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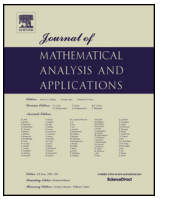

Regular Articles

Rational decay of a multilayered structure-fluid PDE system

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ABSTRACT

In this work, we consider a certain multilayered (thick layer) wave–(thin layer) wave–heat (fluid) interactive PDE system. Such coupled PDE systems have been used in the literature to describe the blood transport process in mammalian vascular systems. In particular, the deformations of the boundary interface (thin layer) are described via the two dimensional elastic equation. The present work constitutes an investigation of the extent of the stabilizing effects of the underlying fluid dissipation – across the boundary interface – upon both the thick and thin structural components. (All three PDE components evolve on their respective geometries.) In this regard, our main result is the derivation of uniform decay rates for classical solutions of this multilayered PDE model. To obtain these estimates, necessary a priori inequalities for certain static multilayered PDE models are generated here to ultimately allow an application of a wellknown resolvent criterion for rational decay.

1. Introduction

The fluid structure interaction (FSI) phenomena constitutes a broad area of research with applications in variety of real world problems [12,26,30]. In particular, mammalian blood vascular walls, being composed of viscoelastic materials, undergo large deformations due to hemodynamic forces generated during the blood transport process. As such, there is a coupling of respective blood flow and wall deformation dynamics. This physiological interaction between arterial walls and blood flow plays a crucial role in the physiology and pathophysiology of the human cardiovascular system, and can be mathematically realized by *multilayered* FSI PDE. In such FSI modeling, the blood flow is governed by the fluid flow PDE component (incompressible Stokes or Navier Stokes); the displacements along the elastic vascular wall are described by the structural PDE component (e.g., systems of elasticity). In this regard, the multilayered FSI modeling with

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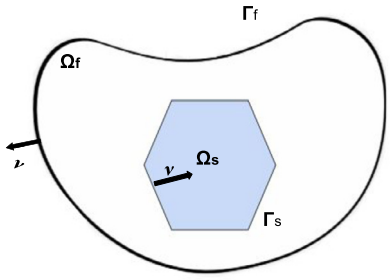

Fig. 1. Multilayered structure-fluid interaction domain.

a view to understanding the incidence of aneurysm caused by arterial wall deformations during the blood transportation process has recently been a topic of great interest, see e.g. [15,39] and reference within.

In this paper, we consider a simplified multilayered structure-fluid interaction (FSI) system where the coupling of the 3D fluid (blood flow) and 3D elastic (structural vascular wall) PDE components is realized via an additional 2D elastic system on the boundary interface.

The PDE Model

Let the fluid geometry $\Omega_f \subseteq \mathbb{R}^3$ be Lipschitz, and the structure domain $\Omega_s \subseteq \mathbb{R}^3$ be a convex *polyhedron* which is strictly contained in Ω_f (see Fig. 1). Moreover, fluid boundary is decomposed via $\partial\Omega_f = \Gamma_f \cup \Gamma_s$, where $\Gamma_s = \partial\Omega_s$, and so $\Gamma_f \cap \Gamma_s = \emptyset$. Thus, Γ_s is the boundary interface between fluid geometry Ω_f and structure geometry Ω_s. The boundary interface is further decomposed via $\Gamma_s = \bigcup_{j=1}^{K} \overline{\Gamma_j}$, where each Γ_j is an open polygonal domain, with $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$. In addition, for $1 \leq j \leq K$, n_j denotes the unit normal vector which is exterior to $\partial\Gamma_j$. Also, as pictured in Fig. 1, $\nu(x)$ denotes the unit outward normal with respect to Ω_f (and so $\nu(x)$ is inward with respect to Ω_s).

For said $\{\Omega_f, \Gamma_s, \Omega_s\}$, the multilayered structure-fluid FSI system in solution variables $u(t,x)$ (corresponding to the fluid velocity), $h_j(t,x)$ $(1 \leq j \leq K)$ (thin layers displacements), and $w(t,x)$ (thick layer displacement) is as follows:

$$\begin{cases} u_t - \Delta u = 0 \ \text{ in } (0,T) \times \Omega_f \\ u|_{\Gamma_f} = 0 \ \text{ on } \ (0,T) \times \Gamma_f \end{cases} \tag{1}$$

$$\begin{cases} \text{For } 1 \leq j \leq K, \\ \frac{\partial^2}{\partial t^2} h_j - \Delta h_j + h_j = \frac{\partial w}{\partial \nu}|_{\Gamma_j} - \frac{\partial u}{\partial \nu}|_{\Gamma_j} \ \text{ on } \ (0,T) \times \Gamma_j \\ \begin{cases} h_j|_{\partial\Gamma_j \cap \partial\Gamma_l} = h_l|_{\partial\Gamma_j \cap \partial\Gamma_l} \\ \frac{\partial h_j}{\partial n_j}\Big|_{\partial\Gamma_j \cap \partial\Gamma_l} = -\frac{\partial h_l}{\partial n_l}\Big|_{\partial\Gamma_j \cap \partial\Gamma_l} \end{cases} \text{ on } (0,T) \times (\partial\Gamma_j \cap \partial\Gamma_l), \ \forall \ 1 \leq l \leq K; \ \ \partial\Gamma_j \cap \partial\Gamma_l \neq \emptyset \end{cases} \tag{2}$$

$$\begin{cases} w_{tt} - \Delta w = 0 \ \text{ in } (0,T) \times \Omega_s \\ w_t|_{\Gamma_j} = \frac{\partial}{\partial t} h_j = u|_{\Gamma_j} \ \text{ on } \ (0,T) \times \Gamma_j, \ \text{ for } j = 1, ..., K \end{cases} \tag{3}$$

$$[u(0), h_1(0), \tfrac{\partial h_1(0)}{\partial t}, ..., h_K(0), \tfrac{\partial h_1(0)}{\partial t}, w(0), w_t(0)] = [u_0, h_{01}, h_{02}, ..., h_{0K}, h_{0K}, w_0, w_1] \in \mathbf{H} \tag{4}$$

where the finite energy space $\mathbf{H}$ is given by

$$\mathbf{H} \equiv \Bigg\{ [u_0, h_{01}, h_{02}, ..., h_{0K}, h_{0K}, w_0, w_1] \in \mathbf{L}^2(\Omega_f) \times \prod_{j=1}^{K} \left[H^1(\Gamma_j) \times L^2(\Gamma_j) \right] \times \mathbf{H}^1(\Omega_s) \times \mathbf{L}^2(\Omega_s) \ : \\ \text{(i)} \ \ w_0|_{\Gamma_j} = h_{0j}; \\ \text{(ii)} \ \ h_j|_{\partial\Gamma_j \cap \partial\Gamma_l} = h_l|_{\partial\Gamma_j \cap \partial\Gamma_l} \ \text{ on } \ (0,T) \times (\partial\Gamma_j \cap \partial\Gamma_l) \ \ \forall \ \ 1 \leq l \leq K \text{ such that } \partial\Gamma_j \cap \partial\Gamma_l \neq \emptyset \Bigg\}. \tag{5}$$

The **H**-inner product is

$$\left(\Phi_0, \tilde{\Phi}_0\right)_{\mathbf{H}} = (u_0, \tilde{u}_0)_{\Omega_f} + \sum_{j=1}^{K} \left[\left(\nabla h_{0j}, \nabla \tilde{h}_{0j}\right)_{\Gamma_j} + \left(h_{0j}, \tilde{h}_{0j}\right)_{\Gamma_j} \right] \\ + \sum_{j=1}^{K} \left(h_{1j}, \tilde{h}_{1j}\right)_{\Gamma_j} + (\nabla w_0, \nabla \tilde{w}_0)_{\Omega_s} + (w_1, \tilde{w}_1)_{\Omega_s} \tag{6}$$

for $\Phi_0 = [u_0, h_{01}, h_{11}(t)...., h_{0K}, h_{1K}, w_0, w_1]$ and $\tilde{\Phi}_0 = \left[\tilde{u}_0, \tilde{h}_{01}, \tilde{h}_{11}(t)...., \tilde{h}_{0K}, \tilde{h}_{1K}, \tilde{w}_0, \tilde{w}_1\right] \in \mathbf{H}$.

