

Fluctuations of Brownian motions on \mathbb{GL}_N

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Abstract. We consider a two parameter family of unitarily invariant diffusion processes on the general linear group \mathbb{GL}_N of $N \times N$ invertible matrices, that includes the standard Brownian motion as well as the usual unitary Brownian motion as special cases. We prove that all such processes have Gaussian spectral fluctuations in high dimension with error of order $O(\frac{1}{N})$; this is in terms of the finite dimensional distributions of the process under a large class of test functions known as trace polynomials. We give an explicit characterization of the covariance of the Gaussian fluctuation field, which can be described in terms of a fixed functional of three freely independent free multiplicative Brownian motions. These results generalize earlier work of Lévy and Maïda, and Diaconis and Evans, on unitary groups.

Résumé. Nous considérons une famille à deux paramètres de processus unitairement invariants sur le groupe général linéaire \mathbb{GL}_N des matrices $N \times N$ inversibles, contenant comme cas particuliers le mouvement brownien standard ainsi que le mouvement brownien unitaire. Nous montrons que tous ces processus ont des fluctuations spectrales gaussiennes d'ordre $O(\frac{1}{N})$ en grande dimension ; ces fluctuations sont établies pour les distributions finies-dimensionnelles du processus sous une classe étendue de fonctions tests appelées polynômes à trace. Nous donnons une expression explicite de la covariance du champ gaussien des fluctuations en fonction d'une fonctionnelle particulière de trois mouvements browniens multiplicatifs librement indépendants. Ces résultats généralisent les précédents travaux de Lévy et Maïda, et de Diaconis et Evans, sur les groupes unitaires.

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1. Introduction

This paper is concerned with the spectral fluctuations of Brownian motions on the general linear groups $\mathbb{GL}_N = \mathrm{GL}(N, \mathbb{C})$, in the large- N limit.

Let \mathbb{M}_N denote the space of $N \times N$ complex matrices. A *random matrix ensemble* or *model* is a sequence of random variables $(B^N)_{N \geq 1}$ such that $B^N \in \mathbb{M}_N$. The first phenomenon typically studied is the convergence in noncommutative distribution (cf. Section 2.4) of B^N , meaning that for each noncommutative polynomial P in two variables, we ask for convergence of $\mathbb{E}[\mathrm{tr}(P(B^N, B^{N*}))]$, where tr is the normalized trace (so that $\mathrm{tr}(I_N) = 1$). In the special case that $B^N = (B^N)^*$ is self-adjoint, this is morally (and usually literally) equivalent to weak convergence in expectation of the empirical spectral distribution of B^N : the random probability measure placing equal masses at each of the random eigenvalues of the matrix. The prototypical example here is Wigner's semicircle law [35]: if B^N is a Wigner ensemble (meaning it is self-adjoint and the upper triangular entries are i.i.d. normal random variables with mean 0 and variance $\frac{1}{N}$) then as $N \rightarrow \infty$ the empirical spectral distribution converges to $\frac{1}{2\pi} \sqrt{(4-x^2)_+} dx$. In fact, the weak convergence is not only in expectation but almost sure.

For non-self-adjoint (and more generally non-normal) ensembles that cannot be characterized by their eigenvalues alone, the noncommutative distribution is the right object to consider. As with Wigner's law, in most cases, we have the stronger result of almost sure convergence of the random variable $\mathrm{tr}(P(B^N, B^{N*}))$ to its asymptotic mean. It is therefore natural to ask for the corresponding central limit theorem: what is the rate of convergence to the mean, and what is the noise profile that remains? More precisely, consider the random variables

$$\mathrm{tr}(P(B^N, B^{N*})) - \mathbb{E}[\mathrm{tr}(P(B^N, B^{N*}))]$$

for each noncommutative polynomial P ; these are known as the spectral *fluctuations*. The question is: what is their order of magnitude, and when appropriately renormalized, what is their limit in distribution as $N \rightarrow \infty$? The standard scaling for this kind of central limit theorem in random matrix theory is well-known to be $\frac{1}{N}$ instead of the classical $\frac{1}{\sqrt{N}}$ (see the fundamental work of Johansson [23]). Thus far, it was known that

$$N(\mathrm{tr}(P(B^N, B^{N*})) - \mathbb{E}[\mathrm{tr}(P(B^N, B^{N*}))])$$

is asymptotically Gaussian when

- B^N is a Wigner random matrix (Cabanal–Duvillard, [8]);
- B^N is an iid (non-Hermitian) matrix (Nourdin and Peccati, [33]; Nouredine, [32]);
- B^N is a unitary random matrix whose distribution is the Haar measure (Diaconis and Evans, [17]);
- B^N is a unitary random matrix arising from a Brownian motion on the unitary group (Lévy and Maïda, [28]) or the orthogonal group (Dahlqvist, [14,15]).

Remark 1.1. The existence of Gaussian fluctuations of a random matrix model is referred to as a *second order distribution* in the work of Mingo, Śniady, and Speicher; cf. [29,30], in which the authors gave the corresponding diagrammatic combinatorial theory of fluctuations. The similar but more complicated combinatorial approach to fluctuations for Haar unitary ensembles was done by Collins and Śniady in [10,12], where it goes under the name *Weingarten calculus*. The more recent work of Dahlqvist [13–15] follows these ideas to provide the combinatorial framework for the finite-time heat kernels on classical compact Lie groups. Since \mathbb{GL}_N is not compact, and consists of mostly non-normal matrices, the approaches of Lévy–Maïda and Dahlqvist do not apply in our setting.

Our main result is of this type, when B^N is sampled from a two-parameter family of random matrix ensembles that may rightly be called *Brownian motions* on \mathbb{GL}_N . Fix $r, s > 0$, and following [25], we will define (in Section 2.1) an (r, s) -Brownian motion $(B_{r,s}^N(t))_{t \geq 0}$ on \mathbb{GL}_N for each dimension $N \geq 1$. This family encompasses the two most well-studied Brownian motions on invertible matrices: the canonical Brownian motion $G^N(t) \equiv B_{\frac{1}{2}, \frac{1}{2}}^N(t)$ on \mathbb{GL}_N , and the canonical Brownian motion $U^N(t) \equiv B_{1,0}^N(t)$ on the unitary group \mathbb{U}_N . These processes are given as solutions to matrix stochastic differential equations

$$dG^N(t) = G^N(t) dZ^N(t), \quad dU^N(t) = iU^N(t) dX^N(t) - \frac{1}{2}U^N(t) dt,$$

where the entries of $Z^N(t)$ are i.i.d. complex Brownian motions of variance $\frac{t}{N}$, and $X^N(t) = \sqrt{2}\Re(Z^N(t))$. The study of the convergence in noncommutative distribution of $U^N(t)$ was completed by Biane [4], and the case of $G^N(t)$ (for fixed $t > 0$) was completed by the first author [9]; the second author introduced the general processes $B_{r,s}^N(t)$ in [25,26] and proved they converge (as processes) a.s. in noncommutative distribution to the relevant free analog, *free multiplicative (r, s) -Brownian motion* (cf. Section 2.4). There has been significant recent interest in these processes – in particular on the large- N limits of their eigenvalue distributions, cf. [19–21]. This naturally leads to the question of the spectral fluctuations of all these processes, which we answer in our Main Theorem 3.3 and Corollary 3.6. We summarize a slightly simplified form of the result here as Theorem 1.2.

Theorem 1.2. *Let $(B_{r,s}^N(t))_{t \geq 0}$ be an (r, s) -Brownian motion on \mathbb{GL}_N . Let $n \in \mathbb{N}$ and $t_1, \dots, t_n \geq 0$; set $\mathbf{T} = (t_1, \dots, t_n)$, and let $B_{r,s}^N(\mathbf{T}) = (B_{r,s}^N(t_1), \dots, B_{r,s}^N(t_n))$. Let P_1, \dots, P_k be noncommutative polynomials in $2n$ variables, and define the random variables*

$$X_j = N[\mathrm{tr}(P_j(B_{r,s}^N(\mathbf{T}), B_{r,s}^N(\mathbf{T})^*)) - \mathbb{E}\mathrm{tr}(P_j(B_{r,s}^N(\mathbf{T}), B_{r,s}^N(\mathbf{T})^*))], \quad 1 \leq j \leq k. \quad (1.1)$$

Then, as $N \rightarrow \infty$, (X_1, \dots, X_k) converges in distribution to a multivariate centered Gaussian.

As mentioned, Theorem 1.2 generalizes the main theorem [28, Theorem 2.6] to general $r, s > 0$ from the $(r, s) = (1, 0)$ case considered there. In fact, even when $(r, s) = (1, 0)$ this is a significant generalization, as the fluctuations proved in [28] were for a single time t – i.e. for a heat-kernel distributed random matrix – while we prove the optimal result for the full process – i.e. for all finite-dimensional distributions.

Remark 1.3.

- (1) In fact, [28, Theorem 8.2] does give a partial generalization to multiple times, in the sense that the argument of P_j in (1.1) is allowed to depend on $B_{1,0}^N(t_j)$ for a j -dependent time; however, it must still be a function of Brownian motion at a single time. Our generalization allows full consideration of all finite-dimensional distributions.
- (2) To be fair, [28] yields Gaussian fluctuations for a larger class of single-variable test functions. In the case of a single time t , the random matrix $U^N(t)$ is normal, and hence ordinary functional calculus makes sense; the fluctuations in [28] extend beyond polynomial test functions to C^1 functions with Lipschitz derivative on the unit circle. Such a generalization is impossible for the generically non-normal matrices in \mathbb{GL}_N .

Theorem 3.3 actually gives a further generalization of Theorem 1.2, as the class of test functions is not just restricted to traces of polynomials, but the much larger algebra of *trace polynomials*, cf. Section 2.2. That is, we may consider more general functions of the form $Y_j = \text{tr}(P_j^1) \cdots \text{tr}(P_j^n)$ (or linear combinations thereof); then the result of Theorem 1.2 applies to the fluctuations $X_j = N[Y_j - \mathbb{E}(Y_j)]$ as well.

Remark 1.4. Moreover, Theorem 3.3 shows that the difference between any mixed moment in X_1, \dots, X_k and the corresponding mixed moment of the limit Gaussian distribution is $O(\frac{1}{N})$. This implies that, in the language of [30], the random matrices $B_{r,s}^N(\mathbf{T})$ possess a second order distribution. (Note we normalize the trace, while [30] uses the unnormalized trace, which accounts for the apparent discrepancy in normalizations.) Since the random matrices $B_{r,s}^N(t)$ are unitarily invariant for each t , it then follows from [29, Theorem 1] that the increments of $(B_{r,s}(t))_{t \geq 0}$ are *asymptotically free of second order*. From the second order distribution of $B_{r,s}^N(\mathbf{T})$ and the unitary invariance, it is also possible to extend the fluctuations from traces of polynomials to more general linear functions of the entries of the matrices. One can use for example [2, Corollary 2.6], and deduce that, if P is any non-commutative polynomial in $2n$ variables (where $\mathbf{T} = (t_1, \dots, t_n)$), then each of the *entries* of the matrix $P(B^N(\mathbf{T}), B^N(\mathbf{T})^*)$ have Gaussian fluctuations.

We can also explicitly describe the covariance of the fluctuations, and thus completely characterize them. The full result is spelled out in Theorem 4.3. Here we state only one result of Corollary 4.5 (which already elucidates how the covariance extends from the unitary $(r, s) = (1, 0)$ case).

Theorem 1.5. *Let $(b_t)_{t \geq 0}$, $(c_t)_{t \geq 0}$, and $(d_t)_{t \geq 0}$ be freely independent free multiplicative (r, s) -Brownian motions in a tracial noncommutative probability space (\mathcal{A}, τ) (for definitions, see Section 2.4). As in Theorem 1.2, let $n = 1$ and $\mathbf{T} = T$, and let $P_1, \dots, P_k \in \mathbb{C}[X]$ be ordinary one-variable polynomials. Then the fluctuations associated to the random variables $\text{tr } P_1, \dots, \text{tr } P_n$ have an asymptotic centered Gaussian distribution with covariance $[\sigma_T(i, j)]_{1 \leq i, j \leq k}$, where*

$$\sigma_T(i, j) = (r + s) \int_0^T \tau \left[\partial P_i(b_t c_{T-t}) (\partial P_j(b_t d_{T-t}))^* \right] dt. \quad (1.2)$$

Here ∂P denotes the derivative of P relative to the unit circle:

$$\partial P(z) = \lim_{\theta \rightarrow 0} \frac{P(ze^{i\theta}) - P(z)}{\theta}.$$

Remark 1.6. In [28], the authors denoted what we call ∂P by P' ; we prefer different notation that is less likely to be confused with the ordinary derivative $\frac{dP}{dz}$ for the polynomial P . In terms of the ordinary derivative, $\partial P(z) = iz \frac{dP}{dz}$.

Eq. (1.2) generalizes [28, Theorem 2.6]. As pointed out there, in the case of the unitary Brownian motion, i.e. $(r, s) = (1, 0)$, the covariances converge as $T \rightarrow \infty$ to the Sobolev $H_{1/2}$ inner-product of the involved polynomials, reproducing the main result of [17] (as it must, since the heat kernel measure on \mathbb{U}_N converges uniformly and exponentially fast to the Haar measure in the large time limit). Theorem 1.5 above, and the more general Theorem 4.3 below, show that, for the general (r, s) -Brownian motions, and for more general trace polynomial test functions, the covariance can always be described by such an integral, involving three freely independent free multiplicative Brownian motions in an input function built out of the *carré du champ* intertwining operator determined by the (r, s) -Laplacian on \mathbb{GL}_N , cf. Section 3.1.

Let us say a few words about the notation used in the formulation of Theorem 3.3. In the first author's paper [9] and the second author's papers [18, 25, 26], two different formalisms were developed to handle general trace polynomial functions. Concretely, two different spaces were defined, namely $\mathbb{C}\{X_j, X_j^* : j \in J\}$ and $\mathcal{P}(J)$; in Theorem 1.2, $J = \{1, \dots, k\}$. Each space leads to a functional calculus adapted to linear combinations of functions from \mathbb{GL}_N^J to \mathbb{C} of the form

$$(G_j)_{j \in J} \mapsto \text{tr}(P_1(G_j, G_j^* : j \in J)) \cdots \text{tr}(P_k(G_j, G_j^* : j \in J)),$$

where P_1, \dots, P_k are noncommutative polynomials in $(G_j)_{j \in J}$ and their adjoints. In the [Appendix](#), we investigate the relationship between these two spaces, demonstrating an explicit algebra isomorphism between a subspace of $\mathbb{C}\{X_j, X_j^*: j \in J\}$ and $\mathcal{P}(J)$ for a given index set J . For notational convenience, most of the calculations throughout this paper (in particular in the proof of Theorem 3.3) are expressed using the space $\mathcal{P}(J)$, but all the results and proofs of this article can be transposed from $\mathcal{P}(J)$ to $\mathbb{C}\{X_j, X_j^*: j \in J\}$ without major modifications.

The rest of the paper is organized as follows. In Section 2, we give the definition of the (r, s) -Brownian motion as well as the definitions of $\mathcal{P}(J)$, and we recall some results from [9,25,26]. Section 3 provides the full statement of our main result Theorem 3.3, an abstract description of the limit covariance matrix, and the proof of Theorem 3.3. In Section 4, we give an alternative description of the limit covariance, using three noncommutative processes in the framework of free probability, extending the results in [28] from the unitary case to the general linear case, and beyond to all (r, s) . Finally, the [Appendix](#) defines the equivalent abstract space $\mathbb{C}\{X_j, X_j^*: j \in J\}$ to encode trace polynomial functional calculus, and investigates the relationship between $\mathbb{C}\{X_j, X_j^*: j \in J\}$ and $\mathcal{P}(J)$.

