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On k-point configuration sets with nonempty interior

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Abstract

We give conditions for k-point configuration sets of thin sets to have nonempty interior, applicable to a wide variety of configurations. This is a continuation of our earlier work (J. Geom. Anal. **31** (2021), 6662–6680) on 2-point configurations, extending a theorem of Mattila and Sjölin (Math. Nachr. **204** (1999), 157–162) for distance sets in Euclidean spaces. We show that for a general class of k-point configurations, the configuration set of a k-tuple of sets, $E_1, \dots, E_k$, has nonempty interior provided that the sum of their Hausdorff dimensions satisfies a lower bound, dictated by optimizing L^2-Sobolev estimates of associated generalized Radon transforms over all nontrivial partitions of the k points into two subsets. We illustrate the general theorems with numerous specific examples. Applications to 3-point configurations include areas of triangles in $\mathbb{R}^2$ or the radii of their circumscribing circles; volumes of pinned parallelepipeds in $\mathbb{R}^3$; and ratios of pinned distances in $\mathbb{R}^2$ and $\mathbb{R}^3$. Results for 4-point configurations include cross-ratios on $\mathbb{R}$, pairs of areas of triangles determined by quadrilaterals in $\mathbb{R}^2$, and dot products of differences in $\mathbb{R}^d$.

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1 | INTRODUCTION

A classical result of Steinhaus [37] states that if $E \subset \mathbb{R}^d$, $d \geqslant 1$, has positive Lebesgue measure, then the difference set $E - E \subset \mathbb{R}^d$ contains a neighborhood of the origin. $E - E$ can interpreted as the set of two-point configurations, $x - y$, of points of E modulo the translation group.

Similarly, in the context of the Falconer distance set problem, a theorem of Mattila and Sjölin [30] states that if $E \subset \mathbb{R}^d$, $d \geqslant 2$, is compact, then the distance set of E, $\Delta(E) := \{\,|x-y| : x, y \in E\} \subset \mathbb{R}$, contains an open interval, that is, has nonempty interior, if the Hausdorff dimension $\dim_{\mathcal{H}}(E) > \frac{d+1}{2}$. This represented a strengthening of Falconer's original result [7], from $\Delta(E)$ merely having positive Lebesgue measure to having nonempty interior, for the same range of $\dim_{\mathcal{H}}(E)$. This was generalized to distance sets with respect to norms on $\mathbb{R}^d$ with positive-curvature unit spheres by Iosevich, Mourgoglou and Taylor [23].

These latter types of result, for two-point configurations in *thin sets*, that is, E allowed to have Lebesgue measure zero but satisfying a lower bound on $\dim_{\mathcal{H}}(E)$, were extended by the current authors to more general settings in [14]: (i) configurations in E as measured by a general class of Φ-*configurations*, which can be vector-valued and nontranslation-invariant; and (ii) *asymmetric* configurations, that is, between points in sets E_1 and E_2 lying in different spaces, for example, between points and circles in $\mathbb{R}^2$, or points and hyperplanes in $\mathbb{R}^d$.

We point out that there are a number of other results that are explicitly, or can be interpreted as being, concerned with establishing conditions under which configuration sets of thin sets have nonempty interior, including [3, 6, 13, 24, 36] in the continuous setting and [4, 5, 34, 35] in finite field analogues.

The purpose of the current paper is to extend our results in [14], from 2-point to quite general k-point configuration sets for $k \geqslant 3$, using that paper's Fourier integral operator (FIO) approach, making use of linear L^2-Sobolev estimates, but now optimizing over all possible nontrivial partitions of the k points into two subsets. The FIO method we describe works in the absence of symmetry and on general manifolds, and indeed, exploring that generality, rather than sharpness of the lower bounds on the Hausdorff dimensions, is the focus of the current work.

However, we will start by illustrating the variety of what can be obtained via this approach with configurations defined by classical geometric quantities in low-dimensional Euclidean spaces. We describe a number of concrete examples, but emphasize that the choice of these specific configurations is arbitrary; our general results can be applied to other configurations of interest, with the Hausdorff dimension threshold guaranteeing that the configuration set has nonempty interior depending on the outcome of optimizing over a family of FIO estimates; see Theorems 2.1 and 5.2 for exact statements. Our first example is the following.

Theorem 1.1 (Areas and circumradii of triangles). *If $E \subset \mathbb{R}^2$ is compact with $\dim_{\mathcal{H}}(E) > 5/3$, then*

(i) *the set of areas of triangles determined by triples of points of E,*

$$\left\{ \frac{1}{2} |\det [x - z,\ y - z]| \ :\ x, y, z \in E \right\} \subset \mathbb{R}, \tag{1.1}$$

contains an open interval; and

(ii) *the set of radii of circles determined by triples of points in E contains an open interval.*

We will see in Remark 4.1 that the FIO method does not yield a pinned version of Theorem 1.1, that is, it says nothing about two-point configuration sets,

$$\left\{ \frac{1}{2}|\det[x-z,\ y-z]| \ :\ x,y \in E \right\}$$

for a fixed $z \in E$. However, in dimension $d \geqslant 3$ it does yield a result for *all* $z \in E$ (or, indeed, any $z \in \mathbb{R}^d$), and we have the following d-point configuration result.

Theorem 1.2 (Strongly pinned† volumes). *Let $d \geqslant 3$. If $E \subset \mathbb{R}^d$ is compact, then for any $x^0 \in \mathbb{R}^d$, the set of volumes of parallelepipeds determined by x^0 and d-tuples of points of E,*

$$V_d^{x^0}(E) := \left\{ \left|\det\left[x^1 - x^0, x^2 - x^0, \dots, x^d - x^0\right]\right| \ :\ x^1, x^2, \dots, x^d \in E \right\}, \tag{1.2}$$

has nonempty interior in $\mathbb{R}$ if $\dim_{\mathcal{H}}(E) > d - 1 + (1/d)$.

Remark 1.3. For $d = 3$, this improves upon an earlier result of the first two authors and Mourgoglou [12], which was that if $\dim_{\mathcal{H}}(E) > 13/5$, then $V_3^0(E)$ has positive Lebesgue measure. (Added in proof: see also [8] for further improvements on lowering the threshold for positive Lebesgue measure of $V_d(E)$.)

Returning to three-point configurations, our method also yields a result about ratios of distances. Mkrtchyan and the first two named authors studied in [11] the existence of similarities of k-point configurations in thin sets. They posed the question of whether, under some lower bound restriction on $\dim_{\mathcal{H}}(E)$, for every $r > 0$ there exist $x, y, z \in E$ such that $|x - z| = r|y - z|$. We can partially address this, showing that the set of such r at least contains an interval.

To put this in perspective, note that an immediate consequence of the result of Mattila and Sjölin [30] is that if $\dim_{\mathcal{H}}(E) > (d + 1)/2$, then

$$\operatorname{int}\left(\left\{ \frac{|w-z|}{|x-y|} \ :\ x, y, z, w \in E \right\} \right) \neq \emptyset. \tag{1.3}$$

(See also [21] for a finite field analogue.) On the other hand, Peres and Schlag [33] showed that if $\dim_{\mathcal{H}}(E) > (d + 2)/2$, a stronger property holds:

$$\text{there exists an } x \in E \text{ such that } \operatorname{int}\left(\left\{ \frac{|x-z|}{|x-y|} \ :\ y, z \in E \right\} \right) \neq \emptyset. \tag{1.4}$$

(See also [22, 25] for extensions of this.) Here we prove a result for a property of intermediate strength, one which implies (1.3) but is in turn implied by (1.4); however, in dimensions $d = 2, 3$ our result is proved for lower $\dim_{\mathcal{H}}(E)$ than the known range for (1.4):

Theorem 1.4 (Ratios of pairs of pinned distances). *Let $d \geqslant 2$ and $E \subset \mathbb{R}^d$ compact. Then, if $\dim_{\mathcal{H}}(E) > (2d + 1)/3$,*

$$\operatorname{int}\left(\left\{ \frac{|x-z|}{|x-y|} \ :\ x, y, z \in E,\ x \neq y \right\} \right) \neq \emptyset. \tag{1.5}$$

† The term *pinned* is often used to refer to estimates for the supremum over $x^0 \in E$ of expressions such as (1.2). Here we obtain a result valid for *all* x^0, hence our adoption of *strongly pinned* for lack of a better term.

We now turn from three-point configurations to a pair of results concerning four-point configurations, in $\mathbb{R}$ and $\mathbb{R}^2$, respectively.

Theorem 1.5 (Cross ratios). *Let $E \subset \mathbb{R}$ be compact with $\dim_{\mathcal{H}}(E) > 3/4$. Then the set of cross ratios of four-tuples of points of E,*

$$\mathrm{Cross}(E) = \left\{ [x_1, x_2; x_3, x_4] = \frac{(x_3 - x_1)(x_4 - x_2)}{(x_3 - x_2)(x_4 - x_1)} \ : \ x_1, x_2, x_3, x_4 \in E \right\} \subset \mathbb{R},$$

contains an open interval.

So far, all of the configurations described have been measured by scalar-valued functions. Returning to $d = 2$, an example of a *vector-valued* configuration is a variation of Theorem 1.1, where one takes four points in the plane, say x, y, z, w, and considers the quadrilateral they generate. Pick one of the two diagonals, say $\overline{yw}$; this splits the quadrilateral into two triangles, and we study the vector-valued configuration consisting of their areas.

Theorem 1.6 (Pairs of areas of triangles). *If $E \subset \mathbb{R}^2$ is a compact set and has $\dim_{\mathcal{H}}(E) > 7/4$, then the set of pairs of areas of triangles determined by 4-tuples of points of E,*

$$\left\{ \left(\frac{1}{2} |\det [x - w, y - w]|, \ \frac{1}{2} |\det [y - w, z - w]| \right) \ : \ x, y, z, w \in E \right\}, \tag{1.6}$$

has nonempty interior in $\mathbb{R}^2$.

Finally, we give two more applications of the FIO method, this time to configurations with a more additive combinatorics flavor. There has been considerable work on products of differences in the discrete or finite field setting; just a few references are [1, 2, 17, 31, 32]. An analogue of some of these results in the continuous setting is the following.

Theorem 1.7. *For $d \geqslant 1$ and $E \subset \mathbb{R}^d$ compact, the set of dot products of differences of points in E,*

$$\{(x - y) \cdot (z - w) \ : \ x, y, z, w \in E\} \subset \mathbb{R},$$

has nonempty interior[†] *if $\dim_{\mathcal{H}}(E) > \frac{d}{2} + \frac{1}{4}$.*

Another result of sum-product type is

Theorem 1.8 (Generalized sum-product sets). *Let $Q_1, \ldots, Q_l$ be nondegenerate, symmetric bilinear forms on $\mathbb{R}^d$. Suppose that $E_i \subset \mathbb{R}^d$ are compact sets with $\dim_{\mathcal{H}}(E_i) > \frac{d}{2} + \frac{1}{2l}$ for all $1 \leqslant i \leqslant 2l$. Then the set of values*

$$\Sigma_{\vec{Q}}(E_1, \ldots, E_{2l}) := \left\{ \sum_{j=1}^{l} Q_j \left(x^{2j-1}, x^{2j} \right) \ : \ x^i \in E_i, \ 1 \leqslant i \leqslant 2l \right\} \subset \mathbb{R} \tag{1.7}$$

† Added in proof: Similarly, if $E \subset \mathbb{R}$ with $\dim_{\mathcal{H}}(E) > \frac{3}{4}$, then $(E + E)(E + E)$ has nonempty interior, answering a question posed to us by Pham; see [27] for related Falconer-type results and references.

has nonempty interior. In particular, taking all of the $Q_j(x,y) = x \cdot y$, *under the same conditions on the* $\dim_{\mathcal{H}}(E_i)$, *the sum-(Euclidean inner) product set of the* E_i,

$$\left\{ \left(x^1 \cdot x^2\right) + \left(x^3 \cdot x^4\right) + \cdots + \left(x^{2l-1} \cdot x^{2l}\right) : x^i \in E_i,\ 1 \leqslant i \leqslant 2l \right\},$$

has nonempty interior.

Remark 1.9. This follows from a more general result allowing the forms to be on spaces of different dimensions; see Theorem 6.1.

2 | THREE-POINT CONFIGURATIONS

To describe a general class of k-point configurations which includes the examples above, we start by recalling the framework of Φ-configuration sets, introduced by Grafakos, Palsson and the first two authors in [9]. We used this approach in the current article's prequel, [14] to establish nonempty interior results for 2-point configuration sets. To minimize the notation, we initially describe these for 3-point configurations, introducing the basic method and results, which will be extended to higher k in Section 5.

A *3-point configuration function* is initially a smooth $\Phi : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^p$ (with $p \leqslant d$); we use the notation $\Phi(x^1, x^2, x^3)$, $x^j \in \mathbb{R}^d$, $j = 1,2,3$. Since, for many problems of interest, there are points, often corresponding to degenerate configurations, where Φ has critical points or fails to be smooth, it is useful to restrict the domain of Φ. Anticipating the extension to k-point configurations later, we label the three copies of $\mathbb{R}^d$ (or open subsets of $\mathbb{R}^d$) as X^1, X^2, X^3. Furthermore, for some applications it is useful to allow the X^j to be manifolds of possibly different dimensions d_j, $j = 1,2,3$. Thus, in general we define a 3-point configuration function to be a mapping $\Phi : X^1 \times X^2 \times X^3 \to T$, where $T \subset\subset \mathbb{R}^p$, or even a p-dimensional manifold, containing the range of Φ on the compact sets of interest. Function spaces on the X^j are with respect to smooth densities, which do not play a significant role and therefore are suppressed in the notation.

