

Twin-Width IV: Ordered Graphs and Matrices*

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ABSTRACT

We establish a list of characterizations of bounded twin-width for hereditary classes of totally ordered graphs: as classes of at most exponential growth studied in enumerative combinatorics, as monadically NIP classes studied in model theory, as classes that do not transduce the class of all graphs studied in finite model theory, and as classes for which model checking first-order logic is fixed-parameter tractable studied in algorithmic graph theory.

This has several consequences. First, it allows us to show that every hereditary class of ordered graphs either has at most exponential growth, or has at least factorial growth. This settles a question first asked by Balogh, Bollobás, and Morris [Eur. J. Comb. '06] on the growth of hereditary classes of ordered graphs, generalizing the Stanley-Wilf conjecture/Marcus-Tardos theorem. Second, it gives a fixed-parameter approximation algorithm for twin-width on ordered graphs. Third, it yields a full classification of fixed-parameter tractable first-order model checking on hereditary classes of ordered binary structures. Fourth, it provides a model-theoretic characterization of classes with bounded twin-width. Finally, it settles our small conjecture [SODA '21] in the case of ordered graphs.

CCS CONCEPTS

• **Theory of computation** → **Finite Model Theory**; • **Mathematics of computing** → **Graph algorithms**.

KEYWORDS

Twin-width, matrices, ordered graphs, enumerative combinatorics, model theory, algorithms, computational complexity, Ramsey theory

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1 INTRODUCTION

A common goal in combinatorics, structural graph theory, complexity theory, and finite model theory, is to delimit the frontier between tractable and intractable classes of graphs, or other structures. Depending on the context, *tractability* may mean, for example, few structures of any given size, small chromatic number, efficient algorithms for problems like computing the clique number; or structural properties of sets definable by logical formulas. In this work, we show that many of such notions of tractability, from different areas of mathematics and computer science, coincide for hereditary classes of ordered graphs, and are exactly captured by the recently introduced notion of bounded twin-width. On the way, we solve several open problems in those areas. Our work has its origins in the Stanley-Wilf conjecture from enumerative combinatorics, which we now briefly recall.

Enumerative combinatorics. The celebrated Stanley-Wilf conjecture from the late 80's states that every proper permutation class has at most exponential growth. To be more specific, an n -permutation may be viewed as an $n \times n$ permutation matrix with 0, 1 entries and exactly one nonzero entry in each row and in each column. A *permutation class* is a class of permutations that is closed under taking subpermutations, that is, under removing from the permutation matrix some nonzero entries, together with the rows and columns containing them. The *growth* of a class of permutations is the function that counts the number of n -permutations in the class, for a given n . The conjecture predicts that either a permutation class contains all permutations, and therefore has growth $n! = 2^{\Omega(n \log n)}$, or otherwise its growth is at most $2^{O(n)}$, that is, bounded by c^n , for some constant c depending only on the class.

The Stanley-Wilf conjecture was confirmed by Marcus and Tardos [37] in 2004, in combination with an earlier result of Klazar [32]. Those results provided powerful tools that were subsequently extended in various directions. In 2006, Balogh, Bollobás and Morris [5, 6] analyzed the growth of classes of ordered structures, and more specifically, *ordered graphs*, that is, graphs equipped with a total order on their vertices. The *growth* of a class $\mathcal{C}$ of structures is

the function that counts the number of n -element structures in the class (up to isomorphism), for a given n . A class $\mathcal{C}$ of structures is *hereditary* if it is closed under taking induced substructures. They conjectured [6, Conjecture 2] that a hereditary class of ordered graphs either has growth at most $2^{O(n)}$, or has growth at least $2^{\Omega(n \log n)}$.

CONJECTURE 1 (BALOGH, BOLLOBÁS AND MORRIS, 2006). *Every hereditary class of ordered graphs either has growth at most $2^{O(n)}$, or has growth at least $\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} k! \geq \lfloor \frac{n}{2} \rfloor! = 2^{\Omega(n \log n)}$.*

The precise lower bound above corresponds to the growth of the class of *ordered partial matchings* with vertices $a_1 < \dots < a_m < b_1 < \dots < b_n$ and edges forming a partial matching between the a_i 's and the b_j 's.

Motivated by this conjecture, the authors confirmed it in special cases, in particular, for *monotone* classes of ordered graphs, which are closed under removing vertices and edges. In a concurrent work, Klazar [33] repeated the above question, and more recently, Gunby and Pálvölgyi [30] observe that establishing the first superexponential jump in the growth of hereditary ordered graph classes is still an open question.

Among other things, we confirm the conjecture of Balogh, Bollobás and Morris. In fact, we prove that if $\mathcal{C}$ has superexponential growth, then it contains one of 25 minimal classes of superexponential growth.

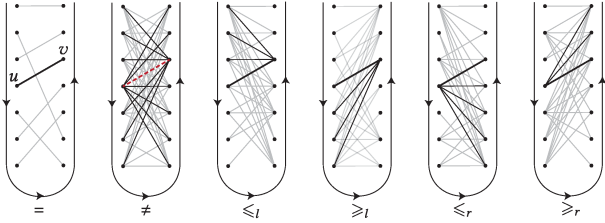


Figure 1: An ordered matching ($=$) induces 5 other ordered bipartite graphs, obtained by replacing edges by non-edges ($\neq$), closing the edges downwards ($\leq_l$) or upwards ($\geq_l$) with respect to the ordering of their left endpoints, and closing the edges downwards ($\leq_r$) or upwards ($\geq_r$) with respect to the ordering of their right endpoints. In each ordered graph, the black edges are those implied by the bold edge uv in the matching. Let $\mathcal{M}_{s,0,0}$ denote the hereditary closure of the class of ordered bipartite graphs obtained from all ordered matchings, as indicated by the parameter $s \in \{=, \neq, \leq_l, \geq_l, \leq_r, \geq_r\}$. More generally, for $\lambda, \rho \in \{0, 1\}$, let $\mathcal{M}_{s,\lambda,\rho}$ denote classes obtained in the same way, but where the left part and/or the right part of each graph are turned into cliques, as indicated by λ and ρ , respectively.

THEOREM 2. *Every hereditary class of ordered graphs either has growth at most $2^{O(n)}$, or contains one of the 24 classes $\mathcal{M}_{s,\lambda,\rho}$ or the class $\mathcal{P}$, and has growth at least $\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} k!$.*

The 24 classes $\mathcal{M}_{s,\lambda,\rho}$ for $s \in \{=, \neq, \leq_l, \geq_l, \leq_r, \geq_r\}$ and $\lambda, \rho \in \{0, 1\}$ are described in Fig. 1. For example, $\mathcal{M}_{=,0,0}$ is the class of

ordered partial matchings, whereas $\mathcal{M}_{\neq,1,1}$ is the class of its edge complements. Additionally, there is the class $\mathcal{P}$ of *ordered permutation graphs*. The ordered permutation graph associated to a permutation π of $[n] = \{1, \dots, n\}$ is the ordered graph G_π with vertices $[n]$ ordered naturally, such that two vertices $i < j$ are adjacent if and only if $\pi(i) > \pi(j)$. Altogether, the 25 classes are mutually incomparable with respect to inclusion, and each of them is minimal: each of its proper hereditary subclasses has at most exponential growth.

Note that the mapping $\pi \mapsto G_\pi$ is injective and the class $\mathcal{P}$ is hereditary, and has growth $n!$. Since $\mathcal{P}$ is minimal, the Stanley-Wilf conjecture/Marcus-Tardos theorem follows as a special case of Theorem 2 applied to the class $\{G_\pi \mid \pi \in \Pi\} \subseteq \mathcal{P}$, where Π is a permutation class.

The resolution of Conjecture 1 is not the main goal of this paper – it is rather a side effect of a bigger theory of twin-width, developed in this and previous papers of the series. This theory has its roots in the Stanley-Wilf conjecture, and therefore yields crops in the field of enumerative combinatorics, but also in other fields. As we show, the notion of twin-width behaves remarkably well for ordered graphs, and for those, draws a line that coincides with several important dividing lines: in logic, algorithms, and combinatorics. Theorem 2 is just one manifestation of the robustness of twin-width for ordered graphs. Let us move on to the relevant dividing lines from logic.

Monadically NIP classes. Model theory typically classifies infinite structures according to the combinatorial complexity of families of sets definable by first-order formulas. This is usually done through the introduction of tameness properties, or dividing lines. The most important such notion is that of stability. A class of structures is stable if no first-order formula $\varphi(x, y)$ encodes arbitrary large half-graphs (bipartite graphs with vertices $u_1, \dots, u_n, v_1, \dots, v_n$ and edges $u_i - v_j$ for $1 \leq i \leq j \leq n$), which roughly means that there is no definable order on arbitrarily large subsets of the structures. Another important dividing line is that of NIP: a class of structures is *NIP* (or *dependent*) if no formula $\varphi(x, y)$ encodes arbitrary bipartite graphs. Equivalently, for every formula $\varphi(x, y)$, the binary relations defined by $\varphi(x, y)$ in the structures from the class have VC-dimension bounded by some constant depending on φ and the considered class. This notion captures the tameness properties of families of sets arising from geometric settings (for instance, sets of points defined by polynomials of bounded degree).

The notion of a *monadically NIP* (or monadically dependent) class is a much stronger requirement, which says that a class of structures is NIP even if the structures can be expanded with arbitrary unary predicates. As we shall see, for hereditary classes of ordered graphs, monadic NIP and NIP coincide. The two notions, NIP and monadic NIP, introduced by Shelah [45, 46] in the 70's and 80's and studied in model theory, are closely related to concepts studied later in finite model theory, namely simple first-order interpretations and transductions.

Interpretations and transductions are a means of producing new structures out of old ones, using formulas. A (simple) interpretation takes an input structure and produces an output structure with the same domain (or a subset of it, defined by a formula $\delta(x)$) while each of its relations is defined by a formula $\varphi(\bar{x})$ interpreted in the input structure. Here, all formulas belong to first-order logic (FO).

For example, there is an interpretation which transforms a given graph G into its edge complement (using the formula $\neg E(x, y)$), and an interpretation which transforms G into its square (using the formula $\exists z. E(x, z) \wedge E(z, y)$). Transductions are a similar notion, but additionally allow to nondeterministically color the input structure before applying an interpretation, which can then use the colors in the formulas. Say that $\mathcal{C}$ *interprets* the class of all graphs if there is an interpretation I such that every (finite) graph G can be obtained as the result of I applied to some structure in $\mathcal{C}$. Replacing interpretations with transductions, we say that $\mathcal{C}$ *transduces* the class of all graphs. If a class is NIP then it does not interpret the class of all graphs (the converse implication requires additional assumptions). Classes that do not transduce the class of all graphs can be equivalently characterized as monadically NIP classes.

The study of transductions in theoretical computer science, more specifically, in finite model theory, originates from the study of word-like and tree-like structures, such as graphs with bounded treewidth [3] or graphs with bounded clique-width [16]. For instance, a result of Courcelle and Oum [18] characterizes classes of bounded clique-width as precisely those which do not transduce the class of all graphs via a transduction of counting monadic second-order logic (CMSO, an extension of FO), and the conjecture of Seese [43] predicts that the same holds for monadic second-order logic MSO. Those questions in finite model theory arose from the study of the tractability of the model checking problem on restricted graph classes.

