosen large enough)

$$\mathbb{E}\left[\int_{t\wedge\tilde{\tau}_N}^{\tilde{\tau}_N} e_s \Phi^n_s(\Theta^n_s)^2 ds\right] \le C \prod_{i\in[\![d]\!]} (N^{-\epsilon} + \widetilde{\mu}_{t_0}[i])^{-1/d} \int_t^T \theta^n_s ds. \tag{5.22}$$

So, taking $t = t_0$ in (5.17), multiplying the whole by $\prod_{i\in[\![d]\!]}(N^{-\epsilon} + \widetilde{\mu}_{t_0}[i])^{1/d}$, and collecting the four bounds (5.17), (5.19), (5.20) and (5.22), we deduce that there exists an exponent $\chi > 0$ such that, for any initial condition $(t_0, \boldsymbol{x}, \boldsymbol{y})$,

$$\prod_{i\in[\![d]\!]} (N^{-\epsilon} + \widetilde{\mu}_{t_0}[i])^{1/d} \Phi(\boldsymbol{y})(z^{N,n} - w^{N,n})^2(t_0, \boldsymbol{x}, \boldsymbol{y}) \le CN^{-\chi} + C\int_{t_0}^T \theta^n_s ds.$$

Taking the supremum over $\boldsymbol{x}$ and $\boldsymbol{y}$, we get $\theta^n_{t_0}$ on the left-hand side. Because t_0 is arbitrary, we can complete the proof of (2.19) by Gronwall's lemma. Q.E.D.

5.2. Proof of Theorem 2.2

Now that Theorem 2.1 has been proved, the proof is pretty straightforward. It is based on a standard diffusion approximation theorem—see for instance Ethier and Kurtz [21, chapter 7, theorem 4.1]—and our plan is, thus, to check the assumption of this latter statement. Throughout the proof, we make use of the following stopping time:

$$\sigma^N_K := \inf\left\{t \in [0,T] : \min_{e\in[\![d]\!]} \mu^N_t[e] \le 1/K \text{ or } \frac{1}{N}\sum_{l\in[\![d]\!]} |Y^{N,l}_s|^\ell \ge K\right\}, \quad \inf\emptyset = +\infty,$$

for $\ell = 3$ and any real $K \ge 0$. We denote by $Y^{N,l}$ the lth coordinate of $\boldsymbol{Y}^N$. A key fact for us is that, by (4.14) and (4.15) with $\iota = 0$ therein,

$$\lim_{K\to\infty} \sup_{N\ge 1} \mathbb{P}(\sigma^N_K \le T) = 0.$$

Note that the initial condition satisfies (2.20) because of the convergence in (2.21) and because $\boldsymbol{y}^N = (1, \ldots, 1)$. The result is important in our analysis. It says that we can easily localize the various conditions appearing in the statement of Ethier and Kurtz [21, chapter 7, theorem 4.1] and just check them up to the stopping time σ^N_K (which is not so different from the fact that, in the latter statement, all the conditions are localized with respect to the stopping τ^r_n, using the same notation as therein).

The next step is to provide a semimartingale expansion of $(\mu^N_t[i])_{0\le t\le T}$ for any $i \in [\![d]\!]$. This is here possible by Itô's lemma (see, for instance, the proof of Theorem 4.1 or Expansion (5.2)). For convenience, we remove the exponent N from most of the notations (for instance, we merely write $(\boldsymbol{X},\boldsymbol{Y})$ for $(\boldsymbol{X}^N,\boldsymbol{Y}^N)$) except when this is clearly needed (say, for instance, when we take a limit over N). With this convention, we get

$$\begin{aligned}
d\mu_t^N[i] = d\left(\frac{1}{N}\sum_{l\in[\![N]\!]} Y_t^l \mathbb{1}_{\{X_t^l=i\}}\right) &= \frac{1}{N}\sum_{l\in[\![N]\!]}\int_{[0,M]^d} Y_{t-}^l(\mathbb{1}_{\{X_{t-}^l+f_{t-}^l(\theta)=i\}} - \mathbb{1}_{\{X_{t-}^l=i\}})\mathcal{N}^l(d\theta,dt)\\
&\quad+\frac{1}{N}\sum_{l\in[\![N]\!]}\int_{[0,\varepsilon N]^{[\![N]\!]^d}} g_{t-}^l(\theta)\mathbb{1}_{\{X_{t-}^l=i\}}\mathcal{N}^0(d\theta,dt)\\
&=\frac{1}{N}\sum_{l\in[\![N]\!]}\int_{[0,M]^d} Y_{t-}^l(\mathbb{1}_{\{X_{t-}^l+f_{t-}^l(\theta)=i\}} - \mathbb{1}_{\{X_{t-}^l=i\}})\nu^l(d\theta)dt\\
&\quad+\frac{1}{N}\sum_{l\in[\![N]\!]}\int_{[0,M]^d} Y_{t-}^l(\mathbb{1}_{\{X_{t-}^l+f_{t-}^l(\theta)=i\}} - \mathbb{1}_{\{X_{t-}^l=i\}})\overline{\mathcal{N}}^l(d\theta,dt)\\
&\quad+\frac{1}{N}\sum_{l\in[\![N]\!]}\int_{[0,\varepsilon N]^{[\![N]\!]^d}} g_{t-}^l(\theta)\mathbb{1}_{\{X_{t-}^l=i\}}\overline{\mathcal{N}}^0(d\theta,dt)\\
&=: dB_t + dM_t + dM_t^0,
\end{aligned}$$

where $\overline{\mathcal{N}}^0,\overline{\mathcal{N}}^1,\cdots,\overline{\mathcal{N}}^N$ denote the compensated Poisson measures and where, in the penultimate line, we use the fact that $\int_{[0,\varepsilon N]^{[\![N]\!]^d}} g_{t-}^l(\theta)d\nu^0(\theta)=0$. By (4.7), we know that

$$\begin{aligned}
dB_t &= \frac{1}{N}\sum_{l\in[\![N]\!]}\sum_{j\in[\![d]\!]} Y_{t-}^l(\delta_{i,j}-\mathbb{1}_{\{X_{t-}^l=i\}})(\varphi(\mu_{t-}^N[j]) + (-\Delta^l w_{t-}^l[j])_+)dt\\
&=\frac{1}{N}\sum_{l\in[\![N]\!]} Y_{t-}^l\left[(\varphi(\mu_{t-}^N[i]) + (-\Delta^l w_{t-}^l[i])_+) - \sum_{j\in[\![d]\!]}\mathbb{1}_{\{X_{t-}^l=i\}}(\varphi(\mu_{t-}^N[j]) + (-\Delta^l w_{t-}^l[j])_+)\right]dt.
\end{aligned}$$

As long as $t<\sigma_K$, we have that

$$\frac{Y_{t-}^l}{N} \le \frac{1}{N}N^{1/m}\left(\frac{1}{N}\sum_{n\in[\![N]\!]}|Y_{t-}^n|^m\right)^{1/m} \le K^{1/m}N^{-1+1/m}. \tag{5.23}$$

Hence, we get, from (4.28), (4.29), and Theorem 2.1,

$$\begin{aligned}
\Delta^l w_{t-}^l[j] &= \Delta^l z_{t-}^l[j] + O(N^{-\chi})\\
&= U^j(t-,\mu_{t-}^N) - U^{X_{t-}^l}(t-,\mu_{t-}^N) + O(N^{-\chi}),
\end{aligned}$$

for a possibly new value of χ. Here, $O(\cdot)$ is the standard Landau notation, it being understood that the underlying constant is deterministic and independent of l, t, and j.

We, thus, end up with

$$\begin{aligned}
dB_t &= \sum_{k\in[\![d]\!]}\mu_{t-}^N[k](\varphi(\mu_{t-}^N[i]) + (U^k-U^j)_+(t,\mu_{t-}^N))\\
&\quad-\sum_{j\in[\![d]\!]}\mu_{t-}^N[i](\varphi(\mu_{t-}^N[j]) + (U^i-U^j)_+(t,\mu_{t-}^N))]dt + O(N^{-\chi})dt\\
&=: b(t,\mu_{t-}^N)dt + O(N^{-\chi})dt
\end{aligned} \tag{5.24}$$

We deduce that (we put an exponent N in B in order to emphasize the dependence on N)

$$\sup_{t\in[0,T\wedge\sigma_K]}\left|B_t^N - \int_0^t b(s,\mu_s^N)ds\right|$$

tends to zero in probability as N tends to ∞, which fits Ethier and Kurtz [21, chapter 7, (4.6)]. By the way, notice that Ethier and Kurtz [21, chapter 7, (4.4)] follows in a straightforward manner.

We now handle the martingale part. By independence of the noises $\mathcal{N}^1,\cdots,\mathcal{N}^N$, it is easy to see that

$$\lim_{N\to\infty}\mathbb{E}\left[\sup_{0\le t\le\sigma_K}|M_t|^2\right]=0.$$

Then, the compensator of $(M_t[i]M_t[j])_{0\le t\le T}$, which we denote by $(A_t^{i,j})_{0\le t\le T}$, satisfies

$$\lim_{N\to\infty}\mathbb{E}\left[\sup_{0\le t\le\sigma_K}|A_t|\right]=0.$$

Next, the compensator of $(M_t^0[i]M_t^0[j])_{0\le t\le T}$ is given by $(A_t^{0,i,j})_{0\le t\le T}$, defined by

$$\begin{aligned}
dA_t^{0,i,j}&=\frac{1}{N^2}\sum_{l,n\in[\![N]\!]}\int_{[0,\varepsilon N]^{[\![N]\!]^d}}g_{t-}^l(\theta)g_{t-}^n(\theta)\mathbb{1}_{\{X_{t-}^l=i\}}\mathbb{1}_{\{X_{t-}^n=j\}}\nu^0(d\theta)dt\\
&=\frac{\varepsilon}{N}\sum_{l,n\in[\![N]\!]}\left(Y_{t-}^lY_{t-}^n\mathbb{1}_{\{X_{t-}^l=i\}}\mathbb{1}_{\{X_{t-}^n=j\}}\mathbf{E}\left[\left(\frac{S_{\mu_{t-}^N}[i]}{N\mu_{t-}^N[i]}-1\right)\left(\frac{S_{\mu_{t-}^N}[j]}{N\mu_{t-}^N[j]}-1\right)\right]\right)dt\\
&=\varepsilon(\mu_{t-}^N[i]\delta_{i,j}-\mu_{t-}^N[i]\mu_{t-}^N[j])dt.
\end{aligned}$$

By combining the last two results, we deduce that the compensator of $((M_t+M_t^0)[i](M_t+M_t^0)[j])_{0\le t\le T}$, which we denote by $(\overline{A}_t^{i,j})_{0\le t\le T}$, satisfies

$$\lim_{N\to\infty}\mathbb{E}\left[\sup_{t\in[0,T\wedge\sigma_K]}|\overline{A}_t^{i,j}-\overline{A}_{t-}^{i,j}|\right]=0,\ \lim_{N\to\infty}\mathbb{E}\left[\sup_{t\in[0,T\wedge\sigma_K]}\left|\overline{A}_t^{i,j}-\int_0^t\varepsilon(\mu_{s-}^N[i]\delta_{i,j}-\mu_{s-}^N[i]\mu_{s-}^N[j])ds\right|\right]=0,$$

which are, respectively, Ethier and Kurtz [21, chapter 7, (4.5) and (4.7)]. The last assumption we need to verify is Ethier and Kurtz [21, chapter 7, (4.3)]. Because the process $(\mu_t^N)_{0\le t\le T}$ takes values in the simplex, it suffices to prove that

$$\forall r>0,\quad\lim_{N\to\infty}\mathbb{P}\left(\sup_{0\le t\le T\wedge\sigma_K^N}|\mu_t^N-\mu_{t-}^N|\ge r\right)=0.\tag{5.25}$$

We may split the jumps into two parts: those that are due to the idiosyncratic noises and those that are due to the common noise. To make it clear, for any $i\in[\![d]\!]$,

$$|\mu_t^N[i]-\mu_{t-}^N[i]|\le\frac{1}{N}\sum_{l\in[\![N]\!]}|Y_t^l-Y_{t-}^l|+\frac{1}{N}\sum_{l\in[\![N]\!]}Y_{t-}\,|\,X_t^l-X_{t-}^l|\,.$$

Because at most one of all the $(X^l)_{l\in[\![N]\!]}$ may jump at a given time, the second term on the right-hand side can be upper bounded in the following way:

$$\frac{1}{N}\sum_{l\in[\![N]\!]}Y_{t-}\,|X_t^l-X_{t-}^l|\le\frac{d}{N}\max_{l\in[\![N]\!]}Y_{t-}^l\le K^{1/m}N^{-1+1/m},$$

the last bound following from (5.23).

Therefore, we can just focus on the jumps induced by $\boldsymbol{Y}$, which is more subtle. The idea is to represent the latter ones as follows. By (2.6), we may indeed represent the jump times of $(\boldsymbol{Y}_t)_{0\le t\le T}$ through the jump times $(\varrho_n)_{n\ge0}$ of a Poisson process of intensity εN on the axis $[0,+\infty)$ (with $\varrho_0=0$). The counting process on $[0,T]$ is denoted by $(R_t=\sum_{n\ge0}\mathbb{1}_{[0,t]}(\tau_n))_{0\le t\le T}$, which is a Poisson process of intensity εN. Then, we use the fact that, when the exponential clock rings (namely, at some time ϱ_n), the jump of $\boldsymbol{Y}$ is given by a multinomial distribution of parameters N and $(\mu_{\varrho_n-}^N[i])_{i\in[\![d]\!]}$. Writing

$$\sup_{0\le t\le T\wedge\sigma_K}\frac{1}{N}\sum_{l\in[\![N]\!]}|Y_t^l-Y_{t-}^l|\le\sup_{n\in\{1,\cdots,R_T\}}\left[\frac{1}{N}\sum_{l\in[\![N]\!]}|Y_{\varrho_n}^l-Y_{\varrho_n-}^l|\,\mathbb{1}_{\{\min_{e\in[\![d]\!]}\mu_{\varrho_n-}^N[e]>1/K\}}\right],$$

we deduce that, for any $r>0$,

$$\begin{aligned}
&\mathbb{P}\left(\sup_{0\le t\le T\wedge\sigma_K}\frac{1}{N}\sum_{l\in[\![N]\!]}|Y_t^l-Y_{t-}^l|\ge r\right)\\
&\le\mathbb{P}(R_T\ge Nr^{-1})+\mathbb{P}\left(\sup_{n\in\{1,\cdots,\lfloor Nr^{-1}\rfloor\}}\left[\frac{1}{N}\sum_{l\in[\![N]\!]}|Y_{\varrho_n}^l-Y_{\varrho_n-}^l\,|\,\mathbb{1}_{\{\min_{e\in[\![d]\!]}\mu_{\varrho_n-}^N[e]>1/K\}}\right]\ge r\right)\\
&\le\varepsilon Tr+Nr^{-1}\sup_{\mu:\min_{e\in[\![d]\!]}\mu[e]>1/K}\mathbb{P}\left(\sup_{i\in[\![d]\!]}\left|\frac{S_\mu[i]}{N}-\mu[i]\right|\ge r\mu[i]\right)\\
&\le\varepsilon Tr+2Nr^{-1}\exp(-2Nr^2K^{-2}),
\end{aligned}$$

the last line following from Hoeffding's inequality. The preceding bound tends to zero as N tends to ∞ first, and then r tends to zero, from which we deduce that, for any $r>0$, the term on the first line tends to zero as N tends to ∞. We get (5.25), which completes the proof. □

6. Proofs of the Auxiliary Estimates of the Weight Process

In this section, we prove the results stated in Section 4.1.

6.1. Proof of Theorem 4.1

6.1.1. First Step. Recall Itô's Equation (4.8) together with the notation (4.9). Given $\varpi>0$, we apply this formula to the function $v(x,y)=\log(\varpi+\mu^N_{x,y}[i])$. We get, for each $i\in[\![d]\!]$, letting $\overline{\Lambda}_t:=\log(\varpi+\overline{\mu}_t[i])$,

$$\begin{aligned}
d\overline{\Lambda}_t &= \sum_{l\in[\![N]\!]}\int_{[0,M]^d}\left[\log\left(\varpi+\overline{\mu}_{t-}[i]+\frac{1}{N}\overline{Y}^l_{t-}\left(\mathbb{1}_{\{\overline{X}^l_{t-}+\overline{f}^l_{t-}(\theta)=i\}}-\mathbb{1}_{\{\overline{X}^l_{t-}=i\}}\right)\right)-\log(\varpi+\overline{\mu}_{t-}[i])\right]\mathcal{N}^l(d\theta,dt)\\
&\quad+\int_{[0,\varepsilon N]^{[\![N]\!]^d}}\left[\log\left(\varpi+\overline{\mu}_{t-}[i]+\frac{1}{N}\sum_{l\in[\![N]\!]}\overline{g}^l_{t-}(\theta)\mathbb{1}_{\{\overline{X}^l_{t-}=i\}}\right)-\log(\varpi+\overline{\mu}_{t-}[i])\right]\mathcal{N}^0(d\theta,dt)\\
&=\sum_{l\in[\![N]\!]}\int_{[0,M]^d}\log\left(1+\frac{\frac{1}{N}\overline{Y}^l_{t-}\left[\mathbb{1}_{\{\overline{X}^l_{t-}+\overline{f}^l_{t-}(\theta)=i\}}-\mathbb{1}_{\{\overline{X}^l_{t-}=i\}}\right]}{\varpi+\overline{\mu}_{t-[i]}}\right)\mathcal{N}^l(d\theta,dt)+\int_{[0,\varepsilon N]^{[\![N]\!]^d}}\log\left(1+\frac{\frac{1}{N}\sum_{l\in[\![N]\!]}\overline{g}^l_{t-}(\theta)\mathbb{1}_{\{\overline{X}^l_{t-}=i\}}}{\varpi+\overline{\mu}_{t-[i]}}\right)\mathcal{N}^0(d\theta,dt),
\end{aligned}$$

and we have

$$\begin{aligned}
-d\overline{\Lambda}_t &= \sum_{l\in[\![N]\!]}\int_{[0,M]^d}\log\left(\frac{\varpi+\overline{\mu}_{t-}[i]}{\varpi+\overline{\mu}_{t-}[i]+\frac{1}{N}\overline{Y}^l_{t-}\left[\mathbb{1}_{\{\overline{X}^l_{t-}+\overline{f}^l_{t-}(\theta)=i\}}-\mathbb{1}_{\{\overline{X}^l_{t-}=i\}}\right]}\right)\mathcal{N}^l(d\theta,dt)\\
&\quad+\int_{[0,\varepsilon N]^{[\![N]\!]^d}}\log\left(\frac{\varpi+\overline{\mu}_{t-}[i]}{\varpi+\overline{\mu}_{t-}[i]+\frac{1}{N}\sum_{l\in[\![N]\!]}\overline{g}^l_{t-}(\theta)\mathbb{1}_{\{\overline{X}^l_{t-}=i\}}}\right)\mathcal{N}^0(d\theta,dt)\\
&=\sum_{l\in[\![N]\!]}\int_{[0,M]^d}\log\left(1-\frac{\frac{1}{N}\overline{Y}^l_{t-}\left[\mathbb{1}_{\{\overline{X}^l_{t-}+\overline{f}^l_{t-}(\theta)=i\}}-\mathbb{1}_{\{\overline{X}^l_{t-}=i\}}\right]}{\varpi+\overline{\mu}_{t-[i]}+\frac{1}{N}\overline{Y}^l_{t-}\left[\mathbb{1}_{\{\overline{X}^l_{t-}+\overline{f}^l_{t-}(\theta)=i\}}-\mathbb{1}_{\{X^l_{t-}=i\}}\right]}\right)\mathcal{N}^l(d\theta,dt)\\
&\quad+\int_{[0,\varepsilon N]^{[\![N]\!]^d}}\log\left(1-\frac{1}{N}\sum_{l\in[\![N]\!]}\overline{g}^l_{t-}(\theta)\frac{\mathbb{1}_{\{\overline{X}^l_{t-}=i\}}}{\varpi+\overline{\mu}_{t-}[i]+\frac{1}{N}\sum_{l\in[\![N]\!]}\overline{g}^l_{t-}(\theta)\mathbb{1}_{\{\overline{X}^l_{t-}=i\}}}\right)\mathcal{N}^0(d\theta,dt).
\end{aligned}$$