For compact sets $E_j \subset X^j$, $j = 1,2,3$, define the 3-point Φ-*configuration set* of E_1, E_2, E_3,

$$\Delta_\Phi(E_1, E_2, E_3) := \left\{ \Phi\left(x^1, x^2, x^3\right) : x^j \in E_j,\ j = 1,2,3 \right\} \subset T. \tag{2.1}$$

The goal is to find conditions on $\dim_{\mathcal{H}}(E_j)$ ensuring that $\mathrm{int}(\Delta_\Phi(E_1, E_2, E_3)) \neq \emptyset$.

If the full differential $D_{x^1,x^2,x^3}\Phi$ has maximal rank ($= p$) everywhere, that is, Φ is a submersion, then Φ is a defining function for a family of smooth surfaces in $X := X^1 \times X^2 \times X^3$, and for each $\mathbf{t} \in T$, the level set

$$Z_{\mathbf{t}} := \left\{ (x^1, x^2, x^3) \in X : \Phi(x^1, x^2, x^3) = \mathbf{t} \right\} \tag{2.2}$$

is smooth and of codimension p in X, and $Z_{\mathbf{t}}$ depends smoothly on $\mathbf{t}$. (For $p = 1$, we denote $\mathbf{t}$ by simply t.)

If $s_j < \dim_{\mathcal{H}}(E_j)$, let μ_j be a Frostman measure on E_j with finite s_j-energy (see the discussion in Section 3). The choice of the μ_j induces a *configuration measure*, ν, on T, having various equivalent definitions, for example, for $g \in C_0(T)$,

$$\int_T g(\mathbf{t})\, d\nu(\mathbf{t}) = \int\int\int_{E_1 \times E_2 \times E_3} g(\Phi(x^1, x^2, x^3))\, d\mu_1(x^1)\, d\mu_2(x^2)\, d\mu_3(x^3). \tag{2.3}$$

If one can show that ν is absolutely continuous with respect to Lebesgue measure, $d\mathbf{t}$; its density function is continuous; and $\Delta_\Phi(E_1, E_2, E_3)$ is nonempty, then it follows that $\mathrm{int}(\Delta_\Phi(E_1, E_2, E_3)) \neq \emptyset$.

Following the general approach of [14], but now exploiting the fact that we are studying 3-point, rather than 2-point, configurations, we will derive the continuity of the density of $d\nu$ (denoted $\nu(\mathbf{t})$) from L^2-Sobolev mapping properties of any of three different families of generalized Radon transforms associated to $Z_\mathbf{t}$, as follows.

Write a nontrivial partition of $\{1,2,3\}$ as $\sigma = (\sigma_L|\sigma_R)$, grouping the variable(s) x^i corresponding to $i \in \sigma_L$ on the left and the variable(s) corresponding to $i \in \sigma_R$ on the right. Due to the symmetry of L^2-Sobolev estimates for FIOs under adjoints, we may assume that $|\sigma_L| = 2$, $|\sigma_R| = 1$; furthermore, permutation within σ_L is irrelevant, so up to interchange of those two indices, there are three such partitions, $\sigma = (12|3)$, $(13|2)$, and $(23|1)$. Corresponding to each of these, for each $\mathbf{t} \in T$, partitioning and permuting the variables according to σ, the surface $Z_\mathbf{t}$ defines incidence relations,

$$\begin{aligned} Z_\mathbf{t}^{(12|3)} &:= \{(x^1, x^2; x^3) : (x^1, x^2, x^3) \in Z_\mathbf{t}\} \subset (X^1 \times X^2) \times X^3, \\ Z_\mathbf{t}^{(13|2)} &:= \{(x^1, x^3; x^2) : (x^1, x^3, x^2) \in Z_\mathbf{t}\} \subset (X^1 \times X^3) \times X^2, \\ Z_\mathbf{t}^{(23|1)} &:= \{(x^2, x^3; x^1) : (x^2, x^3, x^1) \in Z_\mathbf{t}\} \subset (X^2 \times X^3) \times X^1. \end{aligned} \tag{2.4}$$

Each $Z_\mathbf{t}^\sigma$, with $\sigma = (ij|k)$, defines an incidence relation from X^k to $X^i \times X^j$, and to this is associated a generalized Radon transform, $\mathcal{R}_\mathbf{t}^\sigma$; all the $\mathcal{R}_\mathbf{t}^\sigma$ have the "same" Schwartz kernel, namely the singular measure supported on $Z_\mathbf{t}$,

$$\lambda_\mathbf{t} := \chi(x^1, x^2, x^3) \cdot \delta\big(\Phi\big(x^1, x^2, x^3\big) - \mathbf{t}\big),$$

except that the order and grouping of the variables are dictated by σ. That is, the kernel of $\mathcal{R}_\mathbf{t}^{(ij|k)}$ is $K_\mathbf{t}{}^{(ij|k)}(x^i, x^j, x^k) := \lambda(x^1, x^2, x^3)$. (Here χ is a fixed cutoff function $\equiv 1$ on $E_1 \times E_2 \times E_3$ which plays no further role.)

For each σ, we can formulate the double fibration condition, $(DF)_\sigma$, standard in the theory of generalized Radon transforms and originating in the works of Gelfand; Helgason [18]; and Guillemin and Sternberg [15, 16], namely that the two spatial projections from $Z_\mathbf{t}^\sigma$ have maximal rank, namely

$$(DF)_\sigma \qquad \pi_{ij} : Z_\mathbf{t}^\sigma \to X^i \times X^j \text{ and } \pi_k : Z_\mathbf{t}^\sigma \to X^k \text{ are submersions.} \tag{2.5}$$

This implies that not only does $\mathcal{R}_\mathbf{t}^\sigma : \mathcal{D}(X^k) \to \mathcal{E}(X^i \times X^j)$, but also

$$\mathcal{R}_\mathbf{t}^\sigma : \mathcal{E}'(X^k) \to \mathcal{D}'(X^i \times X^j),$$

defined weakly by

$$\mathcal{R}_\mathbf{t}^\sigma f(x^i, x^j) = \int_{\{x^k : \Phi(x^1, x^2, x^3) = \mathbf{t}\}} f(x^k),$$

where the integral is with respect to the surface measure induced by $\lambda_\mathbf{t}$ on the codimension p surface $\{x^k : \Phi(x^1, x^2, x^3) = \mathbf{t}\} \subset X^k$.

An alternate description of the configuration measure defined by (2.3) is in terms of the $\mathcal{R}_{\mathbf{t}}^{\sigma}$; this was stated and proved in the case of 2-point configuration measures in [14, Section 3]. However, the proof there goes over with minor modifications to the case of k-point configurations, and for completeness we give the argument for $k = 3$ in Section 3.4. Namely, as long as the terms in the two arguments of the $\langle\cdot,\cdot\rangle$ pairing below belong to Sobolev spaces on which the bilinear pairing is continuous, ν has a density given by

$$\nu(\mathbf{t}) = \left\langle \mathcal{R}_{\mathbf{t}}^{(ij|k)}(\mu_k), \mu_i \times \mu_j \right\rangle. \tag{2.6}$$

Now, under the double fibration condition $(DF)_\sigma$, the generalized Radon transform $\mathcal{R}_{\mathbf{t}}^{\sigma}$ is a Fourier integral operator (FIO) associated with a canonical relation

$$C_{\mathbf{t}}^{\sigma} \subset \left(T^*\left(X^i \times X^j\right) \setminus 0\right) \times \left(T^*X^k \setminus 0\right),$$

where $C_{\mathbf{t}}^{\sigma} = (N^*Z_{\mathbf{t}}^{\sigma})'$, the (twisted) conormal bundle of $Z_{\mathbf{t}}^{\sigma}$ (see Section 3). All three of $\mathcal{R}_{\mathbf{t}}^{(12|3)}$, $\mathcal{R}_{\mathbf{t}}^{(13|2)}$, $\mathcal{R}_{\mathbf{t}}^{(23|1)}$ are Fourier integral operators of the same order,

$$m = 0 + \frac{1}{2}p - \frac{1}{4}(d_1 + d_2 + d_3) = \frac{p}{2} - \frac{1}{4}d^{tot}, \quad d^{tot} := d_1 + d_2 + d_3.$$

However, due to the (possibly) different dimensions, in order to understand the optimal estimates for the operators $\mathcal{R}_{\mathbf{t}}^{\sigma}$, one knows from standard FIO theory that the estimates are conveniently expressed in terms of what we will call their *effective orders*, m_{eff}^{σ}. These are defined by writing m in three different ways, accounting for the dimension differences $|\dim(X^i \times X^j) - \dim(X^k)|$:

$$m = m_{\text{eff}}^{(12|3)} - \frac{1}{4}|d_1 + d_2 - d_3|,$$

$$m = m_{\text{eff}}^{(13|2)} - \frac{1}{4}|d_1 + d_3 - d_2|, \text{ or}$$

$$m = m_{\text{eff}}^{(23|1)} - \frac{1}{4}|d_2 + d_3 - d_1|.$$

In terms of the m_{eff}^{σ}, the mapping properties of the operators $\mathcal{R}_{\mathbf{t}}^{\sigma}$ can be described as

$$\mathcal{R}_{\mathbf{t}}^{\sigma} : L_r^2 \to L^2_{r - m_{\text{eff}}^{\sigma} - \beta_{\mathbf{t}}^{\sigma}}, \ \forall r \in \mathbb{R},$$

for certain (possible) losses $\beta_{\mathbf{t}}^{\sigma} \geqslant 0$. If, for some value $\mathbf{t}_0 \in T$, $C_{\mathbf{t}_0}^{\sigma}$ is *nondegenerate*, that is, one of its two natural projections to the left or right, π_L or π_R, is of maximal rank (which implies that the other is as well), then $\beta_{\mathbf{t}_0}^{\sigma} = 0$, and by structural stability of submersions this is also true for all $\mathbf{t}$ near $\mathbf{t}_0$ (see Section 3.) Our basic assumption is that, for at least one σ, there is a known $\beta^{\sigma} \geqslant 0$ such that $\mathcal{R}_{\mathbf{t}}^{\sigma} : L_r^2 \to L^2_{r - m_{\text{eff}}^{\sigma} - \beta^{\sigma}}$ uniformly for $\mathbf{t} \in T$.

To simplify the arithmetic, assume that for all the $\sigma = (ij|k)$, we have $d_i + d_j \geqslant d_k$, which includes the equidimensional case, $d_1 = d_2 = d_3$. Then $m_{\text{eff}}{}^{(ij|k)} = (p - d_k)/2$, and thus our basic boundedness assumption is that, for at least one of the σ,

$$\mathcal{R}_{\mathbf{t}}^{\sigma} : L_r^2(X^k) \to L^2_{r + \frac{1}{2}(d_k - p) - \beta_{\mathbf{t}}^{\sigma}}(X^i \times X^j) \text{ uniformly in } \mathbf{t}. \tag{2.7}$$

At the start of the argument, the s_j, $j = 1, 2, 3$, were chosen to be any values such that $\dim_{\mathcal{H}}(E_j) > s_j$, and each μ_j has finite s_j energy, so that $\mu_j \in L^2_{(s_j-d_j)/2}(X^j)$. An easy calculation with Sobolev norms shows that if $u_j \in L^2_{r_j}(\mathbb{R}^{d_j})$ with $r_j \leqslant 0$, $j = 1, 2$, then $u_1 \otimes u_2 \in L^2_{r_1+r_2}(\mathbb{R}^{d_1+d_2})$, and this extends to compactly supported distributions on manifolds (see Proposition 3.2 below). Thus,

$$\mu_1 \times \mu_2 \in L^2_{(s_1+s_2-d_1-d_2)/2}, \; \mu_1 \times \mu_3 \in L^2_{(s_1+s_3-d_1-d_3)/2}, \text{ and } \mu_2 \times \mu_3 \in L^2_{(s_2+s_3-d_2-d_3)/2}.$$

Combining all of these considerations, and focusing on $\sigma = (12|3)$ for the moment, we see that the bilinear pairing in the expression (2.6) for $\nu(\mathbf{t})$ is continuous if

$$(s_3 - d_3)/2 + (d_3 - p)/2 - \beta^{(12|3)} + (s_1 + s_2 - d_1 - d_2)/2 \geqslant 0,$$

that is,

$$s_1 + s_2 + s_3 \geqslant d_1 + d_2 + p + 2\beta^{(12|3)}.$$

The analogous calculation holds for whichever of the $\mathcal{R}^\sigma_{\mathbf{t}}$ one knows estimates for, and the minimum over σ of the right hand sides gives a sufficient condition for $\nu(\mathbf{t})$ to be continuous. Thus, the set where $\nu(\mathbf{t}) > 0$ is an open set; to conclude that $\operatorname{int}(\Delta_\Phi(E_1, E_2, E_3)) \neq \emptyset$, it suffices to show that $\Delta_\Phi(E_1, E_2, E_3)$ itself is nonempty.