Fixed-parameter tractable first-order model checking. Testing if a given FO sentence φ holds in a given structure G takes time $O(|G|^{|\varphi|})$ using a naive algorithm, and it is conjectured that the exponential dependency on $|\varphi|$ cannot be avoided. More precisely, it is conjectured that FO model checking is not *fixed-parameter tractable* (FPT) on the class of all graphs, i.e., does not admit an algorithm with running time $f(\varphi) \cdot |G|^c$, for some computable function $f: \mathbb{N} \rightarrow \mathbb{N}$ and constant c . This is equivalent to the $\text{FPT} \neq \text{AW}[*]$ conjecture from complexity theory [22], as FO model checking is complete for the parameterized complexity class $\text{AW}[*]$.

A very successful line of research in structural and algorithmic graph theory and finite model theory is aimed at identifying classes of graphs, or other structures, for which FO model checking is FPT. This has stemmed from the seminal result of Courcelle [15], which achieves an FPT algorithm for classes of bounded treewidth, concerning the more powerful MSO and CMSO logics. Subsequent examples of graph classes for which FO model checking is FPT include: classes of bounded degree [44], classes of bounded local treewidth [23], classes that (locally) exclude a minor [19], and classes with (locally) bounded expansion [21]. All these examples are monotone, that is, closed under removing vertices and edges, and are therefore *weakly sparse*, that is, exclude some biclique as a subgraph (indeed, model checking is $\text{AW}[*]$ -hard on the class of bipartite graphs).

This line of work, concerning model checking on monotone graph classes, culminated in the result of Grohe, Kreutzer and Siebertz [29]. They proved that for a monotone graph class $\mathcal{C}$, FO model checking is FPT for every *nowhere dense* class $\mathcal{C}$. Nowhere denseness is a general notion of uniform sparsity introduced by Nešetřil and Ossona de Mendez [39], which postulates that the

considered class does not contain arbitrarily large d -subdivisions of cliques as subgraphs, for any fixed d (a d -subdivision is obtained by replacing each edge by a path of length $d + 1$). All the monotone classes listed above are nowhere dense. Conversely, if $\mathcal{C}$ is a monotone class that is not nowhere dense, then FO model checking is $\text{AW}[*]$ -hard [21] on $\mathcal{C}$.

This is not the end of the story, however, since clearly there are hereditary graph classes that are not sparse (nor monotone), and for which FO model checking is still FPT, such as the class of cliques, or edge complements of nowhere dense classes, to name some trivial examples. This observation has motivated further work on dense classes of graphs, or other structures, with FPT model checking. Known such examples include: classes of bounded cliquewidth [17] (here, model checking of formulas of the more powerful CMSO logic is FPT), classes with bounded local clique-width [19, 23], posets of bounded width [24], interpretations of bounded-degree classes [26, 27], some classes of intersection and visibility graphs [31], transductions of bounded expansion classes when a suitable witness is given [28].

As it appears, all known tractable hereditary¹ classes are monadically NIP, equivalently, do not transduce the class of all graphs. This observation is the basis of the following conjecture, that has been circulating in the community for some time:²

CONJECTURE 3. *Let $\mathcal{C}$ be a hereditary class of structures. Then FO model checking is FPT on $\mathcal{C}$ if and only if $\mathcal{C}$ is monadically NIP.*

Both implications of Conjecture 3 are open. This conjecture is confirmed for monotone graph classes [29] (assuming $\text{FPT} \neq \text{AW}[*]$), where monadically NIP classes are precisely nowhere dense classes [1], and the results of [1, 21, 29] in combination yield the following:

THEOREM 4. *The following conditions are equivalent for a monotone class $\mathcal{C}$ of graphs, assuming $\text{FPT} \neq \text{AW}[*]$:*

(1) $\mathcal{C}$ is nowhere dense, (2) FO model checking is FPT on $\mathcal{C}$, (3) $\mathcal{C}$ is monadically NIP.

A similar result holds for bounded treewidth instead of nowhere denseness, and the more powerful logic MSO instead of FO, but under certain additional technical assumptions in place of monotonicity. This relies on the results of [15, 35]. A similar statement may also hold for cliquewidth and MSO, and would follow from the conjectures of Seese [43] and a conjecture of Kreutzer [34, Conjecture 9.2].

Twin-width. The various results concerning FO model checking on dense graph classes mentioned above are apparently of quite different natures, and have evaded attempts of a common generalization. The recently introduced notion of twin-width [9–11] generalizes many, although not all, of those classes. Twin-width is a parameter which, in its basic form, applies to unordered graphs, and is defined as follows. Say that two sets of vertices of a graph G are *pure* if either all edges, or no edges, span across the two sets. A partition of the vertices of G has *width* d if every part is pure with respect to all but at most d other parts. Finally, G has twin-width

¹Tractable classes that are not hereditary include for example the class of all finite Abelian groups [13]

²cf. open problem session at the workshop on Algorithms, Logic and Structure in Warwick in 2016 and [25, Conjecture 8.2].

d if there is a sequence of partitions of its vertices, each of width at most d , which starts with the partition into singletons, where every subsequent partition is obtained from the preceding one by merging two parts into one, and which ends with the partition with a unique part. Twin-width can be also defined for ordered graphs, or more generally, any binary structure, equipped with one or more unary or binary relations.

By definition, twin-width is invariant under edge-complementation, so twin-width plays equally well with sparse and dense graph classes. Furthermore, many well-studied classes of structures have bounded twin-width: planar graphs, and more generally, any class of graphs excluding a fixed minor, cographs, and more generally, any class of bounded clique-width, posets of bounded width, and permutations omitting a fixed permutation pattern (that is, omitting a fixed permutation as a subpermutation).

Despite their generality, classes of bounded twin-width enjoy many remarkable properties of combinatorial, algorithmic, and logical nature. For instance, such classes are small (contain $n! \cdot 2^{O(n)}$ graphs with vertex set $\{1, \dots, n\}$) [10], are χ -bounded (the chromatic number is bounded in terms of the clique number) [9], are monadically NIP (preserved by first-order interpretations and transductions) [11]. Furthermore, FO model checking first-order logic is FPT, assuming the input structure is provided with a *witness* of having bounded twin-width (that is, a sequence of partitions as in the definition of twin-width). More precisely, there is an algorithm that, given an FO sentence φ and a structure G , on n elements, together with a witness that its twin-width is at most d , decides whether φ holds in G in time $f(\varphi, d) \cdot n$ for some computable function f .

For each of the classes $\mathcal{C}$ of bounded twin-width mentioned above there is actually an algorithm that, given a graph $G \in \mathcal{C}$, computes the required witness of low twin-width, in polynomial time [11]. Hence, FO model checking is FPT on these classes (without requiring the witness), generalizing many known results mentioned in the previous paragraphs. It is unknown however whether such witnesses can be computed efficiently for every class $\mathcal{C}$ of bounded twin-width.

As mentioned, all classes of bounded twin-width are monadically NIP, however, the converse fails. For instance, the class of subcubic graphs is nowhere dense (and hence monadically NIP, and model checking is FPT), but has unbounded twin-width [10]. Conversely, cographs are exactly graphs of twin-width 0, but the class of cographs is not nowhere dense, nor can it be obtained from a nowhere dense class by a transduction. Thus, (transductions of) nowhere dense classes and classes of bounded twin-width are incomparable. Moreover, it appears that their definitions are quite dissimilar, and difficult to reconcile.

It transpires from the mere definitions that every graph can be equipped with some total order, resulting in an ordered graph of the same twin-width [11]. Thus, every graph class of bounded twin-width can be turned into a class of ordered graphs of bounded twin-width (and hence a monadically NIP class of ordered graphs). Such orders, while elusive to efficiently find in general, are crucial to most of the combinatorial and algorithmic applications. This suggests that *ordered graphs* of bounded twin-width are a more fundamental object than *unordered graphs* of bounded twin-width. And indeed, as we will see, for classes of ordered graphs, bounded twin-width

precisely aligns with the dividing lines from combinatorics, logic, and parameterized complexity mentioned earlier. In particular, our main result implies the following result, in a striking analogy with Theorem 4:

THEOREM 5. *The following conditions are equivalent for a hereditary class $\mathcal{C}$ of ordered graphs, assuming $\text{FPT} \neq \text{AW}[*]$: (1) $\mathcal{C}$ has bounded twin-width, (2) FO model checking is FPT on $\mathcal{C}$, (3) $\mathcal{C}$ is monadically NIP.*

Therefore, nowhere denseness and bounded twin-width are in fact two facets of the same concept, namely monadic NIP, but this only becomes apparent when the (suitable) order is taken into account in the latter case. In particular, Theorem 5 confirms Conjecture 3 for classes of ordered graphs (assuming $\text{FPT} \neq \text{AW}[*]$), just as Theorem 4 confirms it for monotone classes of graphs. Note that apart from the implication (1) $\rightarrow$ (3), all five remaining implications are new (the implication (1) $\rightarrow$ (2) was only known assuming a witness of bounded twin-width is given).

Informally, Theorem 5 says that FO on hereditary classes of ordered graphs relates to twin-width in the same way as FO on monotone graph classes relates to nowhere denseness, or MSO on monotone graph classes relates to bounded treewidth, or (conjecturally) MSO relates to bounded cliquewidth (see Theorem 4 and the remarks following it).

Theorem 5 in particular provides a model-theoretic characterization of classes of ordered graphs of bounded twin-width, as precisely those that are monadically NIP. This also yields a model-theoretic characterization of classes of *unordered* graphs of bounded twin-width: a class of graphs has bounded twin-width if and only if it can be obtained from some monadically NIP class of *ordered* graphs by forgetting the order.

1.1 Main Result

We are now ready to state our main result in its full form. It gives multiple characterizations of hereditary classes of ordered graphs of bounded twin-width, connecting notions from various areas of mathematics and theoretical computer science, and solving several open problems on the way. It provides a dichotomy result for all such classes: Either they have bounded twin-width, and are therefore well-behaved in many ways, or otherwise, they are very untamable in all those ways.

THEOREM 6. *Let $\mathcal{C}$ be a hereditary class of ordered graphs. Then*

- (i) $\mathcal{C}$ has bounded twin-width
- $\Leftrightarrow$ (ii) $\mathcal{C}$ has bounded grid rank
- $\Leftrightarrow$ (iii) $\mathcal{C}$ has growth $2^{O(n)}$
- $\Leftrightarrow$ (iv) $\mathcal{C}$ does not transduce the class of all graphs
- $\Rightarrow$ (v) FO model checking is FPT on $\mathcal{C}$
- and
- (i') $\mathcal{C}$ has unbounded twin-width
- $\Leftrightarrow$ (ii') $\mathcal{C}$ contains $\mathcal{P}$ or one of the 24 classes $\mathcal{M}_{s,\lambda,\rho}$
- $\Leftrightarrow$ (iii') $\mathcal{C}$ has growth at least $\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} k! \geq \lfloor \frac{n}{2} \rfloor!$
- $\Leftrightarrow$ (iv') $\mathcal{C}$ interprets the class of all graphs
- $\Rightarrow$ (v') FO model checking is $\text{AW}[*]$ -hard on $\mathcal{C}$.