This prompts us to let

$$\overline{\lambda}_t:=\sum_{l\in[\![N]\!]}\int_{[0,M]^d}-\frac{\frac{1}{N}\overline{Y}^l_t\left[\mathbb{1}_{\{\overline{X}^l_{t-}+\overline{f}^l_t(\theta)=i\}}-\mathbb{1}_{\{\overline{X}^l_t=i\}}\right]}{\varpi+\overline{\mu}_{t-}[i]+\frac{1}{N}\overline{Y}^l_{t-}\left[\mathbb{1}_{\{\overline{X}^l_t+\overline{f}^l_t(\theta)=i\}}-\mathbb{1}_{\{\overline{X}^l_t=i\}}\right]}\nu(d\theta)+\int_{[0,\varepsilon N]^{[\![N]\!]^d}}-\frac{\frac{1}{N}\sum_{l\in[\![N]\!]}\overline{g}^l_t(\theta)\mathbb{1}_{\{\overline{X}^l_t=i\}}}{\varpi+\overline{\mu}_t[i]+\frac{1}{N}\sum_{l\in[\![N]\!]}\overline{g}^l_t(\theta)\mathbb{1}_{\{\overline{X}^l_t=i\}}}\nu^0(d\theta),$$

and as a result, we claim that $(Z_t=\exp\{-\overline{\Lambda}_t-\int_0^t\overline{\lambda}_s ds\})_{0\le t\le T}$ is a supermartingale. This follows from the fact that, given a tuple of bounded predictable processes $((F^l_t)_{0\le t\le T})_{l=0,\cdots,N}$, the process

$$\begin{aligned}
Z_t=\exp\Bigg\{&\sum_{l=1}^N\int_0^t\int_{[0,M]^d}\log(1+F^l_t)\mathcal{N}^l(d\theta,dt)+\int_0^t\int_{[0,\varepsilon N]^{[\![N]\!]^d}}\log(1+F^0_t)\mathcal{N}^0(d\theta,dt)\\
&-\sum_{l=1}^N\int_0^t\int_{[0,M]^d}F^l_t\nu(d\theta)dt-\int_0^t\int_{[0,\varepsilon N]^{[\![N]\!]^d}}F^0_t\nu^0(d\theta)dt\Bigg\},
\end{aligned}$$

is a supermartingale because it solves the SDE

$$dZ_t=\sum_{l=1}^N\int_{[0,M]^d}Z_{t-}F^l_t[\mathcal{N}^l(d\theta,dt)-\nu(d\theta)dt]+\int_{[0,\varepsilon N]^{[\![N]\!]^d}}Z_{t-}F^0_t[\mathcal{N}^0(d\theta,dt)-\nu^0(d\theta)dt].$$

In particular, $\mathbb{E}[Z_t]\le(\varpi+\overline{\mu}^i_0)^{-1}$ for any t.

6.1.2. Second Step. By definition of Λ,

$$(\varpi + \overline{\mu}_{t\wedge\overline{\tau}_N}[i])\exp\left(-\overline{\Lambda}_{t\wedge\overline{\tau}_N} - \int_0^{t\wedge\overline{\tau}_N}\overline{\lambda}_s ds\right) = \exp\left(\int_0^{t\wedge\overline{\tau}_N} -\overline{\lambda}_s ds\right), \tag{6.1}$$

and then, using $0 < \varpi < 1$ and $\overline{\mu}_{\overline{\tau}_N}[i] \le 1$, taking expectation and letting t tend to T, we obtain

$$\mathbb{E}\left[\exp\left\{-\int_0^{\overline{\tau}_N}\overline{\lambda}_t dt\right\}\right] \le \frac{2}{\varpi + \overline{\mu}_0[i]}, \tag{6.2}$$

and thus, it remains to bound $-\overline{\lambda}_t$ from below. By (4.3), we obtain, letting $\alpha_t^l(j) := \alpha^l(t, \overline{\mathbf{X}}_t, \overline{\mathbf{Y}}_t)[j]$,

$$\begin{aligned}
-\overline{\lambda}_t \quad &= \frac{1}{N}\sum_{l\in[\![N]\!]}\sum_{j\in[\![d]\!]}[\varphi(\overline{\mu}_t[j]) + \alpha_t^l[j]]\frac{\overline{Y}_t^l(\mathbb{1}_{\{j=i\}} - \mathbb{1}_{\{\overline{X}_t^l=i\}})}{\varpi + \overline{\mu}_t[i] + \frac{1}{N}\overline{Y}_t^l(\mathbb{1}_{\{j=i\}} - \mathbb{1}_{\{\overline{X}}t^l = i\})} \\
&\quad + \varepsilon N\sum_{\boldsymbol{k}\in[\![N]\!]^d}\left(\frac{k^{\overline{X}_t^n}}{N\overline{\mu}_t[\overline{X}_t^n]}\right)^{\iota}\mathcal{M}_{N,\overline{\mu}_t}(\boldsymbol{k})\frac{\frac{1}{N}\sum_{l\in[\![N]\!]}\overline{Y}_t^l\mathbb{1}_{\{\overline{X}_t^l=i\}}\mathbb{1}_{\{\overline{\mu}_t[i]\neq 0\}}\left(\frac{k^i}{N\overline{\mu}_t[i]} - 1\right)}{\varpi + \overline{\mu}_t[i] + \frac{1}{N}\sum_{l\in[\![N]\!]}\overline{Y}_t^l\mathbb{1}_{\{\overline{X}_t^l=i\}}\left(\frac{k^i}{N\overline{\mu}_t[i]} - 1\right)} \\
&= \frac{1}{N}\sum_{l\in[\![N]\!]}\overline{Y}_t^l\sum_{j\in[\![d]\!]}\mathbb{1}_{\{\overline{X}_t^l=j\}}\frac{\varphi(\overline{\mu}_t[i]) + \alpha_t^l[i]}{\varpi + \overline{\mu}_t[i] + \frac{1}{N}\overline{Y}_t^l - \frac{1}{N}\overline{Y}_t^l\mathbb{1}_{\{j=i\}}} - \frac{1}{N}\sum_{l\in[\![N]\!]}\overline{Y}_t^l\mathbb{1}_{\{\overline{X}_t^l=i\}}\sum_{j\in[\![d]\!]}\frac{\varphi(\overline{\mu}_t[j]) + \alpha_t^l[j]}{\varpi + \overline{\mu}_t[i] + \frac{1}{N}\overline{Y}_t^l\mathbb{1}_{\{j=i\}} - \frac{1}{N}\overline{Y}_t^l} \\
&\quad + \varepsilon N\mathbb{1}_{\{\overline{\mu}_t[i]\neq 0\}}\mathbf{E}\left[\left(\frac{S_{\overline{\mu}_t}[\overline{X}_t^n]}{N\overline{\mu}_t[\overline{X}_t^n]}\right)^{\iota}\frac{N^{-1}S_{\overline{\mu}_t}[i] - \overline{\mu}_t[i]}{\varpi + \overline{\mu}_t[i] + N^{-1}S_{\overline{\mu}_t}[i] - \overline{\mu}_t[i]}\right] \\
&= \frac{1}{N}\sum_{l\in[\![N]\!]}\overline{Y}_t^l\sum_{j\neq i}\left(\mathbb{1}_{\{\overline{X}_t^l=j\}}\frac{\varphi(\overline{\mu}_t[i]) + \alpha_t^l[i]}{\varpi + \overline{\mu}_t[i] + \frac{1}{N}\overline{Y}_t^l} - \mathbb{1}_{\{\overline{X}_t^l=i\}}\frac{\varphi(\overline{\mu}_t[j]) + \alpha_t^l[j]}{\varpi + \overline{\mu}_t[i] - \frac{1}{N}\overline{Y}_t^l}\right) \\
&\quad + \varepsilon\mathbb{1}_{\{\overline{\mu}_t[i]\neq 0\}}\sum_{j\in[\![d]\!]}\mathbb{1}_{\{\overline{X}_t^n=j\}}\mathbf{E}\left[\left(\frac{S_{\overline{\mu}_t}[j]}{N\overline{\mu}_t[j]}\right)^{\iota}\frac{S_{\overline{\mu}_t}[i] - N\overline{\mu}_t[i]}{\varpi + N^{-1}S_{\overline{\mu}_t}[i]}\right] =: (\mathrm{I} + \mathrm{II}) + \mathrm{III},
\end{aligned} \tag{6.3}$$

with the same convention as before that the ratio $S_{\overline{\mu}_t}[j]/(N\overline{\mu}_t[j])$ is understood as one if $\overline{\mu}_t[j] = 0$. Importantly, note that the denominators in expressions I and II are positive; indeed, on the event $\{\overline{X}_t^l = i\}$, $\overline{\mu}_t[i] - N^{-1}\overline{Y}_t^l = \overline{\mu}_t[i] - N^{-1}\overline{Y}_t^l\mathbb{1}_{\{\overline{X}_t^l=i\}} = N^{-1}\sum_{j\neq l}\overline{Y}_t^j\mathbb{1}_{\{\overline{X}_t^j=i\}}$, which is nonnegative.

6.1.3. Third Step. We let $\varpi = N^{-\epsilon}$ and consider times $t < \overline{\tau}_N$ so that $\overline{\mu}_t[i] \ge N^{-\epsilon}$ and $\overline{Y}_t^l \le \frac{1}{2}N^{1-\epsilon}$ for any $l \in [\![N]\!]$. We have

$$\varpi + \overline{\mu}_t[i] + \frac{1}{N}\overline{Y}_t^l \le N^{-\epsilon} + \overline{\mu}_t[i] + \frac{1}{2}N^{-\epsilon} \le \frac{5}{2}\overline{\mu}_t[i], \quad \varpi + \overline{\mu}_t[i] - \frac{1}{N}\overline{Y}_t^l \ge \overline{\mu}_t[i] - \frac{1}{2}N^{-\epsilon} \ge \frac{1}{2}\overline{\mu}_t[i],$$

and thus, using the definition of φ and the bounds $0 \le \alpha_t^{i,j} \le 2(T\|f\|_\infty + \|g\|_\infty)$, we obtain (recall (4.2) for the definition of M)

$$\begin{aligned}
&\mathrm{I} \ge \frac{2\kappa}{5}\frac{1}{\overline{\mu}_t[i]}\mathbb{1}_{\{\overline{\mu}_t[i]\le\delta\}}\sum_{j\neq i}\overline{\mu}_t[j] \ge \frac{2}{5}\kappa(1-\delta)\mathbb{1}_{\{\overline{\mu}_t[i]\le\delta\}}\frac{1}{\overline{\mu}_t[i]}, \\
&\mathrm{II} \ge -(d-1)M\overline{\mu}_t[i]\frac{2}{\overline{\mu}_t[i]} = -2(d-1)M.
\end{aligned} \tag{6.4}$$

As to the jth term in III, we note that, for any $l \in [[N]]$, $S_{\overline{\mu}_t}[l] \sim \mathrm{Bin}(N, \overline{\mu}_t[l])$, and applying Hoeffding's inequality, we get

$$\mathbf{P}\left(|S_{\overline{\mu}_t}[l] - N\overline{\mu}_t[l]| \geq \frac{1}{2}N\overline{\mu}_t[l]\right) \leq 2\exp\left\{-\frac{N(\overline{\mu}_t[i])^2}{2}\right\} \leq 2\exp\left\{-\frac{N^{1-2\epsilon}}{2}\right\}. \tag{6.5}$$

For any $j \in [[d]]$, we let

$$\mathrm{III}_j = \varepsilon N \mathbf{E}\left[\left(\frac{S_{\overline{\mu}_t}[j]}{N\overline{\mu}_t[j]}\right)^{\iota} \frac{S_{\overline{\mu}_t}[i] - N\overline{\mu}_t[i]}{\varpi + N^{-1}S_{\overline{\mu}_t}[i]}\right].$$

Observing that, whether $\iota = 0$ or $\iota = 1$, it holds that $\mathbf{E}[(S_{\overline{\mu}_t}[j]/N\overline{\mu}_t[j])^{\iota}] = 1$, we may define the new probability measure

$$\overline{\mathbf{P}}^j := \left(\frac{S_{\overline{\mu}_t}[j]}{N\overline{\mu}_t[j]}\right)^{\iota} \cdot \mathbf{P},$$

and then, denoting by $\overline{\mathbf{E}}^j$ the related expectation, we obtain

$$\begin{aligned}
\mathrm{III}_j &= \varepsilon N \overline{\mathbf{E}}^j\left[\frac{S_{\overline{\mu}_t}[i] - N\overline{\mu}_t[i]}{\varpi + N^{-1}S_{\overline{\mu}_t}[i]}\right] \\
&= \varepsilon N \overline{\mathbf{E}}^j\left[\frac{S_{\overline{\mu}_t}[i] - N\overline{\mu}_t[i]}{N^{1-\epsilon} + S_{\overline{\mu}_t}[i]} \mathbb{1}_{\{|S_{\overline{\mu}_t}[i] - N\overline{\mu}_t[i]| > {}^1\!/\!{}_2 N\overline{\mu}_t[i]\}}\right] + \varepsilon N \overline{\mathbf{E}}^j\left[\frac{S_{\overline{\mu}_t}[i] - N\overline{\mu}_t[i]}{N^{1-\epsilon} + S_{\overline{\mu}_t}[i]} \mathbb{1}_{\{|S_{\overline{\mu}_t}[i] - N\overline{\mu}_t[i]| \leq {}^1\!/\!{}_2 N\overline{\mu}_t[i]\}}\right].
\end{aligned} \tag{6.6}$$

Notice that $\mathbf{E}[(S_{\overline{\mu}_t}[j]/N\overline{\mu}_t[j])^{2\iota}] \leq 1 + (1 - \overline{\mu}_t[j])/(N\overline{\mu}_t[j]) \leq 1 + N^{\epsilon-1} \leq 2$ (the worst case is $\iota = 1$). In particular, for any event $A \in (\Xi, \mathcal{G}, \mathbf{P})$, $\overline{\mathbf{P}}^j(A) \leq \sqrt{2}\mathbf{P}(A)^{1/2}$, and in particular, we have a variant of (6.5) under $\overline{\mathbf{P}}^j$ (up to a multiplicative constant).

Now, the first term in (6.6) can be bounded as follows:

$$\begin{aligned}
&\varepsilon N \left|\overline{\mathbf{E}}^j\left[\frac{S_{\overline{\mu}_t}[i] - N\overline{\mu}_t[i]}{N^{1-\epsilon} + S_{\overline{\mu}_t}[i]} \mathbb{1}_{\{|S_{\overline{\mu}_t}[i] - N\overline{\mu}_t[i]| > {}^1\!/\!{}_2 N\overline{\mu}_t[i]\}}\right]\right| \\
&\leq \varepsilon N \frac{N}{N^{1-\varepsilon}} \overline{\mathbf{P}}^j\left(|\, S_{\overline{\mu}_t}[i] - N\overline{\mu}_t[i]\, | \geq {}^1\!/\!{}_2 N\overline{\mu}_t[i]\right) \\
&\leq 2\varepsilon N^{1+\epsilon} \exp\left\{-\frac{N^{1-2\epsilon}}{4}\right\} \leq C\varepsilon,
\end{aligned} \tag{6.7}$$

with C as in the statement.

