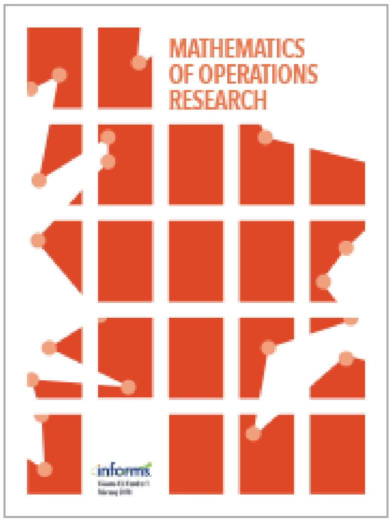

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A Label-State Formulation of Stochastic Graphon Games and Approximate Equilibria on Large Networks

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Abstract. This paper studies stochastic games on large graphs and their graphon limits. We propose a new formulation of graphon games based on a single typical player's label-state distribution. In contrast, other recently proposed models of graphon games work directly with a continuum of players, which involves serious measure-theoretic technicalities. In fact, by viewing the label as a component of the state process, we show in our formulation that graphon games are a special case of mean field games, albeit with certain inevitable degeneracies and discontinuities that make most existing results on mean field games inapplicable. Nonetheless, we prove the existence of Markovian graphon equilibria under fairly general assumptions as well as uniqueness under a monotonicity condition. Most importantly, we show how our notion of graphon equilibrium can be used to construct approximate equilibria for large finite games set on any (weighted, directed) graph that converges in cut norm. The lack of players' exchangeability necessitates a careful definition of approximate equilibrium, allowing heterogeneity among the players' approximation errors, and we show how various regularity properties of the model inputs and underlying graphon lead naturally to different strengths of approximation.

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Keywords: graphon games • mean field games • approximate Nash equilibrium

1. Introduction

This paper is about network-based generalizations of the now-standard *mean field game* (MFG) framework. The latter was introduced in Huang et al. [32] and Lasry and Lions [39] to describe the large-n equilibrium behavior of certain n-player stochastic games. Remarkably, the limiting models in MFG theory are typically quite tractable, and for this reason, MFG theory developed a rich mathematical theory and a broad range of applications. However, the MFG framework is fundamentally limited to games in which *players interact symmetrically*. On the one hand, MFG models can already incorporate heterogeneity in *individual* characteristics (and are often known to economists as *heterogeneous agent models*) in the sense that players may face independent sources of randomness and perhaps, their own *type* parameters. On the other hand, MFG theory is not well suited to modeling heterogeneity in the *interactions* between players, where distinct pairs of players have different interaction strengths. Heterogeneous interactions are the defining feature of *network games*, a well-developed framework that is widely applied in very different contexts from MFG theory; see Jackson [34].

The range of applicability of MFG theory would increase dramatically if it could incorporate nontrivial network structures or heterogeneous interactions while maintaining a tractable limiting (continuum) model. This is a challenging prospect in general because different n-player networks may lead to very different limits as $n \to \infty$, especially in *sparse* networks (Feng et al. [29], Lacker and Soret [38]). A natural first step is to understand the range of network models for which the usual MFG remains the correct limit. There is reason to expect that this is the case for sufficiently *dense* and *approximately regular* networks. This intuition was confirmed in our recent linear-quadratic case study (Lacker and Soret [38]) and by Delarue [25] in a model set on dense Erdős–Rényi graphs; Remark 6 gives a result of this nature as well. Similar ideas appeared in nongame-theoretic models of interacting particle systems with interactions governed by networks, for which recent work has identified a certain universality of the mean field limit. See Bhamidi et al. [9], Coppini [21], Coppini et al. [23], Delattre et al. [26], and Luçon [41] for diffusive dynamics and Basak and Mukherjee [3] for static Ising and Potts models.

There are many network models, however, for which the usual MFG limit is not correct. Several different groups of researchers have recently proposed new continuum models as alternatives to the usual MFG based on the notion of *graphons*. Graphons are natural continuum limits for large dense graphs, and we refer to Lovász [40] for an overview. Essentially, a graphon is a symmetric measurable function $W : [0,1]^2 \to [0, 1]$, with $W(u, v)$ representing the edge density between vertices u and v. For static games based on graphons, we refer to recent work (Carmona et al. [20], Parise and Ozdaglar [42, 43]), and for dynamic games, see Cui and Koeppl [24] and Vasal et al. [48] for discrete time and Aurell et al. [2], Bayraktar et al. [5], Gao et al. [30], and Tangpi and Zhou [47] for continuous time. A related but distinct notion of *graphon mean field games* was developed in a recent series of papers by Caines and Huang [13, 14], in which each node in the network contains a subpopulation with its own mean field of players. There have been similar developments for nongame-theoretic models of interacting diffusions, with recent work (Bayraktar et al. [4], Bet et al. [8]) developing a graphon-based limit theory.

The goal of this paper is to advance the theory of graphon-based analogues of mean field games or *graphon games*. Most importantly, we are able to achieve a level of tractability that is comparable with traditional mean field games in the following sense. The mean field game framework is based on a fixed point problem describing the law of the state process $X = (X_t)_{t\in[0,T]}$ of one "typical" player, which represents a significant dimension reduction when compared with a large n-player game. On the contrary, prior graphon-based models work directly with a *continuum* of players, which arguably does not provide a significant simplification and leads to the serious technical challenges discussed. The graphon game model that we propose is a fixed point problem for the joint law of (U, X), where X is the state process coupled with a Unif[0, 1] random variable U, interpreted as the "vertex" or "label" of the player in the graphon.

In fact, we show that our notion of graphon game is equivalent to a classical MFG model in which $(U, X_t)_{t\in[0,T]}$ is treated as the state process. Whereas the MFG model can be captured by a single forward-backward partial differential equation (PDE) system on $[0, T] \times \mathbb{R}^d$, prior graphon-based models involve a continuum of coupled PDEs, and our model can be captured by a single forward-backward PDE on $[0, T] \times \mathbb{R}^{d+1}$. Despite this equivalence, it is only in special situations that one can directly apply prior theorems from the MFG literature; the coefficients are discontinuous unless the graphon is a continuous function, and the diffusion coefficient of $(U, X_t)_{t\in[0,T]}$ is always degenerate. Hence, although we adapt known MFG methods for our proofs (mainly Lacker [35]), we must tailor them to the graphon setting. Moreover, the finite games we study, which are governed by general interaction matrices that converge in cut norm, are quite different from the finite game naturally associated with the equivalent MFG, and our finite games thus require a significantly more involved construction for approximate equilibria. See Section 3.6 for details.

Working directly with a continuum of players driven by a continuum of independent Brownian motions $(B^u)_{u\in[0,1]}$ raises significant technical difficulties stemming from the fact that $u \longmapsto B^u(\omega)$ is not Lebesgue measurable for a.e. ω. In a linear-quadratic setting, this issue was confronted directly in Aurell et al. [2] via sophisticated measure-theoretic machinery, namely the notion of *Fubini extensions* from Sun [45]. In Bayraktar et al. [4], the issue was carefully avoided by arguing that the *laws* $\mathcal{L}(X^u)$ of the state processes $(X^u)_{u\in[0,1]}$ depend measurably on u, and this is good enough for their purposes. Other works, such as Bet et al. [8], do not explicitly address this issue. By focusing on the joint law of (U, X), we avoid the technical challenges of the continuum. Of course, a joint law of (U, X) with U uniform can be identified with its disintegration (i.e., the conditional law of X given U), but this conditional law is uniquely determined only up to a.e. equality. Our notion of graphon game thus encodes less information than a model with a true continuum of players, as we may make statements about *almost every* player but not about *every* player. However, this minor loss of information brings significant mathematical advantages. First, it avoids the aforementioned measure-theoretic difficulties. Second, it permits a simple topological setting, allowing us to use the weak topology on $\mathcal{P}([0, 1] \times \mathbb{R}^d)$, in which compacts are far more abundant when compared with the uniform or L^p topologies on spaces of functions $[0, 1] \to \mathcal{P}(\mathbb{R}^d)$ employed in some prior works (e.g., Caines and Huang [13]).

Using our new graphon game formulation, we prove several fundamental results under fairly general assumptions on the model inputs. First, we prove the existence of an equilibrium that is *Markovian* in the sense that the control is a function of (t, U, X_t). We also show uniqueness under a graphon version of the Lasry–Lions monotonicity assumption. See Section 3.4 for these results. Our new framework allows us to handle, with relative ease, far more general setups than were considered in prior work. For instance, in prior work, the interactions are *pairwise* in the sense that the effect of the other players $j \neq i$ on a player i is given by a quantity of the form $n^{-1}\sum_{j=1}^n \xi_{ij} h(X^i, X^j)$, where ξ is an $n \times n$ interaction matrix. More generally, we are able to treat higher-order interactions depending on the empirical measure $n^{-1}\sum_{j=1}^n \xi_{ij}\delta_{X^j}$, which admits a simple continuum analogue (defined in Section 2.2) in terms of the joint law of (U, X).

Our most important results justify our new formulation by showing that any graphon game equilibrium can be used to construct approximate equilibria for the n-player game when the latter involves an interaction matrix that converges to the given graphon in the cut norm (or more generally, in the strong operator topology, although this generalization does not complicate our proofs). This is the most challenging part of our work. The precise notion of *approximate equilibrium* can take various forms; the ϵ_i^n error may be different for each player i, and ϵ_i^n may vanish in an averaged or uniform sense, depending on the structural assumptions (such as continuity) imposed on the graphon. See Section 3.5 for precise statements. Prior work on graphon games, with a few exceptions, has assumed the n-player game to be set on a specific exchangeable random graph "sampled" from the graphon in the usual manner, which enjoys particularly strong convergence properties as $n \to \infty$. It is more general and also, arguably more natural to start from an interaction matrix (or graph) for n players and see where it converges rather than constructing a specific n-player network with a desired limit in mind. This in a sense makes the n-player game the starting point of the model rather than the graphon game. This perspective is shared by Bayraktar et al. [5], Cui and Koeppl [24], and Gao et al. [30], although these papers impose various restrictions on the graphs and graphon that our main result (Theorem 2) does not need.

Lastly, to illustrate the relative simplicity of our framework, we study in Section 8 a linear-quadratic model of flocking type similar to Carmona et al. [19] and Lacker and Soret [38]. We explicitly solve the model in terms of a centrality index of a given graphon.

Section 2 introduces the basic notions of kernels and graphons that will be used in the paper. Then, Section 3 presents the main results in full detail.

1.1. Common Notation

We write $[n] := \{1, \dots, n\}$ for $n \in \mathbb{N}$. For a random variable X taking values in a measurable space, we write $\mathcal{L}(X)$ for its law. For a complete separable metric space (E, d), we write $\mathcal{M}_+(E)$ for the space of nonnegative Borel measures of finite variation and $\mathcal{P}(E)$ for the sets of probability measures. We write $\langle \mu, \varphi \rangle = \int_E \varphi \, d\mu$ for $\mu \in \mathcal{M}_+(E)$ and suitably integrable functions φ. We equip $\mathcal{M}_+(E)$ with the usual topology of weak convergence, defined in duality with the space of bounded continuous functions. This topology is also induced by the bounded Lipschitz norm (see Bogachev [10, theorem 8.3.2])

$$\|\mu\|_{BL} := \sup \left\{ \int_E \varphi \, d\mu : \varphi : E \to \mathbb{R}, |\varphi| \le 1, \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{d(x,y)} \le 1 \right\}. \tag{1}$$

We write $C([0, T]; E)$ for the space of continuous functions $[0, T] \to E$, always equipped with the supremum distance $(x, x') \longmapsto \sup_{t \in [0,T]} d(x_t, x'_t)$.

We write Unif[0, 1] to denote the uniform (Lebesgue) measure on [0, 1]. Similarly, Unif(I) denotes the uniform probability measure on any interval I. For a Polish space E, let us also write $\mathcal{P}_{\text{Unif}}([0, 1] \times E)$ for the set of Borel probability measures on $[0, 1] \times E$ with uniform first marginal. Any $\mu \in \mathcal{P}_{\text{Unif}}([0, 1] \times E)$ admits the disintegration $\mu(du, dx) = du\mu_u(dx)$, with $[0, 1] \ni u \longmapsto \mu_u \in \mathcal{P}(E)$ being Borel measurable and uniquely defined up to a.e. equality. The space E will typically be either $\mathbb{R}^d$ or the path space $\mathcal{C}^d := C([0, T]; \mathbb{R}^d)$.

2. Kernels and Graphons

In this section, we give a brief summary of the notion of graphon relevant to our work, most importantly introducing (in Section 2.2) its associated operator, which will play a central role. We borrow most terminology from Lovász [40]. A *graphon* is typically defined as a symmetric measurable function $[0,1]^2 \to [0, 1]$. More generally, a *kernel* is any element of $L^1[0,1]^2$ (i.e., an integrable Borel-measurable real-valued function of $[0,1]^2$).

We work with kernels belonging to $L^1_+[0,1]^2$, the set of nonnegative elements of $L^1[0,1]^2$. We think of [0, 1] as indexing a continuum of possible locations or vertices, with $W(u, v)$ representing the (weighted) edge density between them. We notably do not require our kernels to be graphons (bounded or symmetric), which brings certain advantages in the examples. Also, we work with *labeled* rather than *unlabeled* kernels (Lovász [40], sections 8.2.1 and 8.2.2).

For $n \in \mathbb{N}$, the space of $n \times n$ matrices embeds into the space of kernels as follows. For an $n \times n$ matrix ξ, we introduce the associated *step kernel*

$$\begin{aligned} &W_\xi(u, v) := \xi_{ij}, \quad \text{for } (u, v) \in I_i^n \times I_j^n, \\ &\text{where } I_i^n := [(i-1)/n, i/n), \ \text{for } i = 1, \dots, n-1, \ \text{and} \ \ I_n^n := [(n-1)/n, 1]. \end{aligned} \tag{2}$$

2.1. The Cut Norm

Following Lovász [40, chapter 8.2], we define the *cut norm* on $L^1[0,1]^2$ by

$$\|W\|_\square := \sup_{S_1, S_2} \left| \int_{S_1} \int_{S_2} W(u, v) \, du dv \right|,$$

where the supremum is over pairs of Borel sets $S_1, S_2 \subseteq [0, 1]$. (Technically, this is merely a *seminorm* unless we identify functions that agree a.e.) The cut norm is clearly weaker than the L^1 norm:

$$\|W\|_\square \le \|W\|_{L^1[0,1]^2} := \int_0^1 \int_0^1 |W(u,v)|\, du dv. \tag{3}$$

The cut norm is convenient in part because many natural random graph models converge in cut norm but not in $L^1[0,1]^2$. We provide two examples where the convergence is well known.

Example 1. Let ξ^n be the adjacency matrix of an Erdős–Rényi random graph $G(n, p_n)$. If $p_n = p$ is fixed as $n \to \infty$, then W_{ξ^n} converges in cut norm to the constant graphon $W \equiv p$. Allowing unbounded kernels allows one to treat sparser regimes. Instead of assuming p_n to be constant, assume merely that $np_n \to \infty$ as $n \to \infty$. Then, W_{ξ^n/p_n} converges in cut norm to the constant graphon $W \equiv 1$. See Borgs et al. [11, theorem 2.14(b)] for a more general result.

Example 2. Given a graphon W (i.e., a symmetric and measurable function from $[0,1]^2$ to $[0, 1]$), one can define two natural graphs on the vertex set $[n]$. First, let $U_1, \ldots, U_n \sim \mathrm{Unif}[0, 1]$ be independent, and order them $U_{(1)} < \cdots < U_{(n)}$. Then, for $i \neq j$, either

1. connect vertices (i, j) with probability $W(U_i, U_j)$ or
2. assign weight $W(U_i, U_j)$ to the edge between (i, j).

Note that the latter defines a weighted graph and that the former defines a simple graph. The step kernel associated with the adjacency matrix converges in probability in cut norm to W in either case and in L^1 norm in the latter case. See Borgs et al. [11, theorem 2.14] for the proof along with related sparse graph constructions for kernels W, which are not necessarily bounded.

2.2. Operators Associated with Kernels

To a kernel $W \in L^1[0,1]^2$, we associate the operator $\mathsf{W} : L^\infty[0, 1] \to L^1[0, 1]$, defined by

$$\mathsf{W}\varphi(u) := \int_0^1 W(u,v)\varphi(v)\, dv. \tag{4}$$

The resulting operator norm is equivalent to the cut norm (Lovász [40, lemma 8.11]):

$$\|W\|_\square \le \|\mathsf{W}\|_{\infty\to 1} \le 4\|W\|_\square, \tag{5}$$

$$\text{where } \|\mathsf{W}\|_{\infty\to 1} := \sup\{\|\mathsf{W}\varphi\|_{L^1[0,1]} : \varphi \in L^\infty[0, 1],\ |\varphi| \le 1\}. \tag{6}$$

We work most often with the *strong operator topology* for operators on $L^\infty[0, 1] \to L^1[0, 1]$. We say that a sequence $W_n \in L^1[0,1]^2$ converges in the strong operator topology to $W \in L^1[0,1]^2$ if $\|\mathsf{W}_n\varphi - \mathsf{W}\varphi\|_{L^1[0,1]} \to 0$ for every $\varphi \in L^\infty[0, 1]$. Convergence in cut norm implies convergence in strong operator topology by (5). Although the cut norm is the most common in the graphon literature, working more generally with the strong operator topology leads to no increase in difficulty in any of our proofs.

A key object in our paper is a more general operator associated with a kernel $W \in L^1_+[0,1]^2$. Given a Polish space E and a probability measure m on $[0, 1] \times E$, we define a measure-valued function $\mathsf{W}m : [0, 1] \to \mathcal{M}_+(E)$ by

$$\mathsf{W}m(u) := \int_{[0,1]\times E} W(u,v)\delta_x\, m(dv, dx). \tag{7}$$

To be clear, this measure acts on a bounded measurable function $\varphi : E \to \mathbb{R}$ by

$$\langle \mathsf{W}m(u), \varphi\rangle = \int_{[0,1]\times E} W(u,v)\varphi(x)\, m(dv, dx).$$

Note that if $W \equiv 1$, then $\mathsf{W}m(u)$ is exactly the second marginal of m.

To foreshadow how we will use this operator, think of the measure $\mathsf{W}m(u)$ as representing a continuous version of the neighborhood empirical measure around a vertex u. Indeed, suppose $x_1, \ldots, x_n \in E$ represent state variables of players $1, \ldots, n$, and let $\xi = (\xi_{ij})$ denote an $n \times n$ matrix representing interactions. The influence of the other players on player i is given by the *neighborhood empirical measure*

$$M_i = \frac{1}{n}\sum_{j=1}^n \xi_{ij}\delta_{x_j} \in \mathcal{M}_+(E).$$

Suppose $u_1, \ldots, u_n \in [0,1]$ represent labels of the n players, with $u_i \in I_i^n$ for each i. The label-state empirical measure of the entire population is given by

$$M = \frac{1}{n}\sum_{i=1}^{n} \delta_{(u_i,x_i)} \in \mathcal{P}([0,1]\times E).$$

Using the step kernel from (2), the function $\mathsf{W}_\xi M$ then encodes all of the neighborhood empirical measures in terms of the label-state empirical measure in the sense that

$$\mathsf{W}_\xi M(u_i) = \frac{1}{n}\sum_{j=1}^{n} W_\xi(u_i,u_j)\delta_{x_j} = \frac{1}{n}\sum_{j=1}^{n} \xi_{ij}\delta_{x_j} = M_i.$$

Remark 1. The two operators both denoted W, defined in (4) to act on real-valued functions and in (7) to act on measures, are not as different as they might at first appear. First, note that the former definition extends readily to functions φ with values in suitable vector spaces. Suppose m has uniform first marginal so that by disintegration, we may write $m(du,dx) = du\, m_u(dx)$. We may then write $\mathsf{W}m(u) = \int_0^1 W(u,v)m_v\,dv$, which has the form of (4) but with the measure-valued function $m_\cdot$ in place of the scalar function φ.

Example 3 (Laplacian Matrices). A natural setting, studied for instance in Delarue [25] and Lacker and Soret [38], arises from the so-called *random walk Laplacian* of a connected graph on n vertices. Let us write $i \sim j$ if two vertices i and j are neighbors in this graph, and let d_i denote the degree (number of neighbors) of vertex i. Then, ξ is defined by setting $\xi_{ij} = n/d_i$ if $i \sim j$ and $\xi_{ij} = 0$ otherwise. In this case, $M_i = \frac{1}{d_i}\sum_{j\sim i}\delta_{x_j}$ is the uniform measure over the states of the neighbors of i.

3. Main Results

In this section, we define precisely the n-player and graphon game models. The following assumptions are in force throughout the paper.

3.1. Standing Assumptions

We are given dimensions $d, d_0 \in \mathbb{N}$; a time horizon $T > 0$; a compact metric space A representing the set of actions; and bounded continuous functions

$$\begin{aligned} &b : [0,T]\times\mathbb{R}^d\times A \to \mathbb{R}^d && \sigma : [0,T]\times\mathbb{R}^d \to \mathbb{R}^{d\times d_0}, \\ &f : [0,T]\times\mathbb{R}^d\times\mathcal{M}_+(\mathbb{R}^d)\times A \to \mathbb{R} && g : \mathbb{R}^d\times\mathcal{M}_+(\mathbb{R}^d) \to \mathbb{R}. \end{aligned}$$

Assume that σ is Lipschitz and that $\sigma\sigma^\top$ is uniformly nondegenerate (i.e., bounded from below in semidefinite order by a positive constant times the identity matrix). Assume further that for each $(t,x,m) \in [0,T]\times\mathbb{R}^d\times\mathcal{M}_+(\mathbb{R}^d)$, the following set is convex:

$$\{(b(t,x,a),z) : a \in A,\ z \le f(t,x,m,a)\} \subset \mathbb{R}^d\times\mathbb{R}. \tag{8}$$

Finally, we are given an initial distribution $\lambda \in \mathcal{P}_{\mathrm{Unif}}([0,1]\times\mathbb{R}^d)$.