We should note that the boundary interface does not evolve with time. However, it is well-accepted that if the boundary interface displacements between structure and fluid are small relative to the scale of the geometry, the resulting FSI models are physically relevant and reliable; (see [24,31]).

Notation

Throughout, for a given domain D, the norm of corresponding space $L^2(D)$ will be denoted as $|| \cdot ||_D$ (or simply $|| \cdot ||$ when the context is clear). Inner products in $L^2(\mathcal{O})$ or $\mathbf{L}^2(\mathcal{O})$ will be denoted by $(\cdot, \cdot)_{\mathcal{O}}$, whereas inner products $L^2(\partial\mathcal{O})$ will be written as $\langle \cdot, \cdot \rangle_{\partial\mathcal{O}}$. We will also denote pertinent duality pairings as $\langle \cdot, \cdot \rangle_{X \times X'}$ for a given Hilbert space X. The space $H^s(D)$ will denote the Sobolev space of order s, defined on a domain D; $H_0^s(D)$ will denote the closure of $C_0^\infty(D)$ in the $H^s(D)$-norm $\| \cdot \|_{H^s(D)}$. We make use of the standard notation for the boundary trace of functions defined on $\mathcal{O}$, which are sufficiently smooth: i.e., for a scalar function $\phi \in H^s(\mathcal{O})$, $\frac{1}{2} < s < \frac{3}{2}$, $\gamma(\phi) = \phi\big|_{\partial\mathcal{O}}$, which is a well-defined and surjective mapping on this range of s, owing to the Sobolev Trace Theorem on Lipschitz domains (see e.g., [37], or Theorem 3.38 of [33]). Also, $C > 0$ will denote a generic constant.

2. Literature

Stability analyses of fluid structure interaction (FSI) PDE systems have been an ongoing object of study [2,3,5,6,25,38]. Because of their utility in mathematically describing fluid or flow dynamics as they interact with elastic materials, such FSI models arise in biomedicine, biomechanics and aeroelasticity, see e.g. [12,23]. The main motivation of the current problem comes from the mathematical modeling of vascular blood flow: the corresponding modeling PDE dynamics accounts for the fact that blood-transporting vessels are generally composed of several layers. Such multilayered FSI PDE models have a crucial role in understanding the physiology of the human cardiovascular system [9,28,36].

Examples of single layered FSI – i.e., only one elastic PDE (describing three dimensional bulk elasticity or some lower-dimensional model of plate/shell type) models the structural dynamics; the displacement along the interaction interface is not modeled via any elastic equation – appear extensively in the literature, see e.g. [5,8,10,20,22,24,29,35] and references within. However, many biomedical devices (such as stents) are being developed with the view that vascular wall structures are composed of composite materials and not of single layer; see [16–19,36]. In short, some degree of physical realism is lost if the FSI PDE does not adequately describe arterial wall layers of composite type.

Compared to the extensive work undertaken for single layered FSI, there is a relative paucity of results for multilayered FSI systems. A multilayered FSI (2D heat-1D wave-2D wave) system was initially studied in [36] with a focus on showing wellposedness. Therein, the authors exploited an underlying regularity

which was available by the presence of the additional wave equation. A simplified $1D$ model was studied in [34] where the optimal regularity result was proved. In [7], wellposedness and strong stability of a higher dimensional linear version of the system considered in [36] were studied. In particular, to prove strong decay, the authors of [7] appealed to the wellknown spectral criteria in [1]. Very recently, an alternative *resolvent criterion approach* to strong decay, with respect to a multilayered Lamé-heat system was given in [4]. However, up to the present time, there has not been, to the best of our knowledge, any investigations into *uniform decay properties* – with respect to either finite energy or higher norm – of multilayered FSI PDE models. Accordingly, the question in the present work is, *is the dissipation emanating from the thermal component of the FSI system (1)-(4) strong enough to elicit polynomial decay?*

3. Novelty and challenges

As noted above, despite extensive research activity on single-layer FSI models in the last twenty years or so, a comprehensive long term analysis theory for multilayered FSI – in which the boundary interface coupling between fluid and structure components is realized via an additional elastic equation – is largely absent. Having established in [7] the strong (asymptotic) stability for the multilayered FSI model (1)-(4), the authors in the present work address the issue of obtaining rational decay rates for solutions of the coupled 3D heat-2D wave-3D wave dynamics under consideration. These rational decay rates will pertain to solutions of (1)-(4) which correspond to smooth initial data; i.e., initial data drawn from the domain of the associated heat-wave-wave C_0-semigroup generator $\mathbf{A} : D(\mathbf{A}) \subset \mathbf{H} \to \mathbf{H}$ in (7) below.

Our approach here for solving the rational decay problem will entail an appropriate estimation of the resolvent of the corresponding semigroup generator, with a view of invoking the wellknown resolvent criterion in [13] for polynomial decay. Ultimately, we will obtain an explicit decay rate of $\mathcal{O}(t^{-\frac{2}{11}})$. This is Theorem 3 below. As far as we can tell, this is the first such polynomial stability result obtained for multilayered FSI. By way of obtaining the rational decay result, we will operate in the frequency domain and deal with a static FSI system – which is essentially the *Laplace transformed* version of the original system (1)-(4) – and the resolvent of the generator of the dynamical system. In this regard, challenging issues associated with the analysis are as follows:

(i) Majorizing the solution to the resolvent equation in terms of the heat component. Having obtained in our previous work [7] an understanding of the spectral properties of the corresponding semigroup generator $\mathbf{A} : D(\mathbf{A}) \subset \mathbf{H} \to \mathbf{H}$ (of (7) below), we proceed with considering the resolvent system in (11) below and discern an inherent (static) fluid dissipative relation. Analogous to what the analysis undertaken in the time domain for control of PDE's in general – see e.g., [40] – we will strive here to exploit the static dissipation by *majorizing* the solution Φ of (10) in terms of the heat component $\mathbf{u}$. However the *key* issue here will be an appropriate estimate for the 3D *thick* wave component w.

(ii) Sharpening Poincaré's Inequality for the thermal component. In order to deal with critical boundary trace terms and ultimately majorize them with respect to the static heat dissipation, we will need to refine some of the estimates concerning the boundary term $\frac{\partial u}{\partial \nu}\big|_{\Gamma_s}$. This will require us to prove a sharpening of Poincaré's Inequality; in particular, we will need to properly majorize the L^2-norm of the thermal component in such a way so as to ultimately secure the given decay rate.

(iii) Control of the critical three wave boundary terms. As we pointed out in (i), the main challenge here will be the appropriate estimate for the 3D *thick* wave component w. The use of a certain "Dirichlet" map will ultimately enable us to homogenize this thick wave component of (11), via a new variable z, which has zero Dirichlet boundary trace. To this new variable z, we subsequently apply frequency domain versions of known vector identities for the control of (uncoupled) waves. However, the obtained preliminary estimate on the thick wave component will still contain the problematic boundary term $\frac{\partial z}{\partial \nu}\big|_{\Gamma_s}$, a term which should be controlled in L^2-sense. The desired estimate of this flux term will require the invocation of the thin layer

h-equations in (11) and some *sharp* interpolation inequalities. It is the estimation of this boundary term which ultimately dictates the obtained rational decay rate.