As in [14], this follows by noting that what we have done above already implies the Falconer-type conclusion that $\Delta_\Phi(E_1, E_2, E_3) \subset \mathbb{R}^p$ has positive Lebesgue measure. In fact, if $\{B(\mathbf{t}_j, \epsilon_j)\}$ is any cover of $\Delta_\Phi(E_1, E_2, E_3)$, one has

$$\begin{aligned} 1 &= \mu_1(E_1) \cdot \mu_2(E_2) \cdot \mu_3(E_3) = (\mu_1 \times \mu_2 \times \mu_3)(E_1 \times E_2 \times E_3) \\ &\leqslant (\mu_1 \times \mu_2 \times \mu_3)\left(\Phi^{-1}\left(\bigcup_j B(\mathbf{t}_j, \epsilon_j) \right)\right) \\ &\leqslant \sum_j (\mu_1 \times \mu_2 \times \mu_3)\left(\Phi^{-1}\left(B(\mathbf{t}_j, \epsilon_j)\right)\right) \\ &= \sum_j \nu\left(B(\mathbf{t}_j, \epsilon_j)\right) \leqslant C_\Phi \sum_j \epsilon_j^p, \end{aligned} \tag{2.8}$$

by (3.6) below, so that $\sum_j |B(\mathbf{t}_j, \epsilon_j)|_p \geqslant C'_\Phi$ is bounded below. Hence $\Delta_\Phi(E_1, E_2, E_3)$ has positive p-dimensional Lebesgue measure and is therefore nonempty; by the continuity of $\nu(\mathbf{t})$, it in fact has nonempty interior.

Summarizing, we have established the following method for proving that 3-point configuration sets have nonempty interior:

Theorem 2.1.

(i) *With the notation and assumptions as above, define*

$$s_\Phi = p + \min\left(d_1 + d_2 + 2\beta^{(12|3)},\, d_1 + d_3 + 2\beta^{(13|2)},\, d_2 + d_3 + 2\beta^{(23|1)}\right),$$

where the min *is taken over those of the partitions $\sigma = (ij|k)$ for which*

(a) *the double fibration condition $(DF)_\sigma$ (2.5) holds, and*

(b) *one has uniform boundedness of the generalized Radon transforms $\mathcal{R}_{\mathbf{t}}^\sigma$ with loss of $\leqslant \beta^\sigma$ derivatives (2.7).*

Then, if $E_j \subset X^j$ are compact sets with $\dim_{\mathcal{H}}(E_1) + \dim_{\mathcal{H}}(E_2) + \dim_{\mathcal{H}}(E_3) > s_\Phi$, *it follows that* $\mathrm{int}(\Delta_\Phi(E_1, E_2, E_3)) \neq \emptyset$.

(ii) *In particular, suppose that $X^1 = X^2 = X^3 = X$, with $dim(X) = d$, and there is a partition $\sigma = (ij|k)$ such that (a) holds and the canonical relations $C_{\mathbf{t}}^\sigma$ are nondegenerate (so that $\beta^\sigma = 0$). It follows that, if $E \subset X$ is compact with* $\dim_{\mathcal{H}}(E) > (2d + p)/3$, *then* $\mathrm{int}(\Delta_\Phi(E, E, E)) \neq \emptyset$.

3 | BACKGROUND MATERIAL

We give a brief survey of the relevant facts needed in the paper, referring for more background and further details to Hörmander [19, 20] for Fourier integral operator theory, Mattila [28, 29] for geometric measure theory, and [14] for the case of 2-point configurations.

3.1 | Fourier integral operators

Let X and Y be smooth manifolds of dimensions n_1, n_2, respectively. Then T^*X, T^*Y are each symplectic manifolds, with canonical two-forms denoted ω_{T^*X}, ω_{T^*Y}, respectively. Equip $T^*X \times T^*Y$ with the *difference symplectic form*, $\omega_{T^*X} - \omega_{T^*Y}$. For our purposes, a *canonical relation* will mean a submanifold, $C \subset (T^*X \setminus \mathbf{0}) \times (T^*Y \setminus \mathbf{0})$ (hence of dimension $n_1 + n_2$), which is conic Lagrangian with respect to $\omega_{T^*X} - \omega_{T^*Y}$.

For some $N \geqslant 1$, let $\phi : X \times Y \times (\mathbb{R}^N \setminus \mathbf{0}) \to \mathbb{R}$ be a smooth phase function which is positively homogeneous of degree 1 in $\theta \in \mathbb{R}^N$, that is, $\phi(x, y, \tau\theta) = \tau \cdot \phi(x, y, \theta)$ for all $\tau \in \mathbb{R}_+$. Let Σ_ϕ be the *critical set* of ϕ in the θ variables,

$$\Sigma_\phi := \{(x, y, \theta) \in X \times Y \times (\mathbb{R}^N \setminus \mathbf{0}) : d_\theta \phi(x, y, \theta) = 0\},$$

and

$$C_\phi := \{(x, d_x\phi(x, y, \theta); y, -d_y\phi(x, y, \theta)) : (x, y, \theta) \in \Sigma_\phi\},$$

both of which are conic sets. If we impose the first-order nondegeneracy conditions

$$d_x\phi(x, y, \theta) \neq 0 \text{ and } d_y\phi(x, y, \theta) \neq 0, \forall (x, y, \theta) \in \Sigma_\phi,$$

then $C_\phi \subset (T^*X \setminus \mathbf{0}) \times (T^*Y \setminus \mathbf{0})$. If in addition one demands that

$$\mathrm{rank}[d_{x,y,\theta} d_\theta \phi(x, y, \theta)] = N,\ \forall\, (x, y, \theta) \in \Sigma_\phi,$$

then Σ_ϕ is smooth, m$dim(\Sigma_\phi) = n_1 + n_2$, and the map

$$\Sigma_\phi \ni (x, y, \theta) \to \big(x, d_x\phi(x, y, \theta); y, -d_y\phi(x, y, \theta)\big) \in C_\phi \tag{3.1}$$

is an immersion, whose image is an immersed canonical relation; the phase function ϕ is said to *parametrize* C_ϕ.

For a canonical relation $C \subset (T^*X \setminus \mathbf{0}) \times (T^*Y \setminus \mathbf{0})$ and $m \in \mathbb{R}$, one defines $I^m(X, Y; C) = I^m(C)$, the class of *Fourier integral operators* $A : \mathcal{E}'(Y) \to \mathcal{D}'(X)$ of order m, as the collection of operators whose Schwartz kernels are locally finite sums of oscillatory integrals of the form

$$K(x, y) = \int_{\mathbb{R}^N} e^{i\phi(x,y,\theta)} a(x, y, \theta)\, d\theta,$$

where $a(x, y, \theta)$ is a symbol of order $m - N/2 + (n_1 + n_2)/4$ and ϕ is a phase function as above, parametrizing some relatively open $C_\phi \subset C$.

The FIO relevant for this paper are the *generalized Radon transforms* $\mathcal{R}_{\mathbf{t}}$ determined by defining functions $\Phi : X \times Y \to \mathbb{R}^p$ satisfying the double fibration condition that $D_x\Phi$ and $D_y\Phi$ have maximal rank. The Schwartz kernel of each $\mathcal{R}_{\mathbf{t}}$ is a smooth multiple of $\delta_p(\Phi(x, y) - \mathbf{t})$, where δ_p is the delta distribution on $\mathbb{R}^p$. From the Fourier inversion representation of δ_p, we see that $\mathcal{R}_{\mathbf{t}}$ has kernel

$$K_{\mathbf{t}}(x, y) = \int_{\mathbb{R}^k} e^{i(\Phi(x,y)-\mathbf{t})\cdot\theta}\, b(x, y) \cdot 1(\theta)\, d\theta,$$

where $b \in C_0^\infty$. Since the amplitude is a symbol of order 0, $\mathcal{R}_{\mathbf{t}}$ is an FIO of order $0 + p/2 - (n_1 + n_2)/4 = -(n_1 + n_2 - 2p)/4$ associated with the canonical relation parametrized as in (3.1) by $\phi(x, y, \theta) = (\Phi(x, y) - \mathbf{t}) \cdot \theta$, which is the twisted conormal bundle of the incidence relation $Z_{\mathbf{t}}$,

$$C_{\mathbf{t}} = N^*Z_{\mathbf{t}}' := \left\{ \left(x, \sum_{j=1}^k d_x\Phi_j(x, y)\theta_j; y, -\sum_{j=1}^k d_x\Phi_j(x, y)\theta_j \right) : (x, y) \in Z_{\mathbf{t}},\, \theta \in \mathbb{R}^k \setminus \mathbf{0} \right\}.$$

For T-valued defining functions Φ, as in the general formulation of our results, this discussion is easily modified by introducing local coordinates on T.

For a general canonical relation, C, the natural projections $\pi_L : T^*X \times T^*Y \to T^*X$ and $\pi_R : T^*X \times T^*Y \to T^*Y$ restrict to C, and by abuse of notation, we refer to the restricted maps with the same notation. One can show that, at any point $c_0 = (x_0, \xi_0; y_0, \eta_0) \in C$, one has $\text{corank}(D\pi_L)(c_0) = \text{corank}(D\pi_R)(c_0)$; we say that the canonical relation C is *nondegenerate* if this corank is zero at all points of C, that is, if $D\pi_L$ and $D\pi_R$ are of maximal rank. If $\dim(X) = \dim(Y)$, then C is nondegenerate if and only if π_L, π_R are local diffeomorphisms, and then C is a *local canonical graph*, that is, locally near any $c_0 \in C$ is equal to the graph of a canonical transformation. If $\dim(X) = n_1 > n_2 = \dim(Y)$, then C is nondegenerate if and only if π_L is an immersion and π_R is a submersion. To describe the L^2-Sobolev estimates for FIOs, it is convenient to normalize the order and consider $A \in I^{m_{\text{eff}} - \frac{|m_1 - m_2|}{4}}(C)$. One has

Theorem 3.1 [19, 20]. *Suppose that $C \subset (T^*X \setminus \mathbf{0}) \times (T^*Y \setminus \mathbf{0})$ is a canonical relation, where* $\dim(X) = n_1$, $\dim(Y) = n_2$, *and $A \in I^{m_{\text{eff}} - \frac{|n_1 - n_2|}{4}}$ has a compactly supported Schwartz kernel.*

(i) *If C is nondegenerate, then $A : L^2_s(Y) \to L^2_{s-m_{\mathit{eff}}}(X)$ for all $s \in \mathbb{R}$. Furthermore, the operator norm depends boundedly on a finite number of derivatives of the amplitude and phase function.*

(ii) *If the spatial projections from C to X and to Y are submersions and, for some l, the corank of $D\pi_L$ (and thus that of $D\pi_R$) is $\leqslant l$ at all points of C, then $A : L^2_s(Y) \to L^2_{s-m_{eff}-\frac{l}{2}}(X)$.*

3.2 | Frostman measures and s-energy

Also recall (see Mattila [28, 29]) that if $E \subset \mathbb{R}^d$ is a compact set and $0 < s < d$ satisfies $s < \dim_{\mathcal{H}}(E)$, then there exists a Frostman measure on E relative to s: a probability measure μ, supported on E, satisfying the ball condition

$$\mu(B(x,\delta)) \lesssim \delta^s,\ \forall\, x \in \mathbb{R}^d,\ 0 < \delta < 1, \tag{3.2}$$

and of finite s-energy,

$$\int_E \int_E |x-y|^{-s}\, d\mu(x)\, d\mu(y) < \infty,$$

or equivalently,

$$\int_E |\hat{\mu}(\xi)|^2 \cdot |\xi|^{s-d}\, d\xi < \infty. \tag{3.3}$$

Since μ is of compact support, $\hat{\mu} \in C^\omega$ and thus (3.3) implies

$$\mu \in L^2_{(s-d)/2}(\mathbb{R}^d). \tag{3.4}$$

This also holds in the general setting of $E \subset X$, a compact subset of a d-dimensional manifold X with $\dim_{\mathcal{H}}(E) > s$.

3.3 | Tensor products of Sobolev spaces

We need an elementary result on the tensor products of Sobolev spaces of negative order:

Proposition 3.2. *For $1 \leqslant j \leqslant k$, let X^j be a C^∞ manifold of dimension d_j, and suppose that $u_j \in L^2_{r_j,comp}(X^j)$, $1 \leqslant j \leqslant k$, with each $r_j \leqslant 0$. Then the tensor product $u_1 \otimes \cdots \otimes u_k$ belongs to $L^2_{r,comp}(X^1 \times \cdots \times X^k)$, for $r = \sum_{j=1}^k r_j$.*

Proof. Due to the compact support assumption, we can localize to a coordinate patch on each manifold, reducing the problem to showing that

$$L^2_{r_1}\left(\mathbb{R}^{d_1}\right) \otimes \cdots \otimes L^2_{r_k}\left(\mathbb{R}^{d_k}\right) \hookrightarrow L^2_r\left(\mathbb{R}^{\sum d_j}\right),$$

and this follows from the fact that each $\widehat{u_j}(\xi^j) \cdot \langle \xi^j \rangle^{r_j} \in L^2(\mathbb{R}^{d_j})$, together with the lower bound $\Pi_{j=1}^k \langle \xi^j \rangle^{r_j} \geqslant c \langle \xi^1, \dots, \xi^k \rangle^r$ on $\mathbb{R}^{\sum d_j}$. □

3.4 | Justification of density formula

To justify (2.6), we argue as follows, restricting for simplicity the analysis to the case $k = 3$ discussed in Section 2, when $\Phi : X^1 \times X^2 \times X^3 \to \mathbb{R}^p$. The proof extends to Φ with codomain a general T of dimension p using local coordinates on T, and also extends in a straightforward way to general k.

