DYNAMICS OF LOW-DEGREE RATIONAL INNER SKEW-PRODUCTS ON $\mathbb{T}^2$

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Dedicated to Håkan Hedenmalm on the occasion of his sixtieth birthday

ABSTRACT. We examine iteration of certain skew-products on the bidisk whose components are rational inner functions, with emphasis on simple maps of the form $\Phi(z_1, z_2) = (\phi(z_1, z_2), z_2)$. If ϕ has degree 1 in the first variable, the dynamics on each horizontal fiber can be described in terms of Möbius transformations but the global dynamics on the 2-torus exhibit some complexity, encoded in terms of certain $\mathbb{T}^2$-symmetric polynomials. We describe the dynamical behavior of such mappings Φ and give criteria for different configurations of fixed point curves and rotation belts in terms of zeros of a related one-variable polynomial.

1. INTRODUCTION AND OVERVIEW

Iteration of a rational function $R(z) = \frac{q(z)}{p(z)}$ on the Riemann sphere, that is, the study of

$$z \mapsto R^n(z) = (R \circ R \circ \cdots \circ R)(z) \quad (n = 1, 2, \ldots)$$

on $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$, is a well-known topic in mathematics, discussed in many textbooks (e.g. [5, 12, 23]) and illustrated in beautiful computer images. The theory is quite mature, but important new results are still being discovered. The higher-dimensional theory, addressing iteration of n-variable polynomial or rational mappings R is of later date but is rapidly developing. See, for instance, [14, 16, 29], and the references therein, for basic overviews of dynamics in several complex variables.

In a different direction, considering self-maps of special bounded domains in $\mathbb{C}^n$ (the unit disk in the complex plane, the unit n-ball) allows for the study of iteration of functions that are not necessarily defined throughout $\mathbb{C}^n$, and leads to interesting boundary phenomena not observed in the unbounded setting. An important example in this latter direction is the classical Denjoy-Wolff theorem [12] concerning fixed points of analytic self-maps of the unit disk; there are many subsequent extensions of the original

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result to different settings. See e.g. [6, 19] (and the references therein) for recent developments.

Our investigation concerns the study of the dynamics of certain self-maps of the unit bidisk

$$\mathbb{D}^2 = \{(z_1, z_2) \in \mathbb{C}^2 \colon |z_j| < 1, j = 1, 2\}$$

and seeks to establish some basic facts about their iteration theory. There are many works addressing iteration of analytic self-maps of the bidisk and higher-dimensional polydisks. See, for instance, the papers [1, 4, 13, 17, 18] and the thesis [24] for some background. Our focus is on obtaining detailed results for a restricted class of rational mappings by using elementary means.

We say that a mapping of the form

$$\Phi \colon \mathbb{D}^2 \to \mathbb{D}^2$$

$$(z_1, z_2) \mapsto (\phi_1(z_1, z_2), \phi_2(z_1, z_2))$$

is a *rational inner mapping* (RIM) if each component ϕ_j is a *rational inner function* on $\mathbb{D}^2$. A rational inner function in turn is an analytic function of the form

$$\psi(z_1, z_2) = \frac{q(z_1, z_2)}{p(z_1, z_2)},$$

with $q, p \in \mathbb{C}[z_1, z_2]$ and $p(z) \neq 0$ in $\mathbb{D}^2$, which is bounded in $\mathbb{D}^2$ and has unimodular non-tangential boundary values at almost every point $\zeta \in \mathbb{T}^2 = \{(\zeta_1, \zeta_2) \in \mathbb{C}^2 \colon |\zeta_j| = 1, j = 1, 2\}$. We recall the basic fact that $\mathbb{T}^2$ is the *distinguished boundary* of the bidisk, the subset of the boundary $\partial\mathbb{D}^2$ where most interesting function-theoretic phenomena on $\mathbb{D}^2$ are observed and the maximum modulus principle is supported. By a theorem of Knese [22], if ϕ_j is rational inner then the boundary values $\phi_j^*(\zeta)$ exist as unimodular numbers at *every* point of $\mathbb{T}^2$, and so we can view a rational inner mapping Φ as inducing a map $\Phi \colon \overline{\mathbb{D}^2} \to \overline{\mathbb{D}^2}$, with Φ sending $\mathbb{T}^2$ to $\mathbb{T}^2$. In the same way, the nth iterate of Φ

$$\Phi^n(z_1, z_2) = (\Phi \circ \Phi \circ \cdots \circ \Phi)(z_1, z_2)$$

can be viewed as a mapping of the closure of the bidisk into itself that fixes the distinguished boundary.

Iteration of certain classes of rational inner mappings on the bidisk and on $\mathbb{T}^2$ has been considered in a number of papers. For instance, monomial maps of the form $(z_1, z_2) \mapsto (z_1^{m_1} z_2^{n_1}, z_1^{m_2} z_2^{n_2})$ appear in [15] and in other works. In [26, 25] (see also [30] for some applications), the component maps ϕ_j are assumed to be of the special type

$$\psi_j(z_1, z_2) = B_{j,1}(z_1) \cdot B_{j,2}(z_2)$$

where $B_{j,k}$ are one-variable *finite Blaschke products*. Recall that these are functions $B\colon \mathbb{D} \to \mathbb{D}$ of the form

$$B(z) = e^{i\alpha} z^m \prod_{k=1}^{n} \frac{z - a_k}{1 - \overline{a}_k z},$$

where $\{a_1, \ldots, a_n\} \subset \mathbb{D}$, $\alpha \in \mathbb{R}$, and $m, n \in \mathbb{N}$. It is apparent that a finite Blaschke product extends continuously to the unit circle $\mathbb{T}$, but the dynamics of one-variable Blaschke products nevertheless exhibit complicated features; see for instance [23, Chapter 15]. Similarly, the works [15, 26, 25] uncover a rich dynamical structure associated with monomial maps and two-dimensional Blaschke products.

In contrast with one-variable Blaschke products, a general rational inner function $\psi = q/p$ in $\mathbb{D}^2$ can have boundary singularities: these occur at points $\tau = (\tau_1, \tau_2) \in \mathbb{T}^2$ where $q(\tau_1, \tau_2) = p(\tau_1, \tau_2) = 0$, and represent a genuinely new higher-dimensional phenomenon. The function ψ is in general discontinuous at $\tau \in \mathbb{T}^2$, even though $\psi^*(\tau)$ always exists, meaning that a RIM Φ need not be a continuous self-map of $\overline{\mathbb{D}^2}$. This fact will be the source of several interesting phenomena that we observe in this paper.

Apart from some preliminary observations concerning RIMs, we mostly restrict our attention to *rational inner skew-products* (RISPs). These are rational inner mappings of the special form

$$\Phi(z_1, z_2) = (\phi_1(z_1, z_2), \phi_2(z_2)),$$

with ϕ_1 rational inner in $\mathbb{D}^2$, and ϕ_2 a rational inner function in one variable, viewed as a function on $\mathbb{D}^2$ in the obvious way. Skew-products have been studied extensively in the polynomial setting, see e.g. [20, 27] and the references therein. In fact, we focus on the very simplest case of skew-mappings

$$(z_1, z_2) \mapsto (\phi(z_1, z_2), z_2), \tag{1.1}$$

where ϕ is a rational inner function in $\mathbb{D}^2$ having *bidegree* $(1, n)$, that is, degree 1 in z_1 and degree $n \in \mathbb{N}$ in z_2. (We drop the subscript in the first component to lighten notation.) Skew-products of the form (1.1) fix horizontal lines so that the main dynamics take place along one-dimensional sets, but even this apparently very simple case gives rise to interesting behavior, as is suggested in the images below. The advantage in working with such low-degree RISPs is that the extremely simple classification of fiber dynamics (discussed below) allows us to focus in detail on the global features of iteration on $\mathbb{T}^2$.

We proceed to describe the contents of our paper. We begin by examining three elementary examples of RISPs in Section 2. These examples serve as guides for our general investigations and illustrate most of the features we uncover in a more general setting. In Section 3 we introduce basic definitions for RIMs and RISPs, record some facts about RIFs and their numerator and denominator polynomials, and discuss iteration of Möbius transformations in the unit disk. Section 4 contains the main results of our paper. After some preliminary remarks about general RIMs and certain associated essentially $\mathbb{T}^2$-symmetric polynomials, we focus on degree $(1, n)$ RISPs. We then introduce and study rotation belts associated with a RISP, strips on which the dynamical actions on fibers are conjugate to rotations, and give an estimate on their number. We investigate how different components of the fixed point set of a RISPs can come together at a singular fixed point, and give criteria for different configurations to be present. The main tool we use is a one-variable polynomial Q_α built from the numerator and denominator polynomials of the first-component map ϕ. We conclude in Section 5 by exhibiting further examples of RISPs having more intricate dynamical behavior, and serving as illustrations of our main results.

2. Three examples

A fundamental result due to Rudin and Stout (see [28, Chapter 5]) asserts that any RIF on $\mathbb{D}^2$ can be written as

$$\phi(z) = e^{i\alpha} z_1^{\beta_1} z_2^{\beta_2} \frac{\tilde{p}(z)}{p(z)} \tag{2.1}$$

where $\alpha \in \mathbb{R}$, $\beta_1, \beta_2 \in \mathbb{N}$, the $p \in \mathbb{C}[z_1, z_2]$ has no zeros in $\mathbb{D}^2$, and $\tilde{p}$ is the *reflection* of p, defined as

$$\tilde{p}(z_1, z_2) = z_1^m z_2^n \overline{p\left(\frac{1}{\bar{z}_1}, \frac{1}{\bar{z}_2}\right)}. \tag{2.2}$$

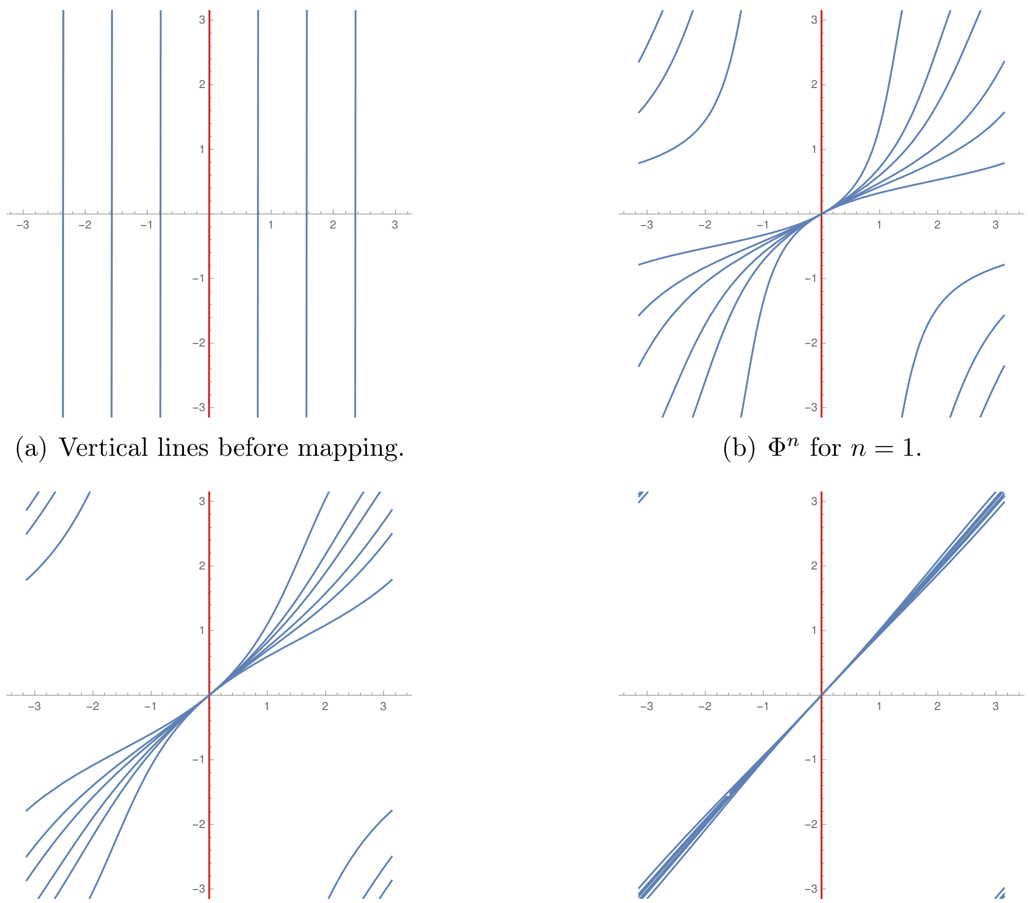
Here the pair (m, n) is the bidegree of p, that is p has degree m in z_1 and degree n in z_2.

This structural result makes examples of rational inner functions particularly easy to construct - all one requires is a polynomial with no zeros in the bidisk. The following such examples can be fruitfully analyzed using elementary means.

Example 2.1. Consider the RISP

$$\Phi(z_1, z_2) = \left(-\frac{2z_1z_2 - z_1 - z_2}{2 - z_1 - z_2}, z_2\right).$$

The first component $\phi = -\frac{\tilde{p}}{p}$ is a degree $(1, 1)$ RIF having a single singularity at $(1, 1)$. A computation reveals that the boundary value $\phi^*(1, 1) = 1$, and so $(1, 1)$ is a fixed point of Φ. In fact, we have $\Phi(z_1, 1) = (1, 1)$, so the entire



(a) Vertical lines before mapping. (b) Φ^n for $n = 1$.

(c) Φ^n for $n = 2$ (d) Φ^n for $n = 5$.

FIGURE 1. *Iteration of* $\Phi = (-\frac{2z_1z_2 - z_1 - z_2}{2 - z_1 - z_2}, z_2)$ *on* $\mathbb{T}^2$.

line $\{z_2 = 1\}$ is mapped to $(1, 1)$. Next, we note that $\phi^*(1, z_2) = 1$ and $\phi^*(z_2, z_2) = z_2$ meaning that both the line $\{z_1 = 1\}$ and the diagonal are comprised of fixed points of Φ. Solving $\phi^*(z) = z_1$, or more precisely, the equation

$$\tilde{p}(z) - z_1 p(z) = 0,$$

confirms that these are all the fixed points of Φ.

We now iterate Φ. We visualize the action of the iterates by viewing the 2-torus as $[-\pi, \pi]^2$ and applying Φ^n to vertical lines of the form $\{(a\pi, t_2)\}$ for several choices of $-1 < a < 1$. Figure 1 suggests that the iterates converge as n grows. Using induction, one can show that, for $n = 1, 2, \ldots$,

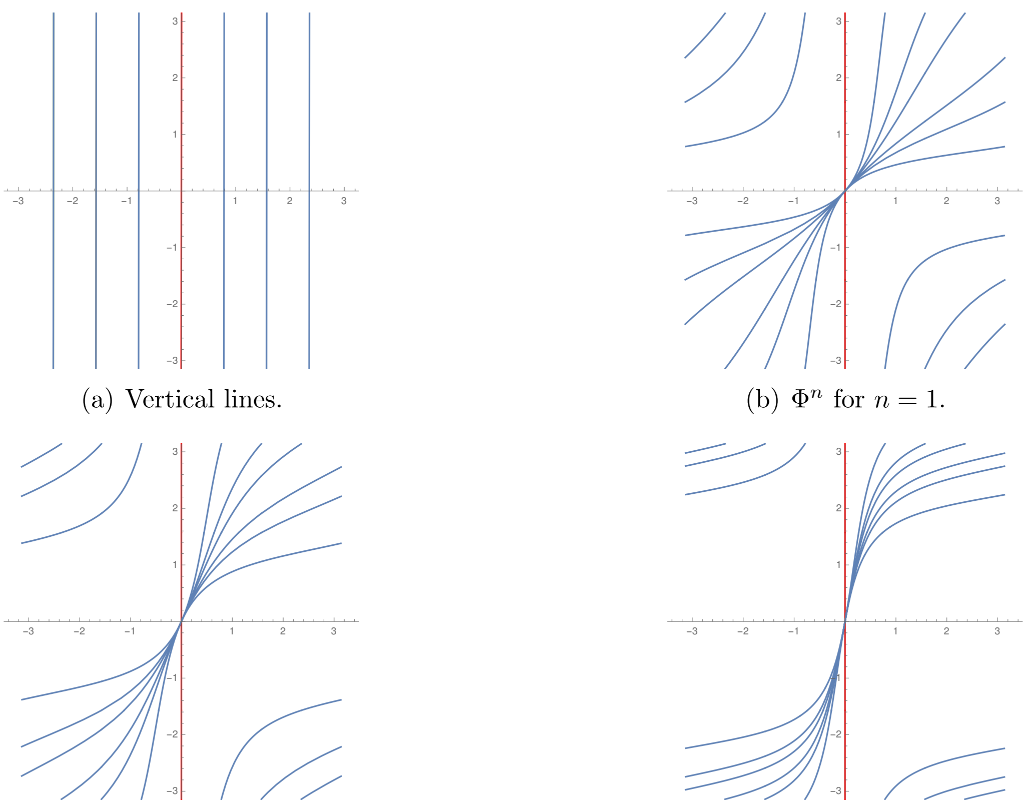
$$\Phi^n(z_1, z_2) = \left(-\frac{2^n z_1 z_2 - z_1 - (2^n - 1)z_2}{2^n - (2^n - 1)z_1 - z_2}, z_2\right).$$

Hence, for all $(z_1, z_2) \in \overline{\mathbb{D}^2} \setminus \{z_1 = 1\}$, we have

$$\Phi^n(z_1, z_2) \to (z_2, z_2) \quad \text{as} \quad n \to \infty.$$

This means that each point on the diagonal is attractive on its corresponding horizontal fiber, while points on $\{z_1 = 1\}$ are repelling fixed points. The special fiber $\{z_2 = 1\}$ is immediately collapsed into $(1,1)$ by Φ, explaining the pinched appearance of the images.

The special role the lines $\{z_1 = 1\}$ and $\{z_2 = 1\}$ play here is related to the fact that they are the level sets of ϕ corresponding to the non-tangential value $\lambda = 1$, which is attained at the singularity of ϕ at $(1,1)$ (see [9]). This will be discussed in detail below.



(a) Vertical lines. (b) Φ^n for $n = 1$. (c) Φ^n for $n = 2$ (d) Φ^n for $n = 5$.

FIGURE 2. *Iteration of* $\Phi = (-\frac{3z_1z_2 - z_1 - z_2 - 1}{3 - z_1 - z_2 - z_1z_2}, z_2)$ *on* $\mathbb{T}^2$.

Example 2.2. Our next example is

$$\Phi(z_1, z_2) = \left(-\frac{3z_1z_2 - z_1 - z_2 - 1}{3 - z_1 - z_2 - z_1z_2}, z_2\right).$$

The first component map $\phi = -\frac{\tilde{p}}{p}$ again has a unique singularity on $\mathbb{T}^2$ at the point $(1,1)$, with $\phi^*(1,1) = 1$, and since $\Phi(z_1, 1) = (1,1)$, the fiber $\{z_2 = 1\}$ collapses. The fixed points of Φ on the 2-torus are determined by $\tilde{p} - z_1 p = 0$, and since $\tilde{p} - z_1 p = -(z_1 - 1)^2(z_2 + 1)$, the fixed points consist of $\{z_1 = 1\} \cup \{z_2 = -1\}$, a union of two lines.

Since $\phi(z_1,-1)=z_1$, we obtain $\Phi(z_1,-1)=(z_1,-1)$, confirming that all the points on $\{z_2=-1\}$ are fixed. By contrast, all other fibers contain a single fixed point at $(1,z_2)$. The dynamics of Φ are shown in Figure 2: we see that when $t_2>0$, points are attracted to the t_2-axis from the right, and when $t_2<0$, the fibers are attracted to the axis from the left. A concrete formula for $\lim_{n\to\infty}\Phi$ is not readily apparent, but a computation shows that

$$\frac{\partial\phi}{\partial z_1}(z_1,z_2)=4\frac{(z_2-1)^2}{(3-z_1-z_2-z_1z_2)^2},$$

hence $\frac{\partial\phi}{\partial z_1}(1,z_2)=1$. As we will see later, this confirms that the fixed points on $\{(1,\zeta_2)\}$ are neither attractive nor repelling in this example: they are so-called parabolic fixed points.

These two examples give an indication of the features (the presence of a collapsing fiber, fixed point curves being smooth, convergence to fixed points) present the general $(1,n)$ RISP setting. However, other phenomena do arise, and mixed behavior is possible, as the next example shows.

Example 2.3. The bidegree $(1,1)$ RISP

$$\Phi(z_1,z_2)=\left(\frac{3z_1z_2-z_1-z_2}{3-z_1-z_2},z_2\right)$$

has $\Phi(1,1)=(1,1)$ so that $(1,1)$ is again a fixed point, but in this example,

$$\phi_1(z_1)=\phi(z_1,1)=\frac{2z_1-1}{2-z_1},$$

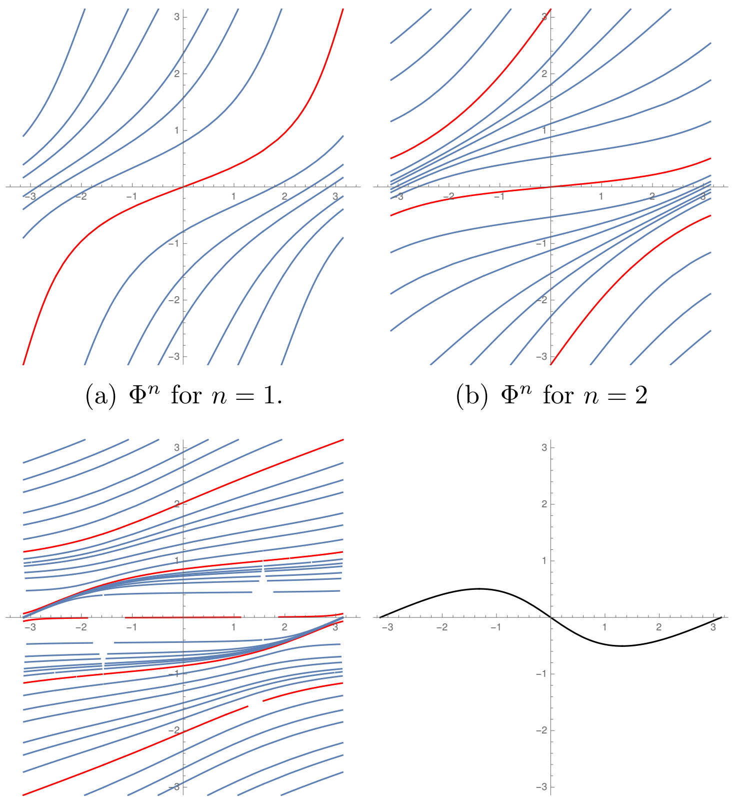
which is non-constant, meaning that $\{z_2=1\}$ does not collapse into a point under Φ. In fact, the component ϕ does not have a singularity at $(1,1)$ or, indeed, at any point of $\mathbb{T}^2$. Instead, the points $(1,1)$ and $(-1,1)$ are fixed by Φ, and since $\phi_1'(1)=3$ and $\phi_1'(-1)=\frac{1}{3}$, the fixed point $(1,1)$ is repelling while $(-1,1)$ is attracting.

We can determine all the fixed points of Φ on $\mathbb{T}^2$ by solving $\tilde{p}(z)-z_1p(z)=0$ for z_2. We obtain

$$z_2=\psi(z_1)=z_1\frac{4-z_1}{4z_1-1},$$

and we note that the right-hand side is unimodular for unimodular z_1, so that we again get a curve of fixed points in $\mathbb{T}^2$. However, ψ is not surjective on $\mathbb{T}$, meaning that there are z_2-fibers without fixed points; see Figure 3(d). For instance, -1 is not in the range of ψ on $\mathbb{T}$, and a computation reveals that $\phi(z_1,-1)=-\frac{4z_1-1}{4-z_1}$ satisfies $\phi^2=\mathrm{id}$. Thus, the fiber dynamics associated with Φ^n on $\{z_2=-1\}$ are those of a rational rotation of order 2. This is clearly visible when looking at the top of the images in Figure 3.

Other fibers that do not intersect the fixed point curve experience rotations of different orders.



(a) Φ^n for $n = 1$. (b) Φ^n for $n = 2$

(c) Φ^n for $n = 5$. (d) Invariant curve.

FIGURE 3. *Iteration of* $\Phi = (\frac{3z_1z_2 - z_1 - z_2}{3 - z_1 - z_2}, z_2)$ *on* $\mathbb{T}^2$.

This third example has some features in common with the first two, but the dynamics varies in nature across fibers. Together, our examples illustrate the three possible dynamical behaviors of a bidegree $(1, n)$ RISP – attracting/repelling fixed points, so-called parabolic fixed points, and rotations on a z_2-fiber. In the investigation below, we explore how the actions on individual fibers fit together in a global picture of the dynamical behavior of a bidegree $(1, n)$ RISP on $\mathbb{T}^2$.

3. Preliminaries

3.1. **Rational inner functions on the bidisk.** Recall that a RIF is of the form

$$\phi(z) = e^{i\alpha} z_1^{\beta_1} z_2^{\beta_2} \frac{\tilde{p}(z)}{p(z)}.$$

In what follows, we shall frequently assume that our RIFs are normalized. We take this to mean $\beta_1 = \beta_2 = 0$, meaning that

$$\phi = e^{i\alpha}\frac{\tilde{p}}{p}$$

for some polynomial p with no zeros in $\mathbb{D}^2$. We also assume that p is *atoral* in the sense of [2], meaning that p and $\tilde{p}$ have no common factors. (If p is not initially atoral, then any toral factors can be canceled with the corresponding toral factors in $\tilde{p}$.)

Let us review some basic facts concerning rational inner functions on the bidisk. By the definition of the reflection operation, any $\tau \in \mathbb{T}^2$ that is a zero of p is a zero of $\tilde{p}$. We say that $\tau \in \mathbb{T}^2$ is a *singular point* of ϕ if $p(\tau) = 0$ (and hence also $\tilde{p}(\tau) = 0$). Next, let us recall *Bézout's theorem* for $\mathbb{C}_\infty \times \mathbb{C}_\infty$, as discussed in [22, Section 12]. Namely, if $P, Q \in \mathbb{C}[z_1, z_2]$ are polynomials with no common factors and with $\deg P = (M_1, N_1)$ and $\deg Q = (M_2, N_2)$, then P and Q have $M_1N_2 + M_2N_1$ common zeros in $\mathbb{C}_\infty \times \mathbb{C}_\infty$. Applying this result to $P = p$ and $Q = \tilde{p}$ shows that any two-variable rational inner function has at most finitely many singularities on $\mathbb{T}^2$.

Since ϕ is a bounded analytic function, Fatou's theorem for polydisks guarantees the existence of non-tangential limits of ϕ: that is,

$$\phi^*(\zeta_1, \zeta_2) = \angle \lim_{\mathbb{D}^2 \ni (z_1, z_2) \to (\zeta_1, \zeta_2)} \phi(z_1, z_2)$$

exists at almost every $\zeta \in \mathbb{T}^2$. Here, $\angle \lim_{z \to \zeta} f(z)$ denotes letting z tend to $\zeta \in \mathbb{T}^2$ with $|z_j - \zeta_j| < c(1 - |z_j|)$, $j = 1, 2$, for some constant $c > 1$. Knese proved [22, Corollary 14.6] that if ϕ is rational inner function, then the non-tangential limit $\phi^*(\zeta)$ exists and is unimodular at *every* point $\zeta \in \mathbb{T}^2$. However, if a normalized ϕ has singular points, then $\phi^*(\zeta)$ is not typically continuous on $\mathbb{T}^2$. See [3, 31, 9, 10] for more background material on RIFs.

Definition 3.1. Let $\Phi = (\phi_1, \phi_2)\colon \mathbb{D}^2 \to \mathbb{D}^2$ be a non-constant normalized RIM. We say that $\tau \in \mathbb{T}^2$ is a *singular fixed point* (SF-point) of Φ if

- $\tau \in \mathbb{T}^2$ is a singular point of at least one of ϕ_1 and ϕ_2;
- τ is a *fixed point*: $\Phi^*(\tau) = \tau$.

Note that, for each denominator p, we can make sure the corresponding component RIFs ϕ_j satisfy the condition $\phi_j^*(\tau) = \tau_j$ by a suitable choice of unimodular factor $e^{i\alpha_j}$. If ϕ_j possesses multiple singularities then we cannot typically normalize all of them to be SF-points simultaneously.

We record a few further facts about rational inner functions. Suppose ϕ is RIF of bidegree (m, n). For a fixed $\zeta_2 \in \mathbb{T}$, the function

$$\phi_{\zeta_2}\colon z_1 \mapsto \phi(z_1, \zeta_2)$$

is a bounded rational function in the unit disk, attaining unimodular boundary values at every point of $\mathbb{T}$. Hence ϕ_{ζ_2} is constant, a monomial, or a finite Blaschke product of degree $1 \leq d \leq m$. Returning to the setting of a RISP $\Phi = (\phi, z_2)$, we note that if λ is a constant, Φ maps the set

$$F_\lambda = \{(z_1, z_2) \in \overline{\mathbb{D}^2} \colon z_2 = \lambda\} \tag{3.1}$$

into itself. The set F_λ is referred to as a *fiber*. On each F_λ, the first component ϕ_λ is generically a finite Blaschke product of degree m but on certain fibers the degree may drop. The most extreme examples are the following.

Definition 3.2. Let $\Phi = (\phi, z_2)$ be a normalized RISP, and let $\lambda \in \mathbb{T}$. We say that F_λ is a *collapsing fiber* for Φ if the one-variable function ϕ_λ is constant.

We often specialize to bidegree $(1, n)$ RIFs. In that case, we write $\phi = e^{i\alpha}\frac{\tilde{p}}{p}$, where

$$p(z) = p_1(z_2) + z_1 p_2(z_2), \tag{3.2}$$

for some $p_1, p_2 \in \mathbb{C}[z_2]$. Then as a consequence $\tilde{p}(z) = \tilde{p}_2(z_2) + z_1\tilde{p}_1(z_2)$, where each $\tilde{p}_j(z_2) = z_2^n \overline{p}(1/\bar{z}_2)$. Note that $p_1(z_2) \neq 0$ for $z_2 \in \mathbb{D}$ since p is assumed to have no zeros in the bidisk, and in fact $p_1(z_2) \neq 0$ for $z \in \overline{\mathbb{D}}$ by [22, Lemma 10.1]. Finally, for $\alpha \in \mathbb{R}$, let us define the polynomial

$$Q_\alpha(z_2) = (p_1(z_2) - e^{i\alpha}\tilde{p}_1(z_2))^2 + 4e^{i\alpha}p_2(z_2)\tilde{p}_2(z_2) \tag{3.3}$$

which we shall later use to analyze the fixed points of the corresponding RISP (ϕ, z_2). Note that if ϕ has bidegree $(1, n)$, then Q_α has degree at most $2n$, and that $\tilde{Q}_\alpha(z_2) = e^{-2i\alpha}Q_\alpha(z_2)$.

When ϕ is a degree $(1, n)$ RIF and $\zeta_2 \in \mathbb{T}$, the one-variable function ϕ_{ζ_2} has $\deg(\phi_{\zeta_2}) = 1$, or is a constant and thus has a collapsing fiber F_{ζ_2}. In this case, we can characterize collapsing fibers using a result from [7]: reversing the roles of z_1 and z_2, [7, Theorem 3.3] yields the following.

Lemma 3.3. *Suppose $\Phi = (\phi, z_2)$ is a normalized bidegree $(1, n)$ RISP. Then F_λ is a collapsing fiber if and only if $(\tau_1, \lambda) \in \mathbb{T}^2$ is a singularity of ϕ for some choice of τ_1. If this is the case, then after multiplication by a unimodular factor $\phi^*(\tau_1, \lambda) = \tau_1$, and (τ_1, λ) is an SF-point of Φ.*

In other words, Lemma 3.3 relates collapsing fibers of a RISP to the level sets of its first component map ϕ. In this regard, two-variable rational inner functions exhibit better behavior than rational inner functions in $\mathbb{D}^n$ when $n \geq 3$. In [10, Theorem 2.8], it is shown that for each fixed $\lambda \in \mathbb{T}$, the unimodular level set

$$\mathcal{C}_\lambda = \{(\zeta_1, \zeta_2) \in \mathbb{T}^2 \colon \tilde{p}(\zeta_1, \zeta_2) - \lambda p(\zeta_1, \zeta_2) = 0\} \tag{3.4}$$

consists of components that can be parametrized using analytic functions. By contrast, unimodular level sets in higher dimensions need not even be continuous [11].

3.2. **Dynamics of Möbius transformations.** Recall that the elements of the conformal automorphism group $\mathrm{Aut}(\mathbb{D})$ of the unit disk are *Möbius transformations* of the form

$$\mathfrak{m}(z) = e^{i\alpha}\frac{a - z}{1 - \overline{a}z},$$

where $\alpha \in \mathbb{T}$ and $a \in \mathbb{D}$. Note that each Möbius transformation extends to the closed disk $\overline{\mathbb{D}}$ and in particular furnishes a smooth diffeomorphism of the unit circle $\mathbb{T}$.

If ϕ is a bidegree $(1, n)$ rational inner function, then for each fixed $\lambda \in \mathbb{T}$, the analytic function $z_1 \mapsto \phi(z_1, \lambda)$ is bounded, satisfies $|\phi(z_1, \lambda)| = 1$, and is of degree 0 or 1. Then either $\phi(\cdot, \lambda)$ is a Möbius transformation for some values of $\alpha(\lambda)$ and $a(\lambda)$, or $\phi(\cdot, \lambda)$ is constant. The latter possibility corresponds to a collapsing fiber, and by Lemma 3.3, this occurs if only if (ζ_1, λ) is a singularity of ϕ for some choice of ζ_1.

In summary, for all but finitely many values

$$\Lambda^\flat = \{\lambda_1^\flat, \ldots, \lambda_N^\flat\} \subset \mathbb{T},$$

whose number is bounded by the degree of ϕ in z_2, the fiber map

$$z_1 \mapsto \phi_\lambda(z_1) = \phi(z_1, \lambda)$$

is a Möbius transformation with parameters $\alpha(\lambda)$ and $a(\lambda)$. The functions $\alpha(\lambda)$ and $a(\lambda)$ are continuous in any open set that does not intersect $\Lambda^\flat$. Associated with $\Lambda^\flat$ is the set

$$\iota(\Lambda^\flat) = \bigcup_{\lambda \in \Lambda^\flat} F_\lambda; \tag{3.5}$$

note that this is a finite union of horizontal lines in $\mathbb{T}^2$.

Let us recall the well-known classification of Möbius transformations depending on their fixed points. See [5, Chapter 1] or [23, Chapter 1] for a background discussion.

Definition 3.4. Suppose $\mathfrak{m} \in \mathrm{Aut}(\mathbb{D}) \setminus \{\mathrm{id}\}$. Then one of the following holds:

- $\mathfrak{m}$ is *elliptic*: $\mathfrak{m}$ has one fixed point in $\mathbb{D}$, and no fixed points on $\mathbb{T}$
- $\mathfrak{m}$ is *parabolic*: $\mathfrak{m}$ has no fixed points in $\mathbb{D}$ and one fixed point on $\mathbb{T}$

- $\mathfrak{m}$ is *hyperbolic*: $\mathfrak{m}$ has no fixed points in $\mathbb{D}$ and two fixed points on $\mathbb{T}$.

Fixed points are classified according to the following scheme.

Definition 3.5. Suppose $t \in \overline{\mathbb{D}}$ is a fixed point of a Möbius transformation $\mathfrak{m}$ and define the *multiplier* $\mathfrak{p} = \mathfrak{m}'(t)$. We say that

- t is *attracting* if $0 < |\mathfrak{p}| < 1$
- t is *repelling* if $|\mathfrak{p}| > 1$
- t is *indifferent* if $|\mathfrak{p}| = 1$.

We further distinguish between *rationally indifferent fixed points* arising when $\mathfrak{p}$ is a root of unity, and *irrationally indifferent fixed points* corresponding to $\mathfrak{p}$ unimodular but not a root of unity.

The dynamics of a Möbius transformation can be described depending on its type in the above list. Recall that two analytic functions $f, g\colon \mathbb{D} \to \mathbb{D}$ are *conformally conjugate* if there exists a conformal map $\mathfrak{n}\colon \mathbb{D} \to \mathbb{D}$ such that

$$g = \mathfrak{n} \circ f \circ \mathfrak{n}^{-1}.$$

See [5, Chapter 2] or [23, Chapters 8-9] for general background. Note that conjugation preserves the multiplier of a Möbius transformation at a fixed point, leading to the following classification.

Proposition 3.6. *Let* $\mathfrak{m} \in \mathrm{Aut}(\mathbb{D}) \setminus \{\mathrm{id}\}$. *Then the following holds:*

- *If* $\mathfrak{m}$ *is elliptic, then* $\mathfrak{m}$ *is conjugate to a rotation* $z \mapsto e^{ia}z$ *with angle* $a = -i \log \mathfrak{p}$, *where* $\mathfrak{p}$ *is the multiplier at the unique fixed point* $t \in \mathbb{D}$.
- *If* $\mathfrak{m}$ *is parabolic then* $\mathfrak{m}$ *has multiplier equal to* 1 *at its unique fixed point* $t \in \mathbb{T}$.
- *If* $\mathfrak{m}$ *hyperbolic then* $\mathfrak{m}$ *has one attracting fixed point* $t_1 \in \mathbb{T}$ *with associated multiplier* $\mathfrak{p} \in \mathbb{D}$ *and one repelling fixed point* $t_2 \in \mathbb{T}$ *with multiplier* $\frac{1}{\mathfrak{p}}$.

Proof. See, for instance, [5, Chapter 1] or [23, Chapter 1]. □

Proposition 3.6 gives us a complete description of the possible dynamics of a bidegree $(1, n)$ RISPs on each fiber F_λ. In the next sections, we investigate how these dynamical properties vary along fibers.

3.3. **Basic notions from complex dynamics.** Our main results can be formulated in terms of the dynamics of Möbius transformations, as discussed in the preceding Subsection 3.2. We have included some comments referencing the *Julia set* of a rational map R on the Riemann sphere, as well as some related notions. As these comments should be viewed as short asides, we do not include full definitions, and refer the reader to [5, 12, 23] for background on rational iteration.

4. RISP dynamics on the boundary

4.1. **General remarks on boundary fixed points.** Consider a bidegree (n_1, n_2) rational inner mapping $\Phi = (\phi_1, \phi_2)$, where $\phi_j = e^{i\alpha_j}\frac{\tilde{p}_j}{p_j}$, $j = 1, 2$, and where as usual each $p_j \in \mathbb{C}[z_1, z_2]$ has no zeros in $\mathbb{D}$. The set of fixed points of Φ in $\overline{\mathbb{D}^2}$ consists of all $z \in \overline{\mathbb{D}^2}$ such that

$$\Phi(z_1, z_2) = (z_1, z_2);$$

recall that Φ is well-defined at each $\eta \in \mathbb{T}^2$ [22, Corollary 14.6].

We first record some facts about fixed points in the general case. For p_j and $\tilde{p}_j$ as above, we define the auxiliary polynomials

$$P_j(z) = e^{i\alpha_j}\tilde{p}_j(z) - z_1 p_j(z), \quad j = 1, 2, \tag{4.1}$$

and we set

$$\Gamma(\Phi) = \mathcal{Z}(P_1) \cap \mathcal{Z}(P_2) \cap \overline{\mathbb{D}^2}.$$

Then $\Gamma(\Phi)$ is comprised of the fixed points of Φ along with all singular points of Φ. As before, we adopt the convention that at least one of these is made into an SF-point by renormalizing the component maps ϕ_j, $j = 1, 2$. Note that

$$\deg P_1 = (n_1 + 1, n_2) \quad \text{and} \quad \deg P_2 = (n_1, n_2 + 1).$$

The polynomials P_j belong to a special class of polynomials that is often singled out in function theory in polydisks. We follow [2] and use the following definition.

Definition 4.1. We say that $P \in \mathbb{C}[z_1, z_2]$ is *essentially* $\mathbb{T}^2$*-symmetric* if $\tilde{P} = e^{i\beta}P$ for some $\beta \in \mathbb{R}$.

We then have the following.

Lemma 4.2. *If Φ is a normalized RIM, then the associated polynomials P_1, P_2 defined in* (4.1) *are essentially $\mathbb{T}^2$-symmetric.*

Proof. This amounts to a computation. Applying the reflection operation (2.2) to the polynomial P_1 and exploiting the fact that reflection is an involution, we obtain

$$\tilde{P}_1(z_1,z_2) = z_1^{n_1+1}z_2^{n_2}e^{-i\alpha_1}\overline{\tilde{p}}_1\left(\frac{1}{\overline{z}_1},\frac{1}{\overline{z}_2}\right) - z_1^{n_1+1}z_2^{n_2}\frac{1}{z_1}\overline{p}_1\left(\frac{1}{\overline{z}_1},\frac{1}{\overline{z}_2}\right)$$
$$= e^{-i\alpha_1}z_1p_1(z_1,z_2) - \tilde{p}_1(z_1,z_2) = -e^{-i\alpha_1}(-z_1p_1(z_1,z_2) + e^{i\alpha_1}\tilde{p}_1(z_1,z_2))$$
$$= -e^{-i\alpha_1}P_1.$$

A similar computation establishes the assertion for P_2. □

Essentially $\mathbb{T}^2$-symmetric polynomials have been studied extensively, see for instance [2, 21, 10] and the references therein. In particular, such polynomials are related to determinantal representations and so-called distinguished varieties. We caution that the focus is often on essentially $\mathbb{T}^2$-symmetric polynomials with no zeros in the bidisk, while the polynomials P_j that we encounter in this work typically *do* have some zeros in $\mathbb{D}^2$, namely at interior fixed points of Φ. As a consequence, some of the known results from the literature (e.g. [8]) no longer apply.

In the pictures in Section 2, we observed the presence of curves of fixed points in the 2-torus. The fixed point set of a general RIM, however, typically consists of isolated points. This can be seen via a standard application of Bézout's theorem. (See [29, Section 1.3] for applications to counting the number of fixed and periodic points in the context of iteration of rational functions in $\mathbb{P}^k$.)

Lemma 4.3. *Suppose $\Phi = (\phi_1, \phi_2)$ is a RIM such that the associated polynomials P_1 and P_2 have no common factors. Then $\Gamma(\Phi)$ is a finite set, with cardinality bounded by $2n_1n_2 + n_1 + n_2$.*

Proof. Lemma 4.3 follows directly from Bézout's theorem as discussed in Section 3. Namely, since P_1 and P_2 have no common zeros, we have

$$\#(\mathcal{Z}(P_1) \cap \mathcal{Z}(P_2)) = (n_1+1)\cdot(n_2+1) + n_2\cdot n_1 = 2n_1n_2 + n_1 + n_2.$$

This is an upper bound on the number of elements of $\Gamma(\Phi)$ since some common zeros could be located in $(\overline{\mathbb{D}^2})^c$. □

Note that the assumption that P_1 and P_2 have no common factor is important: if we set ϕ_1 and ϕ_2 both equal to the RIF in Example 2.1, then P_1 and P_2 are both divisible by $z_1 - z_2$, and the diagonal $\{z_1 = z_2\}$ is fixed by the resulting mapping Φ.

We now focus on fixed points of a RISP $\Phi = (\phi, z_2)$, where $\phi = \frac{\tilde{p}}{p}$, and p has no zeros in $\mathbb{D}^2$. Note that for such functions, the polynomial P_2 in (4.1) becomes identically 0, and so Lemma 4.3 does not apply. Instead, in this case, we get that $z \in \mathbb{T}^2$ satisfies the single condition

$$q(z) - z_1 p(z) = 0, \tag{4.2}$$

and we can expect to see curves of fixed points in $\mathbb{T}^2$. Our goal is to investigate the nature of these fixed point curves.

We begin with a simple consequence of work in [10].

Proposition 4.4. *Let $\Phi = (z_1\phi, z_2)$, where ϕ is a normalized RIF. Then $\Gamma(\Phi) \cap \mathbb{T}^2$ is a union of curves, and each component of $\Gamma(\Phi) \cap \mathbb{T}^2$ can be parametrized by analytic functions.*

The subtlety here is that Φ may have singularities on $\mathbb{T}^2$, so that the implicit function theorem does not apply directly.

Proof. By assumption, the first coordinate of Φ can be written

$$z_1\phi = z_1 e^{i\alpha_1}\frac{\tilde{p}(z_1, z_2)}{p(z_1, z_2)}$$

for some real α_1 and some polynomial p with no zeros on $\mathbb{D}^2$. Then, the fixed point condition (4.2) takes on the form

$$0 = \eta_1 e^{i\alpha}\tilde{p}(\eta) - \eta_1 p(\eta) = \eta_1(e^{i\alpha}\tilde{p}(\eta) - p(\eta)).$$

Since $\eta_1 \neq 0$ for $\eta_1 \in \mathbb{T}$, we get that $\eta \in \mathbb{T}^2$ is a fixed point of Φ if and only if $e^{i\alpha}\tilde{p}(\eta) - p(\eta) = 0$. This latter condition means that η belongs to some unimodular level set of the RIF ϕ. Appealing to [10, Theorem 2.8], we arrive at the desired conclusion. □

In particular, Proposition 4.4 implies that RISPs of the particular form $(z_1\phi, z_2)$ exhibit an abundance of fixed points in the 2-torus. On each fiber, the function $z_1 \mapsto z_1\phi_\lambda(z_1)$ is a Blaschke product of degree at least 1, and for all but finitely many values of λ, the degree is at least 2. This means that the dynamical behavior on a λ-fiber generically does not admit a description in elementary terms.

For instance, $z_1\phi_\lambda(z_1)$ has a fixed point at $z_1 = 0$ for each choice of λ and this fixed point is attracting whenever $z_1\phi_\lambda$ has degree at least 2. Moreover since for generic $\lambda \in \mathbb{T}$, the numerator of $z_1\phi_\lambda(z_1)$ has higher degree than the denominator, the point at infinity of $\mathbb{C}_\infty$ is also an attracting fixed point. This implies [23, Chapter 7] that for generic λ, the Julia set of the rational map $z_1\phi_\lambda(z_1)\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$ is $\mathbb{T}$, the unit circle. We thus have complicated dynamical behavior on each fiber F_λ (see [5, Chapter 6] or [23, Chapter 4]), and for all but finitely many values of $\lambda \in \mathbb{T}$, the points in $F_\lambda \cap \Gamma(\phi)$ are repelling fixed points of the rational map $z_1\phi_\lambda(z_1)$. Proposition 4.4 asserts that these repelling fixed points move in an analytic fashion in the fiber variable λ.

The following example illustrates some of the facts listed above.

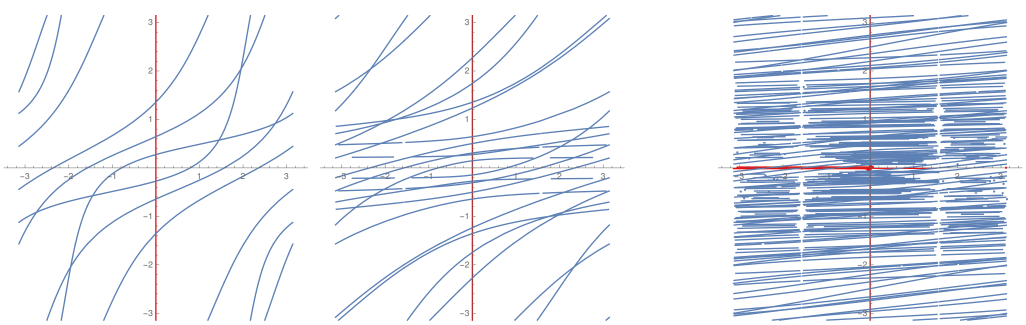
Example 4.5. Consider the RISP

$$\Phi(z) = \left(-z_1\frac{2z_1z_2 - z_1 - z_2}{2 - z_1 - z_2}, z_2\right).$$

As was noted above, the set $\{(0, z_2)\} \subset \overline{\mathbb{D}^2}$ consists of fixed points of Φ. On the fiber $z_2 = 1$, Φ reduces to the identity, and on all other F_λ, the fiber map $z_1\phi_\lambda(z_1)$ has degree 2 and has attracting fixed points at 0 and ∞.

Solving $-(2z_1z_2 - z_1 - z_2) = 2 - z_1 - z_2$, we find that $\{(1, \lambda)\colon \lambda \in \mathbb{T} \setminus \{1\}\} \subset \mathbb{T}^2$ is fixed by Φ, so that $1 \in \mathbb{T}$ is a fixed point on each F_λ. A direct computation shows that $\frac{d}{dz_1}(-z_1\phi_\lambda)(1) = 3$ when $\lambda \neq 1$, confirming the presence of a repelling fixed point.

The dynamics of Φ are visualized in Figure 4 and indeed have a more complicated appearance than the examples in Section 2.



(a) Φ^n for $n = 1$. (b) Φ^n for $n = 2$ (c) Φ^n for $n = 5$.

FIGURE 4. *Iteration of* $\Phi = (-z_1\frac{2z_1z_2 - z_1 - z_2}{2 - z_1 - z_2}, z_2)$ *on* $\mathbb{T}^2$.

4.2. **Rotation belts and parabolic fixed points for simple RISPs.** We return our focus to the simple case where $\Phi = (\phi, z_2)$ and ϕ is a bidegree $(1, n)$ normalized RIF, and where the fiber dynamics are completely described by Proposition 3.6. In this case, since p and $\tilde{p}$ are polynomials of degree 1 in the first variable, (4.2) reduces to the quadratic equation

$$P(z_1, z_2) = e^{i\alpha}\tilde{p}(z_1, z_2) - z_1p(z_1, z_2) = 0.$$

If Φ has a SF-point in $\mathbb{T}^2$ then $\mathcal{Z}(P) \cap \mathbb{T}^2 \neq \emptyset$ but it may well happen that $\mathcal{Z}(P) \cap (F_\lambda \cap \mathbb{T}^2) = \emptyset$ for some values of $\lambda \in \mathbb{T}$. With this in mind, we make the following definition.

Definition 4.6. Let $\lambda_1, \lambda_2 \in [-\pi, \pi]$ and suppose $\lambda_1 < \lambda_2$. We say that the set

$$B(\lambda_1, \lambda_2) = \mathbb{T} \times \{e^{it} \colon \lambda_1 < t < \lambda_2\} \subset \mathbb{T}^2$$

is a *rotation belt* for $\Phi = (\phi, z_2)$ if

- no point belonging to $B(\lambda_1, \lambda_2)$ is a fixed point of Φ
- each of the sets $F_{\lambda_1} \cap \mathbb{T}^2$ and $F_{\lambda_2} \cap \mathbb{T}^2$ contains at least one fixed point of Φ.

The reason for our terminology here is that by Proposition 3.6, each fiber map ϕ_λ associated with fibers contained in $B(\lambda_1, \lambda_2)$ is conjugate to a rotation. The second assumption is essentially a maximality condition. Examples 2.1 and 2.2 illustrate that not all RISPs possess rotation belts. Example 2.3 has a single rotation belt, but in general several rotation belts may be present.

We now give a criterion for a degree $(1, n)$ RISP to possess parabolic fixed points in terms of the one-variable polynomial Q_α. As in (3.2) in Section 3, we write $\phi = e^{i\alpha}\frac{\tilde{p}}{p}$, where

$$p(z) = p_1(z_2) + z_1 p_2(z_2),$$

and $\tilde{p}(z) = \tilde{p}_2(z_2) + z_1\tilde{p}_1(z_2)$ for a pair of one-variable polynomials p_1, p_2. Recall that

$$Q_\alpha = (p_1 - e^{i\alpha}\tilde{p}_1)^2 + 4e^{i\alpha}p_2\tilde{p}_2.$$

We shall need the auxiliary expressions

$$\psi^1_\alpha(z_2) = \frac{1}{2p_2(z_2)}\left(p_1(z_2) - e^{i\alpha}\tilde{p}_1(z_2) + \sqrt{Q_\alpha(z_2)}\right) \tag{4.3}$$

and

$$\psi^2_\alpha(z_2) = \frac{1}{2p_2(z_2)}\left(p_1(z_2) - e^{i\alpha}\tilde{p}_1(z_2) - \sqrt{Q_\alpha(z_2)}\right). \tag{4.4}$$

The functions ψ^j_α parametrize the roots of the polynomial

$$P_\alpha(z) = e^{i\alpha}\tilde{p} - z_1 p = e^{i\alpha}\tilde{p}_2 + e^{i\alpha}z_1\tilde{p}_1 - z_1 p_1 - z_1^2 p_2$$

away from the set $\Lambda^\sharp = \{z_2 \in \mathbb{C} \colon p_2(z_2) = 0\}$. Note that $\Lambda^\flat \cap \Lambda^\sharp = \emptyset$ since $p_1(z_2) \neq 0$ for $z_2 \in \overline{\mathbb{D}}$.

First, let us examine what happens on the set $\Lambda^\sharp$.

Lemma 4.7. *Suppose $\lambda \in \mathcal{Z}(Q_\alpha) \cap (\Lambda^\sharp \cap \mathbb{T})$. Then $\phi_\lambda(z_1) = e^{-i\alpha}z_1$.*

Conversely, if $\phi_\lambda(z_1) = e^{-i\alpha}z_1$ for some $\lambda \in \mathbb{T}$, then $p_2(\lambda) = 0$ and $\lambda \in \mathcal{Z}(Q_\alpha)$.

Proof. By the definition of the reflection operation, $p_2(\lambda) = 0$ implies $\tilde{p}_2(\lambda) = 0$. Next, if also $Q_\alpha(\lambda) = 0$ then $(p_1(\lambda) - e^{i\alpha}\tilde{p}_1(\lambda))^2 = 0$ so that $p_1(\lambda) = e^{i\alpha}\tilde{p}_1(\lambda)$. But then

$$\phi(z_1, \lambda) = \frac{z_1\tilde{p}_1(\lambda) + 0}{p_1(\lambda) + 0} = e^{-i\alpha}z_1.$$

Conversely, suppose

$$e^{-i\alpha}z_1 = \frac{z_1\tilde{p}_1(\lambda) + \tilde{p}_2(\lambda)}{p_1(\lambda) + p_2(\lambda)z_1}$$

for some unimodular λ. Then, comparing coefficients, we get $p_2(\lambda) = 0$ and as a direct consequence that $\tilde{p}_2(\lambda) = 0$. Finally, we read off that $e^{-i\alpha} = \frac{\tilde{p}_1(\lambda)}{p_1(\lambda)}$ and hence $Q_\alpha(\lambda) = 0$, as claimed. □

Having seen what happens on the set $\Lambda^\sharp$, we now examine the behavior of Q_α on $\Lambda^\flat$.

Lemma 4.8. *Suppose $\lambda \in \Lambda^\flat$. Then*

(1) $Q_\alpha(\lambda) = 0$ if and only if $p_1(\lambda) + e^{i\alpha}\tilde{p}_1(\lambda) = 0$;
(2) if $Q_\alpha(\lambda) = 0$ then Q_α vanishes to at least order 2 at λ.

Proof. We first prove (1). By completing the square, we can rewrite Q_α as

$$Q_\alpha(z_2) = (p_1(z_2) + e^{i\alpha}\tilde{p}_1(z_2))^2 + 4e^{i\alpha}(p_2(z_2)\tilde{p}_2(z_2) - p_1(z_2)\tilde{p}_1(z_2)).$$

Since $\lambda \in \Lambda^\flat$, Lemma 3.3 implies that $p(\tau_1, \lambda) = \tilde{p}(\tau_1, \lambda) = 0$ for some $\tau_1 \in \mathbb{T}$. Solving for τ_1, we deduce that $\frac{p_1(\lambda)}{p_2(\lambda)} = \frac{\tilde{p}_2(\lambda)}{\tilde{p}_1(\lambda)}$ and then $p_1(\lambda)\tilde{p}_1(\lambda) - p_2(\lambda)\tilde{p}_2(\lambda) = 0$.

Hence, $Q_\alpha(\lambda) = (p_1(\lambda) + e^{i\alpha}\tilde{p}_1(\lambda))^2$ leading to the desired condition.

We turn to (2). Clearly, any zero of $(p_1 + e^{i\alpha}\tilde{p}_1)^2$ occurs with even multiplicity. Next, we note that $\left|\frac{p_2(z_2)}{p_1(z_2)}\right|^2 \leq 1$ for $z_2 \in \mathbb{D}$, for otherwise p would have a zero in $\mathbb{D}^2$. Now, we note that

$$p_2(e^{it_2})\tilde{p}_2(e^{it_2}) - p_1(e^{it_2})\tilde{p}_1(e^{it_2}) = e^{int_2}\left(|p_2(e^{it_2})|^2 - |p_1(e^{it_2})|^2\right)$$

and since the expression on the right is a non-positive trigonometric polynomial, the Fejér-Riesz theorem implies that it is equal to $-|r(e^{it_2})|^2$ for some polynomial r, with a zero at λ by the proof of (1). □

We shall see later, in Example 5.2, that it may well happen that $Q_\alpha(\lambda) \neq 0$ for certain $\lambda \in \Lambda^\flat$.

Theorem 4.9. *Suppose $Q_\alpha(\lambda) = 0$ for some $\lambda \in \mathbb{T} \setminus (\Lambda^\flat \cup \Lambda^\sharp)$. Then $\psi^1(\lambda)$ and $\psi^2(\lambda)$ coincide as an element of $\mathbb{T}$, and ϕ_λ has a parabolic fixed point at $\psi^1_\alpha(\lambda)$.*

Note that ψ_α^j, $j = 1, 2$, will in general exhibit branch points at points where Q_α vanishes, and so $\Gamma(\Phi)$ does not in general admit a description in terms of analytic functions near parabolic fixed points of Φ in $\mathbb{T}^2$. This is in contrast to the results in [10] which state that level sets of RIFs can be parametrized in such a way at every point of $\mathbb{T}^2$.

Proof. Because of our assumptions, for each $\lambda \in \mathbb{T} \setminus (\Lambda^\flat \cup \Lambda^\sharp)$, the fiber map ϕ_λ is a Möbius transformation not of the form $e^{i\beta} z_1$. Then, by the discussion in Section 3, ϕ_λ either has two hyperbolic fixed points on the unit circle, or a parabolic fixed point on $\mathbb{T}$, or else an elliptic fixed point in $\mathbb{D}$ with another fixed point lying in $\overline{\mathbb{D}^c}$. But $Q_\alpha(\lambda) = 0$ with $\lambda \notin \Lambda^\sharp$ implies that $\psi_\alpha^1(\lambda) = \psi_\alpha^2(\lambda)$. Since $\lambda \notin \Lambda^\flat$, we are in the parabolic case and then necessarily $\psi_\alpha^1(\lambda) \in \mathbb{T}$. □

Example 4.10. Consider the polynomials p and $\tilde{p}$ in appearing in the RIF ϕ in Example 2.3 in Section 2. There, $p_1(z_2) = 3 - z_2$ and $p_2(z_2) = -1$, while $\tilde{p}_1(z_2) = 3z_2 - 1$ and $\tilde{p}_2(z_2) = -z_2$. Then $Q_\alpha(z_2) = 16z_2^2 - 28z_2 + 16$, and this polynomial has conjugate roots on $\lambda_{1,2} = \frac{1}{8}(7 \pm \sqrt{15}i) \in \mathbb{T}$. The corresponding fibers $F_{\lambda_{1,2}}$ bound the rotation belt of Φ.

Example 4.11. It may well happen that $Q_\alpha \equiv 0$, so that most fixed points are parabolic. This can be seen in Example 2.2 from Section 2. In that case, we have $p_1(z_2) = 3 - z_2$ and $p_2(z_2) = -(1 + z_2)$ as well as $\tilde{p}_1(z_2) = 3z_2 - 1$ and $\tilde{p}_2(z_2) = -(1 + z_2)$, and if we insist on the normalized choice $\alpha = \pi$, then indeed Q_α is the zero polynomial. As we saw before, the set $\{(1, e^{it_2}) \colon t_2 \neq 0, \pi\}$ consists of parabolic fixed points of Φ.

This example also illustrates the need to assume $\lambda \notin \Lambda^\flat \cup \Lambda^\sharp$. Namely, $\phi(z_1, 1) = 1$ for $\lambda = 1 \in \Lambda^\flat$ and $\phi(z_1, -1) = -z_1$ for $\lambda = -1 \in \Lambda^\sharp$, and neither case leads to a parabolic fixed point.

Theorem 4.12. *Suppose Φ is a degree $(1, n)$ RISP with associated Q_α not identically zero. Then the number of rotation belts for Φ is less than or equal to*

$$\frac{1}{2} \# \left[(\mathcal{Z}(Q_\alpha) \setminus \Lambda^\flat) \cap \mathbb{T} \right].$$

Proof. Note that $\Gamma(\Phi)$ is parametrized by the analytic functions $\psi_\alpha^{1,2}$ away from the branch points, and that the fiber map derivative $\frac{d}{dz_1}\phi_\lambda$ is continuous off $\iota(\Lambda^\flat)$, see Lemma 4.13 below. Thus, within a rotation belt $B(\lambda_1, \lambda_2)$ the composite map $\lambda \mapsto \mathfrak{p}(\lambda) \in \mathbb{T}$ is continuous. Now if a curve component of $\Gamma(\Phi)$ containing elliptic fixed points were to meet $\mathbb{T}^2$ in a hyperbolic fixed point having $|\mathfrak{p}(\lambda)| \neq 1$, this would force a discontinuity in the multiplier at

λ_1 or λ_2. Hence, the boundary fiber of a rotation belt contains a parabolic point, or is contained in $\iota(\Lambda^\flat \cup \Lambda^\sharp)$.

Next, we observe that neither of the boundary fibers $F_{\lambda_1}, F_{\lambda_2}$ can be collapsing. For if, say, F_{λ_1} was collapsing, then it could contain at most finitely many singularites $(\tau_1, \lambda_1), \ldots, (\tau_m, \lambda)$ of ϕ by Lemma 3.3. But the collapsing of F_{λ_1} would imply that Φ had discontinuities as $\lambda \to \lambda_1$ in $B(\lambda_1, \lambda_2)$ along vertical lines $\{\tau\} \times \mathbb{T} \subset \mathbb{T}^2$ for $\tau \neq \tau_1, \ldots, \tau_m$, which is impossible. Hence, by Theorem 4.9, this implies that the fibers F_{λ_1} and F_{λ_2} associated with a rotation belt $B(\lambda_1, \lambda_2)$ contain a parabolic fixed point, or, if $\lambda \in \mathcal{Z}(Q_\alpha) \cap \Lambda^\sharp$ and $\alpha = 0$, have fiber map $\phi_\lambda = z_1$. □

Note that $\deg Q_\alpha \leq 2n$, so that Q_α has at most $2n$ zeros on $\mathbb{T}$ counting multiplicities. It may well happen, however, that Φ has fewer than n rotation belts if $Q_\alpha(\lambda_j) = 0$ for some $\lambda_j \in \Lambda^\flat$; see Section 5 for an example with $n = 2$ and a single rotation belt. Similarly, examples show that the situation where a rotation belt is bounded by a fiber where the fiber map is equal to the identity can indeed arise.

Lemma 4.13. *Let $\mathfrak{p}(\lambda)$ denote the multiplier of ϕ_λ at a point $\lambda \in \overline{\mathbb{D}}$. Then $\lambda \mapsto \mathfrak{p}(\lambda)$ is an analytic function on each component of $\Gamma(\phi) \setminus [\iota(\mathcal{Z}(Q_\alpha)) \cap \iota(\Lambda^\flat \cup \Lambda^\sharp)]$.*

Proof. Note that for each $\lambda \in \mathbb{T} \setminus (\Lambda^\flat \cup \Lambda^\sharp)$, the Möbius map $\frac{d}{dz_1}\phi_\lambda$ is non-trivial and has no critical points. This implies that $\lambda \mapsto \frac{d}{dz_1}\phi_\lambda(\psi^j_\alpha(\lambda))$ is analytic off $\mathcal{Z}(Q_\alpha)$ and $\Lambda^\flat \cup \Lambda^\sharp$. □

Lemma 4.13 has the following consequence for rotations belts.

Lemma 4.14. *Suppose $B(\lambda_1, \lambda_2)$ is a rotation belt for $\Phi = (\phi, z_2)$ and that $\mathfrak{p}(\lambda)$ is non-constant on (λ_1, λ_2). Then uncountably many ϕ_λ are conjugate to an irrational rotation.*

Proof. By assumption $-i \log(\mathfrak{p}(\lambda))$ is a non-constant smooth function taking values in $[-\pi, \pi]$, and since moreover $-i \log(\mathfrak{p}(\lambda_{1,2})) = 0$, the statement follows from the intermediate value theorem. □

4.3. **Hyperbolic fixed points and SF-points for simple RISPs.** We now examine the dynamical behavior of Φ in the vicinity of an SF-point.

All singularities of a RISP are points of $\mathbb{T}^2$, and lie in collapsing fibers by Lemma 3.3.

Lemma 4.15. *Let $(\tau_1, \lambda_1), \ldots, (\tau_n, \lambda_m)$ be SF-points of Φ. Then $(\tau_j, \lambda_j) \in \Gamma(\Phi) \cap \mathbb{T}^2$ for $j = 1, \ldots, m$.*

The proof of Theorem 4.12 shows that a collapsing fiber cannot be contained in a rotation belt $B(\lambda_1, \lambda_2)$, nor can it coincide with one of the boundary fibers F_{λ_1} or F_{λ_2}. We have already seen that SF-points can be accessed in $\mathbb{T}^2$ via a component of $\Gamma(\Phi) \cap \mathbb{T}^2$ in different ways. An SF-point may belong to a component of $\Gamma(\Phi) \cap \mathbb{T}^2$ consisting of parabolic points (Example 2.2), or to one or more components of $\Gamma(\Phi) \cap \mathbb{T}^2$ made up of hyperbolic fixed points (Example 2.1).

Theorem 4.16. *Let $(\tau, \lambda) \in \mathbb{T}^2$ be an SF-point of a RISP Φ with associated polynomial Q_α not identically zero.*

(1) If $Q_\alpha(\lambda) \neq 0$ then (τ, λ) belongs to a single component of $\Gamma(\Phi) \cap \mathbb{T}^2$. Moreover, this component can be parametrized by an analytic function in some neighborhood of (τ, λ).

(2) If $Q_\alpha(\lambda) = 0$ then (τ, λ) belongs to two components of $\Gamma(\Phi) \cap \mathbb{T}^2$. If Q_α vanishes to even order, then each of these components can be parametrized by an analytic function in some neighborhood of (τ, λ).

Proof. Item (1) follows from the definition of ψ^j, $j = 1, 2$, in (4.3) and (4.4) together with the observation that points in $\Lambda^\flat$ cannot be elements of $\Lambda^\sharp$.

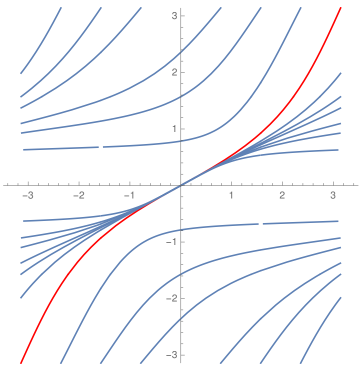
Let us turn to item (2). Lemma 4.8 asserts that Q_α vanishes to order at least 2. If Q_α in fact vanishes to even order at $z_2 = \lambda$, then the function $\sqrt{Q_\alpha(z_2)}$ is analytic in a neighborhood of λ. Hence ψ^j, $j = 1, 2$, are analytic also. □

In other words, a pair of components of $\Gamma(\Phi)$ coming together at an SF-point generically cross transversally. In fact, examination of a wide range of examples (see [32]) suggests that the components of $\Gamma(\Phi)$ can always be parametrized by analytic functions in a neighborhood of an SF-point, but we have not been able to find an elementary proof.

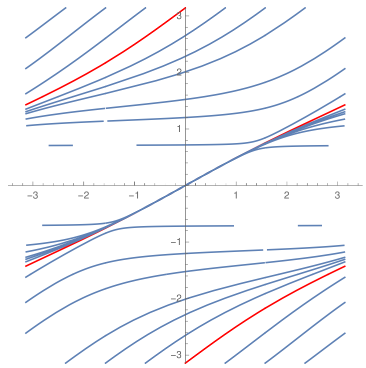
As an example, the RISP in Example 5.2 has two SF-points, with two hyperbolic fixed point curves meeting at one of them, and a single hyperbolic fixed point curve passing through the other.

5. Further examples

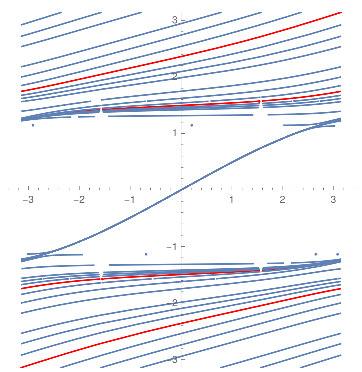
We conclude by examining two examples with more intricate singularities and dynamical behavior. Further examples and images can be found on the webpage [32]. Also included on [32] are images associated with the two-dimensional Blaschke products considered in [26, 25], and with other types of RIMs that are not studied in detail in this paper.



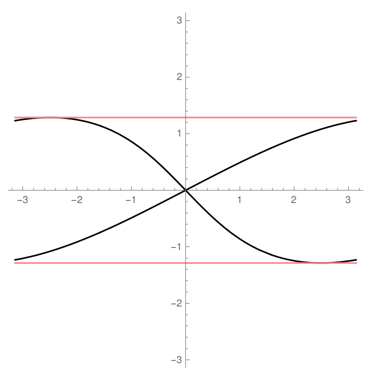
(a) Φ^n for $n = 1$.



(b) Φ^n for $n = 2$.



(c) Φ^n for $n = 5$



(d) $\Gamma(\Phi) \cap \mathbb{T}^2$. Parabolic fibers $F_{\lambda_{2,3}}$ in pink.

FIGURE 5. *Iteration of* Φ *given by* (5.1) *on* $\mathbb{T}^2$. *(Successive images of the vertical axis marked red)*

Example 5.1. Consider the rational inner function

$$\phi(z_1, z_2) = -\frac{\tilde{p}(z)}{p(z)} = -\frac{4z_1z_2^2 - z_2^2 - 3z_1z_2 - z_2 + z_1}{4 - z_1 - 3z_2 - z_1z_2 + z_2^2}, \tag{5.1}$$

which features as an example in the paper [3], and has been further studied in [10, Section 1.3] and [7, Example 5.2] (without the minus sign in front). Since ϕ has a single singularity at $(1, 1)$ and $\phi^*(1, 1) = 1$, the corresponding RISP Φ has a SF-point at $(1, 1)$. We further check that $\phi(1, z_2) = 1$, confirming the presence of a collapsing fiber.

In this example, $\alpha = \pi$, and

$$p(z) = p_1(z_2) + z_1p_2(z_2)$$

with

$$p_1(z_2) = 4 - 3z_2 + z_2^2 \quad \text{and} \quad p_2(z_2) = -(1 + z_2).$$

We have $p_2(-1) = 0$, and as expected $\phi(z_1, -1) = -z_1$, a rotation of order 2. A short computation shows that the associated polynomial Q_α is given

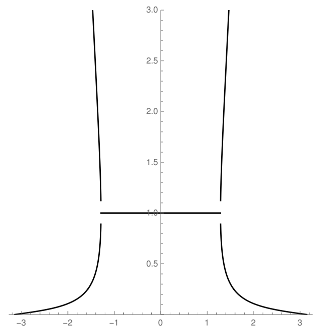


FIGURE 6. Plot of $|\psi^{1,2}(e^{it_2})|$ showing interior/exterior fixed points coming together at parabolic fixed points.

by

$$Q_\alpha(z_2) = 25(z_2-1)^2\left(z_2^2 - \frac{14}{25}z_2 + 1\right), \tag{5.2}$$

which has a double root at $\lambda_1 = 1$, and two roots at $\lambda_{2,3} = \frac{1}{25}(7 \pm 24i)$.

Figure 5 displays the dynamics of Φ on $\mathbb{T}^2$. We notice a single rotation belt bounded by F_{λ_2} and F_{λ_3}, one fewer than the maximum given the degree of Q_α. Also visible are two curves containing hyperbolic fixed points with a normal crossing at $(1,1)$, which is contained in the collapsing fiber. The hyperbolic fixed points are on a pair of curves parametrized by

$$\psi_{-1}^{1,2}(z_2) = \frac{1}{2(1+z_2)}\left(5 - 6z_2 + 5z_2^2 \pm (-1+z_2)\sqrt{25 - 14z_2 + 25z_2^2}\right).$$

These curves exhibit a normal crossing at $(1,1)$, the singularity of the RIF, reflecting the double root of Q_α at $(1,1)$. The branch point nature of the parabolic fixed points at λ_2 and λ_3 can be observed in Figure 6 which displays the absolute value of the functions $\psi^{1,2}(e^{it_2})$.

Example 5.2. We turn to a RIF with multiple singularities on $\mathbb{T}^2$ and multiple rotation belts. Consider the RIF $\phi = \frac{\tilde{p}}{p}$ with

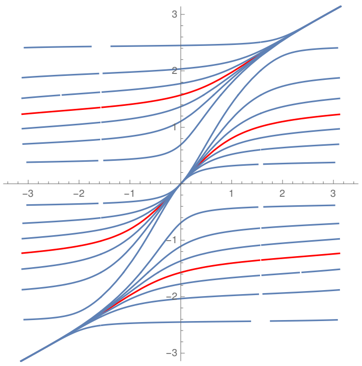
$$p(z) = 4 + z_1 - z_1z_2 + 3z_1z_2^2 + z_1z_2^3 \quad \text{and} \quad \tilde{p}(z) = 4z_1z_2^3 + z_2^3 - z_2^2 + 3z_2 + 1, \tag{5.3}$$

a slightly modified form of [10, Example 7.4] (see also [7, Example 5.4]).

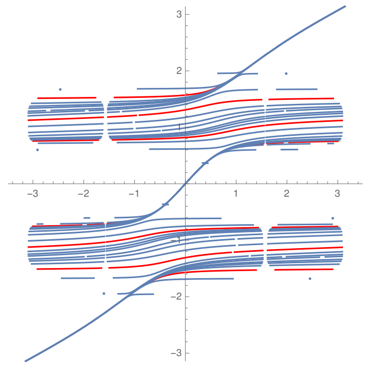
We have $p(-1,1) = 0$ and $p(-1,-1) = 0$, and a computation shows that $\phi(z_1, 1) = 1$ and $\phi(z_1, -1) = -1$, confirming the presence of two collapsing fibers as guaranteed by Lemma 3.3, and so ϕ has two singularities on the 2-torus.

We read off that

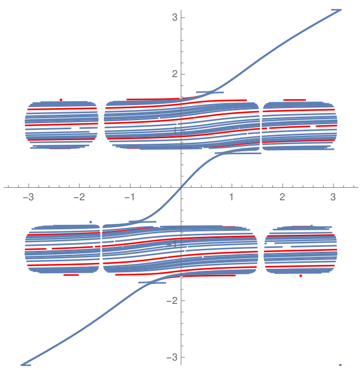
$$p_1(z_2) = 4 \quad \text{and} \quad p_2(z_2) = z_2^3 + 3z_2^2 - z_2 + 1$$



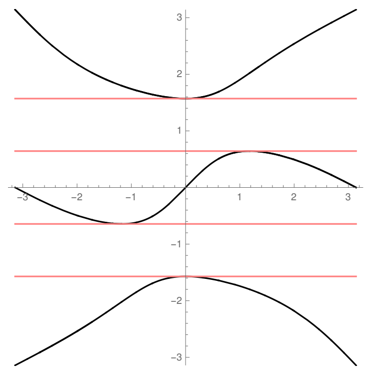
(a) Φ^n for $n = 1$.



(b) Φ^n for $n = 2$.



(c) Φ^n for $n = 5$



(d) $\Gamma(\Phi) \cap \mathbb{T}^2$. Parabolic fibers $F_{\lambda_{2,3}}$ in pink.

FIGURE 7. *Iteration of* Φ *given by* (5.3) *on* $\mathbb{T}^2$. *(Successive images of the vertical axis marked red)*

and

$$\tilde{p}_1(z_2) = 4z_2^3 \quad \text{and} \quad \tilde{p}_2(z_2) = z_2^3 - z_2^2 + 3z_2 + 1,$$

and after some simplifications (using that $\alpha = 0$ here), we find that

$$Q_\alpha(z_2) = 4(z_2 + 1)^2(z_2^2 + 1)(5z_2^2 - 8z_2 + 5). \tag{5.4}$$

The polynomial Q_α has a double root at $z_2 = -1$ (the z_2-coordinate of the SF-point at $(-1, -1)$), and simple roots at $z_2 = \pm i$ and $z_2 = \frac{1}{5}(4 \pm 3i)$. On the other hand $Q_\alpha(1) \neq 0$. This is reflected in the images in Figure 7: $\Gamma(\Phi)$ has two components coming together at $(-1, -1) \in \Lambda^\flat$ with a normal crossing, while $(-1, 1) \in \Lambda^\flat$ is a generic point, in the sense that $\Gamma(\Phi)$ does not have a self-crossing. The simple zeros of Q_α correspond to parabolic points bounding two distinct rotation belts, and these in turn lie between two arrangements of curves with hyperbolic fixed points.

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