

LOGARITHMIC CONCAVITY OF SCHUR AND RELATED POLYNOMIALS

JUNE HUH, JACOB P. MATHERNE, KAROLA MÉSZÁROS, AND AVERY ST. DIZIER

ABSTRACT. We show that normalized Schur polynomials are strongly log-concave. As a consequence, we obtain Okounkov's log-concavity conjecture for Littlewood–Richardson coefficients in the special case of Kostka numbers.

1. INTRODUCTION

Schur polynomials are the characters of finite-dimensional irreducible polynomial representations of the general linear group $GL_m(\mathbb{C})$. Combinatorially, the *Schur polynomial* of a partition λ in m variables is the generating function

$$s_\lambda(x_1, \dots, x_m) = \sum_T x^{\mu(T)}, \quad x^{\mu(T)} = x_1^{\mu_1(T)} \cdots x_m^{\mu_m(T)},$$

where the sum is over all Young tableaux T of shape λ with entries from $[m]$, and

$$\mu_i(T) = \text{the number of } i\text{'s among the entries of } T, \text{ for } i = 1, \dots, m.$$

Collecting Young tableaux of the same weight together, we get

$$s_\lambda(x_1, \dots, x_m) = \sum_\mu K_{\lambda\mu} x^\mu,$$

where $K_{\lambda\mu}$ is the *Kostka number* counting Young tableaux of given shape λ and weight μ [Kos82]. Correspondingly, the *Schur module* $V(\lambda)$, an irreducible representation of the general linear group with highest weight λ , has the weight space decomposition

$$V(\lambda) = \bigoplus_\mu V(\lambda)_\mu \text{ with } \dim V(\lambda)_\mu = K_{\lambda\mu}.$$

June Huh received support from NSF Grant DMS-1638352 and the Ellentuck Fund. Jacob Matherne received support from NSF Grant DMS-1638352, the Association of Members of the Institute for Advanced Study, the Hausdorff Research Institute for Mathematics, and the Deutsche Forschungsgemeinschaft (DFG) under Germany's Excellence Strategy - GZ 2047/1, Projekt-ID 390685813. Karola Mészáros received support from NSF Grant DMS-1501059, CAREER NSF Grant DMS-1847284, and a von Neumann Fellowship funded by the Friends of the Institute for Advanced Study. Avery St. Dizier received support from NSF Grant DMS-2002079.

2020 *Mathematics Subject Classification*: Primary 05E10, 14M12, 17B10.

Key words and phrases: Schur polynomials, log-concavity, Lorentzian polynomials, weight multiplicities.

Schur polynomials were first studied by Cauchy [Cau15], who defined them as ratios of alternants. The connection to the representation theory of $\mathrm{GL}_m(\mathbb{C})$ was found by Schur [Sch01]. For a gentle introduction to these remarkable polynomials, and for all undefined terms, we refer to [Ful97].

We prove several log-concavity properties of Schur polynomials. An operator that turns generating functions into exponential generating functions will play an important role. This linear operator, denoted N , is defined by the condition

$$N(x^\mu) = \frac{x^\mu}{\mu!} = \frac{x_1^{\mu_1}}{\mu_1!} \cdots \frac{x_m^{\mu_m}}{\mu_m!} \text{ for all } \mu \in \mathbb{N}^m.$$

Recall that a *partition* is a weakly decreasing sequence of nonnegative integers.

Theorem 1 (Continuous). For any partition λ , the normalized Schur polynomial

$$N(s_\lambda(x_1, \dots, x_m)) = \sum_{\mu} K_{\lambda\mu} \frac{x^\mu}{\mu!}$$

is either identically zero or its logarithm is concave on the positive orthant $\mathbb{R}_{>0}^m$.

Let e_i be the i -th standard unit vector in \mathbb{N}^m . For $\mu \in \mathbb{Z}^m$ and distinct $i, j \in [m]$, we set

$$\mu(i, j) = \mu + e_i - e_j.$$

We show that the sequence of weight multiplicities of $V(\lambda)$ we encounter is always log-concave if we walk in the weight diagram along any root direction $e_i - e_j$.

Theorem 2 (Discrete). For any partition λ and any $\mu \in \mathbb{N}^m$, we have

$$K_{\lambda\mu}^2 \geq K_{\lambda\mu(i,j)} K_{\lambda\mu(j,i)} \text{ for any } i, j \in [m].$$

For partitions ν, κ, λ , the *Littlewood–Richardson coefficient* $c_{\kappa\lambda}^\nu$ is given by the decomposition

$$V(\kappa) \otimes V(\lambda) \simeq \bigoplus_{\nu} V(\nu)^{\oplus c_{\kappa\lambda}^\nu}.$$

When the skew shape ν/κ has at most one box in each column, $c_{\kappa\lambda}^\nu$ is the Kostka number $K_{\lambda\mu}$, where $\mu = \nu - \kappa$. This equality follows from Pieri's formula

$$h_{\mu_1}(x_1, \dots, x_m) \cdots h_{\mu_m}(x_1, \dots, x_m) = \sum_{\lambda} K_{\lambda\mu} s_{\lambda}(x_1, \dots, x_m),$$

where h_{μ_i} is the μ_i -th complete symmetric function [Ful97, Section 6.1]. When ν/κ has at most one box in each column, the left-hand side is the skew Schur function $s_{\nu/\kappa}$, given by the Littlewood–Richardson rule

$$s_{\nu/\kappa}(x_1, \dots, x_m) = \sum_{\lambda} c_{\kappa\lambda}^\nu s_{\lambda}(x_1, \dots, x_m).$$

Conversely, for any partition λ and any μ , we have

$$K_{\lambda\mu} = c_{\kappa\lambda}^\nu,$$

where ν and κ are the partitions given by $\nu_i = \sum_{j=i}^n \mu_j$ and $\kappa_i = \sum_{j=i+1}^n \mu_j$. Thus Theorem 2 verifies a special case of Okounkov's conjecture that the discrete function

$$(\nu, \kappa, \lambda) \mapsto \log c_{\kappa\lambda}^\nu$$

is concave [Oko03, Conjecture 1]. The conjecture holds in the "classical limit" [Oko03, Section 3], but the general case is refuted in [CDW07]:

$$c_{(3^n, 2^n, 1^n)(2^n, 1^n, 1^n)}^{(4^n, 3^n, 2^n, 1^n)} = \binom{n+2}{2} \text{ and } c_{(6^n, 4^n, 2^n)(4^n, 2^n, 2^n)}^{(8^n, 6^n, 4^n, 2^n)} = \binom{n+5}{5} \text{ for all } n.$$

The same example shows that the log-concavity conjecture for *parabolic Kostka numbers* [Kir04, Conjecture 6.17] also fails.

We point out that, for any fixed λ , the log-concavity of $K_{\lambda\mu}$ along *any* direction is known to hold *asymptotically*. By [Hec82], the *Duistermaat–Heckman measure* obtained from the orbit of λ under SU_m is a translate of the weak limit

$$\lim_{k \rightarrow \infty} \frac{\sum_{\mu} K_{k\lambda\mu} \delta_{\frac{1}{k}\mu}}{\sum_{\mu} K_{k\lambda\mu}},$$

where $\delta_{\frac{1}{k}\mu}$ is the point mass at $\frac{1}{k}\mu$. It follows from [Gra96] that, in this case, the density function of the Duistermaat–Heckman measure is log-concave. We refer to [BGR04, Section 3] for an exposition.

