Compositional Analysis of Interconnected Systems using Delta Dissipativity

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***Abstract*— This paper extends the notion of delta dissipativity, originally introduced in a game theoretic context, to general interconnections of dynamical systems. The main contribution of this paper is a compositionality result that presents conditions under which a large-scale interconnection of delta dissipative systems is delta dissipative. We adapt this result to also analyze stability and asymptotic stability of equilibrium points for the interconnection. Additionally, we formulate a sum-of-squares program for verifying delta dissipativity of a (polynomial) system. The results are illustrated with examples.**

***Index Terms*— Large-scale systems, Lyapunov methods, Stability of nonlinear systems**

I. INTRODUCTION

Existing computational tools for control synthesis and verification do not scale well to large scale, interconnected systems, such as multi-agent robotic systems, traffic flow networks, and biological networks. Recent advances, such as sum-of-squares (SOS) methods, have made it possible to numerically search for Lyapunov functions and to certify measures of performance; however, these procedures are applicable only to problems of modest size. To broaden the applicability of such procedures to large systems, compositional approaches that infer system-level guarantees from appropriate subsystem properties are becoming widespread. In particular, an approach that makes use of dissipativity properties [1] of the subsystems has been developed and combined with computational tools, as summarized in [2].

An obstacle to complete modularity in the dissipativity approach is that the analysis of the subsystems must be performed with the knowledge of the equilibrium of the interconnected system. However, the value of such an equilibrium depends on all other subsystems; if any subsystem is added, removed, or modified, then the equilibrium has to be calculated again and the analysis repeated. This motivates the development of tools that don't require the knowledge of equilibrium. One such tool is *equilibrium independent dissipativity* (EID), where dissipativity is certified with respect to every point that has the *potential* to become an equilibrium [2]–[4]. Although this notion decouples the analysis of subsystems from the equilibrium, the resulting Lyapunov function for the interconnection is equilibrium-dependent.

An alternative approach, called *delta dissipativity*, has recently emerged in the study of a class of evolutionary dynamics arising in population games [5]. A notion of delta *passivity* was introduced in [6] to characterize convergence to Nash equilibria and was further explored in [7], [8]. A more general notion of delta *dissipativity* was introduced in [9] to broaden the admissible evolutionary dynamics.

Delta dissipativity is complementary to EID and offers several advantages. As demonstrated in the references above, it is applicable in classes of evolutionary dynamics where the construction of a storage function verifying EID remains elusive. Furthermore, unlike EID, which generates an equilibrium-dependent Lyapunov function for the interconnection, delta dissipativity builds a Lyapunov function that vanishes when the vector field defining the dynamics vanishes, thereby avoiding an explicit dependence on the equilibrium. This idea is comparable to – but more general than – Krasovskii's construction of Lyapunov functions that are quadratic in the vector field [10].

The idea of composing Lyapunov functions for interconnections goes back to the early literature on large-scale systems (see e.g. [11]) and has been revisited in [12] with small-gain theorems based on input-to-state stability [13]. However, these studies do not account for unknown equilibrium points.

Other related notions include *incremental dissipativity* [14], [15] and *differential dissipativity* [16], which define storage functions in the extended space of pairs of trajectories, and in the tangent bundle of the manifold on which the trajectories evolve, respectively. In contrast, delta dissipativity defines storage functions on the extended space of state and input variables and is interpreted as dissipativity from the time derivative of the input to that of the output.

The aforementioned studies of delta dissipativity focused on dynamical models in population games, which consist of the interconnection of two subsystems: one describing how the distribution of the population into different strategies evolves in response to the payoff of each strategy, and the other describing how the payoff evolves with the population state. In this paper we expand the scope of this notion from the game theoretic context to a general interconnection of dynamical systems. The main contribution of this paper is a compositionality result: conditions under which the composition of delta dissipative systems is delta dissipative.

In Section II, we define delta dissipativity and provide two examples of classes of systems where delta dissipativity can be readily verified. In Section III, we consider an interconnection of subsystems that each satisfy a delta dissipativity property and provide compositionality conditions for delta dissipativity, as well as stability conditions for the equilibrium point of the interconnection. In Section IV, we

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formulate a SOS optimization problem to search for storage functions to verify delta dissipativity.

II. DELTA DISSIPATIVITY

Consider the dynamical system

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t)) \quad (1)$$

$$\mathbf{y}(t) = h(\mathbf{x}(t), \mathbf{u}(t)) \quad (2)$$

with $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{u}(t) \in \mathbb{R}^m$, $\mathbf{y}(t) \in \mathbb{R}^p$, and continuously differentiable $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$. We use bold letters to denote signals ($\mathbf{x} : \mathbb{R} \to \mathbb{R}^n$) and non-bold letters to denote points ($x \in \mathbb{R}^n$).