4. Preliminaries

It is shown in [7] that the multilayered PDE system (1)-(4) can be described via the matrix operator $\mathbf{A} : D(\mathbf{A}) \subset \mathbf{H} \to \mathbf{H}$ defined by

$$\mathbf{A} = \begin{bmatrix} \Delta & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & I & \dots & 0 & 0 & 0 & 0 \\ -\frac{\partial}{\partial \nu}\big|_{\Gamma_1} & (\Delta - I) & 0 & \dots & 0 & 0 & \frac{\partial}{\partial \nu}\big|_{\Gamma_1} & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & I & 0 & 0 \\ -\frac{\partial}{\partial \nu}\big|_{\Gamma_K} & 0 & 0 & \dots & (\Delta - I) & 0 & \frac{\partial}{\partial \nu}\big|_{\Gamma_K} & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & I \\ 0 & 0 & 0 & \dots & 0 & 0 & \Delta & 0 \end{bmatrix}, \tag{7}$$

with

$D(\mathbf{A}) = \{[u_0, h_{01}, h_{11}(t)...., h_{0K}, h_{1K}, w_0, w_1] \in \mathbf{H} :$

(A.i) (a) $u_0 \in H^1(\Omega_f)$, (b) $h_{1j} \in H^1(\Gamma_j)$ for $j = 1, ..., K$, (c) $w_1 \in H^1(\Omega_f)$;

(A.ii) (a) $\Delta u_0 \in L^2(\Omega_f)$, (b) $\Delta w_0 \in L^2(\Omega_s)$, (c) $\Delta h_{0j} - \frac{\partial u_0}{\partial \nu}\big|_{\Gamma_j} + \frac{\partial w_0}{\partial \nu}\big|_{\Gamma_j} \in L^2(\Gamma_j)$ for $j = 1, ..., K$,
(d) $\frac{\partial h_{0j}}{\partial n_j}\Big|_{\partial\Gamma_j} \in H^{-\frac{1}{2}}(\partial\Gamma_j)$ for $j = 1, ..., K$;

(A.iii) (a) $u_0|_{\Gamma_f} = 0$, (b) $u_0|_{\Gamma j} = h_{1j} = w_1|_{\Gamma j}$ for $j = 1, ..., K$;

(A.iv) For $j = 1, ..., K$ such that $\partial\Gamma_j \cap \partial\Gamma_l \neq \emptyset$:
(a) $h_{1j}|_{\partial\Gamma_j \cap \partial\Gamma_l} = h_{1l}|_{\partial\Gamma_j \cap \partial\Gamma_l}$; (b) $\frac{\partial h_{0j}}{\partial n_j}\Big|_{\partial\Gamma_j \cap \partial\Gamma_l} = -\frac{\partial h_{0l}}{\partial n_l}\Big|_{\partial\Gamma_j \cap \partial\Gamma_l}\Big\}$.

This is to say, $\Phi(t) = \big[u(t), h_1(t), \frac{\partial}{\partial t}h_1(t)...., h_K(t), \frac{\partial}{\partial t}h_K, w(t), w_t(t)\big]$ satisfies the PDE model (1)-(4) if and only if these variables solve the following ODE in Hilbert space $\mathbf{H}$:

$$\frac{d}{dt}\Phi(t) = \mathbf{A}\Phi(t) \text{ on } (0,T); \qquad \Phi(0) = \Phi_0 = [u_0, h_{01}, h_{02}, ..., h_{0K}, h_{0K}, w_0, w_1] \in \mathbf{H}. \tag{8}$$

We recall the wellposedness result given in [7]:

Theorem 1. *The linear operator* $\mathbf{A} : D(\mathbf{A}) \subset \mathbf{H} \to \mathbf{H}$, *as defined in* (7), *generates a* C_0*-semigroup* $\{e^{\mathbf{A}t}\}_{t\geq 0}$ *of contractions on* $\mathbf{H}$. *Thus, for* $\Phi_0 = [u_0, h_{01}, h_{02}, ..., h_{0K}, h_{0K}, w_0, w_1] \in \mathbf{H}$, *the solution* $\Phi(t) = \big[u(t), h_1(t), \frac{\partial}{\partial t}h_1(t)...., h_K(t), \frac{\partial}{\partial t}h_K, w(t), w_t(t)\big]$ *of (1)-(4) is given (continuously) by*

$$\Phi(t) = e^{\mathbf{A}t}\Phi_0 \in C([0,T]; \mathbf{H}).$$

5. Main result

Our present work mainly focuses on analyzing the long time behavior of solutions to the given multilayered system (1)-(4), with a view of obtaining rational decay rate of these solutions. Our main proof of stability will be based on an ultimate appeal to wellknown resolvent criterion of A. Borichev and Y. Tomilov [13, Theorem 2.4]:

Theorem 2. *Let $\{T(t)\}_{t\geq 0}$ be a bounded C_0-semigroup on a Hilbert space H with generator A such that $i\mathbb{R}\subset\rho(A)$. Then for fixed $\alpha>0$ the following are equivalent:*
(i) $\|\mathcal{R}(is;A)\|=\mathcal{O}(|s|^{\alpha})$, $|s|\to\infty$;
(ii) $\left\|T(t)A^{-1}x\right\|=o(t^{-\frac{1}{\alpha}})$, $t\to\infty$, $x\in\mathbf{H}$.

Given this operator theoretic result, it will suffice to establish the "frequency domain" PDE estimate in (9); the proof of this estimate will constitute the bulk of the effort in the present paper. Now, we give our main result of polynomial decay for solutions which correspond to smooth initial data as follows:

Theorem 3. *In regard to the multilayered PDE system in (1)-(4) (or equivalently (8)): if $\Phi_0=[u_0,h_{01},h_{02},...,h_{0K},h_{0K},w_0,w_1]\in D(\mathbf{A})$, then the corresponding solution of (1)-(4) (or equivalently (8)) satisfies the estimate,*

$$\|\Phi(t)\|_{\mathbf{H}}\leq\frac{C}{t^{\frac{2}{11}}}\|\Phi_0\|_{D(\mathbf{A})}. \tag{9}$$

That is, the solution to (1)-(4) (or equivalently (8)), which corresponds to smooth initial data, decays at a rate of $\mathcal{O}(t^{-\frac{2}{11}})$.

Proof of Theorem 3

The proof relies on the resolvent criterion given in Theorem 2, and presupposes that there is no intersection of $\sigma(\mathbf{A})$ with the imaginary axis. In fact it was shown in [7, Proposition 7, Lemma 9, Corollary 10] – see Section 4 therein – that $i\mathbb{R}\subset\rho(\mathbf{A})$.

Subsequently, given parameter $\beta\in\mathbb{R}$ and data $\Phi_0^*=[u^*,h_{01}^*,h_{02}^*,...,h_{0K}^*,h_{0K}^*,w_0^*,w_1^*]\in\mathbf{H}$, we consider the resolvent equation

$$[i\beta I-\mathbf{A}]\,\Phi=\Phi_0^*, \tag{10}$$

with solution $\Phi=[u,h_{01},h_{02},...,h_{0K},h_{0K},w_0,w_1]\in D(\mathbf{A})$. From the definition of $\mathbf{A}$, this abstract equation can be written explicitly as

$$\begin{cases} i\beta u-\Delta u=u^* \ \text{ in } \Omega_f\\ u|_{\Gamma_f}=0 \text{ on } \Gamma_f \end{cases}$$

$$\begin{cases} i\beta h_{0j}-h_{1j}=h_{0j}^* \ \text{ in } \ \Gamma_j\\ -\beta^2h_{0j}-\Delta h_{0j}+h_{0j}+\frac{\partial u}{\partial\nu}-\frac{\partial w_0}{\partial\nu}=h_{1j}^*+i\beta h_{0j}^* \ \text{ in } \ \Gamma_j\\ \text{For all } 1\leq l\leq K \text{ such that } \partial\Gamma_j\cap\partial\Gamma_l\neq\emptyset\\ h_{0j}|_{\partial\Gamma_j\cap\partial\Gamma_l}=h_{0l}|_{\partial\Gamma_j\cap\partial\Gamma_l}\\ \frac{\partial h_{0j}}{\partial n_j}\Big|_{\partial\Gamma_j\cap\partial\Gamma_l}=-\frac{\partial h_{0l}}{\partial n_l}\Big|_{\partial\Gamma_j\cap\partial\Gamma_l}, \end{cases} \tag{11}$$

$$\begin{cases} w_1=i\beta w_0-w_0^* \ \text{ in } \Omega_s\\ -\beta^2w_0-\Delta w_0=i\beta w_0^*+w_1^* \ \text{ in } \Omega_s\\ [i\beta w_0-w_0^*]_{\Gamma_j}=h_{1j}=u|_{\Gamma_j} \ \text{ on } \ \Gamma_j. \end{cases}$$