In [BH20], the authors introduce Lorentzian polynomials as a generalization of volume polynomials in algebraic geometry and stable polynomials in optimization theory. See Section 2 for a brief introduction. We show that normalized Schur polynomials are Lorentzian in the sense of [BH20], and deduce Theorems 1 and 2 from the Lorentzian property.

Theorem 3. The normalized Schur polynomial $N(s_{\lambda}(x_1, \dots, x_m))$ is Lorentzian for any λ .

Using general properties of Lorentzian polynomials [BH20, Section 6], Theorem 3 can be strengthened as follows.

Corollary 4. For any sequence of partitions $\lambda^1, \dots, \lambda^\ell$ and any positive integers m_1, \dots, m_ℓ ,

- (1) the normalized product of Schur polynomials $N(\prod_{k=1}^{\ell} s_{\lambda^k}(x_1, \dots, x_{m_k}))$ is Lorentzian, and
- (2) the product of normalized Schur polynomials $\prod_{k=1}^{\ell} N(s_{\lambda^k}(x_1, \dots, x_{m_k}))$ is Lorentzian.

We prove Theorem 3 in Section 2 in a more general context of Schubert polynomials, but the main idea is simple enough to be outlined here. Recall that a Cartier divisor on a complete variety Y is *nef* if it intersects every curve in Y nonnegatively (see [Laz04] for an introduction). The *volume polynomial* of an irreducible complex projective variety Y , with respect to a sequence of nef divisor classes $H = (H_1, \dots, H_m)$, is the homogeneous polynomial

$$\text{vol}_{Y,H}(x_1, \dots, x_m) = \frac{1}{\dim Y!} \int_Y (x_1 H_1 + \dots + x_m H_m)^{\dim Y},$$

where the intersection product of Y is used to expand the integrand. Volume polynomials are prototypical examples of Lorentzian polynomials [BH20, Section 10]. To show that the normalized Schur polynomial of λ is a volume polynomial, we suppose that the partition λ has m parts, and choose a large integer ℓ to get a complementary pair of partitions

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m) \text{ and } \kappa = (\ell, \ell, \dots, \ell) - (\lambda_m, \lambda_{m-1}, \dots, \lambda_1).$$

It is easy to check that the dual of the Schur module $V(\lambda)$ has highest weight $(-\lambda_m, \dots, -\lambda_1)$, see [FH91, Exercise 15.50]. Hence, the Schur polynomials of the partitions λ and κ are related by the identity

$$s_\kappa(x_1, \dots, x_m) = x_1^\ell \cdots x_m^\ell s_\lambda(x_1^{-1}, \dots, x_m^{-1}).$$

Let X be the product of projective spaces $(\mathbb{P}^\ell)^m$, and let Y be a subvariety of X whose fundamental class satisfies

$$[Y] = s_\kappa(H_1, \dots, H_m) \cap [X], \quad H_i = c_1(\pi_i^* \mathcal{O}(1)),$$

where π_i is the i -th projection. The volume polynomial of Y with respect to H is

$$\begin{aligned} \text{vol}_{Y,H}(x_1, \dots, x_m) &= \frac{1}{\dim Y!} \int_Y (x_1 H_1 + \dots + x_m H_m)^{\dim Y} \\ &= \frac{1}{\dim Y!} \int_X s_\kappa(H_1, \dots, H_m) (x_1 H_1 + \dots + x_m H_m)^{\dim Y} = N(s_\lambda(x_1, \dots, x_m)). \end{aligned}$$

Such Y can be constructed from a sequence of generic global sections $\bigoplus_{i=1}^m \pi_i^* \mathcal{O}(1)$ as a degeneracy locus [Ful98, Example 14.3.2], completing the argument.

In Section 2, we introduce Lorentzian polynomials and prove the main results. In Section 3, we present evidence for the ubiquity of Lorentzian polynomials through a series of results and conjectures.

Acknowledgments. We are grateful to Dave Anderson, Alex Fink, Allen Knutson, Thomas Lam, Ricky Liu, Alex Postnikov, Pavlo Pylyavskyy, Vic Reiner, Mark Shimozono, and Alex Yong for fruitful discussions. We thank the Institute for Advanced Study for providing an excellent environment for our collaboration.

2. NORMALIZED SCHUR POLYNOMIALS ARE LORENTZIAN

A subset $J \subseteq \mathbb{Z}^n$ is *M-convex* if, for any index $i \in [n]$ and any $\alpha \in J$ and $\beta \in J$ whose i -th coordinates satisfy $\alpha_i > \beta_i$, there is an index $j \in [n]$ satisfying

$$\alpha_j < \beta_j \text{ and } \alpha - e_i + e_j \in J \text{ and } \beta - e_j + e_i \in J.$$

The notion of M-convexity forms the foundation of discrete convex analysis [Mur03]. The convex hull of an M-convex set is a *generalized permutohedron* in the sense of [Pos09], and conversely, the set of integral points in an integral generalized permutohedron is an M-convex set [Mur03, Theorem 1.9].

Lorentzian polynomials connect discrete convex analysis with many log-concavity phenomena in combinatorics. See [AOGV18, ALOGV19, ALOGV18, BES19, BH18, BH20, EH20] for recent applications. Here we briefly summarize the relevant results, and refer to [BH20] for details.

We fix integers d and $e = d - 2$. By the *support* of a polynomial $h(x_1, \dots, x_n)$, we mean the set of monomials appearing in h , viewed as a subset of \mathbb{N}^n .

Definition 5. Let $h(x_1, \dots, x_n)$ be a degree d homogeneous polynomial. We say that h is *strictly Lorentzian* if all the coefficients of h are positive and

$$\frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_e}} h \text{ has the signature } (+, -, \dots, -) \text{ for any } i_1, \dots, i_e \in [n].$$

We say that h is *Lorentzian* if it satisfies any one of the following equivalent conditions.

(1) All the coefficients of h are nonnegative, the support of h is M-convex, and

$$\frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_e}} h \text{ has at most one positive eigenvalue for any } i_1, \dots, i_e \in [n].$$

(2) All the coefficients of h are nonnegative and, for any $i_1, i_2, \dots \in [n]$ and any positive k ,

the functions h and $\frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_k}} h$ are either identically zero or log-concave on $\mathbb{R}_{>0}^n$.

(3) The polynomial h is a limit of strictly Lorentzian polynomials.

For example, a bivariate polynomial $\sum_{k=0}^d a_k x_1^k x_2^{d-k}$ with nonnegative coefficients is Lorentzian if and only if the sequence a_0, \dots, a_d has no internal zeros and

$$\frac{a_k^2}{\binom{d}{k}^2} \geq \frac{a_{k-1}}{\binom{d}{k-1}} \frac{a_{k+1}}{\binom{d}{k+1}} \text{ for all } 0 < k < d.$$

Polynomials satisfying the second condition of Definition 5, introduced by Gurvits in [Gur09], are called *strongly log-concave*. See [BH20, Section 5] for a proof of the equivalence of the three conditions in Definition 5.

We write \mathcal{S}_n for the group of permutations of $[n]$. The *Schubert polynomial* $\mathfrak{S}_w(x_1, \dots, x_n)$ for $w \in \mathcal{S}_n$ can be defined recursively as follows.