These assumptions can certainly be generalized, particularly the boundedness. We prefer to minimize technicalities in order to focus on the new features of the graphon setting. The final convexity assumption is common in the control literature; it holds when A is a convex subset of a vector space, b is affine in a, and f is concave in a, which includes in particular the setting of *relaxed controls* to which one can always lift the problem if the convexity assumption is not initially satisfied (Lacker [35]).

The most notable restriction is that we do not include any interaction term within the functions b or σ. This significantly simplifies the existence theorem and the approximate equilibrium construction. The former would easily generalize, but the latter would require a satisfactory limit theory for graphon-based interacting stochastic differential equations (SDEs). Such a limit theory is a significant undertaking in its own right and has seen only very recent development, so far only for scalar interactions. By excluding interactions from (b,σ), we avoid this separate issue and focus more on the game-theoretic aspects of graphon models.

We work with Markovian controls throughout the paper, but the framework adapts easily to different kinds of controls, such as open loop.

3.2. Finite Games

Let $n \in \mathbb{N}$ denote the number of players. Each player may choose a control from $\mathcal{A}_n$, the set of measurable functions from $[0,T] \times (\mathbb{R}^d)^n \to A$. For any vector of controls $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathcal{A}_n^n$, there exists a unique solution $\boldsymbol{X}^n = (X^{n,1}, \ldots, X^{n,n})$ of the SDE system

$$dX_t^{n,i} = b(t, X_t^{n,i}, \alpha_i(t, \boldsymbol{X}_t^n))dt + \sigma(t, X_t^{n,i})dB_t^i, \qquad X_0^{n,i} = x_0^{n,i},$$

where $B^1, \ldots, B^n$ are independent d_0-dimensional Brownian motions and $x_0^{n,i}$ are given initial conditions.

The boundedness of b and Lipschitz continuity of σ ensure that this SDE system admits a unique strong solution (Veretennikov [49, theorem 1]).

Let $\xi^n = (\xi_{ij}^n)$ denote an $n \times n$ matrix with nonnegative entries, called the *interaction matrix*. Throughout this paper, we will assume that $\xi_{ii}^n = 0$ for all i; if ξ^n is the adjacency matrix of a (weighted) graph, this is equivalent to assuming that there are no self-loops. This assumption is natural and simplifies the exposition, but it is not hard to generalize. A key role is played by the *neighborhood empirical measures* defined for each player $i \in [n]$ by

$$M_t^{n,i} = \frac{1}{n} \sum_{j=1}^n \xi_{ij}^n \delta_{X_t^{n,j}}, \tag{9}$$

which is a random element of $\mathcal{M}_+(\mathbb{R}^d)$. For $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathcal{A}_n^n$, the objective function of each player $i \in [n]$ is defined by

$$J_i(\boldsymbol{\alpha}) := \mathbb{E}\left[\int_0^T f(t, X_t^{n,i}, M_t^{n,i}, \alpha_i(t, \boldsymbol{X}_t^n))\, dt + g(X_T^{n,i}, M_T^{n,i})\right]. \tag{10}$$

For $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_n) \in [0,\infty)^n$, an $\boldsymbol{\epsilon}$*-Nash equilibrium* is defined as any $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathcal{A}_n^n$ satisfying for all $i \in [n]$

$$J_i(\boldsymbol{\alpha}) \geq \sup_{\beta \in \mathcal{A}_n} J_i(\alpha_1, \ldots, \alpha_{i-1}, \beta, \alpha_{i+1}, \ldots, \alpha_n) - \epsilon_i.$$

We will not state any theorems about n-player games until Section 3.5, but it will inform our definition of the appropriate graphon model in the following section.

3.3. Graphon Games

For a kernel $W \in L_+^1[0,1]^2$, we define the *graphon game* associated with W as follows. Let $\mathcal{A}_U$ denote the set of measurable functions $[0,T] \times [0,1] \times \mathbb{R}^d \to A$. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space supporting a d_0-dimensional $\mathbb{F}$-Brownian motion B and $\mathcal{F}_0$-measurable random variables U and X_0 taking values in $[0, 1]$ and $\mathbb{R}^d$, respectively. The given joint law of (U, X_0) is denoted λ, and its first marginal is assumed to be uniform; that is, $U \sim \text{Unif}[0, 1]$. For $\alpha \in \mathcal{A}_U$, the state process X is the unique solution of the SDE

$$dX_t^\alpha = b(t, X_t^\alpha, \alpha(t, U, X_t^\alpha))dt + \sigma(t, X_t^\alpha)dB_t, \quad X_0^\alpha = X_0. \tag{11}$$

Strong well posedness of this SDE follows easily from Veretennikov [49, theorem 1] under our standing assumptions. Recall in the following the meaning of $\mathsf{W}\mu_t$, defined in (7), as well as the notation $\mathcal{P}_{\text{Unif}}([0, 1] \times E)$ for measures on $[0, 1] \times E$ with uniform first marginal.

Now, to define our notion of equilibrium, suppose we are given a measure flow $\mu_\cdot = (\mu_t)_{t \in [0,T]} \in C([0,T]; \mathcal{P}_{\text{Unif}}([0, 1] \times \mathbb{R}^d))$ representing the label-state joint distribution at each time. In response to this given $\mu_\cdot$, the objective of a typical player is to choose $\alpha \in \mathcal{A}_U$ to maximize

$$J_W(\mu_\cdot, \alpha) := \mathbb{E}\left[\int_0^T f(t, X_t^\alpha, \mathsf{W}\mu_t(U), \alpha(t, U, X_t^\alpha))dt + g(X_T^\alpha, \mathsf{W}\mu_T(U))\right].$$

The measure $\mathsf{W}\mu_t(U)$ here is the natural graphon analogue of the neighborhood empirical measure, as discussed in Section 2.2, when a player is given the uniformly random label U.

Definition 1. We say that $\mu_\cdot \in C([0,T]; \mathcal{P}_{\text{Unif}}([0, 1] \times \mathbb{R}^d))$ is a (*Markovian*) *W equilibrium* (or a *graphon equilibrium* when W is clear from context) if there exists $\alpha^* \in \mathcal{A}_U$ satisfying

$$J_W(\mu_\cdot, \alpha^*) = \sup_{\alpha \in \mathcal{A}_U} J_W(\mu_\cdot, \alpha) \quad \text{and} \quad \mu_t = \mathcal{L}(U, X_t^{\alpha^*}), \ \ \forall t \in [0,T].$$

Any such α^* is called an *equilibrium control for* $\mu_\cdot$.

We might describe this fixed point problem loosely but compactly as follows:

$$\begin{cases} \alpha^* \in \arg\max_{\alpha} \mathbb{E}\left[\int_0^T f(t, X_t^\alpha, \mathsf{W}\mu_t(U), \alpha_t)dt + g(X_T^\alpha, \mathsf{W}\mu_T(U))\right] \\ \quad \text{s.t. } dX_t^\alpha = b(t, X_t^\alpha, \alpha_t)dt + \sigma(t, X_t^\alpha)dB_t, \\ \quad \mu_t = \mathcal{L}(U, X_t^\alpha), \ (U, X_0) \sim \lambda. \end{cases} \tag{12}$$

For comparison, we also state the classical definition of a mean field game equilibrium in the case where there is no graphon present (or $W \equiv 1$). Note that the space $\mathcal{A}_1$ of measurable functions $[0,T] \times \mathbb{R}^d \to A$ may be identified with the subspace of $\mathcal{A}_U$ consisting of controls that do not depend on the uniform variable U (i.e., functions of the form $\alpha(t,u,x) = \widetilde{\alpha}(t,x)$). We say that $\nu_\cdot \in C([0,T];\mathcal{P}(\mathbb{R}^d))$ is a (*Markovian*) *mean field equilibrium* if there exists $\alpha^* \in \mathcal{A}_1$ satisfying

$$J_1(\nu_\cdot, \alpha^*) = \sup_{\alpha \in \mathcal{A}_1} J_1(\nu_\cdot, \alpha) \quad \text{and} \quad \nu_t = \mathcal{L}(X_t^{\alpha^*}) \quad \forall t \in [0,T],$$

where we define

$$J_1(\nu_\cdot, \alpha) := \mathbb{E}\left[\int_0^T f(t, X_t^\alpha, \nu_t, \alpha(t, X_t^\alpha))dt + g(X_T, \nu_T)\right].$$

When $W \equiv 1$, recall that $\mathsf{W}m$ reduces to the second marginal of $m(dv,dx)$; it follows that if $\mu_\cdot$ is a W equilibrium, then the second marginals form a mean field equilibrium. The converse is true but somewhat more subtle because controls for mean field equilibria are allowed to depend on the auxiliary random variable U. See Proposition 1 for a more general relationship between these two equilibrium concepts.

3.4. Existence and Uniqueness of Equilibria

Recall in the following that we are always working under the standing assumptions stated at the beginning of Section 3. The following is proven in Section 4, following the strategy of Lacker [35].

Theorem 1. *Let $W \in L^1_+[0,1]^2$. Then, there exists a W equilibrium.*

For certain W, a mean field equilibrium can be identified with a graphon equilibrium. This is clear when $W \equiv 1$, as noted, but in fact holds more generally.

Proposition 1. *Let $W \in L^1_+[0,1]^2$. Assume that*

$$\int_0^1 W(u,v)\, dv = 1, \quad a.e. \ u \in [0,1]. \tag{13}$$

Suppose $\nu_\cdot \in C([0,T];\mathcal{P}(\mathbb{R}^d))$ is a mean field equilibrium, and let $\alpha^* \in \mathcal{A}_1$ be an equilibrium control for $\nu_\cdot$. Define $\mu_t = \mathrm{Unif}[0,1] \times \nu_t$. Then, $\mu_\cdot = (\mu_t)_{t \in [0,T]}$ is a W equilibrium, and $(t,u,x) \longmapsto \alpha^*(t,x)$ is an equilibrium control for $\mu_\cdot$.

The condition (13) can be interpreted as saying that the graphon W has *constant out degree* or simply, *constant degree* if W is assumed symmetric. A similar principle appeared in the uncontrolled setting in Coppini [22, corollary 2.4].

Example 4. Let us revisit Example 3, where G_n is a simple connected graph on vertex set $[n]$ and $\xi_{ij}^n = (n/d_i)1_{\{i \sim j\}}$. The neighborhood empirical measures become

$$M_t^{n,i} = \frac{1}{n}\sum_{j=1}^n \xi_{ij}^n \delta_{X_t^{n,j}} = \frac{1}{d_i}\sum_{j \sim i} \delta_{X_t^{n,j}}.$$

This models a scenario in which players interact symmetrically with their neighbors in the underlying graph G_n, as in Delarue [25] and Lacker and Soret [38]. It is not clear if there is a simple (e.g., degree-based) characterization of the situations where W_{ξ^n} converges in the strong operator topology (or in cut norm). However, if a limit $W_{\xi^n} \to W$ does exist, then W must satisfy the constant-degree condition of Proposition 1. Indeed, for each $i \in [n]$ and each $u \in I_i^n$, we have

$$\int_0^1 W_{\xi^n}(u,v)dv = \sum_{j=1}^n \int_{I_j^n} W_{\xi^n}(u,v)dv = \sum_{j=1}^n \frac{1}{n}\xi_{ij}^n = \sum_{j=1}^n \frac{1}{d_i}1_{\{i\sim j\}} = 1,$$

and the left-hand side, as a function of u, converges in $L^1[0,1]$ to $\int_0^1 W(u,v)dv$.

We can further show uniqueness of the equilibrium under an additional assumption adapted from the classical Lasry–Lions monotonicity condition.

Proposition 2. *In addition to the standing assumptions of Section 3, assume the following.*

1. *Separable f. There exist two functions f_1, f_2 such that*

$$f(t,x,m,a) = f_1(t,x,a) + f_2(t,x,m).$$

2. *Unique optimal controls. For each $\mu \in C([0,T];\mathcal{P}_{\mathrm{Unif}}([0,1]\times\mathbb{R}^d)$, the supremum in $\sup_{\alpha\in\mathcal{A}_U} J_W(\mu,\alpha)$ is attained uniquely (up to Lebesgue a.e. equality).*

3. *Monotonicity. For each $m_1, m_2 \in \mathcal{P}_{\mathrm{Unif}}([0,1]\times\mathbb{R}^d)$ and $t\in[0,T]$, we have*

$$\int_{[0,1]\times\mathbb{R}^d} (g(x,\mathsf{W}m_1(u)) - g(x,\mathsf{W}m_2(u)))(m_1-m_2)(du,dx) \le 0$$
$$\int_{[0,1]\times\mathbb{R}^d} (f_2(t,x,\mathsf{W}m_1(u)) - f_2(t,x,\mathsf{W}m_2(u)))(m_1-m_2)(du,dx) \le 0. \tag{14}$$

Then, there exists a unique W equilibrium.

The proof is given in Section 4.6 along with a couple of noteworthy examples of functions g satisfying (14) (see Remark 11). The proof follows by reducing the graphon game to a classical mean field game, explained in more detail in Section 3.6.

3.5. Approximate Equilibria

Throughout this section, we are given $W \in L^1_+[0,1]^2$, and we let $\mu_\cdot \in C([0,T];\mathcal{P}_{\mathrm{Unif}}([0,1]\times\mathbb{R}^d))$ denote a W equilibrium and α^* an equilibrium control for $\mu_\cdot$. Also, as in Section 3.2, we are given an arbitrary $n\times n$ matrix ξ^n with positive entries and zeros on the diagonal, $\xi^n_{ii}=0$. We define the step kernel W_{ξ^n} as in (2).

In this section, we explain how the graphon game defined in Section 3.3 gives rise to approximate equilibria for the finite game defined in Section 3.2, when the underlying kernels W_{ξ^n} from (2) converge in a suitable sense to the kernel W. To provide context for the following results, let us briefly recall the analogous construction in mean field game theory. If $\widehat{\alpha}\in\mathcal{A}_1$ denotes a mean field equilibrium control, then players $i\in[n]$ in the n-player game are assigned the controls $\alpha^n_i(t,x_1,\ldots,x_n) = \widehat{\alpha}(t,x_i)$. The vector $(\alpha^n_1,\ldots,\alpha^n_n)$ is then shown to constitute an ϵ^n equilibrium, where $\epsilon^n\to 0$. This strategy dates back to the earliest work on mean field games (Huang et al. [32]); see Carmona and Delarue [18, section 6.1] or Lacker [37, section 2.4] for the closed-loop case.

This strategy requires several adaptations in the present context. First, because players are not exchangeable, we may have a different error ϵ^n_i for each player. Moreover, different modes of convergence to zero can make sense in different contexts, such as $\frac{1}{n}\sum_{i=1}^n \epsilon^n_i \to 0$ or $\max_{i\in[n]}\epsilon^n_i\to 0$. This is also highlighted in our case study (Lacker and Soret [38]).

A second and more delicate point in our setting is in how to deal with labels. A W-equilibrium control $\alpha^*\in\mathcal{A}_U$ depends on an additional Unif[0, 1] variable, which we have interpreted as the label (or vertex) of the player. In order to apply this control α^* in the n-player game, we must specify which labels to assign to each player. In the definition of the step kernel W_{ξ^n}, the player i in the n-player game is associated with the interval I^n_i defined in (2), and it thus makes sense to choose for player i some label $u^n_i\in I^n_i$. We then assign to player i the control

$$\alpha^{n,u^n_i}_i(t,x_1,\ldots,x_n) := \alpha^*(t,u^n_i,x_i). \tag{15}$$

The error $\epsilon^n_i(\boldsymbol{u}^n)$ then depends additionally on the choice of labels $\boldsymbol{u}^n = (u^n_1,\ldots,u^n_n)$, and the question again arises as to the sense in which we can expect these errors to vanish as $n\to\infty$. In general, we only expect these errors to vanish in probability, with respect to a random choice of $\boldsymbol{u}^n$, but we will see that stronger continuity assumptions allow us to strengthen the convergence to be (essentially) uniform in the choice of $\boldsymbol{u}^n$.

Let us define precisely the function $\epsilon^n_i:[0,1]^n\to[0,\infty)$. Fix $\boldsymbol{u}^n=(u^n_1,\ldots,u^n_n)\in[0,1]^n$ in this paragraph. Using the construction (15), define $\boldsymbol{\alpha}^{n,\boldsymbol{u}^n} = (\alpha^{n,u^n_1}_1,\ldots,\alpha^{n,u^n_n}_n)\in\mathcal{A}^n_n$. Recall that $\lambda(du,dx)=du\lambda_u(dx)$ denotes the given joint law of (U,X_0) in the graphon game. Consider the n-player game as described in Section 3.2, with initial conditions $(X^{n,i}_0)_{i=1}^n$ chosen independently with $X^{n,i}_0\sim\lambda_{u^n_i}$. With this choice of initialization, we finally define the nonnegative number

$$\epsilon^n_i(\boldsymbol{u}^n) := \sup_{\beta\in\mathcal{A}_n} J_i\left(\alpha^{n,u^n_1}_1,\ldots,\alpha^{n,u^n_{i-1}}_{i-1},\beta,\alpha^{n,u^n_{i+1}}_{i+1},\ldots,\alpha^{n,u^n_n}_n\right) - J_i(\boldsymbol{\alpha}^{n,\boldsymbol{u}^n}).$$

By definition, $\boldsymbol{\alpha}^{n,\boldsymbol{u}^n}$ is a $\boldsymbol{\epsilon}^n(\boldsymbol{u}^n)$ equilibrium, where $\boldsymbol{\epsilon}^n(\boldsymbol{u}^n) = (\epsilon_1^n(\boldsymbol{u}^n), \dots, \epsilon_n^n(\boldsymbol{u}^n))$. This definition makes sense only if we prespecify a version of the disintegration $u \mapsto \lambda_u$, and otherwise, we should understand $\boldsymbol{\epsilon}^n(\boldsymbol{u}^n)$ to be uniquely defined only up to $\boldsymbol{u}^n$-a.e. equality.

We first show in full generality that $\boldsymbol{\epsilon}^n \to 0$ in an averaged sense. Recall from Section 2.2 the definition of the strong operator topology for operators from $L^\infty[0,1]$ to $L^1[0,1]$, and recall that convergence in this topology is implied by convergence in cut norm.

Theorem 2 (General Kernel). *Assume the disintegration $u \mapsto \lambda_u$ admits a version such that $\{\lambda_u : u \in [0,1]\}$ is tight. Assume W_{ξ^n} converges in the strong operator topology to W, and also,*

$$\lim_{n\to\infty} \frac{1}{n^3} \sum_{i,j=1}^n (\xi_{ij}^n)^2 = 0. \tag{16}$$

Then, if for each $n \in \mathbb{N}$, $(U_1^n, \dots, U_n^n)$ are independent with $U_i^n \sim \mathrm{Unif}(I_i^n)$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n \mathbb{E}\left[\epsilon_i^n(U_1^n, \dots, U_n^n)\right] = 0.$$

The proof of Theorem 2 is given in Section 7 along with the proofs of the two other theorems of this section. The bulk of the analysis is presented first in Section 6 in a more general setting that clarifies the key points.

Remark 2. The assumption (16) is very mild. It holds trivially if $|\xi_{ij}^n|$ are uniformly bounded. If ξ^n is $1/p_n$ times the adjacency matrix of the Erdős–Rényi graph $G(n, p_n)$, then (16) is easily shown to hold in probability when $np_n \to \infty$.

Remark 3. We have assumed f and g to be bounded, which means ϵ_i^n are uniformly bounded. Hence, the conclusion of Theorem 2 is equivalent to saying that $\epsilon_{I_n}^n(U_1^n, \dots, U_n^n) \to 0$ in probability, where $I_n \sim \mathrm{Unif}([n])$. In other words, for randomly assigned labels from $I_1^n \times \cdots \times I_n^n$ and for a randomly chosen player from $[n]$, the error is small. Note that this does not rule out the possibility that certain players and label assignments have large errors ϵ_i^n, but the fraction of such players and label assignments is negligible.

Our next result strengthens the mode of convergence at the price of requiring stronger continuity assumptions both on the graphon and on the optimal state process. Recall in the following that $\mathcal{C}^d = C([0,T]; \mathbb{R}^d)$, and (U, X^{α^*}) is defined as in Section 3.3.