Without loss of generality, we consider $\sigma = (12|3)$. For a $\chi \in C_0^\infty(\mathbb{R}^p)$ supported in a sufficiently small ball, $\chi \equiv 1$ near $\mathbf{0}$, and with $\int \chi \, d\mathbf{t} = 1$, set $\chi_\epsilon(\mathbf{t}) := \epsilon^{-p}\chi(\frac{\mathbf{t}}{\epsilon})$ the associated approximation to the identity, which converges to $\delta(\mathbf{t})$ weakly as $\epsilon \to 0^+$. Define $\mathcal{R}^{(12|3)}_{\mathbf{t},\epsilon}$ to be the operator with Schwartz kernel

$$K^\epsilon_{\mathbf{t}}(x^1, x^2; x^3) := \chi_\epsilon\left(\Phi\left(x^1, x^2, x^3\right) - \mathbf{t}\right).$$

Then $\mathcal{R}^{(12|3)}_{\mathbf{t},\epsilon}(\mu_3) \in C^\infty(X^1 \times X^2)$ and depends smoothly on $\mathbf{t}$, and thus we can represent the measure ν in (2.6) as the weak limit of absolutely continuous measures with smooth densities,

$$\nu(\mathbf{t}) = \lim_{\epsilon\to 0^+} \nu^\epsilon(\mathbf{t}) := \lim_{\epsilon\to 0^+} \langle \mathcal{R}^{(12|3)}_{\mathbf{t},\epsilon}(\mu_3), \mu_1 \times \mu_2 \rangle, \tag{3.5}$$

with ν having a density, which is in fact continuous in $\mathbf{t}$, if the integral represented by the pairing converges. Now, the operators $\mathcal{R}^{(12|3)}_{\mathbf{t},\epsilon} \in I^{-\infty}(C^{(12|3)}_{\mathbf{t}})$, with symbols which converge in the Fréchet topology on the space of symbols as $\epsilon \to 0$ to the symbol of $\mathcal{R}_{\mathbf{t}}$. Since the singular limits $\mathcal{R}^{(12|3)}_{\mathbf{t}}$ satisfy (2.7) (for $\sigma = (12|3)$), so do the $\mathcal{R}^{(12|3)}_{\mathbf{t},\epsilon}$ uniformly in ϵ. Hence, $\nu(\mathbf{t})$, being the uniform limit of smooth functions of $\mathbf{t}$, is continuous. Furthermore, since $\epsilon^p \cdot \chi_\epsilon$ is bounded below by a constant times the characteristic function of the ball of radius ϵ in $\mathbb{R}^p$, we have that

$$\nu(B(\mathbf{t}, \epsilon)) := (\mu_1 \times \mu_2 \times \mu_3)\left(\left\{\left(x^1, x^2, x^3\right) : \left|\Phi(x^1, x^2, x^3) - \mathbf{t}\right| < \epsilon\right\}\right) \leqslant C_\Phi \epsilon^p, \tag{3.6}$$

with constant C_Φ uniform in $\mathbf{t}$, which was used in (2.8) above.

4 | PROOFS OF THEOREMS ON 3-POINT CONFIGURATIONS

We are now able to prove the theorems stated the Introduction that concern 3-point configurations: Theorem 1.1 about the areas of triangles in $\mathbb{R}^2$ and the radii of their circumscribing circles; the three-dimensional case of Theorem 1.2 regarding strongly pinned volumes of parallelepipeds; and Theorem 1.4 on ratios of pinned distances. For all of these, we will show that Theorem 2.1 (ii) applies for appropriate choice of σ.

4.1 | Areas of triangles in $\mathbb{R}^2$

We start with part (i) of Theorem 1.1, on areas. The absolute value of the determinant is irrelevant for the conclusion of nonempty interior; this will also be true for the other results where the configuration measurements have absolute values. Additionally, the $1/2$ can be ignored. So, we start with the scalar-valued configuration function,

$$\Phi(x^1, x^2, x^3) = \det\left[x^1 - x^3,\, x^2 - x^3\right]$$

on $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$. Here, $d = 2$, $p = 1$, and we will show that, using $\sigma = (12|3)$, that the canonical relations C_t^σ are (after localizing) nondegenerate, so that Theorem 2.1 (ii) yields a result for $\dim_{\mathcal{H}}(E) > (2 \cdot 2 + 1)/3 = 5/3$.

We compute the gradient of Φ by noting that, on $\mathbb{R}^2 \times \mathbb{R}^2$,

$$d_{u,v}(\det[u, v]) = \left(-v^\perp, u^\perp\right),$$

where $u^\perp = (-u_2, u_1)$ for $u = (u_1, u_2)$. Hence,

$$d\Phi_{x^1,x^2,x^3} = \left(\left(x^3 - x^2\right)^\perp, \left(x^1 - x^3\right)^\perp, \left(x^2 - x^1\right)^\perp\right). \tag{4.1}$$

Given a compact $E \subset \mathbb{R}^2$ with $\dim_{\mathcal{H}}(E) > 5/3$, pick any s with $5/3 < s < \dim_{\mathcal{H}}(E)$, and take μ to be a Frostman measure on E with finite s-energy. Then we claim that one can find points $x_0^1, x_0^2, x_0^3 \in E$ and a $\delta > 0$ such that[†]

$$\det\left[x_0^1 - x_0^3, x_0^2 - x_0^3\right] \neq 0 \text{ and} \tag{4.2}$$

$$\mu\left(B\left(x_0^j, \delta'\right)\right) > 0,\ j = 1, 2, 3, \forall\, 0 < \delta' < \delta. \tag{4.3}$$

To verify this, suppose not. Then, for every $x^1, x^2, x^3 \in E$ and any $\delta > 0$, either

(i) $\Phi(x^1, x^2, x^3) = 0$, i.e., x^1, x^2, x^3 are collinear, or
(ii) for some $j = 1, 2$ or 3, and some $\delta' < \delta$, $\mu(B(x^j, \delta')) = 0$.

Now, $Z_0 = \{x \in \mathbb{R}^6 : \Phi(x^1, x^2, x^3) = 0\}$ is a five-dimensional algebraic variety. Since $\mu \times \mu \times \mu$ has finite $3s$-energy, and $3s > 5$, it follows that $(\mu \times \mu \times \mu)(Z_0) = 0$, and hence $(\mu \times \mu \times \mu)(E \times E \times E \setminus Z_0) = 1$. We can in fact make this quantitative: Assuming without loss of generality that E is contained in the unit square centered at the origin, for $\epsilon > 0$, let $\mathcal{Z}_\epsilon := \{x \in \mathbb{R}^6 : |x| < 2 \text{ and } |\Phi(x)| < \epsilon\}$. Then, since Z_0 is a rigid motion in $\mathbb{R}^6$ of the Cartesian product of $\mathbb{R}^2$ with a quadratic cone in $\mathbb{R}^4$, one sees that $\mathcal{Z}_\epsilon$ is covered by $\simeq \epsilon^{-5}$ balls of radius ϵ (away from the conical points), together with $\simeq \epsilon^{-1}$ balls of radius $\epsilon^{1/2}$ (covering a tubular neighborhood of the conical points). Since μ satisfies the ball condition (3.2) on $\mathbb{R}^2$, $\mu \times \mu \times \mu$ satisfies the corresponding condition on $\mathbb{R}^6$ with exponent $3s$ and is thus dominated by $3s$-dimensional Hausdorff measure (up to a multiplicative constant). Thus,

$$(\mu \times \mu \times \mu)(\mathcal{Z}_\epsilon) \lesssim \epsilon^{-5} \cdot \epsilon^{3s} + \epsilon^{-1} \cdot \epsilon^{3s/2} \lesssim \epsilon^{3s-5} \to 0 \text{ as } \epsilon \to 0.$$

Thus, if we define $F_\epsilon := E \times E \times E \setminus \mathcal{Z}_\epsilon$, which is compact, and $\tilde{\mu}_\epsilon := (\mu \times \mu \times \mu)|_{F_\epsilon}$, then $\tilde{\mu}_\epsilon(F_\epsilon) > 1/2$ for ϵ sufficiently small. By (ii) above, every $x \in F_\epsilon$ is in a $(\mu \times \mu \times \mu)$-null set which is also relatively open, the intersection of F_ϵ with a set of one of the three forms,

$$B(x^1, \delta') \times \mathbb{R}^2 \times \mathbb{R}^2, \quad \mathbb{R}^2 \times B(x^2, \delta') \times \mathbb{R}^2 \text{ or } \mathbb{R}^2 \times \mathbb{R}^2 \times B(x^3, \delta').$$

Since F_ϵ is compact, it is covered by a finite number of these, and hence it follows that $(\mu \times \mu \times \mu)(F_\epsilon) = \tilde{\mu}_\epsilon(F_\epsilon) = 0$. Contradiction. Hence, there exists an $x_0 = (x_0^1, x_0^2, x_0^3) \in E \times E \times E$ such

[†] Note that (4.3) just says that the x_0^j belong to $\text{supp}(\mu)$ (which by Frostman's Lemma is $\subseteq E$ but can be a proper subset); however, for our purposes, it is useful to express this as (4.3).

that (4.2) and (4.3) hold. We now show that localizing near this base point allows us to apply Theorem 2.1.

Set $t_0 = \Phi(x_0^1, x_0^2, x_0^3) \neq 0$; by continuity of Φ and relabelling there is a $\delta > 0$ with $\Phi(x^1, x^2, x^3) \neq 0$ for $x^j \in X^j := B(x_0^j, \delta)$, $j = 1, 2, 3$. We claim that for $\Phi|_{X^1 \times X^2 \times X^3}$ and t close to t_0, $(DF)_{(12|3)}$ is satisfied and and $C_t^{(12|3)}$ is nondegenerate, so that $\beta^{12,3} = 0$ and the last statement of Theorem 2.1 applies with $d = 2$ and $p = 1$; hence, if $\dim_{\mathcal{H}}(E) > 5/3$, then $\mathrm{int}(A_3(E)) \neq \emptyset$.

That $(DF)_{(12|3)}$ is satisfied is immediate, since all three components of $d_{x^1 x^2, x^3}\Phi$ are nonzero on $X^1 \times X^2 \times X^3$ (the linear independence of the first two by (4.2) and the nonvanishing of third by that linear independence), which is an even stronger condition. As for the canonical relations, one computes

$$C_t^{(12|3)} = \left\{ \left(x^1, x^2, \theta\left(x^3 - x^2\right)^\perp, \theta\left(x^1 - x^3\right)^\perp; x^3, \theta\left(x^2 - x^1\right)^\perp \right) : (x^1, x^2, x^3) \in Z_t^{(12|3)},\ \theta \neq 0 \right\}.$$

Now, shrinking δ if necessary, for $(x^1, x^2, x^3) \in X^1 \times X^2 \times X^3$ and t near t_0, for a smooth, X^1-valued function $y^1(x^2, x^3, t)$ we can parametrize $Z_t^{(12|3)}$ by

$$\left(y^1(x^2, x^3, t) + u(x^2 - x^3),\ x^2,\ x^3 \right), \quad (x^2, x^3) \in X^2 \times X^3,\ u \in \mathbb{R};$$

for example, one can take

$$y^1(x^2, x^3, t) = t\left|x^2 - x^3\right|^{-2} \cdot (x^2 - x^3)^\perp.$$

Thus, (x^2, x^3, u, θ) form coordinates on $C_t^{12,3}$, with respect to which

$$\pi_R\left(x^2, x^3, u, \theta\right) = \left(x^3,\ \theta\left(x^2 - x^1\left(x^2, x^3, u\right)\right)^\perp \right),$$

from which we see that

$$D_{x^3, u, \theta}\pi_R = \begin{bmatrix} I & 0 & 0 \\ * & -\theta(x^2 - x^3)^\perp & (x^2 - x^1)^\perp \end{bmatrix},$$

which is of maximal rank since the last two columns are linearly independent. Thus, π_R is a submersion; by the general properties of canonical relations from Section 3.1, π_L is an immersion and $C_t^{(12|3)}$ is nondegenerate. □

4.2 | Circumradii of triangles in $\mathbb{R}^2$

We now turn to the proof of Theorem 1.1(ii). Changing the notation to denote the vertices of the triangle as $x, y, z \in \mathbb{R}^2$, the circumradius $R(x, y, z)$ of $\triangle xyz$ is the distance from x to the intersection point of the perpendicular bisectors of $\overline{xy}$ and $\overline{xz}$. For computational purposes, we work with

$$\Phi(x, y, z) := 2R^2(x, y, z) = \frac{1}{2}\left(|y - x|^2 + |z - x|^2 + \frac{|z - x|^2((y - x) \cdot (z - x))^2}{\left((y - x)^\perp \cdot (z - x)\right)^2} \right), \tag{4.4}$$

with the homeomorphism $r \to 2r^2$ of $\mathbb{R}_+$ of course preserving nonempty interior. With the same hypersurface Z_0 as in the proof of part (i) corresponds to degenerate triangles, $5/3 < s < \dim_{\mathcal{H}}(E)$ and Frostman measure μ, as in (i) we can find x^0, y^0, $z^0 \in \mathrm{supp}(\mu)$ such that all of the components of $d\Phi$ are nonzero. Picking $\sigma = (13|2)$, the incidence relation $Z_t^\sigma = \{(x, z; y) \,:\, \Phi(x, y, z) = t\}$ satisfies the double fibration condition $(DF)_\sigma$ for t near $t_0 = \Phi(x^0, y^0, z^0)$ and $x \in X^1$, $y \in X^2$, $z \in X^3$, neighborhoods of x^0, y^0, z^0, respectively. We can thus assume that $x_2 \in \mathbb{R}$, $y \in \mathbb{R}^2$, $z \in \mathbb{R}^2$ form coordinates on Z_t^σ, with x_1 a function of x_2, y, z, and rotating in x if necessary, can further assume that $d_z x_1 = 0$ at x^0, y^0, z^0 and is therefore small nearby. On $C_t^\sigma = (N^* Z_t^\sigma)'$, these together with the radial phase variable $\theta \in \mathbb{R} \setminus 0$ are coordinates.