6.1.4. Fourth Step.

In order to bound the second term in III_j, we denote, for $x \geq -N\overline{\mu}_t[i]$,

$$\psi(x) = \frac{x}{(N^{-\epsilon} + \overline{\mu}_t[i])N + x}.$$

We note that ψ is increasing and concave and split

$$\begin{aligned}
\varepsilon N \overline{\mathbf{E}}^j\left[\psi(S_{\overline{\mu}_t}[i] - N\overline{\mu}_t[i]) \mathbb{1}_{\{|S_{\overline{\mu}_t}[i] - N\overline{\mu}_t[i]| \leq {}^1\!/\!{}_2 N\overline{\mu}_t[i]\}}\right] &= \varepsilon N \overline{\mathbf{E}}^j\left[(\psi(S_{\overline{\mu}_t}[i] - N\overline{\mu}_t[i]) + \psi(N\overline{\mu}_t[i] - S_{\overline{\mu}_t}[i])) \mathbb{1}_{\{|S_{\overline{\mu}_t[i]} - N\overline{\mu}_t[i]| \leq {}^1\!/\!{}_2 N\overline{\mu}_t[i]\}}\right] \\
&\quad + \varepsilon N \overline{\mathbf{E}}^j\left[-\psi(N\overline{\mu}_t[i] - S_{\overline{\mu}_t}[i]) \mathbb{1}_{\{|\{S_{\overline{\mu}_t[i]} - N\overline{\mu}_t[i]| \leq {}^1\!/\!{}_2 N\overline{\mu}_t[i]\}}\right] =: (A) + (B).
\end{aligned} \tag{6.8}$$

Notice that, on the event $\{| S_{\overline{\mu}_t}[i] - N\overline{\mu}_t[i] | \le \frac{1}{2} N\overline{\mu}_t[i]\}$, it holds that $N\overline{\mu}_t[i] - S_{\overline{\mu}_t}[i] \ge -\frac{1}{2}N\overline{\mu}_t[i]$, which makes licit the composition by ψ. Thus, Jensen's inequality, under the conditional probability given $\{|S_{\overline{\mu}_t}[i] - N\overline{\mu}_t[i]| \le \frac{1}{2}N\overline{\mu}_t[i]\}$, gives

$$\begin{aligned}
&\overline{\mathbf{E}}^j\left[-\psi\left(N\overline{\mu}_t[i] - S_{\overline{\mu}_t[i]}\right)\Big|\, |S_{\overline{\mu}_t}[i] - N\overline{\mu}_t[i]| \le \frac{1}{2}N\overline{\mu}_t[i]\right] \\
&\ge -\psi\left(\overline{\mathbf{E}}^j\left[N\overline{\mu}_t[i] - S_{\overline{\mu}_t}[i]\Big|\, |S_{\overline{\mu}_t}[i] - N\overline{\mu}_t[i]| \le \frac{1}{2}N\overline{\mu}_t[i]\right]\right) \\
&= -\psi\left(\frac{\overline{\mathbf{E}}^j\left[\left(N\overline{\mu}_t[i] - S_{\overline{\mu}_t}[i]\right)\mathbb{1}_{\left\{|S_{\overline{\mu}_t[i]} - N\overline{\mu}_t[i]| \le \frac{1}{2}N\overline{\mu}_t[i]\right\}}\right]}{\overline{\mathbf{P}}^j\left(|S_{\overline{\mu}_t[i]} - N\overline{\mu}_t[i]| \le \frac{1}{2}N\overline{\mu}_t[i]\right)}\right).
\end{aligned}$$

Now, notice that

$$\overline{\mathbf{E}}^j\left[N\overline{\mu}_t[i] - S_{\overline{\mu}_t}[i]\right] = \mathbf{E}\left[\left(\frac{S_{\overline{\mu}_t}[j]}{N\overline{\mu}_t[j]}\right)^{\iota}\left(N\overline{\mu}_t[i] - S_{\overline{\mu}_t}[i]\right)\right].$$

If $\iota = 0$, then both terms in the formula are obviously zero. Otherwise, $\iota = 1$, and then, at least for $i \neq j$,

$$\begin{aligned}
&\overline{\mathbf{E}}^j[N\overline{\mu}_t[i] - S_{\overline{\mu}_t}[i]] \\
&= \mathbf{E}\left[\frac{S_{\overline{\mu}_t}[j]}{N\overline{\mu}_t[j]}\left(N\overline{\mu}_t[i] - S_{\overline{\mu}_t}[i]\right)\right] \\
&= \frac{1}{N\overline{\mu}_t[j]}\mathbf{E}\left[\left(S_{\overline{\mu}_t}[j] - N\overline{\mu}_t[j]\right)\left(N\overline{\mu}_t[i] - S_{\overline{\mu}_t}[i]\right)\right] = \frac{N\overline{\mu}_t[j]\overline{\mu}_t[i]}{N\overline{\mu}_t[j]} = \overline{\mu}_t[i].
\end{aligned}$$

Noticing that the first term on the third line is negative if $i = j$ and recalling our variant of (6.5), we get in any case (whether $i = j$ or not)

$$\begin{aligned}
&\overline{\mathbf{E}}^j\left[\left(N\overline{\mu}_t[i] - S_{\overline{\mu}_t}[i]\right)\mathbb{1}_{\left\{|S_{\overline{\mu}_t[i]} - N\overline{\mu}_t[i]| \le \frac{1}{2}N\overline{\mu}_t[i]\right\}}\right] \\
&= \overline{\mathbf{E}}^j\left[\left(N\overline{\mu}_t[i] - S_{\overline{\mu}_t}[i]\right)\right] - \overline{\mathbf{E}}^j\left[\left(N\overline{\mu}_t[i] - S_{\overline{\mu}_t}[i]\right)\mathbb{1}_{\left\{|S_{\overline{\mu}_t[i]} - N\overline{\mu}_t[i]| > \frac{1}{2}N\overline{\mu}_t[i]\right\}}\right] \\
&\le \overline{\mu}_t[i] - \overline{\mathbf{E}}^j\left[\left(N\overline{\mu}_t[i] - S_{\overline{\mu}_t}[i]\right)\mathbb{1}_{\left\{|S_{\overline{\mu}_t[i]} - N\overline{\mu}_t[i]| > \frac{1}{2}N\overline{\mu}_t[i]\right\}}\right] \\
&\le \overline{\mu}_t[i] + 2N\overline{\mathbf{P}}^j\left(|S_{\overline{\mu}_t[i]} - N\overline{\mu}_t[i]| > \frac{1}{2}N\overline{\mu}_t[i]\right) \\
&\le \overline{\mu}_t[i] + 2N\exp\left\{-\frac{N^{1-2\epsilon}}{4}\right\},
\end{aligned}$$

so that

$$\begin{aligned}
\frac{\overline{\mathbf{E}}^j\left[\left(N\overline{\mu}_t[i] - S_{\overline{\mu}_t}[i]\right)\mathbb{1}_{\left\{|S_{\overline{\mu}_t[i]} - N\overline{\mu}_t[i]| \le \frac{1}{2}N\overline{\mu}_t[i]\right\}}\right]}{\overline{\mathbf{P}}^j\left(|S_{\overline{\mu}_t[i]} - N\overline{\mu}_t[i]| \le \frac{1}{2}N\overline{\mu}_t[i]\right)} &\le \frac{\overline{\mu}_t[i] + 2N\exp\left\{-\frac{1}{4}N^{1-2\epsilon}\right\}}{1 - 2N\exp\left\{-\frac{1}{4}N^{1-2\epsilon}\right\}} \\
&\le C\overline{\mu}_t[i] + \frac{C}{N^2},
\end{aligned}$$

at least for N large enough (the underlying rank upon which the preceding bound is true only depending on ϵ); by noticing that the left-hand side is upper bounded by N, we can change the constant C accordingly such that the preceding is always true (even for N small). We deduce that

$$\overline{\mathbf{E}}^j\left[-\psi\Big(N\overline{\mu}_t[i]-S_{\overline{\mu}_t[i]}\Big)\,\Big|\,|S_{\overline{\mu}_t}[i]-N\overline{\mu}_t[i]|\le\frac{1}{2}N\overline{\mu}_t[i]\right]\ge-\psi\Big(C\overline{\mu}_t[i]+\frac{C}{N^2}\Big).$$

Now, allowing the constant C to vary from one inequality to another,

$$\psi\Big(C\overline{\mu}_t[i]+\frac{C}{N^2}\Big)\le C\frac{\overline{\mu}_t[i]+N^{-2}}{(N^{-\epsilon}+\overline{\mu}_t[i])N+C\overline{\mu}_t[i]}\le\frac{C}{N}.$$

As a result (recall (6.8) for the definition of (A) and (B)),

$$\begin{aligned}(B)&=\varepsilon N\overline{\mathbf{E}}^j\left[-\psi\Big(N\overline{\mu}_t[i]-S_{\overline{\mu}_t}[i]\Big)\mathbb{1}_{\left\{|S_{\overline{\mu}_t[i]}-N\overline{\mu}_t[i]|\le\frac{1}{2}N\overline{\mu}_t[i]\right\}}\right]\\&=\varepsilon N\overline{\mathbf{E}}^j\left[-\psi\Big(N\overline{\mu}_t[i]-S_{\overline{\mu}_t}[i]\Big)\Big|\Big\{\Big|S_{\overline{\mu}_t[i]}-N\overline{\mu}_t[i]\Big|\le\frac{1}{2}N\overline{\mu}_t[i]\Big\}\right]\\&\quad\times\overline{\mathbf{P}}^j\Big(S_{\overline{\mu}_t[i]}-N\overline{\mu}_t[i]\,|\le\frac{1}{2}N\overline{\mu}_t[i]\Big)\\&\ge-\varepsilon N\frac{C}{N}=-C\varepsilon.\end{aligned}$$

The term (A) is instead

$$\begin{aligned}(A)&=\varepsilon N\mathbf{E}\left[\left(\frac{S_{\overline{\mu}_t}[i]-N\overline{\mu}_t[i]}{(N^{-\epsilon}+\overline{\mu}_t[i])N+S_{\overline{\mu}_t}[i]-N\overline{\mu}_t[i]}+\frac{N\overline{\mu}_t[i]-S_{\overline{\mu}_t}[i]}{(N^{-\epsilon}+\overline{\mu}_t[i])N+N\overline{\mu}_t[i]-S_{\overline{\mu}}[i]}\right)\mathbb{1}_{\left\{|S_{\overline{\mu}_t}[i]-N\overline{\mu}_t[i]|\le\frac{1}{2}N\overline{\mu}_t[i]\right\}}\right]\\&=\varepsilon N\mathbf{E}\left[-\frac{2(S_{\overline{\mu}_t}[i]-N\overline{\mu}_t[i])^2}{(N^{-\epsilon}+\overline{\mu}_t[i])^2N^2-(S_{\overline{\mu}_t}[i]-N\overline{\mu}_t[i])^2}\mathbb{1}_{\left\{|S_{\overline{\mu}_t}[i]-N\overline{\mu}_t[i]|\le\frac{1}{2}N\overline{\mu}_t[i]\right\}}\right]\\&\ge-\varepsilon N\mathbf{E}\left[\frac{2(S_{\overline{\mu}_t}[i]-N\overline{\mu}_t[i])^2}{\frac{3}{4}(\overline{\mu}_t[i])^2N^2}\mathbb{1}_{\left\{|S_{\overline{\mu}_t}[i]-N\overline{\mu}_t[i]|\le\frac{1}{2}N\overline{\mu}_t[i]\right\}}\right]\\&\ge-\varepsilon N\frac{8}{3}\frac{N\overline{\mu}_t[i](1-\overline{\mu}_t[i])}{N^2(\overline{\mu}_t[i])^2}\ge-\frac{8}{3}\varepsilon\frac{1}{\overline{\mu}_t[i]}\ge-\frac{8}{3}\varepsilon\frac{\mathbb{1}_{\{\overline{\mu}_t[i]\le\delta\}}}{\overline{\mu}_t[i]}-\varepsilon\frac{8}{3\delta},\end{aligned}$$

and so, combining with the lower bound for (B), we obtain from (6.6) and (6.7)

$$\mathrm{III}\ge-C\varepsilon-\frac{8}{3}\varepsilon\frac{\mathbb{1}_{\{\overline{\mu}_t[i]\le\delta\}}}{\overline{\mu}_t[i]}-\varepsilon\frac{8}{3\delta}\ge-C_0-\frac{8}{3}\varepsilon\frac{\mathbb{1}_{\{\overline{\mu}_t[i]\le\delta\}}}{\overline{\mu}_t[i]},$$

for a constant C_0 as in the statement, using $\varepsilon<1$.

6.1.5. Conclusion. Putting things together (see (6.3) and (6.4)), we obtain

$$-\overline{\lambda}_t\ge\Big(\frac{2}{5}(1-\delta)\kappa-\frac{8}{3}\varepsilon\Big)\frac{\mathbb{1}_{\{\overline{\mu}_t[i]\le\delta\}}}{\overline{\mu}_t[i]}-C_1,$$

for another C_1 as in the statement. This inequality, applied in (6.2), gives

$$\mathbb{E}\left[\exp\left\{\int_0^{\overline{\tau}_N}\Big(\frac{2}{5}(1-\delta)\kappa-\frac{8}{3}\varepsilon\Big)\frac{\mathbb{1}_{\{\overline{\mu}_t[i]\le\delta\}}}{\overline{\mu}_t[i]}dt\right\}\right]\le\frac{2}{N^{-\epsilon}+\overline{\mu}_0[i]}\exp(C_1T).$$

Therefore (4.11) follows if we choose κ such that

$$\frac{2}{5}(1-\delta)\kappa-\varepsilon\frac{8}{3}\ge\lambda,$$

and then use another value of C.

In order to prove (4.12), we exploit (6.1) to derive

$$\frac{1}{N^{-\epsilon}+\overline{\mu}_{t\wedge\overline{\tau}_N}[i]}\exp\left(\int_0^{t\wedge\overline{\tau}_N}\frac{\lambda}{\overline{\mu}_s[i]}\mathbb{1}_{\{\overline{\mu}_s[i]\le\delta\}}ds-C_1T\right)\le\exp\left(-\overline{\Lambda}_{t\wedge\overline{\tau}_N}-\int_0^{t\wedge\overline{\tau}_N}\overline{\lambda}_s ds\right).$$

Taking expectation, we conclude by recalling that the exponential on the right-hand side has expectation bounded by $1/(N^{-\epsilon}+\overline{\mu}_0[i])$. Q.E.D.

6.2. Proof of Lemma 4.1

6.2.1. First Step. We start with the first line in (4.18). By standard algebra, we get

$$\begin{aligned}\mathbf{E}\left[\left(\frac{S_\mu[i]}{N\mu[i]}\right)^\ell-1\right]&=\frac{1}{N^\ell\mu[i]^\ell}\mathbf{E}\Big[(S_\mu[i]-N\mu[i]+N\mu[i])^\ell\Big]-1\\&=\frac{1}{N^\ell\mu[i]^\ell}\sum_{k=1}^{\ell}\binom{\ell}{k}\mathbf{E}\Big[(S_\mu[i]-N\mu[i])^k\Big](N\mu[i])^{\ell-k}\\&=\binom{\ell}{2}\frac{N\mu[i](1-\mu[i])}{(N\mu[i])^2}+\sum_{k=3}^{\ell}\binom{\ell}{k}\frac{\mathbf{E}[(S_\mu[i]-N\mu[i])^k]}{(N\mu[i])^k}.\end{aligned}\tag{6.9}$$

Recall Rosenthal's inequality (see Petrov [33, theorem 2.9]) for independent integrable random variables $\{Z_k\}_{k=1}^n$ with $\mathbb{E}[Z_k]=0$, and for $p\ge 2$,

$$\mathbb{E}\left|\sum_{k=1}^n Z_k\right|^p\le c(p)\left(\sum_{k=1}^n\mathbb{E}[|Z_k|^p]+\left(\sum_{k=1}^n\mathbb{E}[|Z_k|^2]\right)^{p/2}\right),$$

where $c(p)$ is a positive constant depending only on p. Applying it to the centered sum $S_\mu[i]-N\mu[i]$, we get that, for each real $p\ge 2$,

$$\begin{aligned}&\mathbf{E}[|S_\mu[i]-N\mu[i]|^p]\\&\le C_pN[\mu[i]^p(1-\mu[i])+(1-\mu[i])^p\mu[i]]+C_pN^{p/2}\mu[i]^{p/2}(1-\mu[i])^{p/2}\\&\le C_pN\mu[i]+C_pN^{p/2}\mu[i]^{p/2}\\&\le C_pN^{p/2}\mu[i].\end{aligned}\tag{6.10}$$

And, for a constant C_p depending on p, the value may vary from line to line. Then, by combining the last two inequalities, we obtain the first line in (4.18).

We now turn to the second line in (4.18). Following the analysis of the first line, we have

$$\begin{aligned}&\mathbf{E}\left[\frac{S_\mu[j]}{N\mu[j]}\left\{\left(\frac{S_\mu[i]}{N\mu[i]}\right)^\ell-1\right\}\right]\\&=\frac{1}{N^{\ell+1}\mu[i]^\ell\mu[j]}\sum_{k=1}^{\ell}\binom{\ell}{k}\mathbf{E}\Big[\Big(S_\mu[j]-N\mu[j]\Big)(S_\mu[i]-N\mu[i])^k\Big](N\mu[i])^{\ell-k}\\&\quad+\frac{1}{N^\ell\mu[i]^\ell}\sum_{k=2}^{\ell}\binom{\ell}{k}\mathbf{E}\Big[(S_\mu[i]-N\mu[i])^k\Big](N\mu[i])^{\ell-k}.\end{aligned}$$

When $k=1$ in the first sum, the expectation therein is given by the correlation matrix of the multinomial distribution, namely, $\mathbf{E}[(S_\mu[j]-N\mu[j])(S_\mu[i]-N\mu[i])]=N\mu[i]\delta_{i,j}-N\mu[i]\mu[j]$; when $k=2$ in the second sum, the expectation therein is given by $\mathbf{E}[(S_\mu[i]-N\mu[i])^2]=N\mu[i](1-\mu[i])$. As for the other terms (whatever the sum), we may just invoke the Cauchy–Schwarz inequality and then (6.10) in order to bound the corresponding expectation. As a result, we can find a constant C only depending on ℓ such that

$$\begin{aligned}
&\left|\mathbf{E}\left[\frac{S_\mu[j]}{N\mu[j]}\left\{\left(\frac{S_\mu[i]}{N\mu[i]}\right)^\ell - 1\right\}\right]\right| \\
&\le \frac{\ell}{N\mu[j]} + \frac{\ell(\ell-1)}{2N\mu[i]} + \frac{C}{N^{\ell+1}\mu[i]^\ell\mu[j]}\sum_{k=2}^{\ell} N^{(k+1)/2}(\mu[i])^{1/2}(\mu[j])^{1/2}(N\mu[i])^{\ell-k} \\
&\quad + \frac{C}{N^\ell\mu[i]^\ell}\sum_{k=3}^{\ell} N^{k/2}\mu[i](N\mu[i])^{\ell-k} \\
&\le \frac{\ell(\ell+1)}{2N\min_{e\in[[d]]}\mu[e]} + C\sum_{k=2}^{\ell}\frac{1}{N^{(k+1)/2}(\mu[i])^{k-1/2}(\mu[j])^{1/2}} + C\sum_{k=3}^{\ell}\frac{1}{N^{k/2}(N\mu[i])^{k-1}} \\
&\le \frac{\ell(\ell+1)}{2N\min_{e\in[[d]]}\mu[e]} + C\sum_{k=3}^{\ell+1}\frac{1}{N^{k/2}\min_{e\in[[d]]}\mu[e]^{k-1}},
\end{aligned}$$

which fits the announced inequality.

6.2.2. Second Step.

We now prove (4.19). Following (6.9), we have

$$\begin{aligned}
\left|\left(\frac{S_\mu[i]}{N\mu[i]}\right)^\ell - 1\right| &= \frac{1}{N^\ell\mu[i]^\ell}\left|(S_\mu[i] - N\mu[i] + N\mu[i])^\ell - N^\ell\mu[i]^\ell\right| \\
&\le C\sum_{k=1}^{\ell}|S_\mu[i] - N\mu[i]|^k(N\mu[i])^{-k}.
\end{aligned}\tag{6.11}$$

By the third line in (6.10), we get

$$\begin{aligned}
\mathbf{E}\left[\left|\left(\frac{S_\mu[i]}{N\mu[i]}\right)^\ell - 1\right|^p\right]^{1/p} &\le C\sum_{k=1}^{\ell}(N\mu[i] + N^{kp/2}\mu[i]^{kp/2})^{1/p}(N\mu[i])^{-k} \\
&\le C\left[(N\mu[i])^{1/p}\sum_{k=1}^{\ell}(N\mu[i])^{-k} + \sum_{k=1}^{\ell}(N\mu[i])^{-k/2}\right],
\end{aligned}$$

which completes the proof because $p \ge 2$.

It now remains to address Inequality (4.20). With $\eta > 0$ as in the statement and with C as in (6.11), choose $\eta' = \min(1, \eta/(\ell C))$ and deduce that $|(S_\mu[i]/N\mu[i])^\ell - 1| \le \eta$, on the event $E = \{|S_\mu[i]/N\mu[i] - 1| \le \eta'\}$. Now, Hoeffding's inequality says that $\mathbf{P}(E^\complement) \le 2\exp(-2N\mu[i]^2(\eta')^2)$, which completes the proof. □

6.3. Proof of Proposition 4.1

For a given $\lambda \ge 1$, consider the exponential

$$\mathcal{E}_t := \exp\left\{\lambda\int_0^t \sum_{i\in[[d]]}\frac{\mathbb{1}_{\{\overline{\mu}_s[i]\neq 0\}}}{\overline{\mu}_s[i]}ds\right\}, \quad t\in[0,T].$$

Applying Itô's formula to $(\mathcal{E}^{-1}_{t\wedge\overline{\tau}_N}\sum_{l\in[[N]]}|\overline{Y}^l_{t\wedge\overline{\tau}_N}|^{2\ell})_{0\le t\le T}$ (recall, for instance, (4.7)) and taking expectation, we obtain

$$\begin{aligned}
&\frac{d}{dt}\mathbb{E}\left[\mathcal{E}^{-1}_{t\wedge\overline{\tau}_N}\sum_{l\in[[N]]}|\overline{Y}^l_{t\wedge\overline{\tau}_N}|^{2\ell}\right] + \mathbb{E}\left[\mathbb{1}_{\{t<\overline{\tau}_N\}}\mathcal{E}^{-1}_t\sum_{i\in[[d]]}\frac{\lambda}{\overline{\mu}_t[i]}\sum_{l\in[[N]]}|\overline{Y}^l_t|^{2\ell}\right] \\
&\le \varepsilon N\mathbb{E}\left[\mathbb{1}_{\{t<\overline{\tau}_N\}}\mathcal{E}^{-1}_t\sum_{l\in[[N]]}|\overline{Y}^l_t|^{2\ell}\sum_{i\in[[d]]}\mathbb{1}_{\{\overline{X}^l_t=i\}}\times\sum_{j\in[[d]]}\mathbb{1}_{\{\overline{X}^n_t=j\}}\left|\mathbf{E}\left\{\left(\frac{S_{\overline{\mu}_t}[j]}{N\overline{\mu}_t[j]}\right)^\iota\left(\left(\frac{S_{\overline{\mu}_t}[i]}{N\overline{\mu}_t[i]}\right)^{2\ell} - 1\right)\right\}\right|\right].
\end{aligned}\tag{6.12}$$

We first handle the expectation $\mathbf{E}$ in the right-hand side. The key point is to notice that it can be estimated by the first line in (4.18) when $\iota = 0$ and by the second line in (4.18) when $\iota = 1$.

In any case, we have

$$\left|\mathbf{E}\left\{\left(\frac{S_{\overline{\mu}_t}[j]}{N\overline{\mu}_t[j]}\right)^{\iota}\left(\left(\frac{S_{\overline{\mu}_t}[i]}{N\overline{\mu}_t[i]}\right)^{2\ell}-1\right)\right\}\right| \le \frac{\ell(2\ell+1)}{N\min_{e\in[\![d]\!]}\mu[e]} + C_\ell \sum_{k=3}^{2\ell+1} \frac{1}{N^{k/2}\min_{e\in[\![d]\!]}\overline{\mu}_t[e]^{k-1}}$$
$$\le \sum_{e\in[\![d]\!]} \frac{\ell(2\ell+1)}{N\mu[e]} + C_\ell \sum_{k=3}^{2\ell+1} \frac{1}{N^{k/2}\min_{e\in[\![d]\!]}\overline{\mu}_t[e]^{k-1}},$$

for a constant C_ℓ only depending on ℓ (and the value of which is allowed to vary from line to line). Integrating (6.12) from zero to t, we have (allowing C_ℓ to depend on d)

$$\mathbb{E}\left[\mathcal{E}^{-1}_{t\wedge\overline{\tau}_N}\sum_{l\in[\![N]\!]}|\overline{Y}^l_{t\wedge\overline{\tau}_N}|^{2\ell}\right] + \lambda\mathbb{E}\left[\int_0^{t\wedge\overline{\tau}_N}\mathcal{E}^{-1}_s\sum_{i\in[\![d]\!]}\frac{1}{\overline{\mu}_s[i]}\sum_{l\in[\![N]\!]}|\overline{Y}^l_s|^{2\ell}ds\right]$$
$$\le \sum_{l\in[\![N]\!]}|y^l_0|^{2\ell} + \varepsilon\ell(2\ell+1)\mathbb{E}\left[\int_0^{t\wedge\overline{\tau}_N}\mathcal{E}^{-1}_s\sum_{i\in[\![d]\!]}\frac{1}{\overline{\mu}_s[i]}\sum_{l\in[\![N]\!]}|\overline{Y}^l_s|^{2\ell}ds\right]$$
$$+ \varepsilon C_\ell\mathbb{E}\left[\int_0^{t\wedge\overline{\tau}_N}\mathcal{E}^{-1}_s\sum_{k=3}^{2\ell+1}\frac{1}{N^{k/2}(\min_{e\in[\![d]\!]}\overline{\mu}_s[e])^{k-1}}\sum_{l\in[\![N]\!]}|\overline{Y}^l_s|^{2\ell}ds\right],$$

which implies, if $\lambda \ge \varepsilon\ell(2\ell+1)$,

$$\mathbb{E}\left[\mathcal{E}^{-1}_{t\wedge\overline{\tau}_N}\sum_{l\in[\![N]\!]}|\overline{Y}^l_{t\wedge\overline{\tau}_N}|^{2\ell}\right] \le \sum_{l\in[\![N]\!]}|y^l_0|^{2\ell} + \varepsilon C_\ell\sum_{k=3}^{2\ell+1}\frac{N^{\epsilon(k-1)}}{N^{k/2}}\mathbb{E}\left[\int_0^t\mathcal{E}^{-1}_{s\wedge\overline{\tau}_N}\sum_{l\in[\![N]\!]}|\overline{Y}^l_{s\wedge\overline{\tau}_N}|^{2\ell}ds\right].$$

Hence, if $\epsilon < 1/4$, Gronwall's inequality yields

$$\mathbb{E}\left[\mathcal{E}^{-1}_{t\wedge\overline{\tau}_N}\sum_{l\in[\![N]\!]}|\overline{Y}^l_{t\wedge\overline{\tau}_N}|^{2\ell}\right] \le C\sum_{l\in[\![N]\!]}|y^l_0|^{2\ell},$$

for a constant C as in the statement and whose value may vary from line to line. Now, by the Cauchy–Schwarz inequality,

$$\mathbb{E}\left[\frac{1}{N}\sum_{l\in[\![N]\!]}|\overline{Y}^l_{t\wedge\overline{\tau}_N}|^{\ell}\right] \le \mathbb{E}\left[\mathcal{E}^{-1}_{t\wedge\overline{\tau}_N}\left(\frac{1}{N}\sum_{l\in[\![N]\!]}|\overline{Y}^l_{t\wedge\overline{\tau}_N}|^{\ell}\right)^2\right]^{1/2}\mathbb{E}[\mathcal{E}_{t\wedge\overline{\tau}_N}]^{1/2}$$
$$\le \mathbb{E}\left[\mathcal{E}^{-1}_{t\wedge\overline{\tau}_N}\left(\frac{1}{N}\sum_{l\in[\![N]\!]}|\overline{Y}^l_{t\wedge\overline{\tau}_N}|^{2\ell}\right)\right]^{1/2}\mathbb{E}[\mathcal{E}_{t\wedge\overline{\tau}_N}]^{1/2} \le C\left(\frac{1}{N}\sum_{l\in[\![N]\!]}|y^l_0|^{2\ell}\right)^{1/2}\mathbb{E}[\mathcal{E}_{t\wedge\overline{\tau}_N}]^{1/2}.$$

We invoke (4.13) (assuming throughout that κ is large enough) in order to bound the last term on the right-hand side. We easily get (4.15). Combining (4.14) with (4.15), we get

$$\mathbb{E}\left[\prod_{i\in[\![d]\!]}(N^{-\epsilon}+\overline{\mu}_{t\wedge\overline{\tau}_N}[i])^{-1/d} + \frac{1}{N}\sum_{l\in[\![N]\!]}|\overline{Y}^l_{t\wedge\overline{\tau}_N}|^{\ell}\right]$$
$$\le C\prod_{i\in[\![d]\!]}(N^{-\epsilon}+\overline{\mu}_0[i])^{-1/d} + C\left(\frac{1}{N}\sum_{l\in[\![N]\!]}|y^l_0|^{2\ell}\right)^{1/2}\prod_{i\in[\![d]\!]}(N^{-\epsilon}+\overline{\mu}_0[i])^{-1/(2d)}. \tag{6.13}$$

Choosing $t = T$ and invoking the definition of $\overline{\tau}_N$ in (4.10) (together with the right continuity of the trajectories), we get

$$\mathbb{E}\left[\left(\prod_{i\in[\![d]\!]}(N^{-\epsilon}+\overline{\mu}_{\overline{\tau}_N}[i])^{-1/d} + \frac{1}{N}\sum_{l\in[\![N]\!]}|\overline{Y}^l_{\overline{\tau}_N}|^{\ell}\right)\mathbb{1}_{\{\overline{\tau}_N<T\}}\right]$$
$$\ge \mathbb{P}(\overline{\tau}_N < T)\min\left\{\frac{1}{2}N^{\epsilon/d}, \frac{1}{2^\ell}N^{\ell-1-\ell\epsilon}\right\},$$

where we use the fact that, for any $i\in[\![d]\!]$, $N^{-\epsilon}+\overline{\mu}_{\overline{\tau}_N}[i] \le 2$. Notice that, if $\ell \ge 3$ and $\epsilon < 1/4$, $\ell - 1 - \ell\epsilon \ge 3/2 - 1 \ge 1/2 \ge \epsilon$, we obtain (4.16).

6.4. Proof of Proposition 4.2

6.4.1. First Step. We start with a similar computation to (6.12).

$$
\begin{aligned}
&\frac{d}{dt}\mathbb{E}\left[\mathcal{E}^{-1}_{t\wedge\overline{\tau}_N}\left(\sum_{l\in[\![N]\!]}|\overline{Y}^l_{t\wedge\overline{\tau}_N}|^\ell\right)^{-1}\right]+\mathbb{E}\left[\mathbb{1}_{\{t<\overline{\tau}_N\}}\mathcal{E}^{-1}_t\sum_{i\in[\![d]\!]}\frac{\lambda}{\overline{\mu}_t[i]}\left(\sum_{l\in[\![N]\!]}|\overline{Y}^l_t|^\ell\right)^{-1}\right]\\
&\quad=\varepsilon N\sum_{j\in[\![d]\!]}\mathbb{E}\left[\mathbb{1}_{\{t<\overline{\tau}_N\}}\mathcal{E}^{-1}_t\mathbb{1}_{\{\overline{X}^n_t=j\}}\mathbf{E}\left\{\left(\frac{S_{\overline{\mu}_t}[j]}{N\overline{\mu}_t[j]}\right)^\iota\left[\left(\sum_{i\in[\![d]\!]}\sum_{l\in[\![N]\!]}|\overline{Y}^l_t|^\ell\mathbb{1}_{\{\overline{X}^l_t=i\}}\left(\frac{S_{\overline{\mu}_t}[i]}{N\overline{\mu}_t[i]}\right)^\ell\right)^{-1}-\left(\sum_{l\in[\![N]\!]}|\overline{Y}^l_t|^\ell\right)^{-1}\right]\right\}\right]\\
&\quad=-\varepsilon N\sum_{j\in[\![d]\!]}\mathbb{E}\left[\mathbb{1}_{\{t<\overline{\tau}_N\}}\mathcal{E}^{-1}_t\mathbb{1}_{\{\overline{X}^n_t=j\}}\mathbf{E}\left\{\left(\frac{S_{\overline{\mu}_t}[j]}{N\overline{\mu}_t[j]}\right)^\iota\left(\sum_{l\in[\![N]\!]}|\overline{Y}^l_t|^\ell+D\overline{Y}_t\right)^{-1}\left(\sum_{l\in[\![N]\!]}|\overline{Y}^l_t|^\ell\right)^{-1}D\overline{Y}_t\right\}\right],
\end{aligned}\tag{6.14}
$$

with

$$
D\overline{Y}_t:=\frac{1}{N}\sum_{i\in[\![d]\!]}\sum_{l\in[\![N]\!]}|\overline{Y}^l_t|^\ell\mathbb{1}_{\{\overline{X}^l_t=i\}}\left[\left(\frac{S_{\overline{\mu}_t}[i]}{N\overline{\mu}_t[i]}\right)^\ell-1\right].
$$

Then,

$$
\begin{aligned}
&\mathbf{E}\left\{\left(\frac{S_{\overline{\mu}_t}[j]}{N\overline{\mu}_t[j]}\right)^\iota\left(\sum_{l\in[\![N]\!]}|\overline{Y}^l_t|^\ell+ND\overline{Y}_t\right)^{-1}\left(\sum_{l\in[\![N]\!]}|\overline{Y}^l_t|^\ell\right)^{-1}ND\overline{Y}_t\right\}\\
&=\mathbf{E}\left\{\left(\frac{S_{\overline{\mu}_t}[j]}{N\overline{\mu}_t[j]}\right)^\iota\left(\sum_{l\in[\![N]\!]}|\overline{Y}^l_t|^\ell\right)^{-2}ND\overline{Y}_t\right\}-\mathbf{E}\left\{\left(\frac{S_{\overline{\mu}_t}[j]}{N\overline{\mu}_t[j]}\right)^\iota\left(\sum_{l\in[\![N]\!]}|\overline{Y}^l_t|^\ell+ND\overline{Y}_t\right)^{-1}\left(\sum_{l\in[\![N]\!]}|\overline{Y}^l_t|^\ell\right)^{-2}(ND\overline{Y}_t)^2\right\}.
\end{aligned}\tag{6.15}
$$

By the second line in (4.18) and because $\overline{\mu}_t[i]\geq N^{-\varepsilon}$ for any $i\in[\![d]\!]$ and $t<\overline{\tau}_N$, we notice that, for any $i\in[\![d]\!]$,

$$
\left|\mathbf{E}\left\{\left(\frac{S_{\overline{\mu}_t}[j]}{N\overline{\mu}_t[j]}\right)^\iota\left(\sum_{l\in[\![N]\!]}|\overline{Y}^l_t|^\ell\right)^{-2}ND\overline{Y}_t\right\}\right|=\left(\sum_{l\in[\![N]\!]}|\overline{Y}^l_t|^\ell\right)^{-2}\left|\mathbf{E}\left\{\left(\frac{S_{\overline{\mu}_t}[j]}{N\overline{\mu}_t[j]}\right)^\iota ND\overline{Y}_t\right\}\right|\leq C\left(\sum_{l\in[\![N]\!]}|\overline{Y}^l_t|^\ell\right)^{-1}\left(\frac{1}{N\min_{e\in[\![d]\!]}\overline{\mu}_t[e]}+\frac{1}{N^{3/2-2\epsilon}}\right),\tag{6.16}
$$

for a constant C only depending on ℓ and the value of which may vary from line to line.

6.4.2. Second Step. We now split the expectation $\mathbf{E}\{\cdots\}$ on the last line of (6.15) according to the two events $E:=\cap_{i\in[\![d]\!]}\{|S_{\overline{\mu}_t[i]}/N-\overline{\mu}_t[i]|\leq\eta\overline{\mu}_t[i]\}$ and $E^\complement=\cup_{i\in[\![d]\!]}\{|S_{\overline{\mu}_t[i]}/N-\overline{\mu}_t[i]|>\eta\overline{\mu}_t[i]\}$ for some parameter $\eta>0$ whose value is chosen right after.

