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Poisson geometry and representations of PI 4-dimensional Sklyanin algebras

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Abstract

Take S to be a 4-dimensional Sklyanin (elliptic) algebra that is module-finite over its center Z; thus, S is PI. Our first result is the construction of a Poisson Z-order structure on S such that the induced Poisson bracket on Z is non-vanishing. We also provide the explicit Jacobian structure of this bracket, leading to a description of the symplectic core decomposition of the maximal spectrum Y of Z. We then classify the irreducible representations of S by combining (1) the geometry of the Poisson order structures, with (2) algebro-geometric methods for the elliptic curve attached to S, along with (3) representation-theoretic methods using line and fat point modules of S. Along the way, we improve results of Smith and Tate obtaining a description the singular locus of Y for such S. The classification results for irreducible representations are in turn used to determine the zero sets of the discriminants ideals of these algebras S.

Keywords 4-dimensional Sklyanin algebra · Poisson order · Azumaya locus · Singular locus · Irreducible representation

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1 Introduction

1.1 Overview of Sklyanin algebras and results in the paper

Sklyanin algebras are quadratic algebras that play a major role in the Artin-Schelter-Tate-van den Bergh's classification of noncommutative projective spaces [2–5] and the Feigin-Odeskii's investigation of elliptic algebras [23,24]. These directions were motivated by Sklyanin's work [35] on quantum integrable systems, in which he introduced the algebras now known as *4-dimensional Sklyanin algebras* and studied their representations with applications to the quantum inverse problem method in quantum and statistical mechanics. Since then, n-dimensional versions of Sklyanin's algebras were introduced for $n \geq 3$ and have arisen in numerous areas including: deformation-quantization, non-commutative geometry, and quantum groups and R-matrices; see, for instance, the reviews [8,30,41] and the references within.

In terms of their representation theory, Sklyanin algebras fall into one of two classes– those that are module-finite over their center so that they satisfy a polynomial identity (or, are PI), or otherwise.

(A) In the generic (non-PI) case, a Sklyanin algebra is not finite-module over its center. Its representation theory can be viewed in parallel with the representations of quantum groups at non-roots of unity and Lie algebras over fields of characteristic 0. Sklyanin algebras are more complicated than the other two classes because they

do not have PBW bases in the usual sense and thus one cannot use the classical methods of standard modules (for instance Verma modules in the case of the categories $\mathcal{O}$). However, Artin-Schelter-Tate-van den Bergh have developed powerful projective algebro-geometric methods that can be applied to analyze the representations of these algebras via algebraic geometry of elliptic curves. Such methods were further developed in works of Levasseur–Smith [29], Smith–Staniszkis [38], and Staniszkis [42], to achieve results on the noncommutative projective algebraic geometry of irreducible representations of generic 4-dimensional Sklyanin algebras.

(B) Representations of PI Sklyanin algebras can be viewed in parallel with representations of quantum groups at roots of unity and modular representations of Lie algebras. This case is substantially harder than setting (A). *In this paper we unify the algebro-geometric methods for Sklyanin algebras with Poisson geometric methods for quantum groups at roots of unity, to*

(1) *construct nontrivial structures of Poisson orders on all PI 4-dimensional Sklyanin algebras, and*
(2) *explicitly classify their irreducible representations and describe their dimensions.*

Smith [36] has obtained a number of important results in the direction (2) by exclusively using algebro-geometric methods. Pertaining to direction (1), De Concini-Kac-Procesi [19,20] pioneered the applications of Poisson geometry in representation theory of PI algebras, for the cases of quantized universal enveloping algebras and quantum function algebras at roots of unity. This approach was axiomatized by Brown-Gordon [13] in the theory of *Poisson orders*, and was applied to other families of algebras with PBW bases, such as the symplectic reflection algebras. A Poisson order is an algebra A which is module-finite over its center Z, and such that Z admits a Poisson structure for which all Hamiltonian derivations of Z can be extended to derivations of A. The punchline of the construction is that Brown and Gordon used the latter extension to construct an isomorphism $A/(\mathfrak{m}A) \cong A/(\mathfrak{n}A)$ for $\mathfrak{m}, \mathfrak{n} \in \mathrm{maxSpec}(Z)$ in the same symplectic core, [13,19,20]. This provides general Poisson geometric tools to organize the irreducible representations of A into families that behave in similar ways (e.g., that have the same dimension).

We use previous algebro-geometric results on the PI 4-dimensional Sklyanin algebras S, [2–5,37,39,40], to prove that every such algebra has a nontrivial structure of Poisson order. We classify the symplectic cores of $Z(S)$ and obtain from it a concrete description of the Azumaya locus of S. We then link the developed Poisson geometry back to the algebro-geometric approach to the representation theory of S, [27,29,36], to classify the irreducible representations of S and to describe their dimensions. See Sects. 1.2 and 1.3 for more details.

Poisson structures on commutative algebras which arise as semiclassical limits of elliptic algebras were studied by Odesskiĭ-Feĭgin [23], Pym-Schedler [31], and others. The novelty in our work is the noncommutative extension of Poisson algebras to Poisson orders for all PI 4-dimensional Sklyanin algebras S, which is the key feature that is needed to approach the representation theory of S.

In our previous work [43], we have obtained similar results for the PI 3-dimensional Sklyanin algebras. The 4-dimensional case turns out to be substantially more challenging, and requires new approaches at various steps. We indicate those throughout the paper. On the other hand, we develop methods that can be applied in far greater generality than 4-dimensional Sklyanin algebras, for instance, connected $\mathbb{N}$-graded algebras T containing a regular sequence of central elements $\Omega_1, \ldots, \Omega_m$ such that $T/(T\Omega_1 + \cdots + T\Omega_m)$ is isomorphic to a twisted homogeneous coordinate ring of an elliptic curve; such results are highlighted throughout our work.

We use the solution of the classification problem (2) for irreducible representations of the algebras S to fully determine the zero sets of all of their discriminant ideals. This will likely have further applications in relation to recent work of Bell-Zhang [7], Ceken-Palmieri-Wang-Zhang [16], and Brown-Yakimov [15].

1.2 Poisson orders, classification of symplectic cores and Azumaya loci

We work over an algebraically closed base field $\Bbbk$ of characteristic 0. The projective algebro-geometric data attached to a 4-dimensional Sklyanin algebra S are as follows: an elliptic curve E in $\mathbb{P}^3$, an invertible sheaf $\mathcal{L}$ on E and an automorphism of σ E given by a translation of a point $\tau \in E$. See Sect. 2.1 for details.

One useful fact we have is that S is module-finite over its center, if and only if the automorphism σ attached to S has finite order. In this case, S is PI and the PI degree of S (which is a sharp upper bound on the dimension of irreducible representations of S) is equal to $|\sigma|$; see Proposition 2.12. So, consider the following standing hypothesis and notation used throughout this work.

Hypothesis 1.1 (S, Z, n, s) Let S be a 4-dimensional Sklyanin algebra, and (with the exception of Sects. 2 and 5) assume that S is module-finite over its center Z so that S is PI. For the geometric data $(E, \mathcal{L}, \sigma)$ attached to S mentioned above, let n denote $|\sigma|$, which is equal to the PI degree of S when $|\sigma|$ is finite. Also, let $s := n/(n, 2)$.

In this work, we do not consider the cases when n divides 4, as done typically in works on 4-dimensional Sklyanin algebras [29,38,42]. (See, e.g., [42, proof of Lemma 2.1].)

Smith and Tate [39] proved that the center $Z(S)$ of every PI 4-dimensional Sklyanin algebra S is generated by four elements $z_0, \ldots, z_3$ of degree n and two elements g_1, g_2 of degree 2, subject to two relations F_1 and F_2 of degree $2n$. Set

$$Y := \operatorname{maxSpec}(Z(S)) \quad \text{and} \quad Y_{\gamma_1,\gamma_2} := Y \cap \mathbb{V}(g_1 - \gamma_1, g_2 - \gamma_2), \quad \text{for } \gamma_1, \gamma_2 \in \Bbbk.$$

Denote by Y^{sing} the singular locus of Y. Further, $(Y_{\gamma_1,\gamma_2})^{sing}$ denotes the singular locus of the subvariety Y_{γ_1,γ_2} and

$$(Y^{sing})_{\gamma_1,\gamma_2} := Y^{sing} \cap Y_{\gamma_1,\gamma_2}.$$

Here, $Y^{sing} = \bigcup_{\gamma_1,\gamma_2\in\Bbbk}(Y^{sing})_{\gamma_1,\gamma_2}$, which is contained in but not necessarily equal to $\bigcup_{\gamma_1,\gamma_2\in\Bbbk}(Y_{\gamma_1,\gamma_2})^{sing}$. Moreover, consider the following notation for $\gamma_1, \gamma_2 \in \Bbbk$:

$$(Y_{\gamma_1,\gamma_2})^{smooth} := Y_{\gamma_1,\gamma_2} \setminus (Y_{\gamma_1,\gamma_2})^{sing}, \qquad Y_0^{symp} := \bigcup_{\gamma_1,\gamma_2\in\Bbbk}(Y_{\gamma_1,\gamma_2})^{sing}.$$

Our first main result constructs a nontrivial Poisson order on S, classifies the symplectic cores of its center $Z(S) = \Bbbk[z_0, z_1, z_2, z_3, g_1, g_2]/(F_1, F_2)$, and fully determines the Azumaya locus of S.

Theorem 1.2 (Corollaries 7.5 and 7.8, Theorem 8.1, and Proposition 8.8) *For all 4-dimensional Sklyanin algebras S satisfying Hypothesis 1.1, the following hold:*

(1) *S admits a nontrivial structure of Poisson order for which*
   (a) *the induced Poisson structure on the center $Z(S)$ is of Jacobian form in the sense of (7.4) in terms of two potentials taken to be F_1 and F_2, while*
   (b) *g_1, g_2 lie in the Poisson center of $Z(S)$.*

(2) *The corresponding symplectic core stratification of the Poisson variety Y are*
   (a) *2-dimensional cores: $(Y_{\gamma_1,\gamma_2})^{smooth}$ for $\gamma_1, \gamma_2 \in \Bbbk$;*
   (b) *0-dimensional cores: points in Y_0^{symp}; these are the points on which the Poisson bracket on Y in part (1) vanishes.*

(3) *$Y^{sing} \subseteq Y_0^{symp}$, with strict containment when n is odd, and they both have codimension ≥ 2 in Y;*
(4) *The Azumaya locus of S coincides with the smooth locus $Y \setminus Y^{sing}$ in Y.*

The above result can be thought of as a generalization of the previous work done for PI 3-dimensional Sklyanin algebras. There are a number of complications that arise in the 4-dimensional case compared to the 3-dimensional one. The first one is that in the 4-dimensional case we have $Y^{sing} \subsetneq Y_0^{symp}$ when the order n of σ is odd; in other words the set of symplectic points Y_0^{symp} of the Poisson variety Y is larger than its singular locus. Indeed in that case Y_0^{symp} is the union of Y^{sing} plus four cuspidal curves meeting at the origin; this is proved in Theorem 3.13.

See Fig. 1 in Sect. 3 for an illustration of the n odd case, where $Y^{sing} \subsetneq Y_0^{symp}$ as discussed above. On the other hand, Fig. 2 in Sect. 3 illustrates the n even case, when Y^{sing} and Y_0^{symp} coincide.

As mentioned in Sect. 1.1, the proof of the existence of nontrivial Poisson order structure on S relies on the previous algebro-geometric results for S and a method of higher order specialization developed in [43]. The precise (Jacobian) form of the Poisson structure on $Z(S)$ is derived from the property that g_1, g_2 are in the Poisson center of $Z(S)$ and from Stafford's result [40] that S is a maximal order.

The proof of Theorem 1.2(4) faces a second major difficulty compared to the 3-dimensional case. In the latter case, the symplectic cores of $Z(S)$ of maximal dimension have codimension 1, which was used to show that all of them are in the Azumaya locus of S by incorporating the automorphism group of S. This strategy fails in the 4-dimensional case since the leaves have codimension 2, while the automorphism

group of S is still 1-dimensional. Our strategy for the proof of part (4) is summarized as follows. Firstly, we consider the factors

$$S_{[\kappa_1:\kappa_2]} := S/(\kappa_1 g_1 - \kappa_2 g_2)S, \qquad \text{for } [\kappa_1 : \kappa_2] \in \mathbb{P}^1_{\Bbbk}$$

which all turn out to be Noetherian PI domains of the same PI degree as S. We show that the maximal spectra of their centers are canonically isomorphic to Poisson subvarieties of Y with the property that the symplectic leaves in them of maximal dimension have codimension 1 in Y. Now, by applying Brown and Gordon's Poisson order result [13] (as discussed in Sect. 4.2) to the symplectic core stratification of the centers of the algebras $S_{[\kappa_1:\kappa_2]}$ and using the density of their Azumaya loci, we conclude that all the points in the 2-dimensional cores $(Y_{\gamma_1,\gamma_2})^{smooth}$ lie in the Azumaya locus of S. Then the coincidence of the Azumaya and smooth locus of S follows from a general result of Brown and Goodearl [11] since the non-Azumaya locus of S contained in Y_0^{symp} has codimension ≥ 2 in Y. Thus, Poisson geometry was employed primarily to establish the codimension ≥ 2 fact, for which there are no other known methods.

The novelty of the 4-dimensional Sklyanin algebras of odd PI degree is that their Azumaya loci intersects nontrivially the varieties of symplectic points Y_0^{symp} of Y.

1.3 On irreducible representations of PI 4-dimensional Sklyanin algebras

Recall that n denotes the order of the automorphism σ of the elliptic curve attached to S; it equals the PI degree of S. For $y \in Y$, denote by $\mathfrak{m}_y$ the corresponding maximal ideal of $Z(S)$. It follows from Theorem 1.2(4) that

- For $y \in Y \backslash Y^{sing}$, $S/\mathfrak{m}_y S \cong M_n(\Bbbk)$, and thus, up to isomorphism, there is precisely one irreducible representation of S with central annihilator $\mathfrak{m}_y$ and this representation has dimension n.

It is a simple fact that

- For $y = \underline{0}$, $S/\mathfrak{m}_y S$ is a local algebra, and thus, up to isomorphism, S has precisely one irreducible representation with central annihilator $\mathfrak{m}_{\underline{0}}$ and this representation is the trivial representation of dimension 1.

To achieve results on irreducible representations of intermediate dimension, we first use the explicit structure of the center $Z(S)$ and its link with the geometry of the elliptic curve E to describe the singular locus Y^{sing} of S.

Theorem 1.3 (Theorems 3.13 and 3.16) *Let S be a 4-dimensional Sklyanin algebra satisfying Hypothesis 1.1. Denote by E_2 the subgroup of 2-torsion points of E.*

(1) *If n is odd, then the singular locus of Y is the union of $2(n-1)$ cuspidal curves $C(\omega + k\tau)$,*

$$Y^{sing} = \bigcup_{\substack{\omega \in E_2 \\ 0 \leq k \leq n-2}} C(\omega + k\tau),$$

defined in Lemma 3.12, which only meet at the origin, as depicted in Fig. 1.

(2) *If $n = 2s$ is even, then Y^{sing} is the union of two explicitly defined subvarieties Y_1^{sing} and Y_2^{sing} in Theorem 3.16, which only meet at the origin, as depicted in Fig. 2. When $\Bbbk = \mathbb{C}$, they contain the cuspidal curves*

$$\mathcal{C}(\omega + k\tau) \quad \text{for } \omega \in E_2,\ 0 \leq k \leq s-1$$

described in Notation 8.27.

The proof of Theorem 1.3 relies on the geometry of line modules and fat point modules of S, extending results of [29,39].

Next, we fully classify the irreducible representations of S of intermediate dimension and describe their dimensions. Along the way, using the action of Heisenberg group H_4 of order 64 on S, we determine the exact form of the Smith–Tate defining relations F_1 and F_2 of the center $Z(S)$ when n is even, which is of independent interest (see Proposition 2.23 below).

Theorem 1.4 (Theorems 8.29 and 8.30) *Let S be a complex 4-dimensional Sklyanin algebra satisfying Hypothesis 1.1. Denote by $\mathrm{Irr}_d S$ the isomorphism classes of all d-dimensional irreducible representations of S. Recall the notation above.*

(1) *If $PIdeg(S) = n$ is odd, then we have the following maps via central annihilators.*

$$\mathrm{Irr}_n S \xrightarrow{1:1} Y^{smooth} = (Y \setminus Y^{sing})$$

$$\mathrm{Irr}_{k+1} S \sqcup \mathrm{Irr}_{n-1-k} S \xrightarrow{2:1} (C(\omega + k\tau) = C(\omega + (n-2-k)\tau)) \setminus \{\underline{0}\},\quad 0 \leq k \leq n-2$$

$$\mathrm{Irr}_1 S \xrightarrow{1:1} \{\underline{0}\}$$

(2) *If $PIdeg(S) = n = 2s$ is even, then we have the following maps via central annihilators.*

$$\mathrm{Irr}_n S \xrightarrow{1:1} Y^{smooth} = (Y \setminus Y^{sing})$$

$$\mathrm{Irr}_s S \xrightarrow{2:1} Y^{sing} \setminus \bigcup_{\omega \in E_2, 0 \leq k \leq s-2} \mathcal{C}(\omega + k\tau)$$

$$\mathrm{Irr}_{k+1} S \sqcup \mathrm{Irr}_{s-1-k} S \xrightarrow{4:1} (\mathcal{C}(\omega + k\tau) = \mathcal{C}(\omega + s\tau + (s-2-k)\tau)) \setminus \{\underline{0}\},\quad 0 \leq k \leq s-2;$$

$$\mathrm{Irr}_1 S \xrightarrow{1:1} \{\underline{0}\}$$

Over complex numbers $\Bbbk = \mathbb{C}$, Sklyanin constructed in [35] for each $\omega \in E_2$ and $k \in \mathbb{N} \cup \{0\}$, a representation $V(\omega + k\tau)$ over S in a certain $(k+1)$-dimensional subspace of theta functions of order $2k$; see [38, Section 3] for details. These representations were proved later by Smith and Staniszkis to be irreducible whenever $k < s$ [38, Theorem 3.6]. Our next result shows that every irreducible representation of S of intermediate dimension $< s$ is a scalar twist of $V(\omega + k\tau)^\lambda$ for some $\lambda \in \Bbbk$ such that $V(\omega + k\tau)^\lambda$ equals $V(\omega + k\tau)$ as vector spaces and $s_i \cdot_\lambda v = \lambda^i s_i v$ for any homogeneous element s_i in S of degree i and $v \in V(\omega + k\tau)^\lambda$.

Theorem 1.5 (Theorems 8.29 and 8.30) *Let S be a complex 4-dimensional Sklyanin algebra satisfying Hypothesis 1.1. Then S has irreducible representations of each dimension $1, 2, 3, \ldots, n$ $(= s)$ if n is odd, and of each dimension $1, 2, 3, \ldots, \frac{n}{2}$ $(= s)$, n if n is even. Moreover, the nontrivial irreducible representations of intermediate dimension $< s$ are given by scalar twists of $V(\omega + k\tau)$ for all $\omega \in E_2$ and $0 \leq k \leq s-2$.*

To prove Theorems 1.4 and 1.5, we use the classification of fat points of S by Smith in [36], a number of other representation-theoretic results of Levasseur [29], Le Bruyn [27], Smith [36] and Staniszkis [38], and part (4) of Theorem 1.2 that the non-Azumaya part of $Z(S)$ is Y^{sing}. In particular, we apply the deep connections between fat point modules, $\mathbb{C}^\times \times \mathrm{PGL}_d$-stabilizers of irreducible representations, and $\mathbb{C}^*$-families of irreducible representations of a graded algebra within the framework of the noncommutative projective algebraic geometry of S.

Finally, Theorem 8.31 in Sect. 8.3 contains the aforementioned results on the description of the discriminant ideals of the algebras S.

Remark 1.6 Note that Theorem 1.4 makes precise the representation-theoretic connection between the PI 4-dimensional Sklyanin algebras and the quantized enveloping algebra $U_q(SU(2))$ at q a root of unity, as introduced initially in the physics literature. Namely, the main result of Roche-Arnaudon [33] is that when q is a root of unity of order m, then $U_q(SU(2))$ has d-dimensional irreducible representations for all $1 \leq d \leq m$ when m is odd, and has d-dimensional irreducible representations for all $1 \leq d \leq \frac{m}{2}$ and $d = m$ when m is even. They also show that $U_q(SU(2))$ arises as a 'trigonometric limit' of a 4-dimensional Sklyanin algebra S. Connections between S and $U_q(SU(2))$ (or rather, $U_q(\mathfrak{sl}_2)$) are also discussed in work of Smith and Stafford [37, Section 1].

2 Preliminary results on 4-dimensional Sklyanin algebras *S*

We provide in this section background material on (the noncommutative projective algebraic geometry of) 4-dimensional Sklyanin algebras. This includes a discussion of various ring-theoretic and homological properties of these algebras, which were established by Smith–Stafford [37]. We also provide a detailed analysis of the center and symmetries of 4-dimensional Sklyanin algebras, which will play a key role in the rest of the paper. This analysis extends results of Smith–Tate [39], Smith–Staniszkis [38], and Chirvasitu–Smith [17].

In this section, we do not assume that the 4-dimensional Sklyanin algebras S are PI.

2.1 Noncommutative geometry of S

Here, we recall the definition and properties of the 4-dimensional Sklyanin algebras and the corresponding twisted homogeneous coordinate rings.

Definition 2.1 $(S, S(\alpha,\beta,\gamma), x_i)$ [37] Take $\alpha,\beta,\gamma \in \Bbbk$ so that

$$\alpha+\beta+\gamma+\alpha\beta\gamma = 0 \quad [\Leftrightarrow (1+\alpha)(1+\beta)(1+\gamma) = (1-\alpha)(1-\beta)(1-\gamma)], \tag{2.2}$$

$$(\alpha,\beta,\gamma) \notin \{(-1,1,\gamma),(\alpha,-1,1),(1,\beta,-1)\}. \tag{2.3}$$

Then the (regular) *4-dimensional Sklyanin algebra* $S := S(\alpha,\beta,\gamma)$ are $\Bbbk$-algebras, generated by noncommuting variables x_0, x_1, x_2, x_3 of degree one, subject to the following relations

$$\begin{aligned} x_0x_1 - x_1x_0 &= \alpha(x_2x_3 + x_3x_2), & x_0x_1 + x_1x_0 &= x_2x_3 - x_3x_2,\\ x_0x_2 - x_2x_0 &= \beta(x_3x_1 + x_1x_3), & x_0x_2 + x_2x_0 &= x_3x_1 - x_1x_3,\\ x_0x_3 - x_3x_0 &= \gamma(x_1x_2 + x_2x_1), & x_0x_3 + x_3x_0 &= x_1x_2 - x_2x_1. \end{aligned}$$

These algebras come equipped with geometric data that is used to establish many of their ring-theoretic, homological, and representation-theoretic properties. To start, take a $\mathbb{N}$-graded algebra $A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$ with $A_0 = \Bbbk$, and recall that an *A-point module* is a cyclic, graded left A-module with Hilbert series $(1-t)^{-1}$. These modules play the role of points in noncommutative projective algebraic geometry and the parameterization of A-point modules is referred to as the *point scheme* of A. See [8, Chapter I, Section 3] for more details.

Definition-Lemma 2.4 $(\widehat{E}, E, \phi_1, \phi_2, v_i, e_i, \sigma, \tau)$ [37, Section 2] *The point scheme of the 4-dimensional Sklyanin algebra $S = S(\alpha,\beta,\gamma)$ is given by the union $\widehat{E}$ of an elliptic curve $E := \mathbb{V}(\phi_1,\phi_2) \subseteq \mathbb{P}^3_{[v_0:v_1:v_2:v_3]}$, where*

$$\phi_1 = v_0^2 + v_1^2 + v_2^2 + v_3^2 \quad \text{and} \quad \phi_2 = \tfrac{1-\gamma}{1+\alpha}\, v_1^2 + \tfrac{1+\gamma}{1-\beta}\, v_2^2 + v_3^2,$$

and the four points

$$\{e_0 := [1:0:0:0],\ e_1 := [0:1:0:0],\ e_2 := [0:0:1:0],\ e_3 := [0:0:0:1]\}.$$

The automorphism $\sigma = \sigma_{\alpha\beta\gamma}$ of $\widehat{E}$ attached to S fixes each of the four points e_i, and on E it is defined on a dense open subset by

$$\sigma : \begin{bmatrix} v_0\\ v_1\\ v_2\\ v_3 \end{bmatrix} \mapsto \begin{bmatrix} -2\alpha\beta\gamma v_1v_2v_3 - v_0(-v_0^2 + \beta\gamma v_1^2 + \alpha\gamma v_2^2 + \alpha\beta v_3^2)\\ 2\alpha v_0v_2v_3 + v_1(v_0^2 - \beta\gamma v_1^2 + \alpha\gamma v_2^2 + \alpha\beta v_3^2)\\ 2\beta v_0v_1v_3 + v_2(v_0^2 + \beta\gamma v_1^2 - \alpha\gamma v_2^2 + \alpha\beta v_3^2)\\ 2\gamma v_0v_1v_2 + v_3(v_0^2 + \beta\gamma v_1^2 + \alpha\gamma v_2^2 - \alpha\beta v_3^2) \end{bmatrix}. \tag{2.5}$$

The automorphism σ of E is given by translation by a point of E; call this point τ. The triple $\left(E,\ \mathcal{O}_{\mathbb{P}^3}(1)|_E,\ \sigma\right)$ is referred to as the geometric data of S. □

Using this data, we consider a noncommutative coordinate ring of E; its generators are sections of the invertible sheaf $\mathcal{O}_{\mathbb{P}^3}(1)|_E$ and its multiplication depends on the automorphism σ. The general construction is given as follows.

Definition 2.6 $(\mathcal{L}_i)$ [6] Given a projective scheme X, an invertible sheaf $\mathcal{L}$ on X, and an automorphism σ of X, the *twisted homogeneous coordinate ring* attached to this geometric data is a graded $\Bbbk$-algebra

$$B(X, \mathcal{L}, \sigma) = \bigoplus_{i \geq 0} B_i, \quad \text{where } B_i := H^0(X, \mathcal{L}_i)$$

with $\mathcal{L}_0 = \mathcal{O}_X$, $\mathcal{L}_1 = \mathcal{L}$, and $\mathcal{L}_i = \mathcal{L} \otimes \mathcal{L}^{\sigma} \otimes \cdots \otimes \mathcal{L}^{\sigma^{i-1}}$ for $i \geq 2$. The multiplication map $B_i \otimes B_j \to B_{i+j}$ is defined by $b_i \otimes b_j \mapsto b_i b_j^{\sigma^i}$ using $\mathcal{L}_i \otimes \mathcal{L}_j^{\sigma^i} = \mathcal{L}_{i+j}$.

Notation 2.7 $(B, \mathcal{L})$ Let B denote the twisted homogeneous coordinate ring attached to the geometric data $(E, \mathcal{L}, \sigma)$ from Definition-Lemma 2.4, where $\mathcal{L} := \mathcal{O}_{\mathbb{P}^3}(1)|_E$.