Condition (ii) involves a new width parameter, grid rank, which is pertinent to matrices, and is the key notion binding together

the proof of the theorem. A 0,1-matrix has *grid rank* at least k if its entries can be partitioned into k^2 rectangular zones, using $k - 1$ vertical and $k - 1$ horizontal lines, so that each zone has at least k non-identical rows or at least k non-identical columns. This depends on the order of rows and columns in the matrix. The grid rank of an ordered graph is the grid rank of its adjacency matrix along the order of the graph.

Theorem 6 connects notions from graph theory (i), matrix theory (ii) and Ramsey theory (ii'), enumerative combinatorics (iii),(iii'), finite model theory (iv), model theory (iv'), and parameterized complexity (v),(v'). The lower bound in (iii') is optimal, the 25 classes in (ii') are the fewest possible, and FO model checking is in fact AW[*]-complete in (v').

As a by-product of the proof of Theorem 6 we get an approximation algorithm for twin-width in ordered binary structures.

THEOREM 7. *There is a fixed-parameter algorithm that, given a ordered binary structure G of twin-width k , outputs a witness that G has twin-width at most $2^{O(k^4)}$.*

Theorems 6 and 7 are proved in greater generality for arbitrary classes of ordered binary structures. We also prove an analogue of Theorem 6 for classes of 0,1-matrices that are *submatrix-closed*, that is, closed under removing rows and columns (see Theorem 9). For those, the lower bound in (iii') on the growth is replaced by the tight bound $\sum_{k=0}^n \binom{n}{k}^2 k! \geq n!$, where the *growth* of a class of matrices is the function counting the number of $n \times n$ matrices in the class, for a given n . Additionally, the 25 classes in (ii') are reduced to six classes $\mathcal{F}_s$ of matrices, indexed by a single parameter $s \in \{=, \neq, \leq_R, \geq_R, \leq_C, \geq_C\}$ (see Fig. 2). Those classes are the submatrix closures of: the class of all permutation matrices, the class obtained from permutation matrices by exchanging 0's with 1's, and four classes obtained from permutation matrices by propagating each 1 entry downward/upward/leftward/rightward, respectively.

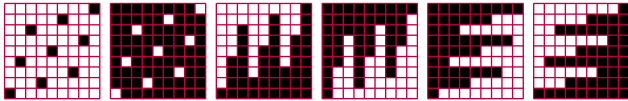


Figure 2: The matrices in $\mathcal{F}_=, \mathcal{F}_{\neq}, \mathcal{F}_{\leq_R}, \mathcal{F}_{\geq_R}, \mathcal{F}_{\leq_C}, \mathcal{F}_{\geq_C}$ (from left to right) for the same permutation matrix (the one to the left). The 1 entries are represented in black, the 0 entries, in white. As is standard with permutation patterns, we always place the first row of the matrix at the bottom.

1.2 Second Main Result

As our second main result, we provide further characterizations of bounded twin-width classes in terms of model-theoretic notions, but which also transpire in algorithmic and structural graph theory. We consider arbitrary *monadically NIP* classes of relational structures, which are not necessarily finite, ordered, or binary. Those can be equivalently characterized as classes which do not transduce the class of all finite graphs. They include all graph classes of bounded twin-width (with or without an order), but also all transductions of *nowhere dense* classes [40], such as classes of bounded maximum degree.

The following theorem generalizes some notions and implications appearing in Theorem 6.

THEOREM 8. *For any class of structures $\mathcal{C}$, we have*

- (1) $\mathcal{C}$ does not transduce the class of all graphs,
- $\Leftrightarrow$ (2) $\mathcal{C}$ is monadically NIP,
- $\Leftrightarrow$ (3) $\mathcal{C}$ does not define large grids (see Definition 19),
- $\Leftrightarrow$ (4) $\mathcal{C}$ is 1-dimensional (see Definition 25),
- $\Rightarrow$ (5) $\mathcal{C}$ is a restrained class (see Definition 17).

For hereditary classes of finite, binary, ordered structures, the above conditions are all equivalent to $\mathcal{C}$ having bounded twin-width.

Defining large grids generalizes the property of containing one of the classes $\mathcal{M}_{s,\lambda,\rho}$ or $\mathcal{P}$ to arbitrary structures, while the notion of a restrained class implies, for classes of ordered graphs, bounded grid rank. In particular, those notions do not require the structures to be ordered, finite, or binary. The notion of 1-dimensionality has a somewhat geometric flavor. It is defined in terms of a variant of forking independence – a central concept in stability theory, generalizing independence in vector spaces or algebraic independence. The equivalence (1) $\leftrightarrow$ (2) is due to Baldwin and Shelah [4], the implications (2) $\rightarrow$ (3) $\rightarrow$ (4) are due to Shelah [46] and the implication (4) $\rightarrow$ (2) is a very recent result of Braunfeld and Laskowski [14]. Our contribution is the implication (4) $\rightarrow$ (5), and the overall equivalence in the case of ordered binary structures. Indeed, the implication (5) $\rightarrow$ (1) follows from the implication (ii) $\rightarrow$ (iv) in Theorem 6. See Section 4 for more details.

We believe that Theorem 8 may be of independent interest, and possibly of broader applicability than just in the context of ordered, binary structures. For example, by Theorem 8, all graph classes of bounded twin-width (without an order) and all transductions of nowhere dense classes are restrained, generalizing the fact that classes of ordered graphs of bounded twin-width have bounded grid rank. We remark that although our proof of Theorem 6 is purely combinatorial, an alternative proof can be derived from Theorem 8 (see our unpublished report [48]). This demonstrates that model-theoretic methods can be used in the context of algorithmic and structural graph theory, and that those two areas are intimately related.

1.3 Consequences and Related Work

As mentioned, Theorem 6 (iii),(iii') immediately yields Theorem 2, thus resolving Conjecture 1 of Balogh, Bollobás, and Morris. Also, Theorem 6 (i),(iv),(iv'),(v),(v') yields Theorem 5, partially confirming Conjecture 3 – characterizing monadically NIP classes as exactly those hereditary classes for which model checking FO is FPT – in the special case of hereditary classes of ordered, binary structures. We now detail some further consequences of our results.

Stanley-Wilf classes. We obtain the following classification of all inclusion-minimal classes of superexponential growth. Call a (submatrix-closed) class of matrices a *Stanley-Wilf class* if it has superexponential growth, but each of its proper subclasses has at most exponential growth, that is, growth $2^{O(n)}$. Then the submatrix-closure of the class of permutation matrices is a Stanley-Wilf class, as shown by Marcus and Tardos. By a similar argument, each of the classes $\mathcal{F}_s$ for $s \in \{=, \neq, \leq_R, \geq_R, \leq_C, \geq_C\}$ is a Stanley-Wilf class. Moreover, these six classes are precisely *all* the Stanley-Wilf classes

of 0, 1-matrices, and every matrix class of superexponential growth contains one of those classes. This is a consequence of our result for matrices, and the fact that the six classes are mutually incomparable.

In the same way, we may define Stanley-Wilf classes of ordered graphs, as those hereditary classes of superexponential growth whose proper hereditary subclasses have at most exponential growth. Then the 25 classes $\mathcal{M}_{s,\lambda,\rho}$ and $\mathcal{P}$ are precisely all the Stanley-Wilf classes of ordered graphs. This statement, for the class $\mathcal{P}$, is equivalent to the original Stanley-Wilf conjecture/Marcus-Tardos theorem.

Small conjecture. Classes of bounded twin-width are small [10], that is, they contain at most $n!c^n$ distinct labeled n -vertex structures, for some constant c . (Actually they further contain at most c^n pairwise non-isomorphic structures [12].) The converse was conjectured for hereditary classes [10]. In the context of classes of totally ordered structures, it is simpler to drop the labeling and to count up to isomorphism. Indeed, every ordered structure has no non-trivial automorphism. Then a class is *small* if, up to isomorphism, it contains $2^{O(n)}$ distinct n -vertex structures. With that in mind, Theorem 6 (i),(iii) confirms the conjecture in the particular case of ordered graphs.

Ramsey theory. Our proofs are based on multiple Ramsey-theoretic arguments, but also our main result, Theorem 6, has a bearing on Ramsey theory. For example, we can conclude the following: *For every ordered matching H there is some cubic graph G such that for every total order $\leq$ on $V(G)$, the resulting ordered graph $G_{\leq}$ contains H as an induced ordered subgraph.* Statements of this form are of interest in Ramsey theory (e.g., [38, 41]).

Here is a proof by contrapositive. If the statement fails then there is some ordered matching H such that every cubic graph G can be equipped with an order in a way which avoids H as an induced ordered subgraph. This way, we obtain a class $\mathcal{C}$ of ordered cubic graphs, which contains an ordering of every cubic graph, and does not contain the class $\mathcal{M}_{=,0,0}$, as it already fails to contain H . Clearly, $\mathcal{C}$ does not contain any of the remaining 24 classes $\mathcal{M}_{s,\lambda,\rho}$ and $\mathcal{P}$, as those have unbounded degree. By Theorem 6 (ii'), (i), $\mathcal{C}$ has bounded twin-width. This implies that also the class of all (unordered) cubic graphs also has bounded twin-width, which we know is false (see [10]). More directly based on Theorem 6, a contradiction can be reached by observing that $\mathcal{C}$ does *not* have growth $2^{O(n)}$.

Grid theorems. Grid theorems are dichotomy results in structural graph theory which state that either a structure has a small *width*, or otherwise, a grid-like obstruction can be found in the structure. For example, this applies to the treewidth parameter and grids occurring as minors [42]. It also applies to clique-width and grids being definable in CMSO [18].

Theorem 6 (i'), (ii') proves an appropriate grid theorem for classes of ordered graphs of unbounded twin-width, and Theorem 8 proves a weaker form for all classes which are not restrained. Indeed, such classes define large grids, which, intuitively, allows to define the 'same row' and 'same column' relations of arbitrarily large grids using first-order formulas in the graphs from the class. From this (also, from Theorem 6 (i'), (iv')) it follows that if a hereditary class has unbounded twin-width then it interprets the class of all graphs.

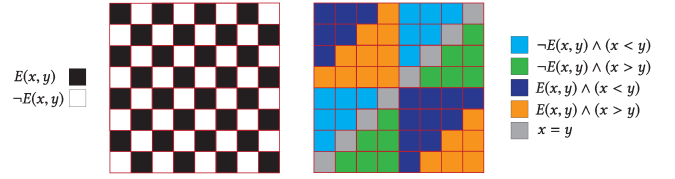


Figure 3: *Left:* The adjacency matrix of the ordered graph G with vertices $1, \dots, n$ and edges ij such that $i + j$ is odd, along the usual order. (The first row is at the bottom.) *Right:* The adjacency matrix along another order, encoding the adjacency as well as the original order. Every 4-division contains some constant zone.