Theorem 3 (Continuous Kernel). *Assume the following.*

1. *The map $[0,1] \ni u \longmapsto W(u,v)dv \in \mathcal{M}_+([0,1])$ is continuous.*
2. *The disintegration $[0,1] \ni u \longmapsto \mathcal{L}(X^{\alpha^*} \mid U = u) \in \mathcal{P}(\mathcal{C}^d)$ admits a continuous version.*

Assume that (16) holds and that W_{ξ^n} converges in the strong operator topology to W. Then,

$$\lim_{n\to\infty} \operatorname*{esssup}_{\boldsymbol{u}^n \in I_1^n \times \cdots \times I_n^n} \frac{1}{n} \sum_{i=1}^n \epsilon_i^n(\boldsymbol{u}^n) = 0.$$

Moreover, if assumption (1) *holds, then assumption* (2) *holds under the following additional conditions.*

- 2a. *The disintegration $[0,1] \ni u \longmapsto \lambda_u \in \mathcal{P}(\mathbb{R}^d)$ admits a continuous version.*
- 2b. *A is a compact convex subset of $\mathbb{R}^k$ for some $k \in \mathbb{N}$.*
- 2c. *$\sigma(t,x) = \sigma$ is constant.*
- 2d. *For each (t,x), $a \longmapsto b(t,x,a)$ is affine, and $a \longmapsto f(t,x,m,a)$ is strictly concave.*

To be clear, the two continuity assumptions in Theorem 3 mean that $\int_0^1 W(u,v)h(v)\,dv$ and $\mathbb{E}[\varphi(X^{\alpha^*}) \mid U = u]$ depend continuously on u for all bounded continuous real-valued functions h and φ on $[0,1]$ and $\mathcal{C}^d$, respectively. In particular, Theorem 3(1) is true if the function $W : [0,1]^2 \to \mathbb{R}$ is itself continuous. These continuity assumptions allow a finer pointwise control over quantities derived from the graphon, ensuring, for instance, that the quantities $\int_0^1 W(u_i^n, v)h(v)\,dv$ and $\mathbb{E}\int_0^1 W(U_i^n, v)h(v)\,dv$ are close, uniformly in the choice of $u_i^n \in I_i^n$, with again $U_i^n \sim \mathrm{Unif}(I_i^n)$. Stronger continuity assumptions on W were used in Bayraktar et al. [4, 5] and Tangpi and Zhou [47].

Remark 4. Assumption (2) in Theorem 3 can be difficult to check, which is why we provide the more tractable sufficient conditions (2a)–(2d). However, assumption (2) is actually automatic in the context of Proposition 1, as $\mathcal{L}(X^{\alpha^*} | U = u) = \mathcal{L}(X^{\alpha^*})$ is constant in u. For an alternative sufficient condition, it is not hard to show that if (1) and (2a) hold and if the control $\alpha^*(t,u,x)$ depends continuously on (u,x) for each t, then assumption (2) holds.

Remark 5. Analogously to Remark 3, the conclusion of Theorem 3 is equivalent to the following. For every $\epsilon, \delta \in (0, 1)$, it holds for sufficiently large n that

$$|\{i \in [n] : \epsilon_i^n(\boldsymbol{u}^n) > \epsilon\}| \le n\delta, \quad \text{for a.e. } \boldsymbol{u}^n \in I_1^n \times \cdots \times I_n^n.$$

In other words, for large enough n and for a.e. choice of labels, we have an (ϵ, δ) equilibrium in the sense of Carmona [16] (also used in Cui and Koeppl [24]); no more than a fraction of δ of the players is further than ϵ from optimality.

Remark 6. Our approximate equilibrium results can be combined with Proposition 1 to yield interesting results on the "universality" of the mean field game approximation. If $W \in L^1_+[0,1]^2$ satisfies (13) and if $W_{\xi^n} \to W$ in the strong operator topology, then a mean field equilibrium (as opposed to a graphon equilibrium) can be used in Theorem 2 to construct approximate equilibria for the n-player games. This justifies the intuition mentioned in Section 1 that the usual MFG approximation remains valid for sufficiently *dense* and *approximately regular* networks. Note as in Remark 4 that condition (2) of Theorem 3 holds automatically in this case; hence, if also $u \longmapsto W(u, v)dv \in \mathcal{M}_+([0, 1])$ is continuous (e.g., if $W \equiv 1$), then we can also apply Theorem 3 as well.

Our final result on approximate equilibria deals with the case where the interaction matrix ξ^n is the weighted adjacency matrix obtained by sampling from the graphon W in a standard manner, as in Example 2(2).

Theorem 4 (Sampling Kernel). *Let $W \in L^1_+[0,1]^2$ be bounded. Assume the disintegration $u \longmapsto \lambda_u$ admits a version such that $\{\lambda_u : u \in [0, 1]\}$ is tight. Then, the following holds for almost every choice of $(u_i)_{i\in\mathbb{N}} \in [0,1]^\infty$, where $[0,1]^\infty$ is equipped with the infinite product measure* $(\mathrm{Unif}[0,1])^\infty$. *Set $\xi_{ij}^n = W(u_i, u_j)1_{i\neq j}$ for $i, j \in [n]$ in the n-player game. Then,*

$$\lim_{n\to\infty} \max_{i\in[n]} \epsilon_i^n(u_1, \ldots, u_n) \to 0.$$

Remark 7. Our connection between the initial conditions $(X_0^{n,i})_{i=1}^n$ and the initial distribution λ covers many natural cases. If $(X_0^{n,i})$ are taken to be i.i.d. $\sim \lambda \in \mathcal{P}(\mathbb{R}^d)$ as is common in the MFG literature, then we can simply choose $\lambda(du, dx) = du\lambda(dx)$.

In general, the initial conditions $X_0^{n,i} \sim \lambda_{u_i^n}$ may be different for each player, although we do still require them to be independent. For another example, a player with label u could have a nonrandom initial position $h(u)$ for some measurable function $h : [0, 1] \to \mathbb{R}^d$, in which case the natural choice is $\lambda(du, dx) = du\delta_{h(u)}(dx)$.

It is natural to expect more general results to be possible, in which we assume merely that the initial empirical measure $\frac{1}{n}\sum_{i=1}^n \delta_{(i/n, X_0^{n,i})}$ converges weakly to λ.

Remark 8. Another approach to justifying our graphon game formulation would be by studying the *convergence problem* (i.e., the problem of analyzing the $n \to \infty$ behavior of the *true* n-player equilibria rather than constructing specific *approximate* equilibria). We do not address this problem in this paper, which was already a difficult problem in mean field game theory (Cardaliaguet et al. [15], Lacker [37]), although we mention the very recent papers (Bayraktar et al. [5], Tangpi and Zhou [47]), which obtain the first results in this direction.

3.6. Graphon Games as Mean Field Games and Their PDE Formulation

This section contains no theorems but illustrates how to recast the graphon equilibrium problem of Section 3.3 as a classical mean field game. The point is simply to view the "label" variable as a state variable with trivial dynamics. For $\alpha \in \mathcal{A}_U$, the $(d+1)$-dimensional process $\overline{X}^\alpha = (U, X^\alpha)$ is the unique solution of the SDE

$$d\overline{X}_t^\alpha = \overline{b}(t, \overline{X}_t^\alpha, \alpha(t, \overline{X}_t^\alpha))\, dt + \overline{\sigma}(t, \overline{X}_t^\alpha)\, dB_t, \tag{17}$$

where $\overline{b} : [0, T] \times \mathbb{R}^{d+1} \times A \to \mathbb{R}^{d+1}$ and $\overline{\sigma} : [0, T] \times \mathbb{R}^{d+1} \to \mathbb{R}^{(d+1)\times d_0}$ are defined by

$$\overline{b}(t, \overline{x}, a) = \begin{pmatrix} 0 \\ b(t, x, a) \end{pmatrix}, \quad \overline{\sigma}(t, \overline{x}) = \begin{pmatrix} 0_{d_0}^\top \\ \sigma(t, x, a) \end{pmatrix},$$

where we write $\overline{x} = (u, x)$ for a generic element of $\mathbb{R}^{d+1} \cong \mathbb{R} \times \mathbb{R}^d$. That is, the vector $\overline{b}$ and matrix $\overline{\sigma}$ simply append an additional zero row. Similarly, define $\overline{f} : [0, T] \times \mathbb{R}^{d+1} \times \mathcal{P}(\mathbb{R}^{d+1}) \times A \to \mathbb{R}$ and $\overline{g} : \mathbb{R}^{d+1} \times \mathcal{P}(\mathbb{R}^{d+1}) \to \mathbb{R}$ by

$$\overline{f}(t, \overline{x}, m, a) = f(t, x, \mathsf{W}m(u), a), \quad \overline{g}(\overline{x}, m) = g(x, \mathsf{W}m(u)).$$

(Define $\overline{f}$ and $\overline{g}$ arbitrarily when $u \notin [0, 1]$.) The graphon equilibrium problem is then nothing but the standard mean field game problem associated with the new coefficients $(\overline{b}, \overline{\sigma}, \overline{f}, \overline{g})$. Indeed, a graphon equilibrium is a measure flow $(\mu_t)_{t\in[0,T]}$ such that there exists $\alpha^* \in \mathcal{A}_U$ satisfying $\mu_t = \mathcal{L}(\overline{X}_t^{\alpha^*})$ for all $t \in [0,T]$ as well as

$$\mathbb{E}\left[\int_0^T \overline{f}(t, \overline{X}_t^\alpha, \mu_t, \alpha(t, \overline{X}_t^\alpha))\,dt + \overline{g}(\overline{X}_T^\alpha, \mu_T)\right] = \sup_{\beta\in\mathcal{A}_U} \mathbb{E}\left[\int_0^T \overline{f}(t, \overline{X}_t^\beta, \mu_t, \beta(t, \overline{X}_t^\beta))\,dt + \overline{g}(\overline{X}_T^\beta, \mu_T)\right].$$

It must be stressed that recasting the graphon equilibrium problem as a classical mean field game in this manner does not significantly simplify its analysis (except in the proof of uniqueness, Proposition 2). There are several reasons that existing theory cannot be applied directly in this framework.

• The kernel $W : [0,1]^2 \to \mathbb{R}$ is not a continuous function in general. It is in some cases, but in many interesting cases, it is not (e.g., the stochastic block model). If W is discontinuous, then $\mathsf{W}\mu_t(u)$ is discontinuous in u, and thus, the objective functions $\overline{f}$ and $\overline{g}$ are discontinuous functions of the state variable $\overline{x}$.

• In the analysis of approximate equilibria, the natural n-player game of Section 3.2 is not equivalent to the one obtained by plugging empirical measures into the objective functions $(\overline{f}, \overline{g})$. The graphon is different in the n-player game, being W_{ξ^n} instead of W, and this makes our convergence analysis more difficult.

• The diffusion matrix $\overline{\sigma}\overline{\sigma}^\top$ of the $(d+1)$-dimensional process $\overline{X}$ is always degenerate, even if that of the original d-dimensional state process X is not.

Although it does not help with our analysis, recasting the graphon model as a mean field game does reveal what the appropriate PDE formulation should be in the spirit of Lasry and Lions [39]. (Similarly, a forward-backward SDE formulation in the spirit of Carmona and Delarue [17, 18] is possible as well, but we omit it here.) Indeed, taking σ to be the identity matrix for simplicity, the value function $v(t,u,x)$ and density flow $\mu(t,u,x)$ should (formally) obey the PDE system

$$\begin{cases} 0 = \partial_t v(t,u,x) + \sup_{a\in A}[b(t,x,a)\cdot\nabla_x v(t,u,x) + f(t,x,\mathsf{W}\mu_t(u),a)] + \frac{1}{2}\Delta_x v(t,u,x) \\ \partial_t \mu(t,u,x) = -\operatorname{div}_x(b(t,x,\widehat{\alpha}(t,u,x))\mu(t,u,x)) + \frac{1}{2}\Delta_x \mu(t,u,x) \\ \text{where } \widehat{\alpha}(t,u,x) = \arg\max_{a\in A}[b(t,x,a)\cdot\nabla_x v(t,u,x) + f(t,x,\mathsf{W}\mu_t(u),a)], \\ \text{and } v(T,u,x) = g(x,\mathsf{W}\mu_T(u)), \quad \mu_0 = \lambda. \end{cases}$$

Notably, there are no derivatives with respect to u. We will not claim to perform any rigorous analysis of this PDE system. However, it is worth noting that a verification theorem for classical solutions only requires v to be once differentiable in t and twice in x, and no differentiability with respect to u is needed. This observation will be used implicitly in our linear-quadratic example in Section 8. Lastly, we mention that the system of PDEs could be formally interpreted as a continuum of conditional measure flows $((t,x)\longmapsto \mu(t,u,x))_{u\in[0,1]}$, which is similar in spirit to the PDE systems discussed in Caines and Huang [13].

3.7. Organization of the Paper

The remaining sections give the proofs of the main theorems with the exception of Section 8, which works out a linear-quadratic example. Section 4 proves the existence and uniqueness as stated in Section 3.4 and may be read independently of Sections 5–7, which deal with approximate equilibria. Similarly, the linear-quadratic example of Section 8 is independent of Sections 4–7. Sections 5 and 6 provide preliminary results for the proofs of Section 7, namely the dependence of the optimal control on the labeling and the convergence of neighborhood empirical measures under various assumptions, respectively. Section 7 is devoted to the proofs of the theorems of Section 3.5.

4. Existence of Graphon Equilibria

This section proves Theorem 1 by adapting the strategy of Lacker [35]. In particular, we will make use of the notion of *relaxed controls* developed in Section 4.2. In this section, we fix a graphon $W \in L^1_+[0,1]^2$. Note that W is not necessarily bounded.

4.1. Continuity of the W Operator

First, we compile some essential continuity properties of the operator W defined in (7). These results will be useful in more general forms, so we work here with a Polish space E, which will later be either $E = \mathbb{R}^d$ or the path space $E = \mathcal{C}^d = C([0,T];\mathbb{R}^d)$. Recall that $\mathcal{P}_{\text{Unif}}([0,1]\times E)$ is the set of probability measures on $[0,1]\times E$ with uniform first marginal, endowed with the topology of weak convergence.

We first recall a well-known fact that continuity assumptions for test functions can be relaxed when dealing with weak convergence of joint distributions with a common marginal.

Lemma 1 (Beiglböck and Lacker [6, lemma 2.1]). *Suppose $h : [0, 1] \times E \to \mathbb{R}$ is bounded and measurable, with $h(u, \cdot)$ continuous on E for a.e. $u \in [0, 1]$. Then, $\mathcal{P}_{\text{Unif}}([0, 1] \times E) \ni \mu \longmapsto \langle \mu, h\rangle$ is continuous.*

The next lemma is the main result of this section. Part (2) will not be needed but is illustrative and not much longer to prove.

Lemma 2. *The following continuity properties hold.*

1. *For a.e. $u \in [0, 1]$, the following map is continuous:*

$$\mathcal{P}_{\text{Unif}}([0, 1] \times E) \ni \mu \longmapsto \mathsf{W}\mu(u) \in \mathcal{M}_+(E).$$

2. *Suppose that the map $[0, 1] \ni u \longmapsto W(u, \cdot) \in (L^1[0, 1], \text{weak})$ is continuous. Then, for each $\mu \in \mathcal{P}_{\text{Unif}}([0, 1] \times E)$ and each bounded measurable function $\varphi : E \to \mathbb{R}$, the map $u \longmapsto \langle \mathsf{W}\mu(u), \varphi\rangle$ is continuous.*

3. *Suppose that the map $[0, 1] \ni u \longmapsto W(u, v)dv \in \mathcal{M}_+([0, 1])$ is continuous. Suppose $\mu \in \mathcal{P}_{\text{Unif}}([0, 1] \times E)$ is such that there exists a version of the disintegration $u \longmapsto \mu_u$, which is continuous. Then, the following map is continuous:*

$$[0, 1] \ni u \longmapsto \mathsf{W}\mu(u) \in \mathcal{M}_+(E).$$

Proof. We first prove Lemma 2(1). Because $W \in L^1[0,1]^2$, it holds by Fubini's theorem that $W(u, \cdot) \in L^1[0, 1]$ for a.e. $u \in [0, 1]$. Fix such a u as well as a bounded continuous $\varphi : \mathbb{R}^d \to \mathbb{R}$. Write

$$\langle \mathsf{W}\mu(u), \varphi\rangle = \int_{[0,1]\times\mathbb{R}^d} W(u, v)\varphi(x)\, \mu(dv, dx) = \int_{\mathbb{R}} x\, F_\mu(dx),$$

where F_μ is the image of μ under the map $(v, x) \longmapsto W(u, v)\varphi(x)$. We first claim that

$$\mathcal{P}_{\text{Unif}}([0, 1] \times \mathbb{R}^d) \ni \mu \longmapsto F_\mu \in \mathcal{P}(\mathbb{R})$$

is continuous. To see this, note for bounded continuous $h : \mathbb{R} \to \mathbb{R}$ that

$$\langle F_\mu, h\rangle = \int_{[0,1]\times\mathbb{R}^d} h(W(u, v)\varphi(x))\, \mu(dv, dx).$$

The bounded function $h(W(u, v)\varphi(x))$ depends continuously on x and measurably on v, and it follows from Lemma 1 that $\mu \longmapsto \langle F_\mu, h\rangle$ is continuous on $\mathcal{P}_{\text{Unif}}([0, 1] \times \mathbb{R}^d)$. To finally deduce that $\int_{\mathbb{R}} x F_\mu(dx)$ depends continuously on $\mu \in \mathcal{P}_{\text{Unif}}([0, 1] \times \mathbb{R}^d)$, simply note that we have the uniform integrability bound

$$\begin{aligned}\int_{\mathbb{R}} |x| 1_{\{|x|\ge r\}}\, F_\mu(dx) &= \int_{[0,1]\times\mathbb{R}^d} |W(u, v)\varphi(x)| 1_{\{|W(u,v)\varphi(x)|\ge r\}}\, \mu(dv, dx)\\ &\le \|\varphi\|_\infty \int_0^1 |W(u, v)| 1_{\{|W(u,v)|\ge r/\|\varphi\|_\infty\}}\, dv\end{aligned}$$

for any $r > 0$, which tends to zero as $r \to \infty$ because $W(u, \cdot) \in L^1[0, 1]$.

To prove Lemma 2(2), fix $\mu \in \mathcal{P}_{\text{Unif}}([0, 1] \times E)$. Let $u_n \to u$ in $[0, 1]$, and let $\varphi : E \to \mathbb{R}$ be bounded and continuous. Let $\psi(v) = \mathbb{E}[\varphi(X) \mid U = v]$, for $(U, X) \sim \mu$. Then, $\psi \in L^\infty[0, 1]$, and so, the weak convergence of $W(u_n, \cdot) \to W(u, \cdot)$ in $L^1[0, 1]$ implies

$$\begin{aligned}\langle \mathsf{W}\mu(u_n), \varphi\rangle &= \int_{[0,1]\times E} W(u_n, v)\varphi(x)\, \mu(dv, dx) = \int_0^1 W(u_n, v)\psi(v)\, dv\\ &\to \int_0^1 W(u, v)\psi(v)\, dv = \langle \mathsf{W}\mu(u), \varphi\rangle.\end{aligned}$$

The proof of Lemma 2(3) is similar to that of Lemma 2(2), except that we must simply note that ψ is continuous in order to justify the convergence. □

4.2. The Relaxed Formulation

A relaxed control is a measure on $[0, T] \times A$ with the first marginal equal to the Lebesgue measure. We will denote $\mathcal{V}$ the set of relaxed controls, equipped with the topology of weak convergence, which makes $\mathcal{V}$ a compact (because A is compact) metric space. For each $q \in \mathcal{V}$, we can identify the measurable map $t \longmapsto q_t \in \mathcal{P}(A)$ that arises from the disintegration $q(dt, da) = dtq_t(da)$, and that is unique up to (Lebesgue) almost everywhere equality. *Strict controls* are relaxed controls $q \in \mathcal{V}$ of the form $q_t = \delta_{\alpha(t)}$ for a.e. t for some measurable $\alpha : [0, T] \to A$.

We will work in this section on the space $\Omega := \mathcal{V} \times [0, 1] \times \mathcal{C}^d$. This Polish space is endowed with its Borel σ field. In the following, a generic element of Ω is denoted (q, u, x), and the coordinate maps on $\mathcal{V}$, $[0, 1]$, and $\mathcal{C}^d$ are denoted Λ, U, and X, respectively. The canonical filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is defined by letting $\mathcal{F}_t$ denote the σ field generated by $\Lambda|_{[0,t]\times A}$, U and $(X_s)_{s\in[0,t]}$.

Let $C_c^\infty(\mathbb{R}^d)$ denote the set of infinitely differentiable functions $\varphi : \mathbb{R}^d \to \mathbb{R}$ with compact support, and let $\nabla\varphi$ and $\nabla^2\varphi$ denote the gradient and the Hessian of φ, respectively. Define the generator L on $\varphi \in C_c^\infty(\mathbb{R}^d)$ by

$$L\varphi(t,x,a) := b(t,x,a)\cdot\nabla\varphi(x) + \frac{1}{2}\mathrm{Tr}\big(\sigma\sigma^\top(t,x)\nabla^2\varphi(x)\big),$$

for $(t,x,a) \in [0,T]\times\mathbb{R}^d\times A$. For $\varphi \in C_c^\infty(\mathbb{R}^d)$, we define a process $N_t^\varphi : \Omega \to \mathbb{R}$ by

$$N_t^\varphi(q,u,x) := \varphi(x_t) - \int_{[0,t]\times A} L\varphi(s,x_s,a)\,q(ds,da), \qquad t\in[0,T].$$

The set of admissible laws $\mathcal{R}$ is defined as the set of $P \in \mathcal{P}(\Omega)$ satisfying

1. $P\circ(U,X_0)^{-1} = \lambda$ and
2. for each $\varphi \in C_c^\infty(\mathbb{R}^d)$, the process $(N_t^\varphi)_{t\in[0,T]}$ is a P martingale.

This gives a martingale problem formulation in the spirit of Stroock and Varadhan [44] for the controlled state processes in (11).