To show that the canonical relation C_t^σ is nondegenerate, it suffices to show that $\pi_R : C_t^\sigma \to T^*X^2$ is a submersion. Since the y coordinate of $\pi_R(x_2, y, z, \theta) = y$, it suffices to show that

$$rank\left[\frac{D\eta}{D(x_2, z, \theta)}\right] = 2.$$

Setting

$$a = |y - x|^2, \quad b = (y - x) \cdot (z - x), \quad c = |z - x|^2, \quad d = (y - x)^{\perp}(z - y),$$

one calculates

$$\frac{D\eta}{Dz} = \frac{cb}{d^2}\left(I - \frac{b}{d}J\right) - \left(1 + \frac{cb}{d^2} - \frac{cb^2}{d^3}\right)\begin{bmatrix} 0 \\ d_z x_1 \end{bmatrix}, \tag{4.5}$$

where $J = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ is the standard 2×2 symplectic matrix, representing the $\perp$ map. The operator pencil $I - \lambda J\ \lambda \in \mathbb{R}$, is nonsingular, while the second term in (4.5) is small near x^0, y^0, z^0, and thus $D\eta/Dz$ is nonsingular, and C_t^σ is nondegenerate. Thus, as for areas, Theorem 2.1 (ii) applies for $s > 5/3$. □

4.3 | Volumes of strongly pinned parallelepipeds in $\mathbb{R}^3$

For the proof of the $d = 3$ case of Theorem 1.2, the configuration function Φ on $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ is

$$\Phi\left(x^1, x^2, x^3\right) = \det\left[x^1,\, x^2,\, x^3\right] = x^1 \cdot \left(x^2 \times x^3\right) = -x^2 \cdot (x^1 \times x^3) = x^3 \cdot (x^1 \times x^2).$$

We will show that Theorem 2.1(ii) applies for $\sigma = (12|3)$, with $k = 3$, $p = 1$ and $d = 3$, giving a positive result for $\dim_{\mathcal{H}}(E) > (2 \cdot 3 + 1)/3 = 7/3$. One computes

$$d\Phi_{x^1, x^2, x^3} = \left(x^2 \times x^3, -x^1 \times x^3, x^1 \times x^2\right).$$

As in the previous proofs, given a compact $E \subset \mathbb{R}^3$, contained in the unit cube and with $\dim_{\mathcal{H}}(E) > 7/3$, pick s with $7/3 < s < \dim_{\mathcal{H}}(E)$ and let μ be a Frostman measure of finite s-energy. We claim there exist x_0^1, x_0^2, $x_0^3 \in E$ and $\delta > 0$ such that

$$x_0^1 \times x_0^2 \neq 0, \,, \; x_0^1 \times x_0^3 \neq 0, \; x_0^2 \times x_0^3 \neq 0, \text{ and} \tag{4.6}$$

$$\mu\left(B\left(x_0^j, \delta'\right)\right) > 0, \; j = 1, 2, 3, \forall\, 0 < \delta' < \delta. \tag{4.7}$$

As before, we proceed with a proof by contradiction: suppose not. Then for every $x = (x^1, x^2, x^3) \in \mathbb{R}^9$ and $\delta > 0$, either

(i) at least one of $x^i \times x^j = 0$, for some $1 \leqslant i < j \leqslant 3$, or
(ii) for some $j = 1, 2$ or 3, and some $\delta' < \delta$, $\mu(B(x^j, \delta')) = 0$.

On $\mathbb{R}^9$, $\mu \times \mu \times \mu$ has finite $3s$-energy and satisfies the ball condition with exponent $3s > 7$. For each $1 \leqslant i < j \leqslant 3$ and $\epsilon > 0$, $\mathcal{W}_\epsilon^{ij} := \{x \in \mathbb{R}^9 : |x^i \times x^j| < \epsilon\}$ is a tubular neighborhood of $\{x : x^i \times x^j = 0\}$, a codimension two quadratic variety in $\mathbb{R}^9$ which is a rigid motion in $\mathbb{R}^9$ of the Cartesian product of $\mathbb{R}^3$ with a 4-dimensional cone in $\mathbb{R}^6$. Following the analysis in the previous proof, each of the $\mathcal{W}_\epsilon^{ij}$ can be covered by $\simeq \epsilon^{-7}$ balls of radius ϵ and ϵ^{-3} balls of radius $\epsilon^{1/2}$, and thus

$$(\mu \times \mu \times \mu)(\mathcal{W}_\epsilon^{ij}) \lesssim \epsilon^{3s-7} + \epsilon^{3s/2-3} \lesssim \epsilon^{3s-7} \to 0 \text{ as } \epsilon \to 0.$$

Hence, if we let $F_\epsilon = E \times E \times E \setminus (\cup_{i,j} \mathcal{W}_\epsilon^{ij})$ and $\tilde{\mu}_\epsilon = (\mu \times \mu \times \mu)|_{F\epsilon}$, then F_ϵ is compact and $\tilde{\mu}_\epsilon(F_\epsilon) > 1/2$ for ϵ sufficiently small. On the other hand, F_ϵ is covered by $\tilde{\mu}_\epsilon$-null and relatively open sets which are intersections of F_ϵ with sets of the three forms

$$B(x^1, \delta') \times \mathbb{R}^3 \times \mathbb{R}^3, \quad \mathbb{R}^3 \times B(x^2, \delta') \times \mathbb{R}^3 \text{ or } \mathbb{R}^3 \times \mathbb{R}^3 \times B(x^3, \delta'),$$

and the compactness of F_ϵ leads to a contradiction. Hence, we can find x_0^1, x_0^2, x_0^3 and δ such that (4.6) and (4.7) hold. Further restricting δ if necessary, we can assume that $x^1 \times x^2$, $x^1 \times x^3$ and $x^2 \times x^3$ are $\neq 0$. for all $x^j \in B(x_0^j, \delta) =: X^j$, $j = 1, 2, 3$.

Restricting Φ to $X^1 \times X^2 \times X^3$, $(DF)_{(12|3)}$ is satisfied; in fact all three components of $d\Phi$ are nonzero. On the incidence relation $Z_t^{(12|3)}$, we can take as coordinates x^2, x^3 and $\vec{u} = (u_2, u_3) \in \mathbb{R}^2$, solving for x^1 with

$$x^1 = y^1(x^2, x^3, t) + u_2 x_2 + u_3 x_3,$$

for some smooth function y^1. Thus,

$$C_t^{(12|3)} = \big\{ \big(x^1, x^2, \theta\big(x^2 \times x^3\big), -\theta\big(x^1 \times x^3\big); \, x^3, -\theta\big(x^1 \times x^2\big)\big) \\ : (x^2, x^3) \in X^2 \times X^3, \, \vec{u} \in \mathbb{R}^2, \, \theta \neq 0 \big\}.$$

Then π_R is a submersion since, with $\xi^3 = -\theta(x^1 \times x^2)$, $D_{x^2, u_3, \theta}\xi^3$ is surjective: one has $D\xi^3(\partial_\theta) = x^1 \times x^2$, $D\xi^3(\partial_{u_3}) = \theta(x^2 \times x^3)$ and the range of $D_{x^2}\xi^3$ is $(x^1)^\perp$; together, these span all of the ∂_{ξ^3} directions.

Since π_R is a submersion, $C_t^{(12|3)}$ is nondegenerate, and Theorem 2.1(ii) applies, this time with $d = 3$ and $p = 1$; hence, if $\dim_{\mathcal{H}}(E) > 7/3$, then $\operatorname{int}(V_3^0(E)) \neq \emptyset$. Q.E.D.

Remark 4.1. The proof for $d \geqslant 4$ will be presented in Section 6.2. On the other hand, Theorem 1.2 does not give a positive result for pinned volumes (areas) in two dimensions,

$$V_2^0(E) := \big\{\det\big[x^1, \, x^2\big] \, : \, x^1, x^2 \in E\big\}.$$

In fact, since this concerns a 2-point configuration, it would already fall under the framework of [14]; however, the projections π_L, π_R from the canonical relation to $T^*\mathbb{R}^2$ both drop rank by

1 everywhere, resulting in a loss of $\beta^{(1|2)} = 1/2$ derivatives. Hence, $\mathcal{R}_t^{(1|2)} \in I^{-\frac{1}{2}}(C_t^{(1|2)})$ is not smoothing on L^2-based Sobolev spaces, and the FIO approach to configuration problems does not imply a result in this case.

4.4 | Ratios of pinned distances

For $E \subset \mathbb{R}^d$, $d \geqslant 2$, we prove Theorem 1.4 concerning the set defined in (1.5). On $(\mathbb{R}^d)^3$, let

$$\Phi(x^1, x^2, x^3) = \frac{|x^1 - x^3|}{|x^1 - x^2|}.$$

We show that, after suitable localization, Theorem 2.1(ii), with $k = 3$, $p = 1$, applies for $\sigma = (12|3)$, implying a nonempty interior result when $\dim_{\mathcal{H}}(E) > (2d+1)/3$.

One computes

$$d\Phi(x^1, x^2, x^3) = \left|x^1 - x^2\right|^{-2} \left((x^2 - x^3), -(x^1 - x^2), -(x^1 - x^3)\right).$$

Let $\dim_{\mathcal{H}}(E) > (2d+1)/3$ and μ be a Frostman measure on E of finite s-energy for some $(2d+1)/3 < s < \dim_{\mathcal{H}}(E)$. Then $\mu \times \mu \times \mu$ is dominated by $3s$-dimensional Hausdorff measure, and $3s > 2d + 1 > 2d$. Since $\{(x^1 - x^2)(x^1 - x^3)(x^2 - x^3) = 0\}$ is a union of three $2d$ dimensional planes, as above one can show that there exist $x_0^1, x_0^2, x_0^3 \in \text{supp}(\mu)$ such that $x_0^i - x_0^j \neq 0$, $i \neq j$. Taking $X^j = B(x_0^j, \delta)$ for suitably small δ, all three components of $d\Phi$ are nonzero and, setting $t_0 = \Phi(x_0^1, x_0^2, x_0^3)$, the double fibration condition $(DF)_\sigma$ is satisfied by Z_t^σ for t close to t_0.

On Z_t^σ, solving for $x^3 = x^1 - t|x^1 - x^2|\omega$ we can take as coordinates $(x^1, x^2, \omega) \in \mathbb{R}^d \times \mathbb{R}^2 \times \mathbb{S}^{d-1}$. Then, the canonical relation of Z_t^σ is

$$\begin{aligned} C_t^\sigma &= (N^* Z_t^\sigma)' \\ &= \left\{\left(\cdot, \cdot, \cdot, \cdot;\, x^1 - t|x^1 - x^2|\omega, \theta t|x^1 - x^2|\omega\right) \,:\, x^1, x^2 \in \mathbb{R}^d,\, \omega \in \mathbb{S}^{d-1},\, \theta \neq 0\right\}, \end{aligned}$$

where we have suppressed the $T^*(X^1 \times X^2)$ components as irrelevant for analyzing π_R. One easily sees that the projection from C_t^σ to $T^* X^3$ is a submersion, so that Theorem 2.1 (ii) applies as claimed.

5 | *k*-POINT CONFIGURATION SETS, GENERAL *k*

To describe results for general k-point configurations, let X^i, $1 \leqslant i \leqslant k$, and T, be smooth manifolds of dimensions d_i and p, respectively. We sometimes denote $X^1 \times \cdots \times X^k$ by X, and set $d_{\text{tot}} := \dim(X) = \sum_{i=1}^k d_i$.

Definition 5.1. Let $\Phi \in C^\infty(X, T)$. Suppose that $E_i \subset X^i$, $1 \leqslant i \leqslant k$, are compact sets. Then the *k-configuration set of the E_i* defined by Φ is

$$\Delta_\Phi(E_1, E_2, \dots, E_k) := \left\{\Phi\left(x^1, \dots, x^k\right) \,:\, x^i \in E_i,\, 1 \leqslant i \leqslant k\right\} \subset T. \tag{5.1}$$

We want to find sufficient conditions on $\dim_{\mathcal{H}}(E_i)$ ensuring that $\Delta_\Phi(E_1, E_2, \dots, E_k)$ has nonempty interior. To this end, now suppose that $\Phi : X \to T$ is an submersion, so that for each $\mathbf{t} \in T, Z_{\mathbf{t}} := \Phi^{-1}(\mathbf{t})$ is a smooth, codimension p submanifold of X, and these vary smoothly with $\mathbf{t}$. For each $\mathbf{t}$, the measure

$$\lambda_{\mathbf{t}} := \delta\big(\Phi\big(x^1, \dots, x^k\big) - \mathbf{t}\,\big) \tag{5.2}$$

is a smooth density on $Z_{\mathbf{t}}$; using local coordinates on T, one sees that this can be represented as an oscillatory integral of the form

$$\int_{\mathbb{R}^p} e^{i\left[\sum_{l=1}^{p}(\Phi_l(x^1,\dots,x^k)-\mathbf{t}_l)\theta_l\right]} a(\mathbf{t}\,)1(\theta)\, d\theta,$$

where the $a(\cdot)$ belongs to a partition of unity on T. Thus, $\lambda_{\mathbf{t}}$ is a Fourier integral distribution on X; in Hörmander's notation [19, 20],

$$\lambda_{\mathbf{t}} \in I^{(2p-d_{\text{tot}})/4}(X; N^*Z_{\mathbf{t}}\,), \tag{5.3}$$

where $N^*Z_{\mathbf{t}} \subset T^*X \setminus 0$ is the conormal bundle of $Z_{\mathbf{t}}$ and the value of the order follows from the amplitude having order zero and the numbers of phase variables and spatial variables being p and d_{tot}, respectively, so that the order is $m := 0 + p/2 - d_{\text{tot}}/4$.