There are other known grid theorems, including the Marcus-Tardos theorem itself. The recent result of Braunfeld and Laskowski [14] characterizes monadically NIP classes as exactly those that do not define large grids, in the sense of Theorem 8.

Related work. There has been a lot of research describing the possible growth rates of the function counting the number of labeled/unlabeled structures with n vertices in hereditary classes of structures. For hereditary classes of labeled, unordered graphs, this has culminated in a series of papers in the 2000's by Balogh, Bollobás, and Weinreich [2, 7, 8]. Laskowski and Terry [36] use model-theoretic methods to generalize some of those results to relational structures. To the best of our knowledge, those results do not yield algorithmic consequences for the model checking problem in first-order logic.

2 PROOF OUTLINE

We now outline the proof of Theorem 6.

Bounded twin-width is already known to imply interesting properties: FO model checking is FPT if a witness of small twin-width is part of the input [11], monadic dependence [11], smallness [10]. Thus, our challenge is to establish that these interesting properties in return imply bounded twin-width. A d -division of a matrix is a partition of its entries into d^2 zones using $d - 1$ vertical and $d - 1$ horizontal separating lines. A central characterization in the first paper of the series [11] goes as follows.

A graph class $\mathcal{C}$ has bounded twin-width if and only if there is a constant d such that every graph $G \in \mathcal{C}$ can be ordered so that the adjacency matrix along that order has no d -division where each zone contains two non-identical rows and two non-identical columns. The backward direction is effective: From such an ordering, we obtain a witness of bounded twin-width in polynomial time. Thus the task of showing bounded twin-width boils down to finding a *good* vertex ordering.

Now that we consider *ordered* graphs, it is tempting to try that very order (part of the structure) to produce a witness of low twin-width. Things are not that simple. Consider the graph G with vertices $1, \dots, n$ ordered naturally, where two vertices are adjacent if and only if they have different parity. The adjacency matrix M of G is the checkerboard matrix to the left in Fig. 3. This matrix is fairly simple and indeed has bounded twin-width (this will be evident once we formally define twin-width). Yet for $d = \lfloor n/2 \rfloor$,

the matrix M has a d -division where each zone has two different rows and columns. A *good reordering* of G puts all the odd-indexed vertices together, followed by all the even-indexed vertices. Then the adjacency matrix M' of G along the new order (Fig. 3, right), where the entries of M' now encode the edges of G as well as the original order, is such that every 4-division contains a constant zone.

Can we find such reorderings automatically? Eventually we can, but a crucial opening step is precisely to nullify the importance of the reordering. We show that matrices have bounded twin-width exactly when they have bounded grid rank. A *rank- k division* of a matrix M is a k -division of M such that every formed zone has at least k distinct rows or at least k distinct columns. The natural strengthening on the condition that zones should satisfy (from combinatorial rank 2 to rank k) exempts us from the need to reorder. Note indeed that the checkerboard matrix does not have any (large) rank- k division already for $k = 3$.

An important intermediate step is provided by the concept of rich divisions (see Section 3.4 for a definition). We first prove that a greedy strategy to find a potential witness of bounded twin-width can only be stopped by the presence of a large rich division; thus, unbounded twin-width implies the existence of arbitrarily large rich divisions. This brings a theme developed in [11] to the ordered world. In turn, leveraging the Marcus-Tardos theorem, we show that huge rich divisions contain large rank- k divisions for large values of k .

By a series of Ramsey-like arguments, we find in large rank divisions more and more structured submatrices encoding universal permutations. Eventually we find at least one of six encodings of all permutations. More precisely, each class of matrices of unbounded grid rank contains one of the classes $\mathcal{F}_s$, for some $s \in \{=, \neq, \leq_R, \geq_R, \leq_C, \geq_C\}$. As each of the classes $\mathcal{F}_s$ has growth $n!$, this chain of implications shows that hereditary classes of matrices with unbounded grid rank have growth at least $n!$. Conversely, classes of matrices of bounded twin-width have growth $2^{O(n)}$ by [10]. That establishes the announced speed gap for matrix classes. Moreover, as each of the classes $\mathcal{F}_s$ interprets the class of all graphs and has an $\text{AW}[*]$ -hard model checking, we obtain the matrix variant of Theorem 6 (see Theorem 9).

Finally, we translate the permutation encodings in the language of ordered graphs. This allows us to refine the growth gap specifically for ordered graphs. We also prove that including a family $\mathcal{F}_s$, or its ordered-graph equivalent $\mathcal{M}_{s,\lambda,\rho}$, is an obstruction to being NIP. This follows from the fact that the class of all permutation graphs is independent. As we get an effectively constructible interpretation to the class of all structures (matrices or ordered graphs), we conclude that FO model checking is not FPT on hereditary classes of unbounded twin-width. This is the end of the road. The remaining implications to establish the equivalences of Theorems 6 and 9 come from [11, Sections 7 and 8], [10, Section 3], and Theorem 7.

Theorem 8 is proved using model-theoretic methods. In particular, it relies on a suitable analogue of forking independence for monadically NIP classes.

For the most part, the proof will be carried out in the language of matrices over a finite alphabet. Matrices are considered ordered, in the sense that they are equipped with a total order on the rows and

on the columns. A *class* of matrices is, by definition, assumed to be closed under taking submatrices, that is, removing rows and/or columns.

For the sake of simplicity, the description below concerns matrices with entries 0 or 1, called 0, 1-matrices. A 0, 1-matrix can be seen as a relational structure whose domain consists of its rows and columns, equipped with two unary predicates marking the rows and the columns, respectively, a total order which places the rows before the columns, and a binary, symmetric relation which relates a row with a column if the entry at their intersection is equal to 1.

Recall that the six matrix classes of unbounded twin-width which arise are: the class $\mathcal{F}_=$ of all permutation matrices, the class $\mathcal{F}_\neq$ obtained from permutation matrices by exchanging 0's with 1's, and four classes $\mathcal{F}_{\leq_R}, \mathcal{F}_{\geq_R}, \mathcal{F}_{\leq_C}, \mathcal{F}_{\geq_C}$ obtained from permutation matrices by propagating each value 1 downward/upward/leftward/rightward, respectively (see Fig. 2). The *growth* of a class of matrices is the function counting the number of distinct (square) $n \times n$ matrices in the class, for a given $n \geq 1$.

Our main result concerning (hereditary) classes of 0, 1-matrices is as follows.

THEOREM 9. *Given a class $\mathcal{M}$ of 0, 1-matrices, the following are equivalent.*

- (i) $\mathcal{M}$ has bounded twin-width.
- (ii) $\mathcal{M}$ has bounded grid rank.
- (iii) $\mathcal{M}$ does not contain any of the six classes $\mathcal{F}_=, \mathcal{F}_\neq, \mathcal{F}_{\leq_R}, \mathcal{F}_{\geq_R}, \mathcal{F}_{\leq_C}, \mathcal{F}_{\geq_C}$.
- (iv) $\mathcal{M}$ does not interpret the class of all graphs.
- (v) $\mathcal{M}$ does not transduce the class of all graphs.
- (vi) $\mathcal{M}$ does not have growth at least $\sum_{k=0}^n \binom{n}{k}^2 k! \geq n!$.
- (vii) $\mathcal{M}$ has growth at most $2^{O(n)}$.
- (viii) FO model checking is FPT on $\mathcal{M}$. (The implication from (viii) holds if $\text{FPT} \neq \text{AW}[*]$.)
- (ix) There is some $r \in \mathbb{N}$ such that no matrix $M \in \mathcal{M}$ admits an r -rich division.

The last condition, (ix), is a technical one, whose definition is deferred to Section 3. This will be a key intermediate step in proving that (ii) implies (i), as well as in getting an approximation algorithm for the twin-width of a matrix. Theorem 9 reads the same for matrices over a finite alphabet A , except that (iii) is replaced by: No selection of $\mathcal{M}$ contains any of the six classes $\mathcal{F}_=, \mathcal{F}_\neq, \mathcal{F}_{\leq_R}, \mathcal{F}_{\geq_R}, \mathcal{F}_{\leq_C}, \mathcal{F}_{\geq_C}$, where for $a \in A$, the *a-selection* of a matrix class $\mathcal{M}$ is the class obtained from the matrices of $\mathcal{M}$ by replacing the letter $a \in A$ with 1 and the remaining letters with 0. In Fig. 4, a class satisfying (iii) is called *pattern-avoiding*.

As mentioned in the introduction, we prove an analogous result (see Theorem 6) for classes of ordered graphs, or more generally for classes of ordered binary structures. In an informal nutshell, the high points of the paper read: For hereditary classes of ordered binary structures, bounded twin-width, small, subfactorial growth, NIP, monadic NIP, and tractability of FO model checking are all equivalent. We conclude by giving a more detailed statement of the approximation algorithm.

THEOREM 7 (MORE PRECISE STATEMENT). *There is a fixed-parameter algorithm, which, given an ordered binary structure G , encoded by a matrix M , and a parameter k , either outputs*

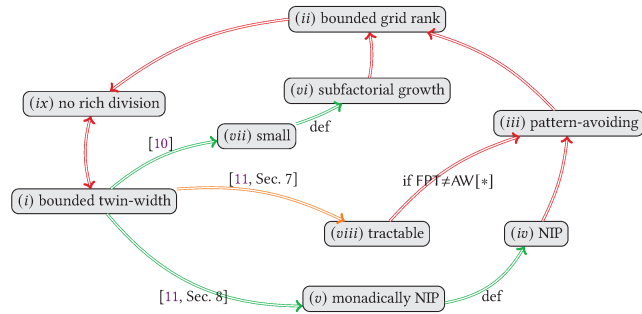


Figure 4: A bird's eye view of the paper. In green, the implications that were already known for general binary structures. In red (other arrows except $(i) \Rightarrow (viii)$), the new implications for matrices on finite alphabets, or ordered binary structures. The effective implication $(i) \Rightarrow (ix)$ is useful for Theorem 7.

- a $2^{O(k^4)}$ -sequence of G , implying that $\text{tw}(G) = 2^{O(k^4)}$, or
- a $2k(k+1)$ -rich division of M , implying that $\text{tw}(G) > k$.

3 CONCEPTS

Everything which is relevant to the rest of the paper will now be properly defined. We may denote by $[i, j]$ the set of integers that are at least i and at most j , and $[i]$ is a short-hand for $[1, i]$. We start with the combinatorial objects.

3.1 Graphs, Orders, Matrices, Permutations

By *graph*, we mean a simple, undirected graph G , and denote its set of vertices $V(G)$ and set of edges $E(G)$. An edge with endpoints u and v is denoted uv or vu . A *total order* on a set X is a binary relation $<$ which is transitive, irreflexive, such that for all $x, y \in X$ either $x < y$ or $y < x$ holds. An *ordered graph* is a graph together with a total order on its vertices. The *edge complement* of a graph (resp. ordered graph) G is the graph (resp. ordered graph) obtained from G by replacing edges by non-edges, and vice-versa.