For $\mu \in \mathcal{P}_{\mathrm{Unif}}([0, 1]\times\mathcal{C}^d)$ representing the fixed population distribution, we write μ_t for the marginal obtained as the image by $(u,x)\longmapsto(u,x_t)$, and we define a random variable $\Gamma^\mu : \Omega\to\mathbb{R}$ by

$$\Gamma^\mu(q,u,x) := g(x_T, \mathsf{W}\mu_T(u)) + \int_{[0,T]\times A} f(t,x_t,\mathsf{W}\mu_t(u),a)\,q(dt,da). \tag{18}$$

Remark 9. Recalling the notation of Section 3.3, if $\alpha\in\mathcal{A}_U$, then $dt\delta_{\alpha(t,U,X_t^\alpha)}(da)$ is a random element of $\mathcal{V}$, and the joint law P^α of $(dt\delta_{\alpha(t,U,X_t^\alpha)}(da), U, X^\alpha)$ defines an element of $\mathcal{R}$. Indeed, the condition $(U, X_0^\alpha)\sim\lambda$ was imposed in Section 3.3, and the defining martingale property (2) of $\mathcal{R}$ follows immediately from Itô's formula. Unpacking the notation, it holds also that

$$J_W(\mu, P^\alpha) = \langle P^\alpha, \Gamma^\mu\rangle. \tag{19}$$

Given $\mu\in\mathcal{P}_{\mathrm{Unif}}([0, 1]\times\mathcal{C}^d)$, a single player's objective is to find

$$\mathcal{R}^*(\mu) := \operatorname*{arg\,max}_{P\in\mathcal{R}}\ \langle P,\Gamma^\mu\rangle := \{P\in\mathcal{R} : \langle P,\Gamma^\mu\rangle \ge \langle Q,\Gamma^\mu\rangle\ \ \forall Q\in\mathcal{R}\}.$$

Our first goal will be to prove the existence of what one might naturally call a *relaxed W equilibrium* defined as a fixed point of the set-valued map $\Phi : \mathcal{P}_{\mathrm{Unif}}([0, 1]\times\mathcal{C}^d)\to 2^{\mathcal{P}_{\mathrm{Unif}}([0,1]\times\mathcal{C}^d)}$ given by

$$\Phi(\mu) := \{P\circ(U,X)^{-1} : P\in\mathcal{R}^*(\mu)\}.$$

That is, a *relaxed W equilibrium* is any $\mu\in\mathcal{P}_{\mathrm{Unif}}([0, 1]\times\mathcal{C}^d)$ satisfying $\mu\in\Phi(\mu)$. We will first prove the existence of such a fixed point in Proposition 3, and then, we will show how to turn it into a true W equilibrium in the sense of Section 3.3.

4.3. Existence of Relaxed Equilibrium

The goal of this section is to prove the following.

Proposition 3. *There exists* $\mu\in\mathcal{P}_{\mathrm{Unif}}([0, 1]\times\mathcal{C}^d)$ *such that* $\mu\in\Phi(\mu)$.

To do so, we will use the following lemma on continuity.

Lemma 3. *The following map is jointly continuous*:

$$\mathcal{P}_{\mathrm{Unif}}([0, 1]\times\mathcal{C}^d)\times\mathcal{R}\ni(\mu,P)\longmapsto\langle P,\Gamma^\mu\rangle\in\mathbb{R}.$$

Proof. Lemma 2(1) and the boundedness and continuity of g and f imply that $\Gamma^\mu(q,u,x)$ is a continuous function of (q,x,μ) for a.e. $u\in[0,1]$; see Lacker [35, corollary A.5]. In addition, Γ^μ is measurable on Ω. The claim follows by applying Lemma 1 with $E = \mathcal{V}\times\mathcal{C}^d$, noting that $\mathcal{R}$ can be viewed as a subset of $\mathcal{P}_{\mathrm{Unif}}([0, 1]\times E)$. □

Proof of Proposition 3. We will apply the Kakutani–Fan–Glicksberg fixed point theorem (Fan [28, theorem 1]), which requires that we identify a nonempty compact convex set $K\subset\mathcal{P}([0, 1]\times\mathcal{C}^d)$ such that

1. $\Phi(\mu) \subset K$ for each $\mu \in K$,
2. $\Phi(\mu)$ is nonempty and convex for each $\mu \in K$, and
3. the graph $\{(\mu, \mu') : \mu \in K, \mu' \in \Phi(\mu)\}$ is closed.

A good choice turns out to be $K := \{P \circ (U, X)^{-1} : P \in \mathcal{R}\}$. Property (1) is then clearly satisfied because $\mathcal{R}^*(\mu) \subset \mathcal{R}$ for all μ.

Let us prove that K is compact and convex, as it is easily seen to be nonempty. First, note that $\mathcal{R}$ is the set of $P \in \mathcal{P}(\Omega)$ satisfying $P \circ (U, X_0)^{-1} = \lambda$ and

$$\langle P, h(N_t^{\varphi} - N_s^{\varphi})\rangle = 0, \tag{20}$$

for all $T \ge t > s \ge 0$, $\varphi \in C_c^{\infty}(\mathbb{R}^d)$, and bounded continuous $\mathcal{F}_s$-measurable functions h (which generate the σ field $\mathcal{F}_s$). This shows clearly that $\mathcal{R}$ is convex, and thus, so is K. To see that $\mathcal{R}$ is closed, note that the continuity of (b, σ) ensures that $L\varphi$ is jointly continuous, and thus, so is $N_t^{\varphi} : \Omega \to \mathbb{R}$ by Lacker [35, corollary A.5]. It follows that (20) is closed under weak limits, and so, $\mathcal{R}$ is a closed set. To see that $\mathcal{R}$ is precompact, we note easily that it is tight because $[0, 1] \times \mathcal{V}$ is compact and because $\{P \circ X^{-1} : P \in \mathcal{R}\}$ is easily seen to be tight as a consequence of the boundedness of (b, σ) (e.g., by Stroock and Varadhan [44, theorem 1.4.6]). The compactness of K follows from compactness of $\mathcal{R}$ because the map $P \longmapsto P \circ (U, X)^{-1}$ is continuous.

Next, for each $\mu \in K$, note that $\mathcal{R}^*(\mu)$ is nonempty as a consequence of the continuity of $P \longmapsto \langle P, \Gamma^{\mu}\rangle$ from Lemma 3 and the compactness of $\mathcal{R}$ shown; it follows that $\Phi(\mu)$ is also nonempty. Convexity of $\mathcal{R}^*(\mu)$ follows from the linearity of $P \longmapsto \langle P, \Gamma^{\mu}\rangle$ and the convexity of $\mathcal{R}$. In turn, convexity of $\Phi(\mu)$ follows from convexity of $\mathcal{R}^*(\mu)$ and linearity of the map $P \longmapsto P \circ (U, X)^{-1}$.

It remains to prove the closedness of the graph of Φ as in (3). By continuity of $P \longmapsto P \circ (U, X)^{-1}$ and compactness of K, it suffices to prove the closedness of

$$\{(\mu, P) : \mu \in K, P \in \mathcal{R}^*(\mu)\}.$$

Suppose $\mu_n \to \mu$ and $P_n \to P$, with $\mu, \mu_n \in K$, $P_n \in \mathcal{R}^*(\mu_n)$, and $P \in \mathcal{R}$. To show that $P \in \mathcal{R}^*(\mu)$, we must show that $\langle P, \Gamma^{\mu}\rangle \ge \langle Q, \Gamma^{\mu}\rangle$ for every $Q \in \mathcal{R}$. This follows easily from the joint continuity of Lemma 3, which yields

$$\langle P, \Gamma^{\mu}\rangle = \lim_n \langle P_n, \Gamma^{\mu_n}\rangle \ge \lim_n \langle Q, \Gamma_n^{\mu}\rangle = \langle Q, \Gamma^{\mu}\rangle,$$

with the inequality coming from the assumption $P_n \in \mathcal{R}^*(\mu_n)$. This completes the proof. □

4.4. Construction of Markovian Equilibrium

We now construct a Markovian equilibrium as defined in Section 3.3, thereby proving Theorem 1. We follow the strategy of the proof of Lacker [35, theorem 3.7] based on Markovian projection (Brunick and Shreve [12]). This section makes heavier use of the notation (Λ, U, X) for the coordinate maps on Ω.

Let μ be any fixed point, $\mu \in \Phi(\mu)$, the existence of which is guaranteed by Proposition 3. Note that $\mu \in \Phi(\mu)$ is equivalent to the existence of $P \in \mathcal{R}^*(\mu)$ such that $\mu = P \circ (U, X)^{-1}$. Because $P \in \mathcal{R}$, the definition of $\mathcal{R}$ and a standard martingale problem argument (e.g., El Karoui et al. [27, theorem 2.5]) show that there exists a P-Brownian motion B such that

$$dX_t = \int_A b(t, X_t, a)\Lambda_t(da)dt + \sigma(t, X_t)dB_t.$$

To handle the additional variable U, we simply note that the $(d+1)$-dimensional process $(U, X_t)_{t\in[0,T]}$ is an Itô process in its own right:

$$d\begin{pmatrix} U \\ X_t \end{pmatrix} = \begin{pmatrix} 0 \\ \int_A b(t, X_t, a)\Lambda_t(da) \end{pmatrix} dt + \begin{pmatrix} 0_{d_0}^{\top} \\ \sigma(t, X_t) \end{pmatrix} dB_t.$$

Consider jointly measurable functions $(\widehat{b}, \widehat{f}) : [0, T] \times [0, 1] \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}$ satisfying

$$\begin{aligned}
\widehat{b}(t, U, X_t) &= \mathbb{E}\left[\int_A b(t, X_t, a)\Lambda_t(da) \mid U, X_t\right], \\
\widehat{f}(t, U, X_t) &= \mathbb{E}\left[\int_A f(t, X_t, \mathsf{W}\mu_t(U), a)\Lambda_t(da) \mid U, X_t\right], \quad P-\text{a.s., a.e. } t \in [0, T].
\end{aligned}$$

Such functions exist by Brunick and Shreve [12, proposition 5.1]. Applying the mimicking theorem (Brunick and Shreve [12, corollary 3.7]), we may find a process $(\widehat{U}_t, \widehat{X}_t)_{t\in[0,T]}$, perhaps on another probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$

with another Brownian motion $\widehat{B}$, solving the SDE

$$d\begin{pmatrix}\widehat{U}_t\\ \widehat{X}_t\end{pmatrix}=\begin{pmatrix}0\\ \widehat{b}(t,\widehat{U},\widehat{X}_t)\end{pmatrix}dt+\begin{pmatrix}0_{d_0}^{\top}\\ \sigma(t,\widehat{X}_t)\end{pmatrix}d\widehat{B}_t \tag{21}$$

and satisfying $(\widehat{U}_t,\widehat{X}_t)\overset{d}{=}(U,X_t)$ for each $t\in[0,T]$. Part of the definition of an SDE solution, of course, is that $\widehat{B}$ is a Brownian motion relative to the filtration $\widehat{\mathbb{F}}=(\widehat{\mathcal{F}}_t)_{t\in[0,T]}$ generated by $(\widehat{U},\widehat{X},\widehat{B})$. From the dynamics (21), we deduce that $\widehat{U}_t=\widehat{U}_0$ for all t, which implies that $\widehat{U}:=\widehat{U}_0$ is Unif[0, 1] because U is. Hence, $\widehat{U}$ is a.s. $\widehat{\mathcal{F}}_0$ measurable and in particular, independent of B.

Now, for $(t,x,m)\in[0,T]\times\mathbb{R}^d\times\mathcal{M}_+(\mathbb{R}^d)$, let $\mathcal{S}(t,x,m)\subset\mathbb{R}^d\times\mathbb{R}$ denote the set defined in (8). From its assumed convexity, we deduce that $(\widehat{b}(t,U,X_t),\widehat{f}(t,U,X_t))$ belongs a.s. to $\mathcal{S}(t,X_t,\mathsf{W}\mu_t(U))$. Thus, using a measurable selection result from Haussmann and Lepeltier [31, theorem A.9], there exist measurable functions $\widehat{\alpha}:[0,T]\times[0,1]\times\mathbb{R}^d\to A$ and $\widehat{z}:[0,T]\times[0,1]\times\mathbb{R}^d\to\mathbb{R}_+$ such that, P-a.s., for a.e. $t\in[0,T]$,

$$\widehat{b}(t,U,X_t)=b(t,X_t,\widehat{\alpha}(t,U,X_t)), \tag{22}$$

$$\widehat{f}(t,U,X_t)=f(t,X_t,\mathsf{W}\mu_t(U),\widehat{\alpha}(t,U,X_t))-\widehat{z}(t,U,X_t). \tag{23}$$

Applying (22), the dynamics (21) can then be written as

$$d\widehat{X}_t=b(t,\widehat{X}_t,\widehat{\alpha}(t,\widehat{U},\widehat{X}_t))dt+\sigma(t,\widehat{X}_t)d\widehat{B}_t.$$

Note that $\widehat{\alpha}$ belongs to $\mathcal{A}_U$, as defined in Section 3.3. By the uniqueness of the SDE, in the notation of Section 3.3, we have

$$(\widehat{U},\widehat{X})\overset{d}{=}(U,X^{\widehat{\alpha}}). \tag{24}$$

As in Remark 9, the joint law $P^{\widehat{\alpha}}$ of $(dt\delta_{\widehat{\alpha}(t,\widehat{U},\widehat{X}_t)}(da),\widehat{U},\widehat{X})$ is thus an element of $\mathcal{R}$. Let $\widehat{\mu}\in\mathcal{P}_{\mathrm{Unif}}([0,1]\times\mathcal{C}^d)$ denote the joint law of $(\widehat{U},\widehat{X})$. Then, as in (19), we have

$$\langle P^{\widehat{\alpha}},\Gamma^{\widehat{\mu}}\rangle=J_W(\widehat{\mu},\widehat{\alpha}). \tag{25}$$

We will complete the proof by showing that, in fact, $\langle P^{\widehat{\alpha}},\Gamma^{\widehat{\mu}}\rangle\ge J_W(\widehat{\mu},\alpha)$ for all $\alpha\in\mathcal{A}_U$. Again, using (19), it suffices to show that $P^{\widehat{\alpha}}\in\mathcal{R}^*(\widehat{\mu})$ (i.e., $\langle P^{\widehat{\alpha}},\Gamma^{\widehat{\mu}}\rangle\ge\langle Q,\Gamma^{\widehat{\mu}}\rangle$ for all $Q\in\mathcal{R}$).

To this end, note that $(\widehat{U},\widehat{X}_t)\overset{d}{=}(U,X_t)$, and thus, $\widehat{\mu}_t=\mu_t$ for each $t\in[0,T]$. Because Γ^{μ}, defined in (18), depends on μ only through its marginals $(\mu_t)_{t\in[0,T]}$,

$$\Gamma^{\mu}(q,u,x)=\Gamma^{\widehat{\mu}}(q,u,x),\quad\text{for all }\ (q,u,x)\in\Omega. \tag{26}$$

Hence,

$$\langle P^{\widehat{\alpha}},\Gamma^{\widehat{\mu}}\rangle=\langle P^{\widehat{\alpha}},\Gamma^{\mu}\rangle=\widehat{\mathbb{E}}\left[\int_0^T f(t,\widehat{X}_t,\mathsf{W}\mu_t(\widehat{U}),\widehat{\alpha}(t,\widehat{U},\widehat{X}_t))dt+g(\widehat{X}_T,\mathsf{W}\mu_T(\widehat{U}))\right],$$

where $\widehat{\mathbb{E}}$ denotes expectation on $(\widehat{\Omega},\widehat{\mathcal{F}},\widehat{\mathbb{P}})$. Using Fubini's theorem and the equality in law $(\widehat{U},\widehat{X}_t)\overset{d}{=}(U,X_t)$ for each $t\in[0,T]$, we find

$$\langle P^{\widehat{\alpha}},\Gamma^{\widehat{\mu}}\rangle=\mathbb{E}\left[\int_0^T f(t,X_t,\mathsf{W}\mu_t(U),\widehat{\alpha}(t,U,X_t))dt+g(X_T,\mathsf{W}\mu_T(U))\right].$$

The identity (23) and the definition of $\widehat{f}$ imply

$$f(t,X_t,\mathsf{W}\mu_t(U),\widehat{\alpha}(t,U,X_t))\ge\mathbb{E}\left[\int_A f(t,X_t,\mathsf{W}\mu_t(U),a)\Lambda_t(da)\,|\,U,X_t\right].$$

Using this, the tower property, and the definition of P, we deduce

$$\langle P^{\widehat{\alpha}},\Gamma^{\widehat{\mu}}\rangle\ge\mathbb{E}\left[\int_0^T\int_A f(t,X_t,\mathsf{W}\mu_t(U),a)\Lambda_t(da)dt+g(X_T,\mathsf{W}\mu_T(U))\right]=\langle P,\Gamma^{\mu}\rangle.$$

We know by assumption that $P\in\mathcal{R}^*(\mu)$. Hence, for any $Q\in\mathcal{R}$, we have $\langle P,\Gamma^{\mu}\rangle\ge\langle Q,\Gamma^{\mu}\rangle$. Using again (26), we deduce finally that $\langle P^{\widehat{\alpha}},\Gamma^{\widehat{\mu}}\rangle\ge\langle Q,\Gamma^{\mu}\rangle=\langle Q,\Gamma^{\widehat{\mu}}\rangle$ for all $Q\in\mathcal{R}$, which completes the proof of Theorem 1 as explained.

Remark 10. The argument in this section shows that, for each μ,

$$\sup_{\beta\in\mathcal{A}_U} J_W(\mu,\beta) = \sup_{P\in\mathcal{R}} \langle P, \Gamma^\mu\rangle. \tag{27}$$

That is, Markovian controls achieve the same value as the more general controls allowed in $\mathcal{R}$, which may depend on additional randomness. See El Karoui et al. [27] and Haussmann and Lepeltier [31] for more general studies of this well-known principle.

4.5. The Case of Constant Degree

Proof of Proposition 1. With $\nu \in C([0,T];,\mathcal{P}(\mathbb{R}^d)$ and $\mu_t = \mathrm{Unif}[0,1]\times\nu_t$ as in the statement of the proposition, the key point is the simple identity $\nu_t = \mathsf{W}\mu_t(u)$. Indeed, for bounded measurable $\varphi:\mathbb{R}^d\to\mathbb{R}$, we have

$$\begin{aligned}\langle \mathsf{W}\mu_t(u),\varphi\rangle &= \int_{[0,1]\times\mathbb{R}^d} W(u,v)\,\varphi(x)\,\mu_t(dv,dx)\\ &= \int_0^1\int_{\mathbb{R}^d} W(u,v)\,\varphi(x)\,\nu_t(dx)\,dv\\ &= \int_{\mathbb{R}^d}\varphi(x)\,\nu_t(dx),\end{aligned}$$

with the last identity following from Fubini's theorem and the assumption (13). Then, $J_1(\nu,\alpha) = J_W(\mu,\alpha)$ for any $\alpha\in\mathcal{A}_1$. Because ν is a mean field equilibrium with control α^*, we have

$$J_W(\mu,\alpha^*) = J_1(\nu,\alpha^*) = \sup_{\alpha\in\mathcal{A}_1} J_1(\nu,\alpha) = \sup_{\alpha\in\mathcal{A}_1} J_W(\mu,\alpha).$$

The only remaining subtlety is to argue that $\sup_{\alpha\in\mathcal{A}_1} J_W(\mu,\alpha) = \sup_{\alpha\in\mathcal{A}_U} J_W(\mu,\alpha)$. That is, the optimal value is the same regardless of whether one allows the controls to depend on an independent uniform U. This can be argued by way of a Markovian projection argument as in Section 4.4 or by directly applying Lacker [35, theorem 3.7]. □

4.6. Uniqueness

This section proves Proposition 2, relying on the recasting of the graphon game as a mean field game as in Section 3.6. The key point is that the monotonicity condition (14) in Proposition 2 translates precisely to the usual Lasry–Lions monotonicity condition for the associated mean field game. Using the same notation as in Section 3.6, Inequality (14) implies

$$\begin{aligned}&\int_{[0,1]\times\mathbb{R}^d} (\overline{g}(\overline{x},m_1) - \overline{g}(\overline{x},m_2)(m_1-m_2)(du,dx)\\ &\qquad = \int_{[0,1]\times\mathbb{R}^d}(g(x,\mathsf{W}m_1(u)) - g(x,\mathsf{W}m_2(u))(m_1-m_2)(du,dx)\le 0\end{aligned}$$

for $m_1, m_2\in\mathcal{P}_{\mathrm{Unif}}([0,1]\times\mathbb{R}^d)$. Similarly, $\overline{f}$ takes the form

$$\overline{f}(t,\overline{x},m,a) = f_1(t,x,a) + \overline{f}_2(t,\overline{x},m),\ \text{ where }\ \overline{f}_2(t,\overline{x},m) = f_2(t,x,Wm(u)),$$

and for all m_1, m_2, we have

$$\int_{[0,1]\times\mathbb{R}^d}(\overline{f}_2(t,\overline{x},m_1) - \overline{f}_2(t,\overline{x},m_2)(m_1-m_2)(du,dx)\le 0.$$

This shows that the mean field game of Section 3.6 satisfies the Lasry–Lions monotonicity condition. The classical uniqueness proof from mean field game theory then applies; see Lacker [36, theorem 8.10] for a short proof, which applies directly in our context.

Remark 11. We mention here two classes of examples of g satisfying (14).

1. Suppose $\varphi:\mathbb{R}^d\to\mathbb{R}$ is bounded and continuous, and let

$$g(x,m) = \left(\varphi(x) - \int_{\mathbb{R}^d}\varphi(y)\,m(dy)\right)^2,\quad x\in\mathbb{R}^d,\ m\in\mathcal{M}_+(\mathbb{R}^d).$$

A straightforward calculation shows that the left-hand side of (14) equals

$$-\int_0^1\int_0^1 W(u,v)\psi(u)\psi(v)\,du\,dv, \quad \text{where } \psi(u) := \int_{\mathbb{R}^d}\varphi(x)\,(m_u^1 - m_u^2)(dx),$$

for $m_i(du,dx) = dum_u^i(dx) \in \mathcal{P}_{\text{Unif}}([0,1]\times\mathbb{R}^d)$, $i=1,2$. Hence, if W is positive semidefinite, we obtain the monotonicity condition (14).