As in the analysis of 3-point configurations in Section 2, we separate the variables $x^1, \dots, x^k$ into groups on the left and right, associating to Φ a collection of families of generalized Radon transforms indexed by the nontrivial partitions of $\{1, \dots, k\}$, with each family then depending on the parameter $\mathbf{t} \in T$. Write such a partition as $\sigma = (\sigma_L \mid \sigma_R)$, with $|\sigma_L|$, $|\sigma_R| > 0$, $|\sigma_L| + |\sigma_R| = k$, and let $\mathcal{P}_k$ denote the set of all such partitions. We will use i and j to refer to elements of σ_L and σ_R, respectively. Define $d_L^\sigma = \sum_{i\in\sigma_L} d^i$ and $d_R^\sigma = \sum_{i\in\sigma_R} d^i$, so that $d_L^\sigma + d_R^\sigma = d_{\text{tot}}$.

For each $\sigma \in \mathcal{P}_k$, $\sigma_L = \{i_1, \dots, i_{|\sigma_L|}\}$ and $\sigma_R = \{j_1, \dots, j_{|\sigma_R|}\}$, where we may assume that $i_1 < \cdots < i_{|\sigma_L|}$ and $j_1 < \cdots < j_{|\sigma_R|}$. with a slight abuse of notation we still refer to as x the permuted version as x,

$$x = (x_L; x_R) := \left(x^{i_1}, \dots, x^{i_{|\sigma_L|}};\, x^{j_1}, \dots, x^{j_{|\sigma_R|}}\right).$$

Write the corresponding reordered Cartesian product as

$$X_L \times X_R := \left(X^{i_1} \times \cdots \times X^{i_{|\sigma_L|}}\right) \times \left(X^{j_1} \times \cdots \times X^{j_{|\sigma_R|}}\right);$$

again by abuse of notation, we sometimes still refer to this as X. The dimensions of the two factors are $\dim(X_L) = d_L^\sigma$ and $\dim(X_R) = d_R^\sigma$, respectively. The choice of σ also defines a permuted version of each $Z_{\mathbf{t}}$,

$$Z_{\mathbf{t}}^\sigma := \{(x_L; x_R) : \Phi(x) = \mathbf{t}\,\} \subset X_L \times X_R, \tag{5.4}$$

with spatial projections to the left and right, $\pi_{X_L} : Z_{\mathbf{t}}^\sigma \to X_L$ and $\pi_{X_R} : Z_{\mathbf{t}}^\sigma \to X_R$. The integral geometric double fibration condition extending (2.5) to general k is the requirement,

$$(DF)_\sigma \qquad \pi_L : Z_{\mathbf{t}}^\sigma \to X_L \text{ and } \pi_R : Z_{\mathbf{t}}^\sigma \to X_R \text{ are submersions.} \tag{5.5}$$

(Note that $(DF)_\sigma$ can only hold for a given σ if $p \leqslant \min(d_L^\sigma, d_R^\sigma)$.)

If $(DF)_\sigma$ holds, then the generalized Radon transform $\mathcal{R}_{\mathbf{t}}^\sigma$, defined weakly by

$$\mathcal{R}_{\mathbf{t}}^\sigma f(x_L) = \int_{\{x_R\,:\,\Phi(x_L,x_R)=\mathbf{t}\}} f(x_R),$$

where the integral is with respect to the surface measure induced by $\lambda_{\mathbf{t}}$ on the codimension p submanifold $\{x_R \,:\, \Phi(x_L, x_R) = \mathbf{t}\} = \{x_R \,:\, (x_L, x_R) \in Z_{\mathbf{t}}^\sigma\} \subset X_R$, extends from mapping $\mathcal{D}(X_R) \to \mathcal{E}(X_L)$ to

$$\mathcal{R}_{\mathbf{t}}^\sigma \,:\, \mathcal{E}'(X_R) \to \mathcal{D}'(X_L).$$

Furthermore,

$$C_{\mathbf{t}}^\sigma := \left(N^* Z_{\mathbf{t}}^\sigma\right)' = \left\{(x_L, \xi_L;\, x_R, \xi_R) \,:\, (x_L, x_R) \in Z_{\mathbf{t}}^\sigma,\, (\xi_L, -\xi_R) \perp T Z_{\mathbf{t}}^\sigma\right\} \tag{5.6}$$

is contained in $(T^*X_L \setminus 0) \times (T^*X_R \setminus 0)$. Thus, $\mathcal{R}_{\mathbf{t}}^\sigma$ is an FIO, $\mathcal{R}_{\mathbf{t}}^\sigma \in I^m(X_L, X_R; C_{\mathbf{t}}^\sigma)$, where the order m is determined by Hörmander's formula, $m = 0 + p/2 - d_{\mathrm{tot}}/4$. Given the possible difference in the dimensions of X_L and X_R, due to the clean intersection calculus, it is useful to express m as

$$m = m_{\mathrm{eff}}^\sigma - \frac{1}{4}\left|d_L^\sigma - d_R^\sigma\right|,$$

where the *effective* order of $\mathcal{R}_{\mathbf{t}}^\sigma$ is defined to be

$$m_{\mathrm{eff}}^\sigma := \left(2p - d_{\mathrm{tot}} + \left|d^\sigma - d^{\hat{\sigma}}\right|\right)/4 = \left(p - \min\left(d_L^\sigma, d_R^\sigma\right)\right)/2. \tag{5.7}$$

As recalled in Section 3.1, if $C_{\mathbf{t}}^\sigma$ is a nondegenerate canonical relation, that is, the cotangent space projections $\pi_L \,:\, C_{\mathbf{t}}^\sigma \to T^*X_L$ and $\pi_R \,:\, C_{\mathbf{t}}^\sigma \to T^*X_R$ have differentials of maximal rank[†], then

$$\mathcal{R}_{\mathbf{t}}^\sigma \,:\, L_r^2(X_R) \to L_{r-m_{\mathrm{eff}}^\sigma}^2(X_L).$$

As in the result concerning 3-point configurations in Theorem 2.1, it is natural to express the estimates for possibly degenerate FIO in terms of possible losses relative to the optimal estimates. As for $k = 3$, our basic assumption is that, for at least one σ, the double fibration condition (5.5) is satisfied and there is a known $\beta^\sigma \geqslant 0$ such that, for all $r \in \mathbb{R}$,

$$\mathcal{R}_{\mathbf{t}}^\sigma \,:\, L_r^2(X_R) \to L_{r-m_{\mathrm{eff}}^\sigma-\beta^\sigma}^2(X_L) \text{ uniformly for } \mathbf{t} \in T. \tag{5.8}$$

Now suppose that, for $1 \leqslant i \leqslant k$, $E_i \subset X^i$ are compact sets. Our goal is to find conditions on the $\dim_{\mathcal{H}}(E_i)$ ensuring that $\Delta_\Phi(E_1, E_2, \dots, E_k)$ has nonempty interior in T. For each i, fix an $s_i < \dim_{\mathcal{H}}(E_i)$ and a Frostman measure μ_i on E_i of finite s_i-energy. Define measures

$$\mu_L := \mu_{i_1} \times \cdots \times \mu_{i_{|\sigma|}} \text{ on } X_L \text{ and } \mu_R := \mu_{j_1} \times \cdots \times \mu_{j_{|\hat{\sigma}|}} \text{ on } X_R.$$

[†] This is a structurally stable condition, so that if $C_{\mathbf{t}_0}^\sigma$ is nondegenerate, then $C_{\mathbf{t}}^\sigma$ is nondegenerate for all $\mathbf{t}$ in some neighborhood of $\mathbf{t}_0$.

By (3.4), each $\mu_i \in L^2_{(s_i-d_i)/2}(X^i)$, so that Proposition 3.2 implies that $\mu_L \in L^2_{r_L}(X_L)$ and $\mu_R \in L^2_{r_R}(X_R)$, where $r_L = \frac{1}{2}\sum_{l=1}^{|\sigma_L|}(s_{i_l} - d_{i_l})$ and $r_R = \frac{1}{2}\sum_{l=1}^{|\sigma_R|}(s_{j_l} - d_{j_l})$, respectively. The analogue of the representation formula (3.5) for $\nu(\mathbf{t})$, justified by a minor modification of the $k = 3$ case in Section 3.4, is

$$\nu(\mathbf{t}) = \left\langle \mathcal{R}^{\sigma}_{\mathbf{t}}(\mu_R), \mu_L \right\rangle. \tag{5.9}$$

Our basic assumption, that (5.8) holds for the σ in question, then implies that $\mathcal{R}^{\sigma}_{\mathbf{t}}(\mu_R) \in L^2_{r_R - m^{\sigma}_{\text{eff}} - \beta^{\sigma}}(X_L)$. Since $\mu_L \in L^2_{r^{\sigma}}(X_L)$, the pairing in (5.9) is bounded, and yields a continuous function of $\mathbf{t}$, if

$$r_R - m^{\sigma}_{\text{eff}} - \beta^{\sigma} + r_L \geqslant 0. \tag{5.10}$$

Noting that

$$r_L + r_R = \frac{1}{2}\left[\left(\sum_{i=1}^{k} s_i\right) - d^{\text{tot}}\right],$$

and using (5.7), we see that (5.10) holds if and only if

$$\begin{aligned}\sum_{i=1}^{k} s_i &\geqslant d^{\text{tot}} + 2\left(m^{\sigma}_{\text{eff}} + \beta^{\sigma}\right)\\ &= d^{\text{tot}} + p - \min(d_L, d_R) + 2\beta^{\sigma}\\ &= \max(d_L, d_R) + p + 2\beta^{\sigma}.\end{aligned}$$

Optimizing over all $\sigma \in \mathcal{P}_k$, we obtain the analogue of Theorem 2.1 for k-point configuration sets:

Theorem 5.2.

(i) *With the notation and assumptions as above, define*

$$s_{\Phi} = \min\left(\max(d_L, d_R) + p + 2\beta^{\sigma}\right),$$

where the min *is taken over those $\sigma \in \mathcal{P}_k$ for which both the double fibration condition (5.5) and the uniform boundedness of the generalized Radon transforms $\mathcal{R}^{\sigma}_{\mathbf{t}}$ with loss of $\leqslant \beta^{\sigma}$ derivatives (5.8) hold.*

Then, if $E_i \subset X^i$, $1 \leqslant i \leqslant k$, are compact sets with $\sum_{i=1}^{k} \dim_{\mathcal{H}}(E_i) > s_{\Phi}$, it follows that $\text{int}(\Delta_{\Phi}(E_1, E_2, \dots, E_k)) \neq \emptyset$.

(ii) *In particular, if $X^1 = \cdots = X^k = X_0$, with $\dim(X_0) = d$, and if $E \subset X_0$ is compact with*

$$\dim_{\mathcal{H}}(E) > \frac{1}{k}\left[\min\left(\max(d_L, d_R) + p\right)\right], \tag{5.11}$$

where the minimum is taken over all $\sigma \in \mathcal{P}_k$ such that the canonical relations $C^{\sigma}_{\mathbf{t}}$ are nondegenerate, then $\text{int}(\Delta_{\Phi}(E, E, \dots, E)) \neq \emptyset$.

6 | PROOFS OF THEOREMS FOR k-POINT CONFIGURATIONS, $k \geqslant 4$

We now use Theorem 5.2 to prove Theorem 1.5 on cross ratios of four-tuples of points in $\mathbb{R}$, Theorem 1.2 concerning strongly pinned volumes of parallelepipeds generated by d-tuples of points in $\mathbb{R}^d$ for $d \geqslant 4$, Theorem 1.6 on pairs of areas of triangles in $\mathbb{R}^2$, Theorem 1.7 on dot products of differences and Theorem 1.8 on generalized sum-product sets.