A *matrix* M over a finite alphabet A is a function $M: R \times C \rightarrow A$, where R is a totally ordered set of rows and C is a totally ordered set of columns. The value $M(r, c)$, also denoted $M_{r,c}$, is the *entry* of M at position (r, c) , or in row r and column c . We may say that M is an $R \times C$ matrix, or an $n \times m$ matrix, where $n = |R|$ and $m = |C|$.

A 0, 1-matrix is a matrix over the alphabet $\{0, 1\}$. A 0, 1-matrix with rows R and columns C can be viewed as an ordered graph with vertices $R \uplus C$, total order $<$ obtained from the orders on R and C by making all the columns larger than all the rows, and edges rc such that $r \in R, c \in C$ and $M(r, c) = 1$.

We distinguish matrices only up to isomorphisms which preserve the order of the rows and columns. A *submatrix* of a matrix M is any matrix obtained from M by deleting a (possibly empty) set of rows and columns. Analogously to permutation classes which are by default closed under taking subpermutations (or patterns), we will define a *class* of matrices as a set of matrices closed under taking submatrices. The *submatrix closure* of a matrix M is the set of all submatrices of M (including M itself). Thus our matrix classes include the submatrix closure of every matrix they contain. On

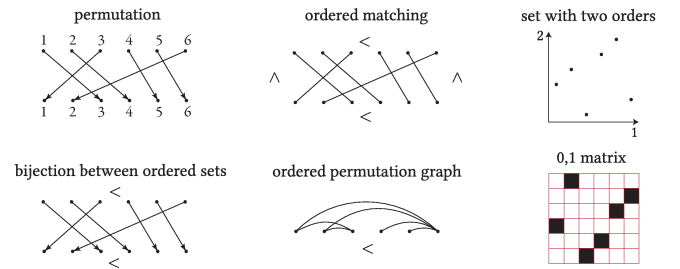


Figure 5: Six different views on the same permutation. We use the convention that the first row of a matrix is at the bottom.

the contrary, classes of (ordered) graphs are only assumed to be closed under isomorphism. A *hereditary* class of (ordered) graphs (resp. binary structures) is one that is closed under taking induced subgraphs (resp. induced substructures).

An n -permutation, for $n \geq 1$, is a bijection $\pi: [n] \rightarrow [n]$. The set of all n -permutations is denoted $\mathfrak{S}_n$. Permutations are of central importance in the theory developed here. Indeed, twin-width has its origins in the Stanley-Wilf conjecture which is precisely about permutations. As we will see, classes of ordered graphs or matrices with unbounded twin-width are exactly those which contain encodings of all permutations, under a suitable encoding.

We will use several views on permutations (see Fig. 5): as bijections between two ordered sets, as sets equipped with two total orders, as ordered matchings, as ordered permutation graphs, and as 0, 1-matrices.

An n -permutation π may be viewed as a bijection π between two totally ordered sets, namely $X = ([n], <_1)$ and $Y = ([n], <_2)$. Conversely, for every bijection $f: X \rightarrow Y$ between two totally ordered sets of size n there is a unique n -permutation π such that $f = i_Y^{-1} \circ \pi \circ i_X$ holds for the unique order-preserving bijections $i_X: X \rightarrow [n]$ and $i_Y: Y \rightarrow [n]$. Using this correspondence, we may define the notion of a *subpermutation*. A subpermutation of an n -permutation π induced by a set $U \subseteq [n]$ is the unique $|U|$ -permutation which corresponds to the restriction $\pi|_U$, treated as a bijection between the ordered sets $U \subseteq [n]$ and $\pi(U) \subseteq [n]$, via the correspondence described above.

Similarly, an n -permutation π defines two orders on $[n]$, namely the usual order $<_1$, and the order $<_2$ such that $i <_2 j$ if and only if $\pi(i) < \pi(j)$. Conversely, every finite set equipped with two total orders is isomorphic to one obtained from a permutation as described above. Via this correspondence, subpermutations correspond exactly to induced substructures of sets equipped with two total orders.

An *ordered matching* is an ordered graph with vertices $a_1 < \dots < a_n < b_1 < \dots < b_n$ such that each a_i is adjacent with exactly one b_j , and vice-versa. Hence, there is a unique n -permutation π such that a_i is adjacent with $b_{\pi(i)}$, for $i \in [n]$.

An *ordered permutation graph* associated with an n -permutation π is the ordered graph G_π with vertices $[n]$ ordered naturally, such that $i < j$ are adjacent if and only if $\pi(i) > \pi(j)$. Note that the isomorphism type of G_π determines the permutation π uniquely.

If σ is a subpermutation of π induced by $U \subseteq [n]$ then G_σ is the ordered subgraph of G_π induced by U . Observe that the edge complement of a permutation graph G_π is also a permutation graph. Namely, if G_π corresponds to two total orders $<_1, <_2$ on $[n]$, as explained above, then the edge complement of G_π corresponds to the orders $<_1, >_2$ on $[n]$.

Finally, n -permutations correspond to $n \times n$ 0, 1-matrices with exactly one 1 in each row and in each column. By convention, the 1 entries in the matrix of permutation $\sigma \in \mathfrak{S}_n$ are at positions $(i, \sigma(i))$ for $i \in [n]$, and, in this context of patterns, the first row is placed at the bottom. A permutation σ is a subpermutation of π if the matrix of σ is a submatrix of the matrix of π .

In fact, we will see even more representations of permutations as matrices or ordered graphs, namely five further matrix classes and twenty-three further classes of ordered graphs.

3.2 Structures

A relational *signature* Σ is a finite set of relation symbols R , each with a specified arity $r \in \mathbb{N}$. A Σ -*structure* A is defined by a set A (the *domain* of A) together with a relation $R^A \subseteq A^r$ for each relation symbol $R \in \Sigma$ with arity r . The syntax and semantics of first-order formulas over Σ , or Σ -*formulas* for brevity, are defined as usual.

A graph is viewed as a structure over the signature with one binary relation E indicating the adjacency between vertices. A total order is viewed as a structure over the signature with one binary relation $<$. An ordered graph is viewed as a structure over the signature with two binary relations, E and $<$. More generally, an *ordered binary structure* is a structure A over a signature Σ consisting of unary and binary relation symbols which includes the symbol $<$, and such that $<$ defines in A a total order on A 's domain.

A matrix M over a finite alphabet A with rows R and columns C is viewed as an ordered binary structure with domain $R \uplus C$, equipped with the following relations:

- unary relations R and C , interpreted as the set of rows and set of columns, respectively,
- a binary relation $<$ which defines a total order on $R \uplus C$, extending the total orders on the rows and columns of M in such a way that the rows precede the columns,
- one binary relation E_a , for each $a \in A$, where $E_a(r, c)$ holds if and only if r is a row, c is a column, and a is the entry of M at row r and column c .

3.3 Twin-Width

In the first paper of the series [11], we define twin-width for general binary structures via *unordered* matrices. The twin-width of (ordered) matrices can be defined this way by encoding the total orders on the rows and on the columns with two binary relations. However we will give an equivalent definition, tailored to ordered structures. This slight shift is already a first step in understanding these structures better, with respect to twin-width. We insist that matrices are always ordered objects, in the current paper. Thus the twin-width of a matrix does *not* coincide with the twin-width of unordered matrices, as defined in [11].

Let M be an $n \times m$ matrix with entries ranging over a fixed finite set. We denote by $R := \{r_1, \dots, r_n\}$ its set of rows and by $C := \{c_1, \dots, c_m\}$ its set of columns. Let S be a non-empty subset

of columns, c_a be the column of S with minimum index a , and c_b , the column of S with maximum index b . The *span* of S is the set of columns $\{c_a, c_{a+1}, \dots, c_{b-1}, c_b\}$. We say that a subset $S \subseteq C$ is in *conflict* with another subset $S' \subseteq C$ if their spans intersect. A partition $\mathcal{P}$ of C is *k-overlapping* if every part of $\mathcal{P}$ is in conflict with at most k other parts of $\mathcal{P}$. The definitions of *span*, *conflict*, and *k-overlapping partition* similarly apply to sets of rows. With that terminology, a *division* is a 0-overlapping partition.

A partition $\mathcal{P}$ is a *contraction* of a partition $\mathcal{P}'$ (defined on the same set) if it is obtained by merging two parts of $\mathcal{P}'$. A *contraction sequence* of M is a sequence of partitions $\mathcal{P}_1, \dots, \mathcal{P}_{n+m-1}$ of the set $R \cup C$ such that $\mathcal{P}_1$ is the partition into $n + m$ singletons, $\mathcal{P}_{i+1}$ is a contraction of $\mathcal{P}_i$ for all $i \in [n + m - 2]$, and $\mathcal{P}_{n+m-1} = \{R, C\}$. In other words, we merge at every step two *column parts* (made exclusively of columns) or two *row parts* (made exclusively of rows), and terminate when all rows and all columns both form a single part. We denote by $\mathcal{P}_i^R$ the partition of R induced by $\mathcal{P}_i$ and by $\mathcal{P}_i^C$ the partition of C induced by $\mathcal{P}_i$. A *contraction sequence* is *k-overlapping* if all partitions $\mathcal{P}_i^R$ and $\mathcal{P}_i^C$ are *k-overlapping* partitions. Note that a 0-overlapping sequence is a sequence of divisions.

If S^R is a subset of R , and S^C is a subset of C , we denote by $S^R \cap S^C$ the submatrix at the intersection of the rows of S^R and of the columns of S^C . Given some column part C_a of $\mathcal{P}_i^C$, the *error value* of C_a is the number of row parts R_b of $\mathcal{P}_i^R$ for which the submatrix $C_a \cap R_b$ of M is not constant. The error value is defined similarly for rows, by switching the role of columns and rows. The *error value* of $\mathcal{P}_i$ is the maximum error value of some part in $\mathcal{P}_i^R$ or in $\mathcal{P}_i^C$. A contraction sequence is a (k, e) -sequence if all partitions $\mathcal{P}_i^R$ and $\mathcal{P}_i^C$ are *k-overlapping* partitions with error value at most e . Strictly speaking, to be consistent with the definitions in the first paper [11], the *twin-width* of a matrix M , denoted by $\text{tww}(M)$, is the minimum $k + e$ such that M has a (k, e) -sequence. This matches, setting $d := k + e$, what we called a *d-sequence* for the binary structure encoding M [11]. We will however not worry about the exact value of twin-width, but merely whether it is bounded or unbounded on a class of structures. Thus, for simplicity, we often consider the minimum integer k such that M has a (k, k) -sequence. This integer is indeed sandwiched between $\text{tww}(M)/2$ and $\text{tww}(M)$.

The twin-width of a matrix class $\mathcal{M}$, denoted by $\text{tww}(\mathcal{M})$, is simply defined as the supremum of $\{\text{tww}(M) \mid M \in \mathcal{M}\}$. We say that $\mathcal{M}$ has *bounded twin-width* if $\text{tww}(\mathcal{M}) < \infty$, or equivalently, if there is a finite integer k such that every matrix $M \in \mathcal{M}$ has twin-width at most k . A class $\mathcal{C}$ of ordered graphs has *bounded twin-width* if all the adjacency matrices of graphs $G \in \mathcal{C}$ along their vertex ordering, or equivalently their submatrix closure, form a set/class with bounded twin-width.