We will give our proof step-wise, estimating each solution component separately:

Step 1: A static dissipation relation for heat component u:

We will start with an inherent (static) fluid dissipative relation which will be the key ingredient for future steps. First, we take the $\mathbf{H}$-inner product of both sides of (10) with respect to pre-image Φ. This gives

$$\begin{aligned} &i\beta \left\|\Phi\right\|_{\mathbf{H}}^2 + \left\|\nabla u\right\|_{\Omega_f}^2 - 2i\sum_{j=1}^{K}\left[\mathrm{Im}\left(\nabla h_{1j}, \nabla h_{0j}\right)_{\Gamma_j} + \mathrm{Im}\left(h_{1j}, h_{0j}\right)_{\Gamma_j}\right] \\ &-2i\mathrm{Im}\left(\nabla w_1, \nabla w_0\right)_{\Omega_s} = \left(\Phi_0^*, \Phi_0\right)_{\mathbf{H}}, \end{aligned} \tag{12}$$

and then the following dissipation relation:

$$\left\|\nabla u\right\|_{\Omega_f}^2 = \mathrm{Re}\left(\Phi_0^*, \Phi\right)_{\mathbf{H}}. \tag{13}$$

In view of relation (13), *we should strive to majorize solution* Φ *of* (10) *in norm in terms of the static heat dissipation*. With this theme in mind, from the mechanical compatibility conditions in (5), and the resolvent relations and matching velocity BC's in (11), we have for $j = 1, ..., K$,

$$\left[i\beta h_{0j} - h_{0j}^*\right]_{\Gamma_j} = h_{1j}|_{\Gamma_j} = u|_{\Gamma_j}. \tag{14}$$

Combining this relation with the Sobolev Trace Theorem (and Poincaré's Inequality), we then have for $j = 1, ..., K$,

$$\left\|\beta h_{0j}\right\|_{H^{\frac{1}{2}}(\Gamma_j)} \leq C\left(\left\|\nabla u\right\|_{\Omega_f} + \left\|\Phi_0^*\right\|_{\mathbf{H}}\right). \tag{15}$$

Moreover, via an integration by parts we get

$$\begin{aligned} \left\|\frac{\partial u}{\partial \nu}\right\|_{H^{-\frac{1}{2}}(\partial\Omega_f)} &\leq C\left(\left\|\Delta u\right\|_{\Omega_f} + \left\|\nabla u\right\|_{\Omega_f}\right) \\ &= C\left(\left\|i\beta u - u^*\right\|_{\Omega_f} + \left\|\nabla u\right\|_{\Omega_f}\right) \end{aligned} \tag{16}$$

which, in turn, gives

$$\left\|\frac{\partial u}{\partial \nu}\right\|_{H^{-\frac{1}{2}}(\partial\Omega_f)} \leq C\left(\left|\beta\right|\left\|u\right\|_{\Omega_f} + \left\|\nabla u\right\|_{\Omega_f} + \left\|\Phi_0^*\right\|_{\mathbf{H}}\right). \tag{17}$$

We should note that this estimate can be refined with respect to β. In fact, for our particular multilayered PDE model, we have the following "sharpening" of Poincaré's Inequality:

Proposition 4. *For* $|\beta| > 0$, *the heat solution component of (11) obeys the estimate*

$$\left|\beta\right|^{\frac{1}{2}}\left\|u\right\|_{\Omega_f} \leq C\left(\left\|\nabla u\right\|_{\Omega_f} + \left\|\Phi_0^*\right\|_{\mathbf{H}}\right). \tag{18}$$

Proof. The details of the proof are taken in large part from Lemma 5.2 of [5] and [6]. Given heat component u of (11), let variable u_1 solve

$$\begin{cases} \Delta u_1 = i\beta u & \text{in } \Omega_f \\ u_1|_{\partial\Omega_f} = 0 & \text{on } \partial\Omega_f. \end{cases} \tag{19}$$

Taking the L^2-inner product of both sides of $(19)_1$ by u_1, we subsequently have, from the heat equation in (11):

$$\begin{aligned} \left\|\nabla u_1\right\|_{\Omega_f}^2 &= -\left(\Delta u_1, u_1\right)_{\Omega_f} \\ &= -\left(i\beta u, u_1\right)_{\Omega_f} \\ &= -\left(\Delta u + u^*, u_1\right)_{\Omega_f} \\ &= \left(\nabla u, \nabla u_1\right)_{\Omega_f} - \left(u^*, u_1\right)_{\Omega_f}; \end{aligned}$$

whence we obtain, via Poincaré's and Young's Inequalities,

$$\|\nabla u_1\|_{\Omega_f} \leq C \left(\|\nabla u\|_{\Omega_f} + \|\Phi_0^*\|_{\mathbf{H}} \right). \tag{20}$$

Since the u_1-equation in (19) gives

$$\|u\|_{H^{-1}(\Omega_f)} = \frac{1}{|\beta|} \|\Delta u_1\|_{H^{-1}(\Omega_f)},$$

using again the Poincaré's Inequality yields

$$\|u\|_{H^{-1}(\Omega_f)} \leq \frac{C}{|\beta|} \|\nabla u_1\|_{\Omega_f} \leq \frac{C}{|\beta|} \left(\|\nabla u\|_{\Omega_f} + \|\Phi_0^*\|_{\mathbf{H}} \right), \tag{21}$$

after using (20). Moreover, it is known – see e.g., Theorem B.8, Theorem 3.30 and Theorem 3.33 of [33] – that

$$L^2(\Omega_f) = \left[H^{-1}(\Omega_f), H^1(\Omega_f) \right]_{\frac{1}{2}}$$

(which is the Lipschitz domain version of the interpolation result Lemma 12.1 of [32]). Combining this with the estimate (21) gives

$$\|u\|_{L^2(\Omega_f)} \leq C \|u\|^{\frac{1}{2}}_{H^{-1}(\Omega_f)} \|\nabla u\|^{\frac{1}{2}}_{\Omega_f} \leq \frac{C}{|\beta|^{\frac{1}{2}}} \left(\|\nabla u\|_{\Omega_f} + \|\Phi_0^*\|_{\mathbf{H}} \right)$$

which yields (18) and finishes the proof of Proposition 4. □

Now, applying the estimate (18) to the right hand side of (17), we then have the following normal derivative trace estimate for the heat component of the static problem (11):

Corollary 5. *For $|\beta| > 1$, the heat solution component of (11) obeys the estimate*

$$\left\| \frac{\partial u}{\partial \nu} \right\|_{H^{-\frac{1}{2}}(\partial\Omega_f)} \leq C |\beta|^{\frac{1}{2}} \left(\|\nabla u\|_{\Omega_f} + \|\Phi_0^*\|_{\mathbf{H}} \right). \tag{22}$$

Step 2: The thick wave displacement w_0

In what follows, we require "Dirichlet" map D, defined by having for given boundary function $g \in H^{\frac{1}{2}}(\Gamma_s)$,

$$\Delta Dg = 0 \ \text{ in } \Omega_s; \qquad Dg|_{\Gamma_s} = g \ \text{ on } \Gamma_s. \tag{23}$$

By the Lax-Milgram Theorem and an argument similar to that which resulted in (17), we have

$$D \in \mathcal{L}\left(H^{\frac{1}{2}}(\Gamma_s), H^1(\Omega_s) \right), \ \ \frac{\partial D}{\partial \nu} \in \mathcal{L}\left(H^{\frac{1}{2}}(\Gamma_s), H^{-\frac{1}{2}}(\Gamma_s) \right). \tag{24}$$

(The latter is the "Dirichlet to Neumann" map.) Therewith, and with respect to the thick wave displacement w_0 in (11), we set

$$z = w_0 + \frac{i}{\beta} D \left(u|_{\Gamma_s} + w_0^*|_{\Gamma_s} \right). \tag{25}$$