6.1 | Cross ratios on $\mathbb{R}$

We prove Theorem 1.5 on the set of cross ratios of four-tuples of points in a set $E \subset \mathbb{R}$. This will be an application of the second part of Theorem 5.2, with $d_j = 1$, $1 \leqslant j \leqslant 4$. Since each of the variables in one dimensional, we use subscripts rather than superscripts. Thus, set $\Phi : \mathbb{R}^4 \to \mathbb{R}$,

$$\Phi(x_1, x_2, x_3, x_4) = [x_1, x_2; x_3, x_4] = \frac{(x_1 - x_3)(x_2 - x_4)}{(x_1 - x_4)(x_2 - x_3)},$$

and introduce the notation

$$x_{ij} = x_i - x_j, \quad 1 \leqslant i < j \leqslant 4.$$

Then one computes

$$d\Phi(x_1, x_2, x_3, x_4) = (x_{14}x_{23})^{-2}(x_{23}x_{24}x_{34},\ -x_{13}x_{14}x_{34},\ x_{14}x_{24}x_{34},\ -x_{12}x_{13}x_{23}).$$

Given a compact $E \subset \mathbb{R}$ with $\dim_{\mathcal{H}}(E) > 3/4$, let s be such that $3/4 < s < \dim_{\mathcal{H}}(E)$ and μ be a Frostman measure on E of finite s-energy. We claim that

$$\left(\exists\, x_1^0,\ x_2^0,\ x_3^0,\ x_4^0 \in supp(\mu)\right) \text{ s.t. } x_i^0 - x_j^0 \neq 0, \qquad \text{for all } 1 \leqslant i < j \leqslant 4, \tag{6.1}$$

so that all four components of $d\Phi$ are nonzero at $x^0 := (x_1^0, x_2^0, x_3^0, x_4^0)$. Arguing as in the proofs for 3-point configurations in Section 4, set

$$\mathcal{Z} = \left\{ x \in \mathbb{R}^4 \ :\ \prod_{1 \leqslant i < j \leqslant 4} x_{ij} = 0 \right\},$$

on the complement of which all of the components of $d\Phi$ are nonzero. Noting that $\mathcal{Z}$ is a union of hyperplanes, $\dim_{\mathcal{H}}(\mathcal{Z}) = 3$; thus, since $\mu \times \mu \times \mu \times \mu$ is dominated by $4s$-dimensional Hausdorff measure and $4s > 3$, $(\mu \times \mu \times \mu \times \mu)(\mathcal{Z}) = 0$. By a slight variant of the reasoning in Theorems 1.1 and 1.2, one obtains (6.1). This actually shows that the conditions in (6.1) hold for a set of full $\mu \times \mu \times \mu \times \mu$ measure, which we use below.

Now let $\sigma = (12|34)$. Setting $t^0 = \Phi(x^0)$, and taking the X^j to be sufficiently small neighborhoods of x_j^0, $1 \leqslant j \leqslant 4$, it follows that for t close to t^0 the double fibration condition (5.5) holds on

$$Z_t^\sigma := \{x \ :\ \Phi(x) = t\} \subset X := (X^1 \times X^2) \times (X^3 \times X^4) =: X_L \times X_R.$$

Since $d_{x_1}\Phi \neq 0$, on Z_t^σ we can solve for x_1 as a function of x_2, x_3, x_4 and the parameter t (possibly again reducing the size of the neighborhoods of the x_j^0),

$$x_1 = y^1(x_2, x_3, x_4, t).$$

Furthermore, using the fact that the factor $(x_{14}x_{23})^{-2}$ in all of the terms of $d\Phi$ can be absorbed into the radial scaling factor, as in (5.6) we define

$$C_t^\sigma := \left(N^* Z_t^\sigma\right)' = \big\{(*, *; x_3, x_4, \theta x_{14}x_{24}x_{34}, -\theta x_{12}x_{13}x_{23}) \\ : (x_2, x_3, x_4) \in X^2 \times X^3 \times X^4, \theta \neq 0\big\},$$

where we have suppressed the $T^*(X^1 \times X^2)$ components on the left. We claim that this is a local canonical graph, so that the family of FIOs lose no derivatives ($\beta^\sigma = 0$); the desired result then follows from Theorem 5.2 (ii), using $d_L = d_R = 2$, $k = 4$ and $p = 1$ in (5.11).

To see that C_t^σ is a local canonical graph, it suffices to show that the differential of $\pi_R : C_t^\sigma \to T^*X_L$ has rank 4. Due to the (x_3, x_4) in the spatial variables, this is equivalent to showing that

$$\frac{D(\xi_3, \xi_4)}{D(\theta, x_2)} = \begin{bmatrix} x_{14}x_{24}x_{34}, & \theta\left(x_{34}\left(x_{14} + x_{24}y^1_{x_2}\right)\right) \\ -x_{12}x_{13}x_{23}, & -\theta\left(-x_{13}x_{23} + x_{12}x_{13} + x_{23}\left(x_{13}y^1_{x_2} + x_{12}y^1_{x_2}\right)\right) \end{bmatrix}$$

is nonsingular. However, the determinant of this is an algebraic function, not identically vanishing on Z_t^σ, so its zero variety is three-dimensional and thus a null set with respect to $4s$-Hausdorff measure, and hence with respect to $\mu \times \mu \times \mu \times \mu$. Thus, choosing our basepoint x^0, and then shrinking the X^j suitably, to avoid this, ensures that C_t^σ is a local canonical graph, finishing the proof of Theorem 1.5.

6.2 | Strongly pinned volumes in $\mathbb{R}^d$, $d \geqslant 4$

With Theorem 5.2 in hand, we now prove Theorem 1.2 concerning pinned volumes in $\mathbb{R}^d$ for $d \geqslant 4$, following the lines of the proof for $d = 3$ in Section 4. On $(\mathbb{R}^d)^d$, let $\Phi(x^1, \ldots, x^d) = \det[x^1, x^2, \ldots, x^d]$. We will show that for $\sigma = (12\ldots(d-1)|d)$, some $t_0 \neq 0$ and with the domain of Φ suitably localized, condition $(DF)_\sigma$ is satisfied and the canonical relation $C_{t_0}^\sigma$ is nondegenerate; these conditions then hold for all t near t_0 by structural stability of submersions. Using just this σ, applying (5.11) with $d_L = d(d-1) > d_R = d$, $p = 1$ and $\beta_\Phi = 0$ shows that if $\dim_{\mathcal{H}}(E) > (1/d)(d(d-1)+1) = d - 1 + (1/d)$ then $\mathrm{int}(\Delta_\Phi(E, \ldots, E)) \neq \emptyset$, proving Theorem 1.2.

To verify the claims for σ, we start by noting that

$$d\Phi = \left(\mathbf{x}^{(1)}, -\mathbf{x}^{(2)}, \ldots, (-1)^{d-1}\mathbf{x}^{(d)}\right),$$

where

$$\mathbf{x}^{(j)} := * \left(x^1 \wedge x^2 \wedge \cdots \wedge x^{j-1} \wedge x^{j+1} \wedge \cdots \wedge x^d\right),$$

where $*$ is the Hodge star operator, which is an isomorphism $*: \Lambda^{d-1}\mathbb{R}^d \to \mathbb{R}^d$. As in the proof for $d = 3$, note that if $d - 1 + (1/d) < s < \dim_{\mathcal{H}}(E)$ and μ is a Frostman measure on E of finite s-energy, one can find $x_0^1, x_0^2, \ldots, x_0^d \in \mathrm{supp}(\mu)$ and $\delta > 0$ such that $\mathbf{x}^{(j)} \neq 0$, $1 \leqslant j \leqslant d$, whenever $x^j \in B(x_0^j, \delta) =: X^j$, $1 \leqslant j \leqslant d$. This follows by a straight-forward modification of the argument in Section 4.3.

For $1 \leqslant j \leqslant d$, each variety $\mathcal{W}^{(j)} := \{x \in \mathbb{R}^{d^2} : \mathbf{x}^{(j)} = 0 \in \mathbb{R}^d\}$ is codimension $(d-1)$, and thus their union is a null set with respect to $\otimes^d \mu$, since $sd > d^2 - (d-1)$. Restricting Φ to $X^1 \times \cdots \times X^d$, $(DF)_\sigma$ is satisfied. In fact, each of the components of $d\Phi$,

$$d_{x^j}\Phi|_{x_0} = (-1)^j \mathbf{x}^{(j)},$$

is nonzero, since when paired against x_0^j it gives $\Phi(x_0^1, \dots, x_0^d) \neq 0$.

Thus, for $\sigma = (12 \dots (d-1)|d)$ and t close to t_0, as coordinates on the incidence relations Z_t^σ we can take $(x^2, \dots, x^d) \in X^2 \times \cdots \times X^d$ and $\vec{u} = (u_2, \dots, u_d) \in \mathbb{R}^{d-1}$, with x^1 determined by

$$x^1 = y^1\left(x^2, \dots, x^d, t\right) + u_2 x^2 + \cdots + u_d x^d,$$

for some smooth function y^1, since the perturbations of any specific x^1 that preserve $\det[x^1, x^2, \dots, x^d]$ are arbitrary translates in the directions spanned by $x^2, \dots, x^d$. Furthermore, by translating by a constant in the s variables, we can assume that at the base point,

$$D_{x^2} y^1(\vec{x}_0, t_0) = 0. \tag{6.2}$$

Thus, in $T^*(X^1 \times \cdots \times X^{d-1}) \times T^*X^d$,

$$\begin{aligned} C_t^\sigma = \Big\{ \big(\cdot, \cdot; x^d, \pm * \theta \big[\big(y^1\left(x^2, \dots, x^d, t\right) \wedge x^2 \wedge \cdots \wedge x^{d-1} \big) + (-1)^d \big(u_d x^2 \wedge \cdots \wedge x^d \big) \big] \big) \\ : (x^2, \dots, x^d) \in X^2 \times \cdots X^d,\ \vec{u} \in \mathbb{R}^{d-1},\ \theta \in \mathbb{R} \setminus 0 \Big\}, \end{aligned}$$

where the first entries, giving the coordinates in $T^*(X^1 \times \cdots X^{d-1})$, have been suppressed because they are not needed to study π_R. In the last, that is, ξ^d, entry, we have used

$$\begin{aligned} \big(y^1(x^2, \dots, x^d, t) + u_2 x^2 + \cdots + u_d x^d\big) \wedge x^2 \wedge \cdots \wedge x^{d-1} \\ = y^1(x^2, \dots, x^d, t) \wedge x^2 \wedge \cdots \wedge x^{d-1} \\ + (-1)^d u_d x^2 \wedge \cdots \wedge x^d. \end{aligned}$$

We claim that $\pi_R : C_t^\sigma \to T^*X^d$ is a submersion, which, as described in Section 3.1, then implies that C_t^σ is nondegenerate and thus σ is one of the competitors in (5.11). Note that

$$D_{x^d}\pi_R = \mathbf{I}_d \oplus \left(\pm * \theta \left[D_{x^d} y^1 \wedge x^2 \wedge \cdots \wedge x^{d-1} + (-1)^d u_d x^2 \wedge \cdots \wedge x^{d-1} \wedge I_d \right] \right)$$

while, for $2 \leqslant j \leqslant d-1$,

$$D_{x^j}\pi_R = \mathbf{0} \oplus \left(\pm * \theta \left[D_{x^j} y^1 \wedge x^2 \wedge \cdots \wedge x^{d-1} + (-1)^d u_d x^2 \wedge \cdots \wedge I_d \wedge \cdots \wedge x^d \right] \right). \tag{6.3}$$

Due to the form of $D_{x^d}\pi_R$, it suffices to show that $D_{x^2}\pi_R$ has rank equal to d. Since $\theta \neq 0$ and $*$ is an isomorphism, we can ignore the $\pm * \theta$ and work directly in the d-dimensional vector space $\Lambda^{d-1}\mathbb{R}^d$. At x_0, the expression in square brackets in (6.3) equals $(-1)^d u_d I_d \wedge (x^3 \wedge \cdots \wedge x^d)$ due to (6.2). Since $x^3 \wedge \cdots \wedge x^d \in \Lambda^{d-2}\mathbb{R}^d - \{0\}$, this last map is an isomorphism $\mathbb{R}^d \to \Lambda^{d-1}\mathbb{R}^d$, thus has rank d, finishing the proof.

6.3 | Pairs of areas in $\mathbb{R}^2$

We now prove Theorem 1.6 concerning the set of pairs of areas of triangles generated by 4-tuples of points in a compact $E \subset \mathbb{R}^2$. Here, $d = 2$, $k = 4$ and $p = 2$. On $(\mathbb{R}^2)^4$, let

$$\Phi\left(x^1, x^2, x^3, x^4\right) = \left(\det\left[x^1 - x^4, x^2 - x^4\right], \det\left[x^2 - x^4, x^3 - x^4\right]\right)$$
$$= ((x^1 - x^4)\cdot(x^2 - x^4)^\perp, (x^2 - x^4)\cdot(x^3 - x^4)^\perp).$$

We will show that, for $\sigma = (13|24)$, although C_t^σ is degenerate, the projections π_L, π_R drop rank by at most 1 everywhere, and therefore, by Theorem 3.1(ii), there is a loss of at most $\beta^\sigma = 1/2$ derivative. Here, $d_L = d_R = 4$, so Theorem 5.2 implies that for

$$4\dim_{\mathcal{H}}(E) > \max(d_L, d_R) + p + 2\beta_\Phi = 4 + 2 + 1 = 7,$$

that is, for $\dim_{\mathcal{H}}(E) > 7/4$, one has $\mathrm{int}(\Delta_\Phi(E,E,E,E)) \neq \emptyset$.

To verify $(DF)_\sigma$ for $\sigma = (13|24)$, we calculate

$$D\Phi = \begin{bmatrix} (x^2 - x^4)^\perp & (x^4 - x^1)^\perp & 0 & (x^1 - x^2)^\perp \\ 0 & (x^3 - x^4)^\perp & (x^4 - x^2)^\perp & (x^2 - x^3)^\perp \end{bmatrix}$$

and note that the first and third columns form a matrix of rank two if $x^2 \neq x^4$, as do the second and fourth columns under the same condition.