We can more generally define the twin-width of ordered binary structures via matrices. The *matrix encoding* of an ordered binary structure A with domain A and binary relations $<, E_1, \dots, E_p$, is the $|A| \times |A|$ matrix over the alphabet $\{-1, 0, 1, 2\}^p$ whose entry at position (x, y) , is the vector $(b_1, \dots, b_p) \in \{-1, 0, 1, 2\}^p$ such that $b_i = 1$ if $E_i(x, y) \wedge \neg E_i(y, x)$ holds, $b_i = -1$ if $\neg E_i(x, y) \wedge E_i(y, x)$ holds, $b_i = 2$ if $E_i(x, y) \wedge E_i(y, x)$ holds, and $b_i = 0$ otherwise. Then the twin-width of an ordered binary structure A is simply

the twin-width of the matrix encoding of $\mathbf{A}$. We choose this particular encoding so that the vector $(b_1, b_2, \dots, b_p)$ at position (x, y) and the one $(b'_1, b'_2, \dots, b'_p)$ at position (y, x) satisfies $b_i = \pm b'_i$ for every $i \in [p]$. We then say that the matrix is *mixed-symmetric* as in each vector some coordinates are symmetric while others are skew-symmetric. This technicality allows to turn a contraction sequence of a matrix encoding into a contraction sequence of its associated ordered binary structure. For more details, see [11, Section 5, Theorem 14].

3.4 Rank Division and Rich Division

We recall that a division $\mathcal{D}$ of a matrix M is a pair $(\mathcal{D}^R, \mathcal{D}^C)$, where $\mathcal{D}^R$ (resp. $\mathcal{D}^C$) is a partition of the rows (resp. columns) of M into (contiguous) intervals, or equivalently, a 0-overlapping partition. A d -division is a division satisfying $|\mathcal{D}^R| = |\mathcal{D}^C| = d$. For every pair $R_i \in \mathcal{D}^R, C_j \in \mathcal{D}^C$, the submatrix $R_i \cap C_j$ may be called *zone* (or *cell*) of $\mathcal{D}$ since it is, by definition, a contiguous submatrix of M . We observe that a d -division defines d^2 zones.

A *rank- k d -division* of M is a d -division $\mathcal{D}$ such that for every $R_i \in \mathcal{D}^R$ and $C_j \in \mathcal{D}^C$ the zone $R_i \cap C_j$ has at least k distinct rows or at least k distinct columns. A *rank- k division* is simply a short-hand for a rank- k k -division. The *grid rank* of a matrix M is the largest integer k such that M admits a rank- k division. A class $\mathcal{M}$ has *bounded grid rank* if there is some integer k such that every matrix $M \in \mathcal{M}$ has grid rank less than k , or equivalently, for every k -division $\mathcal{D}$ of M , there is a zone of $\mathcal{D}$ with less than k distinct rows and less than k distinct columns.

Closely related to rank divisions, a *k -rich division* is a division $\mathcal{D}$ of a matrix M on rows and columns $R \cup C$ such that:

- for every part R_a of $\mathcal{D}^R$ and for every subset Y of at most k parts in $\mathcal{D}^C$, the submatrix $R_a \cap (C \setminus \cup Y)$ has at least k distinct row vectors, and symmetrically
- for every part C_b of $\mathcal{D}^C$ and for every subset X of at most k parts in $\mathcal{D}^R$, the submatrix $(R \setminus \cup X) \cap C_b$ has at least k distinct column vectors.

Informally, in a large rich division (that is, a k -rich division for some large value of k), the diversity in the column vectors within a column part cannot drop too low by removing a controlled number of row parts. And the same applies to the diversity in the row vectors. Observe that a k -rich $k+1$ -division is in particular a rank- k $k+1$ -division.

3.5 Interpretations and Transductions

Let Σ, Γ be signatures. A *simple interpretation* $\mathbf{l}: \Sigma \rightarrow \Gamma$ consists of the following Σ -formulas: a *domain* formula $v(x)$, and for each relation symbol $R \in \Gamma$ of arity r , a formula $\rho_R(x_1, \dots, x_r)$. If $\mathbf{A}$ is a Σ -structure, the Γ -structure $\mathbf{l}(\mathbf{A})$ has domain $v(\mathbf{A}) = \{v \in A : \mathbf{A} \models v(v)\}$ and the interpretation of a relation symbol $R \in \Sigma$ of arity r is $\rho_R(\mathbf{A}) \cap v(\mathbf{A})^r$, that is:

$$R^{\mathbf{l}(\mathbf{A})} = \{(v_1, \dots, v_r) \in v(\mathbf{A})^r : \mathbf{A} \models \rho_R(v_1, \dots, v_r)\}.$$

If $\mathcal{C}$ is a class of Σ -structures then denote $\mathbf{l}(\mathcal{C}) = \{\mathbf{l}(\mathbf{A}) \mid \mathbf{A} \in \mathcal{C}\}$.

An important property of (simple) interpretations is that they can be composed: if $\mathbf{l}: \Sigma \rightarrow \Gamma$ and $\mathbf{j}: \Gamma \rightarrow \Delta$ are interpretations, then there is an interpretation $\mathbf{j} \circ \mathbf{l}: \Sigma \rightarrow \Delta$ (computable from $\mathbf{l}$ and $\mathbf{j}$) such that $(\mathbf{j} \circ \mathbf{l})(\mathbf{A}) = \mathbf{j}(\mathbf{l}(\mathbf{A}))$ for every Σ -structure $\mathbf{A}$. Similarly,

for every Σ -sentence φ there is a sentence $\mathbf{l}^*(\varphi)$ computable from $\mathbf{l}$ and φ such that for every Σ -structure $\mathbf{A}$ and we have

$$\mathbf{l}(\mathbf{A}) \models \varphi \iff \mathbf{A} \models \mathbf{l}^*(\varphi).$$

A class $\mathcal{C}$ *interprets* a class $\mathcal{D}$ if there is an interpretation $\mathbf{l}$ such that $\mathbf{l}(\mathcal{C}) \supseteq \mathcal{D}$. We say that $\mathcal{C}$ *efficiently interprets* $\mathcal{D}$ if additionally there is an algorithm which, given $\mathbf{D} \in \mathcal{D}$, computes in time polynomial in the size of $\mathbf{D}$ a structure $\mathbf{C} \in \mathcal{C}$ such that $\mathbf{l}(\mathbf{C})$ is isomorphic to $\mathbf{D}$. (A structure is represented by the size of its domain written in unary, followed by the adjacency matrices representing each of its relations.) By composition of interpretations, we conclude that if $\mathcal{C}$ efficiently interprets $\mathcal{D}$ and $\mathcal{D}$ efficiently interprets $\mathcal{E}$, then $\mathcal{C}$ efficiently interprets $\mathcal{E}$.

Efficient interpretations are a convenient way for obtaining FPT reductions, as expressed by the following straightforward lemma.

LEMMA 10. *Suppose that $\mathcal{C}$ efficiently interprets a class $\mathcal{D}$. Then there is an FPT reduction of FO model checking on $\mathcal{D}$ to FO model checking on $\mathcal{C}$: there is a computable function f , a constant c , and an algorithm which given a structure $\mathbf{D} \in \mathcal{D}$ and an FO sentence φ computes in time $f(\varphi) \cdot |\mathbf{D}|^c$ a structure $\mathbf{C} \in \mathcal{C}$ and an FO sentence ψ such that*

$$\mathbf{D} \models \varphi \iff \mathbf{C} \models \psi.$$

Since FO model checking on the class of all graphs is AW[*]-hard [20], we get:

COROLLARY 11. *If $\mathcal{C}$ efficiently interprets the class of all graphs then model checking on $\mathcal{C}$ is AW[*]-hard.*

An important class of ordered graphs which efficiently interprets the class of all graphs is the class $\mathcal{M}$ of all ordered matchings. This is expressed by the following folklore result.

LEMMA 12. *The class $\mathcal{M}$ of ordered matchings efficiently interprets the class of all graphs.*

Let $\Sigma \subseteq \Sigma^+$ be relational signatures. The Σ -*reduct* of a Σ^+ -structure $\mathbf{A}$ is the structure obtained from $\mathbf{A}$ by “forgetting” all the relations not in Σ . We denote this interpretation as $\text{Reduct}_\Sigma: \Sigma^+ \rightarrow \Sigma$, or simply Reduct , when Σ is clear from context.

A class $\mathcal{C}$ of Σ -structures *transduces* a class $\mathcal{D}$ if there is a class $\mathcal{C}^+$ of Σ^+ -structures, where Σ^+ is the union of Σ and some unary relation symbols such that $\text{Reduct}_\Sigma(\mathcal{C}^+) = \mathcal{C}$ and $\mathcal{C}^+$ interprets $\mathcal{D}$.

The following result follows from [11].

THEOREM 13. *Let $\mathcal{C}$ be a class of ordered, binary structures, and suppose that $\mathcal{C}$ has bounded twin-width. Then $\mathcal{C}$ does not transduce the class of all graphs.*

This result more generally holds for (non necessarily ordered) binary structures. We only state it in the ordered case, since the definition of twin-width we gave in Section 3.3 only fits ordered binary structures.

Fix a binary signature Σ containing the symbol $<$. An *atomic type* $\tau(x_1, \dots, x_n)$ over Σ is a maximal conjunction of atomic formulas or negated atomic formulas with variables $x_1, \dots, x_n$, which is satisfiable in some ordered Σ -structure. (It is sufficient to verify this condition for structures with n elements, since the formulas

are quantifier-free.) If $\bar{a}$ is an n -tuple of elements of an ordered Σ -structure A then the atomic type of $\bar{a}$ is the unique (up to equivalence) atomic type $\tau(x_1, \dots, x_n)$ satisfied by $\bar{a}$ in A . For an atomic type $\tau(x, y)$ and ordered Σ -structure A let $I_\tau(A)$ be the ordered graph whose domain and order are the same as in A , and where two vertices $u < v$ are adjacent if and only if $\tau(u, v)$ holds in A . Then I_τ is an interpretation from Σ to the signature of ordered graphs.

We formulate a standard lemma reducing the model checking problem for adjacency matrices of structures from a class $\mathcal{C}$ to the model checking problem for $\mathcal{C}$. Let us view here the adjacency matrix $M(A)$ of an ordered Σ -structure A as the $A \times A$ -matrix whose entry at position (a, b) , for $a, b \in A$, is the atomic type of the pair (a, b) in A . Hence, $M(A)$ is a matrix over the alphabet A_Σ consisting of all atomic types $\tau(x, y)$ with two variables.