On the event E, we have $|S_{\overline{\mu}_t[i]}/(N\overline{\mu}_t[i])-1|\leq\eta$, and we can choose η small enough such that $1/2\leq(1-\eta)^\ell\leq 1-\eta\leq 1+\eta\leq(1+\eta)^\ell\leq 3/2$, from which we deduce that, on E,

$$
ND\overline{Y}_t\geq-\frac{1}{2}\sum_{l\in[\![N]\!]}|\overline{Y}^l_t|^\ell.
$$

So, together with Jensen's inequality,

$$
\begin{aligned}
&\mathbf{E}\left\{\left(\frac{S_{\overline{\mu}_t}[j]}{N\overline{\mu}_t[j]}\right)^\iota\left(\sum_{l\in[\![N]\!]}|\overline{Y}^l_t|^\ell+ND\overline{Y}_t\right)^{-1}\left(\sum_{l\in[\![N]\!]}|\overline{Y}^l_t|^\ell\right)^{-2}(ND\overline{Y}_t)^2\mathbb{1}_E\right\}\\
&\leq C\mathbf{E}\left\{\left(\sum_{l\in[\![N]\!]}|\overline{Y}^l_t|^\ell\right)^{-3}(ND\overline{Y}_t)^2\right\}\\
&=C\left(\sum_{l\in[\![N]\!]}|\overline{Y}^l_t|^\ell\right)^{-1}\mathbf{E}\left[\left(\sum_{l\in[\![N]\!]}\frac{|\overline{Y}^l_t|^\ell}{\sum_{k\in[\![N]\!]}|\overline{Y}^k_t|^\ell}\sum_{i\in[\![d]\!]}\mathbb{1}_{\{\overline{X}^l_t=i\}}\left[\left(\frac{S_{\overline{\mu}_t}[i]}{N\overline{\mu}_t[i]}\right)^\ell-1\right]\right)^2\right]\\
&\leq C\left(\sum_{l\in[\![N]\!]}|\overline{Y}^l_t|^\ell\right)^{-1}\sum_{l\in[\![N]\!]}\frac{|\overline{Y}^l_t|^\ell}{\sum_{k\in[\![N]\!]}|\overline{Y}^k_t|^\ell}\sum_{i\in[\![d]\!]}\mathbb{1}_{\{\overline{X}^l_t=i\}}\mathbf{E}\left[\left(\left(\frac{S_{\overline{\mu}_t}[i]}{N\overline{\mu}_t[i]}\right)^\ell-1\right)^2\right]\leq C\left(\sum_{l\in[\![N]\!]}|\overline{Y}^l_t|^\ell\right)^{-1}\frac{1}{N\min_{e\in[\![d]\!]}\overline{\mu}_t[e]},
\end{aligned}\tag{6.17}
$$

the proof of the last line following from (4.19) with $p=2$ and from the fact that $N\min_{e\in[\![d]\!]}\overline{\mu}_t[e]\geq 1$ for $t<\overline{\tau}_N$.

6.4.3. Third Step. We now proceed on the complementary event $E^{\complement}$. We observe that

$$\frac{1}{N}\sum_{l\in[[N]]}|\overline{Y}_t^l|^\ell \geq \left(\frac{1}{N}\sum_{l\in[[N]]}\overline{Y}_t^l\right)^\ell = 1,$$

and the same lower bound holds true for $\frac{1}{N}\sum_{l\in[[N]]}|\overline{Y}_t^l|^\ell + D\overline{Y}_t$ because the global weight is preserved by the dynamics; see Section 2.2.1. Then,

$$\begin{aligned}\mathbf{E}&\left\{\left(\frac{S_{\overline{\mu}_t}[j]}{N\overline{\mu}_t[j]}\right)^\iota\left(\sum_{l\in[[N]]}|\overline{Y}_t^l|^\ell + ND\overline{Y}_t\right)^{-1}\left(\sum_{l\in[[N]]}|\overline{Y}_t^l|^\ell\right)^{-2}(ND\overline{Y}_t)^2\mathbb{1}_{E^{\complement}}\right\}\\&\leq CN^{-1}\mathbf{E}\left\{\left(\frac{S_{\overline{\mu}_t}[j]}{N\overline{\mu}_t[j]}\right)^\iota\left(\sum_{l\in[[N]]}|\overline{Y}_t^l|^\ell\right)^{-2}(ND\overline{Y}_t)^2\mathbb{1}_{E^{\complement}}\right\}.\end{aligned}$$

We then use the fact that $t<\overline{\tau}_N$, which implies that $1/\min_{j\in[[d]]}\overline{\mu}_t[j]\leq N^\epsilon$ and $\max_{l\in[[N]]}\overline{Y}_t^l\leq N^{1-\epsilon}$. This implies that $|ND\overline{Y}_t|\leq(1+N^{\ell\epsilon})\sum_{l\in[[N]]}|\overline{Y}_t^l|^\ell$, from which we get

$$\mathbf{E}\left\{\left(\frac{S_{\overline{\mu}_t}[j]}{N\overline{\mu}_t[j]}\right)^\iota\left(\sum_{l\in[[N]]}|\overline{Y}_t^l|^\ell + ND\overline{Y}_t\right)^{-1}\left(\sum_{l\in[[N]]}|\overline{Y}_t^l|^\ell\right)^{-2}(ND\overline{Y}_t)^2\mathbb{1}_{E^{\complement}}\right\}\leq CN^{(2\ell+1)\epsilon-1}\mathbf{P}\left(E^{\complement}\right). \tag{6.18}$$

By Hoeffding's inequality as in the proof of Lemma 4.1, we can find a constant $c>0$ (depending on η) so that $\mathbf{P}(E^{\complement})\leq C\exp(-2N(\min_{i\in[[d]]}\overline{\mu}_t[i])^2)\leq C\exp(-2N^{1-2\epsilon})$.

6.4.4. Conclusion. By combining (6.14) and (6.16)–(6.18) (multiplying the former by N), we end up with

$$\begin{aligned}\frac{d}{dt}\mathbb{E}&\left[\mathcal{E}^{-1}_{t\wedge\overline{\tau}_N}\left(\frac{1}{N}\sum_{l\in[[N]]}|\overline{Y}^l_{t\wedge\overline{\tau}_N}|^\ell\right)^{-1}\right]+\lambda\mathbb{E}\left[\mathbb{1}_{\{t<\overline{\tau}_N\}}\mathcal{E}_t^{-1}\sum_{i\in[[d]]}\frac{1}{\overline{\mu}_t[i]}\left(\frac{1}{N}\sum_{l\in[[N]]}|\overline{Y}_t^l|^\ell\right)^{-1}\right]\\&\leq C\varepsilon\mathbb{E}\left[\mathbb{1}_{\{t<\overline{\tau}_N\}}\mathcal{E}_t^{-1}\left(\frac{1}{N}\sum_{l\in[[N]]}|\overline{Y}_t^l|^\ell\right)^{-1}\left(\frac{1}{\min_{e\in[[d]]}\overline{\mu}_t[e]}+\frac{1}{N^{(1-\epsilon)/2}}\right)\right]+C\exp(-N^{1-2\epsilon}).\end{aligned}$$

Notice that, because $\epsilon<1/4$, we have $1/N^{(1-\epsilon)/2}\leq 1/N^{1/3}\leq 1/N^\epsilon\leq 1/\min_{e\in[[d]]}\overline{\mu}_t[e]$ (recalling that t is here less than $\overline{\tau}_N$). Hence, by choosing λ large enough with respect to C (which is indeed possible because C only depends on m), we deduce that

$$\mathbb{E}\left[\mathcal{E}^{-1}_{t\wedge\overline{\tau}_N}\left(\frac{1}{N}\sum_{l\in[[N]]}|\overline{Y}^l_{t\wedge\overline{\tau}_N}|^\ell\right)^{-1}\right]\leq\left(\frac{1}{N}\sum_{l\in[[N]]}|y_0^l|^\ell\right)^{-1}+C\exp(-cN^{1-2\epsilon}).$$

It now remains to insert (4.11)–(4.13) (κ being implicitly taken large enough), from which we get

$$\begin{aligned}&\mathbb{E}\left[\left(\frac{1}{N}\sum_{l\in[[N]]}|\overline{Y}^l_{t\wedge\overline{\tau}_N}|^\ell\right)^{-1}\right]\\&\leq\mathbb{E}\left[\mathcal{E}^{-1}_{t\wedge\overline{\tau}_N}\left(\frac{1}{N}\sum_{l\in[[N]]}|\overline{Y}^l_{t\wedge\overline{\tau}_N}|^\ell\right)^{-2}\right]^{1/2}\mathbb{E}[\mathcal{E}_{t\wedge\overline{\tau}_N}]^{1/2}\\&\leq\mathbb{E}\left[\mathcal{E}^{-1}_{t\wedge\overline{\tau}_N}\left(\frac{1}{N}\sum_{l\in[[N]]}|\overline{Y}^l_{t\wedge\overline{\tau}_N}|^\ell\right)^{-1}\right]^{1/2}\mathbb{E}[\mathcal{E}_t]^{1/2}\\&\leq C\left[\left(\frac{1}{N}\sum_{l\in[[N]]}|y_0^l|^\ell\right)^{-1/2}+\exp(-cN^{1-2\epsilon})\right]\prod_{i\in[[d]]}(N^{-\epsilon}+\overline{\mu}_0[i])^{-1/(2d)},\end{aligned}$$

where, in the second line, we used the (already proved) fact that $N^{-1}\sum_{l\in[[N]]}|\overline{Y}_t^l|^\ell\geq 1$. Q.E.D.

7. Proofs of the Estimates Connecting the Nash System with the Master Equation

This section is devoted to the proofs of the various lemmas that enter the demonstration of Proposition 4.3. We recall that, U being defined as an element of $\mathcal{C}^{1+\gamma'/2,2+\gamma'}$, we just know that $\sqrt{p^jp^k}\eth^2_{i,j}U^i$ is bounded and γ'-Hölder continuous in space (for the Wright–Fisher distance) for each $i,j,k \in [\![d]\!]$. By the way, we recall that γ'-Hölder continuity in space for the Wright–Fisher distance implies $\gamma'/2$-Hölder continuity in space for the standard Euclidean distance. We use the latter property quite often in the section. We refer if needed to the monograph Epstein and Mazzeo [20] for a complete review on all these facts

7.1. Proof of Lemma 4.2

Throughout the proof, we fix $t \in [0,T]$ and $(\boldsymbol{x},\boldsymbol{y}) \in \mathcal{T}_N$. This permits us to let $\mu = \mu^N_{\boldsymbol{x},\boldsymbol{y}}$, $\mu^j = \mu^N_{\boldsymbol{x},\boldsymbol{y}}[j]$, and $S^j = S_\mu[j]$. Also, fixing $l \in [\![N]\!]$, we may denote x^l by i, namely, $i := x^l$. Then, notice that

$$u^{N,l}\left(t,\boldsymbol{x},y^1\frac{S_{\mu^N_{\boldsymbol{x},\boldsymbol{y}}}[x^1]}{N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^1]},\dots,y^N\frac{S_{\mu^N_{\boldsymbol{x},\boldsymbol{y}}}[x^N]}{N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^N]}\right) = y^l\frac{S^i}{N\mu^i}U^i\left(t,\frac{S}{N}\right),$$

where, on the right-hand side, we identify the probability measure $\mu^N_{\boldsymbol{x},\boldsymbol{z}}$ for $\boldsymbol{z}$ being given by $z^l = y^l S_{\mu^N_{\boldsymbol{x},\boldsymbol{y}}}[x^l]/(N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^l])$ with S/N, which is made licit by Identity (2.7) (which guarantees that $\mu^N_{\boldsymbol{x},\boldsymbol{z}}$ is indeed a probability measure). We expand by Taylor's formula in the form

$$U^i\left(t,\frac{S}{N}\right) = U^i(t,\mu) + \sum_{j\in[\![d]\!]}\mathfrak{d}_j U^i(t,\mu)\left(\frac{S^j}{N}-\mu^j\right) + \sum_{j,k\in[\![d]\!]}\left(\frac{S^j}{N}-\mu^j\right)\left(\frac{S^k}{N}-\mu^k\right)\int_0^1(1-r)\mathfrak{d}^2_{j,k}U^i\left(t,\mu+r\left(\frac{S}{N}-\mu\right)\right)dr,$$

and then, adding and subtracting $\mathfrak{d}^2_{j,k}U^i(t,\mu)$ inside the integral, we obtain

$$\begin{aligned}
&\frac{1}{y^l}N\mathbf{E}\left[u^{N,l}\left(t,\boldsymbol{x},y^1\frac{S_{\mu^N_{\boldsymbol{x},\boldsymbol{y}}}[x^1]}{N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^1]},\dots,y^N\frac{S_{\mu^N_{\boldsymbol{x},\boldsymbol{y}}}[x^N]}{N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^N]}\right) - u^{N,l}(t,\boldsymbol{x},\boldsymbol{y})\right]\\
&= N\mathbf{E}\left[\frac{S^i}{N\mu^i}\left(U^i\left(t,\frac{S}{N}\right)-U^i(\mu)\right)\right]\\
&= N\mathbf{E}\left[\frac{S^i}{N\mu^i}\sum_{j\in[\![d]\!]}\mathfrak{d}_jU^i(t,\mu)\left(\frac{S^j}{N}-\mu^j\right)\right]\\
&\quad+\frac{1}{2}N\mathbf{E}\left[\frac{S^i}{N\mu^i}\sum_{j,k\in[\![d]\!]}\left(\frac{S^j}{N}-\mu^j\right)\left(\frac{S^k}{N}-\mu^k\right)\mathfrak{d}^2_{j,k}U^i(t,\mu)\right]\\
&\quad+N\mathbf{E}\left[\frac{S^i}{N\mu^i}\sum_{j,k\in[\![d]\!]}\left\{\left(\frac{S^j}{N}-\mu^j\right)\left(\frac{S^k}{N}-\mu^k\right)\int_0^1(1-r)\left[\mathfrak{d}^2_{j,k}U^i\left(t,\mu+r\left(\frac{S}{N}-\mu\right)\right)-\mathfrak{d}^2_{j,k}U^i(t,\mu)\right]dr\right\}\right]\\
&=: \mathrm{I}+\mathrm{II}+R^{N,l}.
\end{aligned}$$

The first term becomes

$$\begin{aligned}
\mathrm{I} &= N\sum_{j\in[\![d]\!]}\mathbf{E}\left[\frac{S^i-N\mu^i}{N\mu^i}\mathfrak{d}_jU^i(t,\mu)\left(\frac{S^j}{N}-\mu^j\right)+\mathfrak{d}_jU^i(t,\mu)\left(\frac{S^j}{N}-\mu^j\right)\right]\\
&= N\sum_{j\in[\![d]\!]}\frac{\mu^i\delta_{j,i}-\mu^i\mu^j}{N\mu^i}\mathfrak{d}_jU^i(t,\mu)+0\\
&= \sum_{j\in[\![d]\!]}(\delta_{j,i}-\mu^j)\mathfrak{d}_jU^i(t,\mu),
\end{aligned}$$

whereas the second term is

$$\begin{aligned}
\mathrm{II} &= \frac{N}{2}\sum_{j,k\in[\![d]\!]}\mathfrak{d}^2_{j,k}U^i(t,\mu)\mathbf{E}\left[\left(\frac{S^j}{N}-\mu^j\right)\left(\frac{S^k}{N}-\mu^k\right)\right]\\
&\quad+\frac{N}{2N\mu^i}\sum_{j,k\in[\![d]\!]}\mathfrak{d}^2_{j,k}U^i(t,\mu)\mathbf{E}\left[(S^i-N\mu^i)\left(\frac{S^j}{N}-\mu^j\right)\left(\frac{S^k}{N}-\mu^k\right)\right]\\
&= \frac{1}{2}\sum_{j,k\in[\![d]\!]}(\mu^j\delta_{j,k}-\mu^j\mu^k)\mathfrak{d}^2_{j,k}U^i(t,\mu)+O\left(\frac{1}{N}\right),
\end{aligned}$$

where in the latter term we use the property of the multinomial distribution

$$\mathbf{E}\Big[(S_i - N\mu_i)(S^j - N\mu^j)\Big(S^k - N\mu^k\Big)\Big] = \begin{cases} N(2(\mu^i)^3 - 3(\mu^i)^2 + \mu^i) & \text{if } i = j = k \\ N\mu^i\mu^j(2\mu^i - 1) & \text{if } i = k \neq j \\ 2N\mu^i\mu^j\mu^k & \text{if } i \neq j \neq k, \end{cases}$$

which is of order N.

It remains to estimate the rest $R^{N,l}$, which is of the form $R^{N,l} = \sum_{j,k \in [\![d]\!]} R_{j,k}$ (we feel better removing the superscripts N, l in the notation). Thus, in the following, we fix j and k and we estimate $|R_{j,k}|$. We use several times Rosenthal's inequality; see (6.10): for any real $p \geq 2$ and any $i \in [\![d]\!]$, it yields

$$\mathbf{E}[|S^i - N\mu^i|^p] \leq N^{p/2}\mu^i. \tag{7.1}$$

In order to use the properties of $[\mathcal{C}_{WF}^{1+\gamma'/2, 2+\gamma'}([0,T] \times \mathcal{S}_{d-1})]^d$, we multiply and divide the integrand in $R_{j,k}$ by

$$Q_{j,k}(r) := \sqrt{(\mu^j + r(S^j/N - \mu^j))(\mu^k + r(S^k/N - \mu^k))},$$

which is not zero because $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{T}_N$ so that we obtain

$$\begin{aligned} R_{j,k} = N\mathbf{E}\Bigg[& \frac{S^i}{N\mu^i}\left(\frac{S^j}{N} - \mu^j\right)\left(\frac{S^k}{N} - \mu^k\right)\int_0^1 \frac{(1-r)}{Q_{j,k}(r)}\left(Q_{j,k}(r)\mathfrak{d}^2_{j,k}U^i\left(t, \mu + r\left(\frac{S}{N} - \mu\right)\right) - \sqrt{\mu^j\mu^k}\,\mathfrak{d}^2_{j,k}U^i(t,\mu)\right. \\ & \left. -(Q_{j,k}(r) - \sqrt{\mu^j\mu^k})\mathfrak{d}^2_{j,k}U^i(t,\mu)\right)dr\Bigg] \\ =: {} & R^1_{j,k} + R^2_{j,k}. \end{aligned}$$