Pick any s with $7/4 < s < \dim_{\mathcal{H}}(E)$ and let μ be a Frostman measure on E of finite s-energy. Arguing as in the earlier proofs, we can pick a four-tuple $x_0 = (x_0^1, x_0^2, x_0^3, x_0^4)$ with each $x_0^j \in \mathrm{supp}(\mu)$ such that

(i) $x_0^2 - x_0^4 \neq 0$; and
(ii) $x_0^1 - x_0^4$ and $x_0^3 - x_0^4$ are linearly independent;

Let $X^j = B(x_0^j, \delta)$, with δ chosen small enough so that (i) and (ii) hold with x_0 replaced by any $x \in X^1 \times X^2 \times X^3 \times X^4$.

Let $\mathbf{t}_0 = (t_0^1, t_0^2) = \Phi(x_0)$. Then, we claim that the projections $\pi_L : C_{\mathbf{t}_0}^\sigma \to T^*X_L$ and $\pi_R : C_{\mathbf{t}_0}^\sigma \to T^*X_R$ drop rank by 1 everywhere; as described in Theorem 3.1(ii), it suffices to show this for one of projections, say π_L. By (ii) above, we can parametrize $Z_{\mathbf{t}_0}^\sigma$ by (x^1, x^3, x^4), with x^2 determined by the nonsingular linear system

$$(x^1 - x^4)\cdot(x^2 - x^4)^\perp = t_0^1, \quad (x^2 - x^4)\cdot(x^3 - x^4)^\perp = t_0^2,$$

whose unique solution we can describe by $x^2 = X^2(x^1, x^3, x^4)$. Then

$$C_{\mathbf{t}_0}^\sigma = \left\{ \left(x^1, x^3, \theta_1\left(X^2 - x^4\right)^\perp, -\theta_2\left(X^2 - x^4\right)^\perp; \dots, \dots\right) : \right.$$
$$\left. (x^1, x^3, x^4) \in X^1 \times X^3 \times X^4, (\theta_1, \theta_2) \in \mathbb{R}^2 \setminus 0 \right\},$$

where the $T^*X^{\hat\sigma}$ components on the right are suppressed because they are not needed for the analysis. One easily sees that $D\pi_L$ drops rank by 1 everywhere, that is, has constant rank equal to 7, with the image of π_L being contained in the hypersurface $\{(x^1, x^3, \xi^1, \xi^3) : \xi^1 \wedge \xi^3 = 0\}$. By a fact valid for general canonical relations, $D\pi_R$ also drops rank by 1 everywhere, as well, and by semicontinuity of the rank, $C_{\mathbf{t}}^\sigma$ drops rank by $k \leqslant 1$ for all $\mathbf{t}$ close to $\mathbf{t}_0$. (Thus, δ above is chosen small enough that all of the values in $\Phi(X)$ are sufficiently close to $\mathbf{t}_0$.) By Theorem 3.1(ii), the $\mathcal{R}_{\mathbf{t}}^\sigma$ lose at most $\beta_\Phi \leqslant 1/2$ derivatives, and we are done.

6.4 | Dot products of differences

To prove the main part of Theorem 1.7, define

$$\Phi : \left(\mathbb{R}^d\right)^4 \to \mathbb{R},\ \Phi(x, y, z, w) = (x - y) \cdot (z - w).$$

We will show that using $\sigma = (13|24)$ results in C_t^σ that are local canonical graphs, so that Theorem 5.2(ii) applies (with $\beta^\sigma = 0$) to yield nonempty interior of the set of dot products of differences for $\dim_{\mathcal{H}}(E) > (d/2) + (1/4)$. By minor modification, the same analysis holds for $\Phi(x, y, z, w) = (x + y) \cdot (z + w)$, yielding the result in the footnote to Theorem 1.7.

One computes

$$d\Phi(x, y, z, w) = (z - w,\ -(z - w),\ x - y,\ -(x - y)),$$

so that $(DF)_\sigma$ is satisfied away from $\mathcal{W} := \{x - y = z - w = 0\}$, which is a codimension $2d$ plane in $\mathbb{R}^{4d}$. If $(d/2) + (1/4) < s < \dim_{\mathcal{H}}(E)$ and μ is a Frostman measure on E of finite s-energy then, arguing as we have above, $\otimes^4\mu$ is dominated by $4s$-dimensional Hausdorff measure, and $4s > 2d + 1$. Since $\mathcal{W}$ is a subspace of dimension $2d$, $(\otimes^4\mu)(\mathcal{W}) = 0$; repeating previous arguments, we can find base points $x^0, y^0, z^0, w^0 \in \text{supp}(\mu)$ and $\epsilon, \delta > 0$ such that $|x - y| + |z - w| > \epsilon$ for $x \in X^1 := B(x^0, \delta)$, $y \in X^2 := B(y^0, \delta)$, $z \in X^3 := B(z^0, \delta)$ and $w \in X^4 := B(w^0, \delta)$, respectively. Thus, $(DF)_\sigma$ is satisfied on $X_L \times X_R$. Furthermore, by relabelling and rotating if necessary, we can assume that $|z_1 - w_1| \neq 0$ on $X_L \times X_R$, so that $d_{x_1}\Phi \neq 0$.

Thus, letting $t^0 = \Phi(x^0, y^0, x^0, w^0)$, for t close to t^0, on the hypersurface Z_t^σ we can solve for x_1 as a smooth function of the other variables: $x_1 = \mathbf{x}_1(x', y, z, w)$, defined for x' in a small ball $B \subset \mathbb{R}^{d-1}$, and then parametrize

$$C_t^\sigma = \big\{\big(\cdot, \cdot, \cdot, \cdot; y,\ w,\ \theta(z - w),\ \theta((\mathbf{x}_1, x') - y)\big) \\ : \ y, z, w \in X^2 \times X^3 \times X^4,\ x' \in B, \theta \neq 0\big\},$$

where we have suppressed the T^*X_L entries as irrelevant for the analysis of $\pi_R : C_t^\sigma \to T^*X_R = T^*(X^2 \times X^4)$. Due to the simple dependence of the T^*X^2 and X^4 entries on the coordinates y, z and w on C_t^σ, and denoting elements of T^*X^4 by (w, ω), we see that

$$\text{rank}(D\pi_R) = 3d + \text{rank}\left(\frac{D\omega}{D(\theta, x')}\right) = 4d.$$

Thus, $C_t^{(13|24)}$ is a local canonical graph for t close to t^0, and Theorem 5.2(ii) applies with $k = 4$, $p = 1$, $d_L = d_R = 2d$, so that for $\dim_{\mathcal{H}}(E) > (1/4)(2d + 1) = (d/2) + (1/4)$,

$$\text{int}(\{(x - y) \cdot (z - w) \ :\ x, y, z, w \in E\}) \neq \emptyset.$$

6.5 | Sum-product sets for bilinear forms

We now state and prove a more general version of Theorem 1.8 on sum-product sets associated to families of bilinear forms.

Theorem 6.1. *Let $\vec{Q} = (Q_1, \dots, Q_l)$, with the Q_j nondegenerate, symmetric bilinear forms on $\mathbb{R}^{n_j}$, $1 \leqslant j \leqslant l$. Define $d_1, \dots, d_{2l}$ by $d_{2j-1} = d_{2j} = n_j$, $1 \leqslant j \leqslant l$. Suppose that $E_i \subset \mathbb{R}^{d_i}$ are compact, $1 \leqslant i \leqslant 2l$, with*

$$\sum_{i=1}^{2l} \dim_{\mathcal{H}}(E_i) > 1 + \frac{1}{2} \sum_{i=1}^{2l} d_i = 1 + \sum_{j=1}^{l} n_j.$$

Then the generalized sum-product set,

$$\Sigma_{\vec{Q}}(E_1, \dots, E_{2l}) := \left\{ \sum_{j=1}^{l} Q_j\left(x^{2j-1}, x^{2j}\right) \,:\, x^i \in E_i,\, 1 \leqslant i \leqslant 2l \right\} \subset \mathbb{R}, \tag{6.4}$$

has nonempty interior.

Define

$$\Phi\left(x^1, \dots, x^{2l}\right) = \sum_{j=1}^{l} Q_j\left(x^{2j-1}, x^{2j}\right) \text{ on } \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_{2l}}.$$

We show that Theorem 6.1 follows from Theorem 5.2 (ii), using $\sigma = (13 \dots (2l-1)|24 \dots (2l))$, so that $d_L = d_R = n := \sum_{j=1}^{l} n_j$, and with $p = 1$. Since we may write $Q_j(x^{2j-1}, x^{2j}) = A^j x^{2j-1} \cdot x^{2j}$ for nonsingular, symmetric $A^j \in R^{n_j \times n_j}$,

$$d_{x^{2j-1}} \Phi = A^j x^{2j} \text{ and } d_{x^{2j}} \Phi = A^j x^{2j-1}.$$

Since the A^j are nonsingular, all of these are nonzero, and thus the double fibration condition (5.5) is satisfied if all x^{2j-1}, $x^{2j} \neq 0$. Letting $X^i = \mathbb{R}^{d_i} \setminus 0$, $1 \leqslant i \leqslant 2l$, it follows that $Z_t^\sigma \subset X := \prod_i X^i$ is a smooth hypersurface, and we need to analyze the canonical relation in $(T^*X_L \setminus 0) \times (T^*X_R \setminus 0)$,

$$C_t^\sigma = \left\{ \left(x^1,\, x^3, \dots, x^{2l-3},\, x^{2l-1}, \theta A^1 x^2,\, \theta A^2 x^4, \dots, \theta A^l x^{2l}; \dots, \dots \right) \,:\, x \in Z_t^\sigma,\, \theta \neq 0 \right\},$$

where the entries on the right, in T^*X_R, are the even variants of the entries on the left and have been suppressed.

For each of the $2l$ sets E_i, let $s_i < \dim_{\mathcal{H}}(E_i)$ and μ_i be a Frostman measure on E_i with finite s_i-energy. Let $E := E_1 \times E_1 \times \cdots \times E_l \times E_l$ and pick a base point $x_0 := (x_0^1, \dots, x_0^{2l}) \in E$, which we can assume has all of its components nonzero and thus belongs to X, and a $0 < \delta_i < |x_0^i|$ such that $\mu_i(B(x_0^i, \delta_i)) > 0$.

Set $t_0 = \Phi(x_0)$. By rotations, if necessary, in $x^1 = (x_1^1, \dots, x_{d_1}^1) =: (x_1^1, (x^1)')$ and x^2, we can assume that $d_{x_1^1} \Phi(x_0) \neq 0$, so that near x^0, $Z_{t_0}^\sigma$ is the graph of a function, $x_1^1 = f((x^1)', x^2, \dots, x^{2l})$, with $d_{x_1^2} f \neq 0$. Hence, we can compute the projection $\pi_L : C_{t_0}^\sigma \to T^*X_L$ with respect to coordinates $(x^1)', x^2, \dots, x^{2l}$, θ. Since A^1 is nonsingular and $\theta \neq 0$, one sees that the map $(x^1)', x^2, \theta$ into the T^*X^1 entries has full rank, as do all of the maps x^{2j-1}, x^{2j} (with θ fixed) to T^*X^{2j-1}, so that $D\pi_L$ has full rank, and $C^\sigma_{t_0}$ is a local canonical graph. Hence, $\beta^\sigma = 0$ and Theorem 5.2(ii) applies, yielding $\text{int}(\Sigma_{\vec{Q}}(E_1, \dots, E_{2l})) \neq \emptyset$ if $\sum_i s_i > n + 1$. that is, if $\sum_i \dim_{\mathcal{H}}(E_i) > 1 + (1/2)\sum d_i$.

7 | FINAL COMMENTS

It would be interesting to know whether the Hausdorff dimension thresholds in any of these theorems are sharp.[†] However, it is worth remarking that the results on pinned volumes and sum-products at least have the correct asymptotic behavior as the dimension or the number of quadratic forms tend to infinity, even for the weaker Falconer problem of positive Lebesgue measure:

In Theorem 1.2, since all of the volumes are zero if x^0 and E both lie in a hyperplane, one cannot take $\dim_{\mathcal{H}}(E) \leqslant d - 1$, and so the restriction $\dim_{\mathcal{H}}(E) > d - 1 + (1/d)$ cannot be improved by more than $1/d$.

Similarly, in Theorems 1.8 and 6.1, if we take E_{2j-1} and E_{2j} to be in Q_j-orthogonal subspaces of $\mathbb{R}^d$ (in the notation of Theorem 1.8), then $\Sigma_{\vec{Q}}(E_1, \dots, E_{2l}) = \{0\}$. Thus, it is necessary that $\dim_{\mathcal{H}}(E_{2j-1}) + \dim_{\mathcal{H}}(E_{2j}) > d$, $1 \leqslant j \leqslant l$, so that the $1/l$ in $\dim_{\mathcal{H}}(E_{2j-1}) + \dim_{\mathcal{H}}(E_{2j}) > d + (1/l)$ cannot be reduced by more than $1/l$.

Finally, we observe that the results here are obtained by extracting as much as possible from standard estimates for linear Fourier integral operators. A number of previous results on translation-invariant Falconer-type configuration problems, such as [9, 10, 12], are based on genuinely bilinear or multi-linear estimates for generalized Radon transforms and FIOs, in settings where the Fourier transform is an effective tool. One can ask whether the thresholds in this paper (and in [14] for 2-point configurations), where the families $\mathcal{R}^\sigma_{\mathbf{t}}$ are typically nontranslation-invariant, can be lowered by obtaining truly multi-linear estimates for FIOs.

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[†] Added in proof: Koh, Pham and Shen have recently shown that even the original result of Mattila and Sjölin on distance sets having nonempty interior, which can be reproved as in [14] using FIO methods with the same thresholds, is not sharp for product sets in dimension $d \geqslant 5$ [26].

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