Then, via (23), the variable z satisfies the following boundary value problem:

$$-\beta^2 z - \Delta z = -i\beta D\left[u|_{\Gamma_s} + w_0^*|_{\Gamma_s}\right] + w_1^* + i\beta w_0^* \text{ in } \Omega_s;$$
$$z = 0 \text{ on } \Gamma_s. \tag{26}$$

Since Ω_s is convex then $z \in H^2(\Omega_s)$ (see e.g., Theorem 3.2.1.2, p. 147 of [27]). Subsequently, we can appeal to the known Sobolev boundary regularity results for polyhedral domains; [11, p. 43, Theorem 6.9]. In short, we have the estimate, for $|\beta| \geq 1$,

$$\|z\|_{H^2(\Omega_s)} + \sum_{j=1}^{K} \left\| \frac{\partial z}{\partial \nu} \right\|_{H^{\frac{1}{2}}(\Gamma_j)} \leq C_0 \left(\beta^2 \|z\|_{\Omega_s} + |\beta| \left\| D\left[u|_{\Gamma_s} + w_0^*|_{\Gamma_s}\right] \right\|_{\Omega_s} + \|w_1^* + i\beta w_0^*\|_{\Omega_s} \right)$$
$$\leq C_0 |\beta| \left(\|\beta z\|_{\Omega_s} + \|\nabla u\|_{\Omega_f} + \|\Phi_0^*\|_{\mathbf{H}} \right), \tag{27}$$

after using (26), (24) and the Sobolev Imbedding Theorem.

With respect to the z-wave equation in (26), we appeal to the "frequency domain" version of the well-known wave identity which is synonymous with boundary control of wave equations; see Proposition 7 (ii) of [2], and also [21],[40]. We adopt those wave identities to our solution component z in the following Proposition:

Proposition 6. *Let $\mathbf{m}(x) = [m_1(x), m_2(x), m_3(x)]$ be an arbitrary real-valued $[C^2(\bar{\Omega}_s)]^3$-vector field, with associated Jacobian matrix $M(x)$. Then the wave component of the solution to the resolvent equation (10) obeys the following relation:*

$$(i) \int_{\Omega_s} \left| M^{\frac{1}{2}}(x) \nabla z \right|^2 d\Omega_s = -Re \int_{\Gamma_s} \frac{\partial z}{\partial \nu} (\mathbf{m} \cdot \nabla \overline{z}) \, d\Gamma_s + \frac{1}{2} \int_{\Gamma_s} \left| \frac{\partial z}{\partial \nu} \right|^2 \mathbf{m} \cdot \nu d\Gamma_s$$
$$+ \frac{1}{2} \int_{\Omega_s} \left\{ |\nabla z|^2 - \beta^2 |z|^2 \right\} div(\mathbf{m}) d\Omega_s h$$
$$- Re \int_{\Omega_s} \left(i\beta D\left[u|_{\Gamma_s} + w_0^*|_{\Gamma_s}\right] - w_1^* - i\beta w_0^* \right) (\mathbf{m} \cdot \nabla \overline{z}) \, d\Omega_s$$
$$= -\frac{1}{2} \int_{\Gamma_s} \left| \frac{\partial z}{\partial \nu} \right|^2 \mathbf{m} \cdot \nu d\Gamma_s + \frac{1}{2} \int_{\Omega_s} \left\{ |\nabla z|^2 - \beta^2 |z|^2 \right\} div(\mathbf{m}) d\Omega_s$$
$$+ i\beta Re \int_{\Omega_s} \nabla \left(D\left[u|_{\Gamma_s} + w_0^*|_{\Gamma_s}\right] - w_0^* \right) \cdot \mathbf{m} \bar{z} d\Omega_s$$
$$+ i\beta Re \int_{\Omega_s} \left(D\left[u|_{\Gamma_s} + w_0^*|_{\Gamma_s}\right] - w_0^* \right) \bar{z} \, div(\mathbf{m}) d\Omega_s$$
$$+ Re \int_{\Omega_s} w_1^* (\mathbf{m} \cdot \nabla \overline{z}) \, d\Omega_s. \tag{28}$$

(ii) If $\tilde{\mathbf{m}}(x)$ is an arbitrary real-valued $[C^2(\bar{\Omega}_s)]^3$-vector field, then the wave component of the solution to the resolvent equation (10) satisfies the relation,

$$\int_{\Omega_s} \left\{ |\nabla z|^2 - \beta^2 |z|^2 \right\} div(\tilde{\mathbf{m}}) d\Omega_s = -Re \int_{\Omega_s} \left(i\beta D\left[u|_{\Gamma_s} + w_0^*|_{\Gamma_s}\right] - w_1^* - i\beta w_0^* \right) \overline{z} \, div(\tilde{\mathbf{m}}) \, d\Omega_s$$

$$-Re\int_{\Omega_s}\left[\nabla z\cdot\nabla div(\tilde{\mathbf{m}})\right]\bar{z}d\Omega_s. \tag{29}$$

(Note that the expressions (28)-(29) each reflect the fact that $z=0$ on Γ_s and/or the unit normal vector $\nu(x)$ is pointing inward with respect to solid geometry Ω_s.)

Now, let the vector fields $\mathbf{m}$ and $\tilde{\mathbf{m}}$ in (28) and (29), respectively, be taken as

$$\mathbf{m}(x)=\tilde{\mathbf{m}}(x)=x. \tag{30}$$

Then, via (24), Young's Inequality and the Sobolev Trace Theorem, we have for $|\beta|\geq 1$,

$$\left|\int_{\Omega_s}\left\{|\nabla z|^2-\beta^2|z|^2\right\}d\Omega_s\right|\leq\epsilon\beta^2\|z\|^2_{\Omega_s}+C_\epsilon\left(\|\nabla u\|^2_{\Omega_f}+\|\Phi_0^*\|^2_{\mathbf{H}}\right). \tag{31}$$

In turn, with vector field $\mathbf{m}$ as specified in (30), applying (31) to the right hand side of (28), using (24), the Sobolev Trace Theorem and again Young's Inequality, we have the following Proposition:

Proposition 7. *For $|\beta|\geq 1$ and $\epsilon>0$, the variable z of (25) and (26) satisfies*

$$\int_{\Omega_s}|\nabla z|^2\,d\Omega_s\leq C^*\int_{\Gamma_s}\left|\frac{\partial z}{\partial\nu}\right|^2d\Gamma_s+\epsilon\left(\|\nabla z\|^2_{\Omega_s}+\beta^2\|z\|^2_{\Omega_s}\right)+C_\epsilon\left(\|\nabla u\|^2_{\Omega_f}+\|\Phi_0^*\|^2_{\mathbf{H}}\right) \tag{32}$$

(with respect to (31) there has also been a rescaling of $\epsilon>0$).

At this point, we also note in the z-wave relation (28) that $\mathbf{m}(x)$ could be specified to be the smooth vector field of Lemma 1.5.1.9, pg. 40 of [27]: That is, for some $\delta>0$, $\mathbf{m}(x)\in[C^\infty(\overline{\Omega}_s)]$ satisfies

$$-\mathbf{m}(x)\cdot\nu\geq\delta\ \text{ a.e. on }\Gamma_s. \tag{33}$$

This gives in (28):

$$\begin{aligned}\frac{\delta}{2}\int_{\Gamma_s}\left|\frac{\partial z}{\partial\nu}\right|^2d\Gamma_s\leq\Bigg|&\int_{\Omega_s}\left|M^{\frac{1}{2}}(x)\nabla z\right|^2d\Omega_s-\frac{1}{2}\int_{\Omega_s}\left\{|\nabla z|^2-\beta^2|z|^2\right\}\mathrm{div}(\mathbf{m})d\Omega_s\\&-i\beta\mathrm{Re}\int_{\Omega_s}\nabla\left(D\left[u|_{\Gamma_s}+w_0^*|_{\Gamma_s}\right]-w_0^*\right)\cdot\mathbf{m}\bar{z}d\Omega_s\\&-i\beta\mathrm{Re}\int_{\Omega_s}\left(D\left[u|_{\Gamma_s}+w_0^*|_{\Gamma_s}\right]-w_0^*\right)\bar{z}\mathrm{div}(\mathbf{m})d\Omega_s-\mathrm{Re}\int_{\Omega_s}w_1^*\left(\mathbf{m}\cdot\nabla\overline{z}\right)d\Omega_s\Bigg|.\end{aligned}$$