The denominator is bounded by $Q_{j,k}(r) \geq (1-r)\sqrt{\mu^j\mu^k}$, and thus, applying the Hölder and Rosenthal inequalities, we get

$$\begin{aligned} |R^1_{j,k}| &\leq CN\mathbf{E}\left[\frac{S^i}{N\mu^i}\left|\frac{S^j}{N} - \mu^j\right|\left|\frac{S^k}{N} - \mu^k\right|\int_0^1 \frac{r}{\sqrt{\mu^j\mu^k}}\sum_{e\in[\![d]\!]}\left|\frac{S^e}{N} - \mu^e\right|^{\frac{\gamma'}{2}} dr\right] \\ &\leq CN\frac{1}{N\mu^i\sqrt{\mu^j\mu^k}}\sum_{e\in[\![d]\!]}\mathbf{E}\left[|S^i - N\mu^i|\left|\frac{S^j}{N} - \mu^j\right|\left|\frac{S^k}{N} - \mu^k\right|\left|\frac{S^e}{N} - \mu^e\right|^{\frac{\gamma'}{2}}\right] \\ &\quad + CN\frac{1}{\sqrt{\mu^j\mu^k}}\sum_{e\in[\![d]\!]}\mathbf{E}\left[\left|\frac{S^j}{N} - \mu^j\right|\left|\frac{S^k}{N} - \mu^k\right|\left|\frac{S^e}{N} - \mu^e\right|^{\frac{\gamma'}{2}}\right] \\ &= \frac{C}{N^{2+\frac{\gamma'}{2}}\mu^i\sqrt{\mu^j\mu^k}}\sum_{e\in[\![d]\!]}\mathbf{E}\Big[|S^i - N\mu^i||S^j - N\mu^j||S^k - N\mu^k||S^e - N\mu^e|^{\frac{\gamma'}{2}}\Big] \\ &\quad + \frac{C}{N^{1+\frac{\gamma'}{2}}\sqrt{\mu^j\mu^k}}\sum_{e\in[\![d]\!]}\mathbf{E}\Big[|S^j - N\mu^j||S^k - N\mu^k||S^e - N\mu^e|^{\frac{\gamma'}{2}}\Big] \\ &\leq \frac{C}{N^{2+\frac{\gamma'}{2}}\mu^i\sqrt{\mu^j\mu^k}}\sum_{e\in[\![d]\!]}\left(\prod_{\ell=i,j,k}\mathbf{E}\Big[|S^\ell - N\mu^\ell|^{\frac{3}{1-\gamma'/6}}\Big]\right)^{\frac{1-\gamma'/6}{3}}(\mathbf{E}\,|S^e - N\mu^e|)^{\frac{\gamma'}{2}} \\ &\quad + \frac{C}{N^{1+\frac{\gamma'}{2}}\sqrt{\mu^j\mu^k}}\sum_{e\in[\![d]\!]}\left(\prod_{\ell=j,k}\mathbf{E}\Big[|S^\ell - N\mu^\ell|^{\frac{2}{1-\gamma'/4}}\Big]\right)^{\frac{1-\gamma'/4}{2}}(\mathbf{E}\,|S^e - N\mu^e|)^{\frac{\gamma'}{2}} \\ &\leq \frac{C}{N^{2+\frac{\gamma'}{2}}\mu^i\sqrt{\mu^j\mu^k}}N^{\frac{3}{2}+\frac{\gamma'}{4}}(\mu^i\mu^j\mu^k)^{\frac{1-\gamma'/6}{3}} + \frac{C}{N^{1+\frac{\gamma'}{2}}\sqrt{\mu^j\mu^k}}N^{1+\frac{\gamma'}{4}}(\mu^j\mu^k)^{\frac{1-\gamma'/4}{2}} \\ &= \frac{C}{N^{\frac{1}{2}+\frac{\gamma'}{4}}(\mu_i)^{\frac{2+\gamma'/6}{3}}(\mu^j\mu^k)^{\frac{1+2\gamma'/6}{6}}} + \frac{C}{N^{\frac{\gamma'}{4}}(\mu^j\mu^k)^{\frac{\gamma'}{8}}} \leq \frac{C}{N^{\frac{1+\gamma'/2}{2}-\epsilon\left(1+\frac{\gamma'}{6}\right)}} + \frac{C}{N^{\frac{\gamma'}{4}-\epsilon\gamma'/2}} \leq \frac{C}{N^\eta}, \end{aligned}$$

for $\epsilon \in (0, 1/4)$ and for $\eta > 0$, depending on ϵ and γ'.

The term $R^2_{j,k}$, using $\sqrt{\mu^j\mu^k}\,|\mathfrak{d}^2_{j,k}U^i(\mu)| \le C$, is bounded as

$$
\begin{aligned}
|R^2_{j,k}| &\le N\mathbf{E}\left[\frac{S^i}{N\mu^i}\left|\frac{S^j}{N}-\mu^j\right|\left|\frac{S^k}{N}-\mu^k\right|\cdot\int_0^1\frac{1}{\sqrt{\mu^j\mu^k}}\sqrt{\left|\left(\mu^j+r\left(\frac{S^j}{N}-\mu^j\right)\right)\left(\mu^k+r\left(\frac{S^k}{N}-\mu^k\right)\right)-\mu^j\mu^k\right|}\,|\mathfrak{d}^2_{j,k}U^i(\mu)|\,dr\right]\\
&\le \frac{CN}{\mu^j\mu^k}\mathbf{E}\left[\frac{S^i}{N\mu^i}\left|\frac{S^j}{N}-\mu^j\right|\left|\frac{S^k}{N}-\mu^k\right|\int_0^1\sqrt{\left|r\left(\frac{S^j}{N}-\mu^j\right)\mu^k+r\left(\frac{S^k}{N}-\mu^k\right)\mu^j\right|}dr\right]\\
&\quad+\frac{CN}{\mu^j\mu^k}\mathbf{E}\left[\frac{S^i}{N\mu^i}\left|\frac{S^j}{N}-\mu^j\right|\left|\frac{S^k}{N}-\mu^k\right|\int_0^1 r\sqrt{\left|\frac{S^j}{N}-\mu^j\right|\left|\frac{S^k}{N}-\mu^k\right|}dr\right]\\
&=: (A)+(B).
\end{aligned}
$$

To estimate (A), we bound the term with $r(S^j/N-\mu^j)\mu^k$ inside the square root; the other term is analogous. We have

$$
\begin{aligned}
&\frac{CN}{\mu^j\mu^k}\mathbf{E}\left[\frac{S^i}{N\mu^i}\left|\frac{S^j}{N}-\mu^j\right|\left|\frac{S^k}{N}-\mu^k\right|\left|\frac{S^j}{N}-\mu^j\right|^{\frac12}\sqrt{\mu^k}\right]\\
&\le \frac{CN}{\mu^j\sqrt{\mu^k}N\mu^i}\mathbf{E}\left[\left|S^i-N\mu^i\right|\left|\frac{S^j}{N}-\mu^j\right|^{\frac32}\left|\frac{S^k}{N}-\mu^k\right|\right]+\frac{CN}{\mu^j\sqrt{\mu^k}}\mathbf{E}\left[\left|\frac{S^j}{N}-\mu^j\right|^{\frac32}\left|\frac{S^k}{N}-\mu^k\right|\right]\\
&= \frac{C}{\mu^i\mu^j\sqrt{\mu^k}N^{\frac52}}\mathbf{E}\left[|S^i-N\mu^i||S^j-N\mu^j|^{\frac32}|\,S^k-N\mu^k\,|\right]+\frac{C}{\mu^j\sqrt{\mu^k}N^{\frac32}}\mathbf{E}\left[|S^j-N\mu^j|^{\frac32}|S^k-N\mu^k|\right]\\
&\le \frac{C}{\mu^i\mu^j\sqrt{\mu^k}N^{\frac52}}\left(\mathbf{E}\left[|S^i-N\mu^i|^3\right]\mathbf{E}\left[|S^j-N\mu^j|^{\frac92}\right]\mathbf{E}\left[|S^k-N\mu^k|^3\right]\right)^{\frac13}+\frac{C}{\mu^j\sqrt{\mu^k}N^{\frac32}}\left(\mathbf{E}\left[|S^j-N\mu^j|^3\right]\mathbf{E}\,|S^k-N\mu^k|^2\right]\right)^{\frac12}\\
&\le \frac{C}{\mu^i\mu^j\sqrt{\mu^k}N^{\frac52}}N^{\frac74}(\mu^i\mu^j\mu^k)^{\frac13}+\frac{C}{\mu^j\sqrt{\mu^k}N^{\frac32}}N^{\frac54}\sqrt{\mu^j\mu^k}\\
&= \frac{C}{N^{\frac34}(\mu^i\mu^j)^{\frac23}(\mu^k)^{\frac16}}+\frac{C}{N^{\frac14}(\mu^j)^{\frac12}}\le\frac{C}{N^{\frac34-\frac32\epsilon}}+\frac{C}{N^{\frac14-\frac12\epsilon}}\le\frac{C}{N^\eta},
\end{aligned}
$$

for $\epsilon\in(0,1/4)$ and for a possibly new value of η. Then, $(A)\le C/N^\eta$. The term (B) is estimated in the same way:

$$
\begin{aligned}
(B) &\le \frac{CN}{\mu^j\mu^k}\mathbf{E}\left[\frac{S^i}{N\mu^i}\left|\frac{S^j}{N}-\mu^j\right|^{\frac32}\left|\frac{S^k}{N}-\mu^k\right|^{\frac32}\right]\\
&\le \frac{CN}{N\mu^i\mu^j\mu^kN^3}\mathbf{E}\left[|S^i-N\mu^i||S^j-N\mu^j|^{\frac32}\,|S^k-N\mu^k|^{\frac32}\right]+\frac{CN}{\mu^j\mu^kN^3}\mathbf{E}\left[|S^j-N\mu^j|^{\frac32}\,|S^k-N\mu^k|^{\frac32}\right]\\
&\le \frac{C}{\mu^i\mu^j\mu^kN^3}N^2(\mu^i\mu^j\mu^k)^{\frac13}+\frac{C}{\mu^j\mu^kN^2}N^{\frac32}(\mu^j\mu^k)^{\frac12}\le\frac{C}{N^{1-2\epsilon}}+\frac{C}{N^{\frac12-\epsilon}}\le\frac{C}{N^\eta},
\end{aligned}
$$

which concludes the proof.

7.2. Proofs of Lemmas 4.3–4.5

Proof of Lemma 4.3. We just prove (4.27); the proof of (4.28) is similar. As in the proof of Lemma 4.2, we fix $t\in[0,T]$ and $(\boldsymbol{x},\boldsymbol{y})\in[\![d]\!]^N\times\mathbb{Y}$. We then let $\mu=\mu^N_{\boldsymbol{x},\boldsymbol{y}}$, and fixing $l,n\in[\![N]\!]$ with $m\ne l$, we denote x^l by i and x^m by k, namely, $i:=x^l$ and $k=x^m$.

We then have the following expansion:

$$
\begin{aligned}
\frac{1}{y^l}\Delta^m u^{N,l}(t,\boldsymbol{x},\boldsymbol{y})[j] &= U^i\left(t,\mu+\frac{y^m}{N}(\delta_j-\delta_k)\right)-U^i(t,\mu)\\
&=\int_0^1\frac{y^m}{N}\left[\mathfrak{d}_jU\left(t,\mu+s\frac{y^m}{N}(\delta_j-\delta_k)\right)-\mathfrak{d}_kU\left(t,\mu+s\frac{y^m}{N}(\delta_j-\delta_k)\right)\right]ds\\
&=\frac{y^m}{N}\left[\mathfrak{d}_jU^i(t,\mu)-\mathfrak{d}_kU^i(t,\mu)\right]\\
&\quad+\int_0^1\frac{y^m}{N}\left[\mathfrak{d}_jU^i\left(t,\mu+s\frac{y^m}{N}(\delta_j-\delta_k)\right)-\mathfrak{d}_jU^i(t,\mu)-\mathfrak{d}_kU^i\left(\mu+s\frac{y^m}{N}(\delta_j-\delta_k)\right)+\mathfrak{d}_kU^i(t,\mu)\right]ds,
\end{aligned}
$$

and the last two lines may be written in the form of a rest $\varrho^{N,l,m}(t,\boldsymbol{x},\boldsymbol{y})[j]$. Because $\mathfrak{D}U^i$ is γ'-Hölder continuous (for the Wright–Fisher distance), this remainder is bounded by

$$|\varrho^{N,l,m}(t,\, \boldsymbol{x},\, \boldsymbol{y})[j]| \le C\frac{(y^m)^{1+\gamma'/2}}{N^{1+\gamma'/2}},$$

which completes the proof.

Proof of Lemma 4.4. We first address the second term in the top line of (4.30), namely, the term containing a^*. As in the proofs of the previous two statements, we fix $t \in [0,T]$, $(\boldsymbol{x},\boldsymbol{y}) \in \mathcal{T}_N$ and $l \in [\![N]\!]$, and we let $\mu = \mu^N_{\boldsymbol{x},\boldsymbol{y}}$ and $i = x^l$.

By the definition of a^* (see (2.11)), we have

$$\frac{1}{y^l}\sum_{m\neq l} a^*\Big(x^m,\, \frac{1}{y^m}u^{N,m}_\bullet\Big)\cdot \Delta^m u^{N,l}[\bullet] = \sum_{m\neq l}\sum_{j\in[\![d]\!]}\frac{1}{y^m}(-\Delta^m u^{N,m}[j])_+\frac{1}{y^l}\Delta^m u^{N,l}[j],$$

and then, by Lemma 4.3,

$$\begin{aligned}
&\frac{1}{y^l}\sum_{m\neq l} a^*\Big(x^m,\, \frac{1}{y^m}u^{N,m}_\bullet\Big)\cdot \Delta^m u^{N,l}[\bullet]\\
&= \sum_{m\neq l}\sum_{j\in[\![d]\!]}\Bigg[\Big((U^{x^m}-U^j)(t,\mu) - \frac{y^m}{N}\Big(\mathfrak{d}_j U^{x^m} - \mathfrak{d}_{x^m}U^{x^m}\Big)(t,\mu) - \varrho^{N,m,m}(t,\boldsymbol{x},\boldsymbol{y})[j]\Big)_+\\
&\quad\times\frac{y^m}{N}\Big(\Big(\mathfrak{d}_j U^i - \mathfrak{d}_{x^m}U^i\Big)(t,\,\mu) + \varrho^{N,l,m}(t,\,\boldsymbol{x},\,\boldsymbol{y})[j]\Big)\Bigg]\\
&= \sum_{m\neq l}\sum_{j,\,k\in[\![d]\!]}\mathbb{1}_{\{x^m=k\}}\Bigg[\Big((U^k-U^j)(t,\,\mu) - \frac{y^m}{N}\Big(\mathfrak{d}_j U^k - \mathfrak{d}_k U^k\Big)(t,\,\mu) - \varrho^{N,m,m}(t,\boldsymbol{x},\boldsymbol{y})[j]\Big)_+\frac{y^m}{N}\Big(\Big(\mathfrak{d}_j U^i - \mathfrak{d}_k U^i\Big)(t,\,\mu) + \varrho^{N,l,m}(t,\,\boldsymbol{x},\,\boldsymbol{y})[j]\Big)\Bigg].
\end{aligned}$$

Then,

$$\begin{aligned}
&\frac{1}{y^l}\sum_{m\neq l} a^*\Big(x^m,\, \frac{1}{y^m}u^{N,m}_\bullet\Big)\cdot \Delta^m u^{N,l}[\bullet]\\
&= \sum_{m\neq l}\sum_{j,\,k\in[\![d]\!]}\mathbb{1}_{\{x^m=k\}}((U^k-U^j)(t,\,\mu))_+\frac{y^m}{N}\Big(\mathfrak{d}_j U^i - \mathfrak{d}_k U^i\Big)(t,\,\mu)\\
&\quad+\sum_{m\neq l}\sum_{j,\,k\in[\![d]\!]}\Big[\mathbb{1}_{\{x^m=k\}}\Big((U^k-U^j)(t,\,\mu) - \frac{y^m}{N}\Big(\mathfrak{d}_j U^k - \mathfrak{d}_k U^k\Big)(t,\,\mu) - \varrho^{N,m,m}(t,\boldsymbol{x},\boldsymbol{y})[j]\Big)_+\varrho^{N,l,m}(t,\,\boldsymbol{x},\,\boldsymbol{y})[j]\Big]\\
&\quad+\sum_{m\neq l}\sum_{j,\,k\in[\![d]\!]}\mathbb{1}_{\{x^n=k\}}\Bigg[\Big((U^k-U^j)(t,\,\mu) - \frac{y^m}{N}\Big(\mathfrak{d}_j U^k - \mathfrak{d}_k U^k\Big)(t,\,\mu) - \varrho^{N,m,m}(t,\boldsymbol{x},\boldsymbol{y})[j]\Big)_+ - ((U^k-U^j)(t,\,\mu))_+\Bigg]\frac{y^m}{N}\Big(\mathfrak{d}_j U^i - \mathfrak{d}_k U^i\Big)(t,\,\mu)\\
&:= \sum_{k,\,j\in[\![d]\!]}\mu^k((U^k-U^j)(t,\,\mu))_+(\mathfrak{d}_j U^i - \mathfrak{d}_k U^i)(t,\,\mu) + R_1 + R_2.
\end{aligned}$$

Using the boundedness of U and $\mathfrak{D}U$, the first remainder is bounded by

$$\begin{aligned}
|R_1| &\le C\sum_{m\in[\![N]\!]}\sup_{j\in[\![d]\!]}|\varrho^{N,l,m}(t,\,\boldsymbol{x},\,\boldsymbol{y})[j]| + C\sum_{m\in[\![N]\!]}\frac{y^m}{N}\sup_{j\in[\![d]\!]}|\varrho^{N,l,m}(t,\,\boldsymbol{x},\,\boldsymbol{y})[j]| + C\sum_{m\in[\![N]\!]}\sup_{j\in[\![d]\!]}|(\varrho^{N,m,m}\varrho^{N,l,m})(t,\,\boldsymbol{x},\,\boldsymbol{y})[j]|\\
&\le C\sum_{m\in[\![N]\!]}\frac{(y^m)^{1+\gamma'/2}}{N^{1+\gamma'/2}} + C\sum_{m\in[\![N]\!]}\frac{(y^m)^{2+\gamma'/2}}{N^{2+\gamma'/2}} + C\sum_{m\in[\![N]\!]}\frac{(y^m)^{2+2\gamma'/2}}{N^{2+2\gamma'/2}},
\end{aligned}$$

whereas the second remainder is bounded by

$$|R_2| \le C\sum_{m\in[\![N]\!]}\Big(\frac{y^m}{N} + \sup_{j\in[\![d]\!]}\Big|\varrho^{N,m,m}(t,\,\boldsymbol{x},\,\boldsymbol{y})[j]\Big|\Big)\frac{y^m}{N} \le C\sum_{m\in[\![N]\!]}\frac{(y^m)^2}{N^2} + C\sum_{m\in[\![N]\!]}\frac{(y^m)^{2+\gamma'/2}}{N^{2+\gamma'/2}}.$$

Finally, because $y^m \le N$, we deduce that $|R_1| + |R_2|$ is less than $C\sum_{m\in[\![N]\!]}(y^m)^{1+\gamma'/2}/N^{1+\gamma'/2}$. By definition of $\mathcal{T}_N$ in (4.22), we then recall that $\max_{m\in[\![N]\!]}y^m \le N^{1-\epsilon}$, which yields

$$\sum_{m\in[\![N]\!]}\frac{(y^m)^{1+\gamma'/2}}{N^{1+\gamma'/2}} \le N^{-\gamma'\epsilon/2}\sum_{m\in[\![N]\!]}\frac{y^m}{N} = N^{-\gamma'\epsilon/2}, \tag{7.2}$$

where we use in the last line the fact that $\sum_{m\in[\![N]\!]} y^m = N$. This shows that the second term on the top line of (4.30) satisfies the bound (4.31).