LEMMA 14. *Let $\mathcal{C}$ be a class of ordered binary structures and let $M = \{M(A) \mid A \in \mathcal{C}\}$ be the class of adjacency matrices of structures in $\mathcal{C}$. Then there is an FPT reduction of the FO model checking problem for M to the FO model checking problem for $\mathcal{C}$. In particular, if the former is AW[*]-hard, so is the latter.*

3.6 Model Theory

Let $\varphi(\bar{x}, \bar{y})$ be a Σ -formula and let $\mathcal{C}$ be a class of Σ -structures. The formula φ is *independent* over $\mathcal{C}$ if for every binary relation $R \subseteq A \times B$ between two finite sets A and B there exists a Σ -structure $C \in \mathcal{C}$, some tuples $(\bar{u}_a)_{a \in A}$ in $C^{|\bar{x}|}$, and $(\bar{v}_b)_{b \in B}$ in $C^{|\bar{y}|}$ such that

$$C \models \varphi(\bar{u}_a, \bar{v}_b) \iff R(a, b) \quad \text{for all } a \in A \text{ and } b \in B.$$

The class $\mathcal{C}$ is *independent* if there is a Σ -formula $\varphi(\bar{x}, \bar{y})$ that is independent over $\mathcal{C}$. Otherwise, the class $\mathcal{C}$ is *dependent* (or *NIP*, for Not the Independence Property). Note that if a class $\mathcal{C}$ interprets the class of all graphs, then it is independent.³

A *monadic lift* of a class $\mathcal{C}$ of Σ -structures is a class $\mathcal{C}^+$ of Σ^+ -structures, where Σ^+ is the union of Σ and a set of unary relation symbols, and $\mathcal{C} = \{\text{Reduct}_{\Sigma}(A) : A \in \mathcal{C}^+\}$. A class $\mathcal{C}$ of Σ -structures is *monadically dependent* (or *monadically NIP*) if every monadic lift of $\mathcal{C}$ is dependent (or NIP).

The following theorem witnesses that transductions are particularly fitting to the study of monadic dependence:

THEOREM 15 (BALDWIN AND SHELAH [4]). *A class $\mathcal{C}$ of Σ -structures is monadically dependent if and only if for every monadic lift $\mathcal{C}^+$ of $\mathcal{C}$ (in Σ^+ -structures), every Σ^+ -formula $\varphi(\bar{x}, \bar{y})$ with $|\bar{x}| = |\bar{y}| = 1$ is dependent over $\mathcal{C}^+$. Consequently, $\mathcal{C}$ is monadically dependent if and only if $\mathcal{C}$ does not transduce the class $\mathcal{G}$ of all finite graphs.*

4 MODEL-THEORETIC CHARACTERIZATIONS

In this section, we present further model-theoretic characterizations of classes of bounded twin-width, as well as prove more general results concerning arbitrary classes of structures, over an arbitrary signature. In particular, we generalize the implications (ii) $\rightarrow$ (iii) $\rightarrow$ (v) from Theorem 9 to arbitrary classes of structures, by

³The converse also holds if the interpretations can use constant symbols, and we can take induced substructures after performing the interpretation, see [47].

proving Theorem 8. Namely, we show that every monadically dependent class of structures excludes certain grid-like patterns, and every class of structures which excludes such grid-like patterns satisfies a property generalizing bounded grid rank.

We start with defining the notion of a restrained class, generalizing the notion of bounded grid rank for matrices. First, we introduce a notion generalizing the concept of the number of distinct rows in a zone of a matrix, in arbitrary structures.

In this section, whenever S is a structure then we identify S with its domain, when writing e.g. $a \in S$ or $A \subseteq S$. We also write $S^{\bar{x}}$ for the set of all *valuations* $\bar{a}$ of a set of variables $\bar{x}$ in S , where a valuation is a function $\bar{a}: \bar{x} \rightarrow S$.

Let $\Delta(\bar{u}; \bar{v})$ be a finite set of formulas $\theta(\bar{u}; \bar{v})$ with free variables contained in $\bar{u}$ and $\bar{v}$. For a structure S , tuple $\bar{a} \in S^{\bar{u}}$ and a set $B \subseteq S$ define the Δ -type of $\bar{a}$ over B as:

$$\text{tp}^\Delta(\bar{a}/B) = \{(\theta, \bar{b}) \in \Delta \times B^{\bar{v}} \mid S \models \theta(\bar{a}; \bar{b})\}.$$

For a set $A \subseteq S$, denote

$$\text{Types}^\Delta(A/B) := \{\text{tp}^\Delta(\bar{a}/B) \mid \bar{a} \in A^{\bar{u}}\}.$$

Example 16. Let M be a 0-1 matrix, viewed as an (ordered) binary structure with the unary predicate $R \subseteq M$ indicating the rows and the binary relation E defining the entries of the matrix. Let $\Delta(u, v) = \{E(u, v)\}$. Let $B \subseteq M \setminus R$ be a set of columns of M . Then, for a row $a \in R$, the set $\text{tp}^\Delta(a/B)$ corresponds to the set of those columns $b \in B$ with a non-zero entry in row a . For a set of rows $A \subseteq R$, $|\text{Types}^\Delta(A/B)|$ is the number of distinct rows in the submatrix of M with rows A and columns B .

Let $\varphi(x; \bar{y})$ be a formula and S a structure. A φ -definable disjoint family is a family $\mathcal{R}$ of pairwise disjoint subsets of S , where for each $R \in \mathcal{R}$ there is $\bar{b} \in S^{\bar{y}}$ with $R = \{a \in S \mid S \models \varphi(a; \bar{b})\}$. For example, if S is a finite ordered structure and $\mathcal{R}$ is a partition of S into convex sets, then $\mathcal{R}$ is a φ -definable family of pairwise disjoint sets, for $\varphi(x; y_1, y_2) = y_1 \leq x \leq y_2$.

Definition 17 (Restrained class). A class $\mathcal{C}$ of structures is *restrained* if the following condition holds. Let $\varphi(x; \bar{y})$ and $\psi(x; \bar{z})$ be formulas over the signature of $\mathcal{C}$, and let $\Delta(\bar{u}; \bar{v})$ be a finite set of formulas. Then there are natural numbers t and k such that for any $S \in \mathcal{C}$ and any φ -definable disjoint family $\mathcal{R}$ and ψ -definable disjoint family $\mathcal{L}$ with $|\mathcal{R}| = |\mathcal{L}| \geq t$ there are $R \in \mathcal{R}$ and $L \in \mathcal{L}$ with $|\text{Types}^\Delta(R/L)| < k$.

The following proposition is an analogue of the statement that matrices of bounded grid rank have bounded twin-width.

PROPOSITION 18. *Let $\mathcal{C}$ be a class of finite, ordered binary structures. If $\mathcal{C}$ is restrained then $\mathcal{C}$ has bounded twin-width.*

We now define a notion which generalizes the notion of avoiding certain patterns. In this case, rather than defining patterns which encode all permutations, it is more convenient to define patterns which encode grids, in the following way.

Fix any signature Σ and a first-order formula $\varphi(\bar{x}, \bar{y}, z)$, where $\bar{x}$ and $\bar{y}$ are sets of variables and z is a single variable. An $m \times n$ grid defined by φ in a structure S is a triple of sets $A \subseteq S^{\bar{x}}$, $B \subseteq S^{\bar{y}}$ and $C \subseteq S$ with $|A| = m$, $|B| = n$ and $|C| = m \times n$, such that the relation

$$\{(\bar{a}, \bar{b}, c) \in A \times B \times C \mid S \models \varphi(\bar{a}, \bar{b}, c)\}$$

is the graph of a bijection from $A \times B$ to C : for each $c \in C$ there is a unique pair $(\bar{a}, \bar{b}) \in A \times B$ such that $S \models \varphi(\bar{a}, \bar{b}, c)$, and for each $(\bar{a}, \bar{b}) \in A \times B$ there is a unique $c \in C$ such that $S \models \varphi(\bar{a}, \bar{b}, c)$.

Definition 19 (Defining large grids). A class of structures $\mathcal{C}$ defines large grids if there is a formula $\varphi(\bar{x}, \bar{y}, z)$ such that for all $n \in \mathbb{N}$, φ defines an $n \times n$ grid in some structure $S \in \mathcal{C}$.

Intuitively, if $\mathcal{C}$ defines large grids then the product of two sets $A \times B$ can be represented by a set of single elements C in some structure $S \in \mathcal{C}$. Hence an arbitrary relation $R \subseteq A \times B$ can be represented by some subset of C , so $\mathcal{C}$ is monadically independent. This is stated in the following lemma, due to Shelah [46] (see [14]).

LEMMA 20. *If $\mathcal{C}$ defines large grids then $\mathcal{C}$ is monadically independent.*

Theorem 8 generalizes the implications (ii) $\rightarrow$ (iii) $\rightarrow$ (v) from Theorem 9 to arbitrary classes of structures – finite or infinite, ordered or unordered, and over an arbitrary signature. It also involves a notion which we call *1-dimensionality*, which is a model-theoretic notion originating from Shelah (it is called *finite satisfiability dichotomy* in [14]), and is defined below (see Def. 25). This is a central tool in the study of monadically dependent classes. It is defined in terms of a variant of forking independence – a key concept in stability theory, generalizing e.g. independence in vector spaces or algebraic independence. Our contribution is to show that every 1-dimensional class is *restrained* (see Def. 17).

We believe Theorem 8 that this may be of independent interest, and possibly of broader applicability than just in the context of ordered structures. For example, all graph classes of bounded twin-width (without an order) and all interpretations of nowhere-dense classes are restrained. Conversely, every class of structures which is not restrained defines large grids.

Theorem 8 allows us to provide further, model theoretic characterizations of hereditary classes of finite, ordered, binary structures of bounded twin-width:

THEOREM 21. *Let $\mathcal{C}$ be a hereditary class of finite, ordered, binary structures. Then the following conditions are equivalent:*

- (1) $\mathcal{C}$ does not transduce the class of all finite graphs,
- (2) $\mathcal{C}$ is monadically dependent,
- (3) $\mathcal{C}$ does not define large grids,
- (4) $\mathcal{C}$ is 1-dimensional,
- (5) $\mathcal{C}$ is a restrained class,
- (6) $\mathcal{C}$ has bounded twin-width,
- (7) $\mathcal{C}$ is dependent.

PROOF. The implications (1) $\rightarrow$ (2) $\rightarrow$ (3) $\rightarrow$ (4) $\rightarrow$ (5) follow from Theorem 8. The implication (5) $\rightarrow$ (6) is proved in Proposition 18. The implication (6) $\rightarrow$ (1) is by [11] (cf. Theorem 13). This proves the equivalence of the first six items.

The implication (2) $\rightarrow$ (7) is immediate, whereas the implication (7) $\rightarrow$ (6) is by Theorem 6, (iv'), (i). $\square$

To prove Theorem 8, it remains to prove that every 1-dimensional class is restrained. First, we need to define 1-dimensionality. It is a wholly model-theoretic notion which can be used to characterize bounded twin-width, but also arbitrary monadically dependent

classes of structures. We first recall some basic notions from model theory.