Estimating right hand side of this relation by means of (24), Young's Inequality and the Sobolev Trace Theorem, we have the control of the trace term $\frac{\partial z}{\partial\nu}|_{\Gamma_s}$ in the following proposition:

Proposition 8. *The variable z of (25) and (26) satisfies*

$$\int_{\Gamma_s}\left|\frac{\partial z}{\partial\nu}\right|^2d\Gamma_s\leq C\left(\|\nabla z\|^2_{L^2(\Omega_s)}+\beta^2\|z\|^2_{\Omega_s}+\|\nabla u\|^2_{\Omega_f}+\|\Phi_0^*\|^2_{\mathbf{H}}\right). \tag{34}$$

We should emphasize that given the right hand side of estimate (32), it is apparent that a useful estimate of the thick wave energy component of (11) will necessitate "decent" control of $\frac{\partial z}{\partial \nu}\big|_{\Gamma_s}$ in L^2-sense: that is, the positive constant C in (34) need not be "small". However, while the estimate (34) does not constitute such needed control, it will serve as an ingredient for attaining that end.

Step 3: The thin wave displacement h_0

With respect to the h-wave equations in (11), we take the L^2-inner product with respect to h_{0j}, for $j = 1, ..., K$. This gives

$$-\left(\Delta h_{0j}, h_{0j}\right)_{\Gamma_j} + \|h_{0j}\|^2_{\Gamma_j} = \left(\beta^2 h_{0j}, h_{0j}\right)_{\Gamma_j} + \left\langle \frac{\partial w_0}{\partial \nu}, h_{0j} \right\rangle_{\Gamma_j} - \left\langle \frac{\partial u}{\partial \nu}, h_{0j} \right\rangle_{\Gamma_j} + \left(h^*_{1j}, h_{0j}\right)_{\Gamma_j} + i\beta \left(h^*_{0j}, h_{0j}\right)_{\Gamma_j}. \tag{35}$$

Subsequently, we invoke the Green's Theorem (for $j = 1, ..., K$) to get

$$\begin{aligned} &\|\nabla h_{0j}\|^2_{\Gamma_j} + \|h_{0j}\|^2_{\Gamma_j} - \left\langle \frac{\partial h_{0j}}{\partial \nu}, h_{0j} \right\rangle_{\partial\Gamma_j} \\ &= \beta^2 \|h_{0j}\|^2_{\Gamma_j} + \left\langle \frac{\partial w_0}{\partial \nu}, h_{0j} \right\rangle_{\Gamma_j} - \left\langle \frac{\partial u}{\partial \nu}, h_{0j} \right\rangle_{\Gamma_j} \\ &\quad + \left(h^*_{1j}, h_{0j}\right)_{\Gamma_j} + i\beta \left(h^*_{0j}, h_{0j}\right)_{\Gamma_j}. \end{aligned} \tag{36}$$

Using the thin layer boundary conditions in (11), we then have upon summation

$$\begin{aligned} &\sum_{j=1}^{K} \left[\|\nabla h_{0j}\|^2_{\Gamma_j} + \|h_{0j}\|^2_{\Gamma_j}\right] \\ &= \sum_{j=1}^{K} \left[\beta^2 \|h_{0j}\|^2_{\Gamma_j} + \left\langle \frac{\partial w_0}{\partial \nu}, h_{0j} \right\rangle_{\Gamma_j} - \left\langle \frac{\partial u}{\partial \nu}, h_{0j} \right\rangle_{\Gamma_j} \right. \\ &\quad \left. + \left(h^*_{1j}, h_{0j}\right)_{\Gamma_j} + i\beta \left(h^*_{0j}, h_{0j}\right)_{\Gamma_j}\right]. \end{aligned}$$

Estimating right hand side by means of (22) and (15), and via the inequality

$$\left\langle \frac{\partial u}{\partial \nu}, h_{0j} \right\rangle_{\Gamma_j} \leq \left(\frac{1}{|\beta|^{\frac{1}{2}}} \left\| \frac{\partial u}{\partial \nu} \right\|_{H^{-\frac{1}{2}}(\Gamma_j)}\right) \left(|\beta|^{\frac{1}{2}} \|h_{0j}\|_{H^{\frac{1}{2}}(\Gamma_j)}\right),$$

we then have for $|\beta| \geq 1$ that the thin wave solution components of (11) obey the following (intermediate) estimate:

$$\sum_{j=1}^{K} \left[\|\nabla h_{0j}\|^2_{\Gamma_j} + \|h_{0j}\|^2_{\Gamma_j}\right] \leq \left| \sum_{j=1}^{K} \left\langle \frac{\partial w_0}{\partial \nu}, h_{0j} \right\rangle_{\Gamma_j} \right| + C \left(\|\nabla u\|^2_{\Omega_f} + \|\Phi^*_0\|^2_{\mathbf{H}}\right). \tag{37}$$

5.1. An appropriate estimate for $\frac{\partial z}{\partial \nu}\big|_{\Gamma_s}$

Using the decomposition (25) and the thin wave equations in (11), we have for $j = 1, ..., K$,

$$\frac{\partial z}{\partial \nu} = -\beta^2 h_{0j} - \Delta h_{0j} + h_{0j} + \frac{\partial u}{\partial \nu} + \frac{i}{\beta} \frac{\partial D}{\partial \nu} \left[u|_{\Gamma_s} + w^*_0|_{\Gamma_s}\right] - h^*_{1j} - i\beta h^*_{0j} \quad \text{in} \quad \Gamma_j. \tag{38}$$

For the first term on right hand side, we use the matching velocities BC and h-resolvent relation in (11) and a wellknown trace moment inequality – see e.g., Theorem 1.6.6, p. 37 of [14] – so as to have

$$\left\|\beta^2 h_{0j}\right\|_{\Gamma_j} = \left\|\beta\, u|_{\Gamma_j} + \beta\, h_{0j}^*|_{\Gamma_j}\right\|_{\Gamma_j} \leq C\,|\beta|\left(\|u\|_{\Omega_f}^{\frac{1}{2}}\,\|\nabla u\|_{\Omega_f}^{\frac{1}{2}} + \|\Phi_0^*\|_{\mathbf{H}}\right). \tag{39}$$

Subsequently invoking the improvement over Poincaré's Inequality in Proposition 4, we have now for $|\beta| \geq 1$,

$$\left\|\beta^2 h_{0j}\right\|_{\Gamma_j} \leq C\left(|\beta|^{\frac{3}{4}}\,\|\nabla u\|_{\Omega_f} + |\beta|\,\|\Phi_0^*\|_{\mathbf{H}}\right).$$

Applying this estimate to the right hand side of (38), along with (37), (22), (24), and the Sobolev Trace Theorem, we then have for $|\beta| \geq 1$ (upon summing over $j = 1, ..., K$),

$$\sum_{j=1}^{K}\left\|\frac{\partial z}{\partial \nu}\right\|_{H^{-1}(\Gamma_j)} \leq C_1\left(\sqrt{\sum_{j=1}^{K}\left|\left\langle \frac{\partial w_0}{\partial \nu}, h_{0j}\right\rangle_{\Gamma_j}\right|} + C\,|\beta|^{\frac{3}{4}}\,\|\nabla u\|_{\Omega_f} + |\beta|\,\|\Phi_0^*\|_{\mathbf{H}}\right). \tag{40}$$