The first term on the top line of (4.30) is handled in the same way, using the additional expansion (4.28) to treat the case when m (the index in the sum) is equal to l. $\square$

Proof of Lemma 4.5. Applying (2.11) and (4.28), we obtain (denote $\mu^N_{\boldsymbol{x},\boldsymbol{y}} = \mu$ and $x^l = i$)

$$
\begin{aligned}
H\Big(x^l, \frac{1}{y^l} u^{N,l}_{\bullet}\Big) &= -\frac{1}{2} \sum_{j\in[\![d]\!]} \Big(-\frac{1}{y^l} \Delta^l u^{N,l}(t,\boldsymbol{x},\boldsymbol{y})[j] \Big)_+^2 \\
&= -\frac{1}{2} \sum_{j\in[\![d]\!]} \Big((U^i - U^j)(t,\mu) - \frac{y^l}{N}\Big(\mathfrak{d}_j U^i - \mathfrak{d}_i U^i\Big)(t,\mu) - \varrho^{N,l,l}(t,\boldsymbol{x},\boldsymbol{y})[j] \Big)_+^2 \\
&= -\frac{1}{2} \sum_{j\in[\![d]\!]} ((U^i - U^j)(t,\mu))_+^2 + r_3^{N,l}.
\end{aligned}
$$

The first term is the Hamiltonian $H(i, U(t,\mu))$. Using (4.29) together with the fact that U and $\mathfrak{D}U$ are bounded and that $y^l \le N$, we can estimate the remainder by

$$
|r_3^{N,l}| \le C \sum_{j\in[\![d]\!]} \Big(1 + \frac{y^l}{N} + \Big|\varrho^{N,l,l}(t,\boldsymbol{x},\boldsymbol{y})[j]\Big|\Big)\Big(\frac{y^l}{N} + \Big|\varrho^{N,l,l}(t,\boldsymbol{x},\boldsymbol{y})[j]\Big|\Big) \le C\frac{y^l}{N} + C\Big(\frac{y^l}{N}\Big)^{1+\gamma'/2} \le C\frac{y^l}{N},
$$

which completes the proof. $\square$

8. Further Prospects.

It is fair to say that, despite our long analysis, we have left open several quite important questions. This is mostly our deliberate choice because we want to keep the paper of a reasonable length.

As we already alluded to in Section 2.4, a first point would be to compare the particle system driven by the equilibrium feedback law $\Delta w^{N,n}$ by the same particle system but driven by the feedback strategy $\Delta z^{N,n}$.

Another problem would be to prove propagation of chaos. Notice indeed that our convergence result Theorem 2.2 does not permit us to prove asymptotic conditional independence of the particles. The usual argument to do so is based on a standard result of Sznitman [34, proposition 2.2], but it does not apply here because the weights are not equal to $1/N$ but to $(Y^l_t/N)_{l\in[\![N]\!]}$. In order to prove that, asymptotically, the particles just interact through the common noise, a possible approach would consist in showing that the distance between $(\mu^N_t)_{0\le t\le T}$ and $(\mathbb{E}[\mu^N_t \mid \mathcal{N}^0])_{0\le t\le T}$ tends to zero. In order to do so, we may think of comparing $(\mu^N_t)_{0\le t\le T}$ and $(\widehat{\mu}^N_t)_{0\le t\le T}$, the latter being obtained by considering the same system as (4.4), driven by the same common noise but by independent copies $\widehat{\mathcal{N}}^1, \cdots, \widehat{\mathcal{N}}^N$ of the Poisson measures $\mathcal{N}^1, \cdots, \mathcal{N}^N$. Alternatively, we might think of a more direct coupling argument; we refer, for instance, to Kurtz and Xiong [26] or to Carmona and Delarue [10, chapter 2] for such a proof of propagation of chaos for diffusive particle systems subjected to a common noise. In this regard, we could use the representation of the MFG equilibrium in the form $P^i_t = \mathbb{E}[Y_t \mathbb{1}_{\{X_t=i\}} \mid \mathcal{F}^W_t]$, where X_t is the state of a tagged player within the population and W is the common noise in (2.1); we refer to Bayraktar et al. [3] for more details on this representation. Still, this would obviously raise subtle questions about a suitable coupling between the Brownian noise W and the Poisson measures $\mathcal{N}^0$.

Finally, we may think of estimating the weak error in the convergence in law proved in Theorem 2.2, at least for the one-dimensional marginals. It seems that, in order to do so, we can adapt Proposition 4.3, choosing, instead of U itself, the solution Z of the equation

$$
\partial_t Z + \sum_{j,k\in[\![d]\!]} p_k\Big[\varphi\Big(p_j\Big) + (U^k - U^j)_+\Big](\partial_{p_j} Z - \partial_{p_k} Z) + \frac{\varepsilon^2}{2} \sum_{j,k\in[\![d]\!]} (p_j\delta_{jk} - p_j p_k)\partial^2_{p_j p_k} Z = 0, Z(T,p) = h(p), \tag{8.1}
$$

for $p \in \mathcal{P}([\![d]\!])$ and for a terminal boundary condition h. Provided h is smooth enough, we know from Epstein and Mazzeo [20, theorem 10.0.2] that the preceding equation has a classic solution. Then, following the proof of Proposition 4.3, we can show that the function $(t,\boldsymbol{x},\boldsymbol{y}) \longmapsto Z(t, \mu^N_{\boldsymbol{x},\boldsymbol{y}})$ is nearly harmonic for the generator of $(\boldsymbol{X}^N, \boldsymbol{Y}^N)$ (which we merely denoted by $(\boldsymbol{X}, \boldsymbol{Y})$ in (4.4) with $\iota = 0$ therein). The fact that the latter mapping is nearly harmonic makes it possible to prove (by Itô's expansion) that $Z(0, \mu^N_{\boldsymbol{x},\boldsymbol{y}})$ and $\mathbb{E}[h(\mu^N_{\boldsymbol{X}^N_T, \boldsymbol{Y}^N_T})]$ get closer as N tends to ∞, whenever $(\boldsymbol{X}^N, \boldsymbol{Y}^N)$ starts from $(\boldsymbol{x}, \boldsymbol{y})$ at time 0. This strategy is reported within the more standard context of

McKean–Vlasov equations in Carmona and Delarue [9, section 5.7.4] and, in a more systematic manner, in the recent contribution Chassagneux et al. [15].

Additionally, one can also construct an asymptotic equilibrium in the N-player game using the master equation; we refer, for instance, to Carmona and Delarue [10, section 6.1.2] for more on this approach in the more standard diffusive setting. Specifically, under the conditions in Theorems 2.1 and 2.2, for any sequence of initial conditions $\{(\boldsymbol{x}^N, \boldsymbol{y}^N) = ((x^{N,l})_{l\in[\![N]\!]}, (1,\ldots,1))\}_N$ satisfying (2.21), the feedback strategy vector $\widehat{\boldsymbol{\alpha}}^* = (\widehat{\alpha}^{*,1}, \ldots, \widehat{\alpha}^{*,N})$ given by

$$\begin{aligned}\widehat{\alpha}^{*,l}(t,\boldsymbol{x},\boldsymbol{y})[j] = a^*(x^l, z_\bullet^{N,l}(t,\boldsymbol{x},\boldsymbol{y}))[j] &:= (z^{N,l}(t,\boldsymbol{x},\boldsymbol{y}) - z^{N,l}(t,(j,\boldsymbol{x}^{-l}),\boldsymbol{y}))_+ \\ &= (-\Delta^l z^{N,l}(t,\boldsymbol{x},\boldsymbol{y})[j])_+, \end{aligned} \tag{8.2}$$

for $l \in [\![N]\!]$ such that $j \neq x^l$, is expected to define an asymptotic Nash equilibrium in Markov feedback form for the N-player game. That means that, for any player $l \in [\![N]\!]$ and any sequence of feedback controls $\{\beta^{N,l}\}_{N\geq l}$, one should have

$$\liminf_{N\to\infty} J^{N,l}(t,\boldsymbol{x},\boldsymbol{y},\beta^l,\widehat{\boldsymbol{\alpha}}^{*,-l}) \geq \limsup_{N\to\infty} J^{N,l}(t,\boldsymbol{x},\boldsymbol{y},\widehat{\boldsymbol{\alpha}}^*),$$

where $J^{N,l}(t,\boldsymbol{x},\boldsymbol{y},\boldsymbol{\alpha})$ denotes the cost when the process $(\boldsymbol{X}^N, \boldsymbol{Y}^N)$ starts at time t with $(\boldsymbol{X}_t^N, \boldsymbol{Y}_t^N) = (\boldsymbol{x}^N, \boldsymbol{y}^N)$ (we here put a superscript N in order to emphasize the dependence on N). Moreover, letting $\widehat{\mu}^N = (\widehat{\mu}_t^N)_{0\leq t\leq T}$ be empirical distribution under $\widehat{\boldsymbol{\alpha}}^*$, $(\widehat{\mu}_t^N)_{0\leq t\leq T}$ is expected to converge in the weak sense on $\mathcal{D}([0,T];\mathbb{R}^d)$ equipped with the $J1$ Skorokhod topology to the solution $(P_t)_{0\leq t\leq T}$ of the SDE (2.4). Notice that, differently from the analysis carried out in the rest of the paper, this forces us to address the asymptotic behavior of the (conditional) mass of a "deviating player" $(\widetilde{X}^{N,l}, \widetilde{Y}^{N,l})$. The latter reads $(Q_t^{N,l}[i] := \mathbb{E}[\widetilde{Y}_t^{N,l} \mathbb{1}_{\{\widetilde{X}_t^{N,l}=i\}} \mid \mathcal{N}^0])_{0\leq t\leq T}$; a peculiarity of it is that, similar to the process $(Q_t)_{0\leq t\leq T}$ in (2.1), it might not take values in the simplex.

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Appendix. Nash Equilibria of the *N*-Player Game

A.1. Proof of Proposition 2.2

A.1.1. Value of $v^{N,l}$ When $y^l = 0$. We first prove that $v^{N,l}(t,\boldsymbol{x},\boldsymbol{y}) = 0$ when $y^l = 0$. To do so, given $\boldsymbol{\alpha}^*$ as in (2.13) and (2.14), we call $(\boldsymbol{X},\boldsymbol{Y})$ the dynamics related to the strategy vector $(0,\boldsymbol{\alpha}^{*,-l})$ and starting at time t from $(\boldsymbol{X}_t,\boldsymbol{Y}_t) = (\boldsymbol{x},\boldsymbol{y})$. By a direct application of Dynkin's formula to $(v^{N,l}(s,\boldsymbol{X}_s,\boldsymbol{Y}_s))_{t\leq s\leq T}$, using in addition the fact that the process Y^l remains equal to zero whenever starting from zero, we get that $v^{N,l}(t,\boldsymbol{x},\boldsymbol{y}) = 0$ when $y^l = 0$.

A.1.2. Verification Argument. Let $\beta(t,\boldsymbol{x},\boldsymbol{y})$ be another feedback control, and then $(\boldsymbol{X},\boldsymbol{Y})$ now denote the dynamics related to the strategy vector $(\beta,\alpha^{*,-l})$ (still starting at time t from $(\boldsymbol{X}_t,\boldsymbol{Y}_t) = (\boldsymbol{x},\boldsymbol{y})$). Fix $l \in [\![N]\!]$. By definition of the Hamiltonian and from the fact that $v^{N,l}(s,\boldsymbol{x},\boldsymbol{y})$ is zero if $y^l = 0$, the Nash system gives

$$\begin{aligned}&\frac{d}{dt}v^{N,l}(s,\boldsymbol{x},\boldsymbol{y}) + \sum_{m\in[\![N]\!]}\varphi\Big(\mu_{\boldsymbol{x},\boldsymbol{y}}^N[\bullet]\Big)\cdot\Delta^m v^{N,l}[\bullet] + \sum_{m\neq l}\alpha^{*,m}(s,\boldsymbol{x},\boldsymbol{y})\cdot\Delta^m v^{N,l}[\bullet] + \beta(s,\boldsymbol{x},\boldsymbol{y})\cdot\Delta^l v^{N,l} \\ &\quad + \varepsilon N\mathbf{E}\left[v^{N,l}\left(s,\boldsymbol{x},y^1\frac{S_{\mu_{\boldsymbol{x},\boldsymbol{y}}^N}[x^1]}{N\mu_{\boldsymbol{x},\boldsymbol{y}}^N[x^1]},\ldots,y^N\frac{S_{\mu_{\boldsymbol{x},\boldsymbol{y}}^N}[x^N]}{N\mu_{\boldsymbol{x},\boldsymbol{y}}^N[x^N]}\right) - v^{N,l}(s,\boldsymbol{x},\boldsymbol{y})\right] \\ &= \frac{1}{2}\sum_{j\neq x^l} y^l|\alpha^{*,l}(s,\boldsymbol{x},\boldsymbol{y})[j]|^2 + \mathbb{1}_{\{y^l=0\}}\beta(s,\boldsymbol{x},\boldsymbol{y})\cdot\Delta^l v^{N,l}[\bullet] - y^l f\Big(x_l,\mu_{\boldsymbol{x},\boldsymbol{y}}^N\Big) \\ &\geq \frac{1}{2}\sum_{j\neq x^l} y^l|\alpha^{*,l}(s,\boldsymbol{x},\boldsymbol{y})[j]|^2 - \mathbb{1}_{\{y^l=0\}}\beta(s,\boldsymbol{x},\boldsymbol{y})\cdot(-\Delta^l v^{N,l})_+[\bullet] - y^l f\Big(x_l,\mu_{\boldsymbol{x},\boldsymbol{y}}^N\Big) \\ &= \frac{1}{2}\sum_{j\neq x^l} y^l|\alpha^{*,l}(s,\boldsymbol{x},\boldsymbol{y})[j]|^2 - y^l\beta(s,\boldsymbol{x},\boldsymbol{y})\cdot\alpha^{*,l}(s,\boldsymbol{x},\boldsymbol{y}) - y^l f\Big(x_l,\mu_{\boldsymbol{x},\boldsymbol{y}}^N\Big) \\ &= \frac{1}{2}\sum_{j\neq x^l} y^l|\alpha^{*,l}(s,\boldsymbol{x},\boldsymbol{y})[j] - \beta(s,\boldsymbol{x},\boldsymbol{y})[j]|^2 - \frac{1}{2}\sum_{j\neq x^l} y^l|\beta(s,\boldsymbol{x},\boldsymbol{y})[j]|^2 - y^l f\Big(x_l,\mu_{\boldsymbol{x},\boldsymbol{y}}^N\Big),\end{aligned}$$

for any $s, \boldsymbol{x}, \boldsymbol{y}$. Applying Dynkin's formula, and the preceding inequality, we obtain

$$\begin{aligned}
v^{N,l}(t,\boldsymbol{x},\boldsymbol{y}) &= \mathbb{E}\Bigg[v^{N,l}(T,\boldsymbol{X}_T,\boldsymbol{Y}_T) - \int_t^T\bigg(\frac{d}{dt}v^{N,l}(s,\boldsymbol{X}_s,\boldsymbol{Y}_s)\\
&\quad+\sum_{m\neq l}\alpha^{*,m}(s,\boldsymbol{X}_s,\boldsymbol{Y}_s)\cdot\Delta^m v^{N,l}(s,\boldsymbol{X}_s,\boldsymbol{Y}_s)+\beta(s,\boldsymbol{X}_s,\boldsymbol{Y}_s)\cdot\Delta^l v^{N,l}(s,\boldsymbol{X}_s,\boldsymbol{Y}_s)\\
&\quad+\varepsilon N\mathbf{E}\bigg[v^{N,l}\bigg(s,\boldsymbol{X}_s,Y_s^1\frac{S_{\mu_s^N}[X_s^1]}{N\mu_s^N[X_s^1]},\ldots,Y_s^N\frac{S_{\mu_s^N}[X_s^N]}{N\mu_s^N[X_s^N]}\bigg)-v^{N,l}(s,\boldsymbol{X}_s,\boldsymbol{Y}_s)\bigg]\bigg)ds\Bigg]\\
&\le \mathbb{E}\bigg[Y_T^l g(X_T^l,\mu_T^N)+\int_t^T Y_s^l(\ell(X_s^l,\beta(s,\boldsymbol{X}_s,\boldsymbol{Y}_s))+f(X_s^l,\mu_s^N))ds\bigg]\\
&\quad-\frac{1}{2}\mathbb{E}\bigg[\int_t^T Y_s^l\sum_{j\neq X_s^l}|\alpha^{*,l}(s,\boldsymbol{X}_s,\boldsymbol{Y}_s)[j]-\beta(s,\boldsymbol{X}_s,\boldsymbol{Y}_s)[j]|^2 ds\bigg],
\end{aligned}$$

which shows that $v^{N,l}(t,\boldsymbol{x},\boldsymbol{y}) \le J^l(t,\boldsymbol{x},\boldsymbol{y},\beta,\boldsymbol{\alpha}^{*,-l})$ (the latter being defined as the cost to l when the system is driven by $[\beta,\boldsymbol{\alpha}^{*,-l}]$ and starts from $(\boldsymbol{x},\boldsymbol{y})$ at time t). Replacing β by $\alpha^{*,l}$, we obtain $v^{N,l}(t,\boldsymbol{x},\boldsymbol{y}) = J^l(t,\boldsymbol{x},\boldsymbol{y},\boldsymbol{\alpha})$. □

A.2. Proof of Insensitivity and Uniqueness of the Nash Equilibrium

We now prove the claims of Proposition 2.3 that we left aside in Section 3. It remains to prove the fact that the equilibrium is uniquely determined (in the sense of (2.17)) and satisfies the insensitivity property (2.16). In fact, we prove both at the same time.

In order to proceed, we assume that we are given a strategy, say $\widehat{\boldsymbol{\alpha}}$, that defines a bounded equilibrium in Markov feedback form.

A.2.1. First Step.

Fix a player l and then identify $\widehat{\alpha}^l$ with the best response of the cost functional (2.10) when all the feedback functions except the lth one are fixed. In order to do so, we may first solve the equation (which is directly inspired from (3.2))

$$\begin{aligned}
&\frac{d}{dt}\widehat{w}^l+\sum_{m\in[\![N]\!]}\varphi\big(\mu_{\boldsymbol{x},\boldsymbol{y}}^N[\bullet]\big)\cdot\Delta^m\widehat{w}^l[\bullet]+\sum_{m\neq l}\widehat{\alpha}^m(t,\boldsymbol{x},\boldsymbol{y})\cdot\Delta^m\widehat{w}^l\\
&+H(x^l,\widehat{w}_\bullet^l)+f(x^l,\mu_{\boldsymbol{x},\boldsymbol{y}}^N)+\varepsilon N\mathbf{E}\bigg[\frac{S_{\mu_{\boldsymbol{x},\boldsymbol{y}}^N}[x^l]}{N\mu_{\boldsymbol{x},\boldsymbol{y}}^N[x^l]}\bigg(\widehat{w}^l\bigg(t,\boldsymbol{x},y^1\frac{S_{\mu_{\boldsymbol{x},\boldsymbol{y}}^N}[x^1]}{N\mu_{\boldsymbol{x},\boldsymbol{y}}^N[x^1]},\ldots,y^N\frac{S_{\mu_{\boldsymbol{x},\boldsymbol{y}}^N}[x^N]}{N\mu_{\boldsymbol{x},\boldsymbol{y}}^N[x^N]}\bigg)-\widehat{w}^l(t,\boldsymbol{x},\boldsymbol{y})\bigg)\bigg]=0,
\end{aligned}$$

with the terminal boundary condition

$$\widehat{w}^l(T,\boldsymbol{x},\boldsymbol{y}) = g(x^l,\mu_{\boldsymbol{x},\boldsymbol{y}}^N). \tag{A.1}$$

Notice that we solve one equation only (and not a system). We then let $\widehat{v}^l(t,\boldsymbol{x},\boldsymbol{y}) := y^l\widehat{w}^l(t,\boldsymbol{x},\boldsymbol{y})$.

Very much in the spirit of the proof of Proposition 2.2 in Section A.1 (with $(\boldsymbol{X},\boldsymbol{Y})$ denoting the same dynamics for some strategy vector $(\beta,\widehat{\alpha}^{-l})$ except that $\widehat{\boldsymbol{\alpha}}$ is now given as some equilibrium), we get that

$$\widehat{v}^l(t,\boldsymbol{x},\boldsymbol{y})+\frac{1}{2}\mathbb{E}\bigg[\int_t^T Y_s^l\sum_{j\neq X_s^l}|a^*\big(X_s^l,\widehat{w}_\bullet^l(s,\boldsymbol{X}_s,\boldsymbol{Y}_s)\big)[j]-\beta(s,\boldsymbol{X}_s,\boldsymbol{Y}_s)[j]|^2ds\bigg]\le J^l\big(t,\boldsymbol{x},\boldsymbol{y},\beta,\widehat{\boldsymbol{\alpha}}^{-l}\big),$$

from which we deduce that $\widehat{\alpha}^l(t,\boldsymbol{x},\boldsymbol{y}) = a^*(x^l,\widehat{w}_\bullet^l(t,\boldsymbol{x},\boldsymbol{y}))$ whenever $y^l>0$. This prompts us to introduce the notation $\mathbb{Y}^l = \{\boldsymbol{y}\in\mathbb{Y} : y^l>0\}$.

Following the proof of Proposition 3.1, we notice that

$$\sup_{t\in[0,T]}\max_{\boldsymbol{x}\in[\![d]\!]^N}\sup_{\boldsymbol{y}\in\mathbb{Y}}\big|\widehat{w}^l(t,\boldsymbol{x},\boldsymbol{y})\big|<\infty.$$

A.2.2. Second Step.

Inspired by the proof of Proposition 3.1, we are to regard $(\widehat{w}^l)_{l\in[\![N]\!]}$ as the fixed point of some mapping $\widehat{\Phi}$. However, for reasons that are made clear subsequently, we construct the mapping $\widehat{\Phi}$ in a slightly different manner than the function Φ in the proof Proposition 3.1. In order to proceed, we introduce a smooth bounded cutoff function ϑ from $\mathbb{R}$ into itself with the property that

$$\vartheta(\widehat{w}^l(t,\boldsymbol{x},\boldsymbol{y})) = \widehat{w}^l(t,\boldsymbol{x},\boldsymbol{y}),\quad t\in[0,T],\,\boldsymbol{x}\in[\![d]\!]^N,\,\boldsymbol{y}\in\mathbb{Y}.$$

Noticing from the first step that $\widehat{\alpha}^l(t,\boldsymbol{x},\boldsymbol{y})\cdot\Delta^m\widehat{w}^l$ always writes in the form

$$\widehat{\alpha}^l(t,\boldsymbol{x},\boldsymbol{y})\cdot\Delta^m\widehat{w}^l[\bullet] = (\widehat{\alpha}^l(t,\boldsymbol{x},\boldsymbol{y})\mathbf{1}_{\{y^l=0\}}+a^*(x^l,\widehat{w}^l(t,\boldsymbol{x},\boldsymbol{y}))\mathbf{1}_{\{y^l>0\}})\cdot\Delta^m\widehat{w}^l[\bullet],$$

we, hence, define $\widehat{\Phi}$ as the mapping that sends an input $(w^l)_{l\in[\![N]\!]}$ onto the solution $(\widetilde{w}^l)_{l\in[\![N]\!]}$ of the system

$$\begin{aligned}\frac{d}{dt}\widetilde{w}^l &+ \sum_{m\in[\![N]\!]}\varphi\Big(\mu^N_{\boldsymbol{x},\boldsymbol{y}}[\bullet]\Big)\cdot\Delta^m\widetilde{w}^l[\bullet] + H\Big(x^l,\vartheta(\widetilde{w}^l)_\bullet\Big) + f\Big(x^l,\mu^N_{\boldsymbol{x},\boldsymbol{y}}\Big)\\ &+\sum_{m\neq l}(\widehat{\alpha}^m(t,\boldsymbol{x},\boldsymbol{y})\mathbb{1}_{\{y^m=0\}} + a^*(x^m,\vartheta(w^m)_\bullet)\mathbb{1}_{\{y^m>0\}})\cdot\Delta^m\vartheta(w^l)[\bullet]\\ &+\varepsilon N\mathbf{E}\left[\frac{S_{\mu^N_{\boldsymbol{x},\boldsymbol{y}}}[x^l]}{N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^l]}\left(w^l\left(t,\boldsymbol{x},y^1\frac{S_{\mu^N_{\boldsymbol{x},\boldsymbol{y}}}[x^1]}{N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^1]},\dots,y^N\frac{S_{\mu^N_{\boldsymbol{x},\boldsymbol{y}}}[x^N]}{N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^N]}\right)-w^l(t,\boldsymbol{x},\boldsymbol{y})\right)\right]=0,\end{aligned}\tag{A.2}$$

with the obvious notation that $\vartheta(w^l)_\bullet$ denotes the vector in $\mathbb{R}^d$ given by $(\vartheta(w^l)(t,\boldsymbol{x},\boldsymbol{y})[j]=\vartheta(w^l)(t,(j,\boldsymbol{x}^{-l}),\boldsymbol{y}))_{j\in[\![d]\!]}$. In other words, $(w^l)_{l\in[\![N]\!]}$ can be approximated by iterating the function $\widehat{\Phi}$.

We then consider an input $(w^l)_{l\in[\![N]\!]}$ with the property that, for any $l,n\in[\![N]\!]$, any $\boldsymbol{y}\in\mathbb{Y}$ such that $y^l>0$ and $y^n=0$, and any $(t,\boldsymbol{x})\in[0,T]\times[\![d]\!]^N$,

$$\Delta^n w^l(t,\boldsymbol{x},\boldsymbol{y})[j]=0,\quad j\in[\![d]\!],\tag{A.3}$$

or, equivalently, for any $\boldsymbol{x}\in[\![d]\!]^N$,

$$w^l(t,\boldsymbol{x},\boldsymbol{y})=w^l(t,(j,\boldsymbol{x}^{-n}),\boldsymbol{y}),\quad j\in[\![d]\!].$$

We, hence, check what the last three terms in (A.2) become, whenever computed at such a point $\boldsymbol{y}\in\mathbb{Y}$ with $y^l>0$ and $y^n=0$, form some $l,n\in[\![N]\!]$. The first point is to notice that, for any $\boldsymbol{x},\boldsymbol{x}'\in[\![d]\!]^N$ with $\boldsymbol{x}^{-n}=\boldsymbol{x}'^{,-n}$,

$$\mu^N_{\boldsymbol{x},\boldsymbol{y}}=\frac{1}{N}\sum_{m=1}^N y^m\delta_{x^m}=\frac{1}{N}\sum_{m\neq n}y^m\delta_{x^m}=\mu^N_{\boldsymbol{x}',\boldsymbol{y}},$$

which shows that

$$f(x^l,\mu^N_{\boldsymbol{x},\boldsymbol{y}})=f(x^l,\mu^N_{\boldsymbol{x}',\boldsymbol{y}}).\tag{A.4}$$

As for the term on the second line of (A.2), we have that $\Delta^m\vartheta(w^l(t,\boldsymbol{x},\boldsymbol{y}))[\bullet]=0$ for any m such that $y^m=0$, and then,

$$\begin{aligned}&\sum_{m\neq l}(\widehat{\alpha}^m(t,\boldsymbol{x},\boldsymbol{y})\mathbb{1}_{\{y^m=0\}}+a^*(x^m,\vartheta(w^m(t,\boldsymbol{x},\boldsymbol{y}))_\bullet)\mathbb{1}_{\{y^m>0\}})\cdot\Delta^m\vartheta(w^l(t,\boldsymbol{x},\boldsymbol{y}))[\bullet]\\&=\sum_{m\neq l}(a^*(x^m,\vartheta(w^m(t,\boldsymbol{x},\boldsymbol{y}))_\bullet)\mathbb{1}_{\{y^m>0\}})\cdot\Delta^m\vartheta(w^l(t,\boldsymbol{x},\boldsymbol{y}))[\bullet].\end{aligned}$$

By assumption, we have that $\vartheta(w^m(t,\boldsymbol{x},\boldsymbol{y}))_\bullet=\vartheta(w^m(t,\boldsymbol{x}',\boldsymbol{y}))_\bullet$ whenever $y^m>0$ (it suffices to replace l by m in (A.3)). By the same argument, $\Delta^m\vartheta(w^l(t,\boldsymbol{x},\boldsymbol{y}))[\bullet]=\Delta^m\vartheta(w^l(t,\boldsymbol{x}',\boldsymbol{y}))[\bullet]$. Therefore,

$$\begin{aligned}&\sum_{m\neq l}\Big(\widehat{\alpha}^m(t,\boldsymbol{x},\boldsymbol{y})\mathbb{1}_{\{y^m=0\}}+a^*(x^m,\vartheta(w^m(t,\boldsymbol{x},\boldsymbol{y}))_\bullet)\mathbb{1}_{\{y^m>0\}}\Big)\cdot\Delta^m\vartheta(w^l(t,\boldsymbol{x},\boldsymbol{y}))[\bullet]\\&=\sum_{m\neq l}\Big(\widehat{\alpha}^m(t,\boldsymbol{x}',\boldsymbol{y})\mathbb{1}_{\{y^m=0\}}+a^*(x^m,\vartheta(w^m(t,\boldsymbol{x}',\boldsymbol{y}))_\bullet)\mathbb{1}_{\{y^m>0\}}\Big)\cdot\Delta^m\vartheta(w^l(t,\boldsymbol{x}',\boldsymbol{y}))[\bullet].\end{aligned}\tag{A.5}$$

We then proceed in a similar manner with the last term in (A.2). Importantly, we recall that, in the expectation therein, the ratio $S_{\mu^N_{\boldsymbol{x},\boldsymbol{y}}}[x^l]/(N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^l])$ is understood as one when $\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^l]=0$. In particular, as long as y^l itself cannot be zero, we always have that $y^lS_{\mu^N_{\boldsymbol{x},\boldsymbol{y}}}[x^l]/(N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^l])>0$. Hence,

$$w^l\left(t,\boldsymbol{x},y^1\frac{S_{\mu^N_{\boldsymbol{x},\boldsymbol{y}}}[x^1]}{N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^1]},\dots,y^N\frac{S_{\mu^N_{\boldsymbol{x},\boldsymbol{y}}}[x^N]}{N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^N]}\right)=w^l\left(t,\boldsymbol{x}',y^1\frac{S_{\mu^N_{\boldsymbol{x}',\boldsymbol{y}}}[x'^{,1}]}{N\mu^N_{\boldsymbol{x}',\boldsymbol{y}}[x'^{,1}]},\dots,y^N\frac{S_{\mu^N_{\boldsymbol{x}',\boldsymbol{y}}}[x^N]}{N\mu^N_{\boldsymbol{x}',\boldsymbol{y}}[x'^{,N}]}\right),$$

from which we deduce that

$$\begin{aligned}&\varepsilon N\mathbf{E}\left[\frac{S_{\mu^N_{\boldsymbol{x},\boldsymbol{y}}}[x^l]}{N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^l]}\left(w^l\left(t,\boldsymbol{x},y^1\frac{S_{\mu^N_{\boldsymbol{x},\boldsymbol{y}}}[x^1]}{N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^1]},\dots,y^N\frac{S_{\mu^N_{\boldsymbol{x},\boldsymbol{y}}}[x^N]}{N\mu^N_{\boldsymbol{x},\boldsymbol{y}}[x^N]}\right)-w^l(t,\boldsymbol{x},\boldsymbol{y})\right)\right]\\&=\varepsilon N\mathbf{E}\left[\frac{S_{\mu^N_{\boldsymbol{x}',\boldsymbol{y}}}[x'^{,l}]}{N\mu^N_{\boldsymbol{x}',\boldsymbol{y}}[x'^{,l}]}\left(w^l\left(t,\boldsymbol{x}',y^1\frac{S_{\mu^N_{\boldsymbol{x}',\boldsymbol{y}}}[x'^{,1}]}{N\mu^N_{\boldsymbol{x}',\boldsymbol{y}}[x'^{,1}]},\dots,y^N\frac{S_{\mu^N_{\boldsymbol{x}',\boldsymbol{y}}}[x^N]}{N\mu^N_{\boldsymbol{x}',\boldsymbol{y}}[x'^{,N}]}\right)-w^l(t,\boldsymbol{x}',\boldsymbol{y})\right)\right].\end{aligned}\tag{A.6}$$

Collecting (A.4)–(A.6), we deduce that, for any $l,n\in[\![N]\!]$ and $\boldsymbol{y}\in\mathbb{Y}$ with $y^l>0$ and $y^n=0$, the last three terms in (A.2) are the same when evaluated at $(t,\boldsymbol{x},\boldsymbol{y})$ and $(t,\boldsymbol{x}',\boldsymbol{y})$ with $\boldsymbol{x}^{-n}=\boldsymbol{x}'^{,-n}$. We finally observe that, for $l,n\in[\![N]\!]$ and for $(\boldsymbol{x},\boldsymbol{y})\in[\![d]\!]^N\times\mathbb{Y}$ fixed with $y^l>0$ and $y^n=0$, (A.2) may be regarded as an ordinary differential equation with $\widetilde{w}^l(\cdot,\boldsymbol{x},\boldsymbol{y})$ as a unique solution. Thus, $\widetilde{w}^l(\cdot, \boldsymbol{x}, \boldsymbol{y}) = \widetilde{w}^l(\cdot, \boldsymbol{x}', \boldsymbol{y})$. Put differently, $\widetilde{w}^l$ satisfies (A.3), which shows that (A.3) is stable by $\widehat{\Phi}$. Therefore, the fixed point $\widehat{w}$ of $\widehat{\Phi}$ also satisfies (A.3) (for any $l \in [\![N]\!]$).

A.2.3. Third Step. We eventually identify $(\widehat{w}^l)_{l \in [\![N]\!]}$ with the solution $(w^{N,l})_{l \in [\![N]\!]}$ given by Proposition 3.1. The proof mostly follows from the argument developed in the previous step. Indeed, we now know that both solutions can be approximated by iterating the mapping $\widehat{\Phi}$ associated with (A.2) with the special feature that all the inputs therein are required to satisfy (A.3).

The analysis achieved in the second step says that, in order to compute $\widetilde{w}^l$ at tuples $(t, \boldsymbol{x}, \boldsymbol{y}) \in [0,T] \times [\![d]\!]^N \times \mathbb{Y}^l$, the precise values of $\widehat{\alpha}^n(t, \boldsymbol{x}, \boldsymbol{y})$ at indices n for which $y^n = 0$ do not matter and, moreover, only the knowledge of each input w^m at points $(t, \boldsymbol{x}, \boldsymbol{y}) \in [0,T] \times [\![d]\!]^N \times \mathbb{Y}^m$, for $m \in [\![N]\!]$, does matter. As a result, for any $l \in [\![N]\!]$, the approximating sequences (as given by the iteration of the mapping $\widehat{\Phi}$ defined through the system (A.2)) of the two solutions $\widehat{w}^l$ and $w^{N,l}$ coincide at any point $(t, \boldsymbol{x}, \boldsymbol{y}) \in [0,T] \times [\![d]\!]^N \times \mathbb{Y}^l$. Therefore,

$$\widehat{w}^l(t, \boldsymbol{x}, \boldsymbol{y}) = w^{N,l}(t, \boldsymbol{x}, \boldsymbol{y}), \quad (t, \boldsymbol{x}, \boldsymbol{y}) \in [0,T] \times [\![d]\!]^N \times \mathbb{Y}^l,$$

which completes the proof.

Endnotes

[1] The interested reader may have a look at Delarue [17] for another construction of a finite state MFG with common noise.

[2] Here, the upper bound for ε is arbitrary and could be replaced by any other finite positive real.

[3] Notice that, in the following formula, we should write $\alpha^l(t, \boldsymbol{X}_t, \boldsymbol{Y}_t)[\bullet]$; for convenience, we remove the bullet $\bullet$.

[4] In fact, there are subtle questions here because the parameter ε in $\overline{\kappa}_0$ only comes through κ_0 itself and does not show up explicitly in our own computations. In other words, the construction of $\overline{\kappa}_0$ follows from constraints that are not the same as those used to define κ_0.

[5] Certainly, we could prove a form of uniqueness of this equilibrium, but it would be rather useless for us. For this reason, we feel better not to address it.

[6] We here use the letter n to denote the generic label of a player in the population, whereas we have used l so far. This is to stay consistent with the notations introduced in Section 3.2, which is used systematically in the sequel of this section.

[7] Note that, even in case $\widetilde{\tau}_N = t_0$ (meaning that the initial condition is outside the domain underpinning the definition of $\widetilde{\tau}_N$ in (5.3)), the bound from (4.16) still holds, which implies (5.18) in this case.

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