Definition 1. *The system (1)-(2) is* ***delta dissipative*** *with supply rate $s : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ if there exists a storage function $S : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ with the following properties:*

Property 1. *$S(x, u) \geq 0$ for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, and*

$$S(x, u) = 0 \;\Leftrightarrow\; f(x, u) = 0$$

Property 2. *For all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $v \in \mathbb{R}^m$,*

$$\nabla_x S(x, u)^\top f(x, u) + \nabla_u S(x, u)^\top v \leq s(v, w(x, u, v))$$

where $w(x, u, v) := \frac{\partial h}{\partial x}(x, u) f(x, u) + \frac{\partial h}{\partial u}(x, u) v$.

We may interpret delta dissipativity as the conventional dissipativity property from the time derivative of the input $\mathbf{u}$ to the time derivative of the output $\mathbf{y}$, that is, dissipativity of the system (1)-(2) augmented with

$$\dot{\mathbf{u}}(t) = \mathbf{v}(t), \quad \dot{\mathbf{y}}(t) = w(\mathbf{x}(t), \mathbf{u}(t), \mathbf{v}(t)),$$

from the new input $\mathbf{v} = \dot{\mathbf{u}}$ to the new output $\dot{\mathbf{y}}$.

In this paper, we restrict our attention to quadratic supply rates of the form

$$s(v, w) = \begin{bmatrix} v \\ w \end{bmatrix}^\top X \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} v \\ w \end{bmatrix}^\top \begin{bmatrix} X^{11} & X^{12} \\ X^{21} & X^{22} \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}, \quad (3)$$

where $X^{11} \in \mathbb{R}^{m \times m}$, and the choice of X defines the type of dissipativity. In the case $m = p$,

$$X = \begin{bmatrix} 0 & \frac{1}{2} I \\ \frac{1}{2} I & -\varepsilon I \end{bmatrix}, \quad \varepsilon \geq 0 \quad (4)$$

corresponds to the notion of delta passivity.

For a memoryless system $\mathbf{y}(t) = h(\mathbf{u}(t))$, we take the storage function to be zero and interpret delta dissipativity as the static inequality

$$s(v, w(u, v)) \geq 0 \quad \forall\, u, v \in \mathbb{R}^m, \quad (5)$$

where $w(u, v) = \frac{\partial h}{\partial u}(u) v$. For a supply rate of the form (3), condition (5) becomes:

$$\begin{bmatrix} I \\ \frac{\partial h(u)}{\partial u} \end{bmatrix}^\top X \begin{bmatrix} I \\ \frac{\partial h(u)}{\partial u} \end{bmatrix} \succeq 0 \quad \forall\, u \in \mathbb{R}^m.$$

Example 1. In this example we show that the system (1)-(2) is delta dissipative with storage function

$$S(x, u) = f(x, u)^\top P f(x, u) \quad (6)$$

if $P = P^\top \succ 0$ satisfies for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$:

$$\begin{bmatrix} P \frac{\partial f}{\partial x}(x, u) + \frac{\partial f}{\partial x}(x, u)^\top P & P \frac{\partial f}{\partial u}(x, u) \\ \frac{\partial f}{\partial u}(x, u)^\top P & 0 \end{bmatrix} \quad (7)$$
$$- \begin{bmatrix} 0 & I \\ \frac{\partial h}{\partial x}(x, u) & \frac{\partial h}{\partial u}(x, u) \end{bmatrix}^\top X \begin{bmatrix} 0 & I \\ \frac{\partial h}{\partial x}(x, u) & \frac{\partial h}{\partial u}(x, u) \end{bmatrix} \preceq 0.$$

First note that $S(x, u)$ is nonnegative definite and vanishes only when $f(x, u) = 0$. Next, if we multiply (7) from the left by $[f(x, u)^\top \; v^\top]$ and from the right by $[f(x, u)^\top \; v^\top]^\top$, and substitute

$$\begin{bmatrix} 0 & I \\ \frac{\partial h}{\partial x}(x, u) & \frac{\partial h}{\partial u}(x, u) \end{bmatrix} \begin{bmatrix} f(x, u) \\ v \end{bmatrix} = \begin{bmatrix} v \\ w(x, u, v) \end{bmatrix},$$

we obtain the inequality

$$f(x, u)^\top \left(P \frac{\partial f}{\partial x}(x, u) + \frac{\partial f}{\partial x}(x, u)^\top P \right) f(x, u)$$
$$+ 2 f(x, u)^\top P \frac{\partial f}{\partial u}(x, u) v \leq \begin{bmatrix} v \\ w(x, u, v) \end{bmatrix}^\top X \begin{bmatrix} v \\ w(x, u, v) \end{bmatrix}.$$

Since the left hand side of this inequality matches $\nabla_x S(x, u)^\top f(x, u) + \nabla_u S(x, u)^\top v$, we conclude delta dissipativity. For a linear system with $f(x, u) = Ax + Bu$, $h(x, u) = Cx + Du$, condition (7) becomes

$$\begin{bmatrix} PA + A^\top P & PB \\ B^\top P & 0 \end{bmatrix} - \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^\top X \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} \preceq 0,$$

which is identical to the standard dissipativity property. Note that the structure of the storage function (6) resembles Krasovskii's construction of Lyapunov functions in the form of a quadratic function of the vector field [10, Exercise 4.10].

