MONOTONICITY OF THE PRINCIPAL PIVOT TRANSFORM

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Abstract. We prove that the principal pivot transform (also known as the partial inverse, sweep operator, or exchange operator in various contexts) maps matrices with positive imaginary part to matrices with positive imaginary part. We show that the principal pivot transform is matrix monotone by establishing Hermitian square representations for the imaginary part and the derivative.

1. Introduction

Suppose that $A \in M_n(\mathbb{C})$ is partitioned into the block matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

and that the block A_{11} is an invertible matrix. The **principal pivot transform** of A is the matrix B given by

$$\mathrm{PPT}(A) = \begin{bmatrix} -A_{11}^{-1} & A_{11}^{-1}A_{12} \\ A_{21}A_{11}^{-1} & A_{22}-A_{21}A_{11}^{-1}A_{12} \end{bmatrix}.$$

Applied to a linear equation $Ax = y$, after partitioning x, y relative to A, the principal pivot transform has the effect of switching the places of the first block of the unknown and the right-hand side and then negating the first block on the right-hand side. That is,

$$A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \text{ if and only if } \mathrm{PPT}(A) \begin{bmatrix} y_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ y_2 \end{bmatrix}. \tag{1.1}$$

We note that some authors use a slightly different version of the principal pivot transform where the top block row is negated. The principal pivot transform has been thoroughly studied in various guises, for example, as the sweep operator [5, 11] and the partial inverse [13, 14]. The PPT is used in many applications (see [10, 12] for a wide list

Date: February 11, 2022.

2020 *Mathematics Subject Classification.* 15A09, 47A56.

Key words and phrases. principal pivot transform, matrix monotone, sweep operator.

† Partially supported by National Science Foundation Mathematical Science Postdoctoral Research Fellowship DMS 1606260 and DMS Analysis Grant 1953963.

‡ Partially supported by National Science Foundation DMS Analysis Grant 2055098 and University of New Haven SRG and Cal Poly startup.

of references including control theory, numerical analysis, and linear regression).

Equation 1.1 implies that

$$\mathrm{PPT}(\mathrm{PPT}(\mathrm{PPT}(\mathrm{PPT}(A)))) = A.$$

Namely, the principal pivot transform algebraically has order 4 and is injective on its domain. Define the **imaginary part** of a matrix A to be

$$\operatorname{Im} A = \frac{A - A^*}{2i},$$

where A^* denotes the adjoint or conjugate transpose of A. Let A, B be self-adjoint matrices. We say that $A \preceq B$ if $B - A$ is positive semidefinite. This ordering on matrices, the so-called Löwner ordering, is related to majorization of eigenvalues and therefore other applications [1].

The PPT is known to have good properties relating to positive definiteness. For example, the PPT preserves P-matrices (which generalize positive semidefinite matrices) under a proper choice of signs (see [12] and [4, Theorem 4.4]). In [10], Poloni and Strabić show that the PPT has a predictable effect on signature. We show that the principal pivot transform is an automorphism of matrices with positive imaginary part.

Theorem 1.1. *The following are true:*

(1) *Suppose A is a block 2 by 2 matrix. If $\operatorname{Im} A$ is positive semidefinite, then $\operatorname{Im} \mathrm{PPT}(A)$ is positive semidefinite.*
(2) *Suppose A, B are self-adjoint matrices of the same size , such that for every $t \in [0, 1]$ the matrix $(1-t)A_{11}+tB_{11}$ is invertible. If $A \preceq B$, then $\mathrm{PPT}(A) \preceq \mathrm{PPT}(B)$.*

Part (1) follows from Proposition 2.1. Part (2) comes integrating a formula for the derivative of the principal pivot transform from Proposition 2.2. (We give a brief proof of (2) after the proof of Proposition 2.2.) In the case where each of the blocks are square matrices, condition (1) implies condition (2) in Theorem 1.1 by the noncommutative Löwner theorem [9, 8, 6, 7]. We note that as the principal pivot transform is an automorphism of the block 2 by 2 matrix "upper half plane", under conjugation by a suitable Cayley transform it is conjugate to an automorphism of block 2 by 2 matrices which was studied in [3]. Matrix monotonicity is important in various contexts including MIMO systems (see, e.g. [15, 2]).

The rather obnoxious assumption in part (2) of Theorem 1.1 that $(1-t)A_{11} + tB_{11}$ is invertible holds trivially if A_{11} and B_{11} are positive definite. The monotonicity of the principal pivot transform on other parts of the domain when such a line segment is contained in the domain is somewhat remarkable. When such a segment is not in the domain, the result fails. For example, the principal pivot transform takes the matrices

$$\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

to each other, but the first is less than the second.

2. Hermitian square representations

The proof relies on the basic observation that if we can write $Y = Z^*XZ$ and X is positive semidefinite, then Y is also positive semidefinite.

Proposition 2.1. *Let A be a square block 2 by 2 matrix such that A_{11} is invertible.*

$$\operatorname{Im}\operatorname{PPT}(A) = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}^* \operatorname{Im} A \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}.$$

Proof. Note

$$2i \operatorname{Im} A = \begin{bmatrix} A_{11}-A_{11}^* & A_{12}-A_{21}^* \\ A_{21}-A_{12}^* & A_{22}-A_{22}^* \end{bmatrix}$$

Now,

$$2i \operatorname{Im} A \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix} = \begin{bmatrix} I-A_{11}^*A_{11}^{-1} & A_{11}^*A_{11}^{-1}A_{12}-A_{21}^* \\ A_{21}A_{11}^{-1}-A_{12}^*A_{11}^{-1} & -A_{21}A_{11}^{-1}A_{12}+A_{12}^*A_{11}^{-1}A_{12}+A_{22}-A_{22}^* \end{bmatrix}.$$

Now considering,

$$\begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}^* = \begin{bmatrix} (A_{11}^*)^{-1} & 0 \\ -A_{12}^*(A_{11}^*)^{-1} & I \end{bmatrix},$$

and calculating,

$$\begin{bmatrix} (A_{11}^*)^{-1} & 0 \\ -A_{12}^*(A_{11}^*)^{-1} & I \end{bmatrix} \begin{bmatrix} I-A_{11}^*A_{11}^{-1} & A_{11}^*A_{11}^{-1}A_{12}-A_{21}^* \\ A_{21}A_{11}^{-1}-A_{12}^*A_{11}^{-1} & -A_{21}A_{11}^{-1}A_{12}+A_{12}^*A_{11}^{-1}A_{12}+A_{22}-A_{22}^* \end{bmatrix},$$

we get

$$\begin{bmatrix} -A_{11}^{-1}+(A_{11}^*)^{-1} & A_{11}^{-1}A_{12}-(A_{11}^*)^{-1}A_{21}^* \\ A_{21}A_{11}^{-1}-A_{12}^*(A_{11}^*)^{-1} & A_{22}-A_{21}A_{11}^{-1}A_{12}-A_{22}^*+A_{12}^*(A_{11}^*)^{-1}A_{21}^* \end{bmatrix},$$

which is exactly $2i \operatorname{Im}\operatorname{PPT}(A)$. □

Proposition 2.2. *Let A, H be like-sized self-adjoint square block 2 by 2 matrices such that A_{11} is invertible.*

$$D\operatorname{PPT}(A)[H] = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}^* H \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}$$

where $D\operatorname{PPT}(A)[H] = \frac{d}{dt}PPT(A+tH)|_{t=0}$.

Thus, if H is positive semidefinite, so is $D\operatorname{PPT}(A)[H]$.

Proof. Evaluate the formula from Proposition 2.1 at $A^{(t)} = A + itH$ to obtain

$$\operatorname{Im} \mathrm{PPT}(A^{(t)}) = \begin{bmatrix} (A^{(t)}_{11})^{-1} & -(A^{(t)}_{11})^{-1}A^{(t)}_{12} \\ 0 & I \end{bmatrix}^* \operatorname{Im} A^{(t)} \begin{bmatrix} (A^{(t)}_{11})^{-1} & -(A^{(t)}_{11})^{-1}A^{(t)}_{12} \\ 0 & I \end{bmatrix}.$$

Dividing by t, we get

$$\frac{\operatorname{Im} \mathrm{PPT}(A^{(t)})}{t} = \begin{bmatrix} (A^{(t)}_{11})^{-1} & -(A^{(t)}_{11})^{-1}A^{(t)}_{12} \\ 0 & I \end{bmatrix}^* \frac{\operatorname{Im} A^{(t)}}{t} \begin{bmatrix} (A^{(t)}_{11})^{-1} & -(A^{(t)}_{11})^{-1}A^{(t)}_{12} \\ 0 & I \end{bmatrix}.$$

Note that

$$\frac{\operatorname{Im} A^{(t)}}{t} = H,$$

and

$$\frac{\operatorname{Im} \mathrm{PPT}(A^{(t)})}{t} = \frac{\mathrm{PPT}(A + itH) - \mathrm{PPT}(A - itH)}{2it}.$$

By taking the limit as t goes to 0, we get

$$D\,\mathrm{PPT}(A)[H] = \begin{bmatrix} A^{-1}_{11} & -A^{-1}_{11}A_{12} \\ 0 & I \end{bmatrix}^* H \begin{bmatrix} A^{-1}_{11} & -A^{-1}_{11}A_{12} \\ 0 & I \end{bmatrix}.$$

□

To see (2) in Theorem 1.1, assume $A \preceq B$. That is, $B - A$ is positive semidefinite. Since $(1-t)A_{11} + tB_{11}$ is invertible for $t \in [0, 1]$, $PPT((1-t)A + tB)$ is well-defined on $[0, 1]$. Now

$$\mathrm{PPT}(B) - \mathrm{PPT}(A) = \int_0^1 D\,\mathrm{PPT}[A + t(B - A)][B - A]\, dt,$$

which is positive semidefinite since $D\,\mathrm{PPT}[A + t(B - A)][B - A]$ is positive semidefinite by Proposition 2.2.

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