To refine the right hand side: Using again the decomposition (25), we have for $|\beta| \geq 1$,

$$\begin{aligned}\sqrt{\sum_{j=1}^{K}\left|\left\langle \frac{\partial w_0}{\partial \nu}, h_{0j}\right\rangle_{\Gamma_j}\right|} &= \sqrt{\sum_{j=1}^{K}\left|\left\langle \frac{\partial}{\partial \nu}\left[z - \frac{i}{\beta}D([u + w_0^*]|_{\Gamma_s})\right], h_{0j}\right\rangle\right|_{\Gamma_j}} \\ &= \sqrt{\sum_{j=1}^{K}\left|\left\langle \frac{1}{\beta}\frac{\partial z}{\partial \nu}, \beta h_{0j}\right\rangle - \frac{i}{\beta}\left\langle \frac{\partial}{\partial \nu}D([u + w_0^*]|_{\Gamma_s}), h_{0j}\right\rangle\right|_{\Gamma_j}}.\end{aligned}$$

Applying (15), (24), Sobolev Trace Theorem and the Young's Inequality, we then obtain for $|\beta| \geq 1$,

$$\sqrt{\sum_{j=1}^{K}\left|\left\langle \frac{\partial w_0}{\partial \nu}, h_{0j}\right\rangle_{\Gamma_j}\right|} \leq \frac{\delta^*}{|\beta|\,C_1}\sum_{j=1}^{K}\left\|\frac{\partial z}{\partial \nu}\right\|_{H^{-\frac{1}{2}}(\Gamma_j)} + C_{\delta^*}\left(\|\nabla u\|_{\Omega_f} + \|\Phi_0^*\|_{\mathbf{H}}\right),$$

where $C_1 > 0$ is the constant in (40). Applying this inequality to (40), we have for $|\beta| \geq 1$,

$$\sum_{j=1}^{K}\left\|\frac{\partial z}{\partial \nu}\right\|_{H^{-1}(\Gamma_j)} \leq \frac{\delta^*}{|\beta|}\sum_{j=1}^{K}\left\|\frac{\partial z}{\partial \nu}\right\|_{H^{-\frac{1}{2}}(\Gamma_j)} + C_{2,\delta^*}\left[|\beta|^{\frac{3}{4}}\,\|\nabla u\|_{\Omega_f} + |\beta|\,\|\Phi_0^*\|_{\mathbf{H}}\right]. \tag{41}$$

Interpolating now between (34) and (41) we have for $j = 1, ..., K$,

$$\begin{aligned}\left\|\tfrac{\partial z}{\partial \nu}\right\|_{H^{-\frac{1}{2}}(\Gamma_j)} &\leq C\left\|\tfrac{\partial z}{\partial \nu}\right\|_{H^{-1}(\Gamma_j)}^{\frac{1}{2}}\left\|\tfrac{\partial z}{\partial \nu}\right\|_{\Gamma_j}^{\frac{1}{2}} \\ &\leq C_3\left\{\left(\tfrac{\delta^*}{|\beta|}\sum_{j=1}^{K}\left\|\tfrac{\partial z}{\partial \nu}\right\|_{H^{-\frac{1}{2}}(\Gamma_j)} + C_{2,\delta^*}\left[|\beta|^{\frac{3}{4}}\,\|\nabla u\|_{\Omega_f} + |\beta|\,\|\Phi_0^*\|_{\mathbf{H}}\right]\right)^{\frac{1}{2}}\right. \\ &\quad \left.\times\left(\|\nabla z\|_{\Omega_s} + \|\beta z\|_{\Omega_s} + \|\nabla u\|_{\Omega_f} + \|\Phi_0^*\|_{\mathbf{H}}\right)^{\frac{1}{2}}\right\}.\end{aligned} \tag{42}$$

Subsequently, for $|\beta| \geq 1$, with constant C_3 as in (42) and using the relation $|ab| \leq \frac{\delta^* K C_3}{2|\beta|}a^2 + \frac{|\beta|}{2\delta^* K C_3}b^2$, we then have

$$\left\|\frac{\partial z}{\partial \nu}\right\|_{H^{-\frac{1}{2}}(\Gamma_j)} \leq \frac{1}{2K}\sum_{j=1}^{K}\left\|\frac{\partial z}{\partial \nu}\right\|_{H^{-\frac{1}{2}}(\Gamma_j)} + \frac{\delta^* K C_3^2}{2|\beta|}\left(\|\nabla z\|_{L^2(\Omega_s)} + \|\beta z\|_{\Omega_s}\right)$$
$$+ C_{\delta^*}\left[|\beta|^{\frac{7}{4}}\|\nabla u\|_{\Omega_f} + |\beta|^2\|\Phi_0^*\|_{\mathbf{H}}\right], \text{ for } j = 1, ..., K.$$

Summing this estimate over j now gives

$$\sum_{j=1}^{K}\left\|\frac{\partial z}{\partial \nu}\right\|_{H^{-\frac{1}{2}}(\Gamma_j)} \leq \frac{1}{2}\sum_{j=1}^{K}\left\|\frac{\partial z}{\partial \nu}\right\|_{H^{-\frac{1}{2}}(\Gamma_j)} + \frac{\delta^*}{|\beta|}C_4\left(\|\nabla z\|_{L^2(\Omega_s)} + \|\beta z\|_{\Omega_s}\right)$$
$$+C_{\delta^*}\left(|\beta|^{\frac{7}{4}}\|\nabla u\|_{\Omega_f} + |\beta|^2\|\Phi_0^*\|_{\mathbf{H}}\right)$$

(where constant C_4 is independent of $\delta^* > 0$). We have then for $j = 1, ..., K$,

$$\left\|\frac{\partial z}{\partial \nu}\right\|_{H^{-\frac{1}{2}}(\Gamma_j)} \leq \frac{\delta^*}{|\beta|}C_5\left(\|\nabla z\|_{\Omega_s} + \|\beta z\|_{\Omega_s}\right) + C_{\delta^*}\left(|\beta|^{\frac{7}{4}}\|\nabla u\|_{\Omega_f} + |\beta|^2\|\Phi_0^*\|_{\mathbf{H}}\right). \tag{43}$$

Subsequently, we interpolate between (43) and (27) to have,

$$\left\|\tfrac{\partial z}{\partial \nu}\right\|_{\Gamma_j} \leq C_6\left\|\tfrac{\partial z}{\partial \nu}\right\|^{\frac{1}{2}}_{H^{-\frac{1}{2}}(\Gamma_j)}\left\|\tfrac{\partial z}{\partial \nu}\right\|^{\frac{1}{2}}_{H^{\frac{1}{2}}(\Gamma_j)}$$

$$\leq C_6\left\{\left[\tfrac{\delta^*}{|\beta|}C_5\left(\|\nabla z\|_{\Omega_s} + \|\beta z\|_{\Omega_s}\right) + C_{\delta^*}\left(|\beta|^{\frac{7}{4}}\|\nabla u\|_{\Omega_f} + |\beta|^2\|\Phi_0^*\|_{\mathbf{H}}\right)\right]\right.$$
$$\left.\times C_0|\beta|\left(\|\beta z\|_{\Omega_s} + \|\nabla u\|_{\Omega_f} + \|\Phi_0^*\|_{\mathbf{H}}\right)\right\}^{\frac{1}{2}}$$

$$\leq C_6\left\{\delta^* C_0 C_5\left(\left[\|\nabla z\|_{\Omega_s} + \|\beta z\|_{\Omega_s} + \|\nabla u\|_{\Omega_f} + \|\Phi_0^*\|_{\mathbf{H}}\right]^2\right)\right.$$
$$\left.+C_{\delta^*}C_0|\beta|\left(|\beta|^{\frac{7}{4}}\|\nabla u\|_{\Omega_f} + |\beta|^2\|\Phi_0^*\|_{\mathbf{H}}\right)\left(\|\nabla z\|_{\Omega_s} + \|\beta z\|_{\Omega_s} + \|\nabla u\|_{\Omega_f} + \|\Phi_0^*\|_{\mathbf{H}}\right)\right\}^{\frac{1}{2}}$$

Invoking once more the relation $|ab| \leq \delta^* a^2 + C_{\delta^*}b^2$, for $\delta^* > 0$, we have now

$$\left\|\frac{\partial z}{\partial \nu}\right\|_{\Gamma_j} \leq \delta^* C_7\left(\|\nabla z\|_{\Omega_s} + \|\beta z\|_{\Omega_s}\right) + C_{\delta^*}\left(|\beta|^{\frac{11}{4}}\|\nabla u\|_{\Omega_f} + |\beta|^3\|\Phi_0^*\|_{\mathbf{H}}\right).$$