The following example constructs a storage function to certify delta passivity for a system with an input nonlinearity. This storage function will be useful for Example 3 below.

Example 2. Consider the dynamical system (1)-(2), with

$$f(x, u) = -\alpha x + g(u), \quad h(x, u) = x, \quad (8)$$

in which $\alpha \geq 0$ and the state dimension n is identical to that of the input and output. We assume $g : \mathbb{R}^n \to \mathbb{R}^n$ is surjective, and $g = \nabla \gamma$ for a strictly convex, continuously differentiable function $\gamma : \mathbb{R}^n \to \mathbb{R}$. Let L be a global Lipschitz constant for $\nabla \gamma$, which we take to be ∞ when $\nabla \gamma$ is not globally Lipschitz. Combined with the convexity of γ, this implies for all $u \in \mathbb{R}^n$, $\hat{u} \in \mathbb{R}^n$,

$$0 \leq (u - \hat{u})^\top (\nabla \gamma(u) - \nabla \gamma(\hat{u})) \leq L \|u - \hat{u}\|^2$$

or, equivalently by [17, Theorem 2.1.5],

$$(u - \hat{u})^\top (\nabla \gamma(u) - \nabla \gamma(\hat{u})) \geq \frac{1}{L} \|\nabla \gamma(u) - \nabla \gamma(\hat{u})\|^2. \quad (9)$$

We now construct a storage function S satisfying

$$\nabla_x S(x, u)^\top f(x, u) + \nabla_u S(x, u)^\top v$$
$$\leq \begin{bmatrix} v \\ f(x, u) \end{bmatrix}^\top \begin{bmatrix} 0 & \frac{1}{2} I \\ \frac{1}{2} I & -\frac{\alpha}{L} I \end{bmatrix} \begin{bmatrix} v \\ f(x, u) \end{bmatrix}, \quad (10)$$

which implies that the system is delta passive with $\varepsilon = \alpha/L$. In constructing S we use the Legendre Transform of γ:

$$\gamma^*(x) = \max_u \left(x^\top u - \gamma(u) \right),$$

which is well defined for all x, since $x^\top u - \gamma(u)$ is strictly concave in u and has a unique maximizer. For each x, the maximizing u satisfies $\nabla\gamma(u) = x$, which has a solution since we assumed surjectivity of $g = \nabla\gamma$. Then the choice

$$S(x,u) = \gamma^*(\alpha x) - (\alpha u^\top x - \gamma(u))$$

satisfies $S(x,u) \geq 0$ by the definition of γ^*. $S(x,u) = 0$ if and only if u is the maximizer of $\alpha x^\top u - \gamma(u)$, which means $\alpha x = \nabla\gamma(u)$ or, equivalently, $f(x,u) = 0$. Note that

$$\begin{aligned} &\nabla_u S(x,u) = -\alpha x + \nabla\gamma(u) = f(x,u) \\ &\nabla_x S(x,u) = \alpha \left(\nabla\gamma^*(\alpha x) - u \right) \\ &\Rightarrow \nabla_x S(x,u)^\top f(x,u) + \nabla_u S(x,u)^\top v = \\ &\alpha \left(\nabla\gamma^*(\alpha x) - u \right)^\top \left(-\alpha x + \nabla\gamma(u) \right) + f(x,u)^\top v, \end{aligned} \tag{11}$$

where we substituted $f(x,u)$ from (8). Next we claim that

$$\left(u - \nabla\gamma^*(\alpha x) \right)^\top \left(\nabla\gamma(u) - \alpha x \right) \geq \frac{1}{L} \| f(x,u) \|^2, \tag{12}$$

which, when substituted back in (11), establishes the delta dissipativity property (10). To see how (12) follows, define $\hat{u} = \nabla\gamma^*(y)$ and note that $y = \nabla\gamma(\hat{u})$ by [18, Theorem 26.5], which states that the inverse function of $\nabla\gamma^*$ is $\nabla\gamma$. Then substitute $\hat{u} = \nabla\gamma^*(y)$ and $\nabla\gamma(\hat{u}) = y$ in (9) to obtain

$$\left(u - \nabla\gamma^*(y) \right)^\top \left(\nabla\gamma(u) - y \right) \geq \frac{1}{L} \| \nabla\gamma(u) - y \|^2. \tag{13}$$

Finally, substitute $y = \alpha x$ in (13) and note from (8) that the right hand side of (13) is then $\frac{1}{L}\|f(x,u)\|^2$. This proves (12) and, thus, the delta dissipativity property (10).

III. COMPOSITIONAL ANALYSIS OF INTERCONNECTED SYSTEMS

Consider the interconnection in Figure 1 with exogenous disturbance input $\mathbf{d}(t) \in \mathbb{R}^m$ and performance output $\mathbf{e}(t) \in \mathbb{R}^p$. The subsystems $G_i,\ i = 1, \ldots, N$, are described by

$$\dot{\mathbf{x}}_i(t) = f_i(\mathbf{x}_i(t), \mathbf{u}_i(t)) \tag{14}$$

$$\mathbf{y}_i(t) = h_i(\mathbf{x}_i(t), \mathbf{u}_i(t)), \tag{15}$$

$\mathbf{x}_i(t) \in \mathbb{R}^{n_i}$, $\mathbf{u}_i(t) \in \mathbb{R}^{m_i}$, $\mathbf{y}_i(t) \in \mathbb{R}^{p_i}$. The matrix M specifies the subsystem inputs and performance output by

$$\begin{bmatrix} u \\ e \end{bmatrix} = M \begin{bmatrix} y \\ d \end{bmatrix} = \begin{bmatrix} M_{uy} & M_{ud} \\ M_{ey} & M_{ed} \end{bmatrix} \begin{bmatrix} y \\ d \end{bmatrix}, \tag{16}$$

where $u := [u_1^\top \cdots u_N^\top]^\top$, $y := [y_1^\top \cdots y_N^\top]^\top$, and the partitioning of M is block conformal. We let $x := [x_1^\top \cdots x_N^\top]^\top$, $f(x,u) := [f_1(x_1,u_1)^\top \cdots f_N(x_N,u_N)^\top]^\top$, $h(x,u) := [h_1(x_1,u_1)^\top \cdots h_N(x_N,u_N)^\top]^\top$, and $n := n_1 + \cdots + n_N$. We assume well-posedness, in the sense that

$$u = M_{uy} h(x,u) + M_{ud} d \tag{17}$$

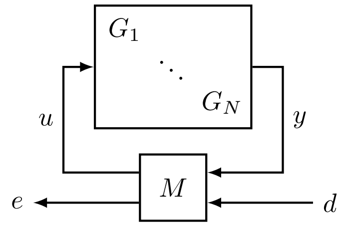

Fig. 1: Interconnected system with exogenous input d and performance output e.

has a unique solution $u = \nu(x,d)$ and, thus, the interconnected system can be written in the monolithic form

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \nu(\mathbf{x}(t), \mathbf{d}(t))) \tag{18}$$

$$\begin{aligned} \mathbf{e}(t) &= M_{ey} h(\mathbf{x}(t), \nu(\mathbf{x}(t), \mathbf{d}(t))) + M_{ed}\mathbf{d}(t) \\ &=: \tilde{h}(\mathbf{x}(t), \mathbf{d}(t)). \end{aligned} \tag{19}$$

Our goal now is to certify the delta dissipativity of the interconnected system with respect to the supply rate

$$s(q,r) = \begin{bmatrix} q \\ r \end{bmatrix}^\top W \begin{bmatrix} q \\ r \end{bmatrix}, \tag{20}$$

where the choice of W defines a performance objective, and q and r play the respective roles of v and w from (3) for the interconnected system.

We assume each subsystem is delta dissipative with a quadratic supply rate defined by a matrix X_i, and we employ the candidate storage function

$$S(x,d) = \sum_{i=1}^N p_i S_i(x_i, \nu_i(x,d)), \tag{21}$$

where $p_i > 0,\ i = 1, \ldots, N$. The theorem below presents a condition under which weights $\{p_i\}_{i=1}^N$ can be found such that (21) serves as a storage function certifying delta dissipativity of the interconnection with respect to the supply rate (20). The main condition (22) is a linear matrix inequality, which can be solved with semidefinite program solvers.

Theorem 1. *Suppose each subsystem (14)-(15) is delta dissipative as in Definition 1, with storage function S_i and quadratic supply rate defined by a matrix X_i. For $j,k = 1,2$, define $\mathbf{X}^{jk}(p_1X_1, \ldots, p_NX_N) =$ blkdiag$(p_1X_1^{jk}, \ldots, p_NX_N^{jk})$ and let*

$$\mathbf{X}(\cdot) = \begin{bmatrix} \mathbf{X}^{11}(\cdot) & \mathbf{X}^{12}(\cdot) \\ \mathbf{X}^{21}(\cdot) & \mathbf{X}^{22}(\cdot) \end{bmatrix}.$$

If there exist $p_i > 0, i = 1, \ldots, N$, such that

$$\begin{bmatrix} \star \end{bmatrix}^\top \begin{bmatrix} \mathbf{X}(p_1X_1, \ldots, p_NX_N) & 0 \\ 0 & -W \end{bmatrix} \begin{bmatrix} M_{uy} & M_{ud} \\ I & 0 \\ 0 & I \\ M_{ey} & M_{ed} \end{bmatrix} \preceq 0, \tag{22}$$

where $[\star]$ is inferred from symmetry, then the interconnection is delta dissipative with respect to the supply rate (20), and (21) is a storage function.

Proof. Suppose there exist $\{p_i\}_{i=1}^N$ such that (22) holds. To show that (21) certifies delta dissipativity with respect to (20), we must show that (21) satisfies Properties 1 and 2 of Definition 1 with input variable d, state equation (18), and output equation (19). Property 1 follows because $S_i \geq 0$ and $p_i > 0$, $i = 1, \ldots, N$, imply $S(x,d) \geq 0$ and

$$S(x,d) = 0 \;\Leftrightarrow\; S_i(x_i, \nu_i(x,d)) = 0,\ \forall\, i$$
$$\Leftrightarrow f_i(x_i, \nu_i(x,d)) = 0,\ \forall\, i \;\Leftrightarrow\; f(x, \nu(x,d)) = 0.$$

The remainder of the proof shows that S satisfies Property 2, i.e., for all $x \in \mathbb{R}^n$, $d \in \mathbb{R}^m$, and $q \in \mathbb{R}^m$,

$$\nabla_x S(x,d)^\top f(x,\nu(x,d)) + \nabla_d S(x,d)^\top q \leq \begin{bmatrix} q \\ r(x,u,q) \end{bmatrix}^\top W \begin{bmatrix} q \\ r(x,u,q) \end{bmatrix}, \tag{23}$$

where $r(x,d,q) := \frac{\partial \tilde{h}}{\partial x}(x,d) f(x,\nu(x,d)) + \frac{\partial \tilde{h}}{\partial d}(x,d) q$. First, we expand the left hand side of this expression as:

$$\nabla_x S(x,d)^\top f(x,\nu(x,d)) = \sum_{i=1}^N p_i \Big(\nabla_{x_i} S_i(x_i,u_i)^\top f_i(x_i,u_i) + \nabla_{u_i} S_i(x_i,u_i)^\top \frac{\partial \nu_i}{\partial x} f(x,u) \Big)\Big|_{u=\nu(x,d)}$$
$$\nabla_d S(x,d)^\top q = \sum_{i=1}^N p_i \Big(\nabla_{u_i} S_i(x_i,u_i)^\top \frac{\partial \nu_i}{\partial d} q \Big)\Big|_{u=\nu(x,d)}.$$

Adding these expressions together, we have

$$\begin{aligned}
&\nabla_x S(x,d)^\top f(x,\nu(x,d)) + \nabla_d S(x,d)^\top q \\
&= \sum_{i=1}^N p_i \Big(\nabla_{x_i} S_i(x_i,u_i)^\top f_i(x_i,u_i) \\
&\qquad + \nabla_{u_i} S_i(x_i,u_i)^\top \Big(\frac{\partial \nu_i}{\partial x} f(x,u) + \frac{\partial \nu_i}{\partial d} q \Big) \Big)\Big|_{u=\nu(x,d)} \\
&\leq \sum_{i=1}^N p_i \begin{bmatrix} v_i \\ w_i(x_i,u_i,v_i)) \end{bmatrix}^\top X_i \begin{bmatrix} v_i \\ w_i(x_i,u_i,v_i)) \end{bmatrix} \\
&= \begin{bmatrix} v(x,d,q) \\ w(x,d,q) \end{bmatrix}^\top \mathbf{X}(p_1 X_1, \ldots, p_N X_N) \begin{bmatrix} v(x,d,q) \\ w(x,d,q) \end{bmatrix},
\end{aligned} \tag{24}$$

where we used the subsystem dissipativity properties in the inequality step, and

$$v(x,d,q) := \frac{\partial \nu}{\partial x}(x,d) f(x,\nu(x,d)) + \frac{\partial \nu}{\partial d}(x,d) q$$
$$w(x,d,q) := \frac{\partial h}{\partial x}(x,u) f(x,u) + \frac{\partial h}{\partial u}(x,u) v(x,d,q)\Big|_{u=\nu(x,d)}.$$

Then, to establish (23), it suffices to show that

$$\begin{bmatrix} v(x,d,q) \\ r(x,d,q) \end{bmatrix} = \begin{bmatrix} M_{uy} & M_{ud} \\ M_{ey} & M_{ed} \end{bmatrix} \begin{bmatrix} w(x,d,q) \\ q \end{bmatrix}. \tag{25}$$

Indeed, if we multiply (22) by $[w^\top \; q^\top]$ from the left and its transpose from the right and use (25) to simplify, then we can upper bound (24) with $\begin{bmatrix} q \\ r \end{bmatrix}^\top W \begin{bmatrix} q \\ r \end{bmatrix}$, which shows (23).

Now we show that (25) holds. Recall from the well-posedness assumption (17) that $\nu(x,d) = M_{uy} h(x,\nu(x,d)) + M_{ud} d$. Then

$$\frac{\partial \nu}{\partial x}(x,d) = M_{uy} \left(\frac{\partial h}{\partial x}(x,u) + \frac{\partial h}{\partial u}(x,u) \frac{\partial \nu}{\partial x}(x,d) \right)\Big|_{u=\nu(x,d)}$$
$$\frac{\partial \nu}{\partial d}(x,d) = M_{uy} \left(\frac{\partial h}{\partial u}(x,u) \frac{\partial \nu}{\partial d}(x,d) \right)\Big|_{u=\nu(x,d)} + M_{ud}.$$

Therefore,

$$\begin{aligned}
v(x,d,q) &= \frac{\partial \nu}{\partial x}(x,d) f(x,\nu(x,d)) + \frac{\partial \nu}{\partial d}(x,d) q \\
&= M_{uy} \Big(\frac{\partial h}{\partial x}(x,u) f(x,u) + \frac{\partial h}{\partial u}(x,u) \frac{\partial \nu}{\partial x}(x,d) f(x,u) \\
&\qquad + \frac{\partial h}{\partial u}(x,u) \frac{\partial \nu}{\partial d}(x,d) q \Big)\Big|_{u=\nu(x,d)} + M_{ud} q \\
&= M_{uy} w(x,d,q) + M_{ud} q.
\end{aligned}$$

Similarly, we have

$$\frac{\partial \tilde{h}}{\partial x}(x,d) = M_{ey} \left(\frac{\partial h}{\partial x}(x,u) + \frac{\partial h}{\partial u}(x,u) \frac{\partial \nu}{\partial x}(x,d) \right)\Big|_{u=\nu(x,d)}$$
$$\frac{\partial \tilde{h}}{\partial d}(x,d) = M_{ey} \left(\frac{\partial h}{\partial u}(x,u) \frac{\partial \nu}{\partial d}(x,d) \right)\Big|_{u=\nu(x,d)} + M_{ed}.$$

Thus, $r(x,d,q) =$

$$\begin{aligned}
&M_{ey} \Big(\frac{\partial h}{\partial x}(x,u) f(x,u) + \frac{\partial h}{\partial u}(x,u) \frac{\partial \nu}{\partial x}(x,d) f(x,u) \\
&\qquad + \frac{\partial h}{\partial u}(x,u) \frac{\partial \nu}{\partial d}(x,d) q \Big)\Big|_{u=\nu(x,d)} + M_{ed} q \\
&= M_{ey} w(x,d,q) + M_{ed} q.
\end{aligned}$$

Hence we have showed (25), concluding the proof. ■

When there are no exogenous inputs ($\mathbf{d} = 0$), we adapt the analysis above to provide a condition for stability.

Corollary 1 (Stability). *Suppose the interconnection (16) with $\mathbf{d} = 0$ admits a unique equilibrium x^* and each subsystem (14)-(15) is delta dissipative as in Definition 1, with storage function S_i and quadratic supply rate defined by a matrix X_i. If there exist $p_i > 0, i = 1, \ldots, N$, such that*

$$\begin{bmatrix} M_{uy} \\ I \end{bmatrix}^\top \mathbf{X}(p_1 X_1, \ldots, p_N X_N) \begin{bmatrix} M_{uy} \\ I \end{bmatrix} \preceq 0, \tag{26}$$

then x^* is stable and $V(x) = S(x,0)$ is a Lyapunov function, with S as in (21).

Proof. Since $\mathbf{d} = 0$, we let $M_{ud} = 0$ without loss of generality. Then, setting $W = 0$, (22) and (26) are equivalent, so the conclusions of Theorem 1 hold for $W = 0$. We will show that $V(x) = S(x,0)$ is a Lyapunov function for the system with $\mathbf{d} = 0$, i.e.,

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \nu(\mathbf{x}(t), 0)).$$

Since S satisfies Property 1 of Definition 1, we have $V(x) \geq 0$ for all $x \in \mathbb{R}^n$ and

$$V(x) = S(x,0) = 0 \Leftrightarrow f(x,\nu(x,0)) = 0 \Leftrightarrow x = x^*. \tag{27}$$

Secondly, since S satisfies Property 2 of Definition 1 with $W = 0$, we have for all $x \in \mathbb{R}^n$, $d \in \mathbb{R}^m$, and $q \in \mathbb{R}^m$,

$$\nabla_x S(x,d)^\top f(x,\nu(x,d)) + \nabla_d S(x,d)^\top q \le 0.$$

Plugging in $d = 0$ and $q = 0$, we have for all $x \in \mathbb{R}^n$,

$$\nabla_x S(x,0)^\top f(x,\nu(x,0)) \le 0. \tag{28}$$

Hence, $V(x) = S(x,0)$ is a Lyapunov function that certifies the stability of x^* [10]. ■

Corollary 2 (Asymptotic Stability)**.** Under the same assumptions of Corollary 1, we can prove *asymptotic* stability of x^* if we replace Property 2 of Definition 1 with Property 2+ below. Furthermore, if $V(x) = S(x,0)$ is radially unbounded, then x^* is globally asymptotically stable.

Property 2+. *For all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $v \in \mathbb{R}^m$,*

$$\begin{aligned}\nabla_x S(x,u)^\top f(x,u) &+ \nabla_u S(x,u)^\top v \\ &\le -\sigma(x,u) + s(v, w(x,u,v)),\end{aligned}$$

where $\sigma : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ satisfies $\sigma(x,u) \ge 0$ for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, and $\sigma(x,u) = 0 \Leftrightarrow f(x,u) = 0$.

Proof. The proof is identical to that of Corollary 1, where now the right hand side of (28) becomes $\sum_{i=1}^N -\sigma_i(x_i, \nu_i(x,0))$. Hence, $\nabla_x S(x,0)^\top f(x,\nu(x,0)) \le 0$ for all $x \in \mathbb{R}^n$ and

$$\begin{aligned}&\nabla_x S(x,0)^\top f(x,\nu(x,0)) = 0 \;\Leftrightarrow\; \sigma_i(x_i,\nu_i(x,0)) = 0 \;\forall\, i \\ &\Leftrightarrow\; f_i(x_i,\nu_i(x,0)) = 0 \;\forall\, i \;\Leftrightarrow\; x = x^*.\end{aligned}$$

Hence, the Lyapunov function $V(x) = S(x,0)$ verifies asymptotic stability of x^*. ■

For an example of Property 2+, consider the system (8) from Example 2. If $\varepsilon > 0$, choose $\hat{\varepsilon} \in (0,\varepsilon)$ and define

$$\sigma(x,u) := (\varepsilon - \hat{\varepsilon})\|f(x,u)\|_2^2. \tag{29}$$

Let $\hat{X}$ be of the form (4) with parameter $\hat{\epsilon}$. Then the system satisfies Property 2+ with supply rate defined by $\hat{X}$.

Now we present an example where a three-state system is decomposed into an interconnection of three single-state systems and a Lyapunov function for the interconnection is constructed using storage functions of the subsystems. The cyclical structure of this network is representative of ring oscillator models in circuits and biology [2], [19]. Although we selected a three-state model for illustration, the method is scalable to any size for which the LMI (26) can be solved.

Example 3. Consider the cyclical feedback interconnection

$$\dot{\mathbf{x}}_1(t) = -\alpha_1 \mathbf{x}_1(t) + \phi_1(\mathbf{x}_3(t)), \tag{30}$$

$$\dot{\mathbf{x}}_2(t) = -\alpha_2 \mathbf{x}_2(t) + \phi_2(\mathbf{x}_1(t)), \tag{31}$$

$$\dot{\mathbf{x}}_3(t) = -\alpha_3 \mathbf{x}_3(t) + \phi_3(\mathbf{x}_2(t)), \tag{32}$$

where $\alpha_i > 0$ and $\phi_i : \mathbb{R} \to \mathbb{R}$ is surjective and strictly decreasing with Lipschitz constant $L_i < \infty$. Using the cyclical structure of the dynamics and the strict decreasing property of the ϕ_i functions, it is simple to show that there exists a unique equilibrium x^*.

To analyze the stability of this equilibrium without having to compute its exact value, we can decompose the system (30)-(32) into three systems of the form (8). Let $h_i(x_i,u_i) = x_i$, $i = 1,2,3$, and let $u = M_{uy} y$, where

$$M_{uy} = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

Define $g_i(u_i) = \phi_i(-u_i),\ i = 1,2,3$. Then each g_i is strictly increasing, so we can find a strictly convex function γ_i satisfying $\nabla \gamma_i = g_i$. Then each subsystem is of the form (8), so it follows from Example 2 that each subsystem is delta dissipative with storage function

$$S_i(x_i,u_i) = \gamma_i^*(\alpha_i x_i) - (\alpha_i u_i^\top x_i - \gamma_i(u_i))$$

and X_i as in (4) with $\varepsilon_i = \frac{\alpha_i}{L_i}$. Then (26) is equivalent to

$$P(M_{uy} - E) + (M_{uy} - E)^\top P \preceq 0, \tag{33}$$

where $P = \mathrm{diag}(p_1,\ldots,p_N)$ and $E = \mathrm{diag}(\epsilon_1,\ldots,\epsilon_N)$. It follows (see [19] or [2, Section 2.3.2]) that $p_i > 0$, $i = 1,2,3$, satisfying (33) exist if and only if $\varepsilon_1\varepsilon_2\varepsilon_3 \ge 1/8$. If this condition holds, then the equilibrium is stable by Corollary 1 and a Lyapunov function is

$$V(x) = \sum_{i=1}^3 p_i S_i(x_i, \nu_i(x)).$$

If $\varepsilon_1\varepsilon_2\varepsilon_3 > 1/8$, we can use the method in Corollary 2 to certify asymptotic stability. Specifically, choose $\hat{\varepsilon}_i \in (0,\varepsilon_i)$ such that $\hat{\varepsilon}_1\hat{\varepsilon}_2\hat{\varepsilon}_3 \ge 1/8$. Each S_i satisfies Property 2+ with σ_i as in (29) and supply rate of the form (4) with parameter $\hat{\varepsilon}_i$. Since $\hat{\varepsilon}_1\hat{\varepsilon}_2\hat{\varepsilon}_3 \ge 1/8$, there exist p_1, p_2, p_3 satisfying (26) so by Corollary 2, x^* is asymptotically stable.

Remark 1. Note that we have a choice when we decompose the system (30)-(32) into three subsystems: we can place the nonlinearity at the input (as in Example 3) or at the output (e.g., $y_1 = \phi_2(x_1)$). When certifying delta dissipativity, Example 2 shows how to construct a storage function when the nonlinearity is at the input, but not when it is at the output. Conversely, when certifying the EID property mentioned in the Introduction, [2] shows how to construct a storage function for an output linearity but not for an input nonlinearity. Decomposing (30)-(32) with the nonlinearity at the output, we can follow the methods of [2] to obtain the Lyapunov function

$$\begin{aligned} V_{\text{EID}}(x) &= \sum_{i=1}^3 p_i V_i(x_i, x_i^*) \\ V_1(x_1,x_1^*) &= \gamma_2(-x_1) - \gamma_2(-x_1^*) + g_2(-x_1^*)(x_1 - x_1^*) \\ V_2(x_2,x_2^*) &= \gamma_3(-x_2) - \gamma_3(-x_2^*) + g_3(-x_2^*)(x_2 - x_2^*) \\ V_3(x_3,x_3^*) &= \gamma_1(-x_3) - \gamma_1(-x_3^*) + g_1(-x_3^*)(x_3 - x_3^*). \end{aligned}$$

Note that, unlike V, V_{EID} depends on the equilibrium x^*. The settings in which these two methods are applicable is an area of further study.

Remark 2. Corollaries 1 and 2 assumed uniqueness of the equilibrium x^* for simplicity. The same idea can be used to establish convergence to a set of equilibria, as the Lyapunov function vanishes on the entire set by construction.

IV. SOS PROGRAM

Consider a system of the form (1)-(2) where f and h are polynomials. Instead of trying to find a storage function by hand, we can use a SOS program to automate the search for a storage function of the form

$$S(x,u) = \psi(x,u)^\top P(x,u)\psi(x,u), \tag{34}$$

where the entries of $P(x,u)$ are polynomial of a fixed order in x and u, and ψ is a user-specified function that has the property $\psi(x,u) = 0 \Leftrightarrow f(x,u) = 0$. One simple choice is $\psi(x,u) = f(x,u)$, in which case the storage function resembles the form of the storage function in Example 1.

In order to certify delta dissipativity, a storage function of this form must satisfy two constraints. First, we require

$$P(x,u) \succ 0 \quad \forall\, x \in \mathbb{R}^n, u \in \mathbb{R}^m.$$

This constraint ensures that $S(x,u)$ satisfies Property 1 of Definition 1. We can encode this condition as a SOS constraint by choosing a small constant $\delta > 0$ and using a dummy variable $l \in \mathbb{R}^n$ as follows [20, Lemma 5]:

$$l^\top (P(x,u) - \delta I)l \in \Sigma[x,u,l], \tag{35}$$

where $\Sigma[\xi]$ is the set of SOS polynomials in ξ. Second, $S(x,u)$ must satisfy Property 2 of Definition 1. We can relax this condition as a SOS constraint as follows:

$$s(v, w(x,u,v)) - \left(\nabla_x S(x,u)^\top f(x,u) + \nabla_n S(x,u)^\top v\right) \in \Sigma[x,u,v], \tag{36}$$

where $w(x,u,v) = \frac{\partial h(x,u)}{\partial x} f(x,u) + \frac{\partial h(x,u)}{\partial u} v$. This is a SOS constraint for any polynomial supply rate s, including the quadratic choice in (3). We combine these constraints into a SOS feasibility program, and we use the toolbox SOSOPT [21] to find $P(x,u)$ satisfying (35) and (36).

Example 4. Consider the dynamical system (1)-(2), with

$$f(x,u) = -x - x^3 + u, \quad h(x,u) = x + u.$$

We aim to certify delta dissipativity for the supply rate defined by (4) with $\varepsilon = 0$. Here, we let $\psi(x,u) = f(x,u)$ and so we search for a storage function of the form $S(x,u) = P(x,u)f(x,u)^2$. When we restrict $P(x,u) = \text{const.}$, the SOS program returns $P = 0.211$, which belongs to the range of feasible values of P in (7) from Example 1. Nonconstant solutions for $P(x,u)$ can also be identified by allowing P to be a polynomial in (x,u).

However, we note that SOS solvers are prone to numerical issues. In particular, there are problems where an equality constraint is implicitly present in the SOS constraint (36). For example, if (36) contains linear terms in v but no quadratic terms in v, then the coefficient of the linear term must equal zero. Identifying these equality constraints and adding them explicitly as constraints in the SOS program can help mitigate numerical issues.

Remark 3 (Global Algorithm)**.** The SOS program above searches for a storage function for a single system. To find a storage function for the interconnection in Figure 1, we can adapt the algorithm from [22] for EID, which uses the alternating direction method of multipliers (ADMM) to separate the search for a storage function into: 1) a global subproblem that searches for candidate subsystem supply rate matrices $\hat{X}_i$ that satisfy (23), which is a semidefinite program; 2) a paralellizable local subproblem that, for each subsystem, searches for a storage function S_i and supply rate matrix X_i that is as close to the proposed supply rate $\hat{X}_i$ as possible. For delta dissipativity, the first step is identical, and the second step is accomplished by modifying the SOS program in the previous section by letting X_i be a decision variable and adding a cost function that penalizes $\|X_i - \hat{X}_i\|$.

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