By a *model* we mean a structure which is typically infinite, as opposed to the structures considered earlier, which were typically finite. The *elementary closure* of a class of structures $\mathcal{C}$ is the class of all models M that satisfy every sentence φ that holds in all structures $S \in \mathcal{C}$. In particular, if $\mathcal{C}$ does not define large grids, then neither does its elementary closure. This is because for any fixed $n \in \mathbb{N}$ the existence of an $n \times n$ -grid defined by a fixed formula $\varphi(\bar{x}, \bar{y}, z)$ can be expressed by a first-order sentence φ' which existentially quantifies $(|\bar{x}| + |\bar{y}|) \cdot n + n^2$ variables, corresponding to sets A, B, C of $\bar{x}$ -tuples, $\bar{y}$ -tuples and single vertices, and then checks that φ defines a bijection between $A \times B$ and C . By the compactness theorem, if $\mathcal{C}$ defines large grids, then its elementary closure contains a structure that defines a grid (A, B, C) with A and B of arbitrarily large infinite cardinalities.

Definition 22 (Elementary extension). Let M, N be two models. Then N is an *elementary extension* of M , written $M < N$, if the domain of M is contained in the domain of N , and for every formula $\varphi(\bar{x})$ and tuple $\bar{a} \in M^{\bar{x}}$ of elements of M ,

$$M \models \varphi(\bar{a}) \quad \text{if and only if} \quad N \models \varphi(\bar{a}).$$

In other words, it does not matter if we evaluate formulas in M or in N . In particular, M and N satisfy the same sentences.

A formula $\varphi(\bar{x})$ with parameters from $C \subseteq N$ is a formula using constant symbols denoting elements from C . Such a formula can be evaluated in N on a tuple $\bar{a} \in N^{\bar{x}}$, as expected. Note that if $M < N$ and $\varphi(\bar{x})$ is a formula with parameters from N and $\bar{a} \in M^{\bar{x}}$ then it is not necessarily the case that $M \models \varphi(\bar{a})$ if and only if $N \models \varphi(\bar{a})$, although this does hold for formulas with parameters from M .

Definition 23 (Independence). Let M be a model and N its elementary extension. For a tuple $\bar{a} \in N^{\bar{x}}$ and a set $B \subseteq N$ say that $\bar{a}$ is *independent* from B over M , denoted $\bar{a} \downarrow_M B$, if for every formula $\varphi(\bar{x})$ with parameters from $B \cup M$ such that $N \models \varphi(\bar{a})$ there is some $\bar{c} \in M^{\bar{x}}$ such that $N \models \varphi(\bar{c})$.

Abusing notation, if B is enumerated by a tuple $\bar{b}$, then we may write $\bar{a} \downarrow_M \bar{b}$. For two sets $A, B \subseteq N$, we write $A \downarrow_M B$ if $\bar{a} \downarrow_M \bar{b}$ for every tuple $\bar{a}$ of elements in A . We write $\nmid_M$ for the negation of the relation $\downarrow_M$. As an example, if $M < N$, then $\bar{a} \nmid_M M$ for every $\bar{a} \in N^{\bar{x}}$.

Example 24. Let N be $(\mathbb{R}, \leq)$ and let M be the union of the open intervals $]0, 1[$ and $]8, 9[$, equipped with the relation $\leq$. Then $M < N$. This is easy to derive from the fact that $(\mathbb{R}, \leq)$ has quantifier elimination, that is, every formula $\varphi(\bar{x})$ is equivalent to a quantifier-free formula. Figure 6 illustrates independence over M .

Definition 25 (1-dimensionality). A model M is *1-dimensional* if for every $M < N$, tuples $\bar{a}, \bar{b}$ of elements of N and a single element $c \in N$, if $\bar{a} \downarrow_M \bar{b}$ then $\bar{a}c \downarrow_M \bar{b}$ or $\bar{a} \downarrow_M \bar{b}c$. A class $\mathcal{C}$ of structures is *1-dimensional* if every model in the elementary closure of $\mathcal{C}$ is 1-dimensional.

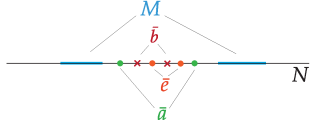


Figure 6: The structures $M < N$, a set B and two tuples, $\bar{a}, \bar{b}$, with $\bar{a} \downarrow_M B$ and $\bar{b} \downarrow_M B$.

Example 26. Any total order $(X, \leq)$ is 1-dimensional. As an illustration, in the situation in Fig. 6, consider the tuples $\bar{a}, \bar{b}$ marked therein. Then $\bar{a} \downarrow_M \bar{b}$. Let $c \in N$. If c belongs to the interval $]b_1, b_2[$ then $\bar{a} \downarrow_M \bar{b}c$. Otherwise, $\bar{a}c \downarrow_M \bar{b}$.

Example 27. Let $N = (\mathbb{R} \times \mathbb{R}, \sim_1, \sim_2)$ where for $i = 1, 2$, the relation $\sim_i$ denotes equality of the i th coordinates. Let M be the induced substructure of N with domain $I \times I$ for some infinite subset $I \subseteq \mathbb{R}$. Then $M < N$. In the situation depicted in Fig. 7, $a \downarrow_M b$ but both $ac \not\downarrow_M b$ and $a \not\downarrow_M bc$. So M is not 1-dimensional.

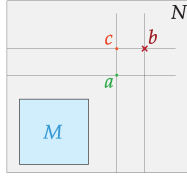


Figure 7: The structures $M < N$ and elements $a, b, c \in N$ with $a \downarrow_M b$, $ac \not\downarrow_M b$ and $a \not\downarrow_M bc$.

The following result is essentially [46, Lemma 2.2] (see also [14]).

PROPOSITION 28. *If a model M does not define large grids then it is 1-dimensional.*

Thus, every class $\mathcal{C}$ that does not define large grids is 1-dimensional, proving the implication (3) $\rightarrow$ (4) in Theorem 8. The proof of the implication (4) $\rightarrow$ (5), which is our contribution, relies on the following lemma. For a formula $\alpha(\bar{x})$ (possibly with parameters) and model N , write $\alpha(N)$ for the set of all tuples in $N^{\bar{x}}$ that satisfy $\alpha(\bar{x})$ in N .

LEMMA 29. *Let $\varphi(x, \bar{y})$ and $\psi(x, \bar{z})$ be formulas, M be a 1-dimensional structure, N its elementary extension, and let $\bar{a}_0, \bar{a}_1 \in N^{\bar{y}}$ and $\bar{b}_0 \in N^{\bar{z}}$ be tuples such that:*

- (1) $\bar{a}_0$ and $\bar{a}_1$ have equal types over M ,
- (2) $N \not\models \exists x. \varphi(x, \bar{a}_0) \wedge \varphi(x, \bar{a}_1)$,
- (3) $M \not\models \exists x. \psi(x, \bar{b}_0)$,
- (4) $\bar{a}_1 \downarrow_M \bar{a}_0 \bar{b}_0$.

Then $\varphi(N; \bar{a}_1) \downarrow_M \psi(N; \bar{b}_0)$.

PROOF. First we prove the following.

Claim 30. Let $a \in \varphi(N; \bar{a}_1)$. Then $\bar{a}_1 \not\downarrow_M \bar{a}_0 a$.

Denote

$$\zeta(\bar{y}; a, \bar{a}_0) := \varphi(a; \bar{y}) \wedge \neg \exists x. \varphi(x; \bar{y}) \wedge \varphi(x; \bar{a}_0).$$

We have that $\zeta(\bar{a}_1; a, \bar{a}_0)$ holds since $\varphi(x; \bar{a}_1) \wedge \varphi(x; \bar{a}_0)$ is not satisfiable in N . Suppose $\zeta(\bar{a}'; a, \bar{a}_0)$ holds for some $\bar{a}' \in M^{\bar{y}}$. Then

$$\exists x. \varphi(x; \bar{a}') \wedge \varphi(x; \bar{a}_1)$$

holds in N , as witnessed by $x = a$. As $\bar{a}_1$ and $\bar{a}_0$ have equal types over M , this implies that

$$\exists x. \varphi(x; \bar{a}') \wedge \varphi(x; \bar{a}_0)$$

holds in N , contradicting $\zeta(\bar{a}'; a, \bar{a}_0)$. Thus $\zeta(\bar{y}; a, \bar{a}_0)$ is not satisfiable in M . In particular, $\bar{a}_1 \not\downarrow_M \bar{a}_0 a$, proving the claim.

We now prove that whenever $\bar{a}$ is a tuple in $\varphi(N; \bar{a}_1)$ and $\bar{b}$ a tuple in $\psi(N; \bar{b}_0)$, then

$$\bar{a}_1 \bar{a} \downarrow_M \bar{a}_0 \bar{b}_0 \bar{b}. \quad (1)$$

In particular, $\bar{a} \downarrow_M \bar{b}$, proving the lemma.

We prove (1) by induction on the length of $\bar{a}$ and $\bar{b}$. The base case where $\bar{a}$ and $\bar{b}$ are empty follows from the fourth assumption of the lemma. In the inductive step, assume we know the result for $\bar{a}, \bar{b}$ and we want to add an element $b \in \psi(N; \bar{b}_0)$ to $\bar{b}$. By 1-dimensionality, one of the two cases holds:

$$\bar{a}_1 \bar{a} b \downarrow_M \bar{a}_0 \bar{b}_0 \bar{b} \quad \text{or} \quad \bar{a}_1 \bar{a} \downarrow_M \bar{a}_0 \bar{b}_0 \bar{b} b.$$

Since $\psi(M; \bar{b}_0) = \emptyset$ by assumption, we have $b \not\downarrow_M \bar{b}_0$, excluding the first case, so the second case must hold, as required.

Now assume we want to add $a \in \varphi(N; \bar{a}_1)$ to $\bar{a}$. By 1-dimensionality,

$$\bar{a}_1 \bar{a} a \downarrow_M \bar{a}_0 \bar{b}_0 \bar{b} \quad \text{or} \quad \bar{a}_1 \bar{a} \downarrow_M \bar{a}_0 \bar{b}_0 \bar{b} a,$$

but the second possibility is excluded by Claim 30, and the first one concludes the inductive step. The lemma follows since we showed that $\bar{a} \downarrow_M \bar{b}$ for all tuples $\bar{a}$ in $\varphi(N; \bar{a}_1)$ and $\bar{b}$ in $\psi(N; \bar{b}_0)$. $\square$

We now sketch the proof of the remaining implication (4) $\rightarrow$ (5). Towards a contradiction, assume $\mathcal{C}$ is not regular, and let $\varphi(x, \bar{y})$, $\psi(x, \bar{z})$ and $\Delta(\bar{u}; \bar{v})$ witness that. As $\Delta(\bar{u}; \bar{v})$ is finite, by a simple counting argument, we may reduce to the case where $\Delta(\bar{u}; \bar{v})$ comprises a single formula $\theta(\bar{u}; \bar{v})$. Using standard model-theoretic tools (like compactness and a variant of Morley sequences) we exhibit a model N in the elementary closure of $\mathcal{C}$, its elementary substructure M , and tuples $\bar{a}_0, \bar{a}_1$ and $\bar{b}_1$ in N satisfying the assumptions of Lemma 29, and such that $\text{Types}^\theta(A/B)$ has cardinality larger than $2^{|M|}$, where $A = \varphi(N, \bar{a})$ and $B = \psi(N, \bar{b})$. This contradicts the conclusion of Lemma 29, that $A \downarrow_M B$.

The details are relegated to the full version, due to space constraints.

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