Summing over $j = 1, ..., K$, and rescaling $\delta^* > 0$, we have now the following desired trace estimate for $\frac{\partial z}{\partial \nu}\big|_{\Gamma_s}$:

Lemma 9. *For $|\beta| \geq 1$ and arbitrary $\delta^* > 0$, the variable z of (25) and (26) obeys the estimate*

$$\left\|\frac{\partial z}{\partial \nu}\right\|_{\Gamma_s} \leq \delta^*\left(\|\nabla z\|_{\Omega_s} + \|\beta z\|_{\Omega_s}\right) + C_{\delta^*}\left(|\beta|^{\frac{11}{4}}\|\nabla u\|_{\Omega_f} + |\beta|^3\|\Phi_0^*\|_{\mathbf{H}}\right). \tag{44}$$

Completion of the Proof of Theorem 3. Applying (44) to right hand side of (32) of Proposition 7 (and subsequently rescaling), we have for $|\beta| \geq 1$,

$$\int_{\Omega_s}|\nabla z|^2\, d\Omega_s \leq \epsilon\left(\|\nabla z\|^2_{\Omega_s} + \beta^2\|z\|^2_{\Omega_s}\right) + C_\epsilon\left(|\beta|^{\frac{11}{2}}\|\nabla u\|^2_{\Omega_f} + |\beta|^6\|\Phi_0^*\|^2_{\mathbf{H}}\right). \tag{45}$$

In turn, we take vector field $\tilde{\mathbf{m}}(x)$ in (29) of Proposition 6 to satisfy $\text{div}(\tilde{\mathbf{m}}) = 1$. Afterwards, we estimate this relation by means of (45), (24), the Sobolev Trace Theorem and the Young's Inequality, $|ab| \leq \rho a^2 + C_\rho b^2$ $(1 > \rho > 0)$ to have

$$\int_{\Omega_s} |\beta z|^2 \, d\Omega_s \leq \frac{\epsilon}{(1-\rho)} \left(\|\nabla z\|^2_{L^2(\Omega_s)} + \beta^2 \|z\|^2_{L^2(\Omega_s)} \right) + C_{\epsilon,\rho} \left(|\beta|^{\frac{11}{2}} \|\nabla u\|^2_{\Omega_f} + |\beta|^6 \|\Phi_0^*\|^2_{\mathbf{H}} \right). \tag{46}$$

In sum, for $|\beta| \geq 1$, (45) and (46) give now,

$$\|z\|^2_{H^1(\Omega_s)} + \|\beta z\|^2_{\Omega_s} + \left\| \frac{\partial z}{\partial \nu} \right\|^2_{\Gamma_s} \leq C \left(|\beta|^{\frac{11}{2}} \|\nabla u\|^2_{\Omega_f} + |\beta|^6 \|\Phi_0^*\|^2_{\mathbf{H}} \right). \tag{47}$$

Subsequently, via the change of variable in (25), (47) and the regularity for D in (24), with respect to the thick layer wave solution components in (11), we have that for $|\beta| \geq 1$,

$$\begin{aligned} \text{(i)} \quad \|w_0\|^2_{H^1(\Omega_s)} &= \left\| z - \frac{i}{\beta} D \left(u|_{\Gamma_s} + w_0^*|_{\Gamma_s} \right) \right\|^2_{H^1(\Omega_s)} \\ &\leq C \left(|\beta|^{\frac{11}{2}} \|\nabla u\|^2_{\Omega_f} + |\beta|^6 \|\Phi_0^*\|^2_{\mathbf{H}} \right); \end{aligned} \tag{48}$$

$$\begin{aligned} \text{(ii)} \quad \|w_1\|^2_{L^2(\Omega_s)} &= \left\| i\beta \left[z - \frac{i}{\beta} D \left(u|_{\Gamma_s} + w_0^*|_{\Gamma_s} \right) \right] - w_0^* \right\|^2_{L^2(\Omega_s)} \\ &\leq C \left(|\beta|^{\frac{11}{2}} \|\nabla u\|^2_{\Omega_f} + |\beta|^6 \|\Phi_0^*\|^2_{\mathbf{H}} \right). \end{aligned} \tag{49}$$

In turn, via (37) and the decomposition (25) we get

$$\begin{aligned} \sum_{j=1}^K \left[\|\nabla h_{0j}\|^2_{\Gamma_j} + \|h_{0j}\|^2_{\Gamma_j} \right] &\leq \left| \sum_{j=1}^K \left\langle \frac{\partial (z - \frac{i}{\beta} D \left(u|_{\Gamma_s} + w_0^*|_{\Gamma_s} \right)}{\partial \nu}, h_{0j} \right\rangle_{\Gamma_j} \right| + C \left(\|\nabla u\|^2_{\Omega_f} + \|\Phi_0^*\|^2_{\mathbf{H}} \right) \\ &\leq C \left(\left\| \frac{\partial z}{\partial \nu} \right\|_{\Gamma_s} + \frac{1}{|\beta|} \left\| \frac{\partial \frac{i}{\beta} D \left(u|_{\Gamma_s} + w_0^*|_{\Gamma_s} \right)}{\partial \nu} \right\|_{H^{-\frac{1}{2}}(\Gamma_s)} \right) \sum_{j=1}^K \|h_{0j}\|_{H^{\frac{1}{2}}(\Gamma_j)}. \end{aligned}$$

Invoking now, (47), (24), the Sobolev Trace Theorem, and (15), we have for $|\beta| \geq 1$,

$$\sum_{j=1}^K \left[\|\nabla h_{0j}\|^2_{\Gamma_j} + \|h_{0j}\|^2_{\Gamma_j} \right] \leq C \left(|\beta|^{\frac{11}{2}} \|\nabla u\|^2_{\Omega_f} + |\beta|^6 \|\Phi_0^*\|^2_{\mathbf{H}} \right). \tag{50}$$

In addition, via the resolvent relation in (11), we have for $j = 1, ..., K$,

$$\|h_{1j}\|^2_{H^{\frac{1}{2}}(\Gamma_j)} = \left\| i\beta h_{0j} - h_{0j}^* \right\|^2_{H^{\frac{1}{2}}(\Gamma_j)},$$

whence by (15),

$$\|h_{1j}\|^2_{H^{\frac{1}{2}}(\Gamma_j)} \leq C \left(\|\nabla u\|^2_{\Omega_f} + \|\Phi_0^*\|^2_{\mathbf{H}} \right). \tag{51}$$

Finally, collecting (48), (49), (50) and (51) we have with the solution variable $\Phi = [u, h_{01}, h_{02}, ..., h_{0K}, h_{0K}, w_0, w_1]$ that

$$\|\Phi\|^2_{\mathbf{H}} \leq C \left(|\beta|^{\frac{11}{2}} \|\nabla u\|^2_{\Omega_f} + |\beta|^6 \|\Phi_0^*\|^2_{\mathbf{H}} \right).$$

Invoking now the static dissipation relation (13) and Young's Inequality one last time, we obtain

$$\|\Phi\|_{\mathbf{H}}^2 \leq C\left(|\beta|^{\frac{11}{2}}\,|(\Phi_0^*,\Phi)_{\mathbf{H}}| + |\beta|^6\,\|\Phi_0^*\|_{\mathbf{H}}^2\right)$$
$$\leq \epsilon\,\|\Phi\|_{\mathbf{H}}^2 + |\beta|^{11}\,\|\Phi_0^*\|_{\mathbf{H}}^2\,.$$

Since data $\Phi_0^* \in \mathbf{H}$ in (10) was arbitrary, this gives the desired resolvent bound

$$\|\mathcal{R}(i\beta;\mathbf{A})\|_{\mathcal{L}(\mathbf{H})} \leq C\,|\beta|^{\frac{11}{2}}\,.$$

An appeal to Theorem 2 now concludes the proof of Theorem 3.

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