(1) If $w = w_\circ$ is the longest permutation $n \ n-1 \ \dots \ 2 \ 1$, then

$$\mathfrak{S}_w(x_1, \dots, x_n) = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^1.$$

(2) If $w(i) > w(i+1)$ for some i and s_i is the adjacent transposition $(i \ i+1)$, then

$$\mathfrak{S}_{ws_i}(x_1, \dots, x_n) = \partial_i \mathfrak{S}_w(x_1, \dots, x_n).$$

The symbol ∂_i stands for the i -th divided difference operator defined by the formula

$$\partial_i \mathfrak{S}_w = \frac{\mathfrak{S}_w - s_i \mathfrak{S}_w}{x_i - x_{i+1}},$$

where $s_i \mathfrak{S}_w$ is the polynomial obtained from \mathfrak{S}_w by interchanging x_i and x_{i+1} . The divided difference operators satisfy the *braid relations*, and it follows that the Schubert polynomials are well-defined [MS05, Exercise 15.3]. For any $w \in \mathcal{S}_n$, we define

$$\mathfrak{S}_w^\vee = N(x_1^{n-1} \cdots x_n^{n-1} \mathfrak{S}_w(x_1^{-1}, \dots, x_n^{-1})).$$

Theorem 6. The polynomial $\mathfrak{S}_w^\vee(x_1, \dots, x_n)$ is Lorentzian for any $w \in \mathcal{S}_n$.

We conjecture that $N(\mathfrak{S}_w(x_1, \dots, x_n))$ is Lorentzian for any $w \in \mathcal{S}_n$, see Section 3.2.

Proof. Recall that the volume polynomial of a projective variety Y , with respect to a sequence of Cartier divisor classes $H = (H_1, \dots, H_n)$, is the homogeneous polynomial

$$\text{vol}_{Y,H}(x_1, \dots, x_n) = \frac{1}{\dim Y!} \int_Y (x_1 H_1 + \cdots + x_n H_n)^{\dim Y}.$$

By [BH20, Theorem 10.1], the volume polynomial is Lorentzian whenever Y is irreducible and H_1, \dots, H_n are nef. We show that \mathfrak{S}_w^\vee is a volume polynomial for suitable $Y = Y_w$ and H .

Let X be the product of projective spaces $(\mathbb{P}^{n-1})^n$. We write $x_{i1}, x_{i2}, \dots, x_{in}$ for the homogeneous coordinates of the i -th projective space, and write π_i for the i -th projection. We consider the map between the rank n vector bundles

$$\Psi : \bigoplus_{i=1}^n \mathcal{O}_X \longrightarrow \bigoplus_{j=1}^n \pi_j^* \mathcal{O}(1), \quad \Psi(x) = (x_{ij})_{1 \leq i \leq n, 1 \leq j \leq n}.$$

For $p, q \in [n]$, the induced map $\bigoplus_{i=1}^p \mathcal{O}_X \rightarrow \bigoplus_{j=1}^q \pi_j^* \mathcal{O}(1)$ will be denoted $\Psi_{p \times q}$. We set

$$Y = Y_w := \left\{ x \in X \mid \text{rank } \Psi_{p \times q}(x) \leq \text{rank } w_{p \times q} \text{ for all } p \text{ and } q \right\},$$

where $w_{p \times q}$ is the $p \times q$ partial permutation matrix with ij -entry 1 for $w(i) = j$. The locus Y is defined by all minors of $(x_{ij})_{1 \leq i \leq p, 1 \leq j \leq q}$ of size one more than the rank of $w_{p \times q}$ for all p and q .

By [Ful92, Theorem 8.2], the fundamental class of Y in the Chow group of X is given by

$$[Y] = \mathfrak{S}_w(H_1, \dots, H_n) \cap [X], \quad H_i = c_1(\pi_i^* \mathcal{O}(1)).$$

An alternative proof of the displayed formula, in a more refined setting, was obtained in [KM05] through an explicit degeneration of Y . An important point for us is that Y is irreducible of expected codimension $\deg \mathfrak{S}_w$ [Ful92]. For an elementary proof that the multi-homogeneous ideal defining Y is prime, see [MS05, Section 16.4]. The volume polynomial of Y with respect to $H = (H_1, \dots, H_n)$ is

$$\begin{aligned} \text{vol}_{Y,H}(x_1, \dots, x_n) &= \frac{1}{\dim Y!} \int_Y (x_1 H_1 + \dots + x_n H_n)^{\dim Y} \\ &= \frac{1}{\dim Y!} \int_X \mathfrak{S}_w(H_1, \dots, H_n) (x_1 H_1 + \dots + x_n H_n)^{\dim Y} = \mathfrak{S}_w^\vee(x_1, \dots, x_n). \end{aligned}$$

The second equality is the projection formula, and the third equality follows from

$$\int_X H^\mu = \begin{cases} 1 & \text{if } \mu = (n-1, \dots, n-1), \\ 0 & \text{if } \mu \neq (n-1, \dots, n-1). \end{cases}$$

Now the Lorentzian property of \mathfrak{S}_w^\vee can be deduced from [BH20, Theorem 10.1]. \square

Lemma 7. For any $\mu \in \mathbb{N}^n$ and any polynomial $f = f(x_1, \dots, x_n)$,

$$N(f) \text{ is Lorentzian if and only if } N(x^\mu f) \text{ is Lorentzian.}$$

Proof. If a polynomial $g(x_1, \dots, x_n)$ is Lorentzian, then so is its partial derivative

$$\partial^\mu g = \left(\frac{\partial}{\partial x_1} \right)^{\mu_1} \cdots \left(\frac{\partial}{\partial x_n} \right)^{\mu_n} g(x_1, \dots, x_n).$$

Therefore, the “if” direction follows from the equality of linear operators

$$\partial^\mu \circ N \circ x^\mu = N.$$

The “only if” direction is a special case of [BH20, Corollary 6.8]. \square

Proof of Theorem 3. As in the introduction, given a partition λ with m parts, we choose a large integer ℓ and write κ for the partition complementary to λ in the $m \times \ell$ rectangle. Choose another large integer n , and let w be the unique element of \mathcal{S}_n satisfying

$$\kappa = (w(m) - m, \dots, w(1) - 1) \text{ and } w(m) > w(m+1) < w(m+2) < \dots < w(n).$$

The element w is the *Grassmannian permutation* in \mathcal{S}_n with the *Lehmer code*

$$L(w) = (w(1) - 1, \dots, w(m) - m, 0, \dots, 0) = (\kappa_m, \dots, \kappa_1, 0, \dots, 0).$$

The Schubert polynomial of w satisfies

$$\mathfrak{S}_w(x_1, \dots, x_n) = s_\kappa(x_1, \dots, x_m) = x_1^\ell \cdots x_m^\ell s_\lambda(x_1^{-1}, \dots, x_m^{-1}),$$

where the first equality is [Man01, Proposition 2.6.8] and the second equality is [FH91, Exercise 15.50]. By Theorem 6, we know that the polynomial \mathfrak{S}_w^\vee is Lorentzian, which is equal to

$$\mathrm{N}(x_1^{n-1} \cdots x_n^{n-1} s_\kappa(x_1^{-1}, \dots, x_m^{-1})) = \mathrm{N}(x^\mu s_\lambda(x_1, \dots, x_m)) \text{ for some } \mu \in \mathbb{N}^n.$$

Therefore, by Lemma 7, the Lorentzian property of \mathfrak{S}_w^\vee implies that of $\mathrm{N}(s_\lambda(x_1, \dots, x_m))$. \square

Proofs of Theorems 1 and 2. Since any nonzero Lorentzian polynomial is log-concave on the positive orthant, Theorem 1 follows from Theorem 3. For Theorem 2, we may suppose that

$$\mu_1 + \cdots + \mu_m = \lambda_1 + \cdots + \lambda_m \geq 2 \text{ and } \kappa := \mu - e_i - e_j \in \mathbb{N}^m.$$

We consider the quadratic form with at most one positive eigenvalue

$$\frac{\partial^{\kappa_1}}{\partial x_1^{\kappa_1}} \cdots \frac{\partial^{\kappa_m}}{\partial x_m^{\kappa_m}} \mathrm{N}(s_\lambda(x_1, \dots, x_m)),$$

viewed as an $m \times m$ symmetric matrix. Its 2×2 principal submatrix corresponding to i and j is either identically zero or has exactly one positive eigenvalue, by Cauchy’s interlacing theorem. The nonpositivity of the 2×2 principal minor gives the conclusion

$$K_{\lambda\mu}^2 \geq K_{\lambda\mu(i,j)} K_{\lambda\mu(j,i)}.$$

\square

Proof of Corollary 4. The first part follows from Theorem 3 and [BH20, Corollary 6.8]. The second part follows from Theorem 3 and [BH20, Corollary 5.5]. \square

In general, if h is a Lorentzian polynomial, then its normalization $N(h)$ is a Lorentzian polynomial [BH20, Corollary 6.7]. We record here that Schur polynomials, before the normalization, need not be Lorentzian.

Example 8. The Schur polynomial of the partition $\lambda = (2, 0)$ in two variables is

$$s_\lambda(x_1, x_2) = x_1^2 + x_1 x_2 + x_2^2.$$

The quadratic form has eigenvalues $\frac{3}{2}$ and $\frac{1}{2}$, and hence s_λ is not Lorentzian.

A polynomial $f(x_1, \dots, x_m)$ is *stable* if f has no zeros in the product of m open upper half planes [Wag11]. Homogeneous stable polynomials with nonnegative coefficients are motivating examples of Lorentzian polynomials [BH20, Proposition 2.2]. We record here that normalized Schur polynomials, although Lorentzian, need not be stable.

Example 9. The normalized Schur polynomial of $\lambda = (3, 1, 1, 1, 1)$ in five variables is

$$N(s_\lambda(x_1, \dots, x_5)) = \frac{1}{12} x_1 x_2 x_3 x_4 x_5 \left(\sum_{1 \leq i < j \leq 5} 3x_i x_j + \sum_{1 \leq i \leq 5} 2x_i^2 \right).$$

By [Wag11, Lemma 2.4], if $N(s_\lambda)$ is stable, then so is its univariate specialization

$$N(s_\lambda)|_{x_2=x_3=x_4=x_5=1} = \frac{1}{6} x_1 (x_1^2 + 6x_1 + 13).$$

However, the displayed cubic has a pair of nonreal zeros, and hence $N(s_\lambda)$ is not stable.

3. UBIQUITY OF LORENTZIAN POLYNOMIALS

It follows from Definition 5 that the support of any Lorentzian polynomial is M -convex. In particular, the Newton polytope of any Lorentzian polynomial is a type A generalized permutohedron in the sense of [Pos09]. Generalized permutohedra have appeared in several works as the supports of multivariate polynomials related to the representation theory of $GL_m(\mathbb{C})$ and its Weyl group S_m , and to the Schubert calculus of the flag variety. These polynomials are natural candidates for the Lorentzian property. We collect results and conjectures for various families of polynomials from this setting.

3.1. Multiplicities of highest weight modules. We point to [Hum08] for background on representation theory of semisimple Lie algebras. Let Λ be the integral weight lattice of the Lie algebra $\mathfrak{sl}_m(\mathbb{C})$, let $\varpi_1, \dots, \varpi_{m-1}$ be the fundamental weights, and let ρ be the sum of the fundamental weights. For $\lambda \in \Lambda$, we write $V(\lambda)$ for the irreducible $\mathfrak{sl}_m(\mathbb{C})$ -module with highest weight λ , and consider its decomposition into finite-dimensional weight spaces

$$V(\lambda) = \bigoplus_{\mu} V(\lambda)_{\mu}.$$

For $\mu \in \Lambda$ and distinct $i, j \in [m]$, we write $\mu(i, j)$ for the element $\mu + e_i - e_j \in \Lambda$.

Conjecture 10. For any $\lambda \in \Lambda$ and any $\mu \in \Lambda$, we have

$$(\dim V(\lambda)_\mu)^2 \geq \dim V(\lambda)_{\mu(i, j)} \dim V(\lambda)_{\mu(j, i)} \text{ for any } i, j \in [m].$$

When λ is dominant, the dimension of the weight space $V(\lambda)_\mu$ is the Kostka number $K_{\lambda\mu}$, and Theorem 2 shows that Conjecture 10 holds in this case. When λ is antidominant [Hum08, Section 4.4], $V(\lambda)$ is the *Verma module* $M(\lambda)$, the universal highest weight module of highest weight λ . We note that Conjecture 10 holds in this case as well.

Proposition 11. For any $\lambda \in \Lambda$ and any $\mu \in \Lambda$, we have

$$(\dim M(\lambda)_\mu)^2 \geq \dim M(\lambda)_{\mu(i, j)} \dim M(\lambda)_{\mu(j, i)} \text{ for any } i, j \in [m].$$

One may deduce Proposition 11 from its stronger variant Proposition 13 below.

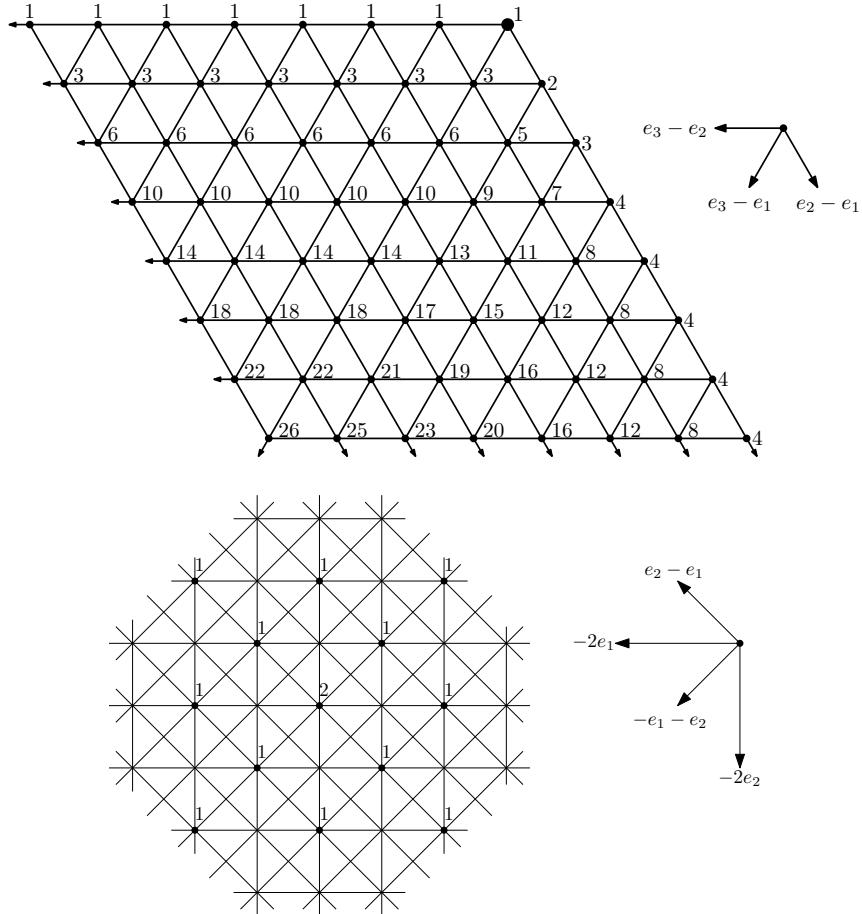
Alternative proof. The Poincaré–Birkhoff–Witt theorem shows that the dimensions of the weight spaces are given by the Kostant partition function p :

$$\dim M(\lambda)_\mu = p(\mu - \lambda) = \text{number of ways to write } \mu - \lambda \text{ as a sum of negative roots.}$$

Lidskij’s volume formula for flow polytopes shows that all Kostant partition function evaluations are mixed volumes of Minkowski sums of polytopes [BV08]. The Alexandrov–Fenchel inequality for mixed volumes [Sch14, Section 7.3] yields the desired log-concavity property. \square

The diagram below shows some of the weight multiplicities of the irreducible $\mathfrak{sl}_4(\mathbb{C})$ -module with highest weight $-2\varpi_1 - 3\varpi_2$. We start from the highlighted vertex $\varpi_1 - 6\varpi_2 - 3\varpi_3$ and walk along negative root directions in the hyperplane spanned by $e_2 - e_1$ and $e_3 - e_2$. In the shown region, the sequence of weight multiplicities along any line is log-concave, as predicted by Conjecture 10.

We note, however, that a naive analog of Conjecture 10 does not hold for symplectic Lie algebras. In the weight diagram of the irreducible representation of $\mathfrak{sp}_4(\mathbb{C})$ with highest weight $2\varpi_2$ shown below, the weight multiplicities along the two diagonals of the square do not form log-concave sequences. On the other hand, the Newton polytope of any homogeneous strongly log-concave polynomial is necessarily a generalized permutohedron of type A : Any edge of the Newton polytope should be parallel to $e_i - e_j$ for some i and j .



To strengthen Conjecture 10, we extend the normalization operator N to the space of Laurent generating functions by the formula

$$N\left(\sum_{\alpha \in \mathbb{Z}^n} c_\alpha x^\alpha\right) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha \frac{x^\alpha}{\alpha!}.$$

For $\lambda \in \Lambda$, we introduce the Laurent generating functions

$$\mathrm{ch}_\lambda(x_1, \dots, x_m) = \sum_{\mu \in \Lambda} \dim V(\lambda)_\mu x^{\mu - \lambda} \text{ and } \underline{\mathrm{ch}}_\lambda(x_1, \dots, x_m) = \sum_{\mu \in \Lambda} \dim M(\lambda)_\mu x^{\mu - \lambda}.$$

Note that every monomial appearing in the shifted characters ch_λ and $\underline{\text{ch}}_\lambda$ is a product of degree zero monomials of the form $x_i x_j^{-1}$ with $i > j$.

We tested the following statement for $\lambda = -w\rho - \rho$ and $\delta = (1, \dots, 1)$, for all permutations w in S_m for $m \leq 6$. We point to <https://github.com/avstdi/Lorentzian-Polynomials> for code supporting the computations in Section 3.

Conjecture 12. The polynomial $N(x^\delta \text{ch}_\lambda(x_1, \dots, x_m))$ is Lorentzian for any $\lambda \in \Lambda$ and $\delta \in \mathbb{N}^m$.

For example, when $m = 4$ and $\lambda = -w\rho - \rho$ for the transposition $w = (1, 2)$, we have

$$\begin{aligned} N(x_1 x_2 x_3 x_4 \text{ch}_\lambda(x_1, x_2, x_3, x_4)) &= \frac{4}{24} x_4^4 + \frac{2}{6} x_1 x_4^3 + \frac{2}{6} x_2 x_4^3 + \frac{4}{6} x_3 x_4^3 + \frac{3}{4} x_3^2 x_4^2 \\ &+ \frac{1}{2} x_1 x_2 x_4^2 + \frac{2}{2} x_1 x_3 x_4^2 + \frac{2}{2} x_2 x_3 x_4^2 + \frac{1}{6} x_3^3 x_4 + \frac{1}{2} x_1 x_3^2 x_4 + \frac{1}{2} x_2 x_3^2 x_4 + \frac{1}{1} x_1 x_2 x_3 x_4, \end{aligned}$$

which is a Lorentzian polynomial. In general, the homogeneous polynomial $N(x^\delta \text{ch}_\lambda)$ can be computed using the Kazhdan–Lusztig theory [Hum08, Chapter 8].

Theorem 3 and Lemma 7 show that Conjecture 12 holds for any δ when λ is dominant. We show that Conjecture 12 holds for any δ when λ is antidominant.

Proposition 13. The polynomial $N(x^\delta \underline{\text{ch}}_\lambda(x_1, \dots, x_m))$ is Lorentzian for any $\lambda \in \Lambda$ and $\delta \in \mathbb{N}^m$.

Proof. Recall that the dimensions of the weight spaces of $M(\lambda)$ are given by the Kostant partition function p . In other words, we have

$$\underline{\text{ch}}_\lambda(x_1, \dots, x_m) = \prod_{i>j} (1 + x_i x_j^{-1} + x_i^2 x_j^{-2} + \dots).$$

It is clear that the product is well-defined. Officially, the product occurs in the ring of formal characters of the category \mathcal{O} of $\mathfrak{sl}_m(\mathbb{C})$ -modules, denoted \mathcal{X} in [Hum08, Section 1.15]. Note that in the expansion of the above product, only the terms of degree at least $-\delta$ contribute to $N(x^\delta \underline{\text{ch}}_\lambda)$. Therefore, we may choose a suitably large $\alpha \in \mathbb{N}^m$ depending on $\delta \in \mathbb{N}^m$ so that

$$N(x^\delta \underline{\text{ch}}_\lambda) = N(x^\delta x^{-\beta} \prod_{i>j} (x_j^{\alpha_j} + x_i x_j^{\alpha_j-1} + \dots + x_i^{\alpha_j})), \text{ where } \beta_i = (m-i)\alpha_i \text{ for all } i.$$

Observe that the right-hand side is the β -th partial derivative of the normalized product of x^δ and $\sum_k x_i^{\alpha_j-k} x_j^k$, whose normalization is the Lorentzian polynomial

$$N(x_j^{\alpha_j} + x_i x_j^{\alpha_j-1} + \dots + x_i^{\alpha_j}) = \frac{1}{\alpha_j!} (x_i + x_j)^{\alpha_j}.$$

The conclusion now follows from [BH20, Corollary 6.8]. □

Conjecture 10 for λ and μ follows from Conjecture 12 for λ and a sufficiently large δ . Conjecture 12 for λ and δ follows from Conjecture 12 for λ and any δ' larger than δ componentwise.

3.2. Schubert polynomials. For $w \in \mathcal{S}_n$ and $\mu \in \mathbb{Z}^n$, we define the number $K_{w\mu}$ by

$$\mathfrak{S}_w(x_1, \dots, x_n) = \sum_{\mu} K_{w\mu} x^{\mu}.$$

As before, for $\mu \in \mathbb{Z}^n$ and distinct $i, j \in [m]$, we set

$$\mu(i, j) = \mu + e_i - e_j.$$

We note that Theorem 2 can be strengthened as follows.

Proposition 14. For any $w \in \mathcal{S}_n$ and any $\mu \in \mathbb{N}^n$, we have

$$K_{w\mu}^2 \geq K_{w\mu(i,j)} K_{w\mu(j,i)} \text{ for any } i, j \in [n].$$

Proof. By Theorem 6, the polynomial \mathfrak{S}_w^{\vee} is Lorentzian. The inequality follows from [BH20, Proposition 9.4] applied to the Lorentzian polynomial \mathfrak{S}_w^{\vee} . \square

Are normalized Schubert polynomials Lorentzian? We tested the following statement for all permutations in \mathcal{S}_n for $n \leq 8$.

Conjecture 15. The polynomial $N(\mathfrak{S}_w(x_1, \dots, x_n))$ is Lorentzian for any $w \in \mathcal{S}_n$.

More generally, we conjecture that, for double Schubert polynomials [MS05, Section 15.5],

$$N(\mathfrak{S}_w(x_1, \dots, x_n, -y_1, \dots, -y_n)) \text{ is Lorentzian for any } w \in \mathcal{S}_n.$$

This would imply that the support of any double Schubert polynomial is M-convex, and hence “saturated” [MTY19, Conjecture 5.2].

Proposition 16. The support of $\mathfrak{S}_w(x_1, \dots, x_n)$ is M-convex for any $w \in \mathcal{S}_n$.

Proposition 16 was conjectured in [MTY19, Conjecture 5.1] and proved in [FMS18] using an explicit description of flagged Schur modules. Here we give an alternative proof based on Theorem 6. A similar argument can be used more generally to show that the supports of single quiver polynomials appearing in [MS05, Section 17.4] are M-convex.

Proof. By Theorem 6, the support of \mathfrak{S}_w^{\vee} is M-convex. It is straightforward to check using the definition of M-convexity the general fact that, if the support of $h(x_1, \dots, x_n)$ is M-convex, then the support of $x^{\mu} h(x_1^{-1}, \dots, x_n^{-1})$ is M-convex for any monomial x^{μ} divisible by all monomials in the support of h . This general fact extends matroid duality [Oxl11, Chapter 2], which is the special case $\mu = (1, \dots, 1)$. \square

Proposition 17. Conjecture 15 holds when $w \in \mathcal{S}_n$ avoids the patterns 1423 and 1432.

Sketch of Proof. By [BH20, Corollary 6.7], the Lorentzian property of \mathfrak{S}_w implies that of $N(\mathfrak{S}_w)$. We deduce the Lorentzian property of \mathfrak{S}_w from known results on Schubert and Lorentzian polynomials, for permutations avoiding 1423 and 1432.

It is shown in [FMS18, Theorem 7] that, for any $w \in \mathcal{S}_n$, the support of \mathfrak{S}_w is the set of integral points in the Minkowski sum of n matroid polytopes. The set J_w of integral points in the Cartesian product of these matroid polytopes is an M-convex subset of $\mathbb{N}^{n \times n}$, and hence the generating function f_w of J_w is a Lorentzian polynomial in n^2 variables x_{ij} [BH20, Theorem 7.1]. Since any nonnegative linear change of coordinates preserves the Lorentzian property [BH20, Theorem 2.10], substituting the variables x_{ij} by x_i in the generating function f_w gives a Lorentzian polynomial. According to [FMS20, Corollary 5.6] and [FG19, Theorem 1.1], this specialization of f_w coincides with \mathfrak{S}_w when w avoids 1423 and 1432, and thus \mathfrak{S}_w is Lorentzian for such permutations. \square

We note that the Schubert polynomials \mathfrak{S}_{1423} and \mathfrak{S}_{1432} are not Lorentzian.

3.3. Degree polynomials. Let $w < w(i, j)$ be a covering relation in the Bruhat order of \mathcal{S}_n labelled by the transposition of $i < j$ in $[n]$. The *Chevalley multiplicity* is the assignment

$$w < w(i, j) \longmapsto \sum_{i \leq k < j} x_k,$$

where x_k are independent variables. The *degree polynomial* of $w \in \mathcal{S}_n$ is the generating function

$$\mathfrak{D}_w(x_1, \dots, x_{n-1}) = \sum_{\mathcal{C}} m_{\mathcal{C}}(x_1, \dots, x_{n-1}),$$

where the sum is over all saturated chains \mathcal{C} from the identity permutation to w , and $m_{\mathcal{C}}$ is the product of Chevalley multiplicities of the covering relations in \mathcal{C} . The degree polynomials were introduced by Bernstein, Gelfand, and Gelfand [BGG73] and studied from a combinatorial perspective by Postnikov and Stanley [PS09].

Proposition 18. The degree polynomial $\mathfrak{D}_w(x_1, \dots, x_{n-1})$ is Lorentzian for any $w \in \mathcal{S}_n$.

Proof. Let B be the group of upper triangular matrices in $GL_n(\mathbb{C})$, and let X_w be the closure of the B -orbit of the permutation matrix corresponding to w in the flag variety $GL_n(\mathbb{C})/B$. By [PS09, Proposition 4.2], the degree polynomial of w is, up to a normalizing constant, the volume polynomial of X_w with respect to the line bundles associated to the fundamental weights $\varpi_1, \dots, \varpi_{n-1}$. The conclusion follows from [BH20, Theorem 10.1]. \square

The same argument shows that the analogous statement holds for Weyl groups in other types.

3.4. Skew Schur polynomials. Let λ/ν be a skew Young diagram. The *skew Schur polynomial* of λ/ν in m variables is the generating function

$$s_{\lambda/\nu}(x_1, \dots, x_m) = \sum_T x^{\mu(T)}, \quad x^{\mu(T)} = x_1^{\mu_1(T)} \cdots x_m^{\mu_m(T)},$$

where the sum is over all Young tableaux T of skew shape λ/ν with entries from $[m]$, and

$$\mu_i(T) = \text{the number of } i\text{'s among the entries of } T, \quad \text{for } i = 1, \dots, m.$$

Are normalized skew Schur polynomials Lorentzian? We tested the following statement for all partitions λ with at most 12 boxes and at most 6 parts.

Conjecture 19. The polynomial $N(s_{\lambda/\nu}(x_1, \dots, x_m))$ is Lorentzian for any λ/ν .

Theorem 3 shows that Conjecture 19 holds when ν is zero, and Corollary 4 provides some further evidence. We remark that the M -convexity of the support of any skew Schur polynomial can be deduced from [MTY19, Proposition 2.9].

In [LPP07], Lam, Postnikov, and Pylyavskyy show that, for two skew shapes λ/μ and ν/ρ ,

$$\left(s_{\frac{\lambda+\nu}{2}/\frac{\mu+\rho}{2}}\right)^2 - s_{\lambda/\mu}s_{\nu/\rho}$$

is a nonnegative linear combination of Schur polynomials, when both $\lambda + \nu$ and $\mu + \rho$ have only even parts. It would be interesting to know if there are relations between different notions of log-concavity for skew Schur polynomials.

3.5. Schur P -polynomials. Let λ be a *strict partition*, that is, a decreasing sequence of positive integers. The *Schur P -polynomial* of λ in m variables is the generating function

$$P_\lambda(x_1, \dots, x_m) = \sum_T x^{\mu(T)}, \quad x^{\mu(T)} = x_1^{\mu_1(T)} \cdots x_m^{\mu_m(T)},$$

where the sum is over all marked shifted Young tableaux of shape λ with entries from $[m]$. See [Mac15, Chapter III] for this and other equivalent definitions of the polynomial P_λ .

Are normalized Schur P -polynomials Lorentzian? We tested the following statement for all strict partitions λ with $\lambda_1 \leq 12$ and at most 4 parts.

Conjecture 20. The polynomial $N(P_\lambda(x_1, \dots, x_m))$ is Lorentzian for any strict partition λ .

The M-convexity of the support of P_λ was observed in [MTY19, Proposition 3.5].

3.6. Grothendieck polynomials. *Grothendieck polynomials* are polynomial representatives of the Schubert classes in the Grothendieck ring introduced by Lascoux and Schützenberger [LS83]. If w is the longest permutation $w_\circ \in \mathcal{S}_n$, then the Grothendieck polynomial of w is the monomial

$$\mathfrak{G}_{w_\circ}(x_1, \dots, x_n) = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^1.$$

In general, if $w(i) > w(i+1)$ for some i and s_i is the adjacent transposition $(i \ i+1)$, then

$$\mathfrak{G}_{ws_i}(x_1, \dots, x_n) = \pi_i \mathfrak{G}_w(x_1, \dots, x_n), \text{ where } \pi_i = \partial_i - \partial_i x_{i+1}.$$

Let $\ell(w)$ be the degree of the Schubert polynomial of w , let $d(w)$ be the degree of the Grothendieck polynomial of w , and let \mathfrak{G}_w^k be the degree $\ell(w) + k$ homogeneous component of the Grothendieck polynomial.

Conjecture 21. The polynomial $(-1)^k N(\mathfrak{G}_w^k(x_1, \dots, x_n))$ is Lorentzian for any $w \in \mathcal{S}_n$ and $k \in \mathbb{N}$.

The M-convexity of the support of \mathfrak{G}_w^k was conjectured in [MS20, Conjecture 5.1] and proved in [EY17] when w is a Grassmannian permutation. Conjecture 21 implies Conjecture 15 because the degree $\ell(w)$ homogeneous component of \mathfrak{G}_w is the Schubert polynomial \mathfrak{S}_w .

We may strengthen Conjecture 21 in terms of the *homogeneous Grothendieck polynomial*

$$\tilde{\mathfrak{G}}_w(x_1, \dots, x_n, z) := \sum_{k=0}^{d(w)-\ell(w)} (-1)^k \mathfrak{G}_w^k(x_1, \dots, x_n) z^{d(w)-\ell(w)-k},$$

where z is a new variable. Are normalized homogeneous Grothendieck polynomials Lorentzian? We tested the following statement for all permutations in \mathcal{S}_n for $n \leq 7$.

Conjecture 22. The polynomial $N(\tilde{\mathfrak{G}}_w(x_1, \dots, x_n, z))$ is Lorentzian for any $w \in \mathcal{S}_n$.

Conjecture 22 implies Conjecture 21 because taking partial derivatives and setting a variable equal to zero preserve the Lorentzian property. We expect an analogous Lorentzian property for double Grothendieck polynomials.

3.7. Key polynomials. *Key polynomials* were introduced by Demazure for Weyl groups [Dem74] and studied by Lascoux and Schützenberger for symmetric groups [LS90]. When $\mu \in \mathbb{N}^n$ is a partition, the key polynomial of μ is the monomial

$$\kappa_\mu(x_1, \dots, x_n) = x^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}.$$

If $\mu_i < \mu_{i+1}$ for some i and s_i is the adjacent transposition $(i \ i+1)$, then

$$\kappa_\mu(x_1, \dots, x_n) = \partial_i x_i \kappa_\nu, \text{ where } \nu = \mu s_i = (\mu_1, \dots, \mu_{i+1}, \mu_i, \dots, \mu_n).$$

We refer to [RS95] for more information about key polynomials.

Are normalized key polynomials Lorentzian? We tested the following statement for all compositions μ with at most 12 boxes and at most 6 parts.

Conjecture 23. The polynomial $N(\kappa_\mu(x_1, \dots, x_n))$ is Lorentzian for any $\mu \in \mathbb{N}^n$.

Theorem 3 shows that Conjecture 23 holds when μ is a weakly increasing sequence of nonnegative integers, because in this case the key polynomial of μ is a Schur polynomial. The M-convexity of the supports of key polynomials was conjectured in [MTY19, Conjecture 3.13] and proved in [FMS18].

We remark that key polynomials [Dem74] and Schubert polynomials [KP87] are both characters of *flagged Schur modules*. Flagged Schur modules are representations of the group of upper triangular matrices in $GL_n(\mathbb{C})$ labelled by *diagrams*. They are also called flagged dual Weyl modules, and, in special cases, key modules. We refer to [RS95, Section 5] and [Magy98, Section 4] for expositions.

It is shown in [FMS18, Theorem 11] that the character of any flagged Schur module has M-convex support. Are normalized characters of flagged Schur modules Lorentzian?

REFERENCES

- [AOGV18] Nima Anari, Shayan Oveis Gharan, and Cynthia Vinzant, *Log-Concave polynomials I: entropy, and a deterministic approximation algorithm for counting bases of matroids*, 2018 IEEE 59th Annual Symposium on Foundations of Computer Science (FOCS), 2018, pp. 35–46.
- [ALOGV19] Nima Anari, Kuikui Liu, Shayan Oveis Gharan, and Cynthia Vinzant, *Log-Concave polynomials II: High-Dimensional Walks and an FPRAS for Counting Bases of a Matroid*, STOC 2019: Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing, 2019, pp. 1–12.
- [ALOGV18] ———, *Log-Concave polynomials III: Mason’s ultra-log-concavity conjecture for independent sets of matroids* (2018), available at [arXiv:1811.01600](https://arxiv.org/abs/1811.01600).
- [BV08] Welleda Baldoni and Michèle Vergne, *Kostant partitions functions and flow polytopes*, Transform. Groups **13** (2008), no. 3-4, 447–469.
- [BES19] Spencer Backman, Christopher Eur, and Connor Simpson, *Simplicial generation of Chow rings of matroids* (2019), available at [arXiv:1905.07114](https://arxiv.org/abs/1905.07114).
- [BGG73] Joseph N. Bernštejn, Israel M. Gel’fand, and Sergei I. Gel’fand, *Schubert cells and the cohomology of the spaces G/P* , Russian Mathematical Surveys **28** (1973), 1–26.
- [BGR04] Sara Billey, Victor Guillemin, and Etienne Rassart, *A vector partition function for the multiplicities of $\mathfrak{sl}_k \mathbb{C}$* , J. Algebra **278** (2004), no. 1, 251–293.

- [BH18] Petter Brändén and June Huh, *Hodge-Riemann relations for Potts model partition functions* (2018), available at [arXiv:1811.01696](https://arxiv.org/abs/1811.01696).
- [BH20] ———, *Lorentzian polynomials*, Ann. of Math. **192** (2020), 821–891.
- [Cau15] Augustin Louis Cauchy, *Mémoire sur les fonctions qui ne peuvent obtenir que deux valeurs égales et de signes contraires par suite des transpositions opérées entre les variables qu'elles renferment*, Œuvres complètes. Series 2. Volume 1, Cambridge University Press, Cambridge, 2009. Reprint of the 1905 original, 1815.
- [CDW07] Calin Chindris, Harm Derksen, and Jerzy Weyman, *Counterexamples to Okounkov's log-concavity conjecture*, Compos. Math. **143** (2007), no. 6, 1545–1557.
- [Dem74] Michel Demazure, *Une nouvelle formule des caractères*, Bull. Sci. Math. (2) **98** (1974), no. 3, 163–172.
- [EY17] Laura Escobar and Alexander Yong, *Newton polytopes and symmetric Grothendieck polynomials*, C. R. Math. Acad. Sci. Paris **355** (2017), no. 8, 831–834 (English, with English and French summaries).
- [EH20] Christopher Eur and June Huh, *Logarithmic concavity for morphisms of matroids*, Adv. Math. **367** (2020).
- [FG19] Neil J. Y. Fan and Peter L. Guo, *Upper bounds of Schubert polynomials* (2019), available at [arXiv:1909.07206](https://arxiv.org/abs/1909.07206).
- [FMS18] Alex Fink, Karola Mészáros, and Avery St. Dizier, *Schubert polynomials as integer point transforms of generalized permutohedra*, Adv. Math. **332** (2018), 465–475.
- [FMS20] ———, *Zero-one Schubert polynomials*, Math. Z. (2020).
- [FH91] William Fulton and Joe Harris, *Representation theory*, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991. A first course; Readings in Mathematics.
- [Ful92] William Fulton, *Flags, Schubert polynomials, degeneracy loci, and determinantal formulas*, Duke Math. J. **65** (1992), no. 3, 381–420.
- [Ful97] ———, *Young tableaux*, London Mathematical Society Student Texts, vol. 35, Cambridge University Press, Cambridge, 1997. With applications to representation theory and geometry.
- [Ful98] ———, *Intersection theory*, 2nd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 2, Springer-Verlag, Berlin, 1998.
- [Gra96] William Graham, *Logarithmic convexity of push-forward measures*, Invent. Math. **123** (1996), no. 2, 315–322.
- [Gur09] Leonid Gurvits, *On multivariate Newton-like inequalities*, Advances in Combinatorial Mathematics, Springer, Berlin, 2009, pp. 61–78.
- [Hec82] Gert J. Heckman, *Projections of orbits and asymptotic behavior of multiplicities for compact connected Lie groups*, Invent. Math. **67** (1982), no. 2, 333–356.
- [Hum08] James E. Humphreys, *Representations of semisimple Lie algebras in the BGG category \mathcal{O}* , Graduate Studies in Mathematics, vol. 94, American Mathematical Society, Providence, RI, 2008.
- [Kar96] Yael Karshon, *Example of a non-log-concave Duistermaat-Heckman measure*, Math. Res. Lett. **3** (1996), no. 4, 537–540.
- [Kir04] Anatol N. Kirillov, *An invitation to the generalized saturation conjecture*, Publ. Res. Inst. Math. Sci. **40** (2004), no. 4, 1147–1239. MR2105706

- [KM05] Allen Knutson and Ezra Miller, *Gröbner geometry of Schubert polynomials*, Ann. of Math. (2) **161** (2005), no. 3, 1245–1318.
- [Kos82] Carl Kostka, *Ueber den Zusammenhang zwischen einigen Formen von symmetrischen Functionen*, J. Reine Angew. Math. **93** (1882), 89–123.
- [KP87] Witold Kraśkiewicz and Piotr Pragacz, *Foncteurs de Schubert*, C. R. Acad. Sci. Paris Sér. I Math. **304** (1987), no. 9, 209–211 (French, with English summary).
- [LPP07] Thomas Lam, Alexander Postnikov, and Pavlo Pylyavskyy, *Schur positivity and Schur log-concavity*, Amer. J. Math. **129** (2007), no. 6, 1611–1622.
- [LS83] Alain Lascoux and Marcel-Paul Schützenberger, *Symmetry and flag manifolds*, Invariant theory (Montecatini, 1982), Lecture Notes in Math., vol. 996, Springer, Berlin, 1983, pp. 118–144.
- [LS90] ———, *Keys & standard bases*, Invariant theory and tableaux (Minneapolis, MN, 1988), IMA Vol. Math. Appl., vol. 19, Springer, New York, 1990, pp. 125–144.
- [Laz04] Robert Lazarsfeld, *Positivity in algebraic geometry. I*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 48, Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series.
- [Mac15] Ian G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed., Oxford Classic Texts in the Physical Sciences, The Clarendon Press, Oxford University Press, New York, 2015.
- [Magy98] Peter Magyar, *Schubert polynomials and Bott-Samelson varieties*, Comment. Math. Helv. **73** (1998), no. 4, 603–636.
- [Man01] Laurent Manivel, *Symmetric functions, Schubert polynomials and degeneracy loci*, SMF/AMS Texts and Monographs, vol. 6, American Mathematical Society, Providence, RI; Société Mathématique de France, Paris, 2001. Translated from the 1998 French original by John R. Swallow; Cours Spécialisés [Specialized Courses], 3.
- [MS20] Karola Mészáros and Avery St. Dizier, *From generalized permutohedra to Grothendieck polynomials via flow polytopes*, Algebr. Comb. **3** (2020), no. 5, 1197–1229.
- [MS05] Ezra Miller and Bernd Sturmfels, *Combinatorial commutative algebra*, Graduate Texts in Mathematics, vol. 227, Springer-Verlag, New York, 2005. MR2110098
- [MTY19] Cara Monical, Neriman Tokcan, and Alexander Yong, *Newton polytopes in algebraic combinatorics*, Sel. Math. New Ser. **25** (2019), no. 66.
- [Mur03] Kazuo Murota, *Discrete convex analysis*, SIAM Monographs on Discrete Mathematics and Applications, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2003.
- [Oko03] Andrei Okounkov, *Why would multiplicities be log-concave?*, The orbit method in geometry and physics (Marseille, 2000), Progr. Math., vol. 213, Birkhäuser Boston, Boston, MA, 2003, pp. 329–347.
- [Oxl11] James Oxley, *Matroid theory*, 2nd ed., Oxford Graduate Texts in Mathematics, vol. 21, Oxford University Press, Oxford, 2011.
- [Pos09] Alexander Postnikov, *Permutohedra, associahedra, and beyond*, Int. Math. Res. Not. IMRN **6** (2009), 1026–1106.
- [PS09] Alexander Postnikov and Richard P. Stanley, *Chains in the Bruhat order*, J. Algebraic Combin. **29** (2009), no. 2, 133–174.
- [RS95] Victor Reiner and Mark Shimozono, *Key polynomials and a flagged Littlewood-Richardson rule*, J. Combin. Theory Ser. A **70** (1995), no. 1, 107–143.

- [Sch14] Rolf Schneider, *Convex bodies: the Brunn-Minkowski theory*, Second expanded edition, Encyclopedia of Mathematics and its Applications, vol. 151, Cambridge University Press, Cambridge, 2014.
- [Sch01] Issai Schur, *Über eine Klasse von Matrizen, die sich einer gegebenen Matrix zuordnen lassen*, Deterich, Berlin; Göttingen (1901).
- [Wag11] David G. Wagner, *Multivariate stable polynomials: theory and applications*, Bull. Amer. Math. Soc. (N.S.) **48** (2011), no. 1, 53–84.
- [Whi86] Neil White (ed.), *Theory of matroids*, Encyclopedia of Mathematics and its Applications, vol. 26, Cambridge University Press, Cambridge, 1986.

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NJ, AND KOREA INSTITUTE FOR ADVANCED STUDY, SEOUL, KOREA.

Email address: `huh@princeton.edu`

MATHEMATICAL INSTITUTE, UNIVERSITY OF BONN, BONN, GERMANY AND MAX PLANCK INSTITUTE FOR MATHEMATICS, BONN, GERMANY.

Email address: `jacobm@math.uni-bonn.de`

DEPARTMENT OF MATHEMATICS, CORNELL UNIVERSITY, ITHACA, NY.

Email address: `karola@math.cornell.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN, URBANA, IL.

Email address: `stdizie2@illinois.edu`