2. Background

In this section, we briefly describe the basic definitions and tools used in this paper. Section 2.1 discusses Brownian motions on \mathbb{GL}_N (including the Brownian motion on \mathbb{U}_N as a special case). Section 2.2 addresses trace polynomial functions, and the two (equivalent) abstract intertwining spaces used to compute with them. Section 2.3 states the main structure theorem for the Laplacian that is used to prove the optimal asymptotic results herein. Finally, Section 2.4 gives a brief primer on free multiplicative Brownian motion. For greater detail on these topics, the reader is directed to the authors' previous papers [9,18,25,26].

2.1. Brownian motions on \mathbb{GL}_N

Fix $r, s > 0$ throughout this discussion. Define the real inner product $\langle \cdot, \cdot \rangle_{r,s}^N$ on \mathbb{M}_N by

$$\langle A, B \rangle_{r,s}^N = \frac{1}{2} \left(\frac{1}{s} + \frac{1}{r} \right) N \Re \operatorname{Tr}(AB^*) + \frac{1}{2} \left(\frac{1}{s} - \frac{1}{r} \right) N \Re \operatorname{Tr}(AB).$$

As discussed in [25], this two-parameter family of metrics encompasses real inner products on $\mathbb{M}_N = \operatorname{Lie}(\mathbb{GL}_N)$ that are invariant under conjugation by \mathbb{U}_N in a strong sense that is natural in our context, and so we restrict our attention to diffusion processes adapted to these metrics. An (r, s) -Brownian motion on \mathbb{GL}_N is a diffusion process starting at the identity and with generator $\frac{1}{2} \Delta_{r,s}^N$, where $\Delta_{r,s}^N$ is the Laplace–Beltrami operator on \mathbb{GL}_N for the left-invariant metric induced by $\langle \cdot, \cdot \rangle_{r,s}^N$. More concretely, if we fix any orthonormal basis $\beta_{r,s}^N$ of \mathbb{M}_N for the inner-product $\langle \cdot, \cdot \rangle_{r,s}^N$, we have

$$\Delta_{r,s}^N = \sum_{\xi \in \beta_{r,s}^N} \partial_\xi^2,$$

where, for $\xi \in \mathbb{M}_N$, ∂_ξ denotes the induced left-invariant vector field on \mathbb{GL}_N :

$$(\partial_\xi f)(G) = \left. \frac{d}{dt} \right|_{t=0} f(Ge^{t\xi}), \quad G \in \mathbb{GL}_N, f \in C^\infty(\mathbb{GL}_N). \quad (2.1)$$

The (r, s) -Brownian motion $B_{r,s}^N(t)$ may also be seen as the solution of a stochastic differential equation, cf. [25, Section 2.1]. Let $W_{r,s}^N(t)$ denote the diffusion on \mathbb{M}_N determined by the (r, s) -metric; in other words, let $W_\xi(t)$ be i.i.d. standard \mathbb{R} -valued Brownian motions for $\xi \in \beta_{r,s}^N$, and take

$$W_{r,s}^N(t) = \sum_{\xi \in \beta_{r,s}^N} W_\xi(t) \xi.$$

This can also be expressed in terms of standard GUE^N -valued Brownian motions:

$$W_{r,s}^N(t) = \sqrt{r}iX^N(t) + \sqrt{s}Y^N(t), \quad (2.2)$$

where $X^N(t)$ and $Y^N(t)$ are independent Hermitian matrices, with all i.i.d. upper triangular entries that are complex Brownian motions of variance $\frac{t}{N}$ above the main diagonal and real Brownian motions of variance $\frac{t}{N}$ on the main diagonal. Then the (r, s) -Brownian motion on \mathbb{GL}_N is the unique (strong) solution of the stochastic differential equation

$$dB_{r,s}^N(t) = B_{r,s}^N(t) dW_{r,s}^N(t) - \frac{1}{2}(r-s)B_{r,s}^N(t) dt \quad (2.3)$$

with $B_{r,s}^N(0) = I_N$, cf. [25, Equation (2.10)].

Fix an index set J ; in this paper J will usually be finite. For all $j \in J$, let $B_{r,s}^{j,N} = (B_{r,s}^{j,N}(t))_{t \geq 0}$ be independent (r, s) -Brownian motions¹ on \mathbb{GL}_N . Set $B^N = (B_{r,s}^{j,N})_{j \in J}$, which is the family of independent (r, s) -Brownian motions on \mathbb{GL}_N indexed by J . The process $(B^N(t))_{t \geq 0}$ is therefore a diffusion process on \mathbb{GL}_N^J . More precisely, $(B^N(t))_{t \geq 0}$ is a Brownian motion on the Lie group \mathbb{GL}_N^J for the metric $(\langle \cdot, \cdot \rangle_{r,s}^N)^{\otimes J}$. The reader is directed to [25, Section 3.1] for a discussion of the Laplace operators on \mathbb{GL}_N^J for the metric $(\langle \cdot, \cdot \rangle_{r,s}^N)^{\otimes J}$. The degenerate $(r, s) = (1, 0)$ case gives the usual Laplacian on \mathbb{U}_N^J , while $(r, s) = (\frac{1}{2}, \frac{1}{2})$ yields the canonical Laplacian on \mathbb{GL}_N^J (induced by the scaled Hilbert–Schmidt inner product on $\mathbb{M}_N = \text{Lie}(\mathbb{GL}_N)$).

For each $j \in J$, let Δ_j^N denote the Laplacian on the j th factor of \mathbb{GL}_N in \mathbb{GL}_N^J . That is to say,

$$\Delta_j^N = \sum_{\xi_j \in \beta_{r,s}^N} \partial_{\xi_j}^2,$$

where $\beta_{r,s}^N$ is an orthonormal basis of \mathbb{M}_N for the inner product $\langle \cdot, \cdot \rangle_{r,s}^N$, and for all $\xi_j \in \beta_{r,s}^N$, ∂_{ξ_j} is the left-invariant vector field which acts only on the j th component of \mathbb{GL}_N^J . For $j \in J$, let $t_j \geq 0$, and set $\mathbf{T} = (t_j)_{j \in J}$. We consider the operator

$$\mathbf{T} \cdot \Delta^N = \sum_{j \in J} t_j \Delta_j^N. \quad (2.4)$$

Definition 2.1. For J finite, denote by $(B^N(t\mathbf{T}))_{t \geq 0}$ the diffusion process on \mathbb{GL}_N^J with generator $\frac{1}{2}\mathbf{T} \cdot \Delta^N$.

We could write down a stochastic differential equation for $B^N(t\mathbf{T})$ similar to (2.3); for our purposes, we only need the fact that it is a diffusion process.

A common computational tool used throughout [9, 18, 25, 26] is the collection of so-called “magic formulas.” In the present context, the form needed is as follows; cf. [25, Equations (2.7) and (3.6)].

Proposition 2.2. Let $\beta_{r,s}^N$ be any orthonormal basis of \mathbb{M}_N for the inner product $\langle \cdot, \cdot \rangle_{r,s}^N$. Then, for any $A, B \in \mathbb{M}_N$, we have

$$\sum_{\xi \in \beta_{r,s}^N} \text{tr}(\xi A) \text{tr}(\xi B) = \sum_{\xi \in \beta_{r,s}^N} \text{tr}(\xi^* A) \text{tr}(\xi^* B) = \frac{1}{N^2}(s-r) \text{tr}(AB)$$

and

$$\sum_{\xi \in \beta_{r,s}^N} \text{tr}(\xi^* A) \text{tr}(\xi B) = \frac{1}{N^2}(s+r) \text{tr}(AB).$$

2.2. The space $\mathcal{P}(J)$

Let J be an index set as above, and let $\mathbf{A} = (A_j)_{j \in J}$ be a collection of matrices in \mathbb{M}_N . A *trace polynomial* function on \mathbb{M}_N is a linear combination of functions of the form

$$\mathbf{A} \mapsto P_0(\mathbf{A}) \text{tr}(P_1(\mathbf{A})) \text{tr}(P_2(\mathbf{A})) \cdots \text{tr}(P_m(\mathbf{A}))$$

for some finite m , where $P_1, \dots, P_m \in \mathbb{C}$ are noncommutative polynomials in $J \times \{1, *\}$ variables (i.e. the polynomials may depend explicitly on A_j and A_j^* for all $j \in J$). Such functions arise naturally in our context: applying the operator

¹We could vary the parameters r, s with $j \in J$ as well, with only trivial modifications to the following; at present, we do not see any advantage in doing so.

$\mathbf{T} \cdot \Delta^N$ to the smooth function $\mathbf{A} \mapsto Q(\mathbf{A})$ for any noncommutative polynomial Q generally results in a trace polynomial function. The vector space of trace polynomial functions is closed under the action of $\mathbf{T} \cdot \Delta^N$, and this is a motivation for defining abstract spaces which encodes them. In [9] and in [18,25], two different spaces are defined, namely $\mathbb{C}\{X_j, X_j^* : j \in J\}$ and $\mathcal{P}(J)$. In this section, we present the space $\mathcal{P}(J)$, and the relation between $\mathbb{C}\{X_j, X_j^* : j \in J\}$ and $\mathcal{P}(J)$ can be found in the [Appendix](#).

First we give the definition of the space $\mathcal{P}(J)$. Let $\mathcal{E}(J) = \bigcup_{n \geq 1} (J \times \{1, *\})^n$ be the set of all words whose letters are pairs of the form $(j, 1)$, or $(j, *)$ for some $j \in J$. Let $\mathbf{v}_J = \{v_\varepsilon : \varepsilon \in \mathcal{E}(J)\}$ be commuting variables indexed by all such words, and let

$$\mathcal{P}(J) \equiv \mathbb{C}[\mathbf{v}_J]$$

be the algebra of *commutative* polynomials in these variables. That is, $\mathcal{P}(J)$ is the vector space with basis 1 together with all monomials

$$v_{\varepsilon(1)} \cdots v_{\varepsilon(k)}, \quad k \in \mathbb{N}, \varepsilon^{(1)}, \dots, \varepsilon^{(k)} \in \mathcal{E}(J),$$

and the (commutative) product on $\mathcal{P}(J)$ is the standard polynomial product.

Remark 2.3. One can think of $\mathcal{P}(J)$ as a particular framework of noncommutative functional calculus. Instead of considering tensor products of $\mathbb{C}\langle X_j, X_j^* : j \in J \rangle$ as usual in free probability, we consider *symmetric* tensor products of $\mathbb{C}\langle X_j, X_j^* : j \in J \rangle$, or equivalently, commutative polynomials in words in $(j, 1)$, or $(j, *)$. It turns out that the commutativity of the product in $\mathcal{P}(J)$ is very convenient in the forthcoming computations.

We present now the following notions of *degree*, *evaluation* and *conjugation* (see [25] for a detailed presentation):

- In [25, Definition 3.2], the notion of *degrees* of elements in the space $\mathcal{P}(J)$ is defined:

$$\deg(v_{\varepsilon(1)} \cdots v_{\varepsilon(k)}) = |\varepsilon^{(0)}| + \cdots + |\varepsilon^{(k)}|,$$

where $|\varepsilon|$ is the length of the string ε .

- Let (\mathcal{A}, τ) be a noncommutative probability space (cf. Section 2.4). For each $\varepsilon = ((j_1, \varepsilon_1), \dots, (j_n, \varepsilon_n))$, there is an evaluation function $[v_\varepsilon]_{(\mathcal{A}, \tau)} : \mathcal{A}^J \rightarrow \mathbb{C}$ given, for each $\mathbf{a} = (a_j)_{j \in J} \in \mathcal{A}^J$, by

$$[v_\varepsilon]_{(\mathcal{A}, \tau)}(\mathbf{a}) = \tau(a_{j_1}^{\varepsilon_1} \cdots a_{j_n}^{\varepsilon_n}).$$

Note, the $*$ is no longer a formal symbol here: a_j^* means the adjoint of a_j in \mathcal{A} . More generally, for all $P \in \mathcal{P}(J) = \mathbb{C}[\mathbf{v}_J]$, we define $[P]_{(\mathcal{A}, \tau)} : \mathcal{A}^J \rightarrow \mathbb{C}$ by saying that, for all $\mathbf{a} \in \mathcal{A}^J$, the maps $P \mapsto [P]_{(\mathcal{A}, \tau)}(\mathbf{a})$ are algebra homomorphisms from $\mathcal{P}(J) = \mathbb{C}[\mathbf{v}_J]$ to \mathbb{C} . Let us emphasize that this implies the following commutativity between the evaluation and the product: for all polynomials $P, Q \in \mathcal{P}(J)$ and $\mathbf{a} \in \mathcal{A}^J$, we have

$$[PQ]_{(\mathcal{A}, \tau)}(\mathbf{a}) = [P]_{(\mathcal{A}, \tau)}(\mathbf{a}) \cdot [Q]_{(\mathcal{A}, \tau)}(\mathbf{a}).$$

In the particular case where $(\mathcal{A}, \tau) = (\mathbb{GL}_N, \text{tr})$, we will simply denote the map $[P]_{(\mathcal{A}, \tau)}$ by $[P]_N$. We finally remark that if $\mathbf{a} = (a_j)_{j \in J}$ with $a_j = 1_{\mathcal{A}}$ for all $j \in J$, then $[P]_{(\mathcal{A}, \tau)}(\mathbf{a})$ does not depend on the space (\mathcal{A}, τ) , and we will simply denote it by

$$P(\mathbf{1}) \equiv [P]_{(\mathcal{A}, \tau)}(\mathbf{a}).$$

- There is a natural notion of conjugation on $\mathcal{P}(J)$: P^* is the result of taking complex conjugates of all coefficients, and reversing $1 \leftrightarrow *$ in all indices. In terms of evaluation as a trace polynomial function, we have $[P^*]_N = \overline{[P]_N}$, cf. [25, Lemma 3.17].

2.3. Computation of the heat kernel

We are now able to see how the Laplacian acts on the space of trace polynomial functions (i.e. functions on \mathbb{M}_N given by evaluations $[P]_N$ of $P \in \mathcal{P}(J)$).

Theorem 2.4. [26, Theorems 3.8 and 3.9] *Let \mathbf{T} be as in (2.4) above. There exist two linear operators $\mathcal{D}^{\mathbf{T}}$ and $\mathcal{L}^{\mathbf{T}}$ on $\mathcal{P}(J)$, independent from N , such that:*

1. $\mathcal{D}^{\mathbf{T}}$ is a first-order operator, i.e. for all $P, Q \in \mathcal{P}(J)$, $\mathcal{D}^{\mathbf{T}}(PQ) = \mathcal{D}^{\mathbf{T}}(P)Q + P\mathcal{D}^{\mathbf{T}}(Q)$;
2. $\mathcal{L}^{\mathbf{T}}$ is a second-order operator, i.e. for all $P, Q, R \in \mathcal{P}(J)$,

$$\mathcal{L}^{\mathbf{T}}(PQR) = \mathcal{L}^{\mathbf{T}}(PQ)R + P\mathcal{L}^{\mathbf{T}}(QR) + \mathcal{L}^{\mathbf{T}}(PR)Q - \mathcal{L}^{\mathbf{T}}(P)QR - P\mathcal{L}^{\mathbf{T}}(Q)R - PQ\mathcal{L}^{\mathbf{T}}(R);$$

3. For all $P \in \mathcal{P}(J)$, $(\mathbf{T} \cdot \Delta^N)([P]_N) = [(\mathcal{D}^{\mathbf{T}} + \frac{1}{N^2}\mathcal{L}^{\mathbf{T}})(P)]_N$.

Remark 2.5. In [26, Section 3.3], there is an inductive definition of $\mathcal{D}^{\mathbf{T}}$ and $\mathcal{L}^{\mathbf{T}}$ which are denoted similarly. In [9, Sections 4.1 and 4.2], there is an explicit definition of $\mathcal{D}^{\mathbf{T}}$ in the simple cases of $J = \{1\}$ and $(r, s) = (1, 0)$ or $(r, s) = (\frac{1}{2}, \frac{1}{2})$, which corresponds respectively to Δ_U and Δ_{GL} , and of $\mathcal{L}^{\mathbf{T}}$ in the same simple cases, which corresponds respectively to $\tilde{\Delta}_U$ and $\tilde{\Delta}_{GL}$. Since we don't need any more details about $\mathcal{D}^{\mathbf{T}}$ and $\mathcal{L}^{\mathbf{T}}$, we refer to [9, 18, 25, 26] for further informations about those operators.

Using Definition 2.1, we deduce the following result from Theorem 2.4.

Corollary 2.6. Let $B^N = (B_{r,s}^{j,N})_{j \in J}$ be a collection of independent (r, s) -Brownian motions on $\mathbb{G}\mathbb{L}_N$. Let $P \in \mathcal{P}(J)$. Then for $t \geq 0$,

$$\mathbb{E}([P]_N(B^N(t\mathbf{T}))) = [e^{\frac{t}{2}(\mathcal{D}^{\mathbf{T}} + \frac{1}{N^2}\mathcal{L}^{\mathbf{T}})}(P)](\mathbf{1}).$$

The exponential of the operator $\mathcal{D}^{\mathbf{T}} + \frac{1}{N^2}\mathcal{L}^{\mathbf{T}}$ on $\mathcal{P}(J)$ makes sense since $\mathcal{P}(J)$ is a union of finite-dimensional subspaces (those trace polynomials of each fixed finite degree) that are invariant under the operator; hence, the exponential can be defined either by matrix exponentiation or by power series. Corollary 2.6 is merely the statement, in the present language, of the fact that the expectation of any function of a diffusion can be computed by applying the associated heat semigroup to the function and evaluating at the starting point.

2.4. Free multiplicative Brownian motion

Here we give a very brief description of free stochastic processes, and free probability in general. For a complete introduction to the tools of free probability, the best source is [31]. For brief summaries of central ideas and tools from free stochastic calculus, the reader is directed to [11, Section 1.2–1.3], [25, Section 2.7], [26, Section 2.4–2.5], and [27, Section 1.1–1.2].

A *noncommutative probability space* is a pair (\mathcal{A}, τ) where \mathcal{A} is a unital algebra of operators on a (complex) Hilbert space, and τ is a (usually tracial) state on \mathcal{A} : a linear functional $\tau: \mathcal{A} \rightarrow \mathbb{C}$ such that $\tau(1) = 1$ and $\tau(ab) = \tau(ba)$. Typical examples are $\mathcal{A} = \mathbb{M}_N$, $\tau = \text{tr}$ (deterministic matrices), or $\mathcal{A} = \mathbb{M}_N \otimes L^{\infty-}(\mathbb{P})$, $\tau = \text{tr} \otimes \mathbb{E}_{\mathbb{P}}$ (random matrices with entries having moments of all orders). In infinite-dimensional cases, it is typical to add other topological and continuity properties to the pair (\mathcal{A}, τ) that we will not elaborate on presently. Elements of the algebra \mathcal{A} are generally called random variables. In any noncommutative probability space, one can speak of the *noncommutative distribution* of a collection of random variables $a_1, \dots, a_n \in \mathcal{A}$: it is simply the collection of all mixed moments in $a_1, \dots, a_n, a_1^*, \dots, a_n^*$; that is, the collection $\tau[P(a_j, a_j^*)_{1 \leq j \leq n}]$ for all noncommutative polynomials P in $2n$ variables. We then speak of convergence in noncommutative distribution: if (\mathcal{A}_N, τ_N) are noncommutative probability spaces, a sequence $(a_1^N, \dots, a_n^N) \in \mathcal{A}_N^n$ converges in distribution to $(a_1, \dots, a_n) \in \mathcal{A}^n$ if

$$\tau[P(a_j^N, (a_j^N)^*)_{1 \leq j \leq n}] \rightarrow \tau[P(a_j, a_j^*)_{1 \leq j \leq n}] \quad \text{as } N \rightarrow \infty, \text{ for each } P.$$

Free independence (sometimes just called freeness) is an independence notion in any noncommutative probability space. Two random variables $a, b \in \mathcal{A}$ are freely independent if, given any $n \in \mathbb{N}$ and any noncommutative polynomials $P_1, \dots, P_n, Q_1, \dots, Q_n$, each in two variables, which are such that $\tau(P_j(a, a^*)) = \tau(Q_j(b, b^*)) = 0$ for each j , it follows that $\tau(P_1(a, a^*)Q_1(b, b^*) \cdots P_n(a, a^*)Q_n(b, b^*)) = 0$. This gives an algorithm for factoring moments: it implies that $\tau(a^n b^m) = \tau(a^n)\tau(b^m)$ for any $n, m \in \mathbb{N}$, just as holds for classically independent random variables, but it also includes higher-order noncommutative polynomial factorizations; for example $\tau(abab) = \tau(a^2)\tau(b)^2 + \tau(a)^2\tau(b^2) - \tau(a)^2\tau(b)^2$. One finds freely independent random variables typically only in infinite-dimensional noncommutative probability spaces, although random matrices often exhibit asymptotic freeness (i.e. they converge in noncommutative distribution to free objects).

In [34], Voiculescu showed that there exists a noncommutative probability space (any free group factor, for example) that possesses limits $x(t), y(t)$ of the matrix-valued diffusion processes $X^N(t), Y^N(t)$ of (2.2) that are freely independent.

Note that this convergence is not just for each t separately, but for the whole process: convergence of the finite-dimensional noncommutative distributions. The one-parameter families $x(t)$, $y(t)$ are known as (free copies of) *additive free Brownian motion*. We refer to them as free stochastic processes, although they are deterministic in the classical sense.

There is an analogous theory of stochastic differential equations in free probability, cf. [5,6]. One may construct stochastic integrals with respect to free additive Brownian motion, precisely mirroring the classical construction. In sufficiently rich noncommutative probability spaces (such as the one Voiculescu dealt with in [34]), free Itô stochastic differential equations of the usual form

$$dm(t) = \mu(t, m(t)) dt + \sigma(t, m(t)) dx(t)$$

have unique long-time solutions with a given initial condition, assuming standard Lipschitz continuity and growth conditions on the drift and diffusion coefficient functions μ, σ (cf. [24]). In particular, letting $w_{r,s}(t) = \sqrt{r}ix(t) + \sqrt{s}y(t)$ (mirroring (2.2)), the free stochastic differential equation analogous to (2.3),

$$db_{r,s}(t) = b_{r,s}(t) dw_{r,s}(t) - \frac{1}{2}(r-s)b_{r,s}(t) dt, \quad b_{r,s}(0) = 1,$$

has a unique solution (that exists for all positive time) which we call *free multiplicative (r, s) -Brownian motion*. In the special case $(r, s) = (1, 0)$, the resulting process takes values in unitary operators and is known as *free unitary Brownian motion*; when $(r, s) = (\frac{1}{2}, \frac{1}{2})$, it is known as (standard) *free multiplicative Brownian motion*. Both were introduced in [3], where it was proven that the process $(B_{1,0}^N(t))_{t \geq 0}$ converges to the process $(b_{1,0}(t))_{t \geq 0}$. The main theorem of [25] is the corresponding convergence result for the general processes $(B_{r,s}^N(t))_{t \geq 0}$ to $(b_{r,s}(t))_{t \geq 0}$.

3. Gaussian fluctuations

In this section, we prove our main Theorem 3.3, which is summarized in the slightly weaker form of Theorem 1.2 in the **Introduction**. To begin, in Section 3.1 we set the stage with the main tool involved in the computation: the *carré du champ* form associated to the Laplacian on \mathbb{GL}_N^J . Section 3.2 then gives the statement of our Main Theorem 3.3 and associated results that together yield the Gaussian fluctuations of the \mathbb{GL}_N Brownian motions. Section 3.3 is devoted to the proof of Theorem 3.3 in the important special case of a product of two factors (i.e. explicit computation of the covariance), to give the reader a self-contained treatment of most of the tools needed for the general proof. Finally, Section 3.4 is devoted to the full proof of Theorem 3.3.

3.1. The carré du champ form

We define the *carré du champ* form of $\mathbf{T} \cdot \Delta^N$ for all twice continuously differentiable $f, g : \mathbb{GL}_N^J \rightarrow \mathbb{C}$ by

$$\Gamma_N^{\mathbf{T}}(f, g) = \frac{1}{2}((\mathbf{T} \cdot \Delta^N)(fg) - (\mathbf{T} \cdot \Delta^N)(f)g - f(\mathbf{T} \cdot \Delta^N)(g)),$$

or equivalently by

$$\Gamma_N^{\mathbf{T}}(f, g) = \frac{1}{2} \sum_{\xi \in \beta_{r,s}^N, j \in J} t_j \cdot (\partial_{\xi_j} f)(\partial_{\xi_j} g). \quad (3.1)$$

This is a version of the carré du champ form introduced by P. Meyer (cf. [16]), and plays a key role in the work of Bakry, Ledoux, and Saloff-Coste (cf. [1]) and many others; it measures the precise defect of a diffusion generator from being first-order. What we define here is really just $\Gamma^{(1)}$; one can iterate the construction to define $\Gamma^{(k)}$ for all $k \in \mathbb{N}$, and these higher-order carré du champ forms contain a lot of information about the underlying diffusion (and the geometry of the space in which it lives). For our purposes, only the first carré du champ will be needed.

As with the operator $\mathbf{T} \cdot \Delta^N$ in Theorem 2.4, the operator $\Gamma_N^{\mathbf{T}}$ is the push forward of an operator on $\mathcal{P}(J)$ as follows. Let us define the symmetric bilinear form on $\mathcal{P}(J) \times \mathcal{P}(J)$ by

$$\Gamma^{\mathbf{T}}(P, Q) = \frac{1}{2}(\mathcal{L}^{\mathbf{T}}(PQ) - \mathcal{L}^{\mathbf{T}}(P)Q - P\mathcal{L}^{\mathbf{T}}(Q)). \quad (3.2)$$

Proposition 3.1. *For all $P, Q \in \mathcal{P}(J)$, we have $N^2\Gamma_N^{\mathbf{T}}([P]_N, [Q]_N) = [\Gamma^{\mathbf{T}}(P, Q)]_N$.*

Proof. Let us denote by $\mathcal{D}_N^{\mathbf{T}}$ the operator $\mathcal{D}^{\mathbf{T}} + \frac{1}{N^2}\mathcal{L}^{\mathbf{T}}$; thus

$$\mathcal{L}^{\mathbf{T}} = N^2(\mathcal{D}^{\mathbf{T}} - \mathcal{D}_N^{\mathbf{T}}).$$

Note from Theorem 2.4 that $\mathcal{D}^{\mathbf{T}}(PQ) - \mathcal{D}^{\mathbf{T}}(P)Q - P\mathcal{D}^{\mathbf{T}}(Q) = 0$. As a consequence,

$$\Gamma^{\mathbf{T}}(P, Q) = \frac{N^2}{2}(\mathcal{D}_N^{\mathbf{T}}(PQ) - \mathcal{D}_N^{\mathbf{T}}(P)Q - P\mathcal{D}_N^{\mathbf{T}}(Q)).$$

Using $(\mathbf{T} \cdot \Delta^N)([P]_N) = [\mathcal{D}_N^{\mathbf{T}}(P)]_N$, we obtained that

$$[\Gamma^{\mathbf{T}}(P, Q)]_N = \frac{N^2}{2} \left((\mathbf{T} \cdot \Delta^N)([PQ]_N) - (\mathbf{T} \cdot \Delta^N)([P]_N) \cdot [Q]_N - [P]_N \cdot (\mathbf{T} \cdot \Delta^N)([Q]_N) \right),$$

which equals $N^2\Gamma_N^{\mathbf{T}}([P]_N, [Q]_N)$, as desired. \square

Since $\mathcal{L}^{\mathbf{T}}$ is a second-order differential operator, we have the following.

Lemma 3.2. *For all $P, Q, R \in \mathcal{P}(J)$,*

$$\Gamma^{\mathbf{T}}(PQ, R) = \Gamma^{\mathbf{T}}(P, R) \cdot Q + P \cdot \Gamma^{\mathbf{T}}(Q, R).$$

Additionally, for all $P_1, \dots, P_k \in \mathcal{P}(J)$,

$$\mathcal{L}^{\mathbf{T}}(P_1 \cdots P_k) = \sum_{i=1}^k P_1 \cdots \widehat{P}_i \cdots P_k \mathcal{L}^{\mathbf{T}}(P_i) + 2 \sum_{1 \leq i < j \leq k} P_1 \cdots \widehat{P}_i \cdots \widehat{P}_j \cdots P_k \Gamma^{\mathbf{T}}(P_i, P_j),$$

where the hats mean that we omit the corresponding factors in the product.

Proof. Using the second-order property of $\mathcal{L}^{\mathbf{T}}$ (cf. Theorem 2.4), we compute

$$\begin{aligned} 2\Gamma^{\mathbf{T}}(PQ, R) &= \mathcal{L}^{\mathbf{T}}(PQR) - \mathcal{L}^{\mathbf{T}}(PQ)R - PQ\mathcal{L}^{\mathbf{T}}(R) \\ &= \mathcal{L}^{\mathbf{T}}(PR)Q - \mathcal{L}^{\mathbf{T}}(P)QR - PQ\mathcal{L}^{\mathbf{T}}(R) \\ &\quad + P\mathcal{L}^{\mathbf{T}}(QR) - P\mathcal{L}^{\mathbf{T}}(Q)R - PQ\mathcal{L}^{\mathbf{T}}(R) \\ &= 2\Gamma^{\mathbf{T}}(P, R) \cdot Q + 2P \cdot \Gamma^{\mathbf{T}}(Q, R). \end{aligned}$$

By a direct induction, we deduce that

$$\begin{aligned} \mathcal{L}^{\mathbf{T}}(P_1 \cdots P_k) &= \mathcal{L}^{\mathbf{T}}(P_1 \cdots P_{k-1})P_k + P_1 \cdots P_{k-1}\mathcal{L}^{\mathbf{T}}(P_k) + 2\Gamma^{\mathbf{T}}(P_1 \cdots P_{k-1}, P_k) \\ &= \mathcal{L}^{\mathbf{T}}(P_1 \cdots P_{k-1})P_k + P_1 \cdots P_{k-1}\mathcal{L}^{\mathbf{T}}(P_k) + 2 \sum_{1 \leq i \leq k} P_1 \cdots \widehat{P}_i \cdots P_{k-1}\Gamma^{\mathbf{T}}(P_i, P_k) \\ &= \cdots \\ &= \sum_{i=1}^k P_1 \cdots \widehat{P}_i \cdots P_k \mathcal{L}^{\mathbf{T}}(P_i) + 2 \sum_{1 \leq i < j \leq k} P_1 \cdots \widehat{P}_i \cdots \widehat{P}_j \cdots P_k \Gamma^{\mathbf{T}}(P_i, P_j). \end{aligned}$$

\square

3.2. Main theorem

For all $P \in \mathcal{P}(J)$, denote by

$$X_P^N = N([P]_N(B^N(\mathbf{T})) - \mathbb{E}([P]_N(B^N(\mathbf{T})))) \quad (3.3)$$

the fluctuation random variable measured by P . For $t \in [0, 1]$, define

$$P_t^{\mathbf{T}} = e^{\frac{1-t}{2}\mathcal{D}^{\mathbf{T}}}P \quad (3.4)$$

and define, for $P, Q \in \mathcal{P}(J)$,

$$\sigma_{\mathbf{T}}(P, Q) = \int_0^1 [e^{\frac{t}{2}\mathcal{D}^{\mathbf{T}}}(\Gamma^{\mathbf{T}}(P_t^{\mathbf{T}}, Q_t^{\mathbf{T}}))](\mathbf{1}) dt. \quad (3.5)$$

Note that $P \in \mathcal{P}(J)$, and the finite-dimensional subspace of elements with degree lower than or equal to the degree of P is invariant under $\mathcal{D}^{\mathbf{T}}$ (cf. [25, Corollary 3.10]). Hence, $\exp(\frac{1-t}{2}\mathcal{D}^{\mathbf{T}})$ makes sense in this context (defined either by matrix exponentiation or power series). The same argument applied twice more shows that the integrand makes sense, and the finite-dimensionality of all involved polynomials yields continuity, so the integral is perfectly well-defined.

The following theorem says that the quantities of the form $\mathbb{E}(X_{P_1}^N \cdots X_{P_k}^N)$ satisfy a Wick formula asymptotically as $N \rightarrow \infty$, with covariances given by $\sigma_{\mathbf{T}}$; this is tantamount to having an asymptotically joint Gaussian distribution, as we now explain. Let us denote by $\mathcal{P}_2(k)$ the set of (unordered) pairings of $\{1, \dots, k\}$.

Theorem 3.3. *For any $P_1, \dots, P_k \in \mathcal{P}(J)$, we have*

$$\mathbb{E}(X_{P_1}^N \cdots X_{P_k}^N) = \sum_{\pi \in \mathcal{P}_2(k)} \prod_{\{i, j\} \in \pi} \sigma_{\mathbf{T}}(P_i, P_j) + O\left(\frac{1}{N}\right).$$

Theorem 3.3 is proved in Section 3.4 below.

A centered jointly Gaussian random vector (X_1, \dots, X_n) satisfies Wick's formula: for any indices p_1, \dots, p_k ,

$$\mathbb{E}(X_{p_1} \cdots X_{p_k}) = \sum_{\pi \in \mathcal{P}_2(k)} \prod_{\{i, j\} \in \pi} \mathbb{E}[X_{p_i} X_{p_j}].$$

Since this relationship also determines all the higher mixed moments in terms of the covariance, Wick's formula also characterizes Gaussians (among those distributions determined by their moments). Theorem 3.3 states that the fluctuation random variables $X_{P_1}^N, \dots, X_{P_k}^N$ satisfy Wick's formula *asymptotically* as $N \rightarrow \infty$. It is therefore natural to expect this means that the joint distribution of this random vector converges to a Gaussian as $N \rightarrow \infty$. This is indeed correct. One convenient way to make this precise is using the language of *Gaussian Hilbert spaces*, as follows.

Lemma 3.4. *There exists a complex Gaussian Hilbert space K (cf. [22]) with some specified random variables $(\gamma_P)_{P \in \mathcal{P}} \in K$ such that $P \mapsto \gamma_P$ is linear, $\mathbb{E}(\gamma_P \gamma_Q) = \sigma_{\mathbf{T}}(P, Q)$ and $\overline{\gamma_P} = \gamma_{P^*}$.*

Proof. Firstly, the map $\sigma_{\mathbf{T}}$ is symmetric, non-negative and bilinear on the subspace \mathcal{P}_{sa} of self-adjoint elements of $\mathcal{P}(J)$, and therefore there exists a real Gaussian Hilbert space H and a linear map $P \mapsto \gamma_P$ from \mathcal{P}_{sa} to H such that $\mathbb{E}(\gamma_P \gamma_Q) = \sigma_{\mathbf{T}}(P, Q)$. Let $K = H_{\mathbb{C}}$, the complexification of H . For all $P \in \mathcal{P}$, we set $\gamma_P = \gamma_{(P+P^*)/2} + i\gamma_{(P-P^*)/2i}$ which is linear in P . By bilinearity of $\sigma_{\mathbf{T}}$, $\mathbb{E}(\gamma_P \gamma_Q) = \sigma_{\mathbf{T}}(P, Q)$. Finally,

$$\overline{\gamma_P} = \gamma_{(P+P^*)/2} - i\gamma_{(P-P^*)/2i} = \gamma_{(P^*+P)/2} + i\gamma_{(P^*-P)/2i} = \gamma_{P^*}. \quad \square$$

Remark 3.5. To clarify: a Gaussian Hilbert space K is a closed subspace of the L^2 space of a probability space with the property that every random variable $\gamma \in K$ has a centered Gaussian distribution; this is sometimes called an *isonormal Gaussian process*. Common alternative language is to index Gaussian random variables $g(\gamma)$ for $\gamma \in K$, in which case we have the isometry property $\mathbb{E}(g(\gamma_1)g(\gamma_2)) = \langle \gamma_1, \gamma_2 \rangle_K$. We presently choose to use the language and notation of Gaussian Hilbert spaces from Janson's book [22], in particular to yield a simple proof of the following corollary to Theorem 3.3.

Corollary 3.6. *As $N \rightarrow \infty$, $(X_P^N)_{P \in \mathcal{P}(J)}$ converges to $(\gamma_P)_{P \in \mathcal{P}(J)}$ in finite-dimensional distribution: for all $P_1, \dots, P_k \in \mathcal{P}(J)$,*

$$(X_{P_1}^N, \dots, X_{P_k}^N) \xrightarrow[N \rightarrow \infty]{(d)} (\gamma_{P_1}, \dots, \gamma_{P_k}).$$

Otherwise stated, in the dual space $\mathcal{P}(J)^$ endowed with the topology of pointwise convergence, the random linear map $X^N : P \mapsto X_P^N$ converge to the random linear map $\gamma : P \mapsto \gamma_P$ in distribution:*

$$X^N \xrightarrow[N \rightarrow \infty]{(d)} \gamma.$$

Note that, for P and Q in $\mathcal{P}(J)$, the asymptotic covariance of X_P^N and X_Q^N , or equivalently the covariance of γ_P and γ_Q , is $\mathbb{E}(\gamma_P \overline{\gamma_Q}) = \mathbb{E}(\gamma_P \gamma_{Q^*}) = \sigma_{\mathbf{T}}(P, Q^*)$, which is different from $\sigma_{\mathbf{T}}(P, Q)$.

Proof. Let $k \in \mathbb{N}$ and $P_1, \dots, P_k \in \mathcal{P}(J)$. Because the vector $(\gamma_{P_1}, \dots, \gamma_{P_k})$ is Gaussian, it suffices to prove the convergence of the $*$ -moments of $(X_{P_1}^N, \dots, X_{P_k}^N)$ to those of $(\gamma_{P_1}, \dots, \gamma_{P_k})$. (This is because the putative limit Gaussian distribution is determined by its moments; cf. [7, Theorem 30.2].)

Let $1 \leq i_1, \dots, i_n, j_1, \dots, j_m \leq k$. First note that

$$\mathbb{E}(X_{P_{i_1}}^N \cdots X_{P_{i_n}}^N \overline{X_{P_{j_1}}^N} \cdots \overline{X_{P_{j_m}}^N}) = \mathbb{E}(X_{P_{i_1}}^N \cdots X_{P_{i_n}}^N X_{P_{j_1}^*}^N \cdots X_{P_{j_m}^*}^N).$$

By Theorem 3.3, this expectation is equal to the Wick formula sum for covariance $\sigma_{\mathbf{T}}$, up to $O(1/N)$; this Wick formula sum describes the moments of the Gaussian random variables $(\gamma_{P_1}, \dots, \gamma_{P_k})$, according to [22, Theorem 3.9] and Lemma 3.4. Thus

$$\mathbb{E}(X_{P_{i_1}}^N \cdots X_{P_{i_n}}^N \overline{X_{P_{j_1}}^N} \cdots \overline{X_{P_{j_m}}^N}) = \mathbb{E}(\gamma_{P_{i_1}} \cdots \gamma_{P_{i_n}} \gamma_{P_{j_1}^*} \cdots \gamma_{P_{j_m}^*}) + O\left(\frac{1}{N}\right).$$

Finally, according to Lemma 3.4, $\gamma_{P_j^*} = \overline{\gamma_{P_j}}$ for each j . Thus, we have shown that

$$\mathbb{E}(X_{P_{i_1}}^N \cdots X_{P_{i_n}}^N \overline{X_{P_{j_1}}^N} \cdots \overline{X_{P_{j_m}}^N}) \xrightarrow{N \rightarrow \infty} \mathbb{E}(\gamma_{P_{i_1}} \cdots \gamma_{P_{i_n}} \overline{\gamma_{P_{j_1}}} \cdots \overline{\gamma_{P_{j_m}}})$$

concluding the proof. □

3.3. Computation of the covariance

Before proceeding with the full proof of Theorem 3.3, it will be instructive to consider the (easier) special case $k = 2$: namely, the leading order terms in the $\frac{1}{N}$ expansion of the covariance $\mathbb{E}(X_P^N X_Q^N)$. Here P, Q are abstract trace polynomials in $\mathcal{P}(J)$, and the random variables X_P, X_Q are the corresponding rescaled fluctuations of a Brownian motion on \mathbb{GL}_N^n , as in (3.3):

$$X_P = N([P]_N(B^N(\mathbf{T})) - \mathbb{E}[P]_N(B^N(\mathbf{T}))),$$

$$X_Q = N([Q]_N(B^N(\mathbf{T})) - \mathbb{E}[Q]_N(B^N(\mathbf{T}))).$$

Denote by $\langle P \rangle$ and $\langle Q \rangle$ the constants

$$\langle P \rangle \equiv \mathbb{E}[P]_N(B^N(\mathbf{T})), \quad \langle Q \rangle \equiv \mathbb{E}[Q]_N(B^N(\mathbf{T})).$$

We will compute the covariance $\mathbb{E}(X_P X_Q)$ to leading order in $\frac{1}{N}$. We do so using the intertwining formula of Corollary 2.6: namely, we use the operators $\mathcal{D} = \mathcal{D}^{\mathbf{T}}$ and $\mathcal{L} = \mathcal{L}^{\mathbf{T}}$ (cf. Theorem 2.4) on the space of abstract trace polynomials satisfying

$$\mathbb{E}[P]_N(B^N(\mathbf{T})) = (e^{\frac{1}{2}(\mathcal{D} + \frac{1}{N^2}\mathcal{L})} P)(\mathbf{1}),$$

where $\mathbf{1}$ means evaluating all the variables in the abstract trace polynomial at 1. The operators \mathcal{D} and \mathcal{L} depend on \mathbf{T} , but we suppress this; they are independent of N . What is important is that \mathcal{D} is first-order, meaning that \mathcal{D} satisfies the ordinary product rule (cf. Theorem 2.4),

$$\mathcal{D}(PQ) = (\mathcal{D}P)Q + P(\mathcal{D}Q).$$

It follows (by a standard power-series argument) that, for any $s \geq 0$, $e^{\frac{s}{2}\mathcal{D}}$ is an algebra homomorphism. If we define a bilinear form Γ on trace polynomials as in (3.2),

$$\Gamma(P, Q) = \frac{1}{2}(\mathcal{L}(PQ) - (\mathcal{L}P)Q - P(\mathcal{L}Q))$$

then, of course, we have

$$\mathcal{L}(PQ) = (\mathcal{L}P)Q + P(\mathcal{L}Q) + 2\Gamma(P, Q). \tag{3.6}$$

This is the “carré du champ” of the second order operator \mathcal{L} .

We may now proceed with the calculation. First, using the intertwining formula, we have

$$\mathbb{E}(X_P X_Q) = N^2 (e^{\frac{1}{2}(\mathcal{D} + \frac{1}{N^2}\mathcal{L})}((P - \langle P \rangle)(Q - \langle Q \rangle)))(\mathbf{1}). \quad (3.7)$$

We start by comparing this to the large- N limit, by separating out the $O(1/N^2)$ term in the exponent. A good way to do this is with Duhamel's formula, which says that, for linear operators A and B on a finite-dimensional vector space,

$$e^A - e^B = \int_0^1 e^{sA} (A - B) e^{(1-s)B} ds.$$

(We give a proof of this below in Lemma 3.8.) Setting $A = \frac{1}{2}(\mathcal{D} + \frac{1}{N^2}\mathcal{L})$ and $B = \frac{1}{2}\mathcal{D}$ (acting on the finite-dimensional subspace of trace polynomials with degree no more than the sum of the degrees of P and Q), this yields

$$e^{\frac{1}{2}(\mathcal{D} + \frac{1}{N^2}\mathcal{L})} = e^{\frac{1}{2}\mathcal{D}} + \frac{1}{2N^2} \int_0^1 e^{\frac{s}{2}\mathcal{D}} \mathcal{L} e^{\frac{1-s}{2}\mathcal{D}} ds.$$

Applying this to (3.7) yields

$$\mathbb{E}(X_P X_Q) = N^2 e^{\frac{1}{2}\mathcal{D}}((P - \langle P \rangle)(Q - \langle Q \rangle))(\mathbf{1}) + \frac{1}{2} \int_0^1 (e^{\frac{s}{2}\mathcal{D}} \mathcal{L} e^{\frac{1-s}{2}\mathcal{D}}((P - \langle P \rangle)(Q - \langle Q \rangle)))(\mathbf{1}) ds.$$

Let's deal with the first term first. Since $e^{\frac{1}{2}\mathcal{D}}$ is an algebra homomorphism, as is evaluation at $\mathbf{1}$,

$$e^{\frac{1}{2}\mathcal{D}}((P - \langle P \rangle)(Q - \langle Q \rangle))(\mathbf{1}) = ((e^{\frac{1}{2}\mathcal{D}} P)(\mathbf{1}) - \langle P \rangle)((e^{\frac{1}{2}\mathcal{D}} Q)(\mathbf{1}) - \langle Q \rangle). \quad (3.8)$$

Now, from the intertwining formula (Corollary 2.6),

$$\langle P \rangle = \mathbb{E}[P_N(B^N(\mathbf{T}))] = (e^{\frac{1}{2}(\mathcal{D} + \frac{1}{N^2}\mathcal{L})} P)(\mathbf{1}),$$

and similarly for $\langle Q \rangle$. Hence, the two terms in the product in (3.8) each have the form

$$(e^{\frac{1}{2}\mathcal{D}} P)(\mathbf{1}) - (e^{\frac{1}{2}(\mathcal{D} + \frac{1}{N^2}\mathcal{L})} P)(\mathbf{1}) = O\left(\frac{1}{N^2}\right), \quad (3.9)$$

which follows simply from the power-series expansion of the exponential (or by Duhamel's formula again, if the reader prefers). Thus, the expression in (3.8) is $O(1/N^2) \cdot O(1/N^2) = O(1/N^4)$, and so even multiplying by N^2 we have

$$\mathbb{E}(X_P X_Q) = \frac{1}{2} \int_0^1 (e^{\frac{s}{2}\mathcal{D}} \mathcal{L} e^{\frac{1-s}{2}\mathcal{D}}((P - \langle P \rangle)(Q - \langle Q \rangle)))(\mathbf{1}) ds + O\left(\frac{1}{N^2}\right). \quad (3.10)$$

Now we proceed with the expression inside the integral. Since $e^{\frac{1-s}{2}\mathcal{D}}$ is a homomorphism, the integrand (before evaluating) can be written as

$$e^{\frac{s}{2}\mathcal{D}} \mathcal{L}((P_s - \langle P \rangle)(Q_s - \langle Q \rangle)),$$

where, for ease of reading, we've denoted $P_s = e^{\frac{1-s}{2}\mathcal{D}} P$ (cf. (3.4)). We now use the "second order product rule" (3.6) for \mathcal{L} . Combined with the fact that \mathcal{L} kills constants, we have

$$\mathcal{L}((P_s - \langle P \rangle)(Q_s - \langle Q \rangle)) = \mathcal{L}(P_s) \cdot (Q_s - \langle Q \rangle) + (P_s - \langle P \rangle) \cdot \mathcal{L}(Q_s) + 2\Gamma(P_s, Q_s),$$

where we've also used the fact that $\Gamma(P + a, Q + b) = \Gamma(P, Q)$ for any constants a, b . Plugging this into the integral in (3.10), this means that there are three terms:

$$\begin{aligned} \mathbb{E}(X_P X_Q) &= \frac{1}{2} \int_0^1 (e^{\frac{s}{2}\mathcal{D}} [\mathcal{L}(P_s) \cdot (Q_s - \langle Q \rangle)])(\mathbf{1}) ds + \frac{1}{2} \int_0^1 (e^{\frac{s}{2}\mathcal{D}} [(P_s - \langle P \rangle) \cdot \mathcal{L}(Q_s)])(\mathbf{1}) ds \\ &\quad + \int_0^1 (e^{\frac{s}{2}\mathcal{D}} \Gamma(P_s, Q_s))(\mathbf{1}) ds + O\left(\frac{1}{N^2}\right). \end{aligned}$$

In the first two terms, we use the homomorphism property of $e^{\frac{s}{2}\mathcal{D}}$ again. Recalling that $Q_s = e^{\frac{1-s}{2}\mathcal{D}}Q$, we have $e^{\frac{s}{2}\mathcal{D}}Q_s = e^{\frac{1}{2}\mathcal{D}}Q$, and so

$$(e^{\frac{s}{2}\mathcal{D}}[\mathcal{L}(P_s) \cdot (Q_s - \langle Q \rangle)])(\mathbf{1}) = (e^{\frac{s}{2}\mathcal{D}}\mathcal{L}(P_s))(\mathbf{1}) \cdot ((e^{\frac{1}{2}\mathcal{D}}Q)(\mathbf{1}) - \langle Q \rangle).$$

The second factor is $O(1/N^2)$, as established in (3.9), and so the whole integral is $O(1/N^2)$. The same argument applies to the second term. Thus, all in all, we have

$$\mathbb{E}(X_P X_Q) = \int_0^1 (e^{\frac{s}{2}\mathcal{D}}\Gamma(P_s, Q_s))(\mathbf{1}) ds + O\left(\frac{1}{N^2}\right).$$

This integral expression is exactly the definition of the limit covariance $\sigma_{\mathbf{T}}(P, Q)$, thus proving the $k = 2$ case of Theorem 3.3 as desired.

Remark 3.7. The preceding calculation demonstrates some, but not all, of the kinds of terms that come up in the full computation of the next section. When one considers expectations of products of 3 or more terms, some of the sub-leading contributions can be $O(1/N)$ instead of $O(1/N^2)$; but the basic idea is the same, it's just a matter of more involved bookkeeping.

3.4. Proof of Theorem 3.3

We now generalize the computation of the preceding section to expectations of longer strings of fluctuation variables X_P . Observing that $(P_1, \dots, P_k) \mapsto \mathbb{E}(X_{P_1}^N \cdots X_{P_k}^N)$ and $(P_1, \dots, P_k) \mapsto \sum_{\pi \in \mathcal{P}_2(k)} \prod_{(i,j) \in \pi} \sigma_{\mathbf{T}}(P_i, P_j)$ are symmetric multilinear forms on $\mathcal{P}(J)$, it suffices by polarization to verify the asymptotic when $P_1 = \dots = P_k = P$ (cf. [22, Appendix D]). In this case, set $Q_N = P - \mathbb{E}[[P]_N(B^N(\mathbf{T}))]$. (Note that Q_N is an element of the abstract space $\mathcal{P}(J)$; it should not be confused with the notation $[Q]_N$ for evaluation as a trace polynomial function on \mathbb{M}_N .) We want to prove that

$$N^k \mathbb{E}([(Q_N)^k]_N(B^N(\mathbf{T}))) = \sum_{\pi \in \mathcal{P}_2(k)} \prod_{(i,j) \in \pi} \sigma_{\mathbf{T}}(P, P) + O\left(\frac{1}{N}\right).$$

To begin, we remark that

$$\mathbb{E}([(Q_N)^k]_N(B^N(\mathbf{T}))) = [e^{\frac{1}{2}(\mathcal{D}^{\mathbf{T}} + \frac{1}{N^2}\mathcal{L}^{\mathbf{T}})}((Q_N)^k)](\mathbf{1}),$$

thanks to Corollary 2.6. The proof will consist in identifying the leading term in the expansion of $e^{\frac{1}{2}(\mathcal{D}^{\mathbf{T}} + \frac{1}{N^2}\mathcal{L}^{\mathbf{T}})}$ in powers of $\frac{1}{N}$.

Appropriate norms. In order to control the negligible terms in the expansion, we will work on finite dimensional spaces. Let $d \in \mathbb{N}$ be the degree of Q_N (which is independent of N). The subalgebra \mathcal{P}_{kd} of elements of $\mathcal{P}(J)$ whose degrees are $\leq kd$ is finite dimensional and we endow it with some fixed unital algebra norm $\|\cdot\|_{(kd)}$. Let us denote by $\|\cdot\|_{(kd)}$ the induced operator norm on the finite dimensional algebra $\text{End}(\mathcal{P}_{kd})$, and by $\|\cdot\|_{(d,d')}$ the induced norm of bilinear maps from $\mathcal{P}_d \times \mathcal{P}_{d'}$ to $\mathcal{P}_{d+d'}$ when $d + d' \leq kd$ (in the following development, we will often omit the indices (kd) or (d, d')). Throughout this proof, we will denote by D , L and Γ the operators $\mathcal{D}^{\mathbf{T}}$, $\mathcal{L}^{\mathbf{T}}$ and $\Gamma^{\mathbf{T}}$ restricted to the finite dimensional algebra \mathcal{P}_{kd} . Let us denote by respectively $O(1/N^2)$, $\mathbf{O}(1/N^2)$ and $\mathbf{O}(1/N^2)$ the class of elements $A(N)$ in respectively \mathbb{C} , \mathcal{P}_{kd} and $\text{End}(\mathcal{P}_{kd})$ such that $|A(N)|$ (resp. $\|A(N)\|_{(kd)}$ and $\|A(N)\|_{(kd)}$) is $\leq C/N^2$ for some constant C . We have the following result.

Lemma 3.8. *For all $t \geq 0$, we have*

$$e^{\frac{1}{2}(D + \frac{1}{N^2}L)} = e^{\frac{1}{2}D} + \frac{1}{2N^2} \int_0^1 e^{\frac{t}{2}(D + \frac{1}{N^2}L)} L e^{\frac{1-t}{2}D} dt. \quad (3.11)$$

In particular, $e^{\frac{1}{2}(D+\frac{1}{N^2}L)} = e^{\frac{1}{2}D} + \mathbf{O}(1/N^2)$. More generally, for all $m \in \mathbb{N}$, we have

$$\begin{aligned} e^{\frac{1}{2}(D+\frac{1}{N^2}L)} &= e^{\frac{1}{2}D} + \sum_{n=1}^m \frac{1}{(2N^2)^n} \int_{0 \leq t_n \leq \dots \leq t_1 \leq 1} e^{\frac{t_n}{2}D} L e^{\frac{t_{n-1}-t_n}{2}D} L \dots L e^{\frac{1-t_1}{2}D} dt_1 \dots dt_n \\ &\quad + \frac{1}{(2N^2)^{m+1}} \int_{0 \leq t_{m+1} \leq \dots \leq t_1 \leq 1} e^{\frac{t_{m+1}}{2}(D+\frac{1}{N^2}L)} L e^{\frac{t_m-t_{m+1}}{2}D} L \dots L e^{\frac{1-t_1}{2}D} dt_1 \dots dt_{m+1}. \end{aligned}$$

Proof. Let us define $S(t) = e^{\frac{t}{2}(D+\frac{1}{N^2}L)} e^{-\frac{t}{2}D}$; then S is differentiable, and

$$S'(t) = \frac{1}{2} e^{\frac{t}{2}(D+\frac{1}{N^2}L)} \left(D + \frac{1}{N^2}L - D \right) e^{-\frac{t}{2}D} = \frac{1}{2N^2} S(t) e^{\frac{t}{2}D} L e^{-\frac{t}{2}D}.$$

Since $S(0) = I_N$, it follows that $S(1) = 1 + \frac{1}{2N^2} \int_0^1 S(t) e^{\frac{t}{2}D} L e^{-\frac{t}{2}D} dt$. Multiplying by $e^{\frac{1}{2}D}$ on the right yields (3.11). We can then compute

$$\|e^{\frac{1}{2}(D+\frac{1}{N^2}L)} - e^{\frac{1}{2}D}\| \leq \frac{1}{2N^2} \int_0^1 \|e^{\frac{t}{2}(D+\frac{1}{N^2}L)} L e^{\frac{1-t_1}{2}D}\| dt \leq \frac{1}{2N^2} e^{\|D\|+\frac{1}{2}\|L\|} \|L\|.$$

The last formula is obtained by induction over m , using at each step the first formula. \square

Remark 3.9. The first formula is often called *Duhamel's formula*: for any operators A, B on some finite dimensional vector space V ,

$$e^A - e^B = \int_0^1 e^{sA} (A - B) e^{(1-s)B} ds.$$

For $n \in \mathbb{N}$, let us denote by $\Delta_n \subset \mathbb{R}^n$ the simplex

$$\Delta_n = \{(t_1, \dots, t_n) \in \mathbb{R}^n : 0 \leq t_n \leq t_{n-1} \leq \dots \leq t_1 \leq 1\}.$$

Using Lemma 3.8 at step $m = [k/2]$, the study of the limit of $N^k [e^{\frac{1}{2}(D+\frac{1}{N^2}L)} (Q_N^k)](\mathbf{1})$ is decomposed into the study of the limits of:

1. $N^k [e^{\frac{1}{2}D} (Q_N^k)](\mathbf{1})$,
2. $\frac{1}{(2N^2)^n} \cdot N^k [\int_{\Delta_n} e^{\frac{t_n}{2}D} L e^{\frac{t_{n-1}-t_n}{2}D} \dots L e^{\frac{t_1-t_2}{2}D} L e^{\frac{1-t_1}{2}D} dt_1 \dots dt_n (Q_N^k)](\mathbf{1})$ for $1 \leq n \leq [k/2]$, and
3. $N^{k-2-[k/2]} [\int_{\Delta_{k+1}} e^{\frac{t_{k+1}}{2}(D+\frac{1}{N^2}L)} L e^{\frac{t_k-t_{k+1}}{2}D} \dots L e^{\frac{t_1-t_2}{2}D} L e^{\frac{1-t_1}{2}D} dt_1 \dots dt_{k+1} (Q_N^k)](\mathbf{1})$,

which we address separately in the following three steps. In the fourth step, we sum up the three convergences to conclude the proof. We will see, using Lemma 3.8, that the only term which does not vanish in the large- N limit is the second term considered when $n = [k/2]$.

Step 1. Since the map $A \mapsto [A(P)](\mathbf{1})$ is linear on $\text{End}(\mathcal{P}_{kd})$, it is therefore bounded and we deduce that

$$[e^{\frac{1}{2}(D+\frac{1}{N^2}L)}(P)](\mathbf{1}) = [e^{\frac{1}{2}D}(P)](\mathbf{1}) + O(1/N^2)$$

from $e^{\frac{1}{2}(D+\frac{1}{N^2}L)} = e^{\frac{1}{2}D} + \mathbf{O}(1/N^2)$. But we have $Q_N = P - \mathbb{E}[[P]_N(B^N(\mathbf{T}))] = P - [e^{\frac{1}{2}(D+\frac{1}{N^2}L)}(P)](\mathbf{1})$ thanks to Corollary 2.6. Consequently, for $k \geq 1$,

$$Q_N^k = (P - [e^{\frac{1}{2}D}(P)](\mathbf{1}))^k + O(1/N^{2k}). \quad (3.12)$$

Since $e^{\frac{1}{2}D}$ is an algebra homomorphism, we therefore deduce that

$$\begin{aligned} [e^{\frac{1}{2}D}(Q_N^k)](\mathbf{1}) &= [e^{\frac{1}{2}D}(P - [e^{\frac{1}{2}D}(P)](\mathbf{1}))^k](\mathbf{1}) + O(1/N^{2k}) = ([e^{\frac{1}{2}D}(P)](\mathbf{1}) - [e^{\frac{1}{2}D}(P)](\mathbf{1}))^k(\mathbf{1}) + O(1/N^{2k}) \\ &= ([e^{\frac{1}{2}D}(P)](\mathbf{1}) - [e^{\frac{1}{2}D}(P)](\mathbf{1}))^k + O(1/N^{2k}) = O(1/N^{2k}). \end{aligned}$$

Hence, we see that $N^k [e^{\frac{1}{2}D}(Q_N^k)](\mathbf{1}) = O(1/N^k)$, and item 1 is negligible in the large- N limit.

Step 2. We are assuming at this step that $2 \leq k$. For all $R \in \mathcal{P}(J)$, $t \geq 0$ and $n \geq 2$, we have by Lemma 3.2

$$\begin{aligned} L((e^{tD} Q_N)^n \cdot R) &= (e^{tD} Q_N)^n L(R) + 2n(e^{tD} Q_N)^{n-1} \Gamma(e^{tD} Q_N, R) \\ &\quad + n(e^{\frac{t}{2}D} Q_N)^{n-1} L(e^{\frac{t}{2}D} Q_N) R + n(n-1)(e^{\frac{t}{2}D} Q_N)^{n-2} \Gamma(e^{\frac{t}{2}D} Q_N, e^{\frac{t}{2}D} Q_N) R. \end{aligned}$$

In others words, for all $d' \leq (k-1)d$, if we define the bilinear map $B_n : (S, R) \mapsto S \cdot L(R) + 2n\Gamma(S, R) + nL(S) \cdot R$ from $\mathcal{P}_d \times \mathcal{P}_{d'}$ to $\mathcal{P}_{d+d'}$, we have, for all $R \in \mathcal{P}_{d'}$,

$$L((e^{\frac{t}{2}D} Q_N)^n \cdot R) = (e^{\frac{t}{2}D} Q_N)^{n-1} B_n(e^{\frac{t}{2}D} Q_N, R) + n(n-1)(e^{\frac{t}{2}D} Q_N)^{n-2} \Gamma(e^{\frac{t}{2}D} Q_N, e^{\frac{t}{2}D} Q_N) R. \quad (3.13)$$

Lemma 3.10. *Denote*

$$G(t) = e^{\frac{t}{2}D} \Gamma(e^{\frac{1-t}{2}D} Q_N, e^{\frac{1-t}{2}D} Q_N) \in \mathcal{P}_{2d}.$$

For all n such that $1 \leq n \leq [k/2]$ and $0 \leq t_n \leq \dots \leq t_0 = 1$, there exists $R_n^N \in \mathcal{P}_{(2n-1)d}$ bounded uniformly in N, t_0, \dots, t_n such that

$$\begin{aligned} &Le^{\frac{t_n-1-t_n}{2}D} L \dots Le^{\frac{1-t_1}{2}D} (Q_N^k) \\ &= \frac{k!}{(k-2n)!} (e^{\frac{1-t_n}{2}D} Q_N)^{k-2n} e^{-\frac{t_n}{2}D} (G(t_1) \dots G(t_n)) + (e^{\frac{1-t_n}{2}D} Q_N)^{k-2n+1} R_n^N. \end{aligned} \quad (3.14)$$

Proof. Indeed, when $n = 1$, setting $R_1^N = kL(e^{\frac{1-t_1}{2}D} Q_N) \in \mathcal{P}_d$, we have

$$Le^{\frac{1-t_1}{2}D} (Q_N^k) = k(k-1)(e^{\frac{1-t_1}{2}D} Q_N)^{k-2n} \Gamma(e^{\frac{1-t_1}{2}D} Q_N, e^{\frac{1-t_1}{2}D} Q_N) + (e^{\frac{1-t_1}{2}D} Q_N)^{k-1} R_1^N.$$

Note that $\|R_1^N\| \leq k \|L\| e^{\frac{1}{2}\|D\|} \|Q_N\|$. Because of (3.12),

$$\|Q_N\| \leq \|P - [e^{\frac{1}{2}D}(P)](\mathbf{1})\| + \|Q_N - P + [e^{\frac{1}{2}D}(P)](\mathbf{1})\| = \|P - [e^{\frac{1}{2}D}(P)](\mathbf{1})\| + O(1/N^2). \quad (3.15)$$

Therefore, Q_N is bounded uniformly in N , and so too is R_1^N . Assume now that $2 \leq n \leq [k/2]$ and that (3.14) has been verified up to level $n-1$. We compute

$$\begin{aligned} &Le^{\frac{t_n-1-t_n}{2}D} L \dots Le^{\frac{1-t_1}{2}D} (Q_N^k) \\ &= Le^{\frac{t_n-1-t_n}{2}D} \left(\frac{k!}{(k-2n+2)!} (e^{\frac{1-t_{n-1}}{2}D} Q_N)^{k-2n+2} e^{-\frac{t_{n-1}}{2}D} (G(t_1) \dots G(t_{n-1})) \right. \\ &\quad \left. + (e^{\frac{1-t_{n-1}}{2}D} Q_N)^{k-2n+3} R_{n-1}^N \right) \\ &= \frac{k!}{(k-2n+2)!} L((e^{\frac{1-t_n}{2}D} Q_N)^{k-2n+2} e^{-\frac{t_n}{2}D} (G(t_1) \dots G(t_{n-1}))) \\ &\quad + L((e^{\frac{1-t_n}{2}D} Q_N)^{k-2n+3} R_{n-1}^N). \end{aligned}$$

We now apply (3.13) to each term. The first term leads to

$$\begin{aligned} &\frac{k!}{(k-2n)!} (e^{\frac{1-t_n}{2}D} Q_N)^{k-2n} e^{-\frac{t_n}{2}D} (G(t_1) \dots G(t_n)) \\ &\quad + \frac{k!}{(k-2n+2)!} (e^{\frac{1-t_n}{2}D} Q_N)^{k-2n+1} B_{k-2n+2}(e^{\frac{1-t_n}{2}D} Q_N, e^{-\frac{t_n}{2}D} (G(t_1) \dots G(t_{n-1}))), \end{aligned}$$

and the second term to

$$\begin{aligned} &(e^{\frac{1-t_n}{2}D} Q_N)^{k-2n+2} B_{k-2n+3}(e^{\frac{1-t_n}{2}D} Q_N, R_{n-1}^N) \\ &\quad + (k-2n+3)(k-2n+2)(e^{\frac{1-t_n}{2}D} Q_N)^{k-2n+1} \Gamma(e^{\frac{1-t_n}{2}D} Q_N, e^{\frac{1-t_n}{2}D} Q_N) R_{n-1}^N. \end{aligned}$$

Thus, $R_n^N \in \mathcal{P}_{(2n-1)d}$ can be defined by

$$\begin{aligned} R_n^N &\equiv \frac{k!}{(k-2n+2)!} B_{k-2n+2}(e^{\frac{1-t_n}{2}D} Q_N, e^{-\frac{t_n}{2}D}(G(t_1) \cdots G(t_{n-1}))) \\ &\quad + (e^{\frac{1-t_n}{2}D} Q_N) B_{k-2n+3}(e^{\frac{1-t_n}{2}D} Q_N, R_{n-1}^N) \\ &\quad + (k-2n+3)(k-2n+2) \Gamma(e^{\frac{1-t_n}{2}D} Q_N, e^{\frac{1-t_n}{2}D} Q_N) R_{n-1}^N, \end{aligned}$$

which verifies (3.14) and which is bounded by

$$\begin{aligned} &\frac{k!}{(k-2n+2)!} \|B_{k-2n+2}\|_{(d,2(n-1)d)} e^{\|D\|} \|Q_N\| \|G(t_1)\| \cdots \|G(t_{n-1})\| \\ &\quad + e^{\|D\|} \|Q_N\|^2 \|B_{k-2n+3}\|_{(d,(2n-1)d)} \|R_{n-1}^N\| \\ &\quad + (k-2n+3)(k-2n+2) \|\Gamma\|_{(d,d)} e^{\|D\|} \|Q_N\|^2 \|R_{n-1}^N\|. \end{aligned}$$

Because of (3.15), it is bounded uniformly in N . We deduce also that

$$G(t_i) = e^{\frac{t_i}{2}D} \Gamma(e^{\frac{1-t_i}{2}D} Q_N, e^{\frac{1-t_i}{2}D} Q_N)$$

is bounded by $\|\Gamma\|_{(d,d)} e^{\|D\|} \|Q_N\|^2$ and consequently is bounded uniformly in N, t_1, \dots, t_n . Thus, R_n^N is bounded uniformly in N, t_1, \dots, t_n , as required. \square

We now use, again, the fact that $e^{\frac{t_n}{2}D}$ is an algebra homomorphism. Applying $e^{\frac{t_n}{2}D}$ to (3.14) on the left, we obtain that, for all $1 \leq n \leq [k/2]$, $N \in \mathbb{N}$, and $(t_1, \dots, t_n) \in \Delta_n$, there exists $R_n \in \mathcal{P}_{(2n-1)d}$ bounded uniformly in N, t_0, \dots, t_n such that

$$e^{\frac{t_n}{2}D} L e^{\frac{t_{n-1}-t_n}{2}D} L \cdots L e^{\frac{1-t_1}{2}D} (Q_N^k) = \frac{k!}{(k-2n)!} (e^{\frac{1}{2}D} Q_N)^{k-2n} G(t_1) \cdots G(t_n) + (e^{\frac{1}{2}D} Q_N)^{k-2n+1} R_n,$$

where $G(t)$ denotes the element $e^{tD} \Gamma(e^{\frac{1-t}{2}D} Q_N, e^{\frac{1-t}{2}D} Q_N) \in \mathcal{P}_{2d}$.

From (3.12), we deduce that we have $[(e^{\frac{1}{2}D} Q_N)^{k-2n}](\mathbf{1}) = O(1/N^{2k-4n})$ and $[(e^{\frac{1}{2}D} Q_N)^{k+1-2n}](\mathbf{1}) = O(1/N^{2k+1-4n})$. We have already remarked in the proof of (3.14) that $G(t_i) = e^{\frac{t_i}{2}D} \Gamma(e^{\frac{1-t_i}{2}D} Q_N, e^{\frac{1-t_i}{2}D} Q_N)$ and Q_N are bounded uniformly in N, t_1, \dots, t_n ; consequently,

$$N^{2k-4n} \frac{k!}{(k-2n)!} [(e^{\frac{1}{2}D} Q_N)^{k-2n} G(t_1) \cdots G(t_n)](\mathbf{1}) \text{ and } N^{k+1-2n} [(e^{\frac{1}{2}D} Q_N)^{k+1-2n} R_n](\mathbf{1})$$

are bounded uniformly in N, t_1, \dots, t_n , and we deduce that

$$N^{k-2n} \left[\int_{\Delta_n} e^{\frac{t_n}{2}D} L e^{\frac{t_{n-1}-t_n}{2}D} L \cdots L e^{\frac{1-t_1}{2}D} dt_1 \cdots dt_n (Q_N^k) \right](\mathbf{1})$$

is $O(1/N)$ if $k > 2n$ and is equal to $k! \int_{\Delta_n} (G(t_1) \cdots G(t_n))(\mathbf{1}) dt_1 \cdots dt_n + O(1/N)$ if $k = 2n$.

In the case where $k = 2n$, because the integrand is symmetric in t_1, \dots, t_n , the remaining term is equal to

$$\frac{k!}{n!} \int_{0 \leq t_1, \dots, t_n \leq 1} [G(t_1) \cdots G(t_n)](\mathbf{1}) dt_1 \cdots dt_n = \frac{k!}{n!} \left(\int_0^1 [G(t)](\mathbf{1}) dt \right)^n = \frac{(2n)!}{n!} \sigma_{\mathbf{T}}(Q_N, Q_N)^n,$$

where the last equality follows from the definition (3.5) of $\sigma_{\mathbf{T}}$ and the definition of $G(t)$ (cf. Lemma 3.14). Note that L kills constants, and similarly $\Gamma(P+c, Q+d) = \Gamma(P, Q)$ for any $c, d \in \mathbb{C}$. As a consequence, $\sigma_{\mathbf{T}}(Q_N, Q_N) = \sigma_{\mathbf{T}}(P, P)$.

To sum up: referring to the second term in Lemma 3.8,

$$\frac{1}{(2N^2)^n} \cdot N^k \left[\int_{\Delta_n} e^{t_n D} L e^{\frac{t_{n-1}-t_n}{2}D} L \cdots L e^{\frac{1-t_1}{2}D} dt_1 \cdots dt_n (Q_N^k) \right](\mathbf{1})$$

is equal to $\frac{(2n)!}{2^n n!} \sigma_{\mathbf{T}}(P, P)^n + O(1/N)$ if $k = 2n$ and $O(1/N)$ if not.

Step 3. We have $Q_N^k = (P - [e^{\frac{1}{2}D}(P)](\mathbf{1}))^k + O(1/N^{2k})$ and

$$\left\| \int_{\Delta_{k+1}} e^{\frac{t_{k+1}}{2}(D + \frac{1}{N^2}L)} L e^{\frac{t_{k+1}-t_k}{2}D} L \dots L e^{\frac{1-t_1}{2}D} dt_1 \dots dt_{k+1} \right\| \leq \|L\|^n e^{\frac{1}{2}(\|L\| + \|D\|)}.$$

Consequently

$$\left[\int_{\Delta_{k+1}} e^{\frac{t_{k+1}}{2}(D + \frac{1}{N^2}L)} L e^{\frac{t_{k+1}-t_k}{2}D} L \dots L e^{\frac{1-t_1}{2}D} dt_1 \dots dt_{k+1} (Q_N^k) \right](\mathbf{1})$$

is bounded uniformly in N . On the other hand, $k - 2([k/2] + 1) \leq -1$ and $N^{k-2([k/2]+1)}$ is therefore $O(1/N)$. Thus, the term studied is $O(1/N)$.

Step 4. Finally, applying Lemma 3.8 with $n = [k/2]$, and using the limits computed in the three previous steps, we have $N^k \mathbb{E}(Q_N^k(B^N(\mathbf{T}))) = \frac{k!}{2^{k/2}(k/2)!} \sigma_{\mathbf{T}}(P, P)^{k/2} + O(1/N)$ if k is even and $O(1/N)$ if not. Because the cardinality of $\mathcal{P}_2(k)$ is $\frac{k!}{2^{k/2}(k/2)!}$ if k is even and 0 if not, we have demonstrated the desired bound,

$$N^k \mathbb{E}(Q_N^k(B^N(\mathbf{T}))) = \sum_{\pi \in \mathcal{P}_2(k)} \prod_{(i,j) \in \pi} \sigma_{\mathbf{T}}(Q, Q) + o\left(\frac{1}{N}\right).$$

This concludes the proof of Theorem 3.3. □

4. Study of the covariance

In [28], Lévy and Maïda established a central limit theorem for random matrices arising from a unitary Brownian motion, which corresponds to our $(r, s) = (1, 0)$ case.

Theorem 4.1. [28, Theorem 2.6] *Let $(U_t^N)_{t \geq 0}$ be a unitary Brownian motion on \mathbb{U}_N ; $U^N(t) = B_{1,0}^N(t)$ in our language. Let $P_1, \dots, P_n \in \mathbb{C}[X, X^{-1}]$, and $T \geq 0$. When $N \rightarrow \infty$, the random vector*

$$N(\text{tr}(P_i(U^N(T))) - \mathbb{E}[\text{tr}(P_i(U^N(T)))]))_{1 \leq i \leq n}$$

converges in distribution to a Gaussian vector.

In fact, the test functions allowed in their approach were not only polynomials but C^1 real-valued functions with Lipschitz derivative on the unit circle. Generalizing to \mathbb{GL}_N does not allow for such functional calculus. The statement above for Laurent polynomials is obtained easily from the real-valued case by linearity.

The limit covariance involves three free unitary Brownian motion $(u_t)_{t \geq 0}$, $(v_t)_{t \geq 0}$ and $(w_t)_{t \geq 0}$ which are freely independent (cf. Section 2.4, in the special case $(r, s) = (1, 0)$). For all $P \in \mathbb{C}[X, X^{-1}]$, we denote by $\partial P \in \mathbb{C}[X, X^{-1}]$ the derivative of P on the unit circle:

$$\partial P(z) = \lim_{\theta \rightarrow 0} \frac{P(ze^{i\theta}) - P(z)}{\theta}, \quad \text{for } z \in \mathbb{U}.$$

(In [28], this was denoted P' .) Concretely, for all $n \in \mathbb{Z}$, if $P = X^n$ then $\partial P = inX^n$. Lévy and Maïda proved that, for all $P, Q \in \mathbb{C}[X, X^{-1}]$, the covariance of the two scaled fluctuation random variables $N(\text{tr}(P(U_T^N)) - \mathbb{E}[\text{tr}(P(U_T^N))])$ and $N(\text{tr}(Q(U_T^N)) - \mathbb{E}[\text{tr}(Q(U_T^N))])$ is asymptotically equal to

$$\int_0^T \tau(\partial P(u_t v_{T-t})(\partial Q(u_t w_{T-t}))^*) dt, \tag{4.1}$$

and moreover that, as $T \rightarrow \infty$, this approaches the Sobolev $H_{1/2}$ inner product of P, Q (cf. [28, Theorem 9.3]). Note that the expression (4.1) is obtained from the expression of the covariance in [28, Definition 2.4] by linearity (since the expression of the covariance in [28, Definition 2.4] is only valid for real-valued functions).

In this section, we relate our result to theirs by giving another expression of the covariance of the fluctuations of the more general processes $B_{r,s}^N$, which naturally generalizes (4.1).

4.1. New characterization of the covariance

Denote by I_N^J the identity element $(I_N, \dots, I_N) \in \mathbb{GL}_N^J$. In the following proposition, we express the covariance with the help of three independent (r, s) -Brownian motions.

Proposition 4.2. *Let B^N, C^N, D^N be three independent (r, s) -Brownian motions on \mathbb{GL}_N^J . For all $P, Q \in \mathcal{P}(J)$, we have*

$$\sigma_{\mathbf{T}}(P, Q) = N^2 \int_0^1 \mathbb{E}[\Gamma_N^{\mathbf{T}}([P]_N(B_{t\mathbf{T}}^N(\cdot)C_{(1-t)\mathbf{T}}^N), [Q]_N(B_{t\mathbf{T}}^N(\cdot)D_{(1-t)\mathbf{T}}^N))(I_N^J)] dt + O\left(\frac{1}{N^2}\right).$$

To be clear on notation: the functions in the arguments of $\Gamma_N^{\mathbf{T}}$ above are

$$G \mapsto [P]_N(B_{t\mathbf{T}}^N G C_{(1-t)\mathbf{T}}^N) \quad \text{and} \quad G \mapsto [Q]_N(B_{t\mathbf{T}}^N G D_{(1-t)\mathbf{T}}^N).$$

We then apply $\Gamma_N^{\mathbf{T}}$ to these functions of G and evaluate at $G = I_N^J$ before taking the expectation value and integrating with respect to t . This (\cdot) notation is used throughout this section.

Proof. For all $P, Q \in \mathcal{P}(J)$, we have

$$\sigma_{\mathbf{T}}(P, Q) = \int_0^1 [e^{\frac{t}{2}\mathcal{D}^{\mathbf{T}}}(\Gamma^{\mathbf{T}}(e^{\frac{1-t}{2}\mathcal{D}^{\mathbf{T}}}P, e^{\frac{1-t}{2}\mathcal{D}^{\mathbf{T}}}Q))](\mathbf{1}) dt.$$

As in the proof of Theorem 3.3, we restrict our computations on a finite-dimensional space \mathcal{P}_d (take d to be the sum of the degrees of P and Q). Because of Lemma 3.11, $N^2(e^{\frac{t}{2}\mathcal{D}^{\mathbf{T}}} - e^{\frac{t}{2}(\mathcal{D}^{\mathbf{T}} + \frac{1}{N^2}\mathcal{L}^{\mathbf{T}})})$ is bounded uniformly in N and t ; consequently, it is straightforward to verify that

$$\sigma_{\mathbf{T}}(P, Q) = \int_0^1 [e^{\frac{t}{2}(\mathcal{D}^{\mathbf{T}} + \frac{1}{N^2}\mathcal{L}^{\mathbf{T}})}(\Gamma^{\mathbf{T}}(e^{\frac{1-t}{2}(\mathcal{D}^{\mathbf{T}} + \frac{1}{N^2}\mathcal{L}^{\mathbf{T}}}P, e^{\frac{1-t}{2}(\mathcal{D}^{\mathbf{T}} + \frac{1}{N^2}\mathcal{L}^{\mathbf{T}}}Q)))](\mathbf{1}) dt + O\left(\frac{1}{N^2}\right).$$

Hence, the proof will be complete once we show that, for $0 \leq t \leq 1$,

$$\begin{aligned} & [e^{\frac{t}{2}(\mathcal{D}^{\mathbf{T}} + \frac{1}{N^2}\mathcal{L}^{\mathbf{T}})}(\Gamma^{\mathbf{T}}(e^{\frac{1-t}{2}(\mathcal{D}^{\mathbf{T}} + \frac{1}{N^2}\mathcal{L}^{\mathbf{T}}}P, e^{\frac{1-t}{2}(\mathcal{D}^{\mathbf{T}} + \frac{1}{N^2}\mathcal{L}^{\mathbf{T}}}Q)))](\mathbf{1}) \\ &= N^2 \mathbb{E}[\Gamma_N^{\mathbf{T}}([P]_N(B_{t\mathbf{T}}^N(\cdot)C_{(1-t)\mathbf{T}}^N), [Q]_N(B_{t\mathbf{T}}^N(\cdot)D_{(1-t)\mathbf{T}}^N))(I_N^J)]. \end{aligned} \quad (4.2)$$

Fix $t \in [0, 1]$. We start from the left side to recover the right side. First of all, using Theorem 2.4, Proposition 3.1 and Definition 2.1, we have

$$\begin{aligned} & [e^{\frac{t}{2}(\mathcal{D}^{\mathbf{T}} + \frac{1}{N^2}\mathcal{L}^{\mathbf{T}})}(\Gamma^{\mathbf{T}}(e^{\frac{1-t}{2}(\mathcal{D}^{\mathbf{T}} + \frac{1}{N^2}\mathcal{L}^{\mathbf{T}}}P, e^{\frac{1-t}{2}(\mathcal{D}^{\mathbf{T}} + \frac{1}{N^2}\mathcal{L}^{\mathbf{T}}}Q)))](\mathbf{1}) \\ &= e^{\frac{t}{2}(\mathbf{T} \cdot \Delta^N)} [\Gamma^{\mathbf{T}}(e^{\frac{1-t}{2}(\mathcal{D}^{\mathbf{T}} + \frac{1}{N^2}\mathcal{L}^{\mathbf{T}}}P, e^{\frac{1-t}{2}(\mathcal{D}^{\mathbf{T}} + \frac{1}{N^2}\mathcal{L}^{\mathbf{T}}}Q))]_N(I_N^J) \\ &= N^2 [e^{\frac{t}{2}(\mathbf{T} \cdot \Delta^N)} (\Gamma_N^{\mathbf{T}}([e^{\frac{1-t}{2}(\mathcal{D}^{\mathbf{T}} + \frac{1}{N^2}\mathcal{L}^{\mathbf{T}}}P]_N, [e^{\frac{1-t}{2}(\mathcal{D}^{\mathbf{T}} + \frac{1}{N^2}\mathcal{L}^{\mathbf{T}}}Q]_N)))](I_N^J) \\ &= N^2 [e^{\frac{t}{2}(\mathbf{T} \cdot \Delta^N)} (\Gamma_N^{\mathbf{T}}(e^{\frac{1-t}{2}(\mathbf{T} \cdot \Delta^N)}[P]_N, e^{\frac{1-t}{2}(\mathbf{T} \cdot \Delta^N)}[Q]_N)))](I_N^J) \\ &= N^2 \mathbb{E}[(\Gamma_N^{\mathbf{T}}(e^{\frac{1-t}{2}(\mathbf{T} \cdot \Delta^N)}[P]_N, e^{\frac{1-t}{2}(\mathbf{T} \cdot \Delta^N)}[Q]_N))(B_{t\mathbf{T}}^N)]. \end{aligned}$$

Referring to the definition (3.1) of $\Gamma_N^{\mathbf{T}}$, this last quantity is equal to the expectation of

$$\frac{1}{2} N^2 \left[\sum_{\xi \in \beta_{r,s}^N, j \in J} t_j (\partial_{\xi_j}(e^{\frac{1-t}{2}(\mathbf{T} \cdot \Delta^N)}[P]_N)) (\partial_{\xi_j}(e^{\frac{1-t}{2}(\mathbf{T} \cdot \Delta^N)}[Q]_N)) \right] (B_{t\mathbf{T}}^N), \quad (4.3)$$

where $\beta_{r,s}^N$ is an orthonormal basis of \mathbb{M}_N for the metric $\langle \cdot, \cdot \rangle_{r,s}^N$, and for all $\xi \in \beta_{r,s}^N$, ξ_j is the left-invariant vector field which acts only on the j th component of \mathbb{GL}_N^J . Consider the first term in the first summand of (4.3) (ignoring the factor

of $\frac{1}{2}N^2t_j$); by definition (2.1) of ∂_{ξ_j} , it is equal to

$$\partial_{\xi_j} \left(e^{\frac{1-t}{2}(\mathbf{T} \cdot \Delta^N)} [P]_N \right) (B_{t\mathbf{T}}^N) = \frac{d}{d\theta} \Big|_{\theta=0} e^{\frac{1-t}{2}(\mathbf{T} \cdot \Delta^N)} [P]_N (B_{t\mathbf{T}}^N e^{\theta \xi_j}). \quad (4.4)$$

Now, the action of the diffusion operator $e^{\frac{1-t}{2}(\mathbf{T} \cdot \Delta^N)}$ on a C^∞ function f is given by

$$(e^{\frac{1-t}{2}(\mathbf{T} \cdot \Delta^N)} f)(G) = \mathbb{E}[f(GC_{(1-t)\mathbf{T}}^N)].$$

We apply this to the right-hand-side of (4.4); since the function $f = [P]_N$ is evaluated at the *random* point $G = B_{t\mathbf{T}}^N e^{\theta \xi_j}$, the result is a conditional expectation:

$$\frac{d}{d\theta} \Big|_{\theta=0} \mathbb{E}[[P]_N (B_{t\mathbf{T}}^N e^{\theta \xi_j} \cdot C_{(1-t)\mathbf{T}}^N) | B_{t\mathbf{T}}^N].$$

By local uniform boundedness, we can move the derivative inside the conditional expectation, which yields

$$\partial_{\xi_j} \left(e^{\frac{1-t}{2}(\mathbf{T} \cdot \Delta^N)} [P]_N \right) (B_{t\mathbf{T}}^N) = \mathbb{E}[\partial_{\xi_j} [P]_N (B_{t\mathbf{T}}^N(\cdot) C_{(1-t)\mathbf{T}}^N) (I_N^J) | B_{t\mathbf{T}}^N]. \quad (4.5)$$

An entirely analogous calculation with the second term in the product in (4.3), this time using the Brownian motion $D_{(1-t)\mathbf{T}}^N$ instead of $C_{(1-t)\mathbf{T}}^N$, yields

$$\partial_{\xi_j} \left(e^{\frac{1-t}{2}(\mathbf{T} \cdot \Delta^N)} [Q]_N \right) (B_{t\mathbf{T}}^N) = \mathbb{E}[\partial_{\xi_j} [Q]_N (B_{t\mathbf{T}}^N(\cdot) D_{(1-t)\mathbf{T}}^N) (I_N^J) | B_{t\mathbf{T}}^N]. \quad (4.6)$$

Taking the product of (4.5) and (4.6), and noting the conditional independence of these terms given $B_{t\mathbf{T}}^N$ (due to the independence of C^N and D^N), this shows that (4.3) is equal to

$$\frac{1}{2}N^2 \left[\sum_{\xi \in \beta_{r,s}^N, j \in J} t_j \mathbb{E}[\partial_{\xi_j} [P]_N (B_{t\mathbf{T}}^N(\cdot) C_{(1-t)\mathbf{T}}^N) \cdot \partial_{\xi_j} [Q]_N (B_{t\mathbf{T}}^N(\cdot) D_{(1-t)\mathbf{T}}^N) (I_N^J) | B_{t\mathbf{T}}^N] \right].$$

Taking the expectation, and referring again to the definition (3.1) of $\Gamma_N^{\mathbf{T}}$, leads to the right side of (4.2). This concludes the proof. \square

We shall now let the dimension tend to infinity in the previous proposition in order to have a new expression of the covariance involving three freely independent free multiplicative (r, s) -Brownian motions.

Theorem 4.3. *For all $P, Q \in \mathcal{P}(J)$, there exists $\tilde{\Gamma}^{\mathbf{T}}(P, Q) \in \mathcal{P}(J^3)$ such that for all $N \in \mathbb{N}$, and all $B, C, D \in \mathbb{GL}_N^J$,*

$$N^2 \Gamma_N^{\mathbf{T}}([P]_N(B(\cdot)C), [Q]_N(B(\cdot)D))(I_N^J) = [\tilde{\Gamma}^{\mathbf{T}}(P, Q)]_N(B, C, D) \quad (4.7)$$

and in this case, taking three families b, c, d of free multiplicative (r, s) -Brownian motions indexed by J which are freely independent in a noncommutative probability space (\mathcal{A}, τ) , we have

$$\sigma_{\mathbf{T}}(P, Q) = \int_0^1 [\tilde{\Gamma}^{\mathbf{T}}(P, Q)]_{(\mathcal{A}, \tau)}(b_{t\mathbf{T}}, c_{(1-t)\mathbf{T}}, d_{(1-t)\mathbf{T}}) dt.$$

This expression for the covariance, albeit instructive, is not explicit, but in the next section, we will compute the function $[\tilde{\Gamma}^{\mathbf{T}}(P, Q)]_N$ explicitly in the one-variable case $J = \{1\}$ and $\mathbf{T} = (T)$.

Proof. Let us suppose first that the polynomials P and Q are given by $P = v_\varepsilon$ and $Q = v_\delta$, with $\varepsilon = ((j_1, \varepsilon_1), \dots, (j_n, \varepsilon_n)) \in \mathcal{E}$ and $\delta = ((k_1, \delta_1), \dots, (k_m, \delta_m)) \in \mathcal{E}$. Hence, for any input $G \in \mathbb{M}_N$,

$$[P]_N(BGC) = \text{tr}((BGC)_{j_1}^{\varepsilon_1} \cdots (BGC)_{j_n}^{\varepsilon_n}), \quad [Q]_N(BGD) = \text{tr}((BGD)_{h_1}^{\delta_1} \cdots (BGD)_{h_m}^{\delta_m}).$$

We can then compute

$$\begin{aligned}\partial_{\xi_j}([P]_N(B(\cdot)C))(I_N^J) &= \sum_{l=1}^n \delta_{j,j_l} \operatorname{tr}((BC)_{j_1}^{\varepsilon_1} \cdots (B\xi C)_{j_k}^{\varepsilon_k} \cdots (BC)_{j_n}^{\varepsilon_n}) \\ &= \sum_{l=1}^n \delta_{j,j_l} \operatorname{tr}(\xi^{\varepsilon_k} \cdot [v_{\varepsilon(l)}]_N(B, C, D)),\end{aligned}$$

where $\varepsilon^{(l)}$ is a word in $\mathcal{E}(J^3)$, which depends on ε and l . Similarly,

$$\partial_{\xi_j}([Q]_N(B(\cdot)D))(I_N^J) = \sum_{h=1}^m \delta_{j,j_h} \operatorname{tr}(\xi^{\delta_h} \cdot [v_{\delta(h)}]_N(B, C, D)),$$

where $\delta^{(h)}$ is a word in $\mathcal{E}(J^3)$, which depends on δ and h . Finally, using the magic formula of Proposition 2.2, we have

$$\begin{aligned}\frac{N^2}{2} \sum_{\xi \in \beta_{r,s}^N, j \in J} t_j \partial_{\xi_j}([P]_N(B(\cdot)C)) \partial_{\xi_j}([Q]_N(B(\cdot)D))(I_N^J) \\ = \frac{1}{2} \sum_{j \in J} t_j \sum_{l=1}^n \sum_{h=1}^m \delta_{j,j_l} \delta_{j,j_h} (s + \sigma_{l,h} r) \operatorname{tr}([v_{\varepsilon(l)}](B, C, D) \cdot [v_{\delta(h)}]_N(B, C, D)),\end{aligned}$$

where $\sigma_{l,h} \in \{\pm 1\}$ depends on ε , δ , l and h . Thus, the element

$$\tilde{\Gamma}^{\mathbf{T}}(v_{\varepsilon}, v_{\delta}) = \frac{1}{2} \sum_{j \in J} t_j \sum_{l=1}^n \sum_{h=1}^m \delta_{j,j_l} \delta_{j,j_h} (s + \sigma_{l,h} r) v_{\varepsilon(l)\delta(h)} \in \mathcal{P}(J^3)$$

satisfies (4.7).

We extend the definition of $\tilde{\Gamma}^{\mathbf{T}}$ to all elements of $\mathcal{P}(J)$ of the form $P_1 \cdots P_k, Q_1 \cdots Q_l \in \mathcal{P}$ by the relation

$$\tilde{\Gamma}^{\mathbf{T}}(P_1 \cdots P_k, Q_1 \cdots Q_l) = \sum_{1 \leq i \leq k} \sum_{1 \leq j \leq l} P_1 \cdots \widehat{P_i} \cdots P_k Q_1 \cdots \widehat{Q_j} \cdots Q_l \tilde{\Gamma}^{\mathbf{T}}(P_i, Q_j),$$

and finally, we extend $\tilde{\Gamma}^{\mathbf{T}}$ to all elements of $\mathcal{P}(J)$ by bilinearity. Because $\Gamma_N^{\mathbf{T}}$ fulfills the same relations, this demonstrates (4.7).

Thanks to Proposition 4.2, we have

$$\begin{aligned}\sigma_{\mathbf{T}}(P, Q) &= N^2 \int_0^1 \mathbb{E}[\Gamma_N^{\mathbf{T}}([P]_N(B_{t\mathbf{T}}^N(\cdot)C_{(1-t)\mathbf{T}}^N), [Q]_N(B_{t\mathbf{T}}^N(\cdot)D_{(1-t)\mathbf{T}}^N))(I_N^J)] dt + O\left(\frac{1}{N^2}\right) \\ &= \mathbb{E}\left[\int_0^1 [\tilde{\Gamma}^{\mathbf{T}}(P, Q)]_N(B_{t\mathbf{T}}^N, C_{(1-t)\mathbf{T}}^N, D_{(1-t)\mathbf{T}}^N) dt\right] + O\left(\frac{1}{N^2}\right) \\ &= \mathbb{E}\left[\left[\int_0^1 (\tilde{\Gamma}^{\mathbf{T}}(P, Q)) dt\right]_N(B_{t\mathbf{T}}^N, C_{(1-t)\mathbf{T}}^N, D_{(1-t)\mathbf{T}}^N)\right] + O\left(\frac{1}{N^2}\right).\end{aligned}$$

In [25], it is proved that, for all $R \in \mathcal{P}(J^3)$, we have

$$\mathbb{E}[[R]_N(B_{t\mathbf{T}}^N, C_{(1-t)\mathbf{T}}^N, D_{(1-t)\mathbf{T}}^N)] = [R]_{(\mathcal{A}, \tau)}(b_{t\mathbf{T}}, c_{(1-t)\mathbf{T}}, d_{(1-t)\mathbf{T}}) + O\left(\frac{1}{N^2}\right),$$

(cf. Remark 4.4). Letting $N \rightarrow \infty$, it follows that

$$\sigma_{\mathbf{T}}(P, Q) = \int_0^1 [\tilde{\Gamma}^{\mathbf{T}}(P, Q)]_{(\mathcal{A}, \tau)}(b_{t\mathbf{T}}, c_{(1-t)\mathbf{T}}, d_{(1-t)\mathbf{T}}) dt.$$

□

Remark 4.4. The main theorem [25, Theorem 1.1] is stated in the special case that R is the trace of a noncommutative polynomial, and moreover only for instances of a single Brownian motion. However, [25, Corollary 5.6] shows how to quickly and easily extend this to the more general setting of convergence of any trace polynomial in instances of any finite family of independent Brownian motions, as we use presently.

4.2. Polynomial test functions

Throughout this section, we investigate the case where $J = \{1\}$ and $\mathbf{T} = (T)$. In this case, we have the injective map from $\mathbb{C}\langle X, X^* \rangle$ to $\text{tr}(\mathbb{C}\{J\}) \cong \mathcal{P}(J)$ denoted by tr , and similarly the injective map from $\mathbb{C}\langle X_1, X_1^*, X_2, X_2^*, X_3, X_3^* \rangle$ to $\text{tr}(\mathbb{C}\{J^3\}) \cong \mathcal{P}(J^3)$ also denoted by tr .

In the case of a polynomial, it is possible to compute explicitly the term $[\tilde{\Gamma}^{\mathbf{T}}(P, Q)]$ of Theorem 4.3, and thus recover the expression for the covariance given by (4.1).

Corollary 4.5. *Let us suppose $J = \{1\}$, $\mathbf{T} = (T)$, and $P, Q \in \mathbb{C}[X]$. Then, following Theorem 4.3,*

$$\begin{aligned} (\tilde{\Gamma}^{\mathbf{T}}(\text{tr } P, \text{tr } Q)) &= T(s - r) \text{tr}(\partial P(X_1 X_2) \partial Q(X_1 X_3)), \\ (\tilde{\Gamma}^{\mathbf{T}}(\text{tr } P, \text{tr } Q^*)) &= T(r + s) \text{tr}(\partial P(X_1 X_2) (\partial Q(X_1 X_3))^*) \end{aligned}$$

and

$$(\tilde{\Gamma}^{\mathbf{T}}(\text{tr } P^*, \text{tr } Q^*)) = T(s - r) \text{tr}((\partial P(X_1 X_2))^* (\partial Q(X_1 X_3))^*).$$

Consequently, taking three free multiplicative (r, s) -Brownian motions b, c, d which are freely independent in a noncommutative probability space (\mathcal{A}, τ) , we have

$$\begin{aligned} \sigma_{\mathbf{T}}(\text{tr } P, \text{tr } Q) &= (s - r) \int_0^T \tau[\partial P(b_t c_{T-t}) \partial Q(b_t d_{T-t})] dt, \\ \sigma_{\mathbf{T}}(\text{tr } P, \text{tr } Q^*) &= (r + s) \int_0^T \tau[\partial P(b_t c_{T-t}) (\partial Q(b_t d_{T-t}))^*] dt \quad \text{and} \\ \sigma_{\mathbf{T}}(\text{tr } P^*, \text{tr } Q^*) &= (s - r) \int_0^T \tau[(\partial P(b_t c_{T-t}))^* (\partial Q(b_t d_{T-t}))^*] dt. \end{aligned}$$

Remark 4.6. Let us make a few comments on this final corollary.

- (1) In the case $(r, s) = (1, 0)$ (the unitary Brownian motion), this result shows that, for all $P, Q \in \mathbb{C}[X]$, the covariance of the random variables $N(\text{tr } P(U_T^N) - \mathbb{E}[\text{tr } P(U_T^N)])$ and $N(\text{tr } Q(U_T^N) - \mathbb{E}[\text{tr } Q(U_T^N)])$ is asymptotically equal to $\sigma_{\mathbf{T}}(\text{tr } P, \text{tr } Q^*)$, which reproduces exactly the expression of (4.1) found by Lévy and Maïda in [28].
- (2) In the case $(r, s) = (\frac{1}{2}, \frac{1}{2})$ (the standard Brownian motion on $\mathbb{G}\mathbb{L}_N$), this result shows that, for all $P \in \mathbb{C}[X]$, the fluctuation random variable $N(\text{tr}(P(G_T^N)) - \mathbb{E}[\text{tr}(P(G_T^N))])$ is asymptotically a rotationally-invariant complex normal distribution of variance $\int_0^T \tau(\partial P(b_t c_{T-t}) (\partial P(b_t d_{T-t}))^*) dt$, where b, c, d are three freely independent standard free multiplicative Brownian motions.

Proof. Let $P = X^n$ and $Q = X^m$. We have $\text{tr } P = v_\varepsilon$ and $\text{tr } Q = v_\delta$ with $\varepsilon = \overbrace{((1, 1), \dots, (1, 1))}^{n \text{ times}} \in \mathcal{E}$ and $\delta = \overbrace{((1, 1), \dots, (1, 1))}^{m \text{ times}} \in \mathcal{E}$. Let $N \in \mathbb{N}$, and $B, C, D \in \mathbb{G}\mathbb{L}_N^J$. Then for all $G \in \mathbb{M}_N$,

$$[\text{tr } P]_N(CGB) = \text{tr}((CGB)^n), \quad [\text{tr } Q]_N(DGB) = \text{tr}((DGB)^m).$$

We compute for all $\xi \in \beta_{r,s}^N$

$$\partial_\xi([\text{tr } P]_N(B(\cdot)C))(I_N^J) = n \text{tr}(\xi(CB)^n)$$

and

$$\partial_\xi([\text{tr } Q]_N(B(\cdot)D))(I_N^J) = m \text{tr}(\xi(DB)^m).$$

Finally, using the magic formula of Proposition 2.2, we have

$$\begin{aligned} mN^2 \sum_{\xi \in \beta_{r,s}^N} T \partial_{\xi} ([\text{tr } P]_N (B(\cdot)C)) \partial_{\xi} ([\text{tr } Q]_N (B(\cdot)D)) (I_N) &= T(s-r)mn \text{tr}((CB)^n (DB)^m) \\ &= (\tilde{\Gamma}^{\mathbf{T}}(\text{tr } P, \text{tr } Q))(B, C, D) \end{aligned}$$

with $(\tilde{\Gamma}^{\mathbf{T}}(\text{tr } P, \text{tr } Q)) = T(s-r) \text{tr}(\partial P(X_1 X_2) \partial Q(X_1 X_3))$. Similar computations lead to the formulas $(\tilde{\Gamma}^{\mathbf{T}}(\text{tr } P, \text{tr } Q^*)) = T(r+s) \text{tr}(\partial P(X_1 X_2) \partial Q(X_1 X_3)^*)$ and $(\tilde{\Gamma}^{\mathbf{T}}(\text{tr } P^*, \text{tr } Q^*)) = T(s-r) \text{tr}(\partial P(X_1 X_2)^* \partial Q(X_1 X_3)^*)$, and we extend the formulas to $P, Q \in \mathbb{C}[X]$ by bilinearity.

Thanks to Proposition 4.3, we know that

$$\begin{aligned} \sigma_{\mathbf{T}}(\text{tr } P, \text{tr } Q) &= \int_0^1 (\tilde{\Gamma}^{\mathbf{T}}(P, Q))(b_{t\mathbf{T}}, c_{(1-t)\mathbf{T}}, d_{(1-t)\mathbf{T}}) dt \\ &= T(s-r) \int_0^1 \tau[\partial P(b_{tT} c_{(1-t)T}) \partial Q(b_{tT} d_{(1-t)T})] dt \\ &= (s-r) \int_0^T \tau[\partial P(b_{tT} c_{T-t}) \partial Q(b_{tT} d_{T-t})] dt, \end{aligned}$$

and the two others cases are treated similarly. \square

Appendix. The intertwining spaces $\mathcal{P}(J)$ and $\mathbb{C}\{J\}$

Let J be an index set. In this appendix, we describe the link between two spaces used to study trace polynomial functions, that is to say linear combination of functions $\mathbb{M}_N^J \rightarrow \mathbb{M}_N$ of the form

$$\mathbf{A} \mapsto P_0(\mathbf{A}) \text{tr}(P_1(\mathbf{A})) \text{tr}(P_2(\mathbf{A})) \cdots \text{tr}(P_m(\mathbf{A}))$$

for some finite m , where $P_1, \dots, P_m \in \mathbb{C}$ are noncommutative $*$ -polynomials in J variables. In Section 2.2, we already defined the space $\mathcal{P}(J)$, introduced in [18,25]. Let us now define the space $\mathbb{C}\{J\}$, another space which was introduced in [9]. Finally, we will see that those two spaces are linked by a natural isomorphism.

The abstract trace polynomial algebra $\mathbb{C}\{J\}$ is a \mathbb{C} -algebra equipped with a center-valued expectation functional $\text{tr}: \mathbb{C}\{J\} \rightarrow Z(\mathbb{C}\{J\})$: a linear map with values in the center of $\mathbb{C}\{J\}$ and satisfying $\text{tr}(1_{\mathbb{C}\{J\}}) = 1_{\mathbb{C}\{J\}}$ and $\text{tr}(\text{tr}(A)B) = \text{tr}(A)\text{tr}(B)$ for all $A, B \in \mathbb{C}\{J\}$. (Note: the symbol tr is presently denoting an abstract function, not necessarily the normalized trace on \mathbb{M}_N .) The algebra $\mathbb{C}\{J\}$ is an extension of $\mathbb{C}\langle J \rangle = \mathbb{C}\langle X_j, X_j^*: j \in J \rangle$, the noncommutative polynomials in J variables and their adjoints, in the sense that we have the injective inclusion $\mathbb{C}\langle J \rangle \subset \mathbb{C}\{J\}$. In [9], it is denoted by

$$\mathbb{C}\{J\} \equiv \mathbb{C}\{X_j, X_j^*: j \in J\}.$$

It is defined by a universal property [9, Universal Property 1.1]: let \mathcal{A} be any \mathbb{C} -algebra equipped with a center-valued trace τ , and specified elements $(A_{(j,\varepsilon)})_{j \in J, \varepsilon \in \{1, *\}}$ in \mathcal{A} . Then there is a unique algebra homomorphism $f: \mathbb{C}\{J\} \rightarrow \mathcal{A}$ such that

- (1) for all $(j, \varepsilon) \in J \times \{1, *\}$, $f(X_j^{\varepsilon}) = A_{(j,\varepsilon)}$; and
- (2) for all $X \in \mathbb{C}\{J\}$, $\tau(f(X)) = f(\text{tr}(X))$.

This property uniquely defines $\mathbb{C}\{J\}$ up to adapted isomorphisms, cf. [9, Proposition-Definition 1.3], but we can also construct explicitly one realization of $\mathbb{C}\{J\}$ as a partially symmetrized tensor algebra over $\mathbb{C}\langle J \rangle$. As a vector space, it has as a basis the set

$$\{M_0 \text{tr } M_1 \cdots \text{tr } M_k, k \in \mathbb{N}, M_0, \dots, M_k \text{ are monomials in } \mathbb{C}\langle J \rangle\}.$$

Following [9], the universal property allows also to define a $\mathbb{C}\{J\}$ -calculus. It is explicitly given as follows: for each $\mathbf{a} = (a_j)_{j \in J} \in \mathcal{A}^J$ and each $P_0, \dots, P_k \in \mathbb{C}\langle J \rangle$, we have

$$(P_0 \text{tr } P_1 \cdots \text{tr } P_k)(\mathbf{a}) = P_0(\mathbf{a}) \cdot \tau(P_1(\mathbf{a})) \cdots \tau(P_k(\mathbf{a})),$$

and $P \mapsto P(\mathbf{a})$ is an algebra homomorphism. As a consequence, the space $\mathbb{C}\{J\}$ can be used to index the trace polynomial function on \mathbb{M}_N .

As we will now see, $\mathcal{P}(J)$ is isomorphic to the “scalar part” $\text{tr}(\mathbb{C}\{J\})$ of $\mathbb{C}\{J\}$.

Lemma A.1. *For any index set J , there is an algebra isomorphism*

$$\Upsilon: \mathbb{C}\langle J \rangle \otimes \mathcal{P}(J) \rightarrow \mathbb{C}\{J\}$$

such that the restriction $\Upsilon|_{1 \otimes \mathcal{P}(J)}$ is an algebra isomorphism onto $\text{tr}(\mathbb{C}\{J\})$. More explicitly, Υ is given as follows: for any monomial $M_0 \in \mathbb{C}\langle J \rangle$ and any words $\varepsilon^{(j)} \in \mathcal{E}$, we have

$$\Upsilon(1) = 1, \quad \Upsilon(M_0 \otimes v_{\varepsilon^{(1)}} \cdots v_{\varepsilon^{(k)}}) = M_0 \text{tr}(X_{\varepsilon^{(1)}}) \cdots \text{tr}(X_{\varepsilon^{(k)}}),$$

where, for all $\varepsilon = ((j_1, \varepsilon_1), \dots, (j_n, \varepsilon_n))$, $X_\varepsilon = X_{j_1}^{\varepsilon_1} \cdots X_{j_n}^{\varepsilon_n}$.

Proof. The homomorphism Υ transforms a basis of $\mathbb{C}\langle J \rangle \otimes \mathcal{P}(J)$ into a basis of $\mathbb{C}\{J\}$, and is therefore a vector space isomorphism. It is simple to check that it is also an algebra homomorphism. Alternatively, $\mathbb{C}\langle J \rangle \otimes \mathcal{P}(J)$ is naturally isomorphic to the construction of $\mathbb{C}\{J\}$ in [9, Appendix] as the partial symmetrization of the tensor algebra over $\mathbb{C}\langle J \rangle$ – the polynomial algebra $\mathcal{P}(J)$ is nothing other than the symmetric tensor algebra over $\mathbb{C}\langle J \rangle$. It is also easy to see that this map defines an algebra isomorphism using the universal property defining the space $\mathbb{C}\langle J \rangle$ in [9], where the center-valued expectation on the algebra $\mathbb{C}\langle J \rangle \otimes \mathcal{P}(J)$ is the tracing map \mathcal{T} of [18, Definition 3.12], defined by

$$\mathcal{T}(X_{\varepsilon^{(0)}} \otimes v_{\varepsilon^{(1)}} \cdots v_{\varepsilon^{(k)}}) = v_{\varepsilon^{(0)}} v_{\varepsilon^{(1)}} \cdots v_{\varepsilon^{(k)}},$$

for any words $\varepsilon^{(0)}, \dots, \varepsilon^{(k)} \in \mathcal{E}$. □

It is immediate that, identifying $\text{tr}(\mathbb{C}\{J\})$ with $\mathcal{P}(J)$ via the isomorphism Υ , the $\mathbb{C}\{J\}$ -calculus is the same as the $\mathcal{P}(J)$ -calculus defined in 2.2: for all $P \in \text{tr}(\mathbb{C}\{J\}) \cong \mathcal{P}(J)$, and all $\mathbf{a} \in \mathcal{A}^J$ we have

$$P(\mathbf{a}) = [P]_{(\mathcal{A}, \tau)}(\mathbf{a}).$$

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