2. Suppose $g(x,m) = \int_{\mathbb{R}^d}\varphi(x,y)m(dy)$, where $\varphi : (\mathbb{R}^d)^2 \to \mathbb{R}$ is bounded and continuous. Then, the left-hand side of (14) equals

$$\int_{[0,1]\times\mathbb{R}^d}\int_{[0,1]\times\mathbb{R}^d} W(u,v)\varphi(x,y)(m_1-m_2)(dv,dy)(m_1-m_2)(du,dx).$$

This is nonpositive if, for instance, W is positive semidefinite and φ is negative semidefinite when viewed as integral operators, so that the tensor product of these two operators is negative definite.

5. On the Dependence of Optimal Controls on *U*

This short section develops two lemmas that will be used solely in the proof of Theorem 3 in Section 7.2. We give these results here because the proofs use the same relaxed formulation of Section 3.4, particularly the Markovian projection of Section 4.4.

For this section, we fix $W \in L_+^1[0,1]^2$ and $\mu_\cdot \in C([0,T];\mathcal{P}_{\text{Unif}}([0,1]\times\mathbb{R}^d))$, and we introduce the following notation. For $u\in[0,1]$, $m\in\mathcal{P}(\mathbb{R}^d)$, and $\alpha\in\mathcal{A}_1$, let $X^{m,\alpha}$ denote the unique in law solution of the SDE

$$dX_t^{m,\alpha} = b(t,X_t^{m,\alpha},\alpha(t,X_t^{m,\alpha}))dt + \sigma(t,X_t^{m,\alpha})dB_t, \qquad X_0^{m,\alpha}\sim m,$$

and define

$$J_W^{u,m}(\mu_\cdot,\alpha) := \mathbb{E}\left[\int_0^T f(t,X_t^{m,\alpha},\mathsf{W}\mu_t(u),\alpha(t,X_t^{m,\alpha}))dt + g(X_T^{m,\alpha},\mathsf{W}\mu_T(u))\right].$$

The first lemma states essentially that, if $\alpha^*\in\mathcal{A}_U$ is optimal for the given $\mu_\cdot$, then the control $(t,x)\longmapsto\alpha^*(t,u,x)$ is still optimal if we freeze the "label" variable $U=u$ for almost every u. Recall in the following that $\lambda\in\mathcal{P}_{\text{Unif}}([0,1]\times\mathbb{R}^d)$ denotes the initial law and $\lambda(du,dx) = du\lambda_u(dx)$ denotes its disintegration.

Lemma 4. *Suppose $\alpha\in\mathcal{A}_U$ satisfies $J_W(\mu_\cdot,\alpha)\ge J_W(\mu_\cdot,\beta)$ for all $\beta\in\mathcal{A}_U$. Then,*

$$J_W^{u,\lambda_u}(\mu_\cdot,\alpha_u) = \sup_{\beta\in\mathcal{A}_1} J_W^{u,\lambda_u}(\mu_\cdot,\beta), \ \ \textit{for a.e. } u\in[0,1],$$

where we define $\alpha_u\in\mathcal{A}_1$ by $\alpha_u(t,x) := \alpha(t,u,x)$.

Proof. Recall the identity (27) from Remark 10. For $u\in[0,1]$ and $m\in\mathcal{P}(\mathbb{R}^d)$, let us define $\mathcal{R}_{u,m}$ as the set of $P\in\mathcal{P}(\Omega)$ such that $P\circ(U,X_0)^{-1} = \delta_u\times m$ and such that $(N_t^\varphi)_{t\in[0,T]}$ is a P martingale for each $\varphi\in C_c^\infty(\mathbb{R}^d)$. The same argument as in Section 4.4 that led to (27) (see also [El Karoui et al. [27, corollary 6.8] or Lacker [35, theorem 3.7]) shows that

$$\sup_{\beta\in\mathcal{A}_1} J_W^{u,m}(\mu_\cdot,\beta) = \sup_{P\in\mathcal{R}_{u,m}}\langle P,\Gamma^{\mu_\cdot}\rangle. \tag{28}$$

It is straightforward to check that $\{(u,P) : u\in[0,1],\, P\in\mathcal{R}_{u,\lambda_u}\}$ is a Borel set in $[0,1]\times\mathcal{P}(\Omega)$. Because the map $\mathcal{P}(\Omega)\ni P\longmapsto\langle P,\Gamma^{\mu_\cdot}\rangle$ is Borel, a standard measurable selection theorem (Bertsekas and Shreve [7, proposition 7.50] then shows that $u\longmapsto\sup_{P\in\mathcal{R}_{u,\lambda_u}}\langle P,\Gamma^{\mu_\cdot}\rangle$ is universally measurable, and

$$\int_0^1 \sup_{P\in\mathcal{R}_{u,\lambda_u}}\langle P,\Gamma^{\mu_\cdot}\rangle\,du = \sup\left\{\int_0^1\langle P_u,\Gamma^{\mu_\cdot}\rangle\,du : P_\cdot \text{ Borel},\ P_u\in\mathcal{R}_{u,\lambda_u} \text{ a.e. } u\right\},$$

where "$P_\cdot$ Borel" means that the map $[0,1]\ni u\longmapsto P_u\in\mathcal{P}(\Omega)$ is Borel measurable. From the definitions and noting that $\int_0^1\delta_u\times\lambda_u\,du = \lambda$, it is straightforward to check that $\int_0^1 P_u\,du$ belongs to $\mathcal{R}$ whenever $P_u\in\mathcal{R}_{u,\lambda_u}$ for a.e. u. Conversely, if $P\in\mathcal{R}$, then the regular conditional measure $P_u := P(\cdot\mid U=u)$ belongs to $\mathcal{R}_{u,\lambda_u}$ for a.e. u. It follows that

$$\sup\left\{\int_0^1\langle P_u,\Gamma^{\mu_\cdot}\rangle\,du : P_\cdot \text{ Borel},\ P_u\in\mathcal{R}_{u,\lambda_u}\text{ a.e. } u\right\} = \sup_{P\in\mathcal{R}}\langle P,\Gamma^{\mu_\cdot}\rangle,$$

and also, that for $P^* \in \mathcal{R}$,

$$P^* \in \arg\max_{P \in \mathcal{R}} \langle P, \Gamma^{\mu_\cdot} \rangle \Longleftrightarrow \text{a.e. } u,\ P^*(\cdot \mid U = u) \in \arg\max_{P \in \mathcal{R}_{u,\lambda_u}} \langle P, \Gamma^{\mu_\cdot} \rangle. \tag{29}$$

Now, let $\alpha \in \arg\max_{\beta \in \mathcal{A}_U} J_W(\mu_\cdot, \beta)$ be the given optimizer. In light of (27), the measure P^α given as in Remark 9 then satisfies $P^\alpha \in \arg\max_{P \in \mathcal{R}} \langle P, \Gamma^{\mu_\cdot} \rangle$. Hence, by (29), the conditional measure $P^\alpha(\cdot \mid U = u)$ belongs to $\arg\max_{P \in \mathcal{R}_{u,\lambda_u}} \langle P, \Gamma^{\mu_\cdot} \rangle$ for a.e. u. However, by well posedness of the SDEs, we have $P^\alpha(\cdot \mid U = u) = P^{\alpha_u}$ for a.e. u (cf. Lacker [37, appendix A]). Using (28), we deduce that $\alpha_u \in \arg\max_{\beta \in \mathcal{A}_1} J_W^{u,\lambda_u}(\mu_\cdot, \beta)$ for a.e. u, as claimed. □

The next lemma and its corollary justify the claim in Theorem 3 that assumptions (1) and (2a)–(2d) imply (2). The lemma is a variation on known arguments, such as Lacker [37, section 5.6]. Essentially, by working with the relaxed formulation, the set-valued map of optimal control laws can be shown to have a closed graph, and the idea is to argue that in certain cases, this set-valued map is singleton valued and thus, necessarily continuous.

Lemma 5. *Suppose conditions* (1) *and* (2a)–(2d) *of Theorem* 3 *hold. Then, for each* $u \in [0, 1]$, *there exists a unique optimizer* α_u^* *for* $\sup_{\alpha \in \mathcal{A}_1} J_W^{u,\lambda_u}(\mu_\cdot, \alpha)$. *Moreover, the law* $\mathcal{L}(X^{\lambda_u, \alpha_u^*})$ *depends continuously on* u.

Proof. We have $b(t, x, a) = b_0(t, x)a + b_1(t, x)$ by assumption. Fix $u \in [0, 1]$ and $m \in \mathcal{P}(\mathbb{R}^d)$. Recall Equation (28) from the proof of Lemma 4. We first claim that any optimizer P on the right-hand side of (28) is necessarily of the form $P = \mathcal{L}(dt\delta_{\alpha(t, X_t^{m,\alpha})}(da), u, X^{m,\alpha})$ for some $\alpha \in \mathcal{A}_1$. To see this, note that we can write $P = \mathcal{L}(\Lambda, u, X)$, where X solves

$$dX_t = (b_0(t, X_t)\int_A a\Lambda_t(da) + b_1(t, X_t))dt + \sigma dB_t, \quad X_0 \sim m. \tag{30}$$

Letting $\widehat{\alpha}(t, X_t) = \mathbb{E}[\int_A a\Lambda_t(da) \mid X_t]$ and applying the Markovian projection (Brunick and Shreve [12, corollary 3.7]), we find that $X_t \stackrel{d}{=} \widehat{X}_t$ for all $t \in [0, T]$, where $\widehat{X}$ solves the SDE

$$d\widehat{X}_t = (b_0(t, \widehat{X}_t)\widehat{\alpha}(t, \widehat{X}_t) + b_1(t, \widehat{X}_t))dt + \sigma dB_t, \quad \widehat{X}_0 \sim m.$$

By Jensen's inequality and strict concavity of $f(t, x, m, a)$ in a, we have

$$\begin{aligned}
\langle P, \Gamma^{\mu_\cdot} \rangle &= \mathbb{E}\left[\int_0^T \int_A f(t, X_t, \mathsf{W}\mu_t(u), a)\Lambda_t(da)dt + g(X_T, \mathsf{W}\mu_T(u))\right] \\
&\le \mathbb{E}\left[\int_0^T f(t, X_t, \mathsf{W}\mu_t(u), \widehat{\alpha}(t, X_t))dt + g(X_T, \mathsf{W}\mu_T(u))\right] \\
&= \mathbb{E}\left[\int_0^T f(t, \widehat{X}_t, \mathsf{W}\mu_t(u), \widehat{\alpha}(t, \widehat{X}_t))dt + g(\widehat{X}_T, \mathsf{W}\mu_T(u))\right] \\
&= J_W^{u,m}(\mu_\cdot, \widehat{\alpha}),
\end{aligned}$$

and this equality is strict unless $\int_A a\Lambda_t(da) = \widehat{\alpha}(t, X_t)$ a.s. a.e. This proves the first claim.

We next claim that, in fact, there is a unique optimizer P on the right-hand side of (28). Because we know the optimizers are Markovian, it suffices to show that the optimal control $\alpha \in \mathcal{A}_1$ on the left-hand side of (28) is unique up to Lebesgue-a.e. equality. To see this, let $\alpha_0, \alpha_1 \in \mathcal{A}_1$ be optimizers. Then, $X^i = X^{m,\alpha_i}$ solves

$$dX_t^i = (b_0(t, X_t^i)\alpha_i(t, X_t^i) + b_1(t, X_t^i))dt + \sigma dB_t^i, \quad X_0^i \sim m.$$

We may assume that X^0 and X^1 are defined on the same probability space, with (X^0, B^0) independent of (X^1, B^1). Let S be a Bernoulli $(1/2)$ random variable, independent of everything else. Then, X^S solves the SDE

$$dX_t^S = (b_0(t, X_t^S)\alpha_S(t, X_t^S) + b_1(t, X_t^S))dt + \sigma dB_t^S, \quad X_0^S \sim m,$$

where we note that B^S is a Brownian motion. Define $\widehat{\alpha}(t, X_t^S) = \mathbb{E}[\alpha_S(t, X_t^S) \mid X_t^S]$. Arguing as via Jensen, we must have $\widehat{\alpha}(t, X_t^S) = \alpha_S(t, X_t^S)$ a.s. a.e., as otherwise, this control would produce a strictly higher reward than α_0 or α_1. This implies $\widehat{\alpha}(t, X_t^0) = \alpha_0(t, X_t^0)$ and $\widehat{\alpha}(t, X_t^1) = \alpha_1(t, X_t^1)$ a.s. a.e. The laws of X_t^0 and X_t^1 have full support for each $t > 0$ by Girsanov's theorem, and we deduce that $\alpha_0 = \alpha_1$ a.e.

Finally, knowing that the optimizer $P_{u,m}^* \in \mathcal{R}_{u,m}$ on the right-hand side of (28) is unique, we will prove that $(u, m) \longmapsto P_{u,m}^*$ is continuous, which implies our claim by composition with the continuous map $u \longmapsto (u, \lambda_u)$. By El Karoui et al. [27, proposition 5.10(b)], the set-valued map $\mathcal{R}_{u,\delta_x}$ is continuous in $(u, x) \in [0, 1] \times \mathbb{R}^d$. By Berge's theorem (Aliprantis and Border [1, theorem 17.31]) and continuity of $P \longmapsto \langle P, \Gamma^{\mu_\cdot} \rangle$, the set-valued map $\mathcal{R}_{u,x}^* := \arg\max_{P \in \mathcal{R}_{u,\delta_x}} \langle P, \Gamma^{\mu_\cdot} \rangle$ has a closed graph. We have just shown it to in fact be singleton valued or $\mathcal{R}_{u,x}^* = \{P_{u,\delta_x}^*\}$ for each (u, x). That is, the function $(u,x) \longmapsto P^*_{u,\delta_x}$ has a closed graph and is thus continuous. To conclude, simply note that $P^*_{u,m} = \int_{\mathbb{R}^d} P^*_{u,\delta_x}\, m(dx)$ (e.g., by El Karoui et al. [27, theorem 5.11(c)]). □

Corollary 1. *Suppose the assumptions of Lemma 5 hold. Assume $\mu_\cdot$ is a W equilibrium, with equilibrium control α^*. Then, the disintegration $[0, 1] \ni u \longmapsto \mathcal{L}(X^{\alpha^*} | U = u) \in \mathcal{P}(\mathcal{C}^d)$ admits a weakly continuous version.*

Proof. Let α denote the equilibrium control corresponding to $\mu_\cdot$. We note again that $u \longmapsto \mathcal{L}(X^{\lambda_u,\alpha_u})$ is a version of the conditional law $\mathcal{L}(X^{\alpha^*} \mid U = u)$. By Lemma 4, the control $\alpha_u(t,x) := \alpha(t,u,x)$ optimizes $J^{u,\lambda_u}_W(\mu_\cdot,\cdot)$ over $\mathcal{A}_1$, for a.e. $u \in [0, 1]$. By Lemma 5, there is a unique (up to Lebesgue a.e. equality) optimizer $\alpha^*_u \in \mathcal{A}_1$ of $J^{u,\lambda_u}_W(\mu_\cdot,\cdot)$. Hence, $\alpha_u = \alpha^*_u$ for a.e. u, and we deduce that $\mathcal{L}(X^{\lambda_u,\alpha_u}) = \mathcal{L}(X^{\lambda_u,\alpha^*_u})$ for a.e. u. The claim now follows from the last statement of Lemma 5. □

6. Convergence of Empirical Measures

In preparation for Section 7, which proves our results about approximate equilibria, we study in this section the general principles underlying these results. These results deal with the convergence of neighborhood empirical measures under various assumptions on the underlying distributions and kernel. We work throughout this section with a general Polish space E. Recall the notation I^n_i from (2).

6.1. General Kernels

Let (U, X) be a random variable taking values in $[0, 1] \times E$, with law $\mu \in \mathcal{P}_{\mathrm{Unif}}([0, 1] \times E)$. Let $n \in \mathbb{N}$, and let $I^n_i = [(i-1)/n, i/n)$ as before for $i \in [n]$. For each $n \in \mathbb{N}$, let $U^n_i \sim \mathrm{Unif}(I^n_i)$, and with

$$\mathcal{L}(X^n_i \mid U^n_i = u) = \mathcal{L}(X \mid U = u), \quad u \in I^n_i.$$

In other words, the law of (U^n_i, X^n_i) is the conditional law of (U, X) given $\{U \in I^n_i\}$. This entails in particular that, for bounded measurable $h : [0, 1] \times E \to \mathbb{R}$,

$$\begin{aligned}\langle \mu, h\rangle = \mathbb{E}[h(U,X)] &= \sum_{i=1}^n \int_{I^n_i} \mathbb{E}[h(u,X) \mid U = u]\, du \\ &= \sum_{i=1}^n \int_{I^n_i} \mathbb{E}[h(u,X^n_i) \mid U^n_i = u]\, du \\ &= \frac{1}{n}\sum_{i=1}^n \mathbb{E}[h(U^n_i, X^n_i)]. \end{aligned} \tag{31}$$

Assume (U, X) and $(U^n_i, X^n_i)^n_{i=1}$ are defined on the same probability space and are independent. Let $W \in L^1_+[0,1]^2$, and recall the definition of $\mathsf{W}\mu(u)$ from (7); with $(U,X) \sim \mu$, note that we may write $\langle \mathsf{W}\mu(u), \varphi\rangle = \mathbb{E}[W(u,U)\varphi(X)]$ for bounded measurable φ. Let (ξ^n_{ij}) again be an $n \times n$ matrix with values in $[0, 1]$ and with zeros on its diagonal. Recall that W_{ξ^n} denotes the associated step kernel, as in (2); we will use repeatedly the fact that $W_{\xi^n}(U^n_i, U^n_j) = \xi^n_{ij}$. Define lastly the (random) empirical measures

$$M^n_i := \frac{1}{n}\sum_{j=1}^n \xi^n_{ij}\delta_{X^n_j} = \frac{1}{n}\sum_{j=1}^n W_{\xi^n}(U^n_i, U^n_j)\delta_{X^n_j}. \tag{32}$$

Recall the definition of the strong operator topology from Section 2.2. The main result of this section is the following theorem, which we will apply only in cases where $h(u,x,m)$ does not depend on x, but the proof of the general case given here is not any more difficult.

Theorem 5. *Assume W_{ξ^n} converges to W in the strong operator topology, and assume (16) holds. Let $h : [0, 1] \times E \times \mathcal{M}_+(E) \to \mathbb{R}$ be a bounded measurable function such that $h(u,x,\cdot)$ is continuous on $\mathcal{M}_+(E)$ for each fixed $(u,x) \in [0, 1] \times E$. Then,*

$$\frac{1}{n}\sum_{i=1}^n \mathbb{E}\left[h\left(U^n_i, X^n_i, M^n_i\right)\right] \to \mathbb{E}[h(U,X,\mathsf{W}\mu(U))]. \tag{33}$$

Proof. We will use several times the following fact. There exists $\overline{A} > 0$ such that

$$\frac{1}{n^2}\sum_{i,j=1}^n \xi^n_{ij} = \|W_{\xi^n}\|_{L^1[0,1]^2} \le \overline{A}, \quad \forall n \in \mathbb{N}. \tag{34}$$

To see this, note that the convergence in strong operator topology $W_{\xi^n} \to W$ implies

$$|\|\mathsf{W}_{\xi^n}\mathbf{1}\|_{L^1[0,1]} - \|\mathsf{W}\mathbf{1}\|_{L^1[0,1]}| \le \|(\mathsf{W}_{\xi^n} - \mathsf{W})\mathbf{1}\|_{L^1[0,1]} \to 0,$$

where $\mathbf{1}$ is the constant function equal to one. Because W and W_{ξ^n} are nonnegative, we have $\|\mathsf{W}\mathbf{1}\|_{L^1[0,1]} = \|W\|_{L^1[0,1]^2}$ and $\|\mathsf{W}_{\xi^n}\mathbf{1}\|_{L^1[0,1]} = \|W_{\xi^n}\|_{L^1[0,1]^2} = \frac{1}{n^2}\sum_{i,j=1}^n \xi_{ij}^n$.

The proof proceeds by a series of simplifications.

Step 1. We first argue that it suffices to prove (33) for h bounded and one Lipschitz. Indeed, suppose this is the case. Define the following probability measures on $[0,1]\times E\times\mathcal{M}_+(E)$:

$$Q_n := \frac{1}{n}\sum_{i=1}^n \mathcal{L}(U_i^n, X_i^n, M_i^n), \qquad Q := \mathcal{L}(U, X, M(U)).$$

We have assumed that $\langle Q_n, h\rangle \to \langle Q, h\rangle$ holds for bounded Lipschitz h. By the Portmanteau theorem, it also holds for bounded continuous h, and in particular, we have $Q_n \to Q$ weakly. The $[0,1]\times E$ marginals of Q_n are all the same (i.e., $\frac{1}{n}\sum_{i=1}^n \mathcal{L}(U_i^n, X_i^n) = \mathcal{L}(U,X) = \mu$ for each n as argued in (31)). Hence, the weak convergence $Q_n \to Q$ also implies the convergence $\langle Q_n, h\rangle \to \langle Q, h\rangle$ for test functions h of the form in the statement of the theorem, with no continuity required in the first two arguments (Beiglböck and Lacker [6, lemma 2.1]).

Step 2. We next claim that it suffices to show that $\mathsf{W}_{\xi^n}\mu_n(U) \to \mathsf{W}\mu(U)$ in probability, where the random probability measure μ_n on $[0,1]\times E$ is defined by

$$\mu_n = \frac{1}{n}\sum_{i=1}^n \delta_{(U_i^n, X_i^n)}.$$

Expanding the notation and applying the definition (7) of the operator W_{ξ^n},

$$\mathsf{W}_{\xi^n}\mu_n(u) = \frac{1}{n}\sum_{j=1}^n W_{\xi^n}(u, U_j^n)\delta_{X_j^n} = M_i^n, \quad \text{for } u \in I_i^n,\ i = 1,\dots,n.$$

Recalling that $W_{\xi^n}(u, U_j^n) = W_{\xi^n}(U_i^n, U_j^n)$ for $u \in I_i^n$, we have

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^n \mathbb{E}[h(U_i^n, X_i^n, M_i^n)] &= \frac{1}{n}\sum_{i=1}^n \mathbb{E}\left[h\left(U_i^n, X_i^n, \frac{1}{n}\sum_{j=1}^n W_{\xi^n}(U_i^n, U_j^n)\delta_{X_j^n}\right)\right] \\
&= \sum_{i=1}^n \int_{I_i^n} \mathbb{E}\left[h\left(u, X_i^n, \frac{1}{n}\sum_{j=1}^n W_{\xi^n}(u, U_j^n)\delta_{X_j^n}\right)\middle| U_i^n = u\right] du,
\end{aligned}$$

with the second step using independence of $(U_i^n, X_i^n)_{i=1}^n$ and the fact that $W_{\xi^n}(u, U_i^n) = W_{\xi^n}(U_i^n, U_i^n) = \xi_{ii}^n = 0$ for $u \in I_i^n$. Because $\mathcal{L}(X_i^n \,|\, U_i^n = u) = \mathcal{L}(X \,|\, U = u)$ for $u \in I_i^n$, this simplifies to

$$\begin{aligned}
&= \sum_{i=1}^n \int_{I_i^n} \mathbb{E}\left[h\left(u, X, \frac{1}{n}\sum_{j=1}^n W_{\xi^n}(u, U_j^n)\delta_{X_j^n}\right)\middle| U = u\right] du \\
&= \int_0^1 \mathbb{E}[h(u, X, \mathsf{W}_{\xi^n}\mu_n(u)) | U = u]\, du \\
&= \mathbb{E}[h(U, X, \mathsf{W}_{\xi^n}\mu_n(U))].
\end{aligned}$$

Here, we also used the assumed independence of (U, X) and $(U_i^n, X_i^n)_{i=1}^n$. Hence, once we know that $\mathsf{W}_{\xi^n}\mu_n(U) \to \mathsf{W}\mu(U)$ in probability, it follows from the bounded convergence theorem that (33) holds for bounded continuous h, which is sufficient by Step 1.

Step 3. We finally prove that $\mathsf{W}_{\xi^n}\mu_n(U) \to \mathsf{W}\mu(U)$ in probability, which will complete the proof as explained in Step 2. Fix a bounded continuous function $\varphi: E \to [-1, 1]$. Expanding the definition,

$$\langle \mathsf{W}_{\xi^n}\mu_n(u), \varphi\rangle = \frac{1}{n}\sum_{j=1}^n W_{\xi^n}(u, U_j^n)\varphi(X_j^n).$$

We must show that $\langle \mathsf{W}_{\xi^n}\mu_n(U), \varphi\rangle \to \langle \mathsf{W}\mu(U), \varphi\rangle$ in probability.

Step 3(a). We first claim that $\langle \mathsf{W}_{\xi^n}\mu_n(U),\varphi\rangle - \mathbb{E}[\langle \mathsf{W}_{\xi^n}\mu_n(U),\varphi\rangle \mid U] \to 0$ in probability and in fact, in L^2. To see this, note for $u \in I_i^n$ that

$$\mathrm{Var}(\langle \mathsf{W}_{\xi^n}\mu_n(U),\varphi\rangle \mid U=u) = \mathrm{Var}\left(\frac{1}{n}\sum_{j=1}^n \xi_{ij}^n \varphi(X_j^n)\right) \le \frac{1}{n^2}\sum_{j=1}^n (\xi_{ij}^n)^2,$$

by independence of $(X_j^n)_{j=1}^n$. Hence,

$$\begin{aligned}\mathbb{E}\left[\left(\langle \mathsf{W}_{\xi^n}\mu_n(U),\varphi\rangle - \mathbb{E}[\langle \mathsf{W}_{\xi^n}\mu_n(U),\varphi\rangle \mid U]\right)^2\right] &= \mathbb{E}\,\mathrm{Var}(\langle \mathsf{W}_{\xi^n}\mu_n(U),\varphi\rangle \mid U) \\ &= \sum_{i=1}^n \int_{I_i^n} \mathrm{Var}(\langle \mathsf{W}_{\xi^n}\mu_n(U),\varphi\rangle \mid U=u)\,du \\ &\le \frac{1}{n^3}\sum_{i,j=1}^n (\xi_{ij}^n)^2,\end{aligned}$$

which vanishes by (16).

Step 3(b). We must finally show that $\mathbb{E}[\langle \mathsf{W}_{\xi^n}\mu_n(U),\varphi\rangle \mid U] \to \langle \mathsf{W}\mu(U),\varphi\rangle$ in probability. To see this, we first use again the independence of $(U_i^n, X_i^n)_{i=1}^n$ to rewrite

$$\begin{aligned}\mathbb{E}[\langle \mathsf{W}_{\xi^n}\mu_n(U),\varphi\rangle \mid U=u] &= \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^n W_{\xi^n}(u,U_i^n)\varphi(X_i^n)\right] = \mathbb{E}[W_{\xi^n}(u,U)\varphi(X)] \\ &= \int_0^1 W_{\xi^n}(u,v)\psi(v)\,dv,\end{aligned}$$

where $\psi(v) := \mathbb{E}[\varphi(X)|U=v]$ and where we again used the fact that $\frac{1}{n}\sum_{i=1}^n \mathcal{L}(U_i^n, X_i^n) = \mathcal{L}(U,X)$ as shown by (31). Similarly, we may write

$$\langle \mathsf{W}\mu(u),\varphi\rangle = \mathbb{E}[W(u,U)\varphi(X)] = \mathbb{E}[W(u,U)\psi(U)] = \int_0^1 W(u,v)\psi(v)\,dv.$$

These identities are to be understood for a.e. $u \in [0,1]$, and combined, they yield

$$\begin{aligned}\mathbb{E}[|\mathbb{E}[\langle \mathsf{W}_{\xi^n}\mu_n(U),\varphi\rangle \mid U] - \langle \mathsf{W}\mu(u),\varphi\rangle|] &= \int_0^1 \left|\int_0^1 (W_{\xi^n}(u,v) - W(u,v))\psi(v)\,dv\right| du \\ &= \|(\mathsf{W}_{\xi^n} - \mathsf{W})\psi\|_{L^1[0,1]},\end{aligned} \tag{35}$$

where we have used the operator notation of (4).

Recalling that φ and thus, ψ are bounded, the right-hand side of (35) converges to zero by the assumption that $W_{\xi^n} \to W$ in the strong operator topology. We deduce that $\mathbb{E}[\langle \mathsf{W}_{\xi^n}\mu_n(U),\varphi\rangle \mid U] \to \langle \mathsf{W}\mu(U),\varphi\rangle$ in L^1 and thus, in probability. This completes the proof of Step 3(b) and thus, the theorem. □

6.2. Continuous Kernels

We now prove an alternative to Theorem 5, which requires stronger assumptions but is, in a sense, uniform in the choice of labels rather than averaged. Fix again $\mu \in \mathcal{P}_{\mathrm{Unif}}([0,1]\times E)$, and assume that there exists a version of the disintegration $\mu(du,dx) = du\mu_u(dx)$ such that $[0,1] \ni u \longmapsto \mu_u \in \mathcal{P}(E)$ is weakly continuous. For $u \in [0,1]$, let X_u denote a random variable with law μ_u. Let us write $\boldsymbol{u} = (u_1,\dots,u_n)$ for a generic element of $I_1^n \times \cdots \times I_n^n$, which we think of as denoting the set of admissible assignments of *labels* to each player $i \in [n]$. For $\boldsymbol{u} \in I_1^n \times \cdots \times I_n^n$, define the (random) empirical measures

$$M_i^{n,\boldsymbol{u}} := \frac{1}{n}\sum_{j=1}^n \xi_{ij}^n \delta_{X_{u_j}} = \frac{1}{n}\sum_{j=1}^n W_{\xi^n}(u_i,u_j)\delta_{X_{u_j}}, \tag{36}$$

where $(X_{u_i})_{i=1}^n$ are assumed independent. Let us stress that (36) and every other expression will involve at most finitely many of the random variables $(X_u)_{u\in[0,1]}$ at a time; at no point must we face any of the complications that accompany a continuum of independent random variables.

Recall the bounded Lipschitz norm $\|\cdot\|_{BL}$ defined in (1).

Theorem 6. *Assume W_{ξ^n} converges to W in the strong operator topology, and assume that (16) holds. Assume also that $[0,1]\ni u\longmapsto W(u,v)dv\in\mathcal{M}_+([0,1])$ is continuous and that there exists a version of the disintegration $\mu(du,dx)=du\mu_u(dx)$ such that the map $[0,1]\ni u\longmapsto \mu_u\in\mathcal{P}(E)$ is weakly continuous.*

Then,

$$\lim_{n\to\infty}\sup_{\boldsymbol{u}=(u_1,\ldots,u_n)\in I_1^n\times\cdots\times I_n^n}\frac{1}{n}\sum_{i=1}^n\mathbb{E}\|M_i^{n,\boldsymbol{u}}-\mathsf{W}\mu(u_i)\|_{BL}=0. \tag{37}$$

Let $h:[0,1]\times\mathcal{M}_+(E)\to\mathbb{R}$ be bounded and measurable, and assume $h(u,\cdot)$ continuous on $\mathcal{M}_+(E)$ uniformly in $u\in[0,1]$, in the sense that

$$\lim_{m'\to m}\sup_{u\in[0,1]}|h(u,m')-h(u,m)|=0,\quad \forall m\in\mathcal{M}_+(E).$$

Then, we have

$$\lim_{n\to\infty}\sup_{\boldsymbol{u}=(u_1,\ldots,u_n)\in I_1^n\times\cdots\times I_n^n}\frac{1}{n}\sum_{i=1}^n\mathbb{E}|h(u_i,M_i^{n,\boldsymbol{u}})-h(u_i,\mathsf{W}\mu(u_i))|=0. \tag{38}$$

Proof. The claim (38) follows immediately from (37) and the assumed uniform continuity of h. As in the proof of Theorem 5, the convergence in cut norm $W_{\xi^n}\to W$ yields $\overline{A}>0$ such that (34) holds.

Step 1. We first prove that

$$\lim_{n\to\infty}\sup_{\boldsymbol{u}=(u_1,\ldots,u_n)\in I_1^n\times\cdots\times I_n^n}\frac{1}{n}\sum_{i=1}^n\mathbb{E}|\langle M_i^{n,\boldsymbol{u}}-\mathsf{W}\mu(u_i),\varphi\rangle|=0, \tag{39}$$

for each Lipschitz function $\varphi:E\to[-1,1]$. Note first for each $i\in[n]$ and $\boldsymbol{u}\in I_1^n\times\cdots\times I_n^n$ that

$$\mathbb{E}|\langle M_i^{n,\boldsymbol{u}}-\mathsf{W}\mu(u_i),\varphi\rangle|\le\mathbb{E}|\langle M_i^{n,\boldsymbol{u}}-\mathbb{E}M_i^{n,\boldsymbol{u}},\varphi\rangle|+|\langle\mathbb{E}M_i^{n,\boldsymbol{u}}-\mathsf{W}\mu(u_i),\varphi\rangle|.$$

For the first term, note that

$$(\mathbb{E}|\langle M_i^{n,\boldsymbol{u}}-\mathbb{E}M_i^{n,\boldsymbol{u}},\varphi\rangle|)^2\le\mathrm{Var}(\langle M_i^{n,\boldsymbol{u}},\varphi\rangle)=\mathrm{Var}\left(\frac{1}{n}\sum_{j=1}^n\xi_{ij}^n\varphi(X_{u_j})\right)\le\frac{1}{n^2}\sum_{j=1}^n(\xi_{ij}^n)^2.$$

Using the assumption (16), we deduce

$$\sup_{\boldsymbol{u}\in I_1^n\times\cdots\times I_n^n}\frac{1}{n}\sum_{i=1}^n\mathbb{E}|\langle M_i^{n,\boldsymbol{u}}-\mathbb{E}M_i^{n,\boldsymbol{u}},\varphi\rangle|\le\left(\frac{1}{n^3}\sum_{i,j=1}^n(\xi_{ij}^n)^2\right)^{1/2}\to 0,$$

and thus, (39) will follow if we show that

$$\lim_{n\to\infty}\sup_{\boldsymbol{u}\in I_1^n\times\cdots\times I_n^n}\frac{1}{n}\sum_{i=1}^n|\langle\mathbb{E}M_i^{n,\boldsymbol{u}}-\mathsf{W}\mu(u_i),\varphi\rangle|=0. \tag{40}$$

Fix $i\in[n]$ and $\boldsymbol{u}\in I_1^n\times\cdots\times I_n^n$ for now. Using $\mathcal{L}(X_{u_j})=\mu_{u_j}$ and the fact that $\xi_{ij}^n=W_{\xi^n}(u_i,u_j)=n\int_{I_j^n}W_{\xi^n}(u_i,v)dv$, we have on the one hand

$$\mathbb{E}\langle M_i^{n,\boldsymbol{u}},\varphi\rangle=\frac{1}{n}\sum_{j=1}^n\xi_{ij}^n\mathbb{E}[\varphi(X_{u_j})]=\sum_{j=1}^n\int_{I_j^n}W_{\xi^n}(u_i,v)\langle\mu_{u_j},\varphi\rangle\,dv.$$

On the other hand,

$$\langle\mathsf{W}\mu(u_i),\varphi\rangle=\int_{[0,1]\times E}W(u_i,v)\varphi(x)\mu(dv,dx)=\int_0^1W(u_i,v)\langle\mu_v,\varphi\rangle\,dv.$$

Hence, to prove (40), we must show equivalently that

$$\lim_{n\to\infty}\sup_{\boldsymbol{u}\in I_1^n\times\cdots\times I_n^n}\frac{1}{n}\sum_{i=1}^n\left|\sum_{j=1}^n\int_{I_j^n}W_{\xi^n}(u_i,v)\langle\mu_{u_j},\varphi\rangle\,dv-\int_0^1W(u_i,v)\langle\mu_v,\varphi\rangle\,dv\right|=0. \tag{41}$$

To prove this, we split the difference into three terms:

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^n\left|\sum_{j=1}^n\int_{I_j^n} W_{\xi^n}(u_i,v)\langle\mu_{u_j},\varphi\rangle\,dv - \int_0^1 W(u_i,v)\langle\mu_v,\varphi\rangle\,dv\right|\\
&\quad\le \frac{1}{n}\sum_{i=1}^n\left|\sum_{j=1}^n\int_{I_j^n}(W_{\xi^n}(u_i,v)\langle\mu_{u_j},\varphi\rangle - W_{\xi^n}(u_i,v)\langle\mu_v,\varphi\rangle)\,dv\right|\\
&\quad+\frac{1}{n}\sum_{i=1}^n\left|\int_0^1 W_{\xi^n}(u_i,v)\langle\mu_v,\varphi\rangle dv - n\int_{I_i^n}\int_0^1 W(u,v)\langle\mu_v,\varphi\rangle dvdu\right|\\
&\quad+\frac{1}{n}\sum_{i=1}^n\left|n\int_{I_i^n}\int_0^1 W(u,v)\langle\mu_v,\varphi\rangle dvdu - \int_0^1 W(u_i,v)\langle\mu_v,\varphi\rangle dv\right|.
\end{aligned}\tag{42}$$

By definition of the step graphon W_{ξ^n}, the first term is equal to

$$\frac{1}{n}\sum_{i=1}^n\left|\sum_{j=1}^n\xi_{ij}^n\int_{I_j^n}\langle\mu_{u_j}-\mu_v,\varphi\rangle\,dv\right|\le\frac{1}{n}\sum_{i,j=1}^n\xi_{ij}^n\int_{I_j^n}\left|\langle\mu_{u_j},\varphi\rangle-\langle\mu_v,\varphi\rangle\right|dv.\tag{43}$$

We deduce from the assumption of weak continuity of $u\longmapsto\mu_u$ that $[0,1]\ni u\longmapsto\langle\mu_u,\varphi\rangle\in\mathbb{R}$ is uniformly continuous. For a given $\epsilon>0$, we can, therefore, choose n large enough so that $|\langle\mu_u-\mu_v,\varphi\rangle|\le\epsilon$ whenever $|u-v|\le 1/n$. Hence, for large-enough n not depending on the choice of $\boldsymbol{u}$, we find that the right-hand side of (43) is bounded by $\overline{A}\epsilon$.

Having dealt with the first term in (42), let us turn to the second. Using the fact that $W_{\xi^n}(u_i,v)=n\int_{I_i^n}W_{\xi^n}(u,v)du$, we can rewrite it as

$$\sum_{i=1}^n\left|\int_{I_i^n}\int_0^1(W_{\xi^n}(u,v)-W(u,v))\langle\mu_v,\varphi\rangle dvdu\right|\le\int_0^1\left|\int_0^1(W_{\xi^n}(u,v)-W(u,v))\langle\mu_v,\varphi\rangle dv\right|du.$$

Because φ is bounded, the right-hand side (which we note does not depend on $\boldsymbol{u}^n$) converges to zero by the assumption that $W_{\xi^n}\to W$ in the strong operator topology.

Finally, the third term in (42) is equal to

$$\sum_{i=1}^n\left|\int_{I_i^n}(\psi(u)-\psi(u_i))du\right|,\tag{44}$$

where we define $\psi(u)=\int_0^1 W(u,v)\langle\mu_v,\varphi\rangle dv$. Recall by assumption that $u\longmapsto W(u,v)dv\in\mathcal{M}_+([0,1])$ is continuous. Because $v\longmapsto\langle\mu_v,\varphi\rangle$ is continuous by assumption, we deduce that ψ is continuous. Therefore, given $\epsilon>0$, we may choose n large enough so that $|\psi(u)-\psi(v)|\le\epsilon$ whenever $|u-v|\le 1/n$, and it follows that (44) is no more than ϵ, regardless of the choice of $\boldsymbol{u}\in I_1^n\times\cdots\times I_n^n$. This concludes the proof of (39).

Step 2. We next show that the set of mean measures $\left\{\frac{1}{n}\sum_{i=1}^n\mathbb{E}M_i^{n,\boldsymbol{u}}:n\ge1,\boldsymbol{u}\in I_1^n\times\cdots\times I_n^n\right\}\subset\mathcal{M}_+(E)$ is tight. The mean measures are given by

$$\frac{1}{n}\sum_{i=1}^n\mathbb{E}M_i^{n,\boldsymbol{u}}=\frac{1}{n^2}\sum_{i,j=1}^n\xi_{ij}^n\mathcal{L}(X_{u_j})=\frac{1}{n^2}\sum_{i,j=1}^n\xi_{ij}^n\mu_{u_j}.$$

Because the map $u\longmapsto\mu_u$ is continuous by assumption, the image $\{\mu_u:u\in[0,1]\}\subset\mathcal{P}(E)$ is compact and thus, tight by Prokhorov's theorem. Hence, for $\epsilon>0$, we may find $K\subset E$ compact such that $\mu_u(K^c)\le\epsilon$ for all $u\in[0,1]$. By (34), $\frac{1}{n}\sum_{i=1}^n\mathbb{E}M_i^{n,\boldsymbol{u}}(K^c)\le\overline{A}\epsilon$.

Step 3. We now prove the claim (37). Let $\mathcal{S}$ denote the set of one-Lipschitz functions $\varphi:E\to[-1,1]$, and let $\epsilon>0$.

By Lemma 2(3), the continuity assumptions on W and the disintegration μ_u imply that the map $[0,1]\ni u\longmapsto\mathsf{W}\mu(u)\in\mathcal{M}_+(E)$ is continuous, and thus, the set of measures $\{\mathsf{W}\mu(u):u\in[0,1]\}\subset\mathcal{M}_+(E)$ is tight. This and Step 2 imply that there exists a compact set $K\subset E$ such that

$$\sup_{u\in[0,1]}\mathsf{W}\mu(u)(K^c)+\sup_{n\in\mathbb{N}}\sup_{\boldsymbol{u}\in I_1^n\times\cdots\times I_n^n}\frac{1}{n}\sum_{i=1}^n\mathbb{E}M_i^{n,\boldsymbol{u}}(K^c)\le\epsilon.\tag{45}$$

The set of one-Lipschitz functions $K \to [-1,1]$ is compact in the uniform topology by Arzelà-Ascoli. We may thus find a finite set $\mathcal{S}_\epsilon \subset \mathcal{S}$ such that $\min_{\psi\in\mathcal{S}_\epsilon}\|(\varphi-\psi)1_K\|_\infty \le \epsilon$ for every $\varphi \in \mathcal{S}$. Now, for any $\varphi,\psi \in \mathcal{S}$ and $u \in [0,1]$, we have

$$|\langle M_i^{n,\boldsymbol{u}} - \mathsf{W}\mu(u),\varphi\rangle| \le |\langle M_i^{n,\boldsymbol{u}} - \mathsf{W}\mu(u),\psi\rangle| + |\langle M_i^{n,\boldsymbol{u}} - \mathsf{W}\mu(u),(\varphi-\psi)1_{K^c}\rangle| \\ + |\langle M_i^{n,\boldsymbol{u}} - \mathsf{W}\mu(u),(\varphi-\psi)1_{K}\rangle|.$$

To estimate the second and third terms, we argue that the total masses of the measures $\frac{1}{n}\sum_{i=1}^n M_i^{n,\boldsymbol{u}}$ and $\mathsf{W}\mu(u)$ are bounded a.s. by some constant $C>0$. Indeed, $\frac{1}{n}\sum_{i=1}^n M_i^{n,\boldsymbol{u}}(E) \le \overline{A}$ a.s. by (34), and the mass $\mathsf{W}\mu(u)(E) = \langle \mathsf{W}\mu(u),1\rangle$ depends continuously on u thanks to Lemma 2(3) and the assumed continuity of W. Hence, for $\boldsymbol{u} \in I_1^n \times\cdots\times I_n^n$,

$$\frac{1}{n}\sum_{i=1}^n \|M_i^{n,\boldsymbol{u}} - \mathsf{W}\mu(u_i)\|_{BL} = \frac{1}{n}\sum_{i=1}^n \sup_{\varphi\in\mathcal{S}}|\langle M_i^{n,\boldsymbol{u}} - \mathsf{W}\mu(u_i),\varphi\rangle| \\ \le \frac{1}{n}\sum_{i=1}^n\left[\max_{\psi\in\mathcal{S}_\epsilon}|\langle M_i^{n,\boldsymbol{u}} - \mathsf{W}\mu(u_i),\psi\rangle| + 2M_i^{n,\boldsymbol{u}}(K^c) + 2\mathsf{W}\mu(u_i)(K^c)\right] + 2C\epsilon.$$

Take expectations, recalling (45), and bound $\max_{\psi\in\mathcal{S}_\epsilon}$ by $\sum_{\psi\in\mathcal{S}_\epsilon}$ to get

$$\frac{1}{n}\sum_{i=1}^n \mathbb{E}\|M_i^{n,\boldsymbol{u}} - \mathsf{W}\mu(u_i)\|_{BL} \le \sum_{\psi\in\mathcal{S}_\epsilon}\frac{1}{n}\sum_{i=1}^n\mathbb{E}|\langle M_i^{n,\boldsymbol{u}} - \mathsf{W}\mu(u_i),\psi\rangle| + 2(2+C)\epsilon,$$

for all $i\in[n]$ and all $\boldsymbol{u} \in I_1^n\times\cdots\times I_n^n$. Send $n\to\infty$ followed by $\epsilon\to 0$ to deduce (37). □

Remark 12. Theorem 6 remains valid under a somewhat weaker convergence assumption than strong operator topology, namely that $\|(\mathsf{W}_{\xi^n}-\mathsf{W})\psi\|_{L^1[0,1]} \to 0$ for $\psi \in C[0,1]$, not necessarily for all $\psi \in L^\infty[0,1]$. This is, of course, what one would call the strong operator topology for the space of operators from $C[0,1] \to L^1[0,1]$. In fact, we do not really need the limit operator W to be an integral operator; it could be something of the form $\mathsf{W}\varphi(u) = \int_{[0,1]}\varphi(v)K_u(dv)$ for some measurable map $u \mapsto K_u \in \mathcal{M}_+(E)$ with $\int_0^1 K_u(E)\,du < \infty$. This is somewhat similar to the (more subtle) notion of *extended graphons* used in the recent study by Jabin et al. [33] of (nongame-theoretic) interacting diffusions, but we will not pursue this generality here.

6.3. Sampling Kernels

The mode of convergence can be further upgraded under the more specific choice of graphon adopted in Theorem 4. Rather than working with a generic matrix ξ^n such that $W_{\xi^n} \to W$, let us now follow a canonical construction in graphon theory. In this section, let us define the empirical measure

$$N_i^{n,\boldsymbol{u}} = \frac{1}{n}\sum_{j=1,\, j\neq i}^n W(u_i,u_j)\delta_{X_{u_j}}, \quad \text{for } \boldsymbol{u} = (u_1,\ldots,u_n)\in[0,1]^n,\ n\in\mathbb{N},$$

where $X_u \sim \mu_u$ for each u are independent as in Section 6.2. In the following, equip $[0,1]^\infty$ with the infinite product measure $(\mathrm{Unif}[0,1])^\infty$.

Theorem 7. *Assume $W:[0,1]^2\to[0,\infty)$ is bounded and measurable. Assume $\{\mu_u : u\in[0,1]\}\subset\mathcal{P}(E)$ is tight. Let $h:[0,1]\times\mathcal{M}_+(E)\to\mathbb{R}$ be bounded and measurable, and assume $h(u,\cdot)$ is continuous on $\mathcal{M}_+(E)$ uniformly in $u\in[0,1]$, in the sense that*

$$\lim_{m'\to m}\sup_{u\in[0,1]}|h(u,m')-h(u,m)| = 0, \quad \forall m\in\mathcal{M}_+(E).$$

Then, for almost every choice of $(u_i)_{i\in\mathbb{N}}\in[0,1]^\infty$, the following holds:

$$\lim_{n\to\infty}\max_{i\in[n]}\mathbb{E}\left|h(u_i,N_i^{n,(u_1,\ldots,u_n)}) - h(u_i,\mathsf{W}\mu(u_i))\right| = 0. \tag{46}$$

Proof. By rescaling, we may assume that $0\le W\le 1$ and $0\le h\le 1$. Let $(u_i)_{i\in\mathbb{N}}$ be arbitrary for now. Let $\varphi:E\to[0,1]$ be measurable, and set $\psi(u)=\mathbb{E}[\varphi(X_u)]$. By the union bound and Hoeffding's inequality,

$$\mathbb{P}\left(\max_{i\in[n]}\left|\frac{1}{n}\sum_{j=1,\,j\neq i}^n W(u_i,u_j)\varphi(X_{u_j}) - \frac{1}{n}\sum_{j=1,\,j\neq i}^n W(u_i,u_j)\psi(u_j)\right| > \delta\right) \le ne^{-2n\delta^2},$$

for each $n \in \mathbb{N}$ and $\delta > 0$. By Borel–Cantelli, we deduce

$$\max_{i\in[n]} \left| \frac{1}{n} \sum_{j=1,\, j\neq i}^{n} W(u_i, u_j)\varphi(X_{u_j}) - \frac{1}{n} \sum_{j=1,\, j\neq i}^{n} W(u_i, u_j)\psi(u_j) \right| \to 0, \text{ a.s.} \tag{47}$$

Next, let $U_i \sim \mathrm{Unif}[0, 1]$ for $i \in \mathbb{N}$ be i.i.d. Again using Hoeffding's inequality, we find

$$\mathbb{P}\left(\left| \frac{1}{n} \sum_{j=1,\, j\neq i}^{n} W(U_i, U_j)\psi(U_j) - \frac{1}{n} \sum_{j=1,\, j\neq i}^{n} \mathbb{E}[W(U_i, U_j)\psi(U_j) | U_i] \right| > \delta \,\middle|\, U_i \right) \leq e^{-2n\delta^2},$$

for each i, a.s. Note that $\frac{1}{n}\sum_{j=1, j\neq i}^{n} \mathbb{E}[W(u, U_j)\psi(U_j)] = \frac{n-1}{n}\mathbb{E}[W(u, U)\psi(U)] = \frac{n-1}{n}\langle \mathsf{W}\mu(u), \varphi\rangle$. Hence, for n large enough that $1/n \leq \delta$, we get

$$\mathbb{P}\left(\left| \frac{1}{n} \sum_{j=1,\, j\neq i}^{n} W(U_i, U_j)\psi(U_j) - \langle \mathsf{W}\mu(U_i), \varphi\rangle \right| > 2\delta \,\middle|\, U_i \right) \leq e^{-2n\delta^2}.$$

Using a union bound and the tower property,

$$\mathbb{P}\left(\max_{i\in[n]} \left| \frac{1}{n} \sum_{j=1,\, j\neq i}^{n} W(U_i, U_j)\psi(U_j) - \langle \mathsf{W}\mu(U_i), \varphi\rangle \right| > 2\delta \right) \leq ne^{-2n\delta^2},$$

again for $n \geq 1/\delta$. Deduce from Borel–Cantelli that

$$\max_{i\in[n]} \left| \frac{1}{n} \sum_{j=1,\, j\neq i}^{n} W(U_i, U_j)\psi(U_j) - \langle \mathsf{W}\mu(U_i), \varphi\rangle \right| \to 0, \text{ a.s.} \tag{48}$$

Combine (47) and (48) to get, for instance,

$$\mathbb{E} \max_{i\in[n]} \left| \frac{1}{n} \sum_{j=1,\, j\neq i}^{n} W(u_i, u_j)\varphi(X_{u_j}) - \langle \mathsf{W}\mu(u_i), \varphi\rangle \right| \to 0,$$

for a.e. choice of $(u_i)_{i\in\mathbb{N}}$. Because we assumed $\{\mu_u : u \in [0, 1]\}$ to be tight, it follows easily from boundedness of W that $\{\mathsf{W}\mu(u) : u \in [0, 1]\} \subset \mathcal{M}_+(E)$ is also tight, and so is $\{\mathbb{E}N_i^{n,\boldsymbol{u}} : n \in \mathbb{N}, i \in [n], \boldsymbol{u} \in [0,1]^n\}$. The latter implies that $\{\mathcal{L}(N_i^{n,\boldsymbol{u}}) : n \in \mathbb{N}, i \in [n], \boldsymbol{u} \in [0,1]^n\} \subset \mathcal{P}(\mathcal{M}_+(E))$ is tight by a well-known argument (Sznitman [46, fact 2.5]), which works not only for probability measures but also, for nonnegative measures of uniformly bounded total mass. We may then argue as in Step 3 of the proof of Theorem 6 that

$$\mathbb{E} \max_{i\in[n]} \| N_i^{n,(u_1,\ldots,u_n)} - \mathsf{W}\mu(u_i)\|_{BL} \to 0,$$

for a.e. choice of $(u_i)_{i\in\mathbb{N}}$. We now easily deduce (46) using the uniform continuity assumption on h. □

7. Approximate Equilibria

In this section, we will prove the results of Section 3.5. Recall that α^* denotes the given W-equilibrium control, X^{α^*} the corresponding state process, and $U \sim \mathrm{Unif}[0, 1]$.

In this section, we will denote $P^{\alpha^*} = \mathcal{L}(U, X^{\alpha^*}) \in \mathcal{P}_{\mathrm{Unif}}([0, 1] \times \mathcal{C}^d)$ the equilibrium joint law, where we recall that $\mathcal{C}^d = C([0, T]; \mathbb{R}^d)$, which is a path space law and which will enable us to use the results proved in Section 6. Let $\mu_\cdot \in C([0, T]; \mathcal{P}_{\mathrm{Unif}}([0, 1] \times \mathbb{R}^d))$ represent the measure flow associated with (U, X^{α^*}) (i.e., $\mu_t := \mathcal{L}(U, X_t^{\alpha^*})$ for all $t \in [0, T]$). Note that μ_t is the time t marginal of P^{α^*}, and thus, $(\mathsf{W}P^{\alpha^*}(u))_t = \mathsf{W}\mu_t(u)$, for each $t \in [0, T]$.

We first elaborate on the notation of Section 3.2 to keep track of the labels (and thus, the controls) assigned to each player. For $n \in \mathbb{N}$ and $\boldsymbol{u}^n := (u_1^n, \ldots, u_n^n) \in [0,1]^n$, let $\boldsymbol{X}^{n,\boldsymbol{u}^n} := (X^{n,u_i^n,i})_{i\in[n]}$ be the process satisfying the dynamics,

$$dX_t^{n,u_i^n,i} = b(t, X_t^{n,u_i^n,i}, \alpha^*(t, u_i^n, X_t^{n,u_i^n,i}))dt + \sigma(t, X_t^{n,u_i^n,i})dB_t^i, \quad X_0^{n,u_i^n,i} \sim \lambda_{u_i^n}, \tag{49}$$

where B^i are independent Brownian motions, and the initial positions $(X_0^{n,u_i^n,i})_{i\in[n]}$ are independent. For each i and each $\beta \in \mathcal{A}_n$, let $X^{n,\beta,u_i^n,i}$ be the process arising when player i switches from the control $\alpha^*(\cdot, u_i^n, \cdot)$ to the control β. More precisely, the process $X^{n,\beta,u_i^n,i}$ is characterized by the dynamics

$$dX_t^{n,\beta,u_i^n,i} = b(t, X_t^{n,\beta,u_i^n,i}, \beta(t, \boldsymbol{X}_t^{n,\beta,\boldsymbol{u}^n,i}))dt + \sigma(t, X_t^{n,\beta,u_i^n,i})dB_t^i, \quad X_0^{n,\beta,u_i^n,i} \sim \lambda_{u_i^n}, \tag{50}$$

where we write $\boldsymbol{X}_t^{n,\beta,\boldsymbol{u}^n,i}$ to denote the vector $\boldsymbol{X}_t^{n,\boldsymbol{u}^n}$ but with the i th component equal to $X_t^{n,\beta,u_i^n,i}$ instead of $X_t^{n,u_i^n,i}$. To simplify the notation, we will sometimes abbreviate $\beta_t = \beta(t, \boldsymbol{X}_t^{n,\beta,\boldsymbol{u}^n,i})$. Let us write also

$$M^{n,\boldsymbol{u}^n,i} := \frac{1}{n}\sum_{j=1}^{n} \xi_{ij}^n \delta_{X^{n,u_j^n,j}}, \tag{51}$$

similarly to (36), for the empirical measure appearing in the objective functions of player i. Note that because $\xi_{ii}^n = 0$, this empirical measure does not depend on the choice of control of player i, and in particular, if player i deviates to β, then the empirical measure (51) does not need to be modified.

Let us introduce some notations that will guide us through the proofs. Recalling the definition of ϵ_i^n, we can bound it by three terms,

$$\epsilon_i^n(\boldsymbol{u}^n) \le \sup_{\beta\in\mathcal{A}_n} \Delta_1^{n,i}(\beta,\boldsymbol{u}^n) + \sup_{\beta\in\mathcal{A}_n} \Delta_2^{n,i}(\beta,\boldsymbol{u}^n) + \Delta_3^{n,i}(\boldsymbol{u}^n),$$

where we defined

$$\begin{aligned}
\Delta_1^{n,i}(\beta,\boldsymbol{u}^n) &:= \mathbb{E}\left[\int_0^T f(t, X_t^{n,\beta,u_i^n,i}, M_t^{n,\boldsymbol{u}^n,i}, \beta_t)dt + g(X_T^{n,\beta,u_i^n,i}, M_T^{n,\boldsymbol{u}^n,i})\right] \\
&\quad - \mathbb{E}\left[\int_0^T f(t, X_t^{n,\beta,u_i^n,i}, \mathsf{W}\mu_t(u_i^n), \beta_t)dt + g(X_T^{n,\beta,u_i^n,i}, \mathsf{W}\mu_T(u_i^n))\right], \\
\Delta_2^{n,i}(\beta,\boldsymbol{u}^n) &:= \mathbb{E}\left[\int_0^T f(t, X_t^{n,\beta,u_i^n,i}, \mathsf{W}\mu_t(u_i^n), \beta_t)dt + g(X_T^{n,\beta,u_i^n,i}, \mathsf{W}\mu_T(u_i^n))\right] \\
&\quad - \mathbb{E}\left[\int_0^T f(t, X_t^{n,u_i^n,i}, \mathsf{W}\mu_t(u_i^n), \alpha^*(t,u_i^n,X_t^{n,u_i^n,i}))dt + g(X_T^{n,u_i^n,i}, \mathsf{W}\mu_T(u_i^n))\right], \\
\Delta_3^{n,i}(\boldsymbol{u}^n) &:= \mathbb{E}\left[\int_0^T f(t, X_t^{n,u_i^n,i}, \mathsf{W}\mu_t(u_i^n), \alpha^*(t,u_i^n,X_t^{n,u_i^n,i}))dt + g(X_T^{n,u_i^n,i}, \mathsf{W}\mu_T(u_i^n))\right] \\
&\quad - \mathbb{E}\left[\int_0^T f(t, X_t^{n,u_i^n,i}, M_t^{n,\boldsymbol{u}^n,i}, \alpha^*(t,u_i^n,X_t^{n,u_i^n,i}))dt + g(X_T^{n,u_i^n,i}, M_T^{n,\boldsymbol{u}^n,i})\right].
\end{aligned}$$

The first term, $\Delta_1^{n,i}$, is the approximation error incurred when player i substitutes the limiting measure $\mathsf{W}\mu_T(u_i^n)$ for the true empirical measure $M_t^{n,\boldsymbol{u}^n,i}$ while using the control β. This is similar for the third term, $\Delta_3^{n,i}$, except now while using the original control $\alpha^*(t,u_i^n,x_i)$. The second term, $\Delta_2^{n,i}$, compares the control β with the control $\alpha^*(t,u_i^n,x_i)$, with the limiting measure in place of the true empirical measure. We will argue that $\Delta_2^{n,i} \le 0$ thanks to the optimality property of α^*, and we will argue that $\Delta_1^{n,i}$ and $\Delta_3^{n,i}$ are small thanks to the convergence of empirical measures.

Lemma 6. *We have* $\sup_{\beta\in\mathcal{A}_n} \Delta_2^{n,i}(\beta,\boldsymbol{u}^n) \le 0$ *for a.e.* $\boldsymbol{u}^n \in [0,1]^n$ *and all* $i \in [n]$.

Proof. Note that $X^{n,u_i^n,i}$ has the same law as $X^{\lambda_{u_i^n},\alpha^*_{u_i^n}}$ as in Lemma 4, where $\alpha_u^*(t,x) := \alpha^*(t,u,x)$. Thus, $\Delta_2^{n,i}(\beta,\boldsymbol{u}^n)$ equals

$$\mathbb{E}\left[\int_0^T f(t, X_t^{n,\beta,u_i^n,i}, \mathsf{W}\mu_t(u_i^n), \beta_t)dt + g(X_T^{n,\beta,u_i^n,i}, \mathsf{W}\mu_T(u_i^n))\right] - J_{\mathsf{W}}^{u_i^n,\lambda_{u_i^n}}(\mu, \alpha_u^*).$$

Recall that $\beta_t = \beta(t, \boldsymbol{X}_t^{n,\beta,\boldsymbol{u}^n,i})$ can depend on all n players' state processes, and for this reason, the claim is not an immediate consequence of Lemma 4. However, this issue is resolved by (28), after noting that the joint law of $(dt\delta_{\beta_t}(da), u_i^n, X^{n,\beta,u_i^n,i})$ belongs to the set $\mathcal{R}_{u_i^n,\lambda_{u_i^n}}$ defined in the proof of Lemma 4. Indeed, we then deduce that

$$\sup_{\beta\in\mathcal{A}_n} \Delta_2^{n,i}(\beta,\boldsymbol{u}^n) \le \sup_{\beta\in\mathcal{A}_1} J_{\mathsf{W}}^{u_i^n,\lambda_{u_i^n}}(\mu,\beta) - J_{\mathsf{W}}^{u_i^n,\lambda_{u_i^n}}(\mu,\alpha_u^*).$$

By Lemma 4, this is ≤ 0 for a.e. $\boldsymbol{u}^n \in [0,1]^n$ and all $i \in [n]$. □

From Lemma 4, we deduce that

$$\epsilon_i^n(\boldsymbol{u}^n) \le \sup_{\beta\in\mathcal{A}_n} \Delta_1^{n,i}(\beta,\boldsymbol{u}^n) + \Delta_3^{n,i}(\boldsymbol{u}^n), \qquad \text{a.e. } \boldsymbol{u}^n.$$

Taking averages, we find

$$\frac{1}{n}\sum_{i=1}^n \epsilon_i^n(\boldsymbol{u}^n) \le \frac{1}{n}\sum_{i=1}^n \sup_{\beta\in\mathcal{A}_n} \Delta_1^{n,i}(\beta,\boldsymbol{u}^n) + \frac{1}{n}\sum_{i=1}^n \Delta_3^{n,i}(\boldsymbol{u}^n). \tag{52}$$

Now that we made use of the optimality of α^*, it remains to use the convergence results of Section 6 to show that the right-hand side of (52) is small.

First, note that $\{\lambda_u : u \in [0,1]\}$ is tight. This is an assumption in Theorems 2 and 4, and in Theorem 3, it is a consequence of the assumed continuity of $u \longmapsto \lambda_u$. By boundedness of b, σ, it is then standard (e.g., using Stroock and Varadhan [44, theorem 1.4.6]) that the set of laws $\{\mathcal{L}(X^{n,\beta,u_i^n,i}) : n \in \mathbb{N}, \beta \in \mathcal{A}_n, \boldsymbol{u}^n \in [0,1]^n, i \in [n]\}$ is a tight subset of $\mathcal{P}(\mathcal{C}^d)$, where we recall that $\mathcal{C}^d := C([0,T];\mathbb{R}^d)$. Letting $\epsilon > 0$, we may then find a compact set $K \subset \mathcal{C}^d$ such that $\sup_{n,\beta,\boldsymbol{u}^n,i} \mathbb{P}(X^{n,\beta,u_i^n,i} \notin K) \le \epsilon$. Define the function $h : [0,1] \times \mathcal{M}_+(\mathcal{C}^d) \to \mathbb{R}$ by

$$h(u,m) = \sup_{a \in A} \sup_{z \in K} \left| \int_0^T (f(t,z_t,\mathsf{W}\mu_t(u),a) - f(t,z_t,m_t,a))dt \right| + \left| g(z_T,\mathsf{W}\mu_T(u)) - g(z_T,m_T) \right|, \tag{53}$$

where $m_t \in \mathcal{M}_+(\mathbb{R}^d)$ denotes the image of a measure $m \in \mathcal{M}_+(\mathcal{C}^d)$ by the coordinate map $x \longmapsto x_t$. Because $f(t,x,m,a)$ and $g(x, m)$ are bounded, measurable, and continuous in (x, m, a), we deduce that function h is bounded and measurable (Aliprantis and Border [1, theorem 18.19]). Moreover, it follows from compactness of A and K that $h(u,\cdot)$ is continuous on $\mathcal{M}_+(\mathcal{C}^d)$ for each $u \in [0,1]$. Note that $h(u,\mathsf{W}\mu(u)) = 0$ for every u. In order to bound (52) in terms of h, let us choose $C > 0$ such that $\max(|f|,|g|) \le C$, and then, note that

$$\frac{1}{n}\sum_{i=1}^n \epsilon_i^n(\boldsymbol{u}^n) \le \frac{2}{n}\sum_{i=1}^n \mathbb{E}|h(u_i^n, M^{n,\boldsymbol{u}^n,i})| + 8\epsilon C. \tag{54}$$

The rest of the argument is different for Theorem 2 versus Theorem 3.

7.1. General Kernels

We first prove Theorem 2. Recall that $U_i^n \sim \mathrm{Unif}(I_i^n)$ are independent, and let $\boldsymbol{U}^n = (U_1^n,\dots,U_n^n)$. Abbreviate $\boldsymbol{I}^n := I_1^n \times \cdots \times I_n^n$, and note that $\boldsymbol{U}^n$ is uniform on $\boldsymbol{I}^n$. Let us also define processes $Y^{n,i}$ such that $(U_i^n, Y^{n,i})_{i\in[n]}$ are independent, with $\mathcal{L}(Y^{n,i} \mid U_i^n = u) = \mathcal{L}(X^{\alpha^*} \mid U = u)$ for $u \in I_i^n$. Let $\boldsymbol{Y}^n = (Y^{n,1},\dots,Y^{n,n})$, and define the neighborhood empirical measures (random measures on $\mathcal{C}^d$)

$$M^{n,i} = \frac{1}{n}\sum_{j=1}^n \xi_{ij}^n \delta_{Y^{n,j}}.$$

Recall that the process $X^{n,u_i^n,i}$ defined in the beginning of the section is such that $\mathcal{L}(X^{n,u_i^n,i}) = \mu_{u_i^n}$. Recalling that (U, X^{α^*}) denotes the equilibrium pair, we have

$$\mathcal{L}(X^{\alpha^*} \mid U = u_i^n) = \mathcal{L}(Y^{n,i} \mid U_i^n = u_i^n) = \mathcal{L}(X^{n,u_i^n,i}). \tag{55}$$

Hence, for a.e. $\boldsymbol{u}^n$ and any bounded measurable function $\varphi : \boldsymbol{I}^n \times (\mathcal{C}^d)^n \to \mathbb{R}$, we can write

$$\mathbb{E}[\varphi(\boldsymbol{u}^n, \boldsymbol{X}^{n,\boldsymbol{u}^n})] = \mathbb{E}[\varphi(\boldsymbol{U}^n,\boldsymbol{Y}^n) \mid \boldsymbol{U}^n = \boldsymbol{u}^n], \quad \text{a.e. } \boldsymbol{u}^n \in \boldsymbol{I}^n.$$

In particular, because the empirical measure $M^{n,\boldsymbol{u}^n,i}$ defined in (51) is a functional of $\boldsymbol{X}^{n,\boldsymbol{u}^n}$, we deduce similarly that

$$\mathbb{E}[\varphi(\boldsymbol{u}^n, M^{n,\boldsymbol{u}^n,i})] = \mathbb{E}[\varphi(\boldsymbol{U}^n, M^{n,i}) \mid \boldsymbol{U}^n = \boldsymbol{u}^n], \quad \text{a.e. } \boldsymbol{u}^n \in \boldsymbol{I}^n,$$

for bounded measurable $\varphi : \boldsymbol{I}^n \times \mathcal{M}_+(\mathcal{C}^d) \to \mathbb{R}$. Applying this in (54), along with the tower property, we deduce

$$\frac{1}{n}\sum_{i=1}^n \mathbb{E}[\epsilon_i^n(\boldsymbol{U}^n)] \le \frac{2}{n}\sum_{i=1}^n \mathbb{E}\,|\,h(U_i^n, M^{n,i})| + 8\epsilon C. \tag{56}$$

The identities (55) put us in the setting of Theorem 5. As noted, h is bounded and continuous in its second variable. Hence, Theorem 5 implies that

$$\frac{1}{n}\sum_{i=1}^n \mathbb{E}|h(U_i^n, M^{n,i})| \to \mathbb{E}|h(U, \mathsf{W}P^{\alpha^*}(U))| = 0,$$

where $P^{\alpha^*} := \mathcal{L}(U, X^{\alpha^*})$, with the last identity using the fact that $h(u, \mathsf{W}P^{\alpha^*}(u)) = 0$ for all u, which is a consequence of the identity of time t marginals $(\mathsf{W}P^{\alpha^*}(u))_t = \mathsf{W}\mu_t$. Applying this in (56) and then, sending $\epsilon \to 0$ complete the proof of Theorem 2.

7.2. Continuous Kernels

We next prove Theorem 3. The fact that (1) and (2a)–(2d) imply (2) is a consequence of Corollary 1. The function $h(u, m)$ from (53) is continuous in m, uniformly in u because

$$\sup_{u\in[0,1]} |h(u,m') - h(u,m)| \le \sup_{a\in A}\sup_{z\in K}\left|\int_0^T (f(t,z_t,m'_t,a) - f(t,z_t,m_t,a))dt\right| + \left|g(z_T,m'_T) - g(z_T,m_T)\right|,$$

and the right-hand side vanishes as $m' \to m$ by compactness of A and K and by joint continuity of f and g. Also using the continuity assumptions of Theorem 3, we are, therefore, in the setting of Theorem 6.

Recalling again that $h(u, \mathsf{W}P^{\alpha^*}(u)) = 0$ for all u where again $P^{\alpha^*} := \mathcal{L}(U, X^{\alpha^*})$, Theorem 6 yields

$$\lim_{n\to\infty}\sup_{\boldsymbol{u}^n\in \boldsymbol{I}^n}\frac{1}{n}\sum_{i=1}^n \mathbb{E}\,|\,h(u_i^n, M^{n,\boldsymbol{u}^n,i})| = 0.$$

Apply this in (54), and then, send $\epsilon \to 0$ to deduce Theorem 3.

7.3. Sampling Kernels

We finally prove Theorem 4. Again, let $P^{\alpha^*} := \mathcal{L}(U, X^{\alpha^*})$, and write $P^{\alpha^*}(du,dx) = duP_u^{\alpha^*}(dx)$ for its disintegration. To prepare for an application of Theorem 4, let us first argue that $\{P_u^{\alpha^*} : u \in [0, 1]\}$ is tight. Note that $P_u^{\alpha^*}$ is the law of the solution of the SDE

$$dX_t = b(t, X_t, \alpha^*(t, u, X_t))dt + \sigma(t, X_t)dB_t, \quad X_0 \sim \lambda_u.$$

Because b and σ are bounded and $\{\lambda_u : u \in [0, 1]\}$ is tight by assumption, the tightness of $\{P_u^{\alpha^*} : u \in [0, 1]\}$ follows easily (e.g., using Stroock and Varadhan [44, theorem 1.4.6]).

Now, recall that $(u_i)_{i\in\mathbb{N}} \in [0,1]^\infty$, where $[0,1]^\infty$ is equipped with $(\mathrm{Unif}[0, 1])^\infty$, and $\xi_{ij}^n = W(u_i, u_j)1_{i\ne j}$ for $i, j \in [n]$ in Theorem 4. As in (54), we have

$$\epsilon_i^n(u_1,\ldots,u_n) \le 2\mathbb{E}|h(u_i, N_i^{n,(u_1,\ldots,u_n)})| + 8\epsilon C, \tag{57}$$

where we define $N_i^{n,(u_1,\ldots,u_n)} = \frac{1}{n}\sum_{j=1,j\ne i}^n W(u_i,u_j)\delta_{X^{n,u_j,j}}$. Recalling that $h(u, \mathsf{W}P^{\alpha^*}(u)) = 0$ for all u, we may thus apply Theorem 7 to get

$$\lim_{n\to\infty}\max_{i\in[n]}\mathbb{E}|h(u_i, N_i^{n,(u_1,\ldots,u_n)})| = 0, \quad \text{for a.e. } (u_i)_{i\in\mathbb{N}} \in [0,1]^\infty.$$

Combine this with (57) and then, send $\epsilon \to 0$ to complete the proof.

Remark 13. Theorem 4 could likely be strengthened to include a rate of convergence if one imposed further continuity assumptions on f and g. The estimates stemming from Hoeffding's inequality in the proof of Theorem 4 could, in principle, be traced through to yield exponential bounds on the measure of the set of $(u_1,\ldots,u_n) \in [0,1]^n$ such that $\max_{i\in[n]}\epsilon_i^n(u_1,\ldots,u_n) > \epsilon$. See Aurell et al. [2, proposition 3] for a related result based on a clever application of the law of the iterated logarithm.

8. A Linear-Quadratic Example

In this section, we study a linear-quadratic model of *flocking behavior*, inspired by Carmona et al. [20] and Lacker and Soret [38], which is simple and yet, rich enough to exhibit an interesting dependence on the structure of the interaction matrix. This will also illustrate the relative simplicity of our formulation of graphon equilibrium. It should be noted that the model in this section does not fit into the standing assumptions imposed for the theoretical developments in Section 3. However, the definitions of the equilibrium require little adaptation for the setting considered.

We work in dimension $d = 1$. We shall now assume $W \in L_+^2[0,1]^2$ (i.e., the kernel is square integrable). For $m \in \mathcal{P}([0, 1] \times \mathbb{R})$, recall the definition of the measure-valued function $\mathsf{W}m : [0, 1] \to \mathcal{M}_+(\mathbb{R})$ from (7). We define its mean $\overline{\mathsf{W}}m : [0, 1] \to \mathbb{R}$ by

$$\overline{\mathsf{W}}m(u) = \int_{[0,1]\times\mathbb{R}} W(u,v)\,x\,m(dv,dx),$$

whenever this integral is well defined. The linear-quadratic model we study can be summarized concisely, as in

(12), as follows:

$$\begin{cases} \inf_{\alpha \in \mathcal{A}} & \frac{1}{2}\mathbb{E}\left[\int_0^T \alpha_t^2 dt + c|X_T - \overline{\mathsf{W}}\mu_T(U)|^2\right] \\ \text{s.t.} & dX_t = \alpha_t dt + \sigma dB_t, \\ & \mu_t = \mathcal{L}(U, X_t),\ (U, X_0) \sim \lambda \in \mathcal{P}_{\text{Unif}}([0,1] \times \mathbb{R}). \end{cases} \tag{58}$$

Note that, in equilibrium, $\overline{\mathsf{W}}\mu_T(u) = \mathbb{E}[W(u,U)X_T]$ for a.e. u. In the notation of Section 3, we are choosing $A = \mathbb{R}$ and

$$b(t,x,a) = a, \quad \sigma(t,x) = \sigma, \quad f(t,x,m,a) = -\frac{1}{2}a^2, \quad g(x,m) = -c\left(x - \int_{\mathbb{R}} x\, m(dx)\right)^2.$$

Proposition 4. *Assume $W \in L^2_+[0,1]^2$ satisfies $\|W\|_{L^2[0,1]^2} < 1 + (cT)^{-1}$. Assume λ has a finite second moment:*

$$\int_{[0,1]\times\mathbb{R}} x^2\, \lambda(du, dx) < \infty.$$

Then, there exists a W equilibrium with associated control given by

$$\alpha(t,u,x) = \frac{c}{c(T-t)+1}(M(u) - x),$$

where $M \in L^2[0,1]$ is defined by

$$M := \frac{1}{cT+1}\mathsf{W}\left(\mathsf{Id} - \frac{cT}{cT+1}\mathsf{W}\right)^{-1}\psi, \tag{59}$$

with $\psi \in L^2[0,1]$ defined by $\psi(u) := \mathbb{E}[X_0 | U = u]$. Here, Id is the identity operator, and W is viewed as an operator on $L^2[0,1]$ as defined in (4).

The assumption that $\|W\|_{L^2[0,1]^2} < 1 + (cT)^{-1}$ ensures the existence of the inverse operator appearing in (59). Equilibria may fail to exist without this assumption. Indeed, if $W \equiv 1 + cT$, then the proof shows that there is no solution, unless $\mathbb{E}[X_0 \mid U] = 0$ a.s., in which case the solution is as before with $M \equiv 0$.

There is a notable appearance here of a common notion of *centrality* used in graph theory.

If X_0 and U are independent, then $\psi \equiv \mathbb{E}[X_0]$, and so,

$$M = \mathbb{E}[X_0]\frac{1}{cT+1}\mathsf{W}\left(\mathsf{Id} - \frac{cT}{cT+1}\mathsf{W}\right)^{-1}\mathbf{1} = \frac{1}{cT}\mathbb{E}[X_0]\left[\left(\mathsf{Id} - \frac{cT}{cT+1}\mathsf{W}\right)^{-1} - \mathsf{Id}\right]\mathbf{1},$$

where $\mathbf{1}$ is the constant function equal to one. The quantity $\left[\left(\mathsf{Id} - \frac{cT}{cT+1}\mathsf{W}\right)^{-1} - \mathsf{Id}\right]\mathbf{1}(u)$ is precisely the *Katz centrality* or *α centrality* of the vertex $u \in [0,1]$, or rather, the infinite-dimensional (graphon) analogue thereof, with parameter $\alpha = cT/(cT+1)$. When X_0 and U are not independent, we have instead a generalization of this centrality concept in which a vertex u receives a weight proportional to the mean initial position $\psi(u)$. Note if $X_0 = h(U)$ is U measurable, then $\psi = h$.

8.1. Derivation of the Solution

We follow roughly the PDE approach discussed in Section 3.6. We fix for now a mean field term and compute the best response. That is, we fix for now a measurable function $M : [0,1] \to \mathbb{R}$ to play the role of the mean function $\overline{\mathsf{W}}\mu_T$. The stochastic control problem in (58) is associated with the Hamilton-Jacobi-Bellman (HJB) equation

$$\partial_t v(t,u,x) - \frac{1}{2}|\partial_x v(t,u,x)|^2 + \frac{\sigma^2}{2}\partial_{xx}v(t,u,x) = 0, \quad v(T,u,x) = c(x - M(u))^2.$$

The corresponding optimal control is $\alpha(t,u,x) = -\partial_x v(t,u,x)$. We solve this PDE explicitly using the ansatz $v(t,u,x) = \psi(t) + \frac{1}{2}\varphi(t)(x - M(u))^2$, where φ and ψ are functions on $[0,T]$ to be determined. Plugging this ansatz into the HJB, we obtain that φ and ψ should satisfy

$$\frac{1}{2}(x - M(u))^2(\varphi'(t) - \varphi(t)^2) + \frac{\sigma^2}{2}\varphi(t) + \psi'(t) = 0,$$

for all $(t,u,x) \in (0,T) \times [0,1] \times \mathbb{R}$, along with the terminal conditions $\varphi(T) = c$ and $\psi(T) = 0$. Matching coefficients, we find

$$\varphi'(t) = \varphi^2(t), \qquad \psi'(t) = -\frac{\sigma^2}{2}\varphi(t).$$

This system is easily solved using the aforementioned boundary conditions:

$$\varphi(t) = \frac{c}{c(T-t)+1}, \qquad \psi(t) = \frac{\sigma^2}{2}\log(c(T-t)+1).$$

The optimal control is thus given by $\alpha^*(t,u,x) = -\partial_x v(t,u,x) = \frac{c}{c(T-t)+1}(M(u)-x)$, and the optimal state process thus satisfies the following dynamics:

$$dX_t = \frac{c}{c(T-t)+1}(M(U)-X_t)dt + \sigma dB_t, \qquad (U, X_0) \sim \lambda.$$

Define $\mu_t = \mathcal{L}(U, X_t)$ for each $t \in [0,T]$. Then, μ is a graphon equilibrium if and only if $M(u) = \overline{\mathsf{W}}\mu_T(u)$ (i.e., $M(u) = \mathbb{E}[W(u,U)X_T]$) for a.e. $u \in [0,1]$. In other words, we will have an equilibrium if we can solve the (McKean–Vlasov) SDE

$$\begin{aligned} dX_t &= \frac{c}{c(T-t)+1}(\overline{\mathsf{W}}\mu_T(U)-X_t)dt + \sigma dB_t, \qquad (U, X_0) \sim \lambda, \\ \mu_t &= \mathcal{L}(U, X_t), \quad t \in [0,T]. \end{aligned} \tag{60}$$

To solve this equation, it is convenient to introduce an independent copy $(\widetilde{B}, \widetilde{U}, \widetilde{X})$ of (B, U, X). As a first step, we find an expression for $\overline{\mathsf{V}}\mu_T(U)$ for every kernel $V \in L^2_+[0,1]^2$, where we note by definition that

$$\overline{\mathsf{V}}\mu_T(U) = \int_{[0,1]\times\mathbb{R}} V(U,v)\, x\, \mu_T(dv,dx) = \mathbb{E}[V(U,\widetilde{U})\widetilde{X}_T|U]. \tag{61}$$

To find an expression for this, note that the SDE (60) implies

$$\widetilde{X}_t = \widetilde{X}_0 + \int_0^t \frac{c}{c(T-s)+1}(\overline{\mathsf{W}}\mu_T(\widetilde{U}) - \widetilde{X}_s)\, ds + \sigma\widetilde{B}_t.$$

Multiply by $V(U,\widetilde{U})$, and take conditional expectations given U, using independence of $\widetilde{B}$ and $\widetilde{U}$, to get

$$\begin{aligned} \overline{\mathsf{V}}\mu_t(U) &= \mathbb{E}[V(U,\widetilde{U})\widetilde{X}_t|U] \\ &= \overline{\mathsf{V}}\mu_0(U) + \int_0^t \frac{c}{c(T-s)+1}(\mathbb{E}[V(U,\widetilde{U})\overline{\mathsf{W}}\mu_T(\widetilde{U})|U] - \overline{\mathsf{V}}\mu_s(U))\, ds. \end{aligned} \tag{62}$$

The second to last term simplifies by Fubini's theorem:

$$\begin{aligned} \mathbb{E}[V(U,\widetilde{U})\overline{\mathsf{W}}\mu_T(\widetilde{U})|U] &= \int_0^1 V(U,\widetilde{u})\overline{\mathsf{W}}\mu_T(\widetilde{u})\, d\widetilde{u} \\ &= \int_0^1 V(U,\widetilde{u})\int_{[0,1]\times\mathbb{R}} W(\widetilde{u},v)\, x\, \mu_T(dv,dx)\, d\widetilde{u} \\ &= \int_{[0,1]\times\mathbb{R}} V\circ W(U,v)\, x\, \mu_T(dv,dx). \end{aligned}$$

Here, we define $V \circ W \in L^2_+[0,1]^2$ by $V\circ W(u,v) := \int_0^1 V(u,\widetilde{u})W(\widetilde{u},v)\, d\widetilde{u}$, which is exactly the kernel of the composition operator $\mathsf{V}\circ\mathsf{W}$, which we abbreviate as VW. We may thus write

$$\mathbb{E}[V(U,\widetilde{U})\overline{\mathsf{W}}\mu_T(\widetilde{U})|U] = \overline{\mathsf{VW}}\mu_T(U).$$

Use this identity and differentiate (62) to find that $(\overline{\mathsf{V}}\mu_t(u))_{t\in[0,T]}$ obeys the differential equation

$$\frac{d}{dt}\overline{\mathsf{V}}\mu_t(u) = \frac{c}{c(T-t)+1}(\overline{\mathsf{VW}}\mu_T(u) - \overline{\mathsf{V}}\mu_t(u)).$$

It follows that $\overline{\mathsf{V}}\mu_t(u)$ must be of the form

$$\overline{\mathsf{V}}\mu_t(u) = \overline{\mathsf{VW}}\mu_T(u) + \kappa(u)(c(T-t)+1), \quad t \in [0,T],$$

for a u-dependent parameter $\kappa(u)$ to be determined by the initial conditions. Setting $t = 0$ implies $\kappa(u) = \frac{1}{cT+1}(\overline{\mathsf{V}}\mu_0(u) - \overline{\mathsf{VW}}\mu_T(u))$, and thus,

$$\overline{\mathsf{V}}\mu_t(u) = \frac{ct}{cT+1}\overline{\mathsf{VW}}\mu_t(u) + \frac{c(T-t)+1}{cT+1}\overline{\mathsf{V}}\mu_0(u). \tag{63}$$

In particular, setting $t = T$ and noting that $\overline{\mathsf{V}}\mu_T(u)$ depends linearly on the operator V, we find

$$\overline{\mathsf{V}\left(\mathsf{Id} - \frac{cT}{cT+1}\mathsf{W}\right)}\mu_T(u) = \frac{1}{cT+1}\overline{\mathsf{V}}\mu_0(u). \tag{64}$$

Choosing $\mathsf{V} = \mathsf{W}\left(\mathsf{Id} - \frac{cT}{cT+1}\mathsf{W}\right)^{-1}$ yields

$$\overline{\mathsf{W}}\mu_T(u) = \frac{1}{cT+1}\overline{\mathsf{W}\left(\mathsf{Id} - \frac{cT}{cT+1}\mathsf{W}\right)^{-1}}\mu_0(u). \tag{65}$$

Note also that $\mu_0 = \mathcal{L}(U, X_0)$, and so, for any kernel V, we have

$$\overline{\mathsf{V}}\mu_0(u) = \mathbb{E}[V(u,U)X_0] = \mathbb{E}[V(u,U)\psi(U)] = \mathsf{V}\psi(u),$$

where $\psi(u) := \mathbb{E}[X_0 | U = u]$. Combining this with (65) shows that $M(u) = \overline{\mathsf{W}}\mu_T(u)$ is given by (59).

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