It is often useful to employ the following embedding of B into a skew-Laurent extension of the function field of E.

Lemma 2.8 [2] *Given $(E, \mathcal{L}, \sigma)$ from Definition-Lemma 2.4, extend σ to an automorphism of the field $\Bbbk(E)$ of rational functions on E by $v^{\sigma}(p) = v(\sigma^{-1} p)$ for $v \in \Bbbk(E)$ and $p \in E$. For any nonzero section w of $\mathcal{L}$, that is, any degree 1 element of B, take D to be the divisor of zeros of w, and let V denote $H^0(E, \mathcal{O}_E(D)) \subset \Bbbk(E)$.*

Then, the vector space isomorphism $vw \mapsto vt$ for $v \in V$ extends to an embedding of B in $\Bbbk(E)[t^{\pm 1}; \sigma]$. Here, $tv = v^{\sigma} t$ for $v \in \Bbbk(E)$. □

Now the first step in obtaining useful properties of 4-dimensional Sklyanin algebras is to use the result below.

Lemma 2.9 (g_1, g_2) [37, Lemma 3.3, Corollary 3.9, Theorem 5.4] *The degree 1 spaces of S and of B are equal and there is a surjective map from S to B, whose kernel is generated by the two central degree 2 elements below*

$$g_1 := -x_0^2 + x_1^2 + x_2^2 + x_3^2 \qquad \text{and} \qquad g_2 := x_1^2 + \tfrac{1+\alpha}{1-\beta}\, x_2^2 + \tfrac{1-\alpha}{1+\gamma}\, x_3^2. \quad (2.10)$$

Moreover, $\{g_1, g_2\}$ is a central regular sequence in S. □

Many good ring-theoretic and homological properties of S are obtained by lifting such properties from the factor B, some of which are listed in the following result.

Proposition 2.11 [37, Theorem 5.5] [28, Corollary 6.7] *The 4-dimensional Sklyanin algebras are Noetherian domains of global dimension 4, they satisfy the Artin-Schelter Gorenstein condition, along with the Auslander-regular and Cohen-Macaulay conditions, and they have Hilbert series $(1-t)^{-4}$.* □

The representation theory of both S and B depend on the geometric data $(E, \mathcal{L}, \sigma)$, as illustrated by the following result.

Proposition 2.12 (*n*) *Both of the algebras S and* $B = B(E, \mathcal{L}, \sigma) \cong S/(Sg_1 + Sg_2)$ *are module-finite over their center, if and only if the automorphism* σ *has finite order. In this case, both S and B satisfy a polynomial identity (i.e., are PI) and are of PI degree* $n := |\sigma| < \infty$. □

Proof Suppose that the automorphism σ has finite order. Since $\mathcal{L}$ is ample and σ-ample, the algebra B is module-finite over its center by [39, Corollary 2.3]. Moreover, so is S by [39, Theorem 3.7(c)].

On the other hand, suppose that the automorphism σ has infinite order. Now the center of S is the polynomial ring $\Bbbk[g_1, g_2]$ by [29, Proposition 6.12]. By comparing the Hilbert series of S to that of $\Bbbk[g_1, g_2]$, we get S cannot be module-finite over its center $\Bbbk[g_1, g_2]$. As a consequence, B is not module-finite over its center as B is a quotient ring of S by a regular sequence of homogenous central elements of degree two by [39, Lemma 3.6(b)].

To verify the last statement, note that B has PI degree n since it has a localization isomorphic to $\Bbbk(E)[t^{\pm 1}; \sigma]$ (see Lemma 2.9), which in turn also has PI degree n. We will also see later in Corollary 8.16 that all nontrivial irreducible representations of B have dimension n, which provides another proof that B has PI degree n.

Finally, we see that the PI degree of S is also n as follows. First, $\mathrm{PIdeg}(S) \geq \mathrm{PIdeg}(B)$ since B is a homomorphic image of S, and recall that $\mathrm{PIdeg}(B) = n$ from above. So, it suffices to show that $\mathrm{PIdeg}(S) \leq n$. Take the central element defined in [36, (4-2)], which is denoted by g later in Notation 3.9. (It is denoted by "c" in [36, Theorem 7.7].) Let M be an irreducible representation of S that is g-torsionfree. By [12, Theorem III.1.7], we know the Azumaya locus of S is dense in $\mathrm{maxSpec}(Z)$. So, we only need to show $\dim M \leq n$. Recall $s = n/(n, 2)$ and note that M is the quotient of a g-torsionfree *fat point module* F of multiplicity $s > 1$ by [38, Lemma 4.1] and [36, Theorem 7.7(c)]. Fat point modules will be discussed in detail in Sect. 8.2; they serve as representatives of the simple objects of the quotient category S-qgr which is the category of graded S-modules modulo those that are bounded above ($M_n = 0$ for all $n \gg 0$). Now if n is odd, then F and its shift $F[1]$ are equivalent in S-qgr by [36, Corollary 8.7]. So, $\dim(M) \leq \mathrm{mult}(F) = s = n$ by [36, discussion after Proposition 3.19]. On the other hand, if n is even, then F and $F[2]$ are equivalent in S-qgr by [36, Proposition 5.4], and hence, $\dim(M) \leq 2\,\mathrm{mult}(F) = 2s = n$ by [36, discussion after Proposition 3.19]. Thus, $\dim(M) \leq n$, as desired. □

Remark 2.13 The claim that the $\mathrm{PIdeg}(S) = n$ first appeared in [36, Theorem 8.8(2)], but the stronger statement that the algebra $S[c^{-1}]$ in [36, Theorem 8.8(1)] is Azumaya (which is used to prove [36, Theorem 8.8(2)]) is incorrect; there, "c" is our "g" in Notation 3.9. We establish later in Theorems 8.1 and 3.16 that the Azumaya locus of S is equal to the smooth locus of $Y := \mathrm{maxSpec}(Z(S))$, and in the case when n is even, the singular locus of Y does not lie in the hypersurface $\mathbb{V}(g)$. That is, there exist g-torsionfree irreducible representations of S of dimension $< n$ when n is even.

2.2 Center of S

As mentioned in the proof of Proposition 2.12, the center of a 4-dimensional Sklyanin algebra S is equal to $\Bbbk[g_1, g_2]$ when $n := |\sigma|$=PIdeg(S) is infinite. On the other hand, one expects that both S and B have a large center when the PI degree of S is finite. We record here several results of Smith–Tate [39] pertaining to the center Z of S when $\mathrm{PIdeg}(S) = n < \infty$.

Lemma 2.14 *(E'')* [39, Corollary 2.8] *Given the geometric data $(E, \mathcal{L}, \sigma)$ from Definition-Lemma 2.4, suppose that $|\sigma| =: n < \infty$. Take $E'' := E/\langle\sigma\rangle$ so that $E \to E''$ is a cyclic étale cover of degree n. Recall Lemma 2.8 and let D'' be the image of D on E'' and let V'' denote $H^0(E'', \mathcal{O}_{E''}(D''))$.*

Then, the center of B is the intersection of B with $\Bbbk(E'')[t^{\pm n}]$, which is equal to $\Bbbk[V''t^n]$, and this is also a twisted homogeneous coordinate ring of E'' for an embedding of $E'' \subseteq \mathbb{P}^3$. □

Central elements of B lift to central elements of S as described below. We will identify $S_1 \cong B_1$ via the canonical projection mentioned in Lemma 2.9.

Definition 2.15 *(s)* [39] Take $|\sigma| =: n < \infty$ and let s be the value $n/(n, 2)$. A section of $B_1 := H^0(E, \mathcal{L})$ is called *good* if its divisor of zeros is invariant under σ^s and consists of distinct points, whose orbits under the group $\langle\sigma\rangle$ do not intersect. A *good basis* of B_1 is a basis that consists of good elements so that the s-th powers of these elements generate B_n if n is odd or generate $(B^{\langle\sigma^2\rangle})_{n/2}$ if n is even.

Notation 2.16 *(ρ)* Take $|\sigma| =: n < \infty$. As mentioned at [39, top of page 31], σ^s fixes the class $[\mathcal{L}]$ in Pic E. So the automorphism σ^s of E induces an automorphism of B_1 via the identification $\mathcal{L}^{\sigma^s} \cong \mathcal{L}$. This automorphism of B will be denoted by ρ.

By [39, Lemma 3.4 and its proof], there is a unique lifting of ρ to a graded automorphism of S and ρ^2 is the identity. Further, the elements x_i^2 and the central elements g_1 and g_2 are all ρ-invariant.

Lemma 2.17 [39, proof of Lemma 3.4, page 46] *When $|\sigma| =: n < \infty$, we can take as a good basis of B_1 the set of generators x_0, x_1, x_2, x_3 of S where each x_i is a ρ-eigenvector. If n is odd, then ρ is the identity. If n is even, then $\rho = \sigma^{n/2}$ of order 2 and each ρ-eigenspace of B_1 is two-dimensional eigenspace with one having eigenvalue 1 and the other having eigenvalue -1.* □

Proposition 2.18 *$(Z, z_i, F_i, u_i, \Phi_i, \ell_i, h_i, f_i, E')$* [39, Theorems 3.7, 4.6, 4.9, 4.10]. *The center Z of S, of PI degree $n < \infty$, is given as follows.*

(1) *The center Z is generated by four algebraically independent elements z_0, z_1, z_2, z_3 of degree n along with g_1, g_2 in (2.10), subject to two relations F_1, F_2 of degree $2n$. In fact, there is a choice of generators z_i of the form*

$$z_i = x_i^n + \textstyle\sum_{1 \le j < n/2} c_{ij} x_i^{n-2j}$$

where $\{x_0, x_1, x_2, x_3\}$ is any good basis of B_1 and $c_{ij} \in \Bbbk[g_1, g_2]_{2j}$.

(2) *If n is even, then there exist elements u_0, u_1, u_2, u_3 of degree $n/2$ whose image in the Veronese subalgebra, $B^{(n/2)}$, of B generate the center. Here, $z_i = u_i^2$.*
(3) *If n is odd, then for $i = 1, 2$,*

$$F_i = \Phi_i(z_0, z_1, z_2, z_3) + h_i(g_1, g_2),$$

where Φ_1, Φ_2 are the quadratic homogeneous defining polynomials of the elliptic curve

$$E'' = E/\langle\sigma\rangle \subset \mathbb{P}^3 = \mathbb{P}(H^0(E'', \mathcal{L}'')^*)$$

with $\mathcal{L}''$ the descent of $\mathcal{L}_n$ to E'', and where h_1, h_2 are homogeneous degree s forms in variables g_1, g_2 having no common factor.
(4) *If n is even, then for $i = 1, 2$,*

$$F_i = \Phi_i(z_0, z_1, z_2, z_3) + \ell_i(z_0, z_1, z_2, z_3)h_i(g_1, g_2) + h_i(g_1, g_2)^2, \quad \text{and}$$
$$f_i(u_0, u_1, u_2, u_3) + h_i(g_1, g_2) = 0 \quad \text{in } Z(S^{(2)}).$$

Here, $E'' = E/\langle\sigma\rangle = \mathbb{V}(\Phi_1, \Phi_2)$, ℓ_1, ℓ_2 are linear forms and f_1, f_2 are linearly independent quadratic forms in the variables u_i defining the elliptic curve

$$E' := E/\langle\sigma^2\rangle \subset \mathbb{P}^3 = \mathbb{P}(H^0(E', \mathcal{L}')^*),$$

where $\mathcal{L}'$ is the descent of $\mathcal{L}_s$ to E', and h_1, h_2 are homogeneous degree s forms in variables g_1, g_2 having no common factor. □

We will use the defining relations of the center Z of PI Sklyanin algebras S mentioned above more extensively in Sect. 3 to figure out the singular locus of maxSpec(Z).

2.3 Symmetries of *S* and of their centers

Now we turn our attention to symmetries of 4-dimensional Sklyanin algebras S and of their respective centers Z. Recall that we do not assume that S satisfies a polynomial identity in this section.

Proposition 2.19 *(H_4, ϵ_1, ϵ_2, ϵ, a, b, c, ξ)* [38, Section 2] [17, Section 2.7] *The group of graded automorphisms of S contains the Heisenberg group H_4 of order 64 which is presented as*

$$H_4 = \langle \epsilon_1, \epsilon_2, \epsilon \ : \ \epsilon_1^4 = \epsilon_2^4 = \epsilon^4 = 1, \ \ \epsilon\epsilon_1 = \epsilon_1\epsilon, \ \ \epsilon\epsilon_2 = \epsilon_2\epsilon, \ \ \epsilon_1\epsilon_2 = \epsilon\epsilon_2\epsilon_1 \rangle.$$

Here, ϵ scales the generators of S by $-i$ and

$$\epsilon_1 : (x_0, x_1, x_2, x_3) \mapsto (b^{\frac{1}{2}} c^{\frac{1}{2}} \xi^{-1} x_1, \quad b^{\frac{-1}{2}} c^{\frac{-1}{2}} \xi x_0, \quad b^{\frac{1}{2}} c^{\frac{-1}{2}} \xi x_3, \ -b^{\frac{-1}{2}} c^{\frac{1}{2}} \xi^{-1} x_2),$$
$$\epsilon_2 : (x_0, x_1, x_2, x_3) \mapsto (a^{\frac{1}{2}} c^{\frac{1}{2}} \xi^{-1} x_2, \ -a^{\frac{1}{2}} c^{\frac{-1}{2}} \xi^{-1} x_3, \ a^{\frac{-1}{2}} c^{\frac{-1}{2}} \xi x_0, \quad a^{\frac{-1}{2}} c^{\frac{1}{2}} \xi x_1)$$

for $a^2 = \alpha,\ b^2 = \beta,\ c^2 = \gamma$, $i := \sqrt{-1} = e^{\frac{\pi i}{2}}$, *and* $\xi = e^{\frac{3\pi i}{4}}$. *In particular,* ϵ^2 *scales the generators of* S *by* -1 *and*

$$\epsilon_1^2 : (x_0, x_1, x_2, x_3) \mapsto (x_0, x_1, -x_2, -x_3)$$
$$\epsilon_2^2 : (x_0, x_1, x_2, x_3) \mapsto (x_0, -x_1, x_2, -x_3).$$

□

The following consequences hold mostly by direct computation. First, consider the following notation.

Notation 2.20 (E_2, ω) Let E_2 be the set of the (four) 2-torsion points of E, and let a point of E_2 be denoted by ω.

Corollary 2.21 (N_4) *Let* N_4 *be the subgroup of* H_4 *generated by* ϵ^2, ϵ_1^2, ϵ_2^2, *which is normal and is isomorphic to* $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. *We have the statements below.*

(1) *The element* ϵ *has four 1-dimensional eigenspaces each with eigenvalue* $-i$.
(2) *The element* ϵ_1^2 *has two 2-dimensional eigenspaces* $\langle x_0, x_1\rangle$ *and* $\langle x_2, x_3\rangle$ *with eigenvalue* 1 *and* -1, *respectively; the element* ϵ_2^2 *has two 2-dimensional eigenspaces* $\langle x_0, x_2\rangle$ *and* $\langle x_1, x_3\rangle$ *with eigenvalue* 1 *and* -1, *respectively; and the element* $\epsilon_1^2\epsilon_2^2$ *has two 2-dimensional eigenspaces* $\langle x_0, x_3\rangle$ *and* $\langle x_1, x_2\rangle$ *with eigenvalue* 1 *and* -1, *respectively.*
(3) *The element* ρ *from Notation 2.16 is an element of the quotient group* $N_4/\langle\epsilon^2\rangle$.
(4) *The subspace* $\Bbbk g_1 + \Bbbk g_2 \subset Z(S)_2$ *is a 2-dimensional irreducible representation of* H_4. *In the basis* $\{g_1, g_2\}$, *the* H_4*-action is given by* $\epsilon \mapsto diag(-1, -1)$, *and*

$$\epsilon_1 \mapsto \frac{i}{bc}\begin{pmatrix} 1 & 1 \\ -1-\beta\gamma & -1 \end{pmatrix}, \quad \epsilon_2 \mapsto \frac{i}{ac}\begin{pmatrix} 1 & \frac{1+\alpha}{1-\beta} \\ -1-\gamma & -1 \end{pmatrix};$$

this implies that the subgroup N_4 *fixes* g_1, g_2.
(5) *The group of graded automorphisms of* B *contains* H_4.
(6) *The subspace* $\Bbbk z_0 + \Bbbk z_1 + \Bbbk z_2 + \Bbbk z_3 \subset Z(S)_n$ *is a 4-dimensional representation of* H_4. *In the basis* $\{z_0, z_1, z_2, z_3\}$, *the* H_4*-action is given by*

$$\epsilon_1 \mapsto \begin{pmatrix} 0 & b^{-\frac{n}{2}}c^{-\frac{n}{2}}\xi^n & 0 & 0 \\ b^{\frac{n}{2}}c^{\frac{n}{2}}\xi^{-n} & 0 & 0 & 0 \\ 0 & 0 & 0 & (-1)^n b^{-\frac{n}{2}}c^{\frac{n}{2}}\xi^{-n} \\ 0 & 0 & b^{\frac{n}{2}}c^{-\frac{n}{2}}\xi^n & 0 \end{pmatrix},$$

$$\epsilon_2 \mapsto \begin{pmatrix} 0 & 0 & a^{-\frac{n}{2}}c^{-\frac{n}{2}}\xi^n & 0 \\ 0 & 0 & 0 & a^{-\frac{n}{2}}c^{\frac{n}{2}}\xi^n \\ a^{\frac{n}{2}}c^{\frac{n}{2}}\xi^{-n} & 0 & 0 & 0 \\ 0 & (-1)^n a^{\frac{n}{2}}c^{-\frac{n}{2}}\xi^{-n} & 0 & 0 \end{pmatrix},$$

and $\epsilon \mapsto \mathrm{diag}((-i)^n, (-i)^n, (-i)^n, (-i)^n)$; *this implies that* N_4 *fixes* z_0, z_1, z_2, z_3 *when* n *is even.*

Proof The first statement about N_4 is clear.

(1) and (2) are clear.

(3) By Lemma 2.17, we only treat the case when $n := |\sigma|$ is even. Here, $\rho = \sigma^{n/2}$ is given by translation by a point ω of E of order 2. We aim to pick off the graded automorphism of $S_1 = B_1$ to which ρ corresponds. To do this, first identify the generators x_i of S with the coordinates v_i of E. Then, take $\Bbbk = \mathbb{C}$ and write E as $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\Lambda)$ for some $\Lambda \in \mathbb{C}$ with $\mathrm{Im}(\Lambda) > 0$ as in [37, Sections 2.9–2.13]. Now ρ corresponds to a translation by a nontrivial 2-torsion point ω of $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\Lambda)$, that is

$$\omega \in \left\{\tfrac{1}{2},\ \tfrac{\Lambda}{2},\ \tfrac{1+\Lambda}{2}\right\}.$$

Using the notation of [37], note that

$$j : \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\Lambda) \to E, \quad z \mapsto [g_{11}(z) : g_{00}(z) : g_{01}(z) : g_{10}(z)]$$

is an isomorphism where

$$g_{pq}(z) = \theta_{pq}(2z)\theta_{pq}(\omega)\gamma_{pq}$$

are holomorphic functions on $\mathbb{C}$ so that the theta functions satisfy the conditions

$$\theta_{pq}(z+1) = (-1)^p\theta_{pq}(z) \quad \text{and} \quad \theta_{pq}(z+\Lambda) = -\exp(\Lambda + 2z + q)\theta_{pq}(z)$$

for $p, q = 0, 1$, and $\gamma_{00} = \gamma_{11} = i$, $\gamma_{01} = \gamma_{10} = 1$. In particular, $\rho(j(z)) = j(z+\omega)$.

Now take $\omega = \frac{1}{2}$. Then,

$$g_{pq}(z+\tfrac{1}{2}) = \theta_{pq}(2z+1)\theta_{pq}(\tfrac{1}{2})\gamma_{pq} = (-1)^p\theta_{pq}(2z)\theta_{pq}(\tfrac{1}{2})\gamma_{pq} = (-1)^p g_{pq}(z).$$

Therefore, we get that

$$\rho([g_{11}(z) : g_{00}(z) : g_{01}(z) : g_{10}(z)]) = [-g_{11}(z) : g_{00}(z) : g_{01}(z) : -g_{10}(z)],$$

which is realized as the coset of $\epsilon_1^2\epsilon_2^2$ in $N_4/\langle\epsilon^2\rangle$ by Proposition 2.19.

For $\omega = \frac{\Lambda}{2}$, we have that

$$\begin{aligned} g_{pq}(z+\tfrac{\Lambda}{2}) &= \theta_{pq}(2z+\Lambda)\theta_{pq}(\frac{\Lambda}{2})\gamma_{pq} \\ &= \exp(-4\pi i z)\exp(-\pi i\Lambda)\exp(-\pi i q)g_{pq}(z). \end{aligned}$$

So,

$$\rho([g_{11}(z) : g_{00}(z) : g_{01}(z) : g_{10}(z)]) = [-g_{11}(z) : g_{00}(z) : -g_{01}(z) : g_{10}(z)],$$

which is realized as the coset of ϵ_2^2 in $N_4/\langle\epsilon^2\rangle$ by Proposition 2.19.

For $\omega = \frac{1+\Lambda}{2}$, we have that

$$\begin{aligned} g_{pq}(z + \tfrac{1+\Lambda}{2}) &= \theta_{pq}(2z + 1 + \Lambda)\theta_{pq}(\frac{1+\Lambda}{2})\gamma_{pq} \\ &= (-1)^p \exp(-4\pi i z)\exp(-\pi i \Lambda)\exp(-\pi i q) g_{pq}(z). \end{aligned}$$

So, ρ is realized as the coset of ϵ_1^2 in $N_4/\langle\epsilon^2\rangle$ by Proposition 2.19.

Finally, collect all of the structure constants to define the automorphism $\rho = \sigma^{n/2}$ in $\mathrm{Aut}(S_1)$ in an algebraically closed field $\Bbbk'$ over $\mathbb{Q}$. Then a standard base change argument shows that ρ is realized as the graded automorphisms $\pm\epsilon_1^2\epsilon_2^2$, $\pm\epsilon_2^2$, $\pm\epsilon_1^2$ of the $\Bbbk'$-algebra S and of the $\Bbbk$-algebra S, respectively, for $\omega = \frac{1}{2}, \frac{\Lambda}{2}, \frac{1+\Lambda}{2}$.

(4) This is a direct calculation using (2.10), (2.2), and Proposition 2.19.

(5) This follows from part (4) and Lemma 2.9.

(6) By part (5) and Proposition 2.18(1), we can consider the action of H_4 on x_0^n, x_1^n, x_2^n, x_3^n. Now this part follows from Proposition 2.19. □

Note that the subgroup $\langle\epsilon_1^2, \epsilon_2^2\rangle$ plays a crucial role in constructing Chirvasitu–Smith's *exotic elliptic algebras of dimension 4* [17]; see also the work of Davies [18] where these algebras appeared independently. On the other hand, similarly to Corollary 2.21(4), the space $\Bbbk g_1 \oplus \Bbbk g_2$ was realized as a representation of H_4 in work of Kevin DeLaet [21] in the case when S is generic.

Notation 2.22 (ρ_1, ρ_2, ρ_3) Recall from the proof of Corollary 2.21(3) that if $n = |\sigma|$ is even, then ρ is identified with a nontrivial element of $N_4/\langle\epsilon^2\rangle$ via an identification with $\{\frac{1}{2}, \frac{\Lambda}{2}, \frac{1+\Lambda}{2}\}$, the set of nontrivial 2-torsion points ω of $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\Lambda) \cong E$.

We take ρ_1 (respectively, ρ_2, ρ_3) to be the automorphism $\rho \in \mathrm{Aut}(E)$ corresponding to $\omega = \frac{1}{2}$ (respectively, $\omega = \frac{\Lambda}{2}$, $\frac{1+\Lambda}{2}$), which in turn is identified with the coset of $\epsilon_1^2\epsilon_2^2$ (respectively, of ϵ_2^2, ϵ_1^2) in $N_4/\langle\epsilon^2\rangle$.

Proposition 2.23 (a_i) *Let $n < \infty$ be even and recall $s = n/2$. Retain the notation above and recall the notation in Proposition 2.18, namely:*

$$\begin{aligned} &Z(S) = \Bbbk[z_0, z_1, z_2, z_3, g_1, g_2]/(F_1, F_2), \\ &\textit{with } \deg(z_i) = n,\ \deg(g_i) = 2,\ \deg(F_i) = 2n, \end{aligned}$$

and for $i = 1, 2$,

$$F_i = \Phi_i(z_0, z_1, z_2, z_3) + \ell_i(z_0, z_1, z_2, z_3)h_i(g_1, g_2) + h_i(g_1, g_2)^2.$$

We obtain the precise description of both Φ_i and $a_i := -\ell_i/2$, as given below. Here, $\lambda, \mu \in \Bbbk$ and $\mu \neq 0$.

(1) *If $\rho = \rho_1$, then*

$$\Phi_1 = a_1^2 - z_0 z_3, \quad \Phi_2 = a_2^2 - z_1 z_2, \quad \textit{with}$$

$$\begin{aligned} a_1 &= \lambda(z_0 + a^s b^s \xi^{3n} z_3) + \mu(z_1 + a^s b^{-s} \xi^n z_2), \\ a_2 &= \mu(b^{-s} c^{-n} \xi^{2n} z_0 + a^s c^{-n} \xi^{3n} z_3) + \lambda(b^s z_1 + a^s \xi^{3n} z_2). \end{aligned}$$

(2) *If* $\rho = \rho_2$, *then* $4|n$ *and*

$$\begin{aligned} &\Phi_1 = a_1^2 - z_0 z_2, \quad \Phi_2 = a_2^2 - z_1 z_3, \quad \textit{with} \\ &a_1 = \lambda(z_0 + a^s c^s \xi^n z_2) + \mu(z_1 + a^s c^{-s} \xi^n z_3), \\ &a_2 = \mu(b^{-n} c^{-s} \xi^n z_0 + a^s b^{-n} z_2) + \lambda(c^s \xi^n z_1 + a^s z_3). \end{aligned}$$

(3) *If* $\rho = \rho_3$, *then* $4|n$ *and*

$$\begin{aligned} &\Phi_1 = a_1^2 - z_0 z_1, \quad \Phi_2 = a_2^2 - z_2 z_3, \quad \textit{with} \\ &a_1 = \lambda(z_0 + b^s c^s \xi^n z_1) + \mu(z_2 + b^s c^{-s} \xi^n z_3), \\ &a_2 = \mu(a^{-n} c^{-s} z_0 + a^{-n} b^s \xi^n z_1) + \lambda(c^s z_2 + b^s \xi^n z_3). \end{aligned}$$

Proof We only show the details for the case (1) and the remaining cases can be verified in a similar fashion. We follow the proof in [39, Theorem 4.10] to get the forms of Φ_1 and Φ_2; namely, ρ_1 has eigenspaces $\langle x_0, x_3\rangle$ and $\langle x_1, x_2\rangle$ by Corollary 2.21(2). Using Corollary 2.21(6), observe that

$$\begin{aligned} &\epsilon_1(z_0 z_3) = c^n \xi^{-2n} z_1 z_2, \ \epsilon_2(z_0 z_3) = c^n z_1 z_2, \quad \epsilon_1\epsilon_2(z_0 z_3) = \xi^{2n} z_0 z_3, \\ &\epsilon_1(z_1 z_2) = c^{-n} \xi^{2n} z_0 z_3, \ \epsilon_2(z_1 z_2) = c^{-n} z_0 z_3, \ \epsilon_1\epsilon_2(z_1 z_2) = \xi^{-2n} z_1 z_2. \end{aligned}$$

Working in B via Proposition 2.18(3,4) and Corollary 2.21(5), we can take $a_1^2 = z_0 z_3$ and $a_2^2 = z_1 z_2$, so

$$\begin{aligned} z_0 z_3 = a_1^2 &= c^n \xi^{-2n} \epsilon_1(a_2)^2 = c^n \epsilon_2(a_2)^2 \ = \xi^{-2n} \epsilon_1\epsilon_2(a_1)^2, \\ z_1 z_2 = a_2^2 &= c^{-n} \xi^{2n} \epsilon_1(a_1)^2 = c^{-n} \epsilon_2(a_1)^2 = \xi^{2n} \epsilon_1\epsilon_2(a_2)^2. \end{aligned}$$

If $a_i = \eta_{i0} z_0 + \eta_{i1} z_1 + \eta_{i2} z_2 + \eta_{i3} z_3$ for $\eta_{ij} \in \Bbbk$, for $i = 1, 2$, then by comparing coefficients of z_k in the equations above we get a_1 and a_2 as claimed. For instance,

$$\begin{aligned} &\eta_{10} = \pm\eta_{13} a^{-s} b^{-s} \xi^n, \ \ \eta_{11} = \pm\eta_{12} a^{-s} b^s \xi^{-n}, \ \ \eta_{12} = \pm\eta_{11} a^s b^{-s} \xi^n, \ \ \eta_{13} = \pm\eta_{10} a^s b^s \xi^{-n}, \\ &\eta_{10} = \pm\eta_{21} b^{-s}, \quad \eta_{11} = \pm\eta_{20} b^s c^n \xi^{-2n}, \quad \eta_{12} = \pm\eta_{23} b^{-s} c^n \xi^{-2n}, \quad \eta_{13} = \pm\eta_{22} b^s, \end{aligned}$$

amongst other conditions on η_{ij} that imply that the $\pm$s are unnecessary.

Finally, we need to show that $\mu \neq 0$ in each case. Suppose $\mu = 0$, then $\Phi_1 = a_1^2 - z_0 z_3$ becomes a quadratic equation in terms of z_0, z_3. Since $\Bbbk$ is algebraically closed, $\mathbb{V}(\Phi_1)$ is an union of two planes, one of which contains E'' since E'' is irreducible and $E'' \subset \mathbb{V}(\Phi_1)$. But E'' is not contained in a hyperplane, so we have a contradiction. □

Now the result below follows immediately from Corollary 2.21(6) and Proposition 2.23.

Corollary 2.24 *When n is even, the vector spaces $\Bbbk a_1 + \Bbbk a_2$ and $\Bbbk h_1 + \Bbbk h_2$ admit the structure of a 2-dimensional irreducible representation of H_4 both via the following actions.*

(1) *If $\rho = \rho_1$, then ϵ acts by $(-1)^s$ and the action of ϵ_1 and ϵ_2 are given by*

$$\epsilon_1 \mapsto \begin{pmatrix} 0 & c^{-s}\xi^n \\ c^s\xi^{-n} & 0 \end{pmatrix} \quad \text{and} \quad \epsilon_2 \mapsto \begin{pmatrix} 0 & c^{-s} \\ c^s & 0 \end{pmatrix}.$$

(2) *If $\rho = \rho_2$, then ϵ acts by $(-1)^s$ and the action of ϵ_1 and ϵ_2 are given by*

$$\epsilon_1 \mapsto \begin{pmatrix} 0 & b^{-s} \\ b^s & 0 \end{pmatrix} \quad \text{and} \quad \epsilon_2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(3) *If $\rho = \rho_3$, then ϵ acts by $(-1)^s$ and the action of ϵ_1 and ϵ_2 are given by*

$$\epsilon_1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \epsilon_2 \mapsto \begin{pmatrix} 0 & a^{-s}\xi^{-n} \\ a^s\xi^n & 0 \end{pmatrix}.$$

□

3 Singular loci of the maximal spectra of the centers of the PI 4-dimensional Sklyanin algebras S

As in Hypothesis 1.1, let S be a 4-dimensional Sklyanin algebra that satisfies a polynomial identity (i.e., is PI) of PI degree n. Recall that $n = |\sigma|$, where $\sigma \in \mathrm{Aut}(E)$ and $\left(E,\ \mathcal{O}_{\mathbb{P}^3}(1)|_E,\ \sigma\right)$ is the projective algebro-geometric data attached to S. The purpose of this section is to provide a detailed description of maxSpec of the center $Z := Z(S)$, and this study depends on the parity of n. The main results are given in Theorems 3.13 and 3.16 below for the cases when n is odd and even, respectively.

Let us set some notation that will be used throughout this section and establish a couple of preliminary results.

Notation 3.1 $(Y,\ Y^{sing},\ Y_{\gamma_1,\gamma_2},\ (Y_{\gamma_1,\gamma_2})^{sing},\ (Y^{sing})_{\gamma_1,\gamma_2},\ Y_0^{symp})$ Let Y denote the affine variety maxSpec$(Z(S))$, let Y^{sing} denote the singular locus of Y, and let

$$Y_{\gamma_1,\gamma_2} := Y \cap \mathbb{V}(g_1 - \gamma_1, g_2 - \gamma_2), \quad \text{for } \gamma_1, \gamma_2 \in \Bbbk.$$

We denote by $(Y_{\gamma_1,\gamma_2})^{sing}$ the singular locus of the subvariety Y_{γ_1,γ_2} and denote

$$(Y^{sing})_{\gamma_1,\gamma_2} := Y^{sing} \cap Y_{\gamma_1,\gamma_2}.$$

Note that $Y^{sing} = \bigcup_{\gamma_1,\gamma_2\in\Bbbk}(Y^{sing})_{\gamma_1,\gamma_2}$. On the other hand, denote

$$Y_0^{symp} := \bigcup_{\gamma_1,\gamma_2\in\Bbbk} (Y_{\gamma_1,\gamma_2})^{sing}.$$

This will be the variety of *symplectic points* of the Poisson bracket on Y that we construct in Sects. 6 and 7 (i.e., where the Poisson structure vanishes, or equivalently the variety of 0-dimensional symplectic cores of the Poisson structure), see Corollary 7.6 for more details. We will also have that $(Y_{\gamma_1,\gamma_2})^{sing} \supseteq (Y^{sing})_{\gamma_1,\gamma_2}$, so $Y_0^{symp} \supseteq Y^{sing}$, but these containments could be strict (see, Theorem 3.13(3)).

Before proceeding with the study of Y^{sing}, we recall some results about line modules of S from Levasseur–Smith [29]. A *line module* of S is a graded cyclic module of S of Hilbert series $(1-t)^{-2}$; such modules are in correspondence with secant lines of the elliptic curve E of the point scheme of S [29, Theorem 4.5].

Notation 3.2 $[M(p,q), \ell_{p,q}, \Omega(z)]$ For each $z \in E$, let

$$\{M(p,q) \mid p,q \in E,\ p+q=z\}$$

denote the family of line modules of S corresponding to the secant line $\ell_{p,q}$ to E at z. All such line modules have a common central annihilator of degree 2 by [29, Corollary 6.6]. Denote the central degree 2 annihilator of $M(p,q)$, with $p+q=z \in E$, by

$$\Omega(z) \in \Bbbk g_1 + \Bbbk g_2.$$

By [29, Corollary 6.9], the only equalities among these annihilators are

$$\Omega(z) = \Omega(-z-2\tau), \quad \text{for } z \in E. \tag{3.3}$$

Recall that τ is the point in E such that the automorphism $\sigma \in \mathrm{Aut}(E)$ is given by $\sigma(p) = p+\tau$ for any $p \in E$.

Now we recall some facts on the geometry of line modules from both [29] and [39].

Notation 3.4 $(\overline{E}, \overline{\mathbb{P}}, \overline{z}, r_i)$ Let $\overline{E}$ denote $E'' = E/\langle\sigma\rangle$ when n is odd, or denote $E' = E/\langle\sigma^2\rangle$ when n is even. Likewise, take $\overline{\mathbb{P}}$ to be $\mathbb{P}(H^0(E'', \mathcal{L}'')^*)$ when n is odd, or $\mathbb{P}(H^0(E', \mathcal{L}')^*)$ when n is even. Take $\overline{z}$ to be the image of the point $z \in E$ in $\overline{E}$.

Recall that $s = n/(n,2)$. By Proposition 2.18, the subalgebra $\Bbbk[u_0, u_1, u_2, u_3, g_1, g_2]$ of S is subject to degree $2s$ relations

$$r_i := f_i(u_0,u_1,u_2,u_3) + h_i(g_1,g_2),$$

for $i = 1, 2$, where f_1, f_2 are linearly independent quadratic forms in u_0, u_1, u_2, u_3 defining the elliptic curve $\overline{E} \subset \overline{\mathbb{P}}$. (We get that $u_i = z_i$, $r_i = F_i$, and $f_i = \Phi_i$ in Proposition 2.18(3), when n is odd.)

By [39, Theorems 3.7 and 4.9], the center of the Veronese subalgebra $S^{(n/s)}$ of S is

$$Z(S^{(n/s)}) = \Bbbk[u_0,u_1,u_2,u_3]^{(n)}[g_1,g_2], \quad \text{subject to relations } r_1, r_2. \tag{3.5}$$

Notation 3.6 $(Q(\overline{z}), \overline{E}_2, \overline{\omega}_i, \overline{e}_i)$ For $\overline{z} \in \overline{E}$, let

$$Q(\overline{z}) = \bigcup \{\ell_{\overline{p},\overline{q}} \mid \overline{p}, \overline{q} \in \overline{E},\ \overline{p} + \overline{q} = \pm\overline{z}\}$$

denote the quadric in $\overline{\mathbb{P}}$ containing $\overline{E}$. By [29, Section 3] and [39, paragraph before Theorem 5.9], we know $Q(\overline{z})$ is singular, if and only if $\overline{z} \in \overline{E}_2$, where $\overline{E}_2$ is the 2-torsion subgroup of $\overline{E}$, and each singular quadric has rank 3 and has only one singular point.

Label the four 2-torsion points on $\overline{E}$ as $\overline{E}_2 = \{\overline{\omega}_i\}_{0\leq i\leq 3}$. For each corresponding singular quadric $Q(\overline{\omega}_i)$, denote its unique singularity by $\overline{e}_i$. We choose representatives for all $\{\overline{e}_i\}_{0\leq i\leq 3}$ as points in the affine space $\mathrm{maxSpec}(\Bbbk[u_0, u_1, u_2, u_3, g_1, g_2])$ as

$$\begin{aligned} \overline{e}_0 &= (1,0,0,0,0,0), \quad \overline{e}_1 = (0,1,0,0,0,0), \\ \overline{e}_2 &= (0,0,1,0,0,0), \quad \overline{e}_3 = (0,0,0,1,0,0), \end{aligned} \tag{3.7}$$

which correspond to the four points $\{e_i\}_{0\leq i\leq 3}$ in the point scheme of S in Definition-Lemma 2.4.

This brings us to the first preliminary result, from Smith–Tate [39].

Lemma 3.8 *$(\pi, f_{\overline{z}}, h_{\overline{z}})$ [39, Lemma 5.7] There is a morphism*

$$\pi : \overline{E} \to \mathbb{P}(\Bbbk r_1 + \Bbbk r_2)$$

such that $\pi(\overline{z}) = f_{\overline{z}} + h_{\overline{z}}$, where $f_{\overline{z}} \in k[u_0, u_1, u_3, u_4]_{2s}$ vanishes on the quadric $Q(\overline{z})$, and $h_{\overline{z}} \in \Bbbk[g_1, g_2]_{2s}$ is a nonzero scalar multiple of

$$\prod_{z\in E,\ \text{preimage of } \overline{z}} \Omega(z) = \prod_{i=0}^{s-1} \Omega(z + i(n/s)\tau).$$

The morphism π is of degree 2 and $\pi(\overline{p}) = \pi(\overline{q})$, if and only if $\overline{p} = \pm\overline{q}$, for $\overline{p}, \overline{q} \in \overline{E}$. □

Finally, we highlight two central elements of S that will play a key role in the description of Y^{sing} in Theorems 3.13 and 3.16 below.

Notation 3.9 $[g, G]$ Consider the elements of $\Bbbk[g_1, g_2]$:

$$g := \prod_{\substack{\omega\in E_2,\\ 0\leq k\leq s-2}} \Omega(\omega + k\tau) \quad \text{and} \quad G := \prod_{\overline{\omega}\in\overline{E}_2} h_{\overline{\omega}} = g \prod_{\omega\in E_2} \Omega(\omega + (s-1)\tau),$$

which are both central in S of degree $8(s-1)$ and $8s$, respectively (since $\deg(g_i) = 2$).

Lemma 3.10 *Retain the notation of Lemma 3.8. Let $\overline{p}, \overline{q} \in \overline{E}$ such that $\overline{p} \neq \pm\overline{q}$. Then, the following sets are the same in* $\mathrm{Proj}(\Bbbk[g_1, g_2])$*:*

(a) *The zero locus* $\mathbb{V}(g)$ *of* g*;*
(b) *The ramification locus of the map* $\mathrm{Proj}(\Bbbk[g_1, g_2]) \to \mathrm{Proj}(\Bbbk[h_{\overline{p}}, h_{\overline{q}}])$ *induced by the natural embedding* $\Bbbk[h_{\overline{p}}, h_{\overline{q}}] \hookrightarrow \Bbbk[g_1, g_2]$*;*
(c) *The zero locus of the determinant of the Jacobian matrix* $\frac{\partial(h_{\overline{p}}, h_{\overline{q}})}{\partial(g_1, g_2)}$.

Proof By Lemma 3.8 and (3.5), we know $\pi(\overline{p})$ and $\pi(\overline{q})$ can be realized as two defining relations of $Z(S^{(n/s)})$. Note that the ramification locus in (b) and the zero locus in (c) do not depend on the choice of $h_{\overline{p}}$ and $h_{\overline{q}}$ as long as they are linearly independent, namely as long as $\overline{p} \neq \pm\overline{q}$.

(a)⇔(b) follows from [39, Lemma 5.8(c)] and the definition of g in Notation 3.9.
(b)⇔(c) This is a standard algebro-geometric fact; see [34, Section 6.3]. □

3.1 For PIdeg(*S*) odd

We assume that the PI degree n of S is odd in this section, so that $n = s$ and $u_i = z_i$ for all $0 \leq i \leq 3$. Fix any $\overline{p}, \overline{q} \in \overline{E} = E/\langle\sigma\rangle$ with $\overline{p} \neq \pm\overline{q}$, and recall by Notation 3.4, (3.5), and Lemma 3.8 that

$$Y := \mathrm{maxSpec}(Z(S)) = \mathbb{V}(\pi(\overline{p}), \pi(\overline{q})).$$

We first establish that Y is smooth outside of the variety $\mathbb{V}(G)$; see Notation 3.9.

Lemma 3.11 *The subvariety* Y_{γ_1,γ_2} *of* Y *is smooth of dimension 2 if* $(\gamma_1, \gamma_2) \notin \mathbb{V}(G)$.

Proof The defining ideal of $Y_{\gamma_1,\gamma_2} \subseteq \mathbb{A}^4_{(z_0,z_1,z_2,z_3)}$ is generated by elements $f_{\overline{p}} + h_{\overline{p}}(\gamma_1, \gamma_2)$ and $f_{\overline{q}} + h_{\overline{q}}(\gamma_1, \gamma_2)$ in $\Bbbk[z_0, z_1, z_2, z_3]$. Recall that Y_{γ_1,γ_2} is nonsingular at a point $P \in Y_{\gamma_1,\gamma_2}$ if the rank of the Jacobian matrix J evaluated at P is $4 - \dim(Y_{\gamma_1,\gamma_2})$; here,

$$J = \begin{bmatrix} \frac{\partial f_{\overline{p}}}{\partial z_0} & \frac{\partial f_{\overline{p}}}{\partial z_1} & \frac{\partial f_{\overline{p}}}{\partial z_2} & \frac{\partial f_{\overline{p}}}{\partial z_3} \\ \frac{\partial f_{\overline{q}}}{\partial z_0} & \frac{\partial f_{\overline{q}}}{\partial z_1} & \frac{\partial f_{\overline{q}}}{\partial z_2} & \frac{\partial f_{\overline{q}}}{\partial z_3} \end{bmatrix}.$$

Moreover, Y_{γ_1,γ_2} is singular at a point P if the rank of J evaluated at P is less than $4 - \dim(Y_{\gamma_1,\gamma_2})$. Hence, it suffices to show that J evaluated at any $P \in Y_{\gamma_1,\gamma_2}$ has rank 2 if $(\gamma_1, \gamma_2) \notin \mathbb{V}(G)$.

Suppose that the rank of J evaluated at some point $P \in Y_{\gamma_1,\gamma_2}$ is less than 2. We will then show that the pair (γ_1, γ_2) lies in $\mathbb{V}(G)$. We can choose $\overline{p}, \overline{q} \in \overline{E}$ so that $h_{\overline{p}}$ and $h_{\overline{q}}$ have no common factors (see Lemma 3.8 and (3.3)). Hence $\mathbb{V}(h_{\overline{p}}, h_{\overline{q}}) = (0, 0)$.

Suppose that $P = (0, 0, 0, 0)$, then $f_{\overline{p}}(0, 0, 0, 0) + h_{\overline{p}}(\gamma_1, \gamma_2) = h_{\overline{p}}(\gamma_1, \gamma_2) = 0$ and $f_{\overline{q}}(0, 0, 0, 0) + h_{\overline{q}}(\gamma_1, \gamma_2) = h_{\overline{q}}(\gamma_1, \gamma_2) = 0$. So, $(\gamma_1, \gamma_2) \in \mathbb{V}(h_{\overline{p}}, h_{\overline{q}})$. Hence, $(\gamma_1, \gamma_2) = (0, 0) \in \mathbb{V}(G)$, and we are done.

Next, take $P \neq (0, 0, 0, 0)$ and we will show that P is the singularity of the following quadric containing $\overline{E}$ defined by

$$f := h_{\overline{q}}(\gamma_1, \gamma_2) f_{\overline{p}} - h_{\overline{p}}(\gamma_1, \gamma_2) f_{\overline{q}} = 0.$$

Consider $\widetilde{P} = (P, \gamma_1, \gamma_2) \in Y$. We only need to check that $\frac{\partial f}{\partial z_i}(P) = 0$ for all $0 \leq i \leq 3$. We have that:

$$\begin{aligned}
\tfrac{\partial f}{\partial z_i}(P) &= \left(h_{\overline{q}}(\gamma_1,\gamma_2)\tfrac{\partial f_{\overline{p}}}{\partial z_i} - h_{\overline{p}}(\gamma_1,\gamma_2)\tfrac{\partial f_{\overline{q}}}{\partial z_i}\right)(P) \\
&= \left(h_{\overline{q}}\tfrac{\partial f_{\overline{p}}}{\partial z_i} - h_{\overline{p}}\tfrac{\partial f_{\overline{q}}}{\partial z_i}\right)(\widetilde{P}) \\
&= \left(-f_{\overline{q}}\tfrac{\partial f_{\overline{p}}}{\partial z_i} + f_{\overline{p}}\tfrac{\partial f_{\overline{q}}}{\partial z_i}\right)(P) \\
&= 2\left(-\textstyle\sum_{0\leq j\leq 3} z_j \tfrac{\partial f_{\overline{q}}}{\partial z_j}\tfrac{\partial f_{\overline{p}}}{\partial z_i} + \sum_{0\leq j\leq 3} z_j \tfrac{\partial f_{\overline{p}}}{\partial z_j}\tfrac{\partial f_{\overline{q}}}{\partial z_i}\right)(P) \\
&= 2\textstyle\sum_{0\leq j\leq 3} z_j \left(\tfrac{\partial f_{\overline{p}}}{\partial z_j}\tfrac{\partial f_{\overline{q}}}{\partial z_i} - \tfrac{\partial f_{\overline{q}}}{\partial z_j}\tfrac{\partial f_{\overline{p}}}{\partial z_i}\right)(P) \\
&= 0.
\end{aligned}$$

We use the identity $f_{\overline{p}} = 2\sum_{0\leq j\leq 3} z_j \frac{\partial f_{\overline{p}}}{\partial z_j}$ since $f_{\overline{p}}$ is homogenous of degree two, and the similar identity for $f_{\overline{q}}$. Also, the last equality above comes from the Jacobian matrix J has rank ≤ 1 at P. Therefore we can set $P = \lambda \overline{e}_k$ for some $\lambda \in \Bbbk^\times$, where $\overline{e}_k$ is the singularity of the corresponding quadric $Q(\overline{\omega}_k)$ containing $\overline{E}$ according to Notation 3.6. Since $\widetilde{P} \in Y$, we then get by Lemma 3.8 that

$$\begin{aligned}
0 = \pi(\overline{\omega}_k)(\widetilde{P}) &= (f_{\overline{\omega}_k} + h_{\overline{\omega}_k})(\widetilde{P}) = f_{\overline{\omega}_k}(P) + h_{\overline{\omega}_k}(\gamma_1,\gamma_2) \\
&= \lambda^2 f_{\overline{\omega}_k}(\overline{e}_k) + h_{\overline{\omega}_k}(\gamma_1,\gamma_2) \qquad = h_{\overline{\omega}_k}(\gamma_1,\gamma_2).
\end{aligned}$$

By Notation 3.9, we conclude that $(\gamma_1, \gamma_2) \in \mathbb{V}(h_{\overline{\omega}_k}) \subset \mathbb{V}(G)$, as desired. □

Next, we show that the union of singular loci of the subvarieties Y_{γ_1,γ_2} for certain points $(\gamma_1, \gamma_2) \in \mathbb{V}(G)$ is equal to a cuspidal curve, denoted by $C(\omega + k\tau)$ below.

Lemma 3.12 *($C(\omega + k\tau)$) For any $\omega \in E_2$ and $0 \leq k \leq n-1$, there exists a nonzero point $\overline{p}_{\omega,k} \in \mathbb{V}(\Omega(\omega + k\tau)) \cap \mathbb{V}(z_0, z_1, z_2, z_3)$ such that*

$$\bigcup_{(\gamma_1,\gamma_2)\in\mathbb{V}(\Omega(\omega+k\tau))} (Y_{\gamma_1,\gamma_2})^{sing} = \{t^n \overline{e}_\omega + t^2 \overline{p}_{\omega,k} \mid t \in \Bbbk\} =: C(\omega + k\tau),$$

where $\overline{\omega} \in \overline{E}_2$ is the image of ω under the isogeny $E \twoheadrightarrow \overline{E} = E/\langle\sigma\rangle$, and $\overline{e}_\omega$ is the singularity of the quadric $Q(\overline{\omega})$. Moreover, we can take $\overline{p}_{\omega,k} = \overline{p}_{\omega,n-2-k}$ with the second index modulo n.

Proof We can choose the generators of the defining ideal of $Y = \mathbb{V}(\pi(\overline{p}), \pi(\overline{q}))$, where $\overline{p} = \overline{\omega} \in \overline{E}_2$ and $\overline{q}$ generic, so that $\pi(\overline{p}) = f_{\overline{\omega}} + h_{\overline{\omega}}$ with $h_{\overline{\omega}} = \prod_{k=0}^{n-1} \Omega(\omega + k\tau)$, and $\pi(\overline{q}) = f_{\overline{q}} + h_{\overline{q}}$. Let $P \in (Y_{\gamma_1,\gamma_2})^{sing}$ and take $\widetilde{P} = (P, \gamma_1, \gamma_2) \in Y$. Since (γ_1, γ_2) is in $\mathbb{V}(\Omega(\omega + k\tau)) \subset \mathbb{V}(h_{\overline{\omega}})$, we have that

$$0 = \pi(\overline{\omega})(\widetilde{P}) = f_{\overline{\omega}}(P) + h_{\overline{\omega}}(\gamma_1,\gamma_2) = f_{\overline{\omega}}(P).$$

By a similar argument in Lemma 3.11, one can further show that P is the singularity of the quadric $Q(\overline{\omega}) = \mathbb{V}(f_{\overline{\omega}})$ containing $\overline{E}$. Hence $P = \lambda \overline{e}_\omega$ for some $\lambda \in \Bbbk$. Pick some nonzero point $\overline{p}_{\omega,k} \in \mathbb{V}(\Omega(\omega + k\tau)) \cap \mathbb{V}(z_0, z_1, z_2, z_3)$. Since $\overline{q} \in \overline{E}$ is generic,

by a possible rescaling of $\overline{p}_{\omega,k}$, we can assume that $f_{\overline{q}}(\overline{e}_\omega) = -h_{\overline{q}}(\overline{p}_{\omega,k}) \neq 0$. Then $\widetilde{P}$ is contained in the following subset of Y:

$$\begin{aligned}\{\alpha\overline{e}_\omega + \beta\overline{p}_{\omega,k} \,|\, \alpha, \beta \in \Bbbk\} \cap Y &= \{\alpha\overline{e}_\omega + \beta\overline{p}_{\omega,k} \,|\, \alpha, \beta \in \Bbbk\} \cap \mathbb{V}(f_{\overline{q}} + h_{\overline{q}}) \\ &= \{\alpha\overline{e}_\omega + \beta\overline{p}_{\omega,k} \,|\, \alpha^2 f_{\overline{q}}(\overline{e}_\omega) + \beta^n h_{\overline{q}}(\overline{p}_{\omega,k}) = 0\} \\ &= \{t^n\overline{e}_\omega + t^2\overline{p}_{\omega,k} \,|\, t \in \Bbbk\}.\end{aligned}$$

Therefore, we have obtained that

$$\bigcup_{(\gamma_1,\gamma_2)\in\mathbb{V}(\Omega(\omega+k\tau))} (Y_{\gamma_1,\gamma_2})^{sing} \subset \{t^n\overline{e}_\omega + t^2\overline{p}_{\omega,k} \,|\, t \in \Bbbk\}.$$

Conversely, take $P = t^n\overline{e}_\omega$ and $\widetilde{P} = t^n\overline{e}_\omega + t^2\overline{p}_{\omega,k}$ for any $t \in \Bbbk$. It is straightforward to check that $P \in (Y_{\gamma_1,\gamma_2})^{sing}$ with $(\gamma_1, \gamma_2) = t^2\overline{p}_{\omega,k}$ since the the first row of the Jacobian matrix vanishes at P:

$$\begin{bmatrix} \frac{\partial f_{\overline{\omega}}}{\partial z_0} & \frac{\partial f_{\overline{\omega}}}{\partial z_1} & \frac{\partial f_{\overline{\omega}}}{\partial z_2} & \frac{\partial f_{\overline{\omega}}}{\partial z_3} \\ \frac{\partial f_{\overline{q}}}{\partial z_0} & \frac{\partial f_{\overline{q}}}{\partial z_1} & \frac{\partial f_{\overline{q}}}{\partial z_2} & \frac{\partial f_{\overline{q}}}{\partial z_3} \end{bmatrix}.$$

Finally, we can set $\overline{p}_{\omega,k} = \overline{p}_{\omega,n-2-k}$ since $\Omega(\omega + k\tau) = \Omega(\omega + (n-2-k)\tau)$ by (3.3). □

Now we show that $Y^{sing} \subset \mathbb{V}(g)$ and we can use the lemma above to get a detailed description of Y^{sing}.

Theorem 3.13 *For n odd, we have*

$$Y^{sing} = Y^{sing} \cap \mathbb{V}(g) = \bigcup_{\substack{\omega\in E_2 \\ 0\le k\le n-2}} C(\omega + k\tau),$$

where $C(\omega+k\tau)$ is the cuspidal curve $\left\{t^n\overline{e}_\omega + t^2\overline{p}_{\omega k} \,|\, t \in \Bbbk\right\}$ defined in Lemma 3.12. As a consequence, Y^{sing} is a union of $2(n-1)$ cuspidal curves in $\mathbb{V}(g)$ meeting at the origin as depicted in Fig. 1.
We also have the following results on the subvarieties $(Y_{\gamma_1,\gamma_2})^{sing}$, $(Y^{sing})_{\gamma_1,\gamma_2}$ of Y:

(1) *If $(\gamma_1, \gamma_2) \notin \mathbb{V}(G)$, then $(Y_{\gamma_1,\gamma_2})^{sing} = (Y^{sing})_{\gamma_1,\gamma_2} = \varnothing$.*
(2) *If $(\gamma_1, \gamma_2) \in \mathbb{V}(g)$, then $(Y_{\gamma_1,\gamma_2})^{sing} = (Y^{sing})_{\gamma_1,\gamma_2}$, which consists of 1 point if $(\gamma_1, \gamma_2) = (0, 0)$, and consists of 2 points otherwise.*
(3) *If $(0, 0) \neq (\gamma_1, \gamma_2) \in \mathbb{V}(G/g)$, then $(Y_{\gamma_1,\gamma_2})^{sing}$ has 2 points while $(Y^{sing})_{\gamma_1,\gamma_2}$ is empty.*
(4) *The group H_4 in Proposition 2.19 fixes Y^{sing} and Y_0^{symp} and*

$$\begin{aligned}\epsilon_1 &: C(\omega + k\tau) \mapsto C(\omega + \xi_1 + k\tau), \quad \epsilon_2 : C(\omega + k\tau) \mapsto C(\omega + \xi_2 + k\tau), \\ \epsilon_1\epsilon_2 &: C(\omega + k\tau) \mapsto C(\omega + \xi_3 + k\tau), \quad \epsilon : C(\omega + k\tau) \mapsto C(\omega + k\tau),\end{aligned}$$

where $\{0, \xi_1, \xi_2, \xi_3\}$ is the 2-torsion subgroup of E as described in [17, Section 2.6].

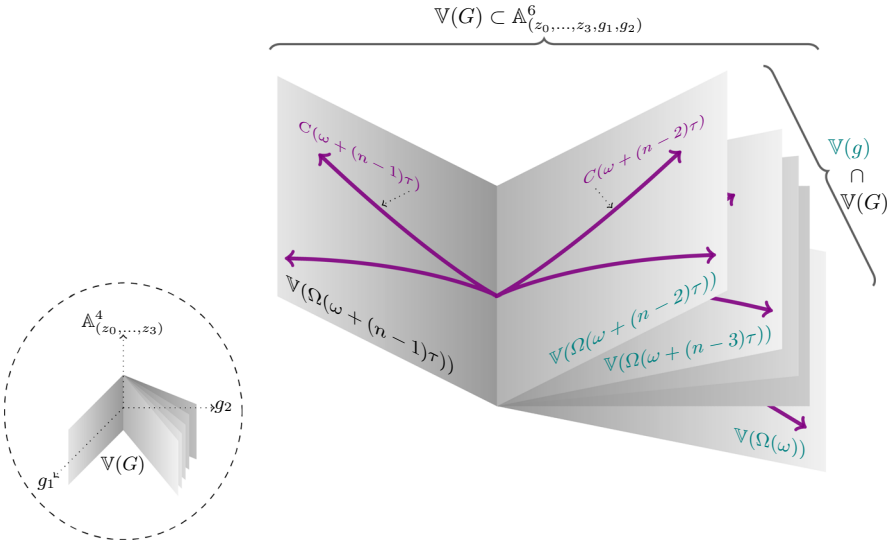

Fig. 1 For n odd: $Y^{sing} = \bigcup_{\gamma_1,\gamma_2\in\mathbb{k}}(Y^{sing})_{\gamma_1,\gamma_2} = \bigcup_{\omega\in E_2,\ 0\le k\le n-2} C(\omega + k\tau)$, and $Y_0^{symp} = \bigcup_{\gamma_1,\gamma_2\in\mathbb{k}}(Y_{\gamma_1,\gamma_2})^{sing} = \bigcup_{\omega\in E_2,\ 0\le k\le n-1} C(\omega + k\tau)$, union of $2n-2$ (resp., $2n$) cuspidal curves meeting at $\{\underline{0}\}$, each with multiplicity 2

As a consequence, $(Y_{\gamma_1,\gamma_2})^{sing}$ consists of only finitely many points, and

$$Y_0^{symp} = \bigcup_{\omega\in E_2} C(\omega + (n-1)\tau) \cup Y^{sing}.$$

Proof of Theorem 3.13 The Jacobian matrix of $Y = \mathbb{V}(\pi(\overline{p}), \pi(\overline{q}))$ is

$$\begin{bmatrix} \frac{\partial f_{\overline{p}}}{\partial z_0} & \frac{\partial f_{\overline{p}}}{\partial z_1} & \frac{\partial f_{\overline{p}}}{\partial z_2} & \frac{\partial f_{\overline{p}}}{\partial z_3} & \frac{\partial h_{\overline{p}}}{\partial g_1} & \frac{\partial h_{\overline{p}}}{\partial g_2} \\ \frac{\partial f_{\overline{q}}}{\partial z_0} & \frac{\partial f_{\overline{q}}}{\partial z_1} & \frac{\partial f_{\overline{q}}}{\partial z_2} & \frac{\partial f_{\overline{q}}}{\partial z_3} & \frac{\partial h_{\overline{q}}}{\partial g_1} & \frac{\partial h_{\overline{q}}}{\partial g_2} \end{bmatrix}.$$

It is clear that if it has rank ≤ 1 at some point $\widetilde{P} = (P, \gamma_1, \gamma_2) \in Y$, then the Jacobian matrix of Y_{γ_1,γ_2}, which is one of its 2×4 minors containing all the partial derivatives of z_i, also has rank ≤ 1 at P. So

$$(Y^{sing})_{\gamma_1,\gamma_2} \subseteq (Y_{\gamma_1,\gamma_2})^{sing}. \tag{3.14}$$

Now if $\widetilde{P} \in Y^{sing}$ then each 2×2 minors of the matrix above vanishes at $\widetilde{P}$. In particular, we have that

$$\det\begin{pmatrix} \frac{\partial h_{\tilde{p}}}{\partial g_1} & \frac{\partial h_{\tilde{p}}}{\partial g_2} \\ \frac{\partial h_{\tilde{q}}}{\partial g_1} & \frac{\partial h_{\tilde{q}}}{\partial g_2} \end{pmatrix}(\widetilde{P}) = \det\left(\frac{\partial(h_{\tilde{p}}, h_{\tilde{q}})}{\partial(g_1, g_2)}\right)(\gamma_1, \gamma_2) = 0.$$

Hence $Y^{sing} \subset \mathbb{V}(g)$ by Lemma 3.10 and

$$Y^{sing} = \bigcup_{(\gamma_1,\gamma_2)\in\mathbb{V}(g)} (Y^{sing})_{\gamma_1,\gamma_2} \subseteq \bigcup_{(\gamma_1,\gamma_2)\in\mathbb{V}(g)} (Y_{\gamma_1,\gamma_2})^{sing} = \bigcup_{\substack{\omega\in E_2\\ 0\le k\le n-2}} C(\omega+k\tau)$$

by Lemma 3.12.

Conversely, we verify that $C(\omega+k\tau) \subset Y^{sing}$ for all $\omega \in E_2$ and $0 \le k \le n-2$ as follows. Let $\pi(\overline{\omega})$ and $\pi(\overline{q})$ be the two defining relations of Y with $q \in E$ generic. It suffices to show that the first row of the Jacobian matrix of Y above vanishes, i.e., we want to show that

$$\begin{aligned}&\left[\tfrac{\partial f_{\overline{\omega}}}{\partial z_0}, \tfrac{\partial f_{\overline{\omega}}}{\partial z_1}, \tfrac{\partial f_{\overline{\omega}}}{\partial z_2}, \tfrac{\partial f_{\overline{\omega}}}{\partial z_3}, \tfrac{\partial h_{\overline{\omega}}}{\partial g_1}, \tfrac{\partial h_{\overline{\omega}}}{\partial g_2}\right](t^n\overline{e}_\omega + t^2\overline{p}_{\omega,k})\\ &= t^{2n}\left[\tfrac{\partial f_{\overline{\omega}}}{\partial z_0}(\overline{e}_\omega), \tfrac{\partial f_{\overline{\omega}}}{\partial z_1}(\overline{e}_\omega), \tfrac{\partial f_{\overline{\omega}}}{\partial z_2}(\overline{e}_\omega), \tfrac{\partial f_{\overline{\omega}}}{\partial z_3}(\overline{e}_\omega), 0, 0\right]\\ &\quad + t^{2n}\left[0,0,0,0, \frac{\partial h_{\overline{\omega}}}{\partial g_1}(\overline{p}_{\omega k}), \frac{\partial h_{\overline{\omega}}}{\partial g_2}(\overline{p}_{\omega k})\right]\end{aligned}$$

vanishes. Since $\overline{e}_\omega$ is the singularity of $Q(\overline{\omega})$, the first summand vanishes. Note that when $0 \le k \le n-2$ we have $\overline{p}_{\omega,k} \in \Omega(\omega+k\tau) = \Omega(\omega+(n-2-k)\tau)$ is a multiple root of $h_{\overline{\omega}} = \prod_{k=0}^{n-1}\Omega(\omega+k\tau)$. So, the second summand vanishes as well. Thus, the first part of the result holds.

(1) This follows from Lemma 3.11.

(2) Suppose $(\gamma_1,\gamma_2) \in \mathbb{V}(g)$. By Lemma 3.12 and the beginning of the result, we have that

$$(Y^{sing})_{\gamma_1,\gamma_2} = (Y_{\gamma_1,\gamma_2})^{sing} = C(\omega+k\tau) \cap \mathbb{V}(g_1-\gamma_1, g_2-\gamma_2)$$

for some $(\gamma_1,\gamma_2) \in \mathbb{V}(\Omega(\omega+k\tau))$ with $0 \le k \le n-2$. If $(\gamma_1,\gamma_2) = (0,0)$, then $(Y^{sing})_{\gamma_1,\gamma_2} = (Y_{\gamma_1,\gamma_2})^{sing} = (0,0,0,0)$. If $(\gamma_1,\gamma_2) \ne (0,0)$, then since n is odd there are two choices of t satisfying the defining equation of $C(\omega+k\tau)$ yielding two different points in $(Y^{sing})_{\gamma_1,\gamma_2} = (Y_{\gamma_1,\gamma_2})^{sing}$.

(3) The argument for $(Y_{\gamma_1,\gamma_2})^{sing}$ is similar as in (2) noting that

$$\bigcup_{(\gamma_1,\gamma_2)\in\mathbb{V}(\Omega(\omega+(n-1)\tau))} (Y_{\gamma_1,\gamma_2})^{sing} = \{t^n\overline{e}_\omega + t^2\overline{p}_{\omega,n-1} \mid t \in \Bbbk\}$$

for any $\omega \in E_2$ by Lemma 3.12.
For $(Y^{sing})_{\gamma_1,\gamma_2}$, recall from Notation 3.9 that $G/g = \prod_{\omega\in E_2}\Omega(\omega+(n-1)\tau)$. Since $g = \prod_{\omega\in E_2, 0\le k\le n-2}\Omega(\omega+k\tau)$, we can see that G/g and g as polynomials in $\Bbbk[g_1,g_2]$ have no common factors by using the only non-trivial identity (3.3) between these central annihilators $\Omega(\omega+k\tau)$. By the argument in the beginning of the proof of this theorem, we know $Y^{sing} \subseteq \mathbb{V}(g)$. This implies that

$$(Y^{sing})_{\gamma_1,\gamma_2} = Y^{sing} \cap \mathbb{V}(g_1-\gamma_1, g_2-\gamma_2) \subset \mathbb{V}(g) \cap \mathbb{V}(g_1-\gamma_1, g_2-\gamma_2).$$

By the assumption on (γ_1, γ_2), we get $\mathbb{V}(g) \cap \mathbb{V}(g_1 - \gamma_1, g_2 - \gamma_2) = \varnothing$ as $\gcd(G/g, g) = 1$ in $\Bbbk[g_1, g_2]$. Therefore, we have $(Y^{sing})_{\gamma_1,\gamma_2} = \varnothing$ if $(0, 0) \neq (\gamma_1, \gamma_2) \in \mathbb{V}(G/g)$.

(4) By Corollary 2.21(4), the group $H_4 \subset \mathrm{Aut}_{gr}(S)$ fixes Y^{sing} and Y_0^{symp}. By [17, Corollary 2.10], we know $\epsilon_i(\Omega(\omega + k\tau)) = \Omega(\omega + \xi_i + k\tau)$ for $i = 1, 2$. Note that $C(\omega + k\tau) \subset \mathbb{V}(\Omega(\omega + k\tau))$. So we have $\epsilon_i(C(\omega + k\tau))$ and $C(\omega + \xi_i + k\tau)$ are included in $\mathbb{V}(\Omega(\omega + \xi_i + k\tau))$. By Lemma 3.12, $\mathbb{V}(\Omega(\omega + \xi_i + k\tau)) \cap Y_0^{symp}$ contains only one cuspidal curve, namely $\{t^n \overline{e}_{\omega+\xi_i} + t^2 \overline{p}_{\omega+\xi_i,k} \mid t \in \Bbbk\}$. So all three cuspidal curves are the same. Finally, ϵ is just a rescaling of the variables of S. So it will fix all the cuspidal curves $C(\omega + k\tau)$. This proves part (4).

Finally, we have

$$\begin{aligned} Y_0^{symp} &= Y_0^{symp} \cap \mathbb{V}(G) \\ &= (Y_0^{symp} \cap \mathbb{V}(G/g)) \cup (Y_0^{symp} \cap \mathbb{V}(g)) \\ &= \textstyle\bigcup_{\omega \in E_2} C(\omega + (n-1)\tau) \cup Y^{sing}. \end{aligned}$$

So, the result follows. □

3.2 For PIdeg(*S*) even

We now assume in this part that the PI degree n of S is even; here, $s = n/2$ and $u_i = z_i^2$ with $\deg u_i = s$ for all $0 \leq i \leq 3$. Recall that the center of the Veronese subalgebra $S^{(2)}$ is

$$Z(S^{(2)}) = \Bbbk[u_0, u_1, u_2, u_3]^{(n)}[g_1, g_2]$$

subject to two defining relations $\pi(\overline{p}) = f_{\overline{p}} + h_{\overline{p}}$ and $\pi(\overline{q}) = f_{\overline{q}} + h_{\overline{q}}$, where the points $\overline{p}, \overline{q} \in \overline{E} = E/\langle\sigma^2\rangle$ satisfy $\overline{p} \neq \pm\overline{q}$. Note that $f_{\overline{p}}$ and $f_{\overline{q}}$ are two linearly independent quadrics in terms of u_0, u_1, u_2, u_3 defining the elliptic curve $\overline{E} \subset \mathbb{P}$. The center Z of S isomorphic to $\Bbbk[z_0, z_1, z_2, z_3, g_1, g_2]$ subject to two defining relations F_1, F_2 of degree $2n$. Moreover, we can write the two defining relations of Z as

$$F_1 = (a_1 - h_1)^2 - z_i z_j, \quad F_2 = (a_2 - h_2)^2 - z_k z_l, \quad \{i, j, k, l\} = \{0, 1, 2, 3\},$$

where $h_1, h_2 \in \Bbbk[g_1, g_2]_n$ have no common factors and a_1, a_2 are linear forms in terms of z_0, z_1, z_2, z_3 given in Proposition 2.23.

Lemma 3.15 *For each pair $(\gamma_1, \gamma_2) \in \Bbbk^2$, we have that $\left(Y_{\gamma_1,\gamma_2}\right)^{sing} \subset \mathbb{V}(z_0 z_1 z_2 z_3)$.*

Proof By way of contradiction, suppose there exists a point

$$\widetilde{P} = (P, \gamma_1, \gamma_2) \in (Y_{\gamma_1,\gamma_2})^{sing} \setminus \mathbb{V}(z_0 z_1 z_2 z_3)$$

for some $\gamma_1, \gamma_2 \in \Bbbk$. Let $\mathfrak{m}_{\widetilde{P}}$ be the maximal ideal of $Z(S)$ corresponding to $\widetilde{P}$. Then by the Lying Over Theorem (for the integral extension $Z(S) \subset Z(S^{(2)})$), there exists

a maximal ideal $\mathfrak{n}$ of $Z(S^{(2)})$ so that $\mathfrak{n} \cap Z(S) = \mathfrak{m}_{\widetilde{P}}$. Moreover, $\mathbb{V}(\mathfrak{n})$ is a point $\widetilde{Q} = (Q, \gamma_1, \gamma_2)$ of $\mathrm{maxSpec}(Z(S^{(2)}))$.

Using the fact that $z_i = u_i^2$ for all $0 \leq i \leq 3$, we get $Q \notin \mathbb{V}(u_0u_1u_2u_3)$ since P is not in $\mathbb{V}(z_0z_1z_2z_3)$. In particular, we have $Q \neq \underline{0}$. Moreover, since $P \in (Y_{\gamma_1,\gamma_2})^{sing}$, we have that Q is a singular point of $\mathrm{maxSpec}(Z(S^{(2)})) \cap \mathbb{V}(g_1 - \gamma_1, g_2 - \gamma_2)$ via the Implicit Function Theorem. Note that the defining relations of $Z(S^{(2)})$ are of the same form as the defining relations of $Z(S)$ in the case when n is odd, we obtain by the argument in the proof of Lemma 3.11 that the point Q is the singular point of some quadric containing $\overline{E}$, namely $\overline{e}_k$ for some $k = 0, \ldots, 3$. This contradicts $Q \notin \mathbb{V}(u_0u_1u_2u_3)$; see (3.7). □

Theorem 3.16 (Y_1^{sing}, Y_2^{sing}) *Take n even and recall the notation of Proposition 2.23. Then, we obtain that the singular locus Y^{sing} of $Y = \mathbb{V}(F_1, F_2) \subset \mathbb{A}^6_{(z_0,z_1,z_2,z_3,g_1,g_2)}$ is the union of subvarieties Y_1^{sing} and Y_2^{sing} of Y defined by*

(i) $Y_1^{sing} = \mathbb{V}(a_1 - h_1, z_0, z_3) \cap Y$ *and* $Y_2^{sing} = \mathbb{V}(a_2 - h_2, z_1, z_2) \cap Y$, *when* $\rho = \rho_1$,
(ii) $Y_1^{sing} = \mathbb{V}(a_1 - h_1, z_0, z_2) \cap Y$ *and* $Y_2^{sing} = \mathbb{V}(a_2 - h_2, z_1, z_3) \cap Y$, *when* $\rho = \rho_2$,
(iii) $Y_1^{sing} = \mathbb{V}(a_1 - h_1, z_0, z_1) \cap Y$ *and* $Y_2^{sing} = \mathbb{V}(a_2 - h_2, z_2, z_3) \cap Y$, *when* $\rho = \rho_3$.

Moreover, we have $Y_1^{sing} \cap Y_2^{sing} = \{\underline{0}\}$.

We also have the following results on the subvarieties $(Y_{\gamma_1,\gamma_2})^{sing}$, $(Y^{sing})_{\gamma_1,\gamma_2}$ of Y:

(1) *The varieties Y_1^{sing} and Y_2^{sing} from Theorem 3.16 are*

$$\text{permuted by } \begin{cases} \epsilon_1 \text{ and } \epsilon_2, \\ \epsilon_1 \text{ and } \epsilon_1\epsilon_2, \\ \epsilon_2 \text{ and } \epsilon_1\epsilon_2, \end{cases} \quad \text{and fixed by } \begin{cases} \epsilon_1\epsilon_2, & \text{when } \rho = \rho_1, \\ \epsilon_2, & \text{when } \rho = \rho_2, \\ \epsilon_1, & \text{when } \rho = \rho_3. \end{cases}$$

where ϵ_i are the group actions in Proposition 2.19.

(2) $(Y^{sing})_{\gamma_1,\gamma_2}$ *has 4 points generically (counting multiplicity), and 1 point, if and only if* $(\gamma_1, \gamma_2) = (0, 0)$.

(3) $(Y_{\gamma_1,\gamma_2})^{sing} = (Y^{sing})_{\gamma_1,\gamma_2} = (Y_1)^{sing}_{\gamma_1,\gamma_2} \cup (Y_2)^{sing}_{\gamma_1,\gamma_2}$, *where*

$$(Y_i)^{sing}_{\gamma_1,\gamma_2} := Y_i^{sing} \cap \mathbb{V}(g_1 - \gamma_1, g_2 - \gamma_2)$$

for $i = 1, 2$.

Proof of Theorem 3.16 We will only treat the case for $\rho = \rho_1$, and other cases are similar. We use the presentation of Z in Proposition 2.23, where we can write the two defining relations of Y as

$$F_1 = (a_1 - h_1)^2 - z_0z_3, \quad F_2 = (a_2 - h_2)^2 - z_1z_2.$$

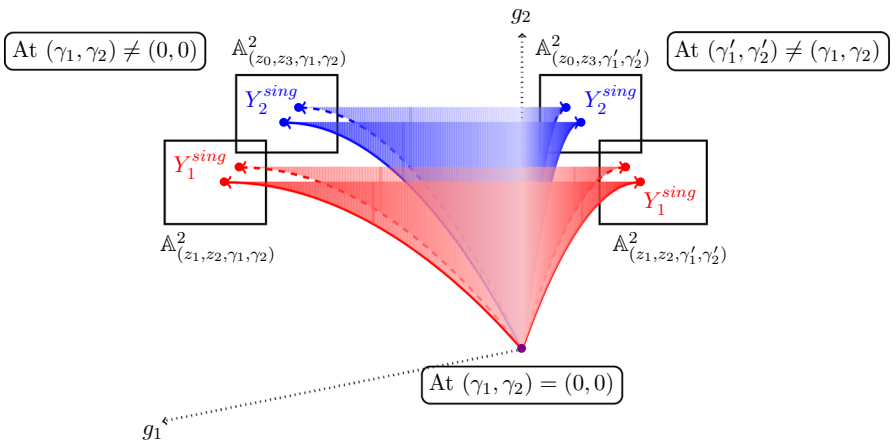

Fig. 2 For n even with $\rho = \rho_1$: $Y^{sing} = Y_0^{symp}$, union of surfaces Y_1^{sing} and Y_2^{sing}

Moreover, we can assume that $h_1, h_2 \in \Bbbk[g_1, g_2]_n$ have no common factors and

$$\begin{aligned} a_1 &= \lambda(z_0 + a^s b^s \xi^{3n} z_3) + \mu(z_1 + a^s b^{-s} \xi^n z_2), \\ a_2 &= \mu(b^{-s} c^{-n} \xi^{2n} z_0 + a^s c^{-n} \xi^{3n} z_3) + \lambda(b^s z_1 + a^s \xi^{3n} z_2), \end{aligned}$$

where $\lambda, \mu \in \Bbbk$ with $\mu \neq 0$ by Proposition 2.23.

Next, it is clear that $(Y^{sing})_{\gamma_1,\gamma_2} \subseteq (Y_{\gamma_1,\gamma_2})^{sing} \subset \mathbb{V}(z_0 z_1 z_2 z_3)$ using Lemma 3.15 for the latter. We now show that Y^{sing} is contained in the union of Y_1^{sing} and Y_2^{sing}. Take a point $\widetilde{P} = (P, \gamma_1, \gamma_2) \in (Y^{sing})_{\gamma_1,\gamma_2}$, where $P = (p_0, \ldots, p_3)$. Then without loss of generality, we can take $p_0 = 0$ and hence $a_1 - h_1 = 0$ at $\widetilde{P}$ in Y^{sing}. Now, let

$$A := a_2 - h_2$$

and consider the following Jacobian matrix (it has rank ≤ 1 at $\widetilde{P}$)

$$\frac{\partial(F_1, F_2)}{\partial(z_0, \ldots, z_3, g_1, g_2)}\Big|_{a_1=h_1} = \begin{bmatrix} -z_3 & 0 & 0 & -z_0 & 0 & 0 \\ 2A\frac{\partial a_2}{\partial z_0} & 2A\frac{\partial a_2}{\partial z_1} - z_2 & 2A\frac{\partial a_2}{\partial z_2} - z_1 & 2A\frac{\partial a_2}{\partial z_3} & -2A\frac{\partial h_2}{\partial g_1} & -2A\frac{\partial h_2}{\partial g_2} \end{bmatrix}.$$

Recall that $p_0 = 0$. Now if $p_3 = 0$, then $\widetilde{P}$ is contained in Y_1^{sing}. On the other hand, suppose $p_3 \neq 0$. Then the determinant of the first two columns of the matrix above, evaluated at $\widetilde{P}$ and set equal to 0, implies that

$$p_2 = 2A\frac{\partial a_2}{\partial z_1}(\widetilde{P}).$$

Likewise, using the first and third columns, and also the first and fourth columns, respectively, we get

$$p_1 = 2A\frac{\partial a_2}{\partial z_2}(\widetilde{P}) \quad \text{and} \quad 2A\frac{\partial a_2}{\partial z_3}(\widetilde{P}) = 0.$$

Since $\frac{\partial a_2}{\partial z_3} \neq 0$, we have that $A = 0$ and thus $p_1 = p_2 = 0$. Therefore, $\widetilde{P} \in Y_2^{sing}$.

Conversely, it is straight-forward to check that Y_1^{sing} and Y_2^{sing} are contained in Y^{sing}. So, $Y^{sing} = Y_1^{sing} \cup Y_2^{sing}$.

Finally, let $\widetilde{P} = (p_0, p_1, p_2, p_3, \gamma_1, \gamma_2) \in Y_1^{sing} \cap Y_2^{sing}$. Then we have $p_k = 0$ for all k (by Lemma 3.15) and $(\gamma_1, \gamma_2) \in \mathbb{V}(h_1, h_2)$. Since h_1, h_2 have no common factors, we get $\gamma_1 = \gamma_2 = 0$ and $\widetilde{P} = 0$, as desired.

(1) follows from Corollary 2.24.

(2) First, assume that $(\gamma_1, \gamma_2) = (0, 0)$. Then $(Y^{sing})_{0,0} \subseteq (Y_{0,0})^{sing}$, where $Y_{0,0} = \Bbbk[z_0, z_1, z_2, z_3]/(\Phi_1, \Phi_2)$ is the affine cone of the smooth elliptic curve $E'' = E/\langle\sigma\rangle$ (see Lemma 2.14). So we get $(Y^{sing})_{0,0} = (Y_{0,0})^{sing} = \{\underline{0}\}$.
Now suppose $(\gamma_1, \gamma_2) \neq (0, 0)$ and $\rho = \rho_1$. By the beginning of the statement, we have $Y_1^{sing} = \mathbb{V}(a_1 - h_1, z_0, z_3, (a_2 - h_2)^2 - z_1z_2)$. So $(Y_1^{sing})_{\gamma_1,\gamma_2}$ is the intersection points of a line $a_1 = h_1$ with a conic $(a_2 - h_2)^2 = z_1z_2$ in the affine space $\mathbb{A}^2_{(z_1,z_2,g_1=\gamma_1,g_2=\gamma_2)}$, which has two points (counting multiplicity). The same argument applies to $(Y_2^{sing})_{\gamma_1,\gamma_2}$ as well. Since $Y_1^{sing} \cap Y_2^{sing} = \{\underline{0}\}$ by the beginning of the statement, we get that $(Y^{sing})_{\gamma_1,\gamma_2} = (Y_1^{sing})_{\gamma_1,\gamma_2} \cup (Y_2^{sing})_{\gamma_1,\gamma_2}$ has 4 points generically.
The argument for $\rho = \rho_2, \rho_3$ follows similarly.

(3) Recall that $(Y_{\gamma_1,\gamma_2})^{sing} \subset \mathbb{V}(z_0z_1z_2z_3)$ by Lemma 3.15. Then, without loss of generality take $\rho = \rho_1$, and take $z_0 = 0$ so that $a_1 = h_1(\underline{\gamma})$, and consider the Jacobian matrix of Y_{γ_1,γ_2}

$$\frac{\partial(F_1(\underline{z}, \underline{\gamma}), F_2(\underline{z}, \underline{\gamma}))}{\partial(z_0, z_1, z_2, z_3)}\Big|_{z_0=0,a_1=h_1} = \begin{bmatrix} -z_3 & 0 & 0 & 0 \\ 2A\frac{\partial a_2}{\partial z_0} & 2A\frac{\partial a_2}{\partial z_1} - z_2 & 2A\frac{\partial a_2}{\partial z_2} - z_1 & 2A\frac{\partial a_2}{\partial z_3} \end{bmatrix},$$

where $A = a_2 - h_2(\overline{\gamma})$. With this matrix, we can show $(Y_{\gamma_1,\gamma_2})^{sing} \subseteq (Y_1)^{sing}_{\gamma_1,\gamma_2} \cup (Y_2)^{sing}_{\gamma_1,\gamma_2}$. Moreover, we can conclude that

$$(Y_1)^{sing}_{\gamma_1,\gamma_2} \cup (Y_2)^{sing}_{\gamma_1,\gamma_2} = (Y^{sing})_{\gamma_1,\gamma_2} \subseteq (Y_{\gamma_1,\gamma_2})^{sing} \subseteq (Y_1)^{sing}_{\gamma_1,\gamma_2} \cup (Y_2)^{sing}_{\gamma_1,\gamma_2}.$$

This proves the result. □

4 Background material on Poisson orders and specialization

We discuss briefly in this section background material on Poisson orders, including the process of specialization mentioned in the introduction, as well as material on

symplectic cores. More details can be found in [43, Sections 2.1 and 2.2] and the references therein.

4.1 Poisson orders and specialization

Here we collect some definitions and facts about Poisson orders and describe an extension of the specialization technique for obtaining such structures. The following definition is due to Brown and Gordon [13].

Definition 4.1 ($\mathrm{Der}(A/C)$, ∂, ∂_z)
Let A be a $\Bbbk$-algebra which is module-finite over a central subalgebra C. Denote by $\mathrm{Der}(A/C)$ the algebra of $\Bbbk$-derivations of A that preserve C.

The algebra A is called a *Poisson C-order* if there exists a $\Bbbk$-linear map

$$\partial : C \to \mathrm{Der}(A/C)$$

such that the induced bracket $\{.,.\}$ on C, given by

$$\{z, z'\} := \partial_z(z'), \quad z, z' \in C, \tag{4.2}$$

makes C a Poisson algebra. The triple $(A, C, \partial\colon C \to \mathrm{Der}(A/C))$ will be also called a *Poisson order* in places where the role of ∂ needs to be emphasized.

As discussed in [13, Section 2.2], specializations of families of algebras give rise to Poisson orders. In our previous work [43] we generalized this construction to obtain Poisson orders from higher degree terms in the derivation ∂; this is reviewed as follows.

Definition 4.3 Let R be an algebra over $\Bbbk$ and $\hbar$ be a central element of R which is regular, i.e., not a zero-divisor of R. We refer to the $\Bbbk$-algebra $R_0 := R/\hbar R$ as the *specialization of* R at $\hbar \in Z(R)$.

Notation 4.4 (θ, ι, N) Retain the notation of Definition 4.3. Let [-, -] denote the commutator of elements of R. Let $\theta : R \twoheadrightarrow R_0$ be the canonical projection; so, $\ker\theta = \hbar R$. Fix a linear map $\iota : Z(R_0) \hookrightarrow R$ such that $\theta \circ \iota = \mathrm{id}_{Z(R_0)}$. Let $N \in \mathbb{Z}_+$ be such that

$$[\iota(z), y] \in \hbar^N R \quad \text{for all} \quad z \in Z(R_0),\ y \in R. \tag{4.5}$$

Note that (4.5) holds for $N = 1$: take $\widetilde{y} \in \theta^{-1}(y)$ for $y \in R_0$ and we get $\theta([\iota(z), \widetilde{y}]) = [z, y] = 0$.

Definition 4.6 Retain the notation above. For $y \in R_0$ and $z \in Z(R_0)$, the *special derivation of level N* is defined (in fact, well-defined) as

$$\partial_z(y) := \theta\left(\frac{[\iota(z), \widetilde{y}]}{\hbar^N}\right), \quad \text{where } \widetilde{y} \in \theta^{-1}(y). \tag{4.7}$$

The next result states that ∂_z is indeed a derivation, and thus specializations yield Poisson orders.

Proposition 4.8 [43, Proposition 2.7 and Corollary 2.8] *Let R be a $\Bbbk$-algebra and $\hbar \in Z(R)$ be a regular element. Assume that $\iota : R_0 := R/(\hbar R) \hookrightarrow R$ is a linear section of the specialization map $\theta : R \twoheadrightarrow R_0$ such that (4.5) holds for some $N \in \mathbb{Z}_+$. Assume that R_0 is module-finite over $Z(R_0)$.*

(1) *If, for all $z \in Z(R_0)$, ∂_z is a special derivation of level N, then*

$$(R_0, Z(R_0), \partial : Z(R_0) \to \mathrm{Der}(R_0/Z(R_0)))$$

is a Poisson order and the map ∂ is a homomorphism of Lie algebras.
(2) *If $C \subset Z(R_0)$ is a Poisson subalgebra of $Z(R_0)$ with respect to the Poisson structure (4.2) and R_0 is module-finite over C, then R_0 is a Poisson C-order via the restriction of ∂ to C.*
(3) *If, in addition to (2), the restricted section $\iota : C \hookrightarrow R$ is an algebra homomorphism, then*

$$\partial_{zz'}(y) = z\partial_{z'}(y) + z'\partial_z(y) \quad \textit{for} \quad z, z' \in C, y \in R_0.$$

We coined such a construction with the following terminology.

Definition 4.9 [43, Definition 2.9] The Poisson order produced in Proposition 4.8 is a *Poisson order of level N* when the level of the special derivation needs to be emphasized.

4.2 Symplectic cores and the Brown–Gordon theorem

Poisson orders can be used to establish isomorphisms for different central quotients of a PI algebra via the result of Brown and Gordon [13] provided below. The result relies on the notion of *symplectic core*, introduced in [13]. We recall some terminology from [13, Section 3.2].

Definition 4.10 ($\mathcal{P}(I)$) Let $(C, \{.,.\})$ be an affine Poisson algebra over a field $\Bbbk$ of characteristic 0. For every ideal I of C, there exists a unique maximal Poisson ideal contained in I, to be denoted by $\mathcal{P}(I)$. If I is prime, then $\mathcal{P}(I)$ is Poisson prime, [25, Lemma 6.2].

(1) We refer to $\mathcal{P}(I)$ above as the *Poisson core* of I.
(2) We say that two maximal ideals $\mathfrak{m}, \mathfrak{n} \in \mathrm{maxSpec}\, C$ of an affine Poisson algebra $(C, \{.,.\})$ are *equivalent* if $\mathcal{P}(\mathfrak{m}) = \mathcal{P}(\mathfrak{n})$.
(3) The equivalence class of $\mathfrak{m} \in \mathrm{maxSpec}\, C$ is referred to as the *symplectic core* of $\mathfrak{m}$. The corresponding partition of $\mathrm{maxSpec}\, C$ is called *symplectic core partition*.

One main benefit of using the symplectic core partition is the powerful result below.

Theorem 4.11 [13, Theorem 4.2] *Assume that $\Bbbk = \mathbb{C}$ and that A is a Poisson C-order which is an affine $\mathbb{C}$-algebra. If $\mathfrak{m}, \mathfrak{n} \in \mathrm{maxSpec}\, C$ are in the same symplectic*

core, then there is an isomorphism between the corresponding finite-dimensional $\mathbb{C}$*-algebras*

$$A/(\mathfrak{m}A) \cong A/(\mathfrak{n}A).$$

□

5 A specialization setting for 4-dimensional Sklyanin algebras

The goal of this section is to produce a setting so that the PI 4-dimensional Sklyanin algebras arise as Poisson orders via specialization; see Sect. 4.1. The section also sets up some of the notation regarding Poisson orders that we will use throughout this work.

Recall that $S := S(\alpha, \beta, \gamma)$ is a 4-dimensional Sklyanin algebra and *we do not necessarily need that* S *is module-finite over its center* Z. In any case, recall that B ($\cong S/(Sg_1 + Sg_2)$) is the corresponding twisted homogeneous coordinate ring.

The reader may wish to view Fig. 3 at this point for a preview of the setting that we will construct for S. Our objective is to produce a degree 0 deformation $S_\hbar$ of S using a formal parameter $\hbar$. The specialization map for S will be realized via a canonical projection $\theta_S : S_\hbar \to S$ given by $\hbar \mapsto 0$. Moreover, $S_\hbar$ will have the structure of a $\Bbbk[[\hbar]]$-algebra.

Notation 5.1 $(\hbar, \widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma})$ To begin, we fix a formal parameter $\hbar$ and let

$$\widetilde{\alpha} := \alpha + \alpha_1\hbar + \alpha_2\hbar^2 + \cdots, \ \widetilde{\beta} := \beta + \beta_1\hbar + \beta_2\hbar^2 + \cdots, \ \widetilde{\gamma} := \gamma + \gamma_1\hbar + \gamma_2\hbar^2 + \cdots,$$

in $\Bbbk[[\hbar]]$ satisfying $\widetilde{\alpha} + \widetilde{\beta} + \widetilde{\gamma} + \widetilde{\alpha}\widetilde{\beta}\widetilde{\gamma} = 0$ (a version of (2.2)).

By our choice (2.3), we know $(\alpha, \beta, \gamma) \notin \{(-1, 1, \gamma), (\alpha, -1, 1), (1, \beta, -1)\}$. Hence, it is clear that

$$(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}) \notin \{(-1, 1, \widetilde{\gamma}), (\widetilde{\alpha}, -1, 1), (1, \widetilde{\beta}, -1)\}.$$

Definition 5.2 $(\widehat{S}_\hbar, S_\hbar)$ Denote by $\widehat{S}_\hbar$ the 4-dimensional Sklyanin algebra over $\overline{\Bbbk((\hbar))}$ with parameters $(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma})$. Define the *formal Sklyanin algebra* to be the $\Bbbk[[\hbar]]$-subalgebra $S_\hbar$ of $\widehat{S}_\hbar$ generated by x_0, x_1, x_2, x_3, that is,

$$S_\hbar := \Bbbk[[\hbar]]\langle x_0, x_1, x_2, x_3\rangle \subset \widehat{S}_\hbar.$$

It is important to point out that $S_\hbar$ is a graded $\Bbbk[[\hbar]]$-algebra with the grading inherited from $\widehat{S}_\hbar$, such that $\deg(\hbar) = 0$ and $\deg(x_i) = 1$ for $0 \le i \le 3$. Notice that by our choice, $1 \pm \widetilde{\alpha}$, $1 \pm \widetilde{\beta}$, $1 \pm \widetilde{\gamma} \notin (\hbar)$. So, these elements are all invertible in the formal power series $\Bbbk[[\hbar]]$. Recall by Lemma 2.9, we obtain the following result.

Lemma 5.3 $(\widetilde{g_1}, \widetilde{g_2})$ *The elements*

$$\widetilde{g_1} = -x_0^2 + x_1^2 + x_2^2 + x_3^2 \quad \text{and} \quad \widetilde{g_2} = x_1^2 + (1+\widetilde{\alpha})(1-\widetilde{\beta})^{-1}x_2^2 + (1-\widetilde{\alpha})(1+\widetilde{\gamma})^{-1}x_3^2$$

form a central regular sequence in $\widehat{S}_\hbar$. □

Lemma 5.4 *The following statements hold for the formal Sklyanin algebra* $S_\hbar$.

(1) $\widehat{S}_\hbar \cong \overline{\Bbbk((\hbar))} \otimes_{\Bbbk[[\hbar]]} S_\hbar$.
(2) *At each degree* d *of* $S_\hbar$, *we get that* $(S_\hbar)_d$ *is a free* $\Bbbk[[\hbar]]$*-module of rank* $\binom{d+3}{3}$.
(3) *The elements* $\widetilde{g_1}, \widetilde{g_2}$ *belong to the center of* $S_\hbar$.
(4) *There is a natural surjection from* $S_\hbar \twoheadrightarrow S$ *via* $\hbar \mapsto 0$ *with kernel equal to* $\hbar S_\hbar$.

Proof (1) This is clear from the definitions of $\widehat{S}_\hbar$ and $S_\hbar$.
(2) Since $\widehat{S}_\hbar$ is a domain, each graded piece $(S_\hbar)_d \subset (\widehat{S}_\hbar)_d$ is a finitely generated torsion-free module over $\Bbbk[[\hbar]]$. Because $\Bbbk[[\hbar]]$ is a PID, this implies that $(S_\hbar)_d$ is a free $\Bbbk[[\hbar]]$-module. By (1), $(S_\hbar)_d$ has rank equal to $\dim(\widehat{S}_\hbar)_d$, which in turn is equal to $\binom{d+3}{3}$ for $\widehat{S}_\hbar$ has Hilbert series $1/(1-t)^4$.
(3) It is easy to check that $\widetilde{g_1}, \widetilde{g_2} \in Z(\widehat{S}_\hbar) \cap S_\hbar \subset Z(S_\hbar)$.
(4) It suffices to show that $S_\hbar/\hbar S_\hbar \cong S$. Clearly there is a surjection $S \twoheadrightarrow S_\hbar/\hbar S_\hbar$. Moreover, it is an isomorphism since on each degree $\dim S_d = \dim(S_\hbar)_d/\hbar(S_\hbar)_d = \binom{d+3}{3}$ by (2). The kernel part follows directly. □

Notation 5.5 (θ_S) Denote by θ_S the corresponding *specialization map for the formal Sklyanin algebra* $S_\hbar$, namely

$$\theta_S : S_\hbar \to S \quad \text{given by} \quad \hbar \mapsto 0.$$

So, the first column in Fig. 3 below is established and we now turn our attention to the second column of that figure.

Definition 5.6 $(E_\hbar, \mathcal{L}_\hbar, \sigma_\hbar, \widehat{B}_\hbar, B_\hbar)$ Denote by $E_\hbar$ the elliptic curve over $\overline{\Bbbk((\hbar))}$, by $\mathcal{L}_\hbar = \mathcal{O}_{\mathbb{P}^3}(1)|_{E_\hbar}$ the invertible sheaf over $E_\hbar$, and by $\sigma_\hbar$ the automorphism of $E_\hbar$ corresponding to $\widehat{S}_\hbar$ as in Definition-Lemma 2.4 with (α, β, γ) replaced by $(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma})$ from Notation 5.1. Let

$$\widehat{B}_\hbar := B(E_\hbar, \mathcal{L}_\hbar, \sigma_\hbar)$$

be the corresponding twisted homogeneous coordinate ring. Its $\Bbbk[[\hbar]]$-subalgebra

$$B_\hbar := \Bbbk[[\hbar]]\langle x_0, x_1, x_2, x_3\rangle \subset \widehat{B}_\hbar,$$

generated by x_0, x_1, x_2, x_3, will be called *formal twisted homogeneous coordinate ring*.

Lemma 5.7 ($\psi_\hbar$) *The canonical projection $\widehat{S}_\hbar \twoheadrightarrow \widehat{B}_\hbar$ induces a surjection $\psi_\hbar : S_\hbar \twoheadrightarrow B_\hbar$, whose kernel is generated by $\widetilde{g}_1, \widetilde{g}_2$. As a consequence, the composition $S_\hbar \overset{\theta_S}{\twoheadrightarrow} S \twoheadrightarrow B$ factors through the map $\psi_\hbar$.*

Proof Note that $\psi_\hbar$ is the composition $S_\hbar \hookrightarrow \widehat{S}_\hbar \twoheadrightarrow \widehat{B}_\hbar$, whose image is $B_\hbar$ by definition. So it remains to show

$$\ker(\psi_\hbar) \;=\; (\widetilde{g}_1\widehat{S}_\hbar + \widetilde{g}_2\widehat{S}_\hbar) \cap S_\hbar \;=\; \widetilde{g}_1 S_\hbar + \widetilde{g}_2 S_\hbar.$$

For simplicity, we write $M := \widetilde{g}_1 S_\hbar$ and $N := \widetilde{g}_1\widehat{S}_\hbar \cap S_\hbar$. It is clear that $M \subseteq N$, which are both graded ideals in $S_\hbar$. By Lemma 5.4(1), we have $M \otimes_{\Bbbk[[\hbar]]} \overline{\Bbbk((\hbar))} = N$. Hence, as finitely generated free modules over $\Bbbk[[\hbar]]$, we get that $\operatorname{rank}(M_d) = \operatorname{rank}(N_d)$ in each degree d. So $(N/M)_d$ is a torsion module over $\Bbbk[[\hbar]]$. Then for any $r \in N_d$, there exists an integer m such that $\hbar^m r = y\widetilde{g}_1$ for some $y \in (S_\hbar)_{d-2}$. Since $\hbar \nmid \widetilde{g}_1$, we get that $\hbar^d \mid y$ and $r \in M_d$. This implies $M_d = N_d$ for every d, and $M = N$. By a similar argument, we can then conclude that $(\widetilde{g}_1\widehat{S}_\hbar + \widetilde{g}_2\widehat{S}_\hbar) \cap S_\hbar = \widetilde{g}_1 S_\hbar + \widetilde{g}_2 S_\hbar$ where the left-hand side is equal to $\ker(\psi_\hbar)$. Finally, the factorization through $\psi_\hbar$ is straight-forward. □

Definition 5.8 (θ_B) Let $\theta_B : B_\hbar \twoheadrightarrow B$ be the map induced by Lemma 5.7, which we call the *specialization map for the formal twisted homogeneous coordinate ring $B_\hbar$*.

Now we complete the verification of Fig. 3 as follows.

Definition 5.9 (w, $L_\hbar$, v_0', v_1', v_2', v_3', $R_\hbar$) Fix a nonzero section $w \in H^0(E, \mathcal{L})$, which is also realized as an element of $H^0(E_\hbar, \mathcal{L}_\hbar)$ via the vector space isomorphism $H^0(E_\hbar, \mathcal{L}_\hbar) \cong \overline{\Bbbk((\hbar))} \otimes_\Bbbk H^0(E, \mathcal{L})$. Denote by $L_\hbar$ the $\Bbbk[[\hbar]]$-subalgebra of the function field $\overline{\Bbbk((\hbar))}(E_\hbar)$ generated by $v_0' := v_0/w$, $v_1' := v_1/w$, $v_2' := v_2/w$ and $v_3' = v_3/w$ satisfying the dehomogenized relations:

$$\begin{aligned}
&\phi_1(v_0', v_1', v_2', v_3') = v_0'^2 + v_1'^2 + v_2'^2 + v_3'^2 = 0,\\
&\phi_2(v_0', v_1', v_2', v_3') = (1-\widetilde{\gamma})(1+\widetilde{\alpha})^{-1}v_1'^2 + (1+\widetilde{\gamma})(1-\widetilde{\beta})^{-1}v_2'^2 + v_3'^2 = 0.
\end{aligned}$$

We call $R_\hbar := (L_\hbar)_{(\hbar)}$ the *integral form* of the field $\overline{\Bbbk((\hbar))}(E_\hbar)$.

Note that the quotient field of $L_\hbar$ is $Q(L_\hbar) \cong \overline{\Bbbk((\hbar))}(E_\hbar)$. Using (2.5) with replacing (α, β, γ) by $(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma})$, one sees that the automorphism $\sigma_\hbar \in \operatorname{Aut}\left(\overline{\Bbbk((\hbar))}(E_\hbar)\right)$ restricts to an automorphism of $R_\hbar$, given by

$$\sigma_\hbar(v_i') = \sigma_\hbar(v_i/w) = \frac{\sigma_\hbar(v_i)/w^2}{\sigma_\hbar(w)/w^2}, \text{ for all } 0 \le i \le 3.$$

Let D and $D_\hbar$ be the divisors of zeros of w for E and $E_\hbar$, respectively. In view of Lemma 2.8, we have the following commutative diagram:

$$\begin{array}{ccccc}
(\widehat{B}_\hbar)_1 = H^0(E_\hbar, \mathcal{L}_\hbar) & \xrightarrow{\cong} & H^0(E_\hbar, \mathcal{O}(D_\hbar))t & \hookrightarrow & \overline{\Bbbk((\hbar))}(E_\hbar)[t^{\pm 1}; \sigma_\hbar] \\
\uparrow & & \uparrow & & \uparrow \\
(B_\hbar)_1 = \Bbbk[[\hbar]] \otimes_\Bbbk H^0(E, \mathcal{L}) & \xrightarrow{\cong} & \Bbbk[[\hbar]] \otimes_\Bbbk H^0(E, \mathcal{O}(D_\hbar))t & \hookrightarrow & R_\hbar[t^{\pm 1}; \sigma_\hbar] \\
\downarrow{\scriptstyle \hbar \mapsto 0} & & \downarrow{\scriptstyle \hbar \mapsto 0} & & \downarrow{\scriptstyle \hbar \mapsto 0} \\
B_1 = H^0(E, \mathcal{L}) & \xrightarrow{\cong} & H^0(E, \mathcal{O}(D))t & \hookrightarrow & \Bbbk(E)[t^{\pm 1}; \sigma]
\end{array}$$

Moreover, we have the canonical embeddings

$$B_\hbar \hookrightarrow R_\hbar[t^{\pm 1}; \sigma_\hbar] \hookrightarrow \overline{\Bbbk((\hbar))}(E_\hbar)[t^{\pm 1}; \sigma_\hbar]. \tag{5.10}$$

The ring $R_\hbar[t^{\pm 1}; \sigma_\hbar]$ is a graded localization of $B_\hbar$ by an Ore set which does not intersect the kernel $\ker \theta_B = \hbar B_\hbar$. Therefore, the following map is well-defined.

Definition 5.11 (θ_R) Let $\theta_R : R_\hbar[t^{\pm 1}; \sigma_\hbar] \twoheadrightarrow \Bbbk(E)[t^{\pm 1}, \sigma]$ be defined by

$$\theta_R(v_i') = v_i', \text{ for all } 0 \le i \le 3, \quad \theta_R(t) = t, \quad \theta_R(\hbar) = 0,$$

which is the extension of θ_B via localization. We also denote by θ_R its restriction to the specialization map $R_\hbar \twoheadrightarrow \Bbbk(E)$. These maps are referred to as the *specialization maps for the integral form of the formal twisted homogeneous coordinate* B.

The commutativity of the cells in Fig. 3 between the second and third column follows directly from the definitions of the maps within this part of the diagram and from the previous commutative diagram.

$$\begin{array}{ccccc}
\widehat{S}_\hbar = S(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}) & \xrightarrow{\mathrm{mod}(\widetilde{g_1}, \widetilde{g_2})} & \widehat{B}_\hbar = B(E_\hbar, \mathcal{L}_\hbar, \sigma_\hbar) & \xrightarrow{\text{gr. quot. ring}} & \overline{\Bbbk((\hbar))}(E_\hbar)[t^{\pm 1}; \sigma_\hbar] \\
\uparrow & & \uparrow & & \uparrow \\
S_\hbar = k[[\hbar]]\langle x_0, x_1, x_2, x_3\rangle & \xrightarrow{\mathrm{mod}(\widetilde{g_1}, \widetilde{g_2})} & B_\hbar = k[[\hbar]]\langle x_0, x_1, x_2, x_3\rangle & \hookrightarrow & R_\hbar[t^{\pm 1}; \sigma_\hbar] \\
\downarrow{\scriptstyle \theta_S} & & \downarrow{\scriptstyle \theta_B} & & \downarrow{\scriptstyle \theta_R} \\
S = S(\alpha, \beta, \gamma) & \xrightarrow{\mathrm{mod}(g_1, g_2)} & B = B(E, \mathcal{L}, \sigma) & \xrightarrow{\text{gr. quot. ring}} & \Bbbk(E)[t^{\pm 1}; \sigma] \\
\uparrow & & \uparrow & & \uparrow \\
Z(S) & \xrightarrow{\mathrm{mod}(g_1, g_2)} & Z(B) & \xrightarrow{\text{gr. quot. ring}} & \Bbbk(E)^\sigma[t^{\pm n}; \sigma]
\end{array}$$

Fig. 3 Specialization setting for Sklyanin algebras: integral forms, Poisson orders, and centers are respectively in the last three rows

6 Poisson orders on PI 4-dimensional Sklyanin algebras S

This section establishes a construction of Poisson orders on PI 4-dimensional Sklyanin algebras with the property that the induced Poisson structure on its center is nontrivial.

Recall that S is a PI 4-dimensional Sklyanin algebra of PI degree $n = |\sigma|$ corresponding to the parameters $\alpha, \beta, \gamma \in \Bbbk$ satisfying (2.2) and (2.3). Let the scalars $\alpha_i, \beta_i, \gamma_i$ defining $\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}$ in Notation 5.1, for $i \geq 1$, be such that

$$\widetilde{\alpha} + \widetilde{\beta} + \widetilde{\gamma} + \widetilde{\alpha}\widetilde{\beta}\widetilde{\gamma} = 0 \quad \text{and} \quad |\sigma_\hbar| = \infty. \tag{6.1}$$

Such scalars exist because $\{(\alpha, \beta, \gamma) \in \mathbb{A}^3 \mid |\sigma_{\alpha\beta\gamma}| = n\}$ is a closed subset of the surface $\mathbb{V}(\alpha + \beta + \gamma + \alpha\beta\gamma) \subset \mathbb{A}^3$.

6.1 Construction of Poisson orders

Throughout the section we will identify the first graded pieces S_1 of S and B_1 of B with each other through vector space isomorphism (Lemma 2.9). In particular, $\{x_0, x_1, x_2, x_3\}$ is a good basis of B_1 as in Lemma 2.17.

Notation 6.2 $(\widetilde{x}_i, \widetilde{z}_i)$ Denote by $\widetilde{x}_i$ the preimage of x_i under the specialization map $\theta_S : S_\hbar \twoheadrightarrow S$ which is given by the same linear combinations of the standard generators of $S_\hbar$ as is x_i given in terms of the standard generators of S. Moreover, set

$$\widetilde{z}_i := \widetilde{x}_i^n + \textstyle\sum_{1 \leq j < n/2} c_{ij} \widetilde{g}^j \widetilde{x}_i^{n-2j} \in S_\hbar, \quad 0 \leq i \leq 3$$

for the polynomials $c_{ij} \in \Bbbk[g_1, g_2]_{2j}$ from Proposition 2.18(1).

Definition 6.3 Consider the terminology below.

(1) A degree 0 section $\iota : Z \hookrightarrow S_\hbar$ of the map $\theta_S : S_\hbar \twoheadrightarrow S$ will be called *good* if

(i) $\iota(z_i) - \widetilde{z}_i \in \widetilde{g}_1 \Bbbk[\widetilde{x}_i, \widetilde{g}_1, \widetilde{g}_2, \hbar] + \widetilde{g}_2 \Bbbk[\widetilde{x}_i, \widetilde{g}_1, \widetilde{g}_2, \hbar]$, and
(ii) $\iota(g_1) = \widetilde{g}_1$ and $\iota(g_2) = \widetilde{g}_2$.

(2) A specialization map $\theta_S : S_\hbar \to S$ will be called a *good specialization of S of level N* if there exists a good section $\iota : Z \hookrightarrow S_\hbar$ such that

$$[\iota(z), y] \in \hbar^N S_\hbar \quad \text{for all } z \in Z,\ y \in S_\hbar. \tag{6.4}$$

Given any section $\iota : Z \hookrightarrow S_\hbar$ of θ_S,

$$[\iota(z), y] \in \hbar S_\hbar \quad \text{for all } z \in Z,\ y \in S_\hbar.$$

Therefore, $N \geq 1$. Next, we prove that for a given PI 4-dimensional Sklyanin algebra S, the levels N of good specializations for S are bounded from above.

Lemma 6.5 *Let S be a PI Sklyanin algebra corresponding to the parameters $\alpha, \beta, \gamma \in \Bbbk$ satisfying (2.2) and (2.3), and choose any formal parameters $\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}$ satisfying (6.1). Then, the set of levels N for good sections of θ_S has an upper bound.*

Proof Consider the algebras $L_\hbar$ and $R_\hbar := (L_\hbar)_{(\hbar)}$ from Definition 5.9 for $w = \widetilde{x}_0$, identified as the preimage of v_0' under θ_B. Since the condition (6.1) is satisfied, the automorphism $\sigma_\hbar$ of $R_\hbar$ has infinite order. This gives

$$R_\hbar^{\sigma_\hbar^n} \subsetneq R_\hbar. \tag{6.6}$$

On the other hand, $\hbar \in R_\hbar$ is a regular element, from which one gets

$$\bigcap_{l\in\mathbb{Z}_+} \hbar^l R_\hbar = 0. \tag{6.7}$$

Combining (6.6) and (6.7), we obtain that there exists a positive integer M such that

$$r^{\sigma_\hbar^n} - r \notin \hbar^M R_\hbar \quad \text{for some} \quad r \in R_\hbar. \tag{6.8}$$

Denote by $M_{\min}$ the least such positive integer, and by $r_{\min} \in R_\hbar$ an element satisfying (6.8) for this integer $M_{\min}$. Although, it will not play a role in the proof, we note that $M_{\min} \geq 2$ because $r^{\sigma_\hbar^n} - r \in \hbar R_\hbar$ for all $r \in R_\hbar$.

In the remainder of the proof, we will show that

$$N < M_{\min}$$

which gives the stated upper bound for levels of good sections. Let $\iota : Z \hookrightarrow S_\hbar$ be one such good section of θ_S, with respect to which the specialization map θ_S has level N. Recall the map $\psi_\hbar : S_\hbar \twoheadrightarrow B_\hbar$ from Lemma 5.7. Then, the condition (6.4) implies that

$$[\psi_\hbar\iota(z), y] \in \hbar^N B_\hbar, \qquad \forall\, z \in Z,\ y \in B_\hbar.$$

From (5.10), we have $B_\hbar \hookrightarrow R_\hbar[t^{\pm 1}; \sigma_\hbar]$ with respect to which $R_\hbar[t^{\pm 1}; \sigma_\hbar]$ is a localization of $B_\hbar$. Hence,

$$[\psi_\hbar\iota(z), y] \in \hbar^N R_\hbar[t^{\pm 1}; \sigma_\hbar], \qquad \forall\, z \in Z,\ y \in R_\hbar[t^{\pm 1}; \sigma_\hbar]. \tag{6.9}$$

Proposition 2.18, combined with the facts that $\ker \psi_\hbar = \widetilde{g}_1 S_\hbar + \widetilde{g}_2 S_\hbar$ (from Lemma 5.7) and that ι is a good section, leads to $\psi_\hbar\iota(z_1) = t^n \in R_\hbar[t^{\pm 1}; \sigma_\hbar]$. By applying (6.9) to $z = z_1$ and $y = r_{\min} \in R_\hbar$, we obtain that

$$[t^n, r_{\min}] = (r_{\min}^{\sigma_\hbar^n} - r_{\min})t^n \in \hbar^N R_\hbar[t^{\pm 1}; \sigma_h].$$

Since t is a unit of $R_\hbar[t^{\pm 1}; \sigma_h]$,

$$r_{\min}^{\sigma_\hbar^n} - r_{\min} \in \hbar^N R_\hbar.$$

The last equation implies that $N < M_{\min}$ because $r_{\min}^{\sigma_\hbar^n} - r_{\min} \notin \hbar^{M_{\min}} R_\hbar$. □

The main theorem of this section establishes the structure of Poisson order on each PI 4-dimensional Sklyanin algebra S for which the Poisson structure on the center of S is nontrivial.

Theorem 6.10 *Let S be a PI 4-dimensional Sklyanin algebra satisfying (2.2)–(2.3) as above. Each Poisson order $(S, Z, \partial : Z \to \mathrm{Der}(S/Z))$ of level N, coming from a good specialization of maximum level N, has the property that the induced Poisson structure on Z is non-zero.*

One interesting direction for further investigation is as follows.

Question 6.11 Given PI 4-dimensional Sklyanin algebra S as used in Theorem 6.10, what is the maximal level N of a good specialization for S?

One approach to this question is to determine the minimal positive integer $M_{\min}$ from the proof of Lemma 6.5 which is explicitly defined in (6.8) and then to use the upper bound $N \leq M_{\min}$.

6.2 Derivations of PI S

We classify a certain type of derivations of S which will play a role in the proof of Theorem 6.10. This description will be obtained in three stages where similar types of derivations of the algebras in Notation 6.12 are classified.

In fact, the techniques below can be generalized easily to obtain similar results for PI algebras T, where some factor of T by a regular central sequence is the twisted homogeneous coordinate ring of an elliptic curve; see Theorem 6.19.

Notation 6.12 $(S', \psi, \psi_1, \psi_2)$ For S a PI 4-dimensional Sklyanin algebra of PI degree $n < \infty$ with center $Z := Z(S)$. Recall $B = S/(g_1S + g_2S)$ and denote $S' := S/(g_1S)$. Moreover, denote the canonical projections

$$\psi_1 : S \to S', \qquad \psi_2 : S' \to B, \qquad \text{and} \quad \psi := \psi_2\psi_1 : S \to B.$$

So,

$$\ker \psi_1 = g_1S, \qquad \ker \psi_2 = g_2S', \qquad \text{and} \quad \ker \psi = g_1S + g_2S.$$

Lemma 6.13 *In the notation above, we have*

$$\psi_1(Z) = Z(S') \quad \textit{and} \quad \psi_2(Z(S')) = Z(B).$$

Proof Since $\psi(Z) = Z(B)$,

$$\psi_2(Z(S')) = Z(B).$$

The map ψ_1 is surjective, thus

$$\psi_1(Z) \subseteq Z(S'). \tag{6.14}$$

Both sides of the inclusion are $\mathbb{N}$-graded algebras with

$$\psi_1(Z) = \bigoplus_{k\in\mathbb{N}} \psi_1(Z_k), \qquad Z(S') = \bigoplus_{k\in\mathbb{N}} Z(S')_k.$$

To prove that (6.14) is an equality, one needs to prove that

$$\psi_1(Z_k) = Z(S')_k, \tag{6.15}$$

which we show by induction on k.

The equality is obvious for $k = 0$. Let k be a positive integer. Assume that (6.15) holds for indices less than k. It follows from the equality $\psi(Z) = Z(B)$ that for each $z' \in Z(S')_k$, there exists $z \in Z_k$ such that

$$z' - \psi_1(z) \in \ker \psi_2 = g_2 S'.$$

Write $z' - \psi_1(z) = g_2 u'$ for some $u' \in S'_{k-2}$. The assumptions on z and z' imply that $g_2 u' \in Z(S')_k$. Therefore, $u' \in Z(S')_{k-2}$ because g_2 is a regular central element of S' of degree 2. It follows from the induction assumption that $u' = \psi_1(u)$ for some $u \in Z_{k-2}$. Finally, using that $g_2 \in Z_2$, we obtain

$$z + g_2 u \in Z(S)_k \qquad \text{and} \qquad z' = \psi_1(z + g_2 u) \in \psi_1(Z_k),$$

which completes the induction and the proof of the lemma. □

Notation 6.16 (ad_r) As usual, for an algebra R and $r \in R$, ad_r will denote the inner derivation of R given by $\mathrm{ad}_r(r') := [r, r'] = rr' - r'r$.

Proposition 6.17 (x, δ) *Assume that S is a PI 4-dimensional Sklyanin algebra of PI degree $n = |\sigma| < \infty$. Let $x \in S_1 \cong B_1$ be a good element.*

(1) *If $\delta \in \mathrm{Der}\, B$ is such that*

$$\text{(i)}\ \delta|_{Z(B)} = 0, \qquad \text{(ii)}\ \delta(x) = 0, \qquad \text{(iii)}\ \deg \delta = l \text{ with } l \le n,$$

then

$$\delta = \begin{cases} 0, & \text{if } l \le 0 \text{ or } l = n \\ \lambda\, \mathrm{ad}_{x^l}, & \text{if } 0 < l < n. \end{cases}$$

for some $\lambda \in \Bbbk$.

(2) *If $\delta \in \mathrm{Der}\, S'$ is such that*

$$\text{(i)}\ \delta|_{Z(S')} = 0, \qquad \text{(ii)}\ \delta(x) = 0, \qquad \text{(iii)}\ \deg \delta = l \text{ with } l \le n,$$

then $\delta = \mathrm{ad}_{p'}$ for some $p' \in \Bbbk[x, g_2]_l$ such that $\deg_x p' < n$.

(3) *If* $\delta \in \mathrm{Der}\, S$ *is such that*

$$\text{(i)}\ \delta|_Z = 0, \qquad \text{(ii)}\ \delta(x) = 0, \qquad \text{(iii)}\ \deg \delta = l \quad \textit{with} \quad l \leq n,$$

then $\delta = \mathrm{ad}_p$ *for some* $p \in \Bbbk[x, g_1, g_2]_l$ *such that* $\deg_x p < n$.

Proof (1) For $l < n$, the statement of this part of the proposition (even without the assumption that $\delta|_{Z(B)} = 0$) was proved by Smith and Tate in [39, Theorem 3.3]. Suppose that $l = n$. Recall that the graded quotient ring of B is identified with $\Bbbk(E)[t^{\pm 1}; \sigma]$ by sending $x \mapsto t$. The derivation $\delta \in \mathrm{Der}(B)$ can be extended to a derivation of $\Bbbk(E)[t; \sigma]$, to be denoted by the same symbol. The conditions (i)–(ii) give that

$$\delta(t) = 0 \quad \text{and} \quad \delta(\Bbbk(E)^\sigma) = 0.$$

Choose $y \in \Bbbk(E)$ and let $q(t) \in \Bbbk(E)^\sigma$ be its minimal polynomial over $\Bbbk(E)^\sigma$. Since the extension $\Bbbk(E)/\Bbbk(E)^\sigma$ is separable, $q'(y) \neq 0$. The assumption that $\deg \delta = n$ implies that $\delta(y) \in \Bbbk(E)x^n$ commutes with y. Therefore $0 = \delta(q(y)) = \delta(y)q'(y)$ and, hence, $\delta(y) = 0$. We have that $\delta(\Bbbk(E)) = 0$ and $\delta(t) = 0$, so $\delta = 0$.

(2) Since $\delta(g_2) = 0$, $\psi\delta : S' \to B$ descends to a derivation of B, which will be denoted by the same symbol.

We prove part (2) by induction on l. If $l \leq 0$, then $\psi_2\delta$ is a homogeneous derivation of B such that

$$\psi_2\delta|_{Z(B)} = 0, \qquad \psi_2\delta(x) = 0, \qquad \textit{and} \quad \deg \psi_2\delta \leq 0$$

because $\psi_2(Z(S')) = Z(B)$ (see Lemma 2.14). Part (1) implies that $\psi_2\delta = 0$. Thus, $\delta(S'_1) \subseteq \ker \psi_2 = g_2S' \subseteq S'_{\geq 2}$. On the other hand, it follows from $\deg \delta \leq 0$ that

$$\delta(S'_1) \subseteq S'_{\leq 1}.$$

This implies that $\delta(S'_1) = 0$ and, consequently that $\delta = 0$ because S' is generated in degree 1.

Let $1 \leq l \leq n$. Assume by way of induction that the statement of part (2) holds for derivations of degree less than l. Let δ be a derivation of S' of degree $l \in [1, n]$ such that $\delta|_{Z(S')} = 0$ and $\delta(x) = 0$. As explained earlier, $\psi_2\delta : S' \to B$ descends to a derivation of B which satisfies

$$\psi_2\delta|_{Z(B)} = 0, \qquad \psi_2\delta(x) = 0, \qquad \textit{and} \quad \deg \psi_2\delta = l \in [1, n]$$

because $\psi_2(Z(S')) = Z(B)$. It follows from part (1) that

$$\psi_2\delta = \lambda\, \mathrm{ad}_{x^l}$$

for some $\lambda \in \Bbbk$ such that $\lambda = 0$ if $l = n$. Therefore,

$$\operatorname{im}(\delta - \lambda \operatorname{ad}_{x^l}) \subseteq S' g_2.$$

Using that g_2 is a regular element of S', we define

$$\delta^- := \tfrac{1}{g_2}(\delta - \lambda \operatorname{ad}_{x^l}) \in \operatorname{Der}(S').$$

The assumptions on δ imply at once that

$$\delta^-|_{Z(S')} = 0, \qquad \delta^-(x) = 0, \qquad \textit{and} \quad \deg \delta^- = l - 2.$$

By the inductive assumption there exists $h' \in \Bbbk[x, g_2]_{l-2}$ such that $\delta^- = \operatorname{ad}_{h'}$, and $\deg_x h' < n$. Therefore,

$$\delta = \operatorname{ad}_{p'} \qquad \text{for} \qquad p' := g_2 h' + \lambda x^l \in \Bbbk[x, g_2]_l.$$

Since $\lambda = 0$ if $l = n$, we have $\deg_x p' < n$. This completes the proof of part (2).

Part (3) is derived from part (2) in the same fashion as (2) is derived from (1) making use of the fact that g_1 is a regular element of S. □

Now the techniques above yield the general statements about derivations of certain PI algebras as follows.

Notation 6.18 (T, Ω_k, $T^{[k]}$, ψ_k) Let T be a connected $\mathbb{N}$-graded algebra, and $\Omega_1, \ldots, \Omega_m$ a regular central sequence of elements of T of positive degree. Let

$$T^{[k]} := T/(\Omega_1 T + \cdots + \Omega_k T)$$

for all $1 \leq k \leq m$ so that $T^{[0]} := T$. Note that $T^{[k]} \cong T^{[k-1]}/(\Omega_k T^{[k-1]})$ and let $\psi_k : T^{[k-1]} \twoheadrightarrow T^{[k]}$ be the canonical projection.

Observe that if $T^{[k]}$ is module-finite over its center, then so is $T^{[i]}$ for all $i < k$ by [39, Lemma 3.6].

Theorem 6.19 *Retain the notation above. Then the following statements hold.*

(1) $\psi_k(Z(T^{[k-1]})) = Z(T^{[k]})$ *for all* $1 \leq k \leq m$.

(2) *Suppose that* $T^{[m]}$ *is isomorphic to* $B_T := B(E_T, \mathcal{L}_T, \sigma_T)$, *a twisted homogeneous coordinate ring of an elliptic curve* $E_T \subset \mathbb{P}^d$ *with* $\mathcal{L}_T = \mathcal{O}_{\mathbb{P}^d}(1)|_{E_T}$ *and* σ_T *in* $\operatorname{Aut}(E_T)$ *given by translation. Assume, further, that* B_T *is module-finite over its center so that* $|\sigma_T| < \infty$. *Let* x_T *be a good element of* B_T *of degree 1.*
If δ *is a derivation of* $T^{[k]}$ *such that*

$$\text{(i)}\ \delta|_{Z(T^{[k]})} = 0, \qquad \text{(ii)}\ \delta(x_T) = 0, \qquad \text{(iii)}\ \deg \delta = l \quad \textit{with} \quad l \leq |\sigma_T|,$$

then $\delta = \operatorname{ad}_p$ *for some* $p \in \Bbbk[x_T, \Omega_{k+1}, \ldots, \Omega_m]_l$ *such that* $\deg_{x_T} p < |\sigma_T|$. □

6.3 Proof of Theorem 6.10, nontriviality of Poisson order structure on S

Let $\iota : Z \to S_\hbar$ be a good section of maximal level N. Let $\partial : Z \to \mathrm{Der}(S/Z)$ be the corresponding Poisson order. We need to prove that the induced Poisson structure on Z is nontrivial. By way of contradiction, assume that this is not the case. Then, for $0 \leq i \leq 3$, we have

$$\text{(i) } \partial_{z_i}|_Z = 0, \qquad \text{(ii) } \partial_{z_i}(x_i) = 0, \qquad \text{(iii) } \deg \partial_{z_i} = n.$$

Property (i) is a restatement of the vanishing of the Poisson structure. Property (ii) follows from the definition of good section, and more precisely the condition in Definition 6.3(1.i) and Notation 6.2. Namely, we have $[\iota(z_i), \widetilde{x}_i] = 0$ and thus,

$$\partial_{z_i}(x_i) = \theta([\iota(z_i), \widetilde{x}_i]/\hbar^N) = 0.$$

The third condition is a consequence of the fact that $\iota(z_i) \in (S_\hbar)_n$. It follows from Proposition 6.17(3) that there exist polynomials p_i for $0 \leq i \leq 3$

$$p_i(x_i, g_1, g_2) \in \Bbbk[x_i, g_1, g_2]_n \qquad \text{and} \qquad \deg_{x_i} p_i(x_i, g_1, g_2) < n, \tag{6.20}$$

such that $\partial_{z_i} = \mathrm{ad}_{p_i(x_i, g_1, g_2)}$. We modify the section $\iota : Z \hookrightarrow S_\hbar$ to form a new good section $\iota^\vee : Z \hookrightarrow S_\hbar$ by first setting

$$\iota^\vee(g_1) := \widetilde{g}_1, \quad \iota^\vee(g_2) := \widetilde{g}_2 \quad \text{and} \quad \iota^\vee(z_i) := \iota(z_i) - \hbar^N p_i(\widetilde{x}_i, \widetilde{g}_1, \widetilde{g}_2)$$

for $0 \leq i \leq 3$. Then we choose a $\Bbbk$-basis of Z of the form

$$\mathcal{B} := \{g_1^{m_1} g_2^{m_2} z_0^{l_0} z_1^{l_1} z_2^{l_2} z_3^{l_3} \mid (m_1, m_2, l_0, l_1, l_2, l_3) \in L\}$$

for some $L \subset \mathbb{N}^6$ and complete the definition of $\iota^\vee$ by setting

$$\iota^\vee\left(g_1^{m_1} g_2^{m_2} z_0^{l_0} z_1^{l_1} z_2^{l_2} z_3^{l_3}\right) = \iota^\vee(g_1)^{m_1} \iota^\vee(g_2)^{m_2} \iota^\vee(z_0)^{l_0} \iota^\vee(z_1)^{l_1} \iota^\vee(z_2)^{l_2} \iota^\vee(z_3)^{l_3} \tag{6.21}$$

for all $(m_1, m_2, l_0, l_1, l_2, l_3) \in L$.

The two properties of the polynomials p_i in (6.20) imply that $\iota^\vee : Z \hookrightarrow S_\hbar$ is a good section of θ_S. Furthermore, we have that

$$[\iota^\vee(z_i), y] \in \hbar^{N+1} S_\hbar, \qquad \forall\, 0 \leq i \leq 3,\ y \in S_\hbar. \tag{6.22}$$

This follows from the equalities

$$[\iota^{\vee}(z_i), y] = \hbar^N \left(\frac{[\iota(z_i), y]}{\hbar^N} + [p_i(\widetilde{x_i}, \widetilde{g_1}, \widetilde{g_2}), y] \right),$$
$$\theta_S \left(\frac{[\iota(z_i), y]}{\hbar^N} + [p_i(\widetilde{x_i}, \widetilde{g_1}, \widetilde{g_2}), y] \right) = \partial_{z_i}(\theta(y)) - \partial_{z_i}(\theta(y)) = 0$$

and the fact that $\ker \theta_S = \hbar S_\hbar$, proved in Lemma 5.4(4).

Combining (6.21) and (6.22), we obtain that

$$[\iota^{\vee}(z), y] \in \hbar^{N+1} S_\hbar, \qquad \forall z \in \mathcal{B},\ 0 \leq i \leq 3,\ y \in S_\hbar.$$

This is a contradiction, since N equals the maximum level of a good section of the projection $\theta_S : S_\hbar \to S$. The contradiction implies that the induced Poisson structure on Z from the Poisson order ∂ is nontrivial. □

7 The Jacobian structure of Poisson orders on PI 4-dimensional Sklyanin algebras

The goal of this part is to describe the Jacobian structure of the (nontrivial) Poisson bracket on the center of the PI 4-dimensional Sklyanin algebras that arise as a Poisson order via good specialization of maximal level. We begin this section with a straightforward result about PI algebras that are Poisson orders via specialization.

Proposition 7.1 *Recall Notation 6.18. For a formal parameter $\hbar$, suppose that there exists an $\Bbbk[[\hbar]]$-algebra $T_\hbar$ which is an $\Bbbk[[\hbar]]$-torsionfree degree 0 deformation of T so that $T_\hbar/(\hbar T_\hbar) \cong T$. Take $\theta_T : T_\hbar \twoheadrightarrow T$, the canonical projection, and let $\widetilde{\Omega}_k \in Z(T_\hbar)$ be a lift of $\Omega_k \in Z(T)$ via θ_T for all $1 \leq k \leq m$.*

Then, for every specialization of T of level N equipped with a section $\iota : Z(T) \hookrightarrow T_\hbar$ with $\iota(\Omega_k) = \widetilde{\Omega}_k$ for all $1 \leq k \leq m$, we have that $\partial_{\Omega_k} = 0$ for the corresponding Poisson order on T. In particular, $\Omega_1, \ldots, \Omega_m$ are in the Poisson center of $Z(T)$.

Proof For $y \in T$ and $\widetilde{y} \in \theta_T^{-1}(y)$, we get that $\partial_{\Omega_k}(y) = \theta_T\left([\widetilde{\Omega}_k, y]/\hbar^N\right)$, which is equal to 0. Therefore, $\partial_{\Omega_k} = 0$, and the second statement holds by the definition of a Poisson order. □

Applying this to the PI 4-dimensional Sklyanin algebras S, we have the statement below.

Corollary 7.2 *The central elements g_1, g_2 lie in the Poisson center of $Z = Z(S)$ when S arises as a Poisson Z-order of level N via specialization.*

Proof Apply Proposition 7.1 by taking $T = S$, $T_\hbar = S_\hbar$, $\{\Omega_1, \ldots, \Omega_m\} = \{g_1, g_2\}$ and by using Lemma 5.4(3). □

Next, we have a general result pertaining to the Poisson structure of the PI algebras T from above. In the following proposition, we use notation $J(\underline{F})_{k,l}$ to denote the determinant of the Jacobian matrix by taking all z_i derivatives except with respect to variables z_k and z_l up to a sign $(-1)^{k+l}$.

Proposition 7.3 ($J(\underline{F})_{k,l}$) *Recall Notation 6.18 and we abuse some notation utilized above as follows. Suppose that T is module-finite over its center with $Z(T)$ generated by algebraically independent homogeneous elements $z_1, \ldots, z_d$, along with homogeneous elements $\Omega_1, \ldots, \Omega_m$, subject to $d-2$ homogeneous relations $F_1, \ldots, F_{d-2}$.*

Suppose that $Y := \mathbb{V}(F_1, \ldots, F_{d-2}) \subset \mathbb{A}^{d+m}$ is an irreducible affine variety and define the Jacobian matrix to be

$$J(\underline{F})_{k,l} := (-1)^{k+l} \det\left(\frac{\partial(F_1, F_2, \ldots, F_{d-3}, F_{d-2})}{\partial(z_1, \ldots, \widehat{z_k}, \ldots, \widehat{z_l}, \ldots, z_d)}\right), \text{ for } k < l$$

and $J(\underline{F})_{l,k} = -J(\underline{F})_{k,l}$. Moreover, suppose that $Z(T) = \Bbbk[Y]$ admits a (homogeneous) Poisson structure of degree 0 so that

(i) *$\Omega_1, \ldots, \Omega_m$ are in the Poisson center,*
(ii) *$J(\underline{F})_{k,l} \neq 0$ for some $1 \leq k, l \leq d$,*
(iii) *$\bigcap_{1 \leq k,l \leq d} \Bbbk[Y][(J(\underline{F})_{k,l})^{-1}] = \Bbbk[Y]$, where the intersection is taken over $J(\underline{F})_{k,l} \neq 0$.*

Then, the Poisson bracket on $Z(T)$ is determined by Jacobian matrices above as follows:

$$\{z_k, z_l\} = \eta J(\underline{F})_{k,l}, \quad \text{for all } 1 \leq k, l \leq d, \text{ and for some } \eta \in \Bbbk[Y]. \tag{7.4}$$

If, further, η has degree 0, then $\eta \in \Bbbk[Y]_0 = \Bbbk$.

Proof By condition (i) we have that

$$\{z_k, F_i\} = \sum_{l=1}^{d} \{z_k, z_l\} \partial F_i / \partial z_l + \sum_{l=1}^{m} \{z_k, \Omega_l\} \partial F_i / \partial \Omega_l = \sum_{l=1}^{d} \{z_k, z_l\} \partial F_i / \partial z_l = 0.$$

Now consider the vector space of all $(d-1)$-tuples over the base field $\Bbbk(Y)$, which is the fraction field of the domain $\Bbbk[Y]$. Note that the vector

$$V = (\{z_k, z_1\}, \ldots, \widehat{\{z_k, z_k\}}, \ldots, \{z_k, z_d\})$$

is perpendicular to vectors $(\partial F_i/\partial z_1, \ldots, \widehat{\partial F_i/\partial z_k}, \ldots, \partial F_i/\partial z_d)$ for all $1 \leq i \leq d-2$. By condition (ii) and Cramer's rule, we know V has to be proportional to the vector

$$\left(J(\underline{F})_{k,1}, \ldots, J(\underline{F})_{k,k-1}, J(\underline{F})_{k,k+1}, \ldots, J(\underline{F})_{k,d}\right).$$

So we can write

$$\{z_k, z_l\} = \eta_{kl} J(\underline{F})_{k,l}, \quad \text{for some } \eta_{kl} \in \Bbbk(Y).$$

By construction, we get $\eta_{kl} = \eta_{kl'}$ and $\eta_{kl} = \eta_{lk}$. Hence, $\eta_{kl} = \eta \in \Bbbk(Y)$ for all $1 \leq k, l \leq d$. Finally, the condition (iii) implies that $\eta \in \Bbbk[Y]$. □

Now we arrive at the result below.

Corollary 7.5 *Recall the notation of Sect. 2 and set* $\{0, 1, 2, 3\} = \{i, j, k, l\}$ *with* $i < j$ *and* $k < l$. *Then, the Poisson structure of the center* Z *of* S *obtained via good specialization is given by*

$$\begin{aligned}\{z_k, z_l\} &= (-1)^{k+l} \eta J(F_1, F_2)_{i,j} \\ &= \begin{cases} (-1)^{k+l} \eta \det \left(\frac{\partial \Phi_p}{\partial z_q} \right)_{p=1,2;\ q=i,j}, & \text{for } |\sigma| =: n \text{ is odd} \\ (-1)^{k+l} \eta \det \left(\frac{\partial \Phi_p}{\partial z_q} + h_p \frac{\partial \ell_p}{\partial z_q} \right)_{p=1,2;\ q=i,j}, & \text{for } |\sigma| =: n \text{ is even} \end{cases}\end{aligned}$$

for some $\eta \in \Bbbk^\times$, *with* g_1, g_2 *in the Poisson center of the Poisson algebra* Z.

Proof. We apply Proposition 7.3 for $T = S$ module-finite over its center Z. In this case, $\{\Omega_1, \ldots, \Omega_m\} = \{g_1, g_2\}$ and $Y = \mathbb{V}(F_1, F_2) \subset \mathbb{A}^6$ from Proposition 2.18. The affine variety Y is irreducible because S is a domain (see Proposition 2.11). First, the degree of η is 0 since the degree of $\det(J(F_1, F_2)_{i,j})$ is $2n$, which is equal to the degree of $\{z_k, z_l\}$. Moreover, Theorem 6.10 yields the nontriviality of the Poisson structure on $Z(S)$ via good specialization. So it suffices to verify conditions (i)-(iii) in Proposition 7.3. First, Corollary 7.2 yields condition (i). Next, for condition (ii), note that

$$Y_0^{symp} \cap \mathbb{V}(g_1 - \gamma_1, g_2 - \gamma_2) = (Y_{\gamma_1,\gamma_2})^{sing}.$$

It follows from Theorem 3.13 (n odd) and Theorem 3.16 (n even) that $(Y_{\gamma_1,\gamma_2})^{sing}$ only consists of finitely many points. The second equality in (7.7) implies that there exist indices $0 \leq k \neq l \leq 3$ such that $J(F_1, F_2)_{k,l}$ does not vanish identically on Y.

Finally, condition (iii) of Proposition 7.3 depends on the singular locus on each slice Y_{γ_1,γ_2} as follows. We employ work of Stafford [40] to understand the structures of Y and its coordinate ring Z; namely, S is a maximal order, so Z is integrally closed and Y is a normal affine variety. So, to get condition (iii), it remains to show that $Y_0^{symp} = \bigcup_{\gamma_1,\gamma_2 \in \Bbbk} (Y_{\gamma_1,\gamma_2})^{sing}$ has codimension ≥ 2 in Y due to [22, discussion after proof of Corollary 11.4]. Indeed, we have that Y_0^{symp} is a union of $2n$ cuspidal curves for n odd by Lemma 3.12 and Theorem 3.13 (see Fig. 1); thus Y_0^{symp} has codimension ≥ 2 in Y. For n even, condition (iii) of Proposition 7.3 follows from Theorem 3.16 since $Y_0^{symp} = Y^{sing}$ is a union of two rational surfaces Y_1^{sing} and Y_2^{sing}, which have codimension ≥ 2 in Y. □

Recall that one says that a Poisson structure on an affine variety vanishes at a point $y \in Y$ if the maximal ideal $\mathfrak{m}_y$ of y is a Poisson ideal of $\Bbbk[Y]$.

Corollary 7.6 *The variety Y_0^{symp} from Notation 3.1 is precisely the subvariety of $Y = maxSpec(Z(S))$ consisting of points on which the Poisson bracket of Corollary 7.5 vanishes.*

Proof By definition,

$$Y_0^{symp} = \bigcup_{\gamma_1,\gamma_2 \in \Bbbk} (Y_{\gamma_1,\gamma_2})^{sing} = \left(\bigcap_{1 \leq k,l \leq 4} \mathbb{V}(J(F_1, F_2)_{k,l})\right) \cap Y. \tag{7.7}$$

Corollary 7.5 implies that the right hand side is precisely the subset of points of Y on which the Poisson bracket of Corollary 7.5 vanishes. □

The following result directly follows from the proof of Corollary 7.5, which will be useful in Sect. 8 for determining the Azumaya locus of S. □

Corollary 7.8 *Let $Y = \text{maxSpec}(Z(S))$ be the Poisson variety with the Poisson bracket given in Corollary 7.5. Then, for $(\gamma_1, \gamma_2) \in \Bbbk^2$, we have that $Y_0^{symp} \cap Y_{\gamma_1,\gamma_2} = (Y_{\gamma_1,\gamma_2})^{sing}$, which consists of finitely many points, and that Y_0^{symp} has codimension ≥ 2 in Y.* □

8 On the representation theory of PI 4-dimensional Sklyanin algebras *S*

The goal of this section is to use the algebro- and Poisson-geometric results presented in the previous sections to study the irreducible representations of PI 4-dimensional Sklyanin algebras S of PI degree $n < \infty$. Recall that such representations are finite-dimensional and their maximum dimension is equal to n [11, Proposition 3.1]. Moreover, the isomorphism classes of irreducible representations (or, simple modules) of S are governed by their central annihilators. We refer the reader to Brown-Goodearl's text [12, Chapter III] for details. In summary, there exists a finite-to-1 map on isomorphism classes of simple S-modules $[M]$ to the vanishing of $\text{ann}_S(M) \cap Z(S)$ in $\text{maxSpec}(Z(S)) =: Y$. The *Azumaya locus* of S consists of maximal ideals $\mathfrak{m} \in Y$ that annihilate irreducible representations of maximum dimension ($= n$). To study the representation theory of S geometrically, it is advantageous to have that the Azumaya locus of S and the smooth part of Y coincide– this is established in Sect. 8.1. In Sect. 8.2 we will then use the singular locus of Y to investigate the irreducible representations of S of intermediate dimension. In Sect. 8.3 we apply these representation theoretic results to describe the zero sets of the discriminant ideals of the PI 4-dimensional Sklyanin algebras.

8.1 Irreducible representations of PI *S* of maximum dimension

We will verify the following result, with the proof presented at the end of this section.

Theorem 8.1 *The Azumaya locus of each PI 4-dimensional Sklyanin algebra S is the smooth part of $Y = \text{maxSpec}(Z(S))$.*

Notation 8.2 ($[\kappa_1 : \kappa_2]$, $S_{[\kappa_1:\kappa_2]}$, g', ψ', ψ'') For $[\kappa_1 : \kappa_2] \in \mathbb{P}^1$, consider the factors

$$S_{[\kappa_1:\kappa_2]} := S/(\kappa_1 g_1 - \kappa_2 g_2)S.$$

Set

$$g' := \begin{cases} g_1, & \kappa_2 \neq 0 \\ g_2, & \text{otherwise.} \end{cases}$$

The images of g_1, g_2, g' in $S_{[\kappa_1:\kappa_2]}$ will be denoted by the same symbols. Denote the projections

$$\psi' : S \twoheadrightarrow S_{[\kappa_1:\kappa_2]} \quad \text{and} \quad \psi'' : S_{[\kappa_1:\kappa_2]} \twoheadrightarrow S/(g_1 S + g_2 S) \cong B.$$

Note that the kernel of ψ' is $g' S_{[\kappa_1:\kappa_2]}$. Moreover, the composition $\psi''\psi'$ equals the projection $\psi : S \twoheadrightarrow B$ from Notation 6.12 because $g_1 S + g_2 S = g' S + (\kappa_1 g_1 - \kappa_2 g_2)S$.

Our first preliminary result is given below.

Proposition 8.3 *For every $[\kappa_1 : \kappa_2] \in \mathbb{P}^1$, the factor $S_{[\kappa_1:\kappa_2]}$ of S is a PI domain of PI degree $n = |\sigma|$ and its center is given by $Z(S_{[\kappa_1:\kappa_2]}) = \psi'(Z(S))$.*

Proof First, $S_{[\kappa_1:\kappa_2]}$ is a domain by [29, Proposition 6.2]. Since the algebra $S_{[\kappa_1:\kappa_2]}$ is a homomorphic image of S, we obtain that the PI degree of $S_{[\kappa_1:\kappa_2]}$ is less or equal to n. From the isomorphism $S_{[\kappa_1:\kappa_2]}/(g' S_{[\kappa_1:\kappa_2]}) \cong B$, we obtain that the PI degree of $S_{[\kappa_1:\kappa_2]}$ is greater or equal than that of B which equals n by Proposition 2.12. Thus, $S_{[\kappa_1:\kappa_2]}$ is a PI domain of PI degree n. The result on the center of $S_{\kappa_1:\kappa_2]}$ follows from a proof similar to that of Lemma 6.13. □

We turn our attention to maxSpec of the center of $S_{[\kappa_1:\kappa_2]}$ next.

Notation 8.4 ($Y_{[\kappa_1:\kappa_2]}$) Denote by $Y_{[\kappa_1:\kappa_2]}$ the subvariety $Y \cap \mathbb{V}(\kappa_1 g_1 - \kappa_2 g_2)$ of Y, which is the disjoint union of subvarieties $\{Y_{\gamma_1,\gamma_2} \mid \gamma_1, \gamma_2 \in \Bbbk,\ \kappa_1\gamma_1 - \kappa_2\gamma_2 = 0\}$ of Y.

It also follows from Proposition 8.3 that

$$Y_{[\kappa_1:\kappa_2]} \cong \operatorname{maxSpec}(Z(S_{[\kappa_1:\kappa_2]})). \tag{8.5}$$

Corollary 8.6 *The varieties $Y_{[\kappa_1:\kappa_2]}$ and Y_{γ_1,γ_2} are irreducible for all $[\kappa_1 : \kappa_2] \in \mathbb{P}^1$ and $\gamma_1, \gamma_2 \in \Bbbk$.*

Proof The first fact follows from the isomorphism (8.5) and the fact that $S_{[\kappa_1:\kappa_2]}$ is a domain (see Proposition 8.3). The set

$$Y^*_{[\kappa_1:\kappa_2]} := Y_{[\kappa_1:\kappa_2]} \backslash \mathbb{V}(g') = Y_{[\kappa_1:\kappa_2]} \backslash \mathbb{V}(g_1, g_2)$$

is dense in $Y_{[\kappa_1:\kappa_2]}$. Thus, it is irreducible too. The $\mathbb{N}$-grading of S gives rise to a $\Bbbk^\times$-action on Y that preserves $Y_{[\kappa_1:\kappa_2]}$. It is easy to see that

$$Y^*_{[\kappa_1:\kappa_2]} \cong \Bbbk^\times \times Y_{\gamma_1,\gamma_2}$$

for $\gamma_1, \gamma_2 \in \Bbbk$ such that $\kappa_1\gamma_1 - \kappa_2\gamma_2 = 0$. This implies that the varieties Y_{γ_1,γ_2} are irreducible for all $\gamma_1, \gamma_2 \in \Bbbk$. □

Next, we study the symplectic cores of Y in terms of the varieties Y_{γ_1,γ_2}. Recall the notation from Sect. 4 and consider the following notation.

Notation 8.7 ($\mathfrak{m}_y$) For $y \in Y$, denote by $\mathfrak{m}_y$ the corresponding maximal ideal of $Z(S)$.

Proposition 8.8 *Consider a Poisson structure on the algebra Z coming from a Poisson Z-order on S of level N via a good specialization, as in Theorem 6.10.*

(1) *For all $\gamma_1, \gamma_2 \in \Bbbk$, Y_{γ_1,γ_2} is a Poisson subvariety of $Y = \mathrm{maxSpec}(Z)$.*
(2) *The symplectic cores of Y are*
   (a) *The points in $(Y_{\gamma_1,\gamma_2})^{sing}$ for $\gamma_1, \gamma_2 \in \Bbbk$ (0-dimensional symplectic cores);*
   (b) *The sets $Y_{\gamma_1,\gamma_2}\backslash(Y_{\gamma_1,\gamma_2})^{sing}$ for $\gamma_1, \gamma_2 \in \Bbbk$ (2-dimensional symplectic cores).*

Proof (1) The ideal $(g_1-\gamma_1, g_2-\gamma_2)$ is a Poisson ideal of Z for all $\gamma_1, \gamma_2 \in \Bbbk$ because g_1 and g_2 are in the Poisson center of $Z(S)$ by Corollary 7.2. This implies that Y_{γ_1,γ_2} is a Poisson subvariety of Y.

(2) It follows from Corollary 7.8 that for $y \in Y$ the ideal $\mathfrak{m}_y$ is Poisson, if and only if $y \in (Y_{\gamma_1,\gamma_2})^{sing}$ for some $\gamma_1, \gamma_2 \in \Bbbk$.
Let $y \in Y_{\gamma_1,\gamma_2}\backslash(Y_{\gamma_1,\gamma_2})^{sing}$ for some $\gamma_1, \gamma_2 \in \Bbbk$. It remains to show that the symplectic core containing y is all of $Y_{\gamma_1,\gamma_2}\backslash(Y_{\gamma_1,\gamma_2})^{sing}$, i.e., that

$$\mathbb{V}(\mathcal{P}(\mathfrak{m}_y)) = Y_{\gamma_1,\gamma_2} \quad \text{for such } y. \tag{8.9}$$

Now [25, Lemma 6.2] implies that the Poisson core $\mathcal{P}(\mathfrak{m}_y)$ is a Poisson prime ideal of $Z(S)$. We have

$$\{y\} \subsetneq \mathbb{V}(\mathcal{P}(\mathfrak{m}_y)) \subseteq Y_{\gamma_1,\gamma_2}$$

The second inclusion follows from part (1) and the first inclusion follows from the fact that $\mathfrak{m}_y$ is not a Poisson ideal for $y \notin (Y_{\gamma_1,\gamma_2})^{sing}$. On the one hand, $\mathbb{V}(\mathcal{P}(\mathfrak{m}_y))$ is an irreducible Poisson subvariety of Y_{γ_1,γ_2}, and on the other hand $Y_{\gamma_1,\gamma_2}\backslash(Y_{\gamma_1,\gamma_2})^{sing}$ is a smooth irreducible symplectic variety. This implies that

$$\mathbb{V}(\mathcal{P}(\mathfrak{m}_y))\backslash(Y_{\gamma_1,\gamma_2})^{sing} = Y_{\gamma_1,\gamma_2}\backslash(Y_{\gamma_1,\gamma_2})^{sing}$$

because a smooth irreducible symplectic variety has no nonempty Poisson subvarieties. Since $(Y_{\gamma_1,\gamma_2})^{sing}$ consists of finitely many points (see Corollary 7.8) and $\mathbb{V}(\mathcal{P}(\mathfrak{m}_y))$ is an irreducible variety properly containing $\{y\}$, we obtain that $\mathbb{V}(\mathcal{P}(\mathfrak{m}_y)) \not\subset (Y_{\gamma_1,\gamma_2})^{sing}$. Therefore, $\mathbb{V}(\mathcal{P}(\mathfrak{m}_y)) = Y_{\gamma_1,\gamma_2}$ for the element $y \in Y_{\gamma_1,\gamma_2}\backslash(Y_{\gamma_1,\gamma_2})^{sing}$ fixed above. So, (8.9) holds as desired.
The dimensions of the symplectic cores follow from Corollary 7.8 and Lemma 3.11. □

Before giving a complete description of the Azumaya locus of S, we prove that it is sufficiently big in the sense that its complement in Y is of codimension ≥ 2; see Corollary 8.15 below.

Proposition 8.10 *The Azumaya locus of S contains the union of*

$$Y_{\gamma_1,\gamma_2}\backslash(Y_{\gamma_1,\gamma_2})^{sing}$$

for $\gamma_1, \gamma_2 \in \Bbbk$.

We first prove the proposition for $\Bbbk = \mathbb{C}$ using Poisson geometry, and then extend it to arbitrary algebraically closed base fields $\Bbbk$ of characteristic 0.

Proof (Proof of Proposition 8.10 for $\Bbbk = \mathbb{C}$) Consider first a pair $(\gamma_1, \gamma_2) \in \mathbb{C}^2\backslash(0,0)$. Fix

$$[\kappa_1 : \kappa_2] \in \mathbb{P}^1 \quad \text{such that} \quad \kappa_1\gamma_1 - \kappa_2\gamma_2 = 0.$$

For $y \in Y_{[\kappa_1:\kappa_2]}$, denote by $\overline{\mathfrak{m}}_y$ the corresponding maximal ideal of $Z(S_{[\kappa_1:\kappa_2]})$. It follows from Proposition 8.3 that

$$S/\mathfrak{m}_y S \cong S_{[\kappa_1:\kappa_2]}/\overline{\mathfrak{m}}_y S_{[\kappa_1:\kappa_2]} \quad \text{for} \quad y \in Y_{[\kappa_1:\kappa_2]}. \tag{8.11}$$

Proposition 8.8 implies that the set

$$\begin{aligned} Y^{**}_{[\kappa_1:\kappa_2]} &:= \bigsqcup\{Y_{\gamma_1',\gamma_2'}\backslash(Y_{\gamma_1',\gamma_2'})^{sing} \mid (\gamma_1', \gamma_2') \in \mathbb{C}^2\backslash(0,0)\} \\ &= \mathbb{C}^\times \cdot (Y_{\gamma_1,\gamma_2}\backslash(Y_{\gamma_1,\gamma_2})^{sing}) \end{aligned}$$

is a single $\mathbb{C}^\times$-orbit of symplectic cores of $Y_{[\kappa_1:\kappa_2]} \cong \operatorname{maxSpec}(Z(S_{[\kappa_1:\kappa_2]}))$. By the Brown–Gordon theorem, Theorem 4.11, we obtain that

$$S_{[\kappa_1:\kappa_2]}/\overline{\mathfrak{m}}_y S_{[\kappa_1:\kappa_2]} \cong S_{[\kappa_1:\kappa_2]}/\overline{\mathfrak{m}}_{y'} S_{[\kappa_1:\kappa_2]} \tag{8.12}$$

for all $y, y' \in Y^{**}_{[\kappa_1:\kappa_2]}$. The Azumaya locus of $S_{[\kappa_1:\kappa_2]}$ and $Y^{**}_{[\kappa_1:\kappa_2]}$ are both dense subsets of the irreducible variety $Y_{[\kappa_1:\kappa_2]}$. Hence, $Y^{**}_{[\kappa_1:\kappa_2]}$ intersects nontrivially the Azumaya locus of $S_{[\kappa_1:\kappa_2]}$. Since the PI degree of $S_{[\kappa_1:\kappa_2]}$ equals n by Proposition 8.3, it follows from (8.12) that $S_{[\kappa_1:\kappa_2]}/\overline{\mathfrak{m}}_y S_{[\kappa_1:\kappa_2]} \cong M_n(\mathbb{C})$ for all $y \in Y_{\gamma_1,\gamma_2}\backslash(Y_{\gamma_1,\gamma_2})^{sing}$. Finally, (8.11) implies that the Azumaya locus of S contains $Y_{\gamma_1,\gamma_2}\backslash(Y_{\gamma_1,\gamma_2})^{sing}$ for $(\gamma_1, \gamma_2) \in \mathbb{C}^2\backslash(0,0)$.

Next, let $(\gamma_1, \gamma_2) = (0,0)$. It follows from Proposition 2.18 that $Y_{0,0} \cong \operatorname{maxSpec}(Z(B))$. For $y \in Y_{0,0}$, denote by $\overline{\overline{\mathfrak{m}}}_y$ the corresponding maximal ideal of $Z(B)$. Applying Proposition 2.18, gives that

$$S/\mathfrak{m}_y S \cong B/\overline{\overline{\mathfrak{m}}}_y B \quad \text{for} \quad y \in Y_{0,0}. \tag{8.13}$$

The set $Y_{0,0}\backslash(Y_{0,0})^{sing}$ is a single symplectic core of $Y_{0,0}$ and thus

$$B/\overline{\overline{\mathfrak{m}}}_y B \cong B/\overline{\overline{\mathfrak{m}}}_{y'} B \quad \text{for} \quad y, y' \in Y_{0,0}\backslash(Y_{0,0})^{sing}.$$

On the other hand, $Y_{0,0}\backslash(Y_{0,0})^{sing}$ is a dense subset of the irreducible variety $Y_{0,0}$; hence, $Y_{0,0}\backslash(Y_{0,0})^{sing}$ intersects the Azumaya locus of B nontrivially. Since the PI degree of B equals n, by Proposition 2.12, $B/\overline{\mathfrak{m}}_y B \cong M_n(\mathbb{C})$ for $y \in Y_{0,0}\backslash(Y_{0,0})^{sing}$. By (8.13),

$$S/\mathfrak{m}_y S \cong M_n(\mathbb{C}) \quad \text{for} \quad y \in Y_{0,0}\backslash(Y_{0,0})^{sing}$$

Therefore, the Azumaya locus of S contains $Y_{0,0}\backslash(Y_{0,0})^{sing}$. □

Proof (Proof of Proposition 8.10 for $\Bbbk$) Fix $y \in Y_{\gamma_1,\gamma_2}\backslash(Y_{\gamma_1,\gamma_2})^{sing}$ for some $\gamma_1, \gamma_2 \in \Bbbk$. Denote by $\Bbbk'$ the algebraically closed subfield of $\Bbbk$ generated by the coordinates of y and $\alpha, \beta, \gamma \in \Bbbk$. In particular, $\gamma_1, \gamma_2 \in \Bbbk'$. Choose a field embedding $\Bbbk' \hookrightarrow \mathbb{C}$.

Let $S_\Bbbk = S$, $S_{\Bbbk'}$, and $S_\mathbb{C}$ be the corresponding Sklyanin algebras over $\Bbbk$, $\Bbbk'$, and $\mathbb{C}$, respectively. Let $Y_\Bbbk = Y$, $Y_{\Bbbk'}$, and $Y_\mathbb{C}$ be the max-spectra of their centers, respectively. Denote by $\mathfrak{m}_{y,\Bbbk} = \mathfrak{m}_y$, $\mathfrak{m}_{\Bbbk',y}$ and $\mathfrak{m}_{\mathbb{C},y}$, the maximal ideals of $Z(S_\Bbbk)$, $Z(S_{\Bbbk'})$, and $Z(S_\mathbb{C})$, corresponding to y considered as a point on $Y_\Bbbk = Y$, $Y_{\Bbbk'}$, and $Y_\mathbb{C}$, respectively. We have

$$(S_{\Bbbk'}/\mathfrak{m}_{\Bbbk',y}S_{\Bbbk'}) \otimes_{\Bbbk'} \Bbbk \cong S_\Bbbk/\mathfrak{m}_{\Bbbk,y}S_\Bbbk \quad \text{and} \quad (S_{\Bbbk'}/\mathfrak{m}_{\Bbbk',y}S_{\Bbbk'}) \otimes_{\Bbbk'} \mathbb{C} \cong S_\mathbb{C}/\mathfrak{m}_{\mathbb{C},y}S_\mathbb{C}. \tag{8.14}$$

Since $y \in Y_{\gamma_1,\gamma_2}\backslash(Y_{\gamma_1,\gamma_2})^{sing}$, we get that $y \notin (Y_{\Bbbk',\gamma_1,\gamma_2})^{sing}$. So, $y \notin (Y_{\mathbb{C},\gamma_1,\gamma_2})^{sing}$. Applying the proposition for $\Bbbk = \mathbb{C}$ established above, we obtain that

$$S_\mathbb{C}/\mathfrak{m}_{\mathbb{C},y}S_\mathbb{C} \cong M_n(\mathbb{C})$$

since all three algebras $S_\Bbbk = S$, $S_{\Bbbk'}$, and $S_\mathbb{C}$ have PI degree n. Now it follows from (8.14) that $S_{\Bbbk'}/\mathfrak{m}_{\Bbbk',y}S_{\Bbbk'} \cong M_n(\Bbbk')$ because $\Bbbk'$ is algebraically closed. Applying (8.14) again gives $S/\mathfrak{m}_y S = S_\Bbbk/\mathfrak{m}_{\Bbbk,y}S_\Bbbk \cong M_n(\Bbbk)$, which completes the proof of the proposition for the field $\Bbbk$. □

Now it follows from Proposition 8.10 and Corollary 7.8 that:

Corollary 8.15 *The complement of the Azumaya locus $\mathcal{A}$ of S in Y is of codimension ≥ 2, that is, $\mathcal{A}_\mathfrak{p}$ is Azumaya over $Z_\mathfrak{p}$ for all height 1 primes $\mathfrak{p}$ of Z.* □

Finally, the proof of the main result of this section is brief, due to the work above.

Proof of Theorem 8.1 This holds by applying a result of Brown-Goodearl [11, Theorem 3.8]; namely, the hypotheses of that result follow from Proposition 2.11 and Corollary 8.15. Alternatively, we can apply the improvement of the theorem obtained by Brown-MacLeod [14, Theorem 3.13]. □

Now the consequence of Theorem 8.1 below follows from Theorem 3.13(2) and Theorem 3.16(2); namely, the slice $Y_{0,0} = Y \cap \mathbb{V}(g_1, g_2)$ has singular locus $\{\underline{0}\}$.

Corollary 8.16 *All nontrivial irreducible representations of the twisted homogeneous coordinate ring B of PI degree n have dimension n.* □

8.2 Irreducible representations of PI S of intermediate dimension

Throughout this subsection, we work over complex numbers $\Bbbk = \mathbb{C}$ (as mentioned in Remark 8.19). The goal of this part is to use the geometry of fat points of S to classify irreducible representations of S of intermediate dimension. Recall that *fat points* of S are simple objects in the quotient category S-qgr. A fat point over S can be represented by a 1-critical graded S-module (called *fat point module*) of Hilbert series $d(1-t)^{-1}$ with multiplicity $d \geq 1$. In particular, when a fat point has multiplicity 1, we usually call it a *point* which is represented by a *point module* of S. We employ work of Smith [36] on the geometry of fat points of S to achieve our goal.

Theorem 8.17 ($M(p)$, $F(\omega + k\tau)$) [36] *All fat points of S have been classified and they come in one of* 3*-parametric families:*

(1) *for each $p \in E$, there is a point module $M(p)$;*
(2) *for any $\omega \in E_2$ and $0 \leq k \leq s-1$, there is a fat point module denoted by $F(\omega + k\tau)$ of multiplicity $k+1$;*
(3) *all other fat point modules over S have multiplicity $s = n/(n,2)$.*

In particular, the fat point $F(\omega + k\tau)$ lies on all the secant lines ℓ_{pq} of E such that $p + q = \omega + k\tau$; namely, there exists a short exact sequence of graded S-modules

$$0 \longrightarrow M(p-(k+1)\tau, q-(k+1)\tau)[-k-1] \longrightarrow M(p,q) \longrightarrow F(\omega+k\tau) \longrightarrow 0. \tag{8.18}$$

□

Note that when $k = 0$, the four point modules $F(\omega)$ correspond to the four points $\{e_i\}_{0 \leq i \leq 3}$ in the point scheme of S.

Next, we recall some facts about the relationship between fat points and irreducible representations of S.

Remark 8.19 ($V(\omega + k\tau)$) Constructed originally by Sklyanin in [35] for each $\omega \in E_2$ and $k \in \mathbb{N} \cup \{0\}$, there exists a representation $V(\omega + k\tau)$ over S in a certain $(k+1)$-dimensional subspace of theta functions of order $2k$. (We require $\Bbbk = \mathbb{C}$ here.) These representations were proved later by Smith and Staniszkis to be irreducible whenever $k < s$ [38, Theorem 3.6]. According to [36, Proposition 3.3 and Section 6], all fat point modules $F(\omega + k\tau)$ of S of multiplicity $< s$ arise as

$$F(\omega + k\tau) = \bigoplus_{i \geq 0} V(\omega + k\tau)t^i, \tag{8.20}$$

where the graded S-action on the right side is given by $s_i \cdot v_j t^j = (s_i v_j)t^{i+j}$ for all $s_i \in S_i$ and $v_j \in V(\omega + k\tau)$. Here, the multiplicity of $F(\omega + k\tau)$ is equal to the dimension of $V(\omega + k\tau)$ which is equal to $k+1$.

Question 8.21 We can extend (8.20) for $k = s-1$, where $F(\omega + (s-1)\tau)$ is a fat point module of multiplicity s. This yields 4 irreducible representations $V(\omega + (s-1)\tau)$ of dimension s for $\omega \in E_2$. What are the forms of the other irreducible representations of S of dimension s?

Moreover, we have the following fact. Recall that S is assumed to be module-finite over its center, so all of its irreducible representations are finite-dimensional.

Lemma 8.22 [38, Lemma 4.1] [36, Sectiosn 3.3 and 5] *Every irreducible representation V of S is the quotient of some fat point module F of S. Here, $\dim_{\Bbbk}(V) \geq mult(F)$. Moreover, if F is isomorphic to a shift $F[e]$ for some $e \geq 1$, then $\dim_{\Bbbk}(V) \leq e \cdot mult(F)$.* □

Notation 8.23 [V^{λ}] Let $\mathrm{Mod}_d(S)$ be the variety of all d-dimensional representations of S. The algebraic group $\mathrm{PGL}_d(\Bbbk) \times \Bbbk^{\times}$ acts on $\mathrm{Mod}_d(S)$ via

$$((T, \lambda).\varphi_V)(a) := \lambda^i T \varphi_V(a) T^{-1}$$

for any $V \in \mathrm{Mod}_d(S)$ with corresponding map $\varphi_V : S \to \mathrm{End}(V)$ and $a \in S_i$. For any $V \in \mathrm{Mod}_d(S)$ and $\lambda \in \Bbbk^{\times}$, we define the *twisted representation*

$$V^{\lambda} := (1, \lambda).V.$$

It is important to point out that $V \cong V^{\lambda}$, if and only if there is some $T \in \mathrm{PGL}_d(\Bbbk)$ such that $(T, \lambda).V = V$.

Proposition 8.24 *Let F be a fat point module over S of multiplicity d, and V be a nontrivial simple quotient of F. Then we have*

(1) $\dim V = de$, *where e is the period of F such that $F \cong F[e]$ and $F \ncong F[i]$ for $1 \leq i \leq e-1$ in S-qgr;*
(2) *the stabilizer of V in $\mathrm{PGL}_{de}(\Bbbk) \times \Bbbk^{\times}$ is conjugate to the subgroup generated by (g_ζ, ζ) with $g_\zeta = \mathrm{diag}(\underbrace{1, \ldots, 1}_{d}, \underbrace{\zeta, \ldots, \zeta}_{d}, \ldots, \underbrace{\zeta^{e-1}, \ldots, \zeta^{e-1}}_{d})$ and ζ is a primitive e-th root of unity;*
(3) *all nontrivial simple quotients of F are in the form of V^{λ}, for some irreducible representation V of S and for some $\lambda \in \Bbbk^{\times}$.* □

Proof Part (1) follows from [27, Proposition 6 and its proof]. Part (2) comes from [9, Lemma 4]. Part (3) holds by the proof of [36, Theorem 3.13]. □

We now turn our attention to the relationship between fat points and Azumaya locus of S, and for this we need some results from noncommutative projective algebraic geometry; see [1] [39, Section 5] for further details. Let

$$\mathcal{R} := \mathrm{Proj}(Z(S)).$$

We consider the sheaf $\mathcal{S}$ of $\mathcal{O}_{\mathcal{R}}$-algebras defined by $\mathcal{S}(\mathcal{R}_{(z)}) = S[z^{-1}]_0$ for any non-zero homogeneous element $z \in Z(S)$. Denote by $\mathcal{Z}$ the *central Proj* of $\mathcal{S}$, which satisfies $\mathcal{Z}(\mathcal{R}_{(z)}) = Z(S[z^{-1}]_0)$ for any non-zero homogeneous element $z \in Z(S)$. Consider the commutative scheme $\mathbf{Spec}(\mathcal{Z})$ defined in [26, Chapter II, Exercise 5.17], and we get by [39, Theorem 5.2] that it is isomorphic to $\mathrm{Proj}(Z(S^{(n/s)}))$. Now fat

point modules of S have support in $\mathbf{Spec}(\mathcal{Z})$; indeed, a fat point module F of S gives rise to a sheaf $\mathcal{F}$ of $\mathcal{S}$-modules such that $\mathcal{F}(\mathcal{R}_{(z)}) = F[z^{-1}]_0$.

We get that there is a quotient morphism

$$\mathbf{Spec}(\mathcal{Z}) \twoheadrightarrow \mathcal{R}, \tag{8.25}$$

induced by the embedding $Z(S) \hookrightarrow Z(S^{(n/s)})$ via $z_i = u_i^{(n/s)}$ and $g_j = g_j$. We obtain that the support of a fat point module F in $\mathcal{R}$ is the image of the support of the corresponding sheaf of $\mathcal{S}$-modules $\mathcal{F}$ in $\mathbf{Spec}(\mathcal{Z})$ via (8.25).

Now, with (3.3) and a result in [39], we obtain the following results on the fat point modules of S of multiplicity $< s$.

Proposition 8.26 [39, Theorem 5.11] *When $0 \leq k \leq s-1$, the support of the fat point modules $F(\omega+k\tau)$ and $F(\omega'+k'\tau)$ coincide in $\mathbf{Spec}(\mathcal{Z})$ if $\omega+k\tau = -\omega'-k'\tau-2\tau$.* □

Notation 8.27 ($\mathcal{C}(\omega + k\tau)$) Recall that for any $\omega \in E_2$ and $0 \leq k \leq s-1$, the simple module $V(\omega + k\tau)$ constructed by Sklyanin gives a non-zero point $(p_0, p_1, p_2, p_3, \gamma_1, \gamma_2)$ in Y via its support, or rather central annihilator, in $Z(S)$. Consider the parametric curve

$$\mathcal{C}(\omega + k\tau) := \left\{ (t^n p_0, t^n p_1, t^n p_2, t^p_3, t^2\gamma_1, t^2\gamma_2) \mid t \in \Bbbk \right\}.$$

Since scaling of the parameter t in $\mathcal{C}(\omega + k\tau)$ respects the degree of the variables, the curve $\mathcal{C}(\omega + k\tau)$ represents a single point in $\mathcal{R} = \mathrm{Proj}(Z(S))$.

Lemma 8.28 *The parametric curve $\mathcal{C}(\omega+k\tau) \subset Y$ is the support of $F(\omega+k\tau)$ in $\mathcal{R}$. As a consequence, $\mathcal{C}(\omega+k\tau) = \mathcal{C}(\omega'+k'\tau)$ if $\omega+k\tau = -\omega'-k'\tau-2\tau$. Moreover, there is a (n/s)-to-1 correspondence between the isomorphism classes of nontrivial simple quotients of $F(\omega+k\tau)$ and the points of $\mathcal{C}(\omega+k\tau)$.*

Proof By Proposition 2.18, $Z(S) = \Bbbk[z_0, z_1, z_2, z_3, g_1, g_2]/(F_1, F_2)$ is a graded algebra with $\deg(z_i) = n$ and $\deg(g_j) = 2$. Hence any scaling of a point $(p_0, p_1, p_2, p_3, \gamma_1, \gamma_2) \in Y$ with respect to the grading still belongs to Y. So $\mathcal{C}(\omega+k\tau) \subset Y$. Now let $\mathcal{I}$ be the smallest graded ideal in $Z(S)$ containing $\mathcal{C}(\omega+k\tau)$. By (8.20), one can check that $\mathcal{I} = \mathrm{Ann}_{Z(S)}(F(\omega+k\tau))$. So $\mathcal{C}(\omega+k\tau)$ represents the support of $F(\omega+k\tau)$ in $\mathcal{R}$ via (8.25).

From the definition of $F(\omega+k\tau)$ in (8.20), we know $V(\omega+k\tau)$ is a simple quotient of $F(\omega+k\tau)$. By Proposition 8.24(3), all the nontrivial simple quotients of $F(\omega+k\tau)$ are given by twisted modules $V(\omega+k\tau)^\lambda$ for some $\lambda \in \Bbbk^\times$. Moreover, it is clear that $F(\omega+k\tau) \cong F(\omega+k\tau)[1]$ in S-qgr. So $F(\omega+k\tau)$ has period 1. (See, also [36, proof of Corollary 8.7 and Proposition 8.1(c)]). As a consequence, $\mathrm{PGL}_{k+1}(\Bbbk) \times \Bbbk^\times$ acts freely on the simple quotients of $F(\omega+k\tau)$ by Proposition 8.24(2). So $V(\omega+k\tau) \ncong V(\omega+k\tau)^\lambda$ for any $\lambda \in \Bbbk^\times$. Let $(p_0, p_1, p_2, p_3, \gamma_1, \gamma_2) \in Y$ be the point corresponding to $V(\omega+k\tau)$. One can check that the point corresponding to $V(\omega+k\tau)^\lambda$ is given by $(\lambda^n p_0, \lambda^n p_1, \lambda^n p_2, \lambda^n p_3, \lambda^2\gamma_1, \lambda^2\gamma_2)$. So $V(\omega+k\tau)^\lambda$ and

$V(\omega + k\tau)^{\lambda'}$ share the same central annihilator, if and only if $\lambda = \lambda'$ if n is odd and $\lambda = \pm\lambda'$ if n is even. This establishes the (n/s)-to-1 correspondence.

Finally, $\mathcal{C}(\omega + k\tau) = \mathcal{C}(\omega' + k'\tau)$ if $\omega + k\tau = -\omega' - k'\tau - 2\tau$ by Proposition 8.26 since they represent the same point in $\mathcal{R}$. □

This brings us to the main results of this section, first for n odd.

Theorem 8.29 *When the PI degree n of S is odd, we have the following statements.*

(1) *The curve $\mathcal{C}(\omega+k\tau)$ is equal to the curve $C(\omega+k\tau)$ from Lemma 3.12, for $\omega \in E_2$ and $0 \le k \le n-2$. Thus, $Y^{sing} = \bigcup_{\omega\in E_2,\ 0\le k\le n-2} \mathcal{C}(\omega + k\tau)$.*

(2) *If $\underline{0} \ne y \in Y^{sing}$ and $0 \le k \le n-2$, then there are exactly 2 irreducible representations of S with central annihilator $\mathfrak{m}_y$. They are simple quotients of $F(\omega+k\tau)$ of dimension $k+1$ and of $F(\omega+(n-2-k)\tau)$ of dimension $n-1-k$, when $y \in C(\omega + k\tau)\backslash\{\underline{0}\}$. Moreover, the origin $\underline{0} \in Y$ corresponds uniquely to the trivial module of S.*

(3) *If $y \in Y^{smooth}$, then there is exactly 1 irreducible representation of S of dimension n with central annihilator $\mathfrak{m}_y$, which is a simple quotient of either the point module $M(p)$ for some $p \in E$ or some fat point module of multiplicity n.*

(4) *S has irreducible representations of each dimension $1, 2, 3, \ldots, n$, where the non-trivial irreducible representations of intermediate dimension $k+1$, for $0 \le k \le n-2$, are given by $V(\omega + k\tau)^{\lambda}$ for $\omega \in E_2$ and $\lambda \in \Bbbk^\times$.*

Proof (1) It suffices to show for all $\omega \in E_2$ and $0 \le k \le n-2$ that the curves $\mathcal{C}(\omega+k\tau)$ and $C(\omega+k\tau)$ share a nonzero common point. Indeed, since the curves are parametrized in the same way, sharing a nonzero point implies that the two sets are equal to each other. Now let $\underline{0} \ne y \in \mathcal{C}(\omega+k\tau)$ correspond to a nontrivial simple quotient $V(\omega + k\tau)$ of a fat point module $F(\omega + k\tau)$ of intermediate multiplicity. Since $\dim V(\omega+k\tau) = k+1$, which is $< n$ by Remark 8.19, the point y lies in the non-Azumaya locus and hence the singular locus of S by Theorem 8.1. Then (8.18) and Lemma 3.12 imply that $y \in Y^{sing} \cap \mathbb{V}(\Omega(\omega+k\tau)) = C(\omega+k\tau)$.

(2) Let $\underline{0} \ne y \in \mathcal{C}(\omega + k\tau) = \mathcal{C}(\omega + (n-2-k)\tau) \subset Y^{sing}$ by Lemma 8.28 for some $\omega \in E_2$ and $0 \le k \le n-2$. Then there are 2 nontrivial irreducible representations V_1 and V_2 of S with central annihilator $\mathfrak{m}_y$ such that V_1 is a simple quotient of $F(\omega + k\tau)$ and V_2 is a simple quotient of $F(\omega + (n-2-k)\tau)$. It is clear that $V_1 \ncong V_2$ since n is odd, $\dim V_1 = \dim V(\omega + k\tau) = k+1$ and $\dim V_2 = \dim V(\omega+(n-2-k)) = n-1-k$ by Remark 8.19. Now if $\{V_i\}_{1\le i\le t}$ is a complete set of all non-isomorphic irreducible representations over S central annihilator $\mathfrak{m}_y$, then applying a result of Braun [10, Proposition 4] yields

$$\begin{aligned} \mathrm{PIdeg}(S) = n \ &\ge \sum_{1\le i\le t} \dim V_i \ \ge \ \dim V_1 + \dim V_2 \\ &= (k+1) + (n-1-k) \ = \ n. \end{aligned}$$

This implies that $t = 2$, as desired.

The statement for the correspondence between the trivial module and the origin of Y is clear.

(3) This can be proved similarly to (2), again using Theorem 8.1.
(4) This follows from (2), (3) and by Remark 8.19.

□

For n even case, the picture for irreducible representations of S of intermediate dimension versus singular locus of Y is somewhat elusive due to the fact that we do not know how the varieties Y_1^{sing} and Y_2^{sing} in Theorem 3.16 compare to the curves $\mathcal{C}(\omega + k\tau)$. But we have the following result.

Theorem 8.30 *Suppose that the PI degree $n = 2s$ of S is even. Then the following statements hold.*

(1) *If $\underline{0} \neq y \in \mathcal{C}(\omega + k\tau)$ for some $\omega \in E_2$ and $0 \leq k \leq s-2$, then there are exactly 4 irreducible representations of S with central annihilator $\mathfrak{m}_y$, two of which are simple quotients of $F(\omega + k\tau)$ of dimension $k+1$ and the other two are simple quotients of $F(\omega + s\tau + (s-k-2)\tau)$ of dimension $s-1-k$.*
(2) *If $\underline{0} \neq y \in \mathcal{C}(\omega + (s-1)\tau)$ for some $\omega \in E_2$, then there are exactly 2 irreducible representations of S of dimension s with central annihilator $\mathfrak{m}_y$, which are simple quotients of $F(\omega + (s-1)\tau)$.*
(3) *If $\underline{0} \neq y \in Y^{sing} \setminus \bigcup_{\omega \in E_2, 0 \leq k \leq s-1} \mathcal{C}(\omega + k\tau)$, then there are exactly 2 irreducible representations of S of dimension s with central annihilator $\mathfrak{m}_y$, which are simple quotients of a fat point module of multiplicity s.*
(4) *If $y \in Y^{smooth}$, then there is exactly 1 irreducible representation of S of dimension n with central annihilator $\mathfrak{m}_y$, which is a simple quotient of a generic fat point module of multiplicity s.*
(5) *S has irreducible representations of each dimension $1, 2, 3, \ldots, s$ and n, where the nontrivial irreducible representations of intermediate dimension $k+1$, for $0 \leq k \leq s-2$, are given by $V(\omega + k\tau)^{\lambda}$ for $\omega \in E_2$ and $\lambda \in \Bbbk^{\times}$.*

Proof (1) Let $\underline{0} \neq y \in \mathcal{C}(\omega + k\tau) = \mathcal{C}(\omega + s\tau + (s-k-2)\tau)$. Then there are two non-isomorphic irreducible representations V and W, with central annihilator $\mathfrak{m}_y$, which are simple quotients of $F(\omega + k\tau)$ and $F(\omega + s\tau + (s-k-2)\tau)$, respectively. Moreover, by Lemma 8.28 there exists another simple quotient V^{-1} (respectively, W^{-1}) of $F(\omega + k\tau)$ (respectively, of $F(\omega + s\tau + (s-k-2)\tau)$) which also corresponds to y and is not isomorphic to V (respectively, to W). So we have in total four non-isomorphic simple modules $V^{\pm 1}$, $W^{\pm 1}$ corresponding to the same point y. Now applying Braun's result [10, Proposition 4] to get that these are the only four irreducible representations with central annihilator $\mathfrak{m}_y$.

(2), (3) and (4) can be proved similarly to part (1).

(5) Let V be a nontrivial irreducible representation over S, which is a simple quotient of some fat point F by Lemma 8.22. If $F = M(p)$ for some $p \in E$, then V is a module over the twisted homogeneous coordinate ring B and $\dim V = n$ by [29, Lemma 5.8(c)] and Proposition 8.24(3). Suppose $F = F(\omega + k\tau)$. Then $\dim V = \dim V(\omega + k\tau) = k+1$ by Remark 8.19. Finally, if V is a simple quotient of a fat point of multiplicity s. Then by Proposition 8.24(1), we have $\dim V = es$, where e is the period of F. By [11, Proposition 3.1(a)], we know $\dim V \leq \mathrm{PIdeg}(S) = n$. So we have $e = 1, 2$. Now our result follows from Proposition 8.24(3). □

8.3 Discriminant ideals of the PI 4-dimensional Sklyanin algebras

Discriminant ideals of PI algebras play an important role in the study of maximal orders [32], the automorphism and isomorphism problems for families of algebras [16], the Zariski cancellation problem [7], and the description of dimensions of irreducible representations [15].

Let A be an algebra and $C \subseteq Z(A)$ be a central subalgebra. A *trace map* on A is a nonzero map $\mathrm{tr}\colon A \to C$ which is cyclic ($\mathrm{tr}(xy) = \mathrm{tr}(yx)$ for $x, y \in A$) and C-linear. For a positive integer ℓ, the ℓ-th *discriminant ideal* $D_\ell(A/C)$ and the ℓ-th *modified discriminant ideal* $MD_\ell(A/C)$ of A over C are the ideals of C with generating sets

$$\{\det([\mathrm{tr}(y_i y_j)]_{i,j=1}^{\ell}) \mid y_1, \ldots, y_\ell \in A\} \text{ and}$$
$$\{\det([\mathrm{tr}(y_i y'_j)]_{i,j=1}^{\ell}) \mid y_1, y'_1, \ldots, y_\ell, y'_\ell \in A\}.$$

Every maximal order A in a central simple algebra admits the *reduced trace map* $\mathrm{tr} : A \to Z(A)$, see [32, Section 9]. Stafford [40] proved that the PI 4-dimensional Sklyanin algebras S are maximal orders in central simple algebras.

The next theorem describes the zero sets of the discriminant ideals of a PI Sklyanin algebra S of PI degree n. Denote the quadric function

$$q(k) := (k^2 + (s-k)^2)n/s,$$

keeping in mind that $s = n$ if n is odd, and $s = n/2$ if n is even.

Theorem 8.31 *Let S be a PI 4-dimensional Sklyanin algebra of PI degree n with reduced trace map $\mathrm{tr} : S \to Z$. For all positive integers ℓ, the zero sets of the ℓ-th discriminant and ℓ-th modified discriminant ideals of S over its center coincide,*

$$\mathbb{V}(D_\ell(S/Z(S)), \mathrm{tr}) = \mathbb{V}(MD_\ell(S/Z(S)), \mathrm{tr});$$

denote this set by $\mathbb{V}_\ell \subset Y := \mathrm{maxSpec}(Z)$. For the top discriminant ideal of S, we have

$$\mathbb{V}_{n^2} = Y^{sing}.$$

For the lower level discriminant ideals of S and base field $\Bbbk = \mathbb{C}$, the following hold:

(1) *If n is odd, then*

$$\mathbb{V}_\ell = \begin{cases} Y^{sing}, & \ell \in \left[q\,(n-1)+1,\ n^2\right] \\ \bigcup_{k\in[0,n-2], q(k+1)<\ell} C(\omega + k\tau), & \ell \in \left[q(\lfloor \frac{n}{2} \rfloor)+1,\ q\,(n-1)\right] \\ \underline{0}, & \ell \in \left[2,\ q(\lfloor \frac{n}{2} \rfloor)\right] \\ \varnothing, & \ell = 1. \end{cases}$$

(2) *If n is even, then*

$$\mathbb{V}_\ell = \begin{cases} Y^{sing}, & \ell \in \left[2s^2+1,\ n^2\right] \\ \bigcup_{k\in[0,s-2],q(k+1)<\ell} \mathcal{C}(\omega+k\tau), & \ell \in \left[q\left(\lfloor \frac{s}{2} \rfloor\right)+1,\ 2s^2\right] \\ \underline{0}, & \ell \in \left[2,\ q\left(\lfloor \frac{s}{2} \rfloor\right)\right] \\ \varnothing, & \ell = 1. \end{cases}$$

□

Proof Let A be a maximal order in a central simple algebra over a field of characteristic 0 with reduced trace map $\mathrm{tr} : A \to Z(A)$. For $\mathfrak{m} \in \mathrm{maxSpec}(Z)$, denote by $\mathrm{Irr}_{\mathfrak{m}}(A)$ the set of isomorphism classes of irreducible representations of A with central annihilator $\mathfrak{m}$, and set

$$d(\mathfrak{m}) := \sum_{V \in \mathrm{Irr}_{\mathfrak{m}}(A)} (\dim_{\Bbbk} V)^2.$$

By [15, Main Theorem (a),(e)] and Theorem 8.1, we have that $\mathbb{V}_{n^2} = Y^{sing}$, and for all $\ell \in \mathbb{Z}$ that

$$\mathbb{V}_\ell = \left\{ \mathfrak{m} \in \mathrm{maxSpec} Z(A) \mid d(\mathfrak{m}) < \ell \right\}. \tag{8.32}$$

Theorems 8.29 and 8.30 imply that for the PI 4-dimensional Sklyanin algebra S, the square-dimension function $d : \mathrm{maxSpec}(Z) \to \mathbb{Z}$ is given by

$$d(\mathfrak{m}_y) = \begin{cases} n^2, & y \in Y^{smooth} \\ q(k+1), & y \in C(\omega+k\tau)\backslash\{\underline{0}\},\ k \in [0, n-2] \\ 1, & y = \underline{0} \end{cases}$$

for n odd, and by

$$d(\mathfrak{m}_y) = \begin{cases} n^2, & y \in Y^{smooth} \\ q(0) = 2s^2, & y \in Y^{sing} \backslash \left(\bigcup_{k\in[0,s-2]} \mathcal{C}(\omega+k\tau)\right) \\ q(k+1), & y \in \mathcal{C}(\omega+k\tau)\backslash\{\underline{0}\},\ k \in [0, s-2] \\ 1, & y = \underline{0} \end{cases}$$

for n even. The statement of the theorem follows by combining the formulas for $d(\mathfrak{m}_y)$ with (8.32). Namely for n odd, the smallest value of $d(\mathfrak{m}_y)$ for $y \in C(\omega+k\tau)\backslash\{\underline{0}\}$, with $0 \le k \le n-2$, is $q(\lfloor \frac{n}{2} \rfloor)$ and the largest value for such $d(\mathfrak{m}_y)$ is $q(n-1)$. Moreover, for n even, the smallest value of $d(\mathfrak{m}_y)$ for $y \in \mathcal{C}(\omega+k\tau)\backslash\{\underline{0}\}$, with $0 \le k \le s-2$, is $q(\lfloor \frac{s}{2} \rfloor)$ and the largest value for such $d(\mathfrak{m}_y)$ is $q(s-1)$; note that $q(0) > q(s-1)$. □

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