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The Iwasawa Main Conjectures for GL_2 and derivatives of p-adic L-functions

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We prove under mild hypotheses the three-variable Iwasawa Main Conjecture for p-ordinary modular forms base changed to an imaginary quadratic field $\mathcal{K}$ in which p splits in the indefinite setting (in the definite setting this is a result due to Skinner–Urban). Being in a setting encompassing Heegner points and their variation in p-adic families, our main result has new applications to Greenberg's nonvanishing conjecture for central derivatives of p-adic L-functions of Hida families with root number -1.

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1. Introduction

Fix a positive integer N and a prime $p \nmid 6N$. Let $\boldsymbol{f} = \sum_{n=1}^{\infty} \boldsymbol{a}_n q^n \in \mathbb{I}[\![q]\!]$ be a Hida family of tame level N, where $\mathbb{I}$ is a finite flat extension of the one-variable Iwasawa algebra $\mathfrak{O}_L[\![T]\!]$ with coefficients in the ring of integers $\mathfrak{O}_L$ of a finite extension L of $\mathbf{Q}_p$. Let

$$\rho_{\boldsymbol{f}} : G_{\mathbf{Q}} := \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \mathrm{Aut}_{F_{\mathbb{I}}}(V_{\boldsymbol{f}}) \simeq \mathrm{GL}_2(F_{\mathbb{I}}),$$

where $F_{\mathbb{I}}$ denotes the fraction field of $\mathbb{I}$, be the Galois representation associated to $\boldsymbol{f}$ (which we take to be the contragradient of the Galois representation first constructed in [25]), and let $\bar{\rho}_{\boldsymbol{f}} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\kappa_{\mathbb{I}})$, where $\kappa_{\mathbb{I}} = \mathbb{I}/\mathfrak{m}_{\mathbb{I}}$ is the residue field of $\mathbb{I}$, be the associated semi-simple residual representation. By work of Mazur and Wiles [44,66], upon restriction to a decomposition group $D_p \subset G_{\mathbf{Q}}$ at p we have

$$\bar{\rho}_{\boldsymbol{f}}|_{D_p} \sim \begin{pmatrix} \bar{\varepsilon} & * \\ & \bar{\delta} \end{pmatrix}$$

where the character $\bar{\delta}$ is unramified. We assume that

$$\bar{\rho}_{\boldsymbol{f}} \text{ is absolutely irreducible,} \tag{irred}$$

and fix a $G_{\mathbf{Q}}$-stable lattice $T_{\boldsymbol{f}} \subset V_{\boldsymbol{f}}$ which is free of rank two over $\mathbb{I}$. Denote by $\mathscr{F}^- T_{\boldsymbol{f}}$ the I_p-coinvariants of $T_{\boldsymbol{f}}$, where $I_p \subset D_p$ is the inertia subgroup, and set

$$A_{\boldsymbol{f}} := T_{\boldsymbol{f}} \otimes_{\mathbb{I}} \mathbb{I}^{\vee}, \qquad \mathscr{F}^- A_{\boldsymbol{f}} := (\mathscr{F}^- T_{\boldsymbol{f}}) \otimes_{\mathbb{I}} \mathbb{I}^{\vee},$$

where $\mathbb{I}^{\vee} = \mathrm{Hom}_{\mathrm{cts}}(\mathbb{I}, \mathbf{Q}_p/\mathbf{Z}_p)$ is the Pontryagin dual of $\mathbb{I}$.

Let $\mathcal{K}$ be an imaginary quadratic field of discriminant prime to Np, and let $\Gamma_{\mathcal{K}} = \mathrm{Gal}(\mathcal{K}_\infty/\mathcal{K})$ be the Galois group of the maximal $\mathbf{Z}_p^2$-extension of $\mathcal{K}$ unramified outside p. The *Greenberg Selmer group* of $A_{\boldsymbol{f}}$ over $\mathcal{K}_\infty$ is defined by

$$\mathrm{Sel}_{\mathtt{Gr}}(\mathcal{K}_\infty, A_{\boldsymbol{f}}) := \ker\left\{\mathrm{H}^1(\mathcal{K}_\infty, A_{\boldsymbol{f}}) \longrightarrow \prod_{w\nmid p}\mathrm{H}^1(I_w, A_{\boldsymbol{f}}) \times \prod_{w\mid p}\mathrm{H}^1(\mathcal{K}_{\infty,w}, \mathscr{F}^- A_{\boldsymbol{f}})\right\}, \quad (1.1)$$

where w runs over the corresponding places of $\mathcal{K}_\infty$. The Pontryagin dual

$$X_{\mathtt{Gr}}(\mathcal{K}_\infty, A_{\boldsymbol{f}}) := \mathrm{Hom}_{\mathrm{cts}}(\mathrm{Sel}_{\mathtt{Gr}}(\mathcal{K}_\infty, A_{\boldsymbol{f}}), \mathbf{Q}_p/\mathbf{Z}_p)$$

is well-known to be a finitely generated $\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]$-module.

Assume also that

$$\bar{\rho}_{\boldsymbol{f}} \text{ is } p\text{-distinguished, i.e., } \bar{\varepsilon} \neq \bar{\delta}. \quad \text{(dist)}$$

Thanks to [67], it follows that $\mathscr{F}^- T_{\boldsymbol{f}}$ is $\mathbb{I}$-free of rank one. Moreover, from the work of Hida [27] there exists a 3-variable p-adic L-function $L_p^{\mathtt{Hi}}(\boldsymbol{f}/\mathcal{K}) \in \mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]$ uniquely characterized by the interpolation of the critical values for the Rankin–Selberg L-function $L(\boldsymbol{f}_\phi/\mathcal{K}, \chi, s)$ attached to the classical specializations $\boldsymbol{f}_\phi$ (base changed to $\mathcal{K}$) of $\boldsymbol{f}$ twisted by finite order characters $\chi : \Gamma_{\mathcal{K}} \to \mu_{p^\infty}$.

An instance of the Iwasawa–Greenberg main conjectures formulated in [23] then predicts the following. From now on in this Introduction and in our main results we shall assume that $\mathbb{I}$ is regular.

Iwasawa–Greenberg Main Conjecture. *The module $X_{\mathtt{Gr}}(\mathcal{K}_\infty, A_{\boldsymbol{f}})$ is $\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]$-torsion, and*

$$\mathrm{Char}_{\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]}(X_{\mathtt{Gr}}(\mathcal{K}_\infty, A_{\boldsymbol{f}})) = (L_p^{\mathtt{Hi}}(\boldsymbol{f}/\mathcal{K}))$$

as ideals in $\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]$.

Many cases of this conjecture are known by the work of Skinner–Urban [57] and [38]. As we shall explain below, in this paper we place ourselves in a setting complementary to that in [57]. Write

$$N = N^+N^-$$

with N^- being the largest factor of N divisible only by primes inert in $\mathcal{K}$. The following is our main result towards the Iwasawa–Greenberg Main Conjecture.

Theorem A. *In addition to* (irred) *and* (dist), *assume that:*

- *N is squarefree,*

- *some specialization* $\boldsymbol{f}_\phi$ *is the* p*-stabilization of a newform* $f \in S_2(\Gamma_0(N))$*,*
- N^- *is the product of a positive even number of primes,*
- $\bar{\rho}_{\boldsymbol{f}}$ *is ramified at every prime* $q|N^-$*,*
- p *splits in* $\mathcal{K}$*.*

Then $X_{\mathtt{Gr}}(\mathcal{K}_\infty, A_{\boldsymbol{f}})$ *is* $\mathbb{I}[\![\Gamma_\mathcal{K}]\!]$*-torsion, and*

$$\mathrm{Char}_{\mathbb{I}[\![\Gamma_\mathcal{K}]\!]}(X_{\mathtt{Gr}}(\mathcal{K}_\infty, A_{\boldsymbol{f}})) = (L_p^{\mathtt{Hi}}(\boldsymbol{f}/\mathcal{K}))$$

as ideals in $\mathbb{I}[\![\Gamma_\mathcal{K}]\!] \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$*.*

As in [57], the fact that $X_{\mathtt{Gr}}(\mathcal{K}_\infty, A_{\boldsymbol{f}})$ is $\mathbb{I}[\![\Gamma_\mathcal{K}]\!]$-torsion follows easily from Kato's work [38], and the proof of Theorem A is reduced to establishing the divisibility "$\subseteq$" as ideals in $\mathbb{I}[\![\Gamma_\mathcal{K}]\!]$ predicted by the main conjecture. For the proof of this divisibility, in [57] the authors study congruences between p-adic families of cuspidal automorphic forms and Eisenstein series on $\mathrm{GU}(2,2)$, and their method (in particular, their application of Vatsal's nonvanishing results [61]) relies crucially on their hypothesis that N^- is the squarefree product of an *odd* number of primes. In contrast, when N^- is divisible by an even number of primes as in Theorem A, the central L-values studied in [61] all vanish for sign reasons, and another approach is needed.

Our main idea for the proof of Theorem A is to use Beilinson–Flach classes and their explicit reciprocity laws [41,40] to link the Iwasawa–Greenberg Main Conjecture for $L_p^{\mathtt{Hi}}(\boldsymbol{f}/\mathcal{K})$ to the main conjecture for a different p-adic L-function $\mathscr{L}_\mathfrak{p}(\boldsymbol{f}/\mathcal{K})$ studied by the second-named author [64] using Eisenstein congruences on $\mathrm{GU}(3,1)$, and then exploit our assumption on N^- to prove the latter main conjecture using Heegner points and their variation in p-adic families [35,19,12,13].

As a consequence of our approach, we also obtain an application to Greenberg's conjecture (see [47, §0] and [22]) on the generic order of vanishing at the center of the p-adic L-functions attached to cusp form in Hida families. To state this, assume for simplicity that $\mathbb{I}$ is just $\mathfrak{O}_L[\![T]\!]$, and for each $k \in \mathbf{Z}_{\geqslant 2}$ let $\boldsymbol{f}_k$ be the p-stabilized newform on $\Gamma_0(Np)$ obtained by setting $T = (1+p)^{k-2} - 1$ in $\boldsymbol{f}$. One can show that the p-adic L-functions $L_p^{\mathtt{MTT}}(\boldsymbol{f}_k, s)$ of [43] satisfy a functional equation

$$L_p^{\mathtt{MTT}}(\boldsymbol{f}_k, s) = -wL_p^{\mathtt{MTT}}(\boldsymbol{f}_k, k-s)$$

with a sign $w = \pm 1$ independent of $k \in \mathbf{Z}_{>2}$ with $k \equiv 2 \pmod{p-1}$.

Greenberg's nonvanishing conjecture. *Let* $e \in \{0,1\}$ *be such that* $-w = (-1)^e$*. Then*

$$\left.\frac{L_p^{\mathtt{MTT}}(\boldsymbol{f}_k, s)}{(s-k/2)^e}\right|_{s=k/2} \neq 0,$$

for all but finitely many $k \in \mathbf{Z}_{\geqslant 2}$ *with* $k \equiv 2 \pmod{p-1}$*.*

In other words, for all but finitely many k as above, the order of vanishing of $L_p^{\mathtt{MTT}}(\boldsymbol{f}_k, s)$ at the center should be the least allowed by the sign in the functional equation.

To state our result in the direction of this conjecture, let

$$T_{\boldsymbol{f}}^{\dagger} := T_{\boldsymbol{f}} \otimes \Theta^{-1}$$

be the self-dual twist of $T_{\boldsymbol{f}}$. By work of Plater [51] (and more generally, Nekovář [46]) there is a cyclotomic $\mathbb{I}$-adic height pairing

$$\langle -, - \rangle_{\mathcal{K},\mathbb{I}}^{\mathrm{cyc}} : \mathrm{Sel}_{\mathtt{Gr}}(\mathcal{K}, T_{\boldsymbol{f}}^{\dagger}) \times \mathrm{Sel}_{\mathtt{Gr}}(\mathcal{K}, T_{\boldsymbol{f}}^{\dagger}) \longrightarrow F_{\mathbb{I}} \tag{1.2}$$

interpolating the p-adic height pairings for the classical specialization of $\boldsymbol{f}$ as constructed by Perrin-Riou [49]. It is expected that $\langle -, - \rangle_{\mathcal{K},\mathbb{I}}^{\mathrm{cyc}}$ is non-degenerate, in the sense that its kernel on either side should reduce to $\mathbb{I}$-torsion submodule of $\mathrm{Sel}_{\mathtt{Gr}}(\mathcal{K}, T_{\boldsymbol{f}}^{\dagger})$.

Theorem B. *In addition to* (irred) *and* (dist)*, assume that:*

- *N is squarefree,*
- *$\boldsymbol{f}_2$ is old at p,*
- *there are at least two primes $\ell \| N$ at which $\bar{\rho}_{\boldsymbol{f}}$ is ramified.*

If $\mathrm{Sel}_{\mathtt{Gr}}(\mathbf{Q}, T_{\boldsymbol{f}}^{\dagger})$ *has* $\mathbb{I}$*-rank one and* $\langle -, - \rangle_{\mathcal{K},\mathbb{I}}^{\mathrm{cyc}}$ *is non-degenerate, then*

$$\frac{d}{ds} L_p^{\mathtt{MTT}}(\boldsymbol{f}_k, s)\Big|_{s=k/2} \neq 0,$$

for all but finitely many $k \in \mathbf{Z}_{\geqslant 2}$ with $k \equiv 2 \pmod{p-1}$.

Remark 1.1. The counterpart to Theorem B in rank zero, i.e., the implication

$$\mathrm{rank}_{\mathbb{I}}\ \mathrm{Sel}_{\mathtt{Gr}}(\mathbf{Q}, T_{\boldsymbol{f}}^{\dagger}) = 0 \quad \Longrightarrow \quad L_p^{\mathtt{MTT}}(\boldsymbol{f}_k, k/2) \neq 0, \tag{1.3}$$

for all but finitely many k as above, follows easily from [57] (see Theorem 5.10).

Remark 1.2. By the control theorem for $\mathrm{Sel}_{\mathtt{Gr}}(\mathbf{Q}, T_{\boldsymbol{f}}^{\dagger})$ (see e.g. [46, Prop. 12.7.13.4(i)]) and the p-parity conjecture for classical Selmer groups, the hypothesis that $\mathrm{Sel}_{\mathtt{Gr}}(\mathbf{Q}, T_{\boldsymbol{f}}^{\dagger})$ has $\mathbb{I}$-rank $e \in \{0, 1\}$ implies that $-w = (-1)^e$. Conversely, it is expected that

$$\mathrm{rank}_{\mathbb{I}}\ \mathrm{Sel}_{\mathtt{Gr}}(\mathbf{Q}, T_{\boldsymbol{f}}^{\dagger}) \overset{?}{=} \begin{cases} 1 & \text{if } w = 1, \\ 0 & \text{if } w = -1, \end{cases}$$

and this is known to follow from Howard's "horizontal nonvanishing conjecture" (see [35, Cor. 3.4.3]).

Remark 1.3. For certain Hida families $\boldsymbol{f}$ with CM (a case that is excluded by our hypotheses), the analogue of Theorem B is due to Agboola–Howard and Rubin [1, Thm. B]. (See also [59] and [10] for more general CM cases.) In this case, the analogue of the rank one and the non-degeneracy assumptions in Theorem B follow from Greenberg's nonvanishing results [21] and a transcendence result of Bertrand [4]. In rank zero, the analogue of (1.3) in the CM case follows from [21] and Rubin's proof of the Iwasawa main conjecture for imaginary quadratic fields [53].

We conclude this Introduction with some more details on the ingredients that go into the proofs of the above results.

Denote by $\hat{\mathbf{Z}}_p^{\mathrm{ur}}$ the completion of the ring of integers of the maximal unramified extension of $\mathbf{Q}_p$. The proof of Theorem A builds on the link that we establish in §3 between different instances of the Iwasawa–Greenberg main conjectures involving Selmer groups with different local conditions above p. In particular, letting $\mathfrak{p}$ be the prime of $\mathcal{K}$ above p determined by a fixed embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$, a central role is played by the Selmer group defined by

$$\mathrm{Sel}_{\emptyset,0}(\mathcal{K}_\infty, A_{\boldsymbol{f}}) := \ker\Big\{\mathrm{H}^1(\mathcal{K}_\infty, A_{\boldsymbol{f}}) \longrightarrow \prod_{w\nmid p} \mathrm{H}^1(I_w, A_{\boldsymbol{f}}) \times \prod_{w\mid\overline{\mathfrak{p}}} \mathrm{H}^1(\mathcal{K}_{\infty,w}, A_{\boldsymbol{f}})\Big\}.$$

The Pontryagin dual of $\mathrm{Sel}_{\emptyset,0}(\mathcal{K}_\infty, A_{\boldsymbol{f}})$ is conjectured to be $\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]$-torsion, with characteristic ideal generated by a p-adic L-function

$$\mathscr{L}_{\mathfrak{p}}(\boldsymbol{f}/\mathcal{K}) \in \mathbb{I}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!], \qquad \text{where } \ \mathbb{I}^{\mathrm{ur}} := \mathbb{I}\hat{\otimes}_{\mathbf{Z}_p}\hat{\mathbf{Z}}_p^{\mathrm{ur}},$$

interpolating the critical values of the Rankin–Selberg L-function $L(\boldsymbol{f}_\phi/\mathcal{K}, \chi, s)$ with χ running over characters of $\Gamma_{\mathcal{K}}$ corresponding to theta series of weight *higher* than the weight of $\boldsymbol{f}_\phi$. This second instance of the main conjecture can be related on the one hand to the Iwasawa–Greenberg Main Conjecture for $L_p^{\mathtt{Hi}}(\boldsymbol{f}/\mathcal{K})$ by building on the explicit reciprocity laws for the Rankin–Eisenstein classes of Kings–Loeffler–Zerbes [40], and on the other hand (after anticyclotomic descent) its specialization in weight two is directly related to the main conjecture of the p-adic L-function of Bertolini–Darmon–Prasanna [2], allowing us to take the results of [64] and [12] towards the proof of those different main conjectures to bring to bear on the Iwasawa–Greenberg Main Conjecture for $L_p^{\mathtt{Hi}}(\boldsymbol{f}/\mathcal{K})$.

On the other hand, a key ingredient in the proof of Theorem B is the Birch and Swinnerton-Dyer type formula for $L_p^{\mathtt{Hi}}(\boldsymbol{f}/\mathcal{K})$ along the anticyclotomic Iwasawa algebra $\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]$ that we obtain in Theorem 5.8 by building on the earlier results of the paper, leading to a Gross–Zagier type formula for Howard's system of big Heegner points $\mathfrak{Z}_\infty$ that we then apply for a suitably chosen imaginary quadratic field $\mathcal{K}$.

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2. p-adic L-functions

2.1. Hida families

Let $\mathbb{I}$ be a local reduced normal extension of $\mathfrak{O}_L[\![T]\!]$, where $\mathfrak{O}_L$ is the ring of integers of a finite extension L of $\mathbf{Q}_p$, and denote by $\mathcal{X}_a(\mathbb{I}) \subset \mathrm{Hom}_{\mathrm{cts}}(\mathbb{I}, \overline{\mathbf{Q}}_p)$ the set of continuous $\mathfrak{O}_L$-algebra homomorphisms $\phi : \mathbb{I} \to \overline{\mathbf{Q}}_p$ satisfying

$$\phi(1+T) = \zeta(1+p)^{k-2}$$

for some p-power root of unity $\zeta = \zeta_\phi$ and some integer $k = k_\phi \in \mathbf{Z}_{\geqslant 2}$ called the *weight* of ϕ. We shall refer to the elements of $\mathcal{X}_a(\mathbb{I})$ as *arithmetic* primes of $\mathbb{I}$, and let $\mathcal{X}_a^o(\mathbb{I})$ denote the set consisting of arithmetic primes ϕ with $\zeta_\phi = 1$ and weight $k_\phi \equiv 2 \pmod{p-1}$.

Let N be a positive integer prime to p, let χ be an even Dirichlet character modulo Np taking values in L, and let $\boldsymbol{f} = \sum_{n=1}^\infty \boldsymbol{a}_n q^n \in \mathbb{I}[\![q]\!]$ be an ordinary $\mathbb{I}$-adic cusp eigenform of tame level N and character χ, as defined in [57, §3.3.9]. In particular, for every $\phi \in \mathcal{X}_a(\mathbb{I})$ of weight k we have

$$\boldsymbol{f}_\phi := \sum_{n=1}^\infty \phi(\boldsymbol{a}_n) q^n \in S_k(\Gamma_0(p^t N), \chi\omega^{2-k}\psi_\zeta),$$

where

- $t \geqslant 1$ is such that ζ is a primitive p^{t-1}-st root of unity,
- ω is the Teichmüller character, and
- $\psi_\zeta : (\mathbf{Z}/p^t\mathbf{Z})^\times \to \overline{\mathbf{Q}}_p^\times$ is determined by $\psi_\zeta(1+p) = \zeta$.

Denote by $S^{\mathrm{ord}}(N, \chi; \mathbb{I})$ the space of such $\mathbb{I}$-adic eigenforms $\boldsymbol{f}$. If in addition $\boldsymbol{f}_\phi$ is N-new for all $\phi \in \mathcal{X}_a(\mathbb{I})$, we say that $\boldsymbol{f}$ is a *Hida family* of tame level N and character χ.

We refer to $\boldsymbol{f}_\phi$ as the specialization of $\boldsymbol{f}$ at ϕ. More generally, if $\phi \in \mathrm{Hom}_{\mathrm{cts}}(\mathbb{I}, \overline{\mathbf{Q}}_p)$ is such that $\boldsymbol{f}_\phi$ is a classical eigenform, we say that $\boldsymbol{f}_\phi$ is a classical specialization of

$\boldsymbol{f}$; this includes the specializations of $\boldsymbol{f} \in S^{\mathrm{ord}}(N, \chi; \mathbb{I})$ at $\phi \in \mathcal{X}_a(\mathbb{I})$, but possibly also specializations in weight 1, for example.

2.2. Congruence modules

We recall the notion of congruence modules following the treatment of [57, §12.2] and [36, §3.3]. Let $\boldsymbol{f}$ be a Hida family of tame level N and character χ defined over $\mathbb{I}$. Letting $\mathbb{T}(N, \chi, \mathbb{I})$ be the Hecke algebra acting $S^{\mathrm{ord}}(N, \chi; \mathbb{I})$, the Hida family $\boldsymbol{f}$ defines an algebra homomorphism $\lambda_{\boldsymbol{f}} : \mathbb{T}(N, \chi, \mathbb{I}) \to \mathbb{I}$ which factors through a local component of $\mathbb{T}(N, \chi, \mathbb{I})$ denoted $\mathbb{T}_{\mathfrak{m}_{\boldsymbol{f}}}$. Then, since $\boldsymbol{f}$ is N-new, upon extension of scalars to the fraction field $F_{\mathbb{I}}$ of $\mathbb{I}$ there is an algebra direct sum decomposition

$$\lambda : \mathbb{T}_{\mathfrak{m}_{\boldsymbol{f}}} \otimes_{\mathbb{I}} F_{\mathbb{I}} \simeq F_{\mathbb{I}} \times \mathbb{T}'$$

with the projection onto the first factor given by $\lambda_{\boldsymbol{f}}$. The *congruence ideal* $C(\boldsymbol{f}) \subset \mathbb{I}$ is defined by

$$C(\boldsymbol{f}) := \lambda_{\boldsymbol{f}} \left(\mathbb{T}_{\mathfrak{m}_{\boldsymbol{f}}} \cap \lambda^{-1}(F_{\mathbb{I}} \times \{0\} \right).$$

As in [40, §7.7], we shall also consider the fractional ideal $J_{\boldsymbol{f}} := C(\boldsymbol{f})^{-1} \subset F_{\mathbb{I}}$. As noted in [40], it follows from [27, Thm. 4.2] that elements of $J_{\boldsymbol{f}}$ define meromorphic functions on $\mathrm{Spec}(\mathbb{I})$ which are regular at all arithmetic points.

2.3. Rankin–Selberg p-adic L-functions

Let Γ be the Galois group of the cyclotomic $\mathbf{Z}_p^\times$-extension of $\mathbf{Q}$, and set

$$\Lambda_\Gamma = \mathbf{Z}_p[\![\Gamma]\!].$$

Note that if $j \in \mathbf{Z}$ and χ is a Dirichlet character of p-power conductor, there is a unique $\phi \in \mathrm{Hom}_{\mathrm{cts}}(\Lambda_\Gamma, \overline{\mathbf{Q}}_p^\times)$ extending the character $z \mapsto z^j \chi(z)$ on $\mathbf{Z}_p^\times$.

Theorem 2.1. *Let $\boldsymbol{f}_1, \boldsymbol{f}_2$ be Hida families of tame levels N_1, N_2, respectively, and let $N = \mathrm{lcm}(N_1, N_2)$. Then there is an element*

$$L_p(\boldsymbol{f}_1, \boldsymbol{f}_2) \in \left(J_{\boldsymbol{f}_1} \hat{\otimes}_{\mathbf{Z}_p} \mathbb{I}_{\boldsymbol{f}_2} \hat{\otimes}_{\mathbf{Z}_p} \Lambda_\Gamma \right) \otimes_{\mathbf{Z}} \mathbf{Z}[\mu_N]$$

uniquely characterized by the following interpolation property. Let f_1, f_2 be classical specializations of $\boldsymbol{f}_1$, $\boldsymbol{f}_2$ of weights k_1, k_2, respectively, with $k_1 > k_2 \geqslant 1$, let j be an integer in the range $k_2 \leqslant j \leqslant k_1 - 1$, and let χ be a Dirichlet character of p-power conductor. Suppose that the local component at p of the automorphic representation π_{f_1} is a principal series representation $\pi(\eta_1, \eta_1')$ with η_1 unramified and $\eta_1(p)$ a p-adic unit. Then the *value of $L_p(\boldsymbol{f}_1, \boldsymbol{f}_2)$ at the corresponding specialization $\phi \in \mathrm{Spec}(\mathbb{I}_{\boldsymbol{f}_1}\hat{\otimes}_{\mathbf{Z}_p}\mathbb{I}_{\boldsymbol{f}_2}\hat{\otimes}_{\mathbf{Z}_p}\Lambda_\Gamma)$ is given by*

$$\phi(L_p(\boldsymbol{f}_1, \boldsymbol{f}_2)) = \frac{\mathcal{E}(f_1, f_2, \chi, j)}{\mathcal{E}(f_1)\mathcal{E}^*(f_1)} \cdot \frac{\Gamma(j)\Gamma(j-k_2+1)}{\pi^{2j+1-k_2}(-i)^{k_1-k_2}2^{2j+k_1-k_2}\left\langle f_1, f_1^c|_{k_1}\left(\begin{smallmatrix} & -1 \\ p^{t_1}N_1 & \end{smallmatrix}\right)\right\rangle_{\Gamma_0(p^{t_1}N_1)}} \times L(f_1, f_2, \chi^{-1}, j),$$

where

- *α_i and β_i are the roots of the Hecke polynomial of f_i at p, with α_i a p-adic unit,*
- *denoting by p^t the conductor of χ,*

$$\mathcal{E}(f_1, f_2, \chi, j) = \begin{cases} \left(1 - \frac{p^{j-1}}{\alpha_1\alpha_2}\right)\left(1 - \frac{p^{j-1}}{\alpha_1\beta_2}\right)\left(1 - \frac{\beta_1\alpha_2}{p^j})(1 - \frac{\beta_1\beta_2}{p^j}\right) & \text{if } t = 0, \\ G(\chi)^2 \cdot \left(\frac{p^{2j-2}}{\alpha_1^2\alpha_2\beta_2}\right)^t & \text{if } t \geqslant 1, \end{cases}$$

  with $G(\chi)$ is the Gauss sum of χ,
- *denoting by p^{t_1} the p-part of the conductor of η_1', then*

$$\mathcal{E}(f_1)\mathcal{E}^*(f_1) = \begin{cases} \left(1 - \frac{\beta_1}{p\alpha_1}\right)\left(1 - \frac{\beta_1}{\alpha_1}\right) & \text{if } t_1 = 0, \\ G(\chi_1) \cdot \eta_1'\eta_1^{-1}(p^{t_1})p^{-t_1} & \text{if } t_1 \geqslant 1, \end{cases}$$

  where χ_1 is the nebentypus of f_1.

Proof. This follows from [27, Thm. 5.1], which we have stated adopting the formulation in [40, Thm. 7.7.2] (slightly extended to include more general specializations of the dominant Hida family $\boldsymbol{f}_1$). □

We shall consider the p-adic L-functions $L_p(\boldsymbol{f}_1, \boldsymbol{f}_2)$ of Theorem 2.1 in the cases where either $\boldsymbol{f}_1$ or $\boldsymbol{f}_2$ has CM. Thus we let $\boldsymbol{f}$ be a fixed Hida family of tame level N defined over $\mathbb{I}$, which we assume contains all the N-th root of unity, and assume that $\bar{\rho}_{\boldsymbol{f}}$ satisfies hypotheses (irred) and (dist) from the Introduction. On the other hand, let $\mathcal{K}$ be an imaginary quadratic field of discriminant $-D_{\mathcal{K}} < 0$ prime to pN such that

$$p = \mathfrak{p}\overline{\mathfrak{p}} \text{ splits in } \mathcal{K},$$

with $\mathfrak{p}$ the prime above p induced by our fixed embedding $\imath_p : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$. Let $\mathcal{K}_\infty$ be $\mathbf{Z}_p^2$-extension of $\mathcal{K}$ as in the Introduction, and denote by $\Gamma_{\mathfrak{p}} \simeq \mathbf{Z}_p$ the Galois group over $\mathcal{K}$ of the maximal subfield of $\mathcal{K}_\infty$ unramified outside $\overline{\mathfrak{p}}$. We then let

$$\boldsymbol{g}=\sum_{n=1}^{\infty}\boldsymbol{b}_n q^n\in\mathbb{I}_{\boldsymbol{g}}[\![q]\!] \tag{2.1}$$

be the canonical Hida family of CM forms constructed in [37, §5.2], where $\mathbb{I}_{\boldsymbol{g}}=\mathbf{Z}_p[\![\Gamma_{\mathfrak{p}}]\!]$. Specifically, denoting by $\theta_{\mathfrak{p}}:\mathbb{A}_{\mathcal{K}}^{\times}\to\Gamma_{\mathfrak{p}}$ the composition of the Artin reciprocity map $\mathrm{rec}_{\mathcal{K}}:\mathbb{A}_{\mathcal{K}}^{\times}\to G_{\mathcal{K}}^{\mathrm{ab}}$ with the natural projection $G_{\mathcal{K}}^{\mathrm{ab}}\twoheadrightarrow\Gamma_{\mathfrak{p}}$, we have

$$\boldsymbol{b}_n=\sum_{N(\mathfrak{a})=n,(\mathfrak{a},\overline{\mathfrak{p}})=1}\theta_{\mathfrak{p}}(x_{\mathfrak{a}}),$$

where the sum is over integral ideals $\mathfrak{a}\subset\mathcal{O}_{\mathcal{K}}$, and $x_{\mathfrak{a}}\in\mathbb{A}_{\mathcal{K}}^{\infty,\times}$ is any finite idèle of $\mathcal{K}$ with $\mathrm{ord}_w(x_{\mathfrak{a},w})=\mathrm{ord}_w(\mathfrak{a})$ for all finite places w of $\mathcal{K}$.

2.4. Non-dominant CM: $(\boldsymbol{f}_1,\boldsymbol{f}_2)=(\boldsymbol{f},\boldsymbol{g})$

Since $\bar{\rho}_{\boldsymbol{f}}$ satisfies hypotheses (irred) and (dist), by [67], the local ring $\mathbb{T}_{\mathfrak{m}_{\boldsymbol{f}}}$ introduced in §2.2 is known to be Gorenstein, and by Hida's results [26] it follows that the congruence ideal $C(\boldsymbol{f})$ is principal.

Denote by Γ^{cyc} the Galois group of the cyclotomic $\mathbf{Z}_p$-extension of $\mathbf{Q}$.

Definition 2.2. Let $c_{\boldsymbol{f}}\in C(\boldsymbol{f})$ be a generator, and set

$$L_p^{\mathtt{Hi}}(\boldsymbol{f}/\mathcal{K}):=c_{\boldsymbol{f}}\cdot e_1 L_p(\boldsymbol{f},\boldsymbol{g}),$$

where $e_1L_p(\boldsymbol{f},\boldsymbol{g})\in J_{\boldsymbol{f}}\hat{\otimes}_{\mathbf{Z}_p}\mathbb{I}_{\boldsymbol{g}}[\![\Gamma^{\mathrm{cyc}}]\!]$ is the natural projection of $L_p(\boldsymbol{f},\boldsymbol{g})$ via $\Gamma\twoheadrightarrow\Gamma^{\mathrm{cyc}}$.

We will often identify Γ^{cyc} with the Galois group $\Gamma_{\mathcal{K}}^{\mathrm{cyc}}$ of the cyclotomic $\mathbf{Z}_p$-extension of $\mathcal{K}$. Letting $\Gamma_{\mathcal{K}}$ be the Galois group $\mathrm{Gal}(\mathcal{K}_\infty/\mathcal{K})$, note that the canonical projections to $\Gamma_{\mathfrak{p}}$ and $\Gamma_{\mathcal{K}}^{\mathrm{cyc}}$ induce an isomorphism

$$\Gamma_{\mathcal{K}}\simeq\Gamma_{\mathfrak{p}}\times\Gamma_{\mathcal{K}}^{\mathrm{cyc}}.$$

Since $\mathbb{I}_{\boldsymbol{g}}=\mathbf{Z}_p[\![\Gamma_{\mathfrak{p}}]\!]$, we may thus consider $L_p^{\mathtt{Hi}}(\boldsymbol{f}/\mathcal{K})$ as an element in $\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]$.

On the other hand, the action of complex conjugation yields a decomposition

$$\Gamma_{\mathcal{K}}\simeq\Gamma_{\mathcal{K}}^{\mathrm{ac}}\times\Gamma_{\mathcal{K}}^{\mathrm{cyc}},$$

where $\Gamma_{\mathcal{K}}^{\mathrm{ac}}$ denotes the Galois group of the anticyclotomic $\mathbf{Z}_p$-extension of $\mathcal{K}$. We next study the projections of $L_p^{\mathtt{Hi}}(\boldsymbol{f}/\mathcal{K})$ to $\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]$ and $\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{cyc}}]\!]$.

2.4.1. Anticyclotomic restriction of $L_p^{\mathtt{Hi}}(\boldsymbol{f}/\mathcal{K})$

Assume that $\boldsymbol{f}$ has trivial tame character, and following [35, Def. 2.1.3] define the *critical character* $\Theta:G_{\mathbf{Q}}\to\mathbb{I}^{\times}$ by

$$\Theta := [\langle \varepsilon_{\mathrm{cyc}} \rangle^{1/2}], \tag{2.2}$$

where $\varepsilon_{\mathrm{cyc}} : G_{\mathbf{Q}} \to \mathbf{Z}_p^\times$ is the cyclotomic character, $\langle - \rangle : \mathbf{Z}_p^\times \to 1 + p\mathbf{Z}_p$ is the natural projection, and

$$[-] : 1 + p\mathbf{Z}_p \hookrightarrow \mathbf{Z}_p[\![1 + p\mathbf{Z}_p]\!]^\times \simeq \mathbf{Z}_p[\![T]\!]^\times \longrightarrow \mathbb{I}^\times$$

is the composition of the obvious maps. This induces the $\mathbb{I}$-linear twist map

$$\mathrm{tw}_{\Theta^{-1}} : \mathbb{I}[\![\Gamma_{\mathcal{K}}]\!] \longrightarrow \mathbb{I}[\![\Gamma_{\mathcal{K}}]\!] \tag{2.3}$$

defined by $\gamma \mapsto \Theta^{-1}(\gamma)\gamma$ for $\gamma \in \Gamma_{\mathcal{K}}$. (This map, which will appear repeatedly throughout the paper, will be used to restrict to the "central critical line" in the weight-cyclotomic space.)

Write N as the product

$$N = N^+ N^-$$

with N^+ (resp. N^-) divisible only by primes which are split (resp. inert) in $\mathcal{K}$, and consider the following generalized *Heegner hypothesis*:

$$N^- \text{ is the squarefree product of an even number of primes.} \tag{gen-H}$$

Whenever we assume that $\mathcal{K}$ satisfies (gen-H), we fix an integral ideal $\mathfrak{N}^+ \subset \mathcal{O}_{\mathcal{K}}$ with $\mathcal{O}_{\mathcal{K}}/\mathfrak{N}^+ \simeq \mathbf{Z}/N^+\mathbf{Z}$.

Proposition 2.3. *Let $L_p^{\mathtt{Hi}}(\boldsymbol{f}^\dagger/\mathcal{K})_{\mathrm{ac}}$ be the image of $\mathrm{tw}_{\Theta^{-1}}(L_p^{\mathtt{Hi}}(\boldsymbol{f}/\mathcal{K}))$ under the natural projection $\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!] \to \mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]$. If $\mathcal{K}$ satisfies (gen-H), then $L_p^{\mathtt{Hi}}(\boldsymbol{f}^\dagger/\mathcal{K})_{\mathrm{ac}}$ is identically zero.*

Proof. Let $\phi \in \mathrm{Spec}(\mathbb{I}_{\boldsymbol{f}} \hat{\otimes}_{\mathbf{Z}_p} \mathbb{I}_{\boldsymbol{g}} \hat{\otimes}_{\mathbf{Z}_p} \mathbf{Z}_p[\![\Gamma^{\mathrm{cyc}}]\!]) = \mathrm{Spec}(\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!])$ be a specialization in the range specified in Theorem 2.1, with $f_1 = \boldsymbol{f}_\phi$ the p-stabilization of a newform $f \in S_k(\Gamma_0(N))$ of weight $k \geqslant 2$ and $f_2 = \boldsymbol{g}_\phi$ a classical weight 1 specialization. By the interpolation property, the value $\phi(L_p^{\mathtt{Hi}}(\boldsymbol{f}/\mathcal{K}))$ is a multiple of

$$L(f_1, f_2, \chi^{-1}, j) = L(f/\mathcal{K}, \psi, j),$$

with ψ a finite order character of $\Gamma_{\mathcal{K}}$ and $1 \leqslant j \leqslant k-1$, and so $\phi(\mathrm{tw}_{\Theta^{-1}}(L_p^{\mathtt{Hi}}(\boldsymbol{f}/\mathcal{K})))$ is also a multiple of $L(f/\mathcal{K}, \psi', k/2)$ for a finite order character ψ' of $\Gamma_{\mathcal{K}}$. If ψ' factors through the projection $\Gamma_{\mathcal{K}} \twoheadrightarrow \Gamma_{\mathcal{K}}^{\mathrm{ac}}$, then the L-function $L(f/\mathcal{K}, \psi', s)$ is self-dual, with a functional equation relating its values at s and $k - s$, and if $\mathcal{K}$ satisfies the hypothesis (gen-H), then the sign in this functional equation is -1 (see e.g. [17, §1]). Thus $L(f/\mathcal{K}, \psi', k/2) = 0$, and letting ϕ vary, the result follows. □

2.4.2. Cyclotomic restriction of $L_p^{\mathtt{Hi}}(\boldsymbol{f}/\mathcal{K})$

As above, we denote by $\Gamma_{\mathcal{K}}^{\mathrm{cyc}}$ the Galois group of the cyclotomic $\mathbf{Z}_p$-extension of $\mathcal{K}$, which we shall identify with Γ^{cyc}, and let $\gamma \in \Gamma^{\mathrm{cyc}}$ be a topological generator.

For $f = \boldsymbol{f}_\phi$ the specialization of $\boldsymbol{f}$ at some $\phi \in \mathcal{X}_a^o(\mathbb{I})$ of weight $k \geqslant 2$ defined over a finite extension $L/\mathbf{Q}_p$ with ring of integers $\mathfrak{O}_L$, and ϵ a primitive (in our application, quadratic) L-valued Dirichlet character of conductor C prime to p, we let $L_p^{\mathtt{MTT}}(f\otimes\epsilon) \in \mathfrak{O}_L[\![\Gamma]\!]$ be the cyclotomic p-adic L-function attached to $f\otimes\epsilon$ in [43]. This is characterized by the following interpolation property. If $\phi' \in \mathcal{X}_a^o(\mathfrak{O}_L[\![\Gamma^{\mathrm{cyc}}]\!])$ is given by $\phi'(\gamma) = \zeta(1+m)^m$ with $0 \leqslant m \leqslant k-2$ and ζ a primitive p^{t-1}-st root of unity, then

$$\begin{aligned} L_p^{\mathtt{MTT}}(f\otimes\epsilon)(\phi') &= \phi(\boldsymbol{a}_p)^{-t}\left(1-\frac{\omega^{-m}\psi_\zeta^{-1}\epsilon(p)p^{k-2-m}}{\phi(\boldsymbol{a}_p)}\right)\left(\frac{1-\omega^m\psi_\zeta\bar{\epsilon}(p)p^m}{\phi(\boldsymbol{a}_p)}\right)\\ &\quad\times\frac{(p^{t'}C)^{m+1}\cdot\Gamma(m+1)}{(-2\pi i)^m\cdot G(\epsilon\omega^{-m}\psi_\zeta^{-1})\cdot\Omega_f^{(-1)^m\epsilon(-1)}}\cdot L(f\otimes\epsilon,\omega^{-m}\psi_\zeta^{-1},m+1), \end{aligned} \tag{2.4}$$

where ω is the Teichmüller character, ψ_ζ is as in §2.1, $t' = \max\{1,t\}$, and $\Omega_f^{\pm} \in \mathbf{C}^\times$ are Shimura's periods, normalized up to a unit in $\mathfrak{O}_L^\times$ as in [57, §3.3.3].

Theorem 2.4. *Let $L_p^{\mathtt{Hi}}(\boldsymbol{f}/\mathcal{K})_{\mathrm{cyc}}$ be the image of $L_p^{\mathtt{Hi}}(\boldsymbol{f}/\mathcal{K})$ under the natural projection $\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!] \to \mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{cyc}}]\!]$. Then for every $\phi \in \mathcal{X}_a^o(\mathbb{I})$, we have*

$$\phi(L_p^{\mathtt{Hi}}(\boldsymbol{f}/\mathcal{K})_{\mathrm{cyc}}) = L_p^{\mathtt{MTT}}(\boldsymbol{f}_\phi)\cdot L_p^{\mathtt{MTT}}(\boldsymbol{f}_\phi\otimes\epsilon_{\mathcal{K}})$$

up to a unit, where $\epsilon_{\mathcal{K}}$ is the quadratic character associated to $\mathcal{K}$.

Proof. Since we assume that $\bar{\rho}_{\boldsymbol{f}}$ satisfies hypotheses (irred) and (dist) from the Introduction, by [26, Thm. 0.1] (see also [57, Lem. 12.1]) for every $\phi \in \mathcal{X}_a^o(\mathbb{I})$ of weight $k \geqslant 2$ (hence of trivial nebentypus) we have the period relation

$$\phi(c_{\boldsymbol{f}}) = u\cdot p^{k/2-1}\cdot\frac{2^{-3}(2i)^{k+1}\langle\boldsymbol{f}_\phi,\boldsymbol{f}_\phi\rangle_{\Gamma_0(N)}}{\Omega_{\boldsymbol{f}_\phi}^{+}\cdot\Omega_{\boldsymbol{f}_\phi}^{-}}$$

where $u \in \phi(\mathbb{I})^\times$. Moreover, we have that $\Omega_{\boldsymbol{f}_\phi}^{\pm} = \Omega_{\boldsymbol{f}_\phi\otimes\epsilon_{\mathcal{K}}}^{\mp}$ up to a unit (see [58, Lem. 9.6] for example). In light of the factorization

$$L(\boldsymbol{f}_\phi/\mathcal{K},\psi_\zeta^{-1},1) = L(\boldsymbol{f}_\phi,\psi_\zeta^{-1},1)\cdot L(\boldsymbol{f}_\phi\otimes\epsilon_{\mathcal{K}},\psi_\zeta^{-1},1),$$

the result thus follows from a direct comparison of the interpolation properties in Theorem 2.1 (with $k_1 = k$ and $j = k_2 = 1$) and (2.4) with $m = 0$. $\square$

2.5. Dominant CM: $(\boldsymbol{f}_1, \boldsymbol{f}_2) = (\boldsymbol{g}, \boldsymbol{f})$

As in §2.4, we let $\boldsymbol{f} \in \mathbb{I}[\![q]\!]$ be a fixed Hida family of tame level N and trivial tame character, and $\boldsymbol{g}$ be the CM Hida family in (2.1).

Let $\hat{\mathbf{Z}}_p^{\mathrm{ur}}$ be the completion of the ring of integers of the maximal unramified extension of $\mathbf{Q}_p$, and set

$$\mathbb{I}^{\mathrm{ur}} := \mathbb{I} \hat{\otimes}_{\mathbf{Z}_p} \hat{\mathbf{Z}}_p^{\mathrm{ur}}.$$

By [39, §5.3.0] (see also [18, Thm. II.4.14]) there exists a p-adic L-function $\mathscr{L}_{\mathfrak{p}}(\mathcal{K}) \in \hat{\mathbf{Z}}_p^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!]$ such that if ψ is a character of $\Gamma_{\mathcal{K}}$ corresponding to an algebraic Hecke character of $\mathcal{K}$ crystalline at the primes above p and infinity type (a, b) with $0 \leqslant -b < a$, then

$$\mathscr{L}_{\mathfrak{p}}(\mathcal{K})(\psi) = \left(\frac{\sqrt{D_{\mathcal{K}}}}{2\pi}\right)^b \cdot \Gamma(b) \cdot (1 - \psi(\mathfrak{p})) \cdot (1 - p^{-1}\psi^{-1}(\overline{\mathfrak{p}})) \cdot \frac{\Omega_p^{b-a}}{\Omega_{\mathcal{K}}^{b-a}} \cdot L(\psi, 0), \tag{2.5}$$

where $\Omega_{\mathcal{K}} \in \mathbf{C}^\times$ and $\Omega_p \in \mathbf{C}_p^\times$ are certain CM periods (as defined in e.g. [14, §2.5]).

Definition 2.5. Let $h_{\mathcal{K}}$ be the class number of $\mathcal{K}$, and set

$$\mathscr{L}_{\mathfrak{p}}(\boldsymbol{f}/\mathcal{K}) := \big(h_{\mathcal{K}} \cdot \mathscr{L}_{\mathfrak{p}}(\mathcal{K})_{\mathrm{ac}}\big) \cdot L_p(\boldsymbol{g}, \boldsymbol{f}), \tag{2.6}$$

where $\mathscr{L}_{\mathfrak{p}}(\mathcal{K})_{\mathrm{ac}}$ is the anticyclotomic projection of $\mathscr{L}_{\mathfrak{p}}(\mathcal{K})$.

Remark 2.6. As in [32] here we view $\mathscr{L}_{\mathfrak{p}}(\mathcal{K})_{\mathrm{ac}}$ as an element in $\hat{\mathbf{Z}}_p^{\mathrm{nr}}[\![\Gamma_{\mathcal{K}}]\!]$ via the map sending $\gamma \in \Gamma_{\mathcal{K}}^{\mathrm{ac}}$ to $\tilde{\gamma}^{c-1}$, where $\tilde{\gamma} \in \Gamma_{\mathcal{K}}$ is any lift of γ, and c denotes the action of complex conjugation.

Note that *a priori* $\mathscr{L}_{\mathfrak{p}}(\boldsymbol{f}/\mathcal{K})$ is an element in $J_{\boldsymbol{g}} \otimes_{\mathbb{I}} \mathbb{I}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!]$.

Proposition 2.7. *The p-adic L-function in* (2.6) *is integral, i.e.,* $\mathscr{L}_{\mathfrak{p}}(\boldsymbol{f}/\mathcal{K}) \in \mathbb{I}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!]$.

Proof. By construction, if $H(\boldsymbol{g}) \in \mathbb{I}_{\boldsymbol{g}}$ is a congruence power series for $\boldsymbol{g}$ (i.e., a generator of the principal ideal $C(\boldsymbol{g}) \subset \mathbb{I}_{\boldsymbol{g}}$), then the product $H(\boldsymbol{g}) \cdot L_p(\boldsymbol{g}, \boldsymbol{f})$ is integral, so it suffices to show that $h_{\mathcal{K}} \cdot \mathscr{L}_{\mathfrak{p}}(\mathcal{K})_{\mathrm{ac}}$ is divisible by $H(\boldsymbol{g})$. By [32, Thm. 0.3] and Rubin's proof [53] of the Iwasawa main conjecture for $\mathcal{K}$, one has that such divisibility holds up to powers of the augmentation ideal $(\gamma_{\mathfrak{p}} - 1) \subset \mathbb{I}_{\boldsymbol{g}}$; since by [5, Thm. A(i)] one knows that $H(\boldsymbol{g})$ is not divisible by $\gamma_{\mathfrak{p}} - 1$, the result follows. □

The later arguments in this paper will exploit the close link between $\mathscr{L}_{\mathfrak{p}}(\boldsymbol{f}/\mathcal{K})$ and the 3-variable p-adic L-function constructed in [64] and that we now recall. Fix a finite set Σ of places $\mathcal{K}$ outside p and containing all the places dividing $ND_{\mathcal{K}}$. Then by the results in [64, §7.5] there exists an element

$$\mathfrak{L}_{\mathfrak{p}}^{\Sigma}(\boldsymbol{f}/\mathcal{K}) \in \mathbb{I}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!]$$

characterized by the following interpolation property. For a Zariski dense set of points $\phi \in \mathrm{Spec}(\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!])$, corresponding to pairs $(\boldsymbol{f}_{\phi}, \boldsymbol{\psi}_{\phi})$ with $\boldsymbol{f}_{\phi}$ of weight 2 and conductor $p^t N$ generating a unitary automorphic representation $\pi_{\boldsymbol{f}_{\phi}}$ whose component at p is isomorphic to $\pi(\chi_{1,p}, \chi_{2,p})$ with $v_p(\chi_{1,p}(p)) = -\frac{1}{2}$ and $v_p(\chi_{2,p}(p)) = \frac{1}{2}$, and $\boldsymbol{\psi}_{\phi}$ a Hecke character of $\mathcal{K}$ of infinity type $(-n, 0)$ for some $n \geqslant 3$ and conductor p^t, we have

$$\begin{aligned}\phi(\mathfrak{L}_{\mathfrak{p}}^{\Sigma}(\boldsymbol{f}/\mathcal{K})) = p^{(n-3)t}\boldsymbol{\psi}_{\phi,\mathfrak{p}}^{2}\chi_{1,p}^{-1}\chi_{2,p}^{-1}(p^{-t})G(\boldsymbol{\psi}_{\phi,\mathfrak{p}}\chi_{1,p}^{-1})G(\boldsymbol{\psi}_{\phi,\mathfrak{p}}\chi_{2,p}^{-1})\Gamma(n)\Gamma(n-1)\Omega_p^{2n} \\ \times \frac{L^{\Sigma}(\boldsymbol{f}_{\phi}, \chi_{\phi}^{-1}\boldsymbol{\psi}_{\phi}, 0)}{(2\pi i)^{2n-1}\Omega_{\mathcal{K}}^{2n}},\end{aligned} \tag{2.7}$$

where χ_{ϕ} is the nebentypus of $\boldsymbol{f}_{\phi}$ and $L^{\Sigma}(\boldsymbol{f}_{\phi}, \chi_{\phi}^{-1}\boldsymbol{\psi}_{\phi}, 0)$ is the Σ-imprimitive Rankin–Selberg L-value.

Proposition 2.8. *Let*

$$\mathfrak{L}_{\mathfrak{p}}(\boldsymbol{f}/\mathcal{K}) := \mathfrak{L}_{\mathfrak{p}}^{\Sigma}(\boldsymbol{f}/\mathcal{K}) \times \prod_{w\in\Sigma} P_w(\Psi_{\mathcal{K}}(\mathrm{Frob}_w))^{-1}, \tag{2.8}$$

where P_w is the Euler factor at w and $\Psi_{\mathcal{K}} : G_{\mathcal{K}} \twoheadrightarrow \Gamma_{\mathcal{K}}$ is the natural projection. Then $\mathfrak{L}_{\mathfrak{p}}(\boldsymbol{f}/\mathcal{K}) = \mathscr{L}_{\mathfrak{p}}(\boldsymbol{f}/\mathcal{K})$ up to a unit.

Proof. We begin by noting that $\mathfrak{L}_{\mathfrak{p}}(\boldsymbol{f}/\mathcal{K})$ satisfies the same interpolation property as in (2.7) but with $L^{\Sigma}(\boldsymbol{f}_{\phi}, \chi_{\phi}^{-1}\boldsymbol{\psi}_{\phi}, 0)$ replaced by the primitive counterpart $L(\boldsymbol{f}_{\phi}, \chi_{\phi}^{-1}\boldsymbol{\psi}_{\phi}, 0)$. Now, any character $\boldsymbol{\psi}_{\phi}$ as above can be written as the product $\boldsymbol{\psi}_{\phi}' \cdot \boldsymbol{\psi}_{\phi}''$, with $\boldsymbol{\psi}_{\phi}'$ cyclotomic (i.e., factoring through $\Gamma_{\mathcal{K}} \twoheadrightarrow \Gamma_{\mathcal{K}}^{\mathrm{cyc}}$), and $\boldsymbol{\psi}_{\phi}''$ corresponding to a Hecke character unramified at $\overline{\mathfrak{p}}$ and of infinity type $(-n, 0)$. We then have that $\chi_{\phi}^{-1}\boldsymbol{\psi}_{\phi}'$ (resp. the theta series of $\boldsymbol{\psi}_{\phi}''$) corresponds to $\chi|\cdot|^j$ (resp. $f_1 = \boldsymbol{g}_{\phi}$) in the notation of Theorem 2.1, and

$$L(\boldsymbol{f}_{\phi}, \chi_{\phi}^{-1}\boldsymbol{\psi}_{\phi}, 0) = L(f_1, f_2, \chi^{-1}, j)$$

with $f_2 = \boldsymbol{f}_{\phi}$. Letting $p^t D_{\mathcal{K}}$ be the conductor of $\boldsymbol{g}_{\phi}$, by [31, Thm. 7.1] a direct calculation shows that the product

$$\mathcal{E}(\boldsymbol{g}_{\phi})\mathcal{E}^*(\boldsymbol{g}_{\phi}) \cdot \left\langle \boldsymbol{g}_{\phi}, \boldsymbol{g}_{\phi}|_2 \begin{pmatrix} & -1 \\ p^t D_{\mathcal{K}} & \end{pmatrix} \right\rangle_{\Gamma_0(p^t D_{\mathcal{K}})}$$

in Theorem 2.1 agrees, up to a p-adic unit independent of ϕ, with

$$\frac{\Gamma(n)G(\boldsymbol{\psi}_{\phi,\overline{\mathfrak{p}}}''^{-1})L(\boldsymbol{\psi}_{\phi}''(\boldsymbol{\psi}_{\phi}'')^{-c}, 1)}{(-2\pi i)^n} \cdot \frac{L(\epsilon_{\mathcal{K}}, 1)}{-2\pi i}, \tag{2.9}$$

where $\epsilon_{\mathcal{K}}$ is the quadratic character associated with $\mathcal{K}/\mathbf{Q}$. By the class number formula, the second factor in this product is given by $h_{\mathcal{K}}$ up to a p-adic unit, while by the interpolation property of the Katz p-adic L-function, the first factor multiplied by $(\Omega_p/\Omega_{\mathcal{K}})^{2n}$ is interpolated by $\mathscr{L}_{\mathfrak{p}}(\mathcal{K})_{\mathrm{ac}}$ for varying ϕ. Comparing the interpolation formulas in Theorem 2.1 and (2.7) therefore yields the result. □

We conclude this section by discussing the anticyclotomic restriction of $\mathscr{L}_{\mathfrak{p}}(\boldsymbol{f}/\mathcal{K})$.

Theorem 2.9. *Assume that $\mathcal{K}$ satisfies (gen-H), and if $N^- > 1$ assume in addition that N is squarefree. Then there exists an element $\mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(\boldsymbol{f}/\mathcal{K}) \in \mathbb{I}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]$ such that for every $\phi \in \mathcal{X}_a^o(\mathbb{I})$ of weight $k \geqslant 2$, and every crystalline character ψ of $\Gamma_{\mathcal{K}}^{\mathrm{ac}}$ corresponding to a Hecke character of infinity type $(n, -n)$ with $n \geqslant 0$, we have*

$$\begin{aligned}\phi(\mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(\boldsymbol{f}/\mathcal{K})^2)(\psi) &= \mathcal{E}_{\mathfrak{p}}(\boldsymbol{f}_\phi, \psi)^2 \cdot \psi(\mathfrak{N}^+)^{-1} \cdot 2^3 \cdot \varepsilon(\boldsymbol{f}_\phi) \cdot w_{\mathcal{K}}^2 \sqrt{D_{\mathcal{K}}} \\ &\quad \cdot \Gamma(k+n)\Gamma(n+1)\Omega_p^{2k+4n} \times \frac{L(\boldsymbol{f}_\phi/\mathcal{K}, \psi, k/2) \cdot \alpha(\boldsymbol{f}_\phi, \boldsymbol{f}_\phi^B)^{-1}}{(2\pi)^{k+2n+1} \cdot (\mathrm{Im}\,\boldsymbol{\theta})^{k+2n} \cdot \Omega_{\mathcal{K}}^{2k+4n}},\end{aligned}$$

where $\mathcal{E}_{\mathfrak{p}}(\boldsymbol{f}_\phi, \psi) = (1 - \phi(\boldsymbol{a}_p)\psi_{\overline{\mathfrak{p}}}(p)p^{-k/2})(1 - \phi(\boldsymbol{a}_p)^{-1}\psi_{\overline{\mathfrak{p}}}(p)p^{k/2-1})$, $\varepsilon(\boldsymbol{f}_\phi)$ is the global root number of $\boldsymbol{f}_\phi$, $w_{\mathcal{K}} := |\mathcal{O}_{\mathcal{K}}^\times|$, $\Omega_p \in \mathbf{C}_p^\times$ and $\Omega_{\mathcal{K}} \in \mathbf{C}^\times$ are CM periods attached to $\mathcal{K}$ as [14, §2.5], $\boldsymbol{\theta} \in \mathcal{K}$ is as in (4.1) below, and

$$\alpha(\boldsymbol{f}_\phi, \boldsymbol{f}_\phi^B) = \frac{\langle \boldsymbol{f}_\phi, \boldsymbol{f}_\phi \rangle}{\langle \boldsymbol{f}_\phi^B, \boldsymbol{f}_\phi^B \rangle}$$

is a ratio of Petersson norms of $\boldsymbol{f}_\phi$ and its transfer $\boldsymbol{f}_\phi^B$ to a quaternion algebra B, normalized as in [52, §2.2].

Proof. When $N^- = 1$, this is [13, Thm. 2.11] (in which case $\alpha(\boldsymbol{f}_\phi, \boldsymbol{f}_\phi^B) = 1$). In the following we sketch how to extend that result to include the more general Heegner hypothesis (gen-H). Some of the notations used here will be introduced later in §4.

Let $\mathcal{O}_B$ be a maximal order of B, and let Ig_{N^+,N^-} be the Igusa scheme over $\mathbf{Z}_{(p)}$ classifying abelian surfaces with $\mathcal{O}_B$-multiplication and U_∞-level structure (here U_∞ is the open compact $U_r \subset \hat{R}_r^\times$ in §4.1 with $r = \infty$). For any valuation ring W finite flat over $\mathbf{Z}_p$, denote by $V_p(W)$ the module of formal functions on Ig_{N^+,N^-} (i.e., p-adic modular forms) defined over W, and set

$$V_p(\mathbb{I}) := V_p(W_0)\hat{\otimes}_{W_0}\mathbb{I},$$

where $W_0 = W(\kappa_{\mathbb{I}})$ is the ring of Witt vectors of the residue field of $\mathbb{I}$. For every $\mathcal{O}_{\mathcal{K}}$-ideal $\mathfrak{a}$ prime to $\mathfrak{N}^+\mathfrak{p}$, the construction of $\varsigma^{(s)}$ (for arbitrary $s \geqslant 0$) in §4.2 determines CM points $x(\mathfrak{a}) \in \mathrm{Ig}_{N^+,N^-}$, and the argument in [28, Thm. 3.2.16] with the use of q-expansions and the q-expansion principle replaced by Serre–Tate-expansions and the resulting t-expansion principle around any such $x(\mathfrak{a})$ (see e.g. [30, p. 107]) shows that every element $\boldsymbol{f}^B \in V_p(\mathbb{I})$ defines a p-adic family (in fact, finite collections of such, since $\mathbb{I}$ is finite over $W_0[\![T]\!]$) of p-adic modular forms $\boldsymbol{f}_z^B = \boldsymbol{f}^B(u^z - 1) \in V_p(W_0)$, where $u = 1 + p$, indexed by $z \in \mathbf{Z}_p$.

The Hida family $\boldsymbol{f}$ corresponds to minimal prime in the localized universal p-ordinary Hecke algebra $\mathbb{T}^{\rm ord}_{\infty,\mathfrak{m}}$, and by the integral Jacquet–Langlands correspondence (see the discussion in [42, §5.3] for example), there exists a p-adic family $\boldsymbol{f}_B$ as above corresponding to $\boldsymbol{f}$, which we normalize by requiring that some Serre–Tate expansion $\boldsymbol{f}_z^B(t)$ does not vanish modulo p.

There are U- and V-operators acting on $\boldsymbol{f}_B$ defined as in [6, §3.6], and we set

$$\boldsymbol{f}_B^\flat := \boldsymbol{f}_B|(VU - UV).$$

With these, one can define $\mathbb{I}^{\rm ur}$-valued measures $\mu_{\boldsymbol{f}_B, x(\mathfrak{a})}$ and $\mu_{\boldsymbol{f}^\flat_{B,\mathfrak{a}}}$ on $\mathbf{Z}_p$ (with the latter supported on $\mathbf{Z}_p^\times$ by [6, Prop. 4.17]) as in [13, §2.7], and an $\mathbb{I}^{\rm ur}$-valued measure $\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\boldsymbol{f}/\mathcal{K})$ on $\mathrm{Gal}(H_{p^\infty}/\mathcal{K})$ by

$$\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\boldsymbol{f}/\mathcal{K})(\phi) = \sum_{[\mathfrak{a}]\in\mathrm{Pic}(\mathcal{O}_\mathcal{K})} \boldsymbol{\xi}\boldsymbol{\chi}^{-1}(\mathfrak{a})\mathbf{N}(\mathfrak{a})^{-1} \int_{\mathbf{Z}_p^\times} (\phi|[\mathfrak{a}])(z)\mathrm{d}\mu_{\boldsymbol{f}^\flat_{B,\mathfrak{a}}}(z)$$

for all $\phi : \mathrm{Gal}(H_{p^\infty}/\mathcal{K}) \to \mathcal{O}_{\mathbf{C}_p}^\times$, where, if $\sigma_\mathfrak{a}$ corresponds to $\mathfrak{a}$ under the Artin reciprocity map, $\phi|[\mathfrak{a}]$ is the character on $z \in \mathbf{Z}_p^\times$ given by $\phi(\sigma_\mathfrak{a}\mathrm{rec}_\mathfrak{p}(z))$ for the local reciprocity map $\mathrm{rec}_\mathfrak{p} : \mathcal{K}_\mathfrak{p}^\times \to G_\mathcal{K}^{\rm ab} \to \Gamma_\mathcal{K}^{\rm ac}$, $\boldsymbol{\chi} : \mathcal{K}^\times\backslash\mathbb{A}_\mathcal{K}^\times \to \mathbb{I}^\times$ is the character given by $x \mapsto \Theta(\mathrm{rec}_\mathbf{Q}(\mathrm{N}_{\mathcal{K}/\mathbf{Q}}(x)))$ for the reciprocity map $\mathrm{rec}_\mathbf{Q} : \mathbf{Q}^\times\backslash\mathbb{A}^\times \to G_\mathbf{Q}^{\rm ab}$, and $\boldsymbol{\xi}$ is the auxiliary anticyclotomic $\mathbb{I}$-adic character constructed in [13, Def. 2.8].

Still denoting by $\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\boldsymbol{f}/\mathcal{K})$ its image under the natural projection $\mathbb{I}^{\rm ur}[\![\mathrm{Gal}(H_{p^\infty}/\mathcal{K})]\!] \to \mathbb{I}^{\rm ur}[\![\Gamma_\mathcal{K}^{\rm ac}]\!]$, and setting

$$\mathscr{L}_\mathfrak{p}^{\mathsf{BDP}}(\boldsymbol{f}/\mathcal{K}) = \mathrm{tw}_{\boldsymbol{\xi}^{-1}}(\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\boldsymbol{f}/\mathcal{K})),$$

one then readily checks as in the proof of [13, Thm. 2.11] that for every $\phi \in \mathcal{X}_a^o(\mathbb{I})$, the specialization $\phi(\mathscr{L}_\mathfrak{p}(\boldsymbol{f}/\mathcal{K}))$ agrees with the measure constructed in [6, §8.4] (in a formulation germane to that in [9, §5.2]) for the newform associated with $\boldsymbol{f}_\phi$, from where the stated interpolation property follows from [6, Prop. 8.9]. □

Corollary 2.10. *With hypotheses as in Theorem 2.9, denote by $\mathscr{L}_\mathfrak{p}(\boldsymbol{f}^\dagger/\mathcal{K})_{\rm ac}$ the image of $\mathrm{tw}_{\Theta^{-1}}(\mathscr{L}_\mathfrak{p}(\boldsymbol{f}/\mathcal{K}))$ under the natural projection $\mathbb{I}^{\rm ur}[\![\Gamma_\mathcal{K}]\!] \to \mathbb{I}^{\rm ur}[\![\Gamma_\mathcal{K}^{\rm ac}]\!]$. Then*

$$\mathscr{L}_\mathfrak{p}(\boldsymbol{f}^\dagger/\mathcal{K})_{\rm ac} = \mathscr{L}_\mathfrak{p}^{\mathsf{BDP}}(\boldsymbol{f}/\mathcal{K})^2 \tag{2.10}$$

up to a unit in $\mathbb{I}^{\rm ur}[\![\Gamma_\mathcal{K}^{\rm ac}]\!][1/p]^\times$. In particular, $\mathscr{L}_\mathfrak{p}(\boldsymbol{f}^\dagger/\mathcal{K})_{\rm ac}$ is nonzero.

Proof. In light of Proposition 2.8, the claimed equality up to a unit follows from a direct comparison of the respective interpolation formulas (*cf.* [37, §3.3]). On the other hand, for every $\phi \in \mathcal{X}_a^o(\mathbb{I})$ the p-adic L-function $\mathscr{L}_{\mathfrak{p}}^{\mathsf{BDP}}(\boldsymbol{f}/\mathcal{K})$ specializes at ϕ to the p-adic L-functions constructed in [14, §3.3] (for $N^- = 1$), and in [9, §5.2] and [6, §8] (for $N^- > 1$); since the latter are nonzero by [14, Thm. 3.9] and [9, Thm. 5.7], the last claim in the theorem follows. □

3. Iwasawa theory

Throughout this section, we fix a positive integer N and a prime $p \nmid 6N$, and let $\boldsymbol{f} = \sum_{n=1}^{\infty} \boldsymbol{a}_n q^n \in \mathbb{I}[\![q]\!]$ be a Hida family of tame level N and trivial tame character, and let $\mathcal{K}$ be an imaginary quadratic field of discriminant prime of Np in which $p = \mathfrak{p}\overline{\mathfrak{p}}$ splits.

3.1. Selmer groups

Let $T_{\boldsymbol{f}}$ be the big Galois representation associated to $\boldsymbol{f}$, for which we shall take the geometric realization denoted $M(\boldsymbol{f})^*$ in [40, Def. 7.2.5]. Thus $T_{\boldsymbol{f}}$ is a locally free $\mathbb{I}$-module of rank 2, and letting $D_p \subset G_{\mathbf{Q}}$ be the decomposition group at p determined by our fixed embedding $\iota_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$, it fits in an exact sequence of $\mathbb{I}[\![D_p]\!]$-modules

$$0 \longrightarrow \mathscr{F}^{+}T_{\boldsymbol{f}} \longrightarrow T_{\boldsymbol{f}} \longrightarrow \mathscr{F}^{-}T_{\boldsymbol{f}} \longrightarrow 0 \tag{3.1}$$

with $\mathscr{F}^{\pm}T_{\boldsymbol{f}}$ locally free of rank 1 over $\mathbb{I}$, and with the D_p-action on the quotient $\mathscr{F}^{-}T_{\boldsymbol{f}}$ given by the unramified character sending an arithmetic Frobenius to $\boldsymbol{a}_p \in \mathbb{I}^{\times}$.

Let $k_{\mathbb{I}} := \mathbb{I}/\mathfrak{m}_{\mathbb{I}}$ be the residue field of $\mathbb{I}$, and denote by $\bar{\rho}_{\boldsymbol{f}} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\kappa_{\mathbb{I}})$ the semi-simple residual representation associated with $T_{\boldsymbol{f}}$, which by (3.1) is conjugate to an upper-triangular representation upon restriction to D_p:

$$\bar{\rho}_{\boldsymbol{f}}|_{D_p} \sim \begin{pmatrix} \bar{\varepsilon} & * \\ & \bar{\delta} \end{pmatrix}.$$

Assume that $\bar{\rho}_{\boldsymbol{f}}$ is absolutely irreducible and that $\bar{\varepsilon} \neq \bar{\delta}$. Then by work of Wiles [66] (see also [40, Thm. 7.2.8]), $T_{\boldsymbol{f}}$ is free of rank 2 over $\mathbb{I}$, and each $\mathscr{F}^{\pm}T_{\boldsymbol{f}}$ is free of rank 1.

Recall that $\Gamma_{\mathcal{K}}$ denotes the Galois group of the $\mathbf{Z}_p^2$-extension $\mathcal{K}_\infty/\mathcal{K}$, and consider the $\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]$-module

$$\mathbf{T} := T_{\boldsymbol{f}} \otimes_{\mathbb{I}} \mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]$$

equipped with the $G_{\mathcal{K}}$-action via $\rho_{\boldsymbol{f}} \otimes \Psi_{\mathcal{K}}$, where $\rho_{\boldsymbol{f}}$ is the $G_{\mathbf{Q}}$-representation afforded by $T_{\boldsymbol{f}}$, and $\Psi_{\mathcal{K}}$ is the tautological character $G_{\mathcal{K}} \twoheadrightarrow \Gamma_{\mathcal{K}} \hookrightarrow \mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]^{\times}$. Replacing $\Gamma_{\mathcal{K}}$ by $\Gamma_{\mathcal{K}}^{\mathrm{ac}}$ (resp. $\Gamma_{\mathcal{K}}^{\mathrm{cyc}}$), we define the $G_{\mathcal{K}}$-module $\mathbf{T}^{\mathrm{ac}}$ (resp. $\mathbf{T}^{\mathrm{cyc}}$) similarly.

As in [35], we also define the critical twist

$$T_{\boldsymbol{f}}^{\dagger} := T_{\boldsymbol{f}} \otimes \Theta^{-1}, \tag{3.2}$$

where $\Theta : G_{\mathbf{Q}} \to \mathbb{I}^{\times}$ is the character (2.2), and define its deformations $\mathbf{T}^{\dagger}, \mathbf{T}^{\dagger,\mathrm{ac}}$, and $\mathbf{T}^{\dagger,\mathrm{cyc}}$ similarly as before.

In the definitions that follow, we let M denote either of the above Galois modules, for which we naturally define $\mathscr{F}^{\pm}M$ using (3.1). We also let Σ be a finite set of places of $\mathbf{Q}$ containing ∞ and the primes dividing Np, and for any number field F, let $\mathfrak{G}_{F,\Sigma}$ be the Galois group of the maximal extension of F unramified outside the places above Σ.

Consider the *p-relaxed* Selmer group defined by

$$\mathrm{Sel}^{\{p\}}(F, M) = \ker\left\{\mathrm{H}^1(\mathfrak{G}_{F,\Sigma}, M) \longrightarrow \prod_{v\in\Sigma,\ v\nmid p} \frac{\mathrm{H}^1(F_v, M)}{\mathrm{H}^1_{\mathrm{ur}}(F_v, M)}\right\},$$

where $\mathrm{H}^1_{\mathrm{ur}}(F_v, M) = \ker\{\mathrm{H}^1(F_v, M) \to \mathrm{H}^1(F_v^{\mathrm{ur}}, M)\}$ is the unramified local condition.

Definition 3.1. For $v|p$ and $\mathscr{L}_v \in \{\emptyset, \mathtt{Gr}, 0\}$, set

$$\mathrm{H}^1_{\mathscr{L}_v}(F_v, M) := \begin{cases} \mathrm{H}^1(F_v, M) & \text{if } \mathscr{L}_v = \emptyset, \\ \ker\{\mathrm{H}^1(F_v, M) \longrightarrow \mathrm{H}^1(F_v^{\mathrm{ur}}, \mathscr{F}^{-}M)\} & \text{if } \mathscr{L}_v = \mathtt{Gr}, \\ \{0\} & \text{if } \mathscr{L}_v = 0, \end{cases}$$

and for $\mathscr{L} = \{\mathscr{L}_v\}_{v|p}$, define

$$\mathrm{Sel}_{\mathscr{L}}(F, M) := \ker\left\{\mathrm{Sel}^{\{p\}}(F, M) \longrightarrow \prod_{v|p} \frac{\mathrm{H}^1(F_v, M)}{\mathrm{H}^1_{\mathscr{L}_v}(F_v, M)}\right\}.$$

Thus, for example $\mathrm{Sel}_{0,\emptyset}(\mathcal{K}, M)$ is the subspace of $\mathrm{Sel}^{\{p\}}(\mathcal{K}, M)$ consisting of classes which satisfy no condition (resp. are locally trivial) at $\overline{\mathfrak{p}}$ (resp. $\mathfrak{p}$). For the ease of notation, we let $\mathrm{Sel}_{\mathtt{Gr}}(F, M)$ denote the Selmer group $\mathrm{Sel}_{\mathscr{L}}(F, M)$ given by $\mathscr{L}_v = \mathtt{Gr}$ for all $v|p$.

We shall also need to consider Selmer groups for the discrete module

$$A_{\boldsymbol{f}} := \mathrm{Hom}_{\mathrm{cts}}(T_{\boldsymbol{f}}, \mu_{p^\infty}).$$

To define these, we note that by Shapiro's lemma there is a canonical isomorphism

$$\mathrm{H}^1(\mathcal{K}, \mathbf{T}) \simeq \varprojlim_{\mathcal{K}\subset_{\mathrm{f}} F \subset \mathcal{K}_\infty} \mathrm{H}^1(F, T_{\boldsymbol{f}}), \tag{3.3}$$

where F runs over the finite extensions of $\mathcal{K}$ contained in $\mathcal{K}_\infty$ and the limit is with respect to the corestriction maps. The isomorphism (3.3) is compatible with the local restriction maps (see e.g. [57, §3.1.2]), and therefore the Selmer groups $\mathrm{Sel}_{\mathscr{L}}(\mathcal{K}, \mathbf{T})$ are defined by local conditions $\mathrm{H}^1_{\mathscr{L}_v}(F_v, T_{\boldsymbol{f}}) \subset \mathrm{H}^1(F_v, T_{\boldsymbol{f}})$ for all primes v (with the unramified local condition for $v \nmid p$). Thus we may let

$$\mathrm{Sel}_{\mathscr{L}}(\mathcal{K}_\infty, A_{\boldsymbol{f}}) \subset \varinjlim_{\mathcal{K} \subset_{\mathrm{f}} F \subset \mathcal{K}_\infty} \mathrm{H}^1(F, A_{\boldsymbol{f}})$$

be the submodule cut out by the orthogonal complements of $\mathrm{H}^1_{\mathscr{L}_v}(F_v, T_{\boldsymbol{f}})$ under the perfect Tate duality

$$\mathrm{H}^1(F_v, T_{\boldsymbol{f}}) \times \mathrm{H}^1(F_v, A_{\boldsymbol{f}}) \longrightarrow \mathbf{Q}_p/\mathbf{Z}_p.$$

This also defines the Selmer groups $\mathrm{Sel}_{\mathscr{L}}(F, A_{\boldsymbol{f}}) \subset \mathrm{H}^1(F, A_{\boldsymbol{f}})$ for any number field F, and we shall also consider their variants for the twisted module

$$A^\dagger_{\boldsymbol{f}} := \mathrm{Hom}_{\mathbf{Z}_p}(T^\dagger_{\boldsymbol{f}}, \mu_{p^\infty}),$$

or their specializations. Finally, if W denotes any of the preceding discrete modules, we set

$$X_{\mathscr{L}}(F, W) := \mathrm{Hom}_{\mathbf{Z}_p}(\mathrm{Sel}_{\mathscr{L}}(F, W), \mathbf{Q}_p/\mathbf{Z}_p),$$

which we simply denote by $X_{\mathtt{Gr}}(F, W)$ when $\mathscr{L}_v = \mathtt{Gr}$ for all $v|p$.

We now record a number of lemmas for our later use.

Lemma 3.2. *Assume that $\bar{\rho}_{\boldsymbol{f}}|_{G_F}$ is absolutely irreducible. Then $\mathrm{Sel}_{\mathtt{Gr}}(F, T^\dagger_{\boldsymbol{f}})$ and $X_{\mathtt{Gr}}(F, A^\dagger_{\boldsymbol{f}})$ have the same $\mathbb{I}$-rank.*

Proof. For any height one prime $\mathfrak{P} \subset \mathbb{I}$, let $\mathbb{I}_{\mathfrak{P}}$ be the localization of $\mathbb{I}$ at $\mathfrak{P}$, and let $F_{\mathfrak{P}} = \mathbb{I}_{\mathfrak{P}}/\mathfrak{P}$ be the residue field. It suffices to show that for all but finitely many $\mathfrak{P} \in \mathcal{X}_a(\mathbb{I})$, the spaces $\mathrm{Sel}_{\mathtt{Gr}}(F, T^\dagger_{\boldsymbol{f}})_{\mathfrak{P}}/\mathfrak{P}$ and $X_{\mathtt{Gr}}(F, A^\dagger_{\boldsymbol{f}})_{\mathfrak{P}}/\mathfrak{P}$ have the same $F_{\mathfrak{P}}$-dimension.

As noted in [46, §12.7.5] (see also [35, Lem. 2.1.6]), Hida's results imply that the localization $\mathbb{I}_{\mathfrak{P}}$ of $\mathbb{I}$ at any $\mathfrak{P} \in \mathcal{X}_a(\mathbb{I})$ is a discrete valuation ring. Let $\pi \in \mathbb{I}_{\mathfrak{P}}$ be a uniformizer. From Nekovář's theory (see [46, Prop. 12.7.13.4(i)]) and the identification [35, (21)], multiplication by π induces natural maps

$$\mathrm{Sel}_{\mathtt{Gr}}(F, T^\dagger_{\boldsymbol{f}})_{\mathfrak{P}}/\pi \hookrightarrow \mathrm{Sel}_{\mathtt{Gr}}(F, T^\dagger_{\boldsymbol{f},\mathfrak{P}}/\pi),$$
$$\mathrm{Sel}_{\mathtt{Gr}}(F, A^\dagger_{\boldsymbol{f},\mathfrak{P}}[\pi]) \twoheadrightarrow \mathrm{Sel}_{\mathtt{Gr}}(F, A^\dagger_{\boldsymbol{f}})_{\mathfrak{P}}[\pi]$$

which are isomorphisms for all but finitely many $\mathfrak{P} \in \mathcal{X}_a(\mathbb{I})$. Since by [33, Lem. 1.3.3] the spaces $\mathrm{Sel}_{\mathtt{Gr}}(F, T^\dagger_{\boldsymbol{f},\mathfrak{P}}/\pi)$ and $\mathrm{Sel}_{\mathtt{Gr}}(F, A^\dagger_{\boldsymbol{f},\mathfrak{P}}[\pi])$ have the same $F_{\mathfrak{P}}$-dimension, the result follows. □

Lemma 3.3. *Assume that $\bar{\rho}_{\boldsymbol{f}}|_{G_{\mathcal{K}}}$ is absolutely irreducible. Then $\mathrm{H}^1(\mathfrak{G}_{\mathcal{K},\Sigma}, \mathbf{T}^\dagger)$ and $\mathrm{H}^1(\mathfrak{G}_{\mathcal{K},\Sigma}, \mathbf{T}^{\dagger,\mathrm{ac}})$ are torsion-free over $\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]$ and $\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]$, respectively.*

Proof. This follows from [50, §1.3.3], since $\mathrm{H}^0(\mathcal{K}_\infty, \bar{\rho}_{\boldsymbol{f}}) = \mathrm{H}^0(\mathcal{K}_\infty^{\mathrm{ac}}, \bar{\rho}_{\boldsymbol{f}}) = \{0\}$ by the hypothesis. □

Lemma 3.4. *We have $\mathrm{rank}_{\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]}(X_{\mathtt{Gr},\emptyset}(\mathcal{K}_\infty^{\mathrm{ac}}, A_{\boldsymbol{f}}^\dagger)) = 1 + \mathrm{rank}_{\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]}(X_{\mathtt{Gr},0}(\mathcal{K}_\infty^{\mathrm{ac}}, A_{\boldsymbol{f}}^\dagger))$. Moreover, if $\mathbb{I}$ is regular then*

$$\mathrm{Char}_{\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]}(X_{\mathtt{Gr},\emptyset}(\mathcal{K}_\infty^{\mathrm{ac}}, A_{\boldsymbol{f}}^\dagger)_{\mathrm{tors}}) = \mathrm{Char}_{\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]}(X_{0,\mathtt{Gr}}(\mathcal{K}_\infty^{\mathrm{ac}}, A_{\boldsymbol{f}}^\dagger)_{\mathrm{tors}}),$$

where the subscript tors *denotes the $\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]$-torsion submodule.*

Proof. The first claim follows from an argument similar to that in Lemma 3.2 using part (2) of [12, Lem. 2.3]. For the second, note that the regularity of $\mathbb{I}$ implies that of $\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]$. Thus by [19, Lem. 6.18] the second claim follows from part (3) of [12, Lem. 2.3]. □

We conclude this section with the following useful commutative algebra lemma from [57], which will be used repeatedly in the proof of our main results.

Lemma 3.5. *Let R be a local ring and $\mathfrak{a} \subset R$ a proper ideal such that $R/\mathfrak{a}$ is a domain. Let $I \subset R$ be an ideal and $\mathcal{L}$ an element of R with $I \subset (\mathcal{L})$. Denote by a 'bar' the image under the reduction map $R \to R/\mathfrak{a}$. If $\overline{\mathcal{L}} \in R/\mathfrak{a}$ is nonzero and $\overline{\mathcal{L}} \in \overline{I}$, then $I = (\mathcal{L})$.*

Proof. This is a special case of [57, Lem. 3.2]. □

3.2. Explicit reciprocity laws

Let $G_{\mathbf{Q}}$ act on the cyclotomic Iwasawa algebra Λ_Γ introduced in §2.3 via the tautological character $G_{\mathbf{Q}} \twoheadrightarrow \Gamma \hookrightarrow \Lambda_\Gamma^\times$. In [40], Kings–Loeffler–Zerbes constructed Beilinson–Flach elements

$${}_c\mathcal{BF}_m^{\boldsymbol{f},\boldsymbol{g}} \in \mathrm{H}^1(\mathbf{Q}(\mu_m), T_{\boldsymbol{f}} \hat{\otimes}_{\mathbf{Z}_p} T_{\boldsymbol{g}} \hat{\otimes}_{\mathbf{Z}_p} \Lambda_\Gamma)$$

attached to pairs of Hida families $\boldsymbol{f}, \boldsymbol{g}$, and related the image of ${}_c\mathcal{BF}_1^{\boldsymbol{f},\boldsymbol{g}}$ under a Perrin-Riou big logarithm map to the p-adic L-functions $L_p(\boldsymbol{f}, \boldsymbol{g})$ and $L_p(\boldsymbol{g}, \boldsymbol{f})$ of Theorem 2.1. In this section we describe the variant of their results that we shall need.

Since $\mathbf{T}^\dagger = \mathbf{T} \otimes \Theta^{-1}$ by definition, the twist map $\mathrm{tw}_{\Theta^{-1}} : \mathbb{I}[\![\Gamma_{\mathcal{K}}]\!] \to \mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]$ of (2.3) induces a $\mathbb{I}$-linear isomorphism

$$\widetilde{\mathrm{tw}}_{\Theta^{-1}} : \mathrm{H}^1(\mathcal{K}, \mathbf{T}) \longrightarrow \mathrm{H}^1(\mathcal{K}, \mathbf{T}^\dagger)$$

satisfying $\widetilde{\mathrm{tw}}_{\Theta^{-1}}(\lambda x) = \mathrm{tw}_{\Theta^{-1}}(\lambda)\widetilde{\mathrm{tw}}_{\Theta^{-1}}(x)$ for all $\lambda \in \mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]$.

Theorem 3.6 *(Kings–Loeffler–Zerbes). There exists a class* $\mathcal{BF}^\dagger \in \mathrm{Sel}_{\mathtt{Gr},\emptyset}(\mathcal{K},\mathbf{T}^\dagger)$ *and* $\mathbb{I}[\![\Gamma_\mathcal{K}]\!]$*-linear injections with pseudo-null cokernel*

$$\begin{aligned}\mathrm{Col}^{(1),\dagger} : \mathrm{H}^1(\mathcal{K}_{\overline{\mathfrak{p}}},\mathscr{F}^-\mathbf{T}^\dagger) &\longrightarrow J_{\boldsymbol{f}}\otimes_{\mathbb{I}}\mathbb{I}[\![\Gamma_\mathcal{K}]\!],\\ \mathrm{Col}^{(2),\dagger} : \mathrm{H}^1(\mathcal{K}_{\mathfrak{p}},\mathscr{F}^+\mathbf{T}^\dagger) &\longrightarrow J_{\boldsymbol{g}}\otimes_{\mathbb{I}}\mathbb{I}]\!]\Gamma_\mathcal{K}[\![,\end{aligned}$$

where $\boldsymbol{g}$ *is the CM Hida family in* (2.1)*, such that*

$$\begin{aligned}\mathrm{Col}^{(1),\dagger}(\mathrm{loc}_{\overline{\mathfrak{p}}}(\mathcal{BF}^\dagger)) &= \mathrm{tw}_{\Theta^{-1}}(L_p(\boldsymbol{f},\boldsymbol{g}))\\ \mathrm{Col}^{(2),\dagger}(\mathrm{loc}_{\mathfrak{p}}(\mathcal{BF}^\dagger)) &= \mathrm{tw}_{\Theta^{-1}}(L_p(\boldsymbol{g},\boldsymbol{f})).\end{aligned}$$

In particular, for every prime v *of* $\mathcal{K}$ *above* p*, the class* $\mathrm{loc}_v(\mathcal{BF}^\dagger) \in \mathrm{H}^1(\mathcal{K}_v,\mathbf{T}^\dagger)$ *is non-torsion over* $\mathbb{I}[\![\Gamma_\mathcal{K}]\!]$*.*

Proof. This follows from the results of [40], as explained in [12, Thm. 2.4], to which one needs to add some of the analysis in [8] and [3].

Indeed, taking $m = 1$ in [40, Def. 8.1.1] (and using [41, Lem. 6.8.9] to dispense with an auxiliary $c > 1$ needed for the construction), one obtains a cohomology class

$$\mathcal{BF}^{\boldsymbol{f},\boldsymbol{g}} \in \mathrm{H}^1(\mathbf{Q}, T_{\boldsymbol{f}}\hat{\otimes}_{\mathbf{Z}_p}T_{\boldsymbol{g}}\hat{\otimes}_{\mathbf{Z}_p}\Lambda_\Gamma)$$

attached to our fixed Hida family $\boldsymbol{f}$ and a second Hida family $\boldsymbol{g}$. Denote by $e_1\mathcal{BF}^{\boldsymbol{f},\boldsymbol{g}}$ the image of $\mathcal{BF}^{\boldsymbol{f},\boldsymbol{g}}$ under the natural map

$$e_1 : \mathrm{H}^1(\mathbf{Q}, T_{\boldsymbol{f}}\hat{\otimes}_{\mathbf{Z}_p}T_{\boldsymbol{g}}\hat{\otimes}_{\mathbf{Z}_p}\Lambda_\Gamma) \to \mathrm{H}^1(\mathbf{Q}, T_{\boldsymbol{f}}\hat{\otimes}_{\mathbf{Z}_p}T_{\boldsymbol{g}}\hat{\otimes}_{\mathbf{Z}_p}\mathbf{Z}_p[\![\Gamma^{\mathrm{cyc}}]\!])$$

induced by the projection $\Gamma \twoheadrightarrow \Gamma^{\mathrm{cyc}}$. Taking $\boldsymbol{g}$ to be the canonical CM Hida family in (2.1), by [8] (see also [3, Prop. 4.1]) we have a $G_\mathbf{Q}$-module isomorphism

$$T_{\boldsymbol{g}} \simeq \mathrm{Ind}_\mathcal{K}^\mathbf{Q}\mathbf{Z}_p[\![\Gamma_\mathfrak{p}]\!]$$

where the $G_\mathcal{K}$-action on $\mathbf{Z}_p[\![\Gamma_\mathfrak{p}]\!]$ is given by the tautological character $G_\mathcal{K} \twoheadrightarrow \Gamma_\mathfrak{p} \hookrightarrow \mathbf{Z}_p[\![\Gamma_\mathfrak{p}]\!]^\times$. By Shapiro's lemma, $e_1\mathcal{BF}^{\boldsymbol{f},\boldsymbol{g}}$ therefore defines a class $\mathcal{BF} \in \mathrm{H}^1(\mathcal{K},\mathbf{T})$ whose image under $\widetilde{\mathrm{tw}}_{\Theta^{-1}}$ defines a class $\mathcal{BF}^\dagger$ with the desired properties.

More precisely, the inclusion $\mathcal{BF}^\dagger \in \mathrm{Sel}_{\mathtt{Gr},\emptyset}(\mathcal{K},\mathbf{T}^\dagger)$ follows from [40, Prop. 8.1.7], and by the explicit reciprocity law of [40, Thm. 10.2.2], the maps

$$\mathrm{Col}^{(1)} := \langle\mathcal{L}(-),\eta_{\boldsymbol{f}}\otimes\omega_{\boldsymbol{g}}\rangle, \quad \mathrm{Col}^{(2)} := \langle\mathcal{L}(-),\eta_{\boldsymbol{g}}\otimes\omega_{\boldsymbol{f}}\rangle$$

described in the proof of [12, Thm. 2.4] send the restriction at $\overline{\mathfrak{p}}$ and $\mathfrak{p}$ of $\mathcal{BF}^{\boldsymbol{f},\boldsymbol{g}}$ to the p-adic L-functions $L_p(\boldsymbol{f},\boldsymbol{g})$ and $L_p(\boldsymbol{g},\boldsymbol{f})$, respectively, and are injective with pseudo-null cokernel by [40, Thm. 8.2.3]. Thus letting $\mathrm{Col}^{(1),\dagger}$ and $\mathrm{Col}^{(2),\dagger}$ be the $\mathbb{I}[\![\Gamma_\mathcal{K}]\!]$-linear maps defined by the commutative diagrams

$$\begin{array}{ccc} \mathrm{H}^1(\mathcal{K}_{\overline{\mathfrak{p}}}, \mathscr{F}^-\mathbf{T}) & \xrightarrow{\mathrm{Col}^{(1)}} & J_{\boldsymbol{f}} \otimes_{\mathbb{I}} \mathbb{I}[\![\Gamma_{\mathcal{K}}]\!] \\ \downarrow{\scriptstyle \widetilde{\mathrm{tw}}_{\Theta^{-1}}} & & \downarrow{\scriptstyle \mathrm{tw}_{\Theta^{-1}}} \\ \mathrm{H}^1(\mathcal{K}_{\overline{\mathfrak{p}}}, \mathscr{F}^-\mathbf{T}^\dagger) & \xrightarrow{\mathrm{Col}^{(1),\dagger}} & J_{\boldsymbol{f}} \otimes_{\mathbb{I}} \mathbb{I}[\![\Gamma_{\mathcal{K}}]\!] \end{array} \qquad \begin{array}{ccc} \mathrm{H}^1(\mathcal{K}_{\mathfrak{p}}, \mathscr{F}^+\mathbf{T}) & \xrightarrow{\mathrm{Col}^{(2)}} & J_{\boldsymbol{g}} \otimes_{\mathbb{I}} \mathbb{I}[\![\Gamma_{\mathcal{K}}]\!] \\ \downarrow{\scriptstyle \widetilde{\mathrm{tw}}_{\Theta^{-1}}} & & \downarrow{\scriptstyle \mathrm{tw}_{\Theta^{-1}}} \\ \mathrm{H}^1(\mathcal{K}_{\mathfrak{p}}, \mathscr{F}^+\mathbf{T}^\dagger) & \xrightarrow{\mathrm{Col}^{(2),\dagger}} & J_{\boldsymbol{g}} \otimes_{\mathbb{I}} \mathbb{I}[\![\Gamma_{\mathcal{K}}]\!], \end{array}$$

the result follows, with the last claim being an immediate consequence of the nonvanishing of the p-adic L-functions $L_p(\boldsymbol{f}, \boldsymbol{g})$ and $L_p(\boldsymbol{g}, \boldsymbol{f})$ (see e.g. [12, Rem. 1.3]). □

We shall also need to consider anticyclotomic variants of the maps $\mathrm{Col}^{(i),\dagger}$ in Theorem 3.6. Letting $\mathcal{I}_{\mathrm{cyc}}$ be the kernel of the natural projection $\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!] \to \mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]$, the map

$$\mathrm{Col}_{\mathrm{ac}}^{(1),\dagger} : \mathrm{H}^1(\mathcal{K}_{\overline{\mathfrak{p}}}, \mathscr{F}^-\mathbf{T}^{\dagger,\mathrm{ac}}) \longrightarrow J_{\boldsymbol{f}} \otimes_{\mathbb{I}} \mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]$$

is defined by reducing $\mathrm{Col}^{(1),\dagger}$ modulo the ideal $\mathcal{I}_{\mathrm{cyc}}$, using the fact that by the vanishing of $\mathrm{H}^0(\mathcal{K}_{\overline{\mathfrak{p}}}, \mathscr{F}^-\mathbf{T}^{\dagger,\mathrm{ac}})$ the restriction map induces a natural isomorphism

$$\mathrm{H}^1(\mathcal{K}_{\overline{\mathfrak{p}}}, \mathscr{F}^-\mathbf{T}^\dagger)/\mathcal{I}_{\mathrm{cyc}} \simeq \mathrm{H}^1(\mathcal{K}_{\overline{\mathfrak{p}}}, \mathscr{F}^-\mathbf{T}^{\dagger,\mathrm{ac}}).$$

The map $\mathrm{Col}_{\mathrm{ac}}^{(2),\dagger} : \mathrm{H}^1(\mathcal{K}_{\mathfrak{p}}, \mathscr{F}^+\mathbf{T}^{\dagger,\mathrm{ac}}) \to J_{\boldsymbol{g}} \hat{\otimes}_{\mathbf{Z}_p} \mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]$ is defined in the same manner.

Note that since the maps $\mathrm{Col}^{(i),\dagger}$ are injective with pseudo-null cokernel, the same is true for the maps $\mathrm{Col}_{\mathrm{ac}}^{(i),\dagger}$.

Corollary 3.7. *Let $\mathcal{BF}^{\dagger,\mathrm{ac}}$ be the image of the class $\mathcal{BF}^\dagger$ under the natural map $\mathrm{H}^1(\mathcal{K}, \mathbf{T}^\dagger) \to \mathrm{H}^1(\mathcal{K}, \mathbf{T}^{\dagger,\mathrm{ac}})$, and assume that $\mathcal{K}$ satisfies (gen-H). Then we have the inclusion*

$$\mathrm{loc}_{\overline{\mathfrak{p}}}(\mathcal{BF}^{\dagger,\mathrm{ac}}) \in \ker\{\mathrm{H}^1(\mathcal{K}_{\overline{\mathfrak{p}}}, \mathbf{T}^{\dagger,\mathrm{ac}}) \longrightarrow \mathrm{H}^1(\mathcal{K}_{\overline{\mathfrak{p}}}, \mathscr{F}^-\mathbf{T}^{\dagger,\mathrm{ac}})\};$$

in particular, $\mathcal{BF}^{\dagger,\mathrm{ac}} \in \mathrm{Sel}_{\mathfrak{Gr}}(\mathcal{K}, \mathbf{T}^{\dagger,\mathrm{ac}})$. Moreover, if we assume in addition that N is squarefree when $N^- > 1$, then the class $\mathrm{loc}_{\mathfrak{p}}(\mathcal{BF}^{\dagger,\mathrm{ac}})$ is non-torsion over $\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]$.

Proof. The combination of Theorem 3.6 and Proposition 2.3 yields the vanishing of the image of $\mathrm{loc}_{\overline{\mathfrak{p}}}(\mathcal{BF}^{\dagger,\mathrm{ac}})$ under the map $\mathrm{Col}_{\mathrm{ac}}^{(1),\dagger}$, so the first claim follows from its injectivity. The second claim follows from Theorem 3.6 together with the nonvanishing result of Corollary 2.10. □

3.3. Iwasawa main conjectures

We now use the explicit reciprocity laws of Theorem 3.6 to relate different variants of the Iwasawa Main Conjecture for Rankin–Selberg convolutions.

Theorem 3.8. *Assume that $\bar{\rho}_{\boldsymbol{f}}|_{G_{\mathcal{K}}}$ is irreducible. Then the following are equivalent:*

(i) *$X_{\mathtt{Gr},0}(\mathcal{K}_\infty, A_{\boldsymbol{f}}^\dagger)$ is $\mathbb{I}[\![\Gamma_\mathcal{K}]\!]$-torsion, $\mathrm{Sel}_{\mathtt{Gr},\emptyset}(\mathcal{K}, \mathbf{T}^\dagger)$ has $\mathbb{I}[\![\Gamma_\mathcal{K}]\!]$-rank one, and*

$$\mathrm{Char}_{\mathbb{I}[\![\Gamma_\mathcal{K}]\!]}(X_{\mathtt{Gr},0}(\mathcal{K}_\infty, A_{\boldsymbol{f}}^\dagger)) = \mathrm{Char}_{\mathbb{I}[\![\Gamma_\mathcal{K}]\!]}\left(\frac{\mathrm{Sel}_{\mathtt{Gr},\emptyset}(\mathcal{K}, \mathbf{T}^\dagger)}{\mathbb{I}[\![\Gamma_\mathcal{K}]\!] \cdot \mathcal{BF}^\dagger}\right)$$

up to powers of p.

(ii) *Both $X_{\emptyset,0}(\mathcal{K}_\infty, A_{\boldsymbol{f}}^\dagger)$ and $\mathrm{Sel}_{0,\emptyset}(\mathcal{K}, \mathbf{T}^\dagger)$ are $\mathbb{I}[\![\Gamma_\mathcal{K}]\!]$-torsion, and*

$$\mathrm{Char}_{\mathbb{I}[\![\Gamma_\mathcal{K}]\!]}(X_{\emptyset,0}(\mathcal{K}_\infty, A_{\boldsymbol{f}}^\dagger)) \cdot \mathbb{I}^{\mathrm{ur}}[\![\Gamma_\mathcal{K}]\!] = (\mathrm{tw}_{\Theta^{-1}}(\mathscr{L}_\mathfrak{p}(\boldsymbol{f}/\mathcal{K})))$$

up to powers of p.

(iii) *Both $X_{\mathtt{Gr}}(\mathcal{K}_\infty, A_{\boldsymbol{f}}^\dagger)$ and $\mathrm{Sel}_{\mathtt{Gr}}(\mathcal{K}, \mathbf{T}^\dagger)$ are $\mathbb{I}[\![\Gamma_\mathcal{K}]\!]$-torsion, and*

$$\mathrm{Char}_{\mathbb{I}[\![\Gamma_\mathcal{K}]\!]}(X_{\mathtt{Gr}}(\mathcal{K}_\infty, A_{\boldsymbol{f}}^\dagger)) = (\mathrm{tw}_{\Theta^{-1}}(L_p^{\mathtt{Hi}}(\boldsymbol{f}/\mathcal{K}))).$$

up to powers of p.

Moreover, if in addition $\mathcal{K}$ satisfies (gen-H), with N being squarefree when $N^- > 1$, then the following are equivalent:

(i)' *$X_{\mathtt{Gr},0}(\mathcal{K}_\infty^{\mathrm{ac}}, A_{\boldsymbol{f}}^\dagger)$ is $\mathbb{I}[\![\Gamma_\mathcal{K}^{\mathrm{ac}}]\!]$-torsion, $\mathrm{Sel}_{\mathtt{Gr},\emptyset}(\mathcal{K}, \mathbf{T}^{\dagger,\mathrm{ac}})$ has $\mathbb{I}[\![\Gamma_\mathcal{K}^{\mathrm{ac}}]\!]$-rank one, and*

$$\mathrm{Char}_{\mathbb{I}[\![\Gamma_\mathcal{K}^{\mathrm{ac}}]\!]}(X_{\mathtt{Gr},0}(\mathcal{K}_\infty^{\mathrm{ac}}, A_{\boldsymbol{f}}^\dagger)) = \mathrm{Char}_{\mathbb{I}[\![\Gamma_\mathcal{K}^{\mathrm{ac}}]\!]}\left(\frac{\mathrm{Sel}_{\mathtt{Gr},\emptyset}(\mathcal{K}, \mathbf{T}^{\dagger,\mathrm{ac}})}{\mathbb{I}[\![\Gamma_\mathcal{K}^{\mathrm{ac}}]\!] \cdot \mathcal{BF}^{\dagger,\mathrm{ac}}}\right)$$

up to powers of p.

(ii)' *Both $X_{\emptyset,0}(\mathcal{K}_\infty^{\mathrm{ac}}, A_{\boldsymbol{f}}^\dagger)$ and $\mathrm{Sel}_{0,\emptyset}(\mathcal{K}, \mathbf{T}^{\dagger,\mathrm{ac}})$ are $\mathbb{I}[\![\Gamma_\mathcal{K}^{\mathrm{ac}}]\!]$-torsion, and*

$$\mathrm{Char}_{\mathbb{I}[\![\Gamma_\mathcal{K}^{\mathrm{ac}}]\!]}(X_{\emptyset,0}(\mathcal{K}_\infty^{\mathrm{ac}}, A_{\boldsymbol{f}}^\dagger)) \cdot \mathbb{I}^{\mathrm{ur}}[\![\Gamma_\mathcal{K}^{\mathrm{ac}}]\!] = (\mathscr{L}_\mathfrak{p}^{\mathtt{BDP}}(\boldsymbol{f}/\mathcal{K})^2)$$

up to powers of p.

Proof. Consider the exact sequence coming form Poitou–Tate duality

$$0 \longrightarrow \mathrm{Sel}_{0,\emptyset}(\mathcal{K}, \mathbf{T}^\dagger) \longrightarrow \mathrm{Sel}_{\mathtt{Gr},\emptyset}(\mathcal{K}, \mathbf{T}^\dagger) \xrightarrow{\mathrm{loc}_\mathfrak{p}} \mathrm{H}^1_{\mathtt{Gr}}(\mathcal{K}_\mathfrak{p}, \mathbf{T}^\dagger)$$
$$\longrightarrow X_{\emptyset,0}(\mathcal{K}_\infty, A_{\boldsymbol{f}}^\dagger) \longrightarrow X_{\mathtt{Gr},0}(\mathcal{K}_\infty, A_{\boldsymbol{f}}^\dagger) \longrightarrow 0.$$

By Theorem 3.6, the cokernel of the map $\mathrm{loc}_\mathfrak{p}$ is $\mathbb{I}[\![\Gamma_\mathcal{K}]\!]$-torsion, and so the equivalence between the claimed ranks in (i) and (ii) follows. By Lemma 3.3, if $\mathrm{Sel}_{0,\emptyset}(\mathcal{K}, \mathbf{T}^\dagger)$ is $\mathbb{I}[\![\Gamma_\mathcal{K}]\!]$-torsion then it is trivial, and so the above yields

$$0 \longrightarrow \frac{\mathrm{Sel}_{\mathtt{Gr},\emptyset}(\mathcal{K},\mathbf{T}^\dagger)}{\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]\cdot\mathcal{BF}^\dagger} \xrightarrow{\mathrm{loc}_{\mathfrak{p}}} \frac{\mathrm{H}^1_{\mathtt{Gr}}(\mathcal{K}_{\mathfrak{p}},\mathbf{T}^\dagger)}{\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]\cdot\mathrm{loc}_{\mathfrak{p}}(\mathcal{BF}^\dagger)} \longrightarrow X_{\emptyset,0}(\mathcal{K}_\infty,A_{\boldsymbol{f}}^\dagger) \longrightarrow X_{\mathtt{Gr},0}(\mathcal{K}_\infty,A_{\boldsymbol{f}}^\dagger) \longrightarrow 0. \tag{3.4}$$

As noted in the proof of Proposition 2.7, the congruence power series $H(\boldsymbol{g})$ of the CM Hida family $\boldsymbol{g}$ in (2.1) is divisible by $h_K\cdot\mathscr{L}_{\mathfrak{p}}(\mathcal{K})_{\mathrm{ac}}$; together with [32, Thm. 0.2] it follows that the congruence ideal of $\boldsymbol{g}$ is generated by $h_K\cdot\mathscr{L}_{\mathfrak{p}}(\mathcal{K})_{\mathrm{ac}}$ after inverting p. Therefore by Theorem 3.6 and definition (2.6), the map $\mathrm{Col}^{(2),\dagger}$ multiplied by this generator yields an injection

$$\frac{\mathrm{H}^1_{\mathtt{Gr}}(\mathcal{K}_{\mathfrak{p}},\mathbf{T}^\dagger)\cdot\mathbb{I}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!][1/p]}{\mathbb{I}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!][1/p]\cdot\mathrm{loc}_{\mathfrak{p}}(\mathcal{BF}^\dagger))} \hookrightarrow \frac{\mathbb{I}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!][1/p]}{(\mathrm{tw}_{\Theta^{-1}}(\mathscr{L}_{\mathfrak{p}}(\boldsymbol{f}/\mathcal{K})))}$$

with pseudo-null cokernel, which combined with (3.4) completes the proof of the equivalence (i)$\Longleftrightarrow$(ii). The equivalence (i)'$\Longleftrightarrow$(ii)' when $\mathcal{K}$ satisfies (gen-H) is shown in the same way, using the nonvanishing of $\mathrm{loc}_{\mathfrak{p}}(\mathcal{BF}^{\dagger,\mathrm{ac}})$ from Corollary 3.7.

Now consider the exact sequence

$$0 \longrightarrow \mathrm{Sel}_{\mathtt{Gr}}(\mathcal{K},\mathbf{T}^\dagger) \to \mathrm{Sel}_{\mathtt{Gr},\emptyset}(\mathcal{K},\mathbf{T}^\dagger) \xrightarrow{\mathrm{loc}_{\overline{\mathfrak{p}}}} \frac{\mathrm{H}^1(\mathcal{K}_{\overline{\mathfrak{p}}},\mathbf{T}^\dagger)}{\mathrm{H}^1_{\mathtt{Gr}}(\mathcal{K}_{\overline{\mathfrak{p}}},\mathbf{T}^\dagger)} \simeq \mathrm{H}^1(\mathcal{K}_{\overline{\mathfrak{p}}},\mathscr{F}^-\mathbf{T}^\dagger) \longrightarrow X_{\mathtt{Gr}}(\mathcal{K}_\infty,A_{\boldsymbol{f}}^\dagger) \longrightarrow X_{\mathtt{Gr},0}(\mathcal{K}_\infty,A_{\boldsymbol{f}}^\dagger) \longrightarrow 0,$$

which similarly as before implies the equivalence between the claimed $\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]$-ranks in (ii) and (iii), and by Theorem 3.6 and Lemma 3.3 yields the exact sequence

$$0 \longrightarrow \frac{\mathrm{Sel}_{\mathtt{Gr},\emptyset}(\mathcal{K},\mathbf{T}^\dagger)}{\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]\cdot\mathcal{BF}^\dagger} \xrightarrow{\mathrm{loc}_{\overline{\mathfrak{p}}}} \frac{\mathrm{H}^1(\mathcal{K}_{\overline{\mathfrak{p}}},\mathscr{F}^-\mathbf{T}^\dagger)}{\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]\cdot\mathrm{loc}_{\mathfrak{p}}(\mathcal{BF}^\dagger)} \longrightarrow X_{\mathtt{Gr}}(\mathcal{K}_\infty,A_{\boldsymbol{f}}^\dagger) \longrightarrow X_{\mathtt{Gr},0}(\mathcal{K}_\infty,A_{\boldsymbol{f}}^\dagger) \longrightarrow 0.$$

Lastly, since by Theorem 3.6 and Definition 2.2 the map $\mathrm{Col}^{(1),\dagger}$ multiplied by a generator of the congruence ideal $C(\boldsymbol{f})$ yields an injection $\mathrm{H}^1(\mathcal{K}_{\overline{\mathfrak{p}}},\mathscr{F}^-\mathbf{T}^\dagger)\to\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]$ with pseudo-null cokernel sending $\mathrm{loc}_{\mathfrak{p}}(\mathcal{BF}^\dagger)$ into $\mathrm{tw}_{\Theta^{-1}}(L_p^{\mathtt{Hi}}(\boldsymbol{f}/\mathcal{K}))$ up to a unit in $\mathbb{I}^\times$, the equivalence (ii)$\Longleftrightarrow$(iii) follows. □

3.4. Rubin's height formula

Recall the decomposition $\Gamma_{\mathcal{K}}\simeq\Gamma_{\mathcal{K}}^{\mathrm{cyc}}\times\Gamma_{\mathcal{K}}^{\mathrm{ac}}$. Fix a topological generator $\gamma_{\mathrm{cyc}}\in\Gamma_{\mathcal{K}}^{\mathrm{cyc}}$, and using the identification $\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]\simeq(\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!])[\![\Gamma_{\mathcal{K}}^{\mathrm{cyc}}]\!]$, expand

$$\mathrm{tw}_{\Theta^{-1}}(L_p^{\mathtt{Hi}}(\boldsymbol{f}/\mathcal{K})) = L_{p,0}^{\mathtt{Hi}}(\boldsymbol{f}^\dagger/\mathcal{K})_{\mathrm{ac}} + L_{p,1}^{\mathtt{Hi}}(\boldsymbol{f}^\dagger/\mathcal{K})_{\mathrm{ac}}\cdot(\gamma_{\mathrm{cyc}}-1)+\cdots \tag{3.5}$$

as a power series in $\gamma_{\rm cyc}-1$. Note that the constant term $L_{p,0}^{\tt Hi}(\boldsymbol{f}^\dagger/\mathcal{K})_{\rm ac}$ in this expansion corresponds to the image of ${\rm tw}_{\Theta^{-1}}(L_p^{\tt Hi}(\boldsymbol{f}/\mathcal{K}))$ under the natural projection $\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]\rightarrow\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\rm ac}]\!]$.

By Shapiro's lemma, we may consider the Beilinson–Flach class $\mathcal{BF}^\dagger\in{\rm Sel}_{{\tt Gr},\emptyset}(\mathcal{K},\mathbf{T}^\dagger)$ of Theorem 3.6 as a norm-compatible system of classes $\mathcal{BF}_F^\dagger\in{\rm Sel}_{{\tt Gr},\emptyset}(F,T_{\boldsymbol{f}}^\dagger)$ with F running over the finite extensions of $\mathcal{K}$ contained in $\mathcal{K}_\infty$. For any (possibly infinite) intermediate extension $\mathcal{K}\subset L\subset\mathcal{K}_\infty$, we then put

$$\mathcal{BF}^\dagger(L):=\varprojlim_F\mathcal{BF}_F^\dagger$$

with F running over the finite extensions of $\mathcal{K}$ contained in L, so in particular $\mathcal{BF}^\dagger(\mathcal{K}_\infty)$ is nothing but $\mathcal{BF}^\dagger$.

Denote by $\mathcal{K}_n^{\rm ac}$ the subextension of $\mathcal{K}_\infty^{\rm ac}$ with $[\mathcal{K}_n^{\rm ac}:\mathcal{K}]=p^n$, define $\mathcal{K}_k^{\rm cyc}$ similarly, and set $L_{n,k}=\mathcal{K}_n^{\rm ac}\mathcal{K}_k^{\rm cyc}$ for all $k\leqslant\infty$.

Lemma 3.9. *Assume that $\mathcal{K}$ satisfies (gen-H) and that $\bar\rho_{\boldsymbol{f}}|_{G_{\mathcal{K}}}$ is irreducible. Then there is a unique*

$$\beta_n^\dagger\in{\rm H}^1(\mathcal{K}_{n,\overline{\mathfrak{p}}}^{\rm ac},\mathscr{F}^-\mathbf{T}^{\dagger,{\rm cyc}})$$

such that ${\rm loc}_{\overline{\mathfrak{p}}}(\mathcal{BF}^\dagger(L_{n,\infty}))=(\gamma_{\rm cyc}-1)\beta_n^\dagger$. Furthermore, for varying n the images $\beta_n^\dagger(\mathbb{1})$ of $\beta_n^\dagger$ under the corestriction map ${\rm H}^1(\mathcal{K}_{n,\overline{\mathfrak{p}}}^{\rm ac},\mathscr{F}^-\mathbf{T}^{\dagger,{\rm cyc}})\rightarrow{\rm H}^1(\mathcal{K}_{n,\overline{\mathfrak{p}}}^{\rm ac},\mathscr{F}^-T_{\boldsymbol{f}}^\dagger)$ are norm-compatible, defining a class

$$\varprojlim_n\beta_n^\dagger(\mathbb{1})\in\varprojlim_n{\rm H}^1(\mathcal{K}_{n,\overline{\mathfrak{p}}}^{\rm ac},\mathscr{F}^-T_{\boldsymbol{f}}^\dagger)\simeq{\rm H}^1(\mathcal{K}_{\overline{\mathfrak{p}}},\mathscr{F}^-\mathbf{T}^{\dagger,{\rm ac}})$$

that is sent to the linear term $L_{p,1}^{\tt Hi}(\boldsymbol{f}^\dagger/\mathcal{K})_{\rm ac}$ in the expansion (3.5) under the map ${\rm Col}_{\rm ac}^{(1),\dagger}$.

Proof. After Theorem 3.6, the first claim follows from the vanishing of $L_{p,0}^{\tt Hi}(\boldsymbol{f}^\dagger/\mathcal{K})_{\rm ac}$ (see Proposition 2.3) and the injectivity of ${\rm Col}^{(1),\dagger}$, with the uniqueness claim being an immediate consequence of Lemma 3.3. The other claims are a direct consequence of the definitions of $\beta_n^\dagger$ and $L_{p,1}^{\tt Hi}(\boldsymbol{f}^\dagger/\mathcal{K})_{\rm ac}$. □

Let $\mathcal{I}^{\rm cyc}=(\gamma_{\rm cyc}-1)$ be the augmentation ideal in $\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\rm cyc}]\!]$, and put $\mathcal{J}^{\rm cyc}=\mathcal{I}^{\rm cyc}/(\mathcal{I}^{\rm cyc})^2$. By work of Plater [51] (*cf.* Nekovář [46, §11] more generally), for every n there is an $\mathbb{I}$-adic height pairing

$$\langle-,-\rangle_{\mathcal{K}_n^{\rm ac},\mathbb{I}}^{\rm cyc}:{\rm Sel}_{\tt Gr}(\mathcal{K}_n^{\rm ac},T_{\boldsymbol{f}}^\dagger)\times{\rm Sel}_{\tt Gr}(\mathcal{K}_n^{\rm ac},T_{\boldsymbol{f}}^\dagger)\longrightarrow\mathcal{J}^{\rm cyc}\otimes_{\mathbb{I}}F_{\mathbb{I}}. \tag{3.6}$$

(Note that the local indecomposability hypothesis (H1) in [51, p. 107] is only used to ensure the existence of well-defined sub and quotients at the places above p, which for

$T_{\boldsymbol{f}}^\dagger$ is automatic, while hypotheses (H2) and (H3) in [51] follow from [35, Lem. 2.4.4] for $T_{\boldsymbol{f}}^\dagger$.)

Indeed, keeping the notations introduced in §3.1, in light of [51, Lem. 5.8] Plater's definition (which we shall briefly recall in the proof of Proposition 3.10 below) gives a $\mathcal{J}^{\rm cyc} \otimes_{\mathbb{I}} F_{\mathbb{I}}$-valued height pairing on the modified Selmer group

$$\widetilde{\mathfrak{Sel}}(\mathcal{K}_n^{\rm ac}, T_{\boldsymbol{f}}^\dagger) := \ker\left\{ {\rm Sel}^{\{p\}}(\mathcal{K}_n^{\rm ac}, T_{\boldsymbol{f}}^\dagger) \longrightarrow \prod_{v|p} \frac{{\rm H}^1(\mathcal{K}_n^{\rm ac}, T_{\boldsymbol{f}}^\dagger)}{{\rm H}^1_{\tt Gr}(\mathcal{K}_{n,v}^{\rm ac}, T_{\boldsymbol{f}}^\dagger)^{\rm sat}} \right\},$$

where ${\rm H}^1_{\tt Gr}(\mathcal{K}_{n,v}^{\rm ac}, T_{\boldsymbol{f}}^\dagger)^{\rm sat}$ is the saturation of ${\rm H}^1_{\tt Gr}(\mathcal{K}_{n,v}^{\rm ac}, T_{\boldsymbol{f}}^\dagger)$ in ${\rm H}^1(\mathcal{K}_{n,v}^{\rm ac}, T_{\boldsymbol{f}}^\dagger)$, taking $\mathcal{J}^{\rm cyc}$-values on the submodule $\mathfrak{Sel}(\mathcal{K}_n^{\rm ac}, T_{\boldsymbol{f}}^\dagger)$ of $\widetilde{\mathfrak{Sel}}(\mathcal{K}_n^{\rm ac}, T_{\boldsymbol{f}}^\dagger)$ consisting of classes x with

$${\rm loc}_v(x) \in \bigcap_k {\rm cor}_{L_{n,k,v}/\mathcal{K}_{n,v}^{\rm ac}}({\rm H}^1_{\tt Gr}(L_{n,k,v}, T_{\boldsymbol{f}}^\dagger)^{\rm sat})$$

for all primes v above p. Since by the same argument as in [51, Lem. 5.8] (using [45, Lem. 6.3]) the quotient $\widetilde{\mathfrak{Sel}}(\mathcal{K}_n^{\rm ac}, T_{\boldsymbol{f}}^\dagger)/\mathfrak{Sel}(\mathcal{K}_n^{\rm ac}, T_{\boldsymbol{f}}^\dagger)$ is $\mathbb{I}$-torsion, killed by a nonzero element of $\mathbb{I}$ independent of n, from the obvious inclusion ${\rm Sel}_{\tt Gr}(\mathcal{K}_n^{\rm ac}, T_{\boldsymbol{f}}^\dagger) \subset \widetilde{\mathfrak{Sel}}(\mathcal{K}_n^{\rm ac}, T_{\boldsymbol{f}}^\dagger)$ we get a pairing as in (3.6) with denominators bounded independently of n.

The next result generalizes the height formula of [54, Thm. 3.2(ii)] to our context.

Proposition 3.10. *Assume that $\mathcal{K}$ satisfies (gen-H) and that $\bar\rho_{\boldsymbol{f}}|_{G_{\mathcal{K}}}$ is irreducible. Then the classes $\mathcal{BF}^\dagger_{\mathcal{K}_n^{\rm ac}}$ land in ${\rm Sel}_{\tt Gr}(\mathcal{K}_n^{\rm ac}, T_{\boldsymbol{f}}^\dagger)$, and for every $x \in {\rm Sel}_{\tt Gr}(\mathcal{K}_n^{\rm ac}, T_{\boldsymbol{f}}^\dagger)$ we have*

$$\langle \mathcal{BF}^\dagger_{\mathcal{K}_n^{\rm ac}}, x\rangle^{\rm cyc}_{\mathcal{K}_n^{\rm ac},\mathbb{I}} = (\beta_n^\dagger(\mathbb{1}), {\rm loc}_{\overline{\mathfrak{p}}}(x))_{\mathcal{K}_{n,\overline{\mathfrak{p}}}^{\rm ac}} \otimes (\gamma_{\rm cyc} - 1), \tag{3.7}$$

where $(-,-)_{\mathcal{K}_{n,\overline{\mathfrak{p}}}^{\rm ac}}$ is the local Tate pairing

$$\frac{{\rm H}^1(\mathcal{K}_{n,\overline{\mathfrak{p}}}^{\rm ac}, T_{\boldsymbol{f}}^\dagger)}{{\rm H}^1_{\tt Gr}(\mathcal{K}_{n,\overline{\mathfrak{p}}}^{\rm ac}, T_{\boldsymbol{f}}^\dagger)} \times {\rm H}^1_{\tt Gr}(\mathcal{K}_{n,\overline{\mathfrak{p}}}^{\rm ac}, T_{\boldsymbol{f}}^\dagger) \longrightarrow \mathbb{I}.$$

Proof. The first claim follows from the explicit reciprocity law of Theorem 3.6, the vanishing of $L_{p,0}^{\tt Hi}(\boldsymbol{f}^\dagger/\mathcal{K})_{\rm ac}$, and the injectivity of ${\rm Col}^{(1),\dagger}$. On the other hand, the proof of formula (3.7) could be deduced from the general result [46, (11.3.14)], but shall give a proof following the more direct generalization of Rubin's formula contained in [60, §3].

We begin by recalling Plater's definition of the $\mathbb{I}$-adic height pairing (itself a generalization of Perrin-Riou's [49, §1.2] in the p-adic setting). Let λ be the isomorphism $\Gamma_{\mathcal{K}}^{\rm cyc} \simeq \mathcal{J}^{\rm cyc}$ sending $\gamma_{\rm cyc}$ to the class of $\gamma_{\rm cyc} - 1$. Composing with the natural isomorphism ${\rm Gal}(L_{n,\infty}/\mathcal{K}_n^{\rm ac}) \simeq \Gamma^{\rm cyc}$ the map λ defines a class in ${\rm H}^1(\mathcal{K}_n^{\rm ac}, \mathcal{J}^{\rm cyc})$, where we equip $\mathcal{J}^{\rm cyc}$ with the trivial Galois action, and so taking cup product we get

$$\rho_v : \mathrm{H}^1(K^{\mathrm{ac}}_{n,v}, \mathbb{I}(1)) \xrightarrow{\cup \mathrm{loc}_v(\lambda)} \mathrm{H}^2(K^{\mathrm{ac}}_{n,v}, \mathcal{J}^{\mathrm{cyc}}(1)) \simeq \mathcal{J}^{\mathrm{cyc}}$$

for every place v.

Denote by $\mathrm{Sel}_{\mathtt{Gr}}(\mathcal{K}^{\mathrm{ac}}_n, T^\dagger_{\boldsymbol{f}})^{\mathrm{univ}}$ the submodule of $\mathrm{Sel}_{\mathtt{Gr}}(\mathcal{K}^{\mathrm{ac}}_n, T^\dagger_{\boldsymbol{f}})$ consisting of classes lying in $\mathrm{H}^1_{\mathtt{Gr}}(\mathcal{K}^{\mathrm{ac}}_{n,v}, T^\dagger_{\boldsymbol{f}})^{\mathrm{univ}}$ for all primes v above p, and let $x, y \in \mathrm{Sel}_{\mathtt{Gr}}(\mathcal{K}^{\mathrm{ac}}_n, T^\dagger_{\boldsymbol{f}})^{\mathrm{univ}}$. Then x corresponds to an extension of Galois modules

$$0 \longrightarrow T^\dagger_{\boldsymbol{f}} \longrightarrow X \longrightarrow \mathbb{I} \longrightarrow 0.$$

The Kummer dual of this sequence induces maps on cohomology

$$\mathrm{H}^1(\mathcal{K}^{\mathrm{ac}}_n, X^*(1)) \longrightarrow \mathrm{H}^1(\mathcal{K}^{\mathrm{ac}}_n, T^\dagger_{\boldsymbol{f}}) \xrightarrow{\delta} \mathrm{H}^2(\mathcal{K}^{\mathrm{ac}}_n, \mathbb{I}(1))$$

such that $\delta(y) = 0$ (since $\mathrm{H}^2(\mathcal{K}^{\mathrm{ac}}_n, \mathbb{I}(1))$ injects into $\bigoplus_v \mathrm{H}^2(\mathcal{K}^{\mathrm{ac}}_{n,v}, \mathbb{I}(1))$ and the v-th component of $\delta(y)$ is given by $\mathrm{loc}_v(y) \cup \mathrm{loc}_v(x) = 0$ by the self-duality of Greenberg's local conditions). Thus y is the image of some $y^{\mathrm{glob}} \in \mathrm{H}^1(\mathcal{K}^{\mathrm{ac}}_n, X^*(1))$.

On the other hand, if v is any place of $\mathcal{K}^{\mathrm{ac}}_n$, for every k we can write $\mathrm{loc}_v(y) = \mathrm{cor}_{L_{n,k,v}/\mathcal{K}^{\mathrm{ac}}_{n,v}}(y_{k,v})$ for some $y_{k,v} \in \mathrm{H}^1_{\mathtt{Gr}}(L_{n,k,v}, T^\dagger_{\boldsymbol{f}})^{\mathrm{sat}}$, and by a similar argument as above there exists a class $\widetilde{y}_{k,v} \in \mathrm{H}^1(L_{n,k,v}, X^*(1))$ lifting $y_{k,v}$ under the natural map π_v in the exact sequence

$$\mathrm{H}^1(L_{n,k,v}, X^*(1)) \xrightarrow{\pi_v} \mathrm{H}^1(L_{n,k,v}, T^\dagger_{\boldsymbol{f}}) \xrightarrow{\delta_v} \mathrm{H}^2(L_{n,k,v}, \mathbb{I}(1)). \tag{3.8}$$

The difference $\mathrm{loc}_v(y^{\mathrm{glob}}) - \mathrm{cor}_{L_{n,k,v}/\mathcal{K}^{\mathrm{ac}}_{n,v}}(\widetilde{y}_{k,v})$ is then the image of some class $w_{k,v} \in \mathrm{H}^1(\mathcal{K}^{\mathrm{ac}}_{n,v}, \mathbb{I}(1))$, and we define

$$\langle y, x\rangle^{\mathrm{cyc}}_{\mathcal{K}^{\mathrm{ac}}_n, \mathbb{I}} := \lim_{k\to\infty} \sum_v \rho_v(w_{k,v}),$$

a limit which is easily checked to exist and be independent of all choices. If in addition $y = y_0$ is the base class of a compatible system of classes

$$y_\infty = \varprojlim_k y_k \in \mathrm{H}^1(\mathcal{K}^{\mathrm{ac}}_n, \mathbf{T}^{\dagger,\mathrm{cyc}}) = \varprojlim_k \mathrm{H}^1(L_{n,k}, T^\dagger_{\boldsymbol{f}}),$$

then one easily checks (see e.g. [1, Lem. 3.2.2]) that there are classes $y^{\mathrm{glob}}_k \in \mathrm{H}^1(L_{n,k}, X^*(1))$ lifting y_k. Similarly as above, for every place v of $L_{n,k}$ the corestriction of $\mathrm{loc}_v(y^{\mathrm{glob}}_k) - \widetilde{y}_{k,v}$ to $\mathrm{H}^1(K^{\mathrm{ac}}_{n,v}, X^*(1))$ is the image of a class $w'_{k,v} \in \mathrm{H}^1(K^{\mathrm{ac}}_{n,v}, \mathbb{I}(1))$, and with these choices we see that the above expression for $\langle y, x\rangle^{\mathrm{cyc}}_{\mathcal{K}^{\mathrm{ac}}_n, \mathbb{I}}$ reduces to

$$\langle y, x\rangle^{\mathrm{cyc}}_{\mathcal{K}^{\mathrm{ac}}_n, \mathbb{I}} = \lim_{k\to\infty} \sum_{v|p} \rho_v(w'_{k,v}). \tag{3.9}$$

As in [60, §3.8], division by $\gamma_{\rm cyc}-1$ defines a natural *derivative map*

$$\mathfrak{Der}:\mathrm{H}^1(\mathcal{K}_{n,v}^{\rm ac},T_{\boldsymbol{f}}^\dagger\otimes_{\mathbb{I}}\mathcal{I}^{\rm cyc})\longrightarrow\mathrm{H}^1(\mathcal{K}_{n,v}^{\rm ac},T_{\boldsymbol{f}}^\dagger)$$

whose composition with the natural projection $\mathrm{H}^1(\mathcal{K}_{n,v}^{\rm ac},T_{\boldsymbol{f}}^\dagger)\to\mathrm{H}^1(\mathcal{K}_{n,v}^{\rm ac},\mathscr{F}^-T_{\boldsymbol{f}}^\dagger)$ factors as

$$\begin{array}{ccc}\mathrm{H}^1(\mathcal{K}_{n,v}^{\rm ac},T_{\boldsymbol{f}}^\dagger\otimes_{\mathbb{I}}\mathcal{I}^{\rm cyc}) & \twoheadrightarrow & \mathrm{H}^1(\mathcal{K}_{n,v}^{\rm ac},\mathscr{F}^-T_{\boldsymbol{f}}^\dagger\otimes_{\mathbb{I}}\mathcal{I}^{\rm cyc}) \\ \downarrow{\scriptstyle\mathfrak{Der}} & & \downarrow{\scriptstyle\mathfrak{Der}_-} \\ \mathrm{H}^1(\mathcal{K}_{n,v}^{\rm ac},T_{\boldsymbol{f}}^\dagger) & \twoheadrightarrow & \mathrm{H}^1(\mathcal{K}_{n,v}^{\rm ac},\mathscr{F}^-T_{\boldsymbol{f}}^\dagger).\end{array} \tag{3.10}$$

Letting $\mathrm{pr}_{\mathbb{1}}$ be the natural projection $\mathrm{H}^1(\mathcal{K}_{n,v}^{\rm ac},X^*(1)\otimes_{\mathbb{I}}\mathbb{I}[\![\Gamma^{\rm cyc}]\!])\to\mathrm{H}^1(\mathcal{K}_{n,v}^{\rm ac},X^*(1))$, the expression (3.9) for $\langle y,x\rangle_{\mathcal{K}_n^{\rm ac},\mathbb{I}}^{\rm cyc}$ can be rewritten as

$$\langle y,x\rangle_{\mathcal{K}_n^{\rm ac},\mathbb{I}}^{\rm cyc}=\sum_{v|p}\mathrm{pr}_{\mathbb{1}}(\mathrm{loc}_v(y_\infty^{\rm glob})-\widetilde{y}_{\infty,v}),$$

where $\mathrm{loc}_v(y_\infty^{\rm glob})-\widetilde{y}_{\infty,v}\in\mathrm{H}^1(\mathcal{K}_{n,v}^{\rm ac},X^*(1)\otimes_{\mathbb{I}}\mathbb{I}[\![\Gamma^{\rm cyc}]\!])$ is a lift of $\mathrm{loc}_v(y_\infty)-y_{\infty,v}\in\mathrm{H}^1(\mathcal{K}_{n,v}^{\rm ac},T_{\boldsymbol{f}}^\dagger\otimes\mathcal{I}^{\rm cyc})$, and hence by [60, Prop. 3.10] we obtain

$$\begin{aligned}\langle y,x\rangle_{\mathcal{K}_n^{\rm ac},\mathbb{I}}^{\rm cyc}&=\sum_{v|p}\delta_v\left(\mathfrak{Der}(\mathrm{loc}_v(y_\infty)-y_{\infty,v})\right)\otimes(\gamma^{\rm cyc}-1)\\&=\sum_{v|p}\left(\mathfrak{Der}(\mathrm{loc}_v(y_\infty)-y_{\infty,v}),\mathrm{loc}_v(x)\right)_{\mathcal{K}_{n,v}^{\rm ac}}\otimes(\gamma^{\rm cyc}-1)\\&=\sum_{v|p}\left(\mathfrak{Der}_-(\mathrm{loc}_v(y_\infty)),\mathrm{loc}_v(x)\right)_{\mathcal{K}_{n,v}^{\rm ac}}\otimes(\gamma^{\rm cyc}-1),\end{aligned}\tag{3.11}$$

where the last equality follows from the commutativity of (3.10) and the fact that $y_{\infty,v}=\{y_{k,v}\}_k$ has trivial image in $\mathrm{H}^1(\mathcal{K}_{n,v}^{\rm ac},\mathscr{F}^-\mathbf{T}^{\dagger,\rm cyc})$.

Now taking $y_\infty=\mathcal{BF}^\dagger(L_{n,\infty})$ in (3.11) we see that the contribution to $\langle\mathcal{BF}_{\mathcal{K}_n^{\rm ac}}^\dagger,x\rangle_{\mathcal{K}_n^{\rm ac},\mathbb{I}}^{\rm cyc}$ from $\mathfrak{p}$ is zero, since $\mathcal{BF}^\dagger(L_{n,\infty})\in\mathrm{Sel}_{\mathrm{Gr},\emptyset}(\mathcal{K}_n^{\rm ac},\mathbf{T}^{\dagger,\rm cyc})$ is finite at the places above $\mathfrak{p}$, while at $\overline{\mathfrak{p}}$ chasing through the definitions we see that

$$\mathfrak{Der}_-(\mathrm{loc}_{\overline{\mathfrak{p}}}(\mathcal{BF}^\dagger(L_{n,\infty}))=\beta_n^\dagger(\mathbb{1}),$$

thus concluding the proof of the height formula (3.7). □

4. Big Heegner points

In this section, we explain the construction of big Heegner points and classes. The results in this section are essentially a reformulation (influenced by [16] and [15]) of work of Longo–Vigni [42] and Fouquet [19], extending to Shimura curves Howard's original construction for modular curves [35].

Fix a positive integer N and a prime $p \nmid 6N$. Let $\mathcal{K}$ be an imaginary quadratic field with ring of integers $\mathcal{O}_{\mathcal{K}}$ and discriminant $-D_{\mathcal{K}} < 0$ prime to Np, and write

$$N = N^+N^-$$

with N^+ (resp. N^-) divisible only by primes which are split (resp. inert) in $\mathcal{K}$. Throughout, we assume the following *generalized Heegner hypothesis*:

$$N^- \text{ is the squarefree product of an even number of primes,} \tag{gen-H}$$

and fix an integral ideal $\mathfrak{N}^+$ of $\mathcal{K}$ with $\mathcal{O}_{\mathcal{K}}/\mathfrak{N}^+ \simeq \mathbf{Z}/N^+\mathbf{Z}$.

4.1. Towers of Shimura curves

Let $B/\mathbf{Q}$ be an indefinite quaternion algebra of discriminant N^-. We fix a $\mathbf{Q}$-algebra embedding $\iota_{\mathcal{K}} : \mathcal{K} \hookrightarrow B$, which we shall use to identify $\mathcal{K}$ with a subalgebra of B. Let $z \mapsto \overline{z}$ be the non-trivial automorphism of $\mathcal{K}$, and choose a basis $\{1, j\}$ of B over $\mathcal{K}$ such that:

- $j^2 = \beta \in \mathbf{Q}^\times$ with $\beta < 0$ and $jt = \bar{t}j$ for all $t \in \mathcal{K}$,
- $\beta \in (\mathbf{Z}_q^\times)^2$ for $q \mid pN^+$, and $\beta \in \mathbf{Z}_q^\times$ for $q \mid D_{\mathcal{K}}$.

Fix a square-root $\delta = \sqrt{-D_{\mathcal{K}}}$, and define $\boldsymbol{\theta} \in \mathcal{K}$ by

$$\boldsymbol{\theta} := \frac{D'_{\mathcal{K}} + \delta}{2}, \quad \text{where } D'_{\mathcal{K}} := \begin{cases} D_{\mathcal{K}} & \text{if } 2 \nmid D_{\mathcal{K}}, \\ D_{\mathcal{K}}/2 & \text{if } 2 \mid D_{\mathcal{K}}, \end{cases} \tag{4.1}$$

so that $\mathcal{O}_{\mathcal{K}} = \mathbf{Z} + \boldsymbol{\theta}\mathbf{Z}$. For every prime $q \mid pN^+$, define the isomorphism $i_q : B_q := B \otimes_{\mathbf{Q}} \mathbf{Q}_q \simeq \mathrm{M}_2(\mathbf{Q}_q)$ by

$$i_q(\boldsymbol{\theta}) = \begin{pmatrix} \mathrm{Tr}(\boldsymbol{\theta}) & -\mathrm{Nm}(\boldsymbol{\theta}) \\ 1 & 0 \end{pmatrix}, \qquad i_q(j) = \sqrt{\beta} \begin{pmatrix} -1 & \mathrm{Tr}(\boldsymbol{\theta}) \\ 0 & 1 \end{pmatrix},$$

where Tr and Nm are the reduced trace and norm maps on B. For primes $q \nmid Np$, we fix any isomorphism $i_q : B_q \simeq \mathrm{M}_2(\mathbf{Q}_q)$ with $i_q(\mathcal{O}_{\mathcal{K}} \otimes_{\mathbf{Z}} \mathbf{Z}_q) \subset \mathrm{M}_2(\mathbf{Z}_q)$.

Let $\hat{\mathbf{Z}}$ be the profinite completion of $\mathbf{Z}$, and for any abelian group M set $\hat{M} = M \otimes_{\mathbf{Z}} \hat{\mathbf{Z}}$. For each $r \geqslant 0$, let R_r be the Eichler order of B of level N^+p^r with respect to the isomorphisms $\{i_q : B_q \simeq \mathrm{M}_2(\mathbf{Q}_q)\}_{q \nmid N^-}$, and let $U_r \subset \hat{R}_r^\times$ be the compact open subgroup defined by

$$U_r := \left\{ (x_q)_q \in \hat{R}_r^\times \ : \ i_p(x_p) \equiv \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \pmod{p^r} \right\}.$$

Consider the double coset spaces

$$X_r = B^\times \backslash \big(\mathrm{Hom}_{\mathbf{Q}}(\mathcal{K}, B) \times \hat{B}^\times / U_r\big), \tag{4.2}$$

where $b \in B^\times$ acts on $(\Psi, g) \in \mathrm{Hom}_{\mathbf{Q}}(\mathcal{K}, B) \times \hat{B}^\times$ by

$$b \cdot (\Psi, g) = (b\Psi b^{-1}, bg),$$

and U_r acts on $\hat{B}^\times$ by right multiplication. As is well-known (see e.g. [42, §§2.1-2]), X_r can be identified with a set of algebraic points on the Shimura curve with complex uniformization

$$X_r(\mathbf{C}) = B^\times \backslash \big(\mathrm{Hom}_{\mathbf{Q}}(\mathbf{C}, B) \times \hat{B}^\times / U_r\big).$$

Let $\mathrm{rec}_{\mathcal{K}} : \mathcal{K}^\times \backslash \hat{\mathcal{K}}^\times \to \mathrm{Gal}(\mathcal{K}^{\mathrm{ab}}/\mathcal{K})$ be the reciprocity map of class field theory. By Shimura's reciprocity law, if $P \in X_r$ is the class of a pair (Ψ, g), then $\sigma \in \mathrm{Gal}(\mathcal{K}^{\mathrm{ab}}/\mathcal{K})$ acts on P by

$$P^\sigma := [(\Psi, \hat{\Psi}(a)g)],$$

where $a \in \mathcal{K}^\times \backslash \hat{\mathcal{K}}^\times$ is such that $\mathrm{rec}_{\mathcal{K}}(a) = \sigma$, and $\hat{\Psi} : \hat{\mathcal{K}} \to \hat{B}$ is the adelization of Ψ. We extend this to an action of $G_{\mathcal{K}} := \mathrm{Gal}(\overline{\mathbf{Q}}/\mathcal{K})$ in the obvious manner.

The curves X_r are also equipped with natural actions of Hecke operators T_ℓ for $\ell \nmid Np$, U_ℓ for $\ell | Np$, and diamond operators $\langle d \rangle$ for $d \in (\mathbf{Z}/p^r\mathbf{Z})^\times$, as described in [42, §2.4] and [16, §2.1], for example.

4.2. Compatible systems of Heegner points

For each $c \geqslant 1$, let $\mathcal{O}_c = \mathbf{Z} + c\mathcal{O}_{\mathcal{K}}$ be the order of $\mathcal{K}$ of conductor c and denote by H_c the ring class field of $\mathcal{K}$ of that conductor, so that $\mathrm{Pic}(\mathcal{O}_c) \simeq \mathrm{Gal}(H_c/\mathcal{K})$ by class field theory. In particular, H_1 is the Hilbert class field of $\mathcal{K}$.

Definition 4.1. A point $P \in X_r$ is a *Heegner point of conductor c* if it is the class of a pair (Ψ, g) with

$$\Psi(\mathcal{O}_c) = \Psi(\mathcal{K}) \cap (B \cap g\hat{R}_r g^{-1})$$

and

$$\Psi_p\left((\mathcal{O}_c \otimes \mathbf{Z}_p)^\times \cap (1 + p^r \mathcal{O}_c \otimes \mathbf{Z}_p)^\times\right) = \Psi_p\left((\mathcal{O}_c \otimes \mathbf{Z}_p)^\times\right) \cap g_p U_{r,p} g_p^{-1},$$

where Ψ_p and $U_{r,p}$ denote the p-components of Ψ and U_r, respectively.

For each prime $q \neq p$ define

- $\varsigma_q = 1$, if $q \nmid N^+$,
- $\varsigma_q = \delta^{-1}\begin{pmatrix} \boldsymbol{\theta} & \overline{\boldsymbol{\theta}} \\ 1 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathcal{K}_\mathfrak{q}) = \mathrm{GL}_2(\mathbf{Q}_q)$, if $q = \mathfrak{q}\overline{\mathfrak{q}}$ splits with $\mathfrak{q} \mid \mathfrak{N}^+$,

and for each $s \geqslant 0$, let

- $\varsigma_p^{(s)} = \begin{pmatrix} \boldsymbol{\theta} & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} p^s & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathcal{K}_\mathfrak{p}) = \mathrm{GL}_2(\mathbf{Q}_p)$, if $p = \mathfrak{p}\overline{\mathfrak{p}}$ splits in $\mathcal{K}$,
- $\varsigma_p^{(s)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} p^s & 0 \\ 0 & 1 \end{pmatrix}$, if p is inert in $\mathcal{K}$.

Remark 4.2. We shall ultimately assume that p splits in $\mathcal{K}$, but it is worth-noting that, just as in [35,42], the constructions in this section also allow the case p inert in $\mathcal{K}$.

Set $\varsigma^{(s)} = \varsigma_p^{(s)}\prod_{q\neq p}\varsigma_q$, which we view as an element in $\hat{B}^\times$ via the isomorphisms $\{i_q : B_q \simeq \mathrm{M}_2(\mathbf{Q}_q)\}_{q\nmid N^-}$ introduced in §4.1. With the $\mathbf{Q}$-algebra embedding $\iota_\mathcal{K} : \mathcal{K} \hookrightarrow B$ fixed there, one easily checks that for all $s \geqslant r$ the points

$$P_{s,r} := [(\iota_\mathcal{K}, \varsigma^{(s)})] \in X_r$$

are Heegner points of conductor p^s in the sense of Definition 4.1 with the following properties:

- *Field of definition*: $P_{s,r} \in H^0(H_{p^s}(\mu_{p^r}), X_r)$.
- *Galois equivariance*: For all $\sigma \in \mathrm{Gal}(H_{p^s}(\mu_{p^r})/H_{p^s})$,

$$P_{s,r}^\sigma = \langle\vartheta(\sigma)\rangle \cdot P_{s,r},$$

  where $\vartheta : \mathrm{Gal}(H_{p^s}(\mu_{p^r})/H_{p^s}) \to \mathbf{Z}_p^\times/\{\pm 1\}$ is such that $\vartheta^2 = \varepsilon_{\mathrm{cyc}}$.
- *Horizontal compatibility*: If $s \geqslant r > 1$, then

$$\sum_{\sigma\in\mathrm{Gal}(H_{p^s}(\mu_{p^r})/H_{p^{s-1}}(\mu_{p^r}))} \alpha_r(P_{s,r}^\sigma) = U_p \cdot P_{s,r-1},$$

  where $\alpha_r : X_r \to X_{r-1}$ is the map induced by the inclusion $U_r \subset U_{r-1}$.
- *Vertical compatibility*: If $s \geqslant r \geqslant 1$, then

$$\sum_{\sigma\in\mathrm{Gal}(H_{p^s}(\mu_{p^r})/H_{p^{s-1}}(\mu_{p^r}))} P_{s,r}^\sigma = U_p \cdot P_{s-1,r}.$$

(See [15, Thm. 1.2] and the references therein.)

4.3. Big Heegner points

Let $\mathbb{B}_r$ the $\mathbf{Z}_p$-algebra generated by the Hecke operators T_ℓ, U_ℓ, and $\langle a\rangle$ acting on the Shimura curve X_r from §4.1, let $\mathfrak{h}_r$ be the $\mathbf{Z}_p$-algebra generated by the usual Hecke operators T_ℓ, U_ℓ, and $\langle a\rangle$ acting on the space $S_2(\Gamma_{0,1}(N,p^r))$ of classical modular form of level $\Gamma_{0,1}(N,p^r) := \Gamma_0(N)\cap\Gamma_1(p^r)$, and let $\mathbb{T}_{N,r}^{N^-}$ be the quotient of $\mathfrak{h}_r$ acting faithfully on the subspace of $S_2(\Gamma_{0,1}(N,p^r))$ consisting of N^--new forms.

The Jacquet–Langlands correspondence yields $\mathbf{Z}_p$-algebra isomorphisms

$$\mathbb{B}_r \simeq \mathbb{T}_{N,r}^{N^-} \tag{4.3}$$

(see [29, §2.4]). In particular, letting $e_{\mathrm{ord}} = \lim_{n\to\infty} U_p^{n!}$ be Hida's ordinary projector, the $\mathbf{Z}_p$-module

$$\mathfrak{D}_r^{\mathrm{ord}} := e_{\mathrm{ord}}(\mathrm{Div}(X_r)\otimes_{\mathbf{Z}}\mathbf{Z}_p)$$

is naturally endowed with an action of $\mathbb{T}_r^{\mathrm{ord}} := e_{\mathrm{ord}}\mathbb{T}_{N,r}^{N^-}$.

Denote by $\mathbb{T}_r^\dagger$ the free $\mathbb{T}_r^{\mathrm{ord}}$-module of rank one equipped with the Galois action via the inverse of the critical character Θ, and set $\mathfrak{D}_r^\dagger := \mathfrak{D}_r^{\mathrm{ord}}\otimes_{\mathbb{T}_r^{\mathrm{ord}}}\mathbb{T}_r^\dagger$.

Let $P_{s,r}\in X_r$ be the Heegner point of conductor p^s $(s\geqslant r)$ constructed in §4.2, and denote by $\mathcal{P}_{s,r}$ the image of $e_{\mathrm{ord}}P_{s,r}$ in $\mathfrak{D}_r^{\mathrm{ord}}$. It follows from the Galois-equivariance property of $P_{s,r}$ that

$$\mathcal{P}_{s,r}^\sigma = \Theta(\sigma)\cdot\mathcal{P}_{s,r}$$

for all $\sigma\in\mathrm{Gal}(H_{p^s}(\mu_{p^r})/H_{p^s})$ (see [42, §7.1]), and hence $\mathcal{P}_{s,r}$ defines an element

$$\mathcal{P}_{s,r}\otimes\zeta_r \in \mathrm{H}^0(H_{p^s},\mathfrak{D}_r^\dagger). \tag{4.4}$$

Let $\mathrm{Pic}(X_r)$ be the Picard variety of X_r, and set

$$\mathfrak{J}_r^{\mathrm{ord}} := e_{\mathrm{ord}}(\mathrm{Pic}(X_r)\otimes_{\mathbf{Z}}\mathbf{Z}_p), \qquad \mathfrak{J}_r^\dagger := \mathfrak{J}_r^{\mathrm{ord}}\otimes_{\mathbb{T}_r^{\mathrm{ord}}}\mathbb{T}_r^\dagger.$$

Since the U_p-operator has degree p, taking ordinary parts yields an isomorphism $\mathfrak{D}_r^{\mathrm{ord}}\simeq\mathfrak{J}_r^{\mathrm{ord}}$, and so we may also view (4.4) as $\mathcal{P}_{s,r}\otimes\zeta_r\in\mathrm{H}^0(H_{p^s},\mathfrak{J}_r^\dagger)$.

Let $t\geqslant 0$, and denote by $\mathfrak{G}_{H_{p^t}}$ the Galois group of the maximal extension of H_{p^t} unramified outside the primes above pN. Consider the twisted Kummer map

$$\mathrm{Kum}_r : \mathrm{H}^0(H_{p^t},\mathfrak{J}_r^\dagger) \longrightarrow \mathrm{H}^1(\mathfrak{G}_{H_{p^t}},\mathrm{Ta}_p(\mathfrak{J}_r^\dagger))$$

as explicitly defined in [35, p. 101]. This map is equivariant for the Galois- and U_p-actions, and hence by horizontal compatibility the classes

$$\mathfrak{X}_{p^t,r} := \mathrm{Kum}_r(\mathrm{Cor}_{H_{p^{r+t}}/H_{p^t}}(\mathcal{P}_{r+t,r}\otimes\zeta_r)) \tag{4.5}$$

satisfy $\alpha_{r,*}(\mathfrak{X}_{p^t,r}) = U_p \cdot \mathfrak{X}_{p^t,r-1}$ for all $r > 1$, where

$$\alpha_{r,*} : \mathrm{H}^1(\mathfrak{G}_{H_{p^t}}, \mathrm{Ta}_p(\mathfrak{J}_r^\dagger)) \longrightarrow \mathrm{H}^1(\mathfrak{G}_{H_{p^t}}, \mathrm{Ta}_p(\mathfrak{J}_{r-1}^\dagger))$$

is the map induced by the covering $X_r \to X_{r-1}$ by Albanese functoriality.

Now let $\boldsymbol{f} \in \mathbb{I}[\![q]\!]$ be a Hida family of tame level N. In order to define big Heegner points attached to $\boldsymbol{f}$ from the system of Heegner classes (4.5) for varying r, we need to recall the following result realizing the big Galois representation $T_{\boldsymbol{f}}$ attached to $\boldsymbol{f}$ in the étale cohomology of the p-tower of Shimura curves

$$\cdots \longrightarrow X_r \longrightarrow X_{r-1} \longrightarrow \cdots$$

(rather than classical modular curves, as implicitly taken in §3.1).

Let $\kappa_{\mathbb{I}} = \mathbb{I}/\mathfrak{m}_{\mathbb{I}}$ be the residue field of $\mathbb{I}$, and denote by $\bar{\rho}_{\boldsymbol{f}} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\kappa_{\mathbb{I}})$ the associated semi-simple residual representation. Set

$$\mathbb{T}_\infty^{\mathrm{ord}} := \varprojlim_r \mathbb{T}_r^{\mathrm{ord}}.$$

By (4.3) (see also the discussion in [42, §5.3]), there is a maximal ideal $\mathfrak{m} \subset \mathbb{T}_\infty^{\mathrm{ord}}$ associated with $\bar{\rho}_{\boldsymbol{f}}$, and $\boldsymbol{f}$ corresponds to a minimal prime in the localization $\mathbb{T}_{\infty,\mathfrak{m}}^{\mathrm{ord}}$.

Theorem 4.3. *Assume that:*

(i) $\bar{\rho}_{\boldsymbol{f}}$ *is absolutely irreducible and p-distinguished,*
(ii) $\bar{\rho}_{\boldsymbol{f}}$ *is ramified at every prime $\ell | N^-$ with $\ell \equiv \pm 1 \pmod p$,*

and let $\mathfrak{m} \subset \mathbb{T}_\infty^{\mathrm{ord}}$ be the maximal ideal associated with $\bar{\rho}_{\boldsymbol{f}}$. Then the module

$$\mathbf{Ta}_{\mathfrak{m}}^{\mathrm{ord}} := \left(\varprojlim_r \mathrm{Ta}_p(\mathfrak{J}_r^{\mathrm{ord}})\right) \otimes_{\mathbb{T}_\infty^{\mathrm{ord}}} \mathbb{T}_{\infty,\mathfrak{m}}^{\mathrm{ord}}$$

is free of rank 2 over $\mathbb{T}_{\infty,\mathfrak{m}}^{\mathrm{ord}}$, and if $\boldsymbol{f}$ corresponds to the minimal prime $\mathfrak{a} \subset \mathbb{T}_{\infty,\mathfrak{m}}^{\mathrm{ord}}$, then there is an isomorphism

$$T_{\boldsymbol{f}} \simeq \mathbf{Ta}_{\mathfrak{m}}^{\mathrm{ord}} \otimes_{\mathbb{T}_{\infty,\mathfrak{m}}^{\mathrm{ord}}} \mathbb{T}_{\infty,\mathfrak{m}}^{\mathrm{ord}}/\mathfrak{a}$$

as $(\mathbb{T}_{\infty,\mathfrak{m}}^{\mathrm{ord}}/\mathfrak{a})[G_{\mathbf{Q}}]$-modules.

Proof. This is shown in [19, Thm. 3.1] assuming the "mod p multiplicity one" hypothesis in [19, Prop. 3.7]. Since by [24, Cor. 8.11] that hypothesis is ensured by our ramification condition on $\bar{\rho}_{\boldsymbol{f}}$, the result follows. □

Let $\mathfrak{m} \subset \mathbb{T}_{\infty}^{\mathrm{ord}}$ be a maximal ideal satisfying the hypotheses of Theorem 4.3, and suppose that the Hida family $\boldsymbol{f}$ corresponds to a minimal prime of $\mathbb{T}_{\infty,\mathfrak{m}}^{\mathrm{ord}}$, so by Theorem 4.3 there is a quotient map $\mathbf{Ta}_{\mathfrak{m}}^{\mathrm{ord}} \to T_{\boldsymbol{f}}$. Note also that immediately from the definitions there are natural maps $\mathrm{Ta}_p(\mathfrak{J}_r^{\dagger}) \to \mathbf{Ta}_{\mathfrak{m}}^{\mathrm{ord}} \otimes \Theta^{-1} \to T_{\boldsymbol{f}}^{\dagger}$.

Definition 4.4. The *big Heegner point* of conductor p^t is the class

$$\mathfrak{X}_{p^t} \in \mathrm{H}^1(H_{p^t}, T_{\boldsymbol{f}}^{\dagger})$$

given by the image of $\varprojlim_r U_p^{-r} \cdot \mathfrak{X}_{p^t,r}$ under the composite map

$$\varprojlim_r \mathrm{H}^1(\mathfrak{G}_{H_{p^t}}, \mathrm{Ta}_p(\mathfrak{J}_r^{\dagger})) \longrightarrow \mathrm{H}^1(\mathfrak{G}_{H_{p^t}}, \mathbf{Ta}_{\mathfrak{m}}^{\mathrm{ord}} \otimes \Theta^{-1}) \longrightarrow \mathrm{H}^1(H_{p^t}, T_{\boldsymbol{f}}^{\dagger}).$$

We conclude this section with the following result due to Howard, showing that the big Heegner points are Selmer classes under mild hypotheses.

Proposition 4.5. *Assume that $\bar{\rho}_{\boldsymbol{f}}$ is ramified at every prime $\ell | N^-$. Then the classes $\mathfrak{X}_{p^t}$ lie in $\mathrm{Sel}_{\mathtt{Gr}}(H_{p^t}, T_{\boldsymbol{f}}^{\dagger})$.*

Proof. The argument in [35, Prop. 2.4.5] (see also [42, Prop. 10.1]) shows that for every prime w of H_{p^t} the localization $\mathrm{loc}_w(\mathfrak{X}_{p^t})$ lies in the subspace $\mathrm{H}^1_{\mathtt{Gr}}(H_{p^t,w}, T_{\boldsymbol{f}}^{\dagger}) \subset \mathrm{H}^1(H_{p^t,w}, T_{\boldsymbol{f}}^{\dagger})$ defining $\mathrm{Sel}_{\mathtt{Gr}}(H_{p^t}, T_{\boldsymbol{f}}^{\dagger})$, except when $w|\ell|N^-$, in which case it is shown that

$$\mathrm{loc}_w(\mathfrak{X}_{p^t}) \in \ker\left\{\mathrm{H}^1(H_{p^t,w}, T_{\boldsymbol{f}}^{\dagger}) \longrightarrow \frac{\mathrm{H}^1(H_{p^t,w}^{\mathrm{ur}}, T_{\boldsymbol{f}}^{\dagger})}{\mathrm{H}^1(H_{p^t,w}^{\mathrm{ur}}, T_{\boldsymbol{f}}^{\dagger})_{\mathrm{tors}}}\right\},$$

where $\mathrm{H}^1(H_{p^t,w}^{\mathrm{ur}}, T_{\boldsymbol{f}}^{\dagger})_{\mathrm{tors}}$ denotes the $\mathbb{I}$-torsion submodule of $\mathrm{H}^1(H_{p^t,w}^{\mathrm{ur}}, T_{\boldsymbol{f}}^{\dagger})$. However, such primes ℓ are inert in $\mathcal{K}$, so $H_{p^t,w} = \mathcal{K}_{\ell}$, and since our hypothesis on $\bar{\rho}_{\boldsymbol{f}}$ implies that $\mathrm{H}^1(\mathcal{K}_{\ell}^{\mathrm{ur}}, T_{\boldsymbol{f}}^{\dagger})$ is $\mathbb{I}$-torsion free (see e.g. [11, Lem. 3.12]), the result follows. □

Recall that $\mathcal{K}_{\infty}^{\mathrm{ac}}$ is the anticyclotomic $\mathbf{Z}_p$-extension of $\mathcal{K}$, and $\mathcal{K}_n^{\mathrm{ac}}$ denotes the subextension of $\mathcal{K}_{\infty}^{\mathrm{ac}}$ with $[\mathcal{K}_n^{\mathrm{ac}} : \mathcal{K}] = p^n$. Similarly as in [35, §3.3] and [42, §10.3], we set

$$\mathfrak{Z}_n := \mathrm{Cor}_{H_{p^t}/\mathcal{K}_n^{\mathrm{ac}}}(U_p^{-t} \cdot \mathfrak{X}_{p^t}) \in \mathrm{H}^1(\mathcal{K}_n^{\mathrm{ac}}, T_{\boldsymbol{f}}^{\dagger}),$$

where $t \gg 0$ is chosen so that $\mathcal{K}_n^{\mathrm{ac}} \subset H_{p^t}$. By horizontal compatibility, the definition of $\mathfrak{Z}_n$ is independent of the choice of t, and for varying n they define a system

$$\mathfrak{Z}_{\infty} := \varprojlim_n \mathfrak{Z}_n \in \varprojlim_n \mathrm{H}^1(\mathcal{K}_n^{\mathrm{ac}}, T_{\boldsymbol{f}}^{\dagger}) \simeq \mathrm{H}^1(\mathcal{K}, \mathbf{T}^{\dagger,\mathrm{ac}}).$$

By the work of Cornut–Vatsal [17] (see also [35, Cor. 3.1.2], which naturally extends to quaternionic setting considered here) the class $\mathfrak{Z}_{\infty}$ is not $\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]$-torsion.

5. Main results

In this section we conclude the proof of the main results of this paper. Fix a positive integer N and a prime $p \nmid 6N$ and let

$$\boldsymbol{f} = \sum_{n=1}^{\infty} \boldsymbol{a}_n q^n \in \mathbb{I}[\![q]\!]$$

be a primitive Hida family of tame level N. Let $\mathcal{K}$ be an imaginary quadratic field of discriminant prime to Np satisfying the generalized Heegner hypothesis (gen-H) relative to N. Our results will require some of the technical hypotheses below, which we record here for our later reference.

(h0) $\mathbb{I}$ is regular,
(h1) some specialization $\boldsymbol{f}_\phi$ is the p-stabilization of a newform $f \in S_2(\Gamma_0(N))$,
(h2) $\bar{\rho}_{\boldsymbol{f}}$ is irreducible,
(h3) N is squarefree,
(h4) $N^- \neq 1$,
(h5) $\bar{\rho}_{\boldsymbol{f}}$ is ramified at every prime $\ell | N^-$,
(h6) p splits in $\mathcal{K}$.

As usual, here N^- denotes the largest factor of N divisible only by primes which are inert in $\mathcal{K}$.

5.1. Proof of Theorem A

The following is Theorem A in the Introduction.

Theorem 5.1. *Assume hypotheses* (h0)–(h6). *Then* $X_{\mathtt{Gr}}(\mathcal{K}_\infty, A_{\boldsymbol{f}})$ *is* $\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]$*-torsion, and*

$$\mathrm{Char}_{\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]}(X_{\mathtt{Gr}}(\mathcal{K}_\infty, A_{\boldsymbol{f}})) = (L_p^{\mathtt{Hi}}(\boldsymbol{f}/\mathcal{K}))$$

as ideals in $\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!] \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$.

Proof. It suffices to show that the twisted module $X_{\mathtt{Gr}}(\mathcal{K}_\infty, A_{\boldsymbol{f}}^\dagger)$ is $\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]$-torsion, with characteristic ideal generated by $\mathrm{tw}_{\Theta^{-1}}(L_p^{\mathtt{Hi}}(\boldsymbol{f}/\mathcal{K}))$ after inverting p (see the twisting lemma [55, Lem. 6.1.2]). In light of Theorem 3.8, this will follow from showing that $X_{\emptyset,0}(\mathcal{K}_\infty, A_{\boldsymbol{f}}^\dagger)$ is $\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]$-torsion, with characteristic ideal generated by $\mathrm{tw}_{\Theta^{-1}}(\mathscr{L}_{\mathfrak{p}}(\boldsymbol{f}/\mathcal{K}))$ after extending scalars to $\mathbb{I}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!]$; this is what we shall prove below.

From [64, Thm. 1.1] (see also Remark 5.3 below) we obtain the divisibility

$$\mathrm{Char}_{\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]}(X_{\emptyset,0}(\mathcal{K}_\infty, A_{\boldsymbol{f}}^\dagger)) \cdot \mathbb{I}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!] \subset (\mathrm{tw}_{\Theta^{-1}}(\mathscr{L}_{\mathfrak{p}}(\boldsymbol{f}/\mathcal{K}))) \quad \text{in } \mathbb{I}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!], \tag{5.1}$$

which by descent via $\Gamma_{\mathcal{K}} \twoheadrightarrow \Gamma_{\mathcal{K}}^{\mathrm{ac}}$, Corollary 2.10, and [9, Thm. B] yields the divisibility

$$\mathrm{Char}_{\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]}(X_{\emptyset,0}(\mathcal{K}_\infty^{\mathrm{ac}}, A_{\boldsymbol{f}}^\dagger)) \cdot \mathbb{I}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!] \subset (\mathscr{L}_{\mathfrak{p}}^{\mathsf{BDP}}(\boldsymbol{f}/\mathcal{K})^2) \quad \text{in } \mathbb{I}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]. \tag{5.2}$$

Now let $\phi \in \mathcal{X}_a(\mathbb{I})$ be such that $\boldsymbol{f}_\phi$ is the ordinary p-stabilization of a newform $f \in S_2(\Gamma_0(N))$ defined over $\mathfrak{O}_L$, and put $\mathfrak{O}_L^{\mathrm{ur}} = \hat{\mathbf{Z}}\hat{\otimes}_{\mathbf{Z}_p}\mathfrak{O}_L$. By the construction in Theorem 2.9, the p-adic L-function $\mathscr{L}_{\mathfrak{p}}^{\mathsf{BDP}}(\boldsymbol{f}/\mathcal{K})$ specializes at ϕ to the p-adic L-function $\mathscr{L}_{\mathfrak{p}}^{\mathsf{BDP}}(f/K) \in \mathfrak{O}_L^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]$ of [12, Thm. 1.5] (see also [7, §3.1]). Since by [12, Thm. 3.4] the module $X_{\emptyset,0}(\mathcal{K}_\infty^{\mathrm{ac}}, A_f)$ is $\mathfrak{O}_L^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]$-torsion, with

$$\mathrm{Char}_{\mathfrak{O}_L^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]}(X_{\emptyset,0}(\mathcal{K}_\infty^{\mathrm{ac}}, A_f)) \cdot \mathfrak{O}_L^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!] = (\mathscr{L}_{\mathfrak{p}}^{\mathsf{BDP}}(f/\mathcal{K})^2) \quad \text{in } \mathfrak{O}_L^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!], \tag{5.3}$$

by Lemma 3.5 we deduce that $X_{\emptyset,0}(\mathcal{K}_\infty^{\mathrm{ac}}, A_{\boldsymbol{f}}^{\dagger,\mathrm{ac}})$ is $\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]$-torsion and that the divisibility (5.2) is an equality. With this equality at hand, another application of Lemma 3.5 yields equality in (5.1), concluding the proof of the theorem. □

For our later reference, we record the following results shown in the course of the proof of Theorem 5.1.

Theorem 5.2. *Assume hypotheses* (h0)–(h6). *Then the modules* $X_{\emptyset,0}(\mathcal{K}_\infty, A_{\boldsymbol{f}})$ *and* $X_{\emptyset,0}(\mathcal{K}_\infty^{\mathrm{ac}}, A_{\boldsymbol{f}}^\dagger)$ *are torsion over* $\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]$ *and* $\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]$, *respectively, and the following equalities hold:*

$$\mathrm{Char}_{\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]}(X_{\emptyset,0}(\mathcal{K}_\infty, A_{\boldsymbol{f}})) \cdot \mathbb{I}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!] = (\mathscr{L}_{\mathfrak{p}}(\boldsymbol{f}/\mathcal{K})) \quad \textit{in } \mathbb{I}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!],$$
$$\mathrm{Char}_{\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]}(X_{\emptyset,0}(\mathcal{K}_\infty^{\mathrm{ac}}, A_{\boldsymbol{f}}^\dagger)) \cdot \mathbb{I}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!] = (\mathscr{L}_{\mathfrak{p}}^{\mathsf{BDP}}(\boldsymbol{f}/\mathcal{K})^2) \quad \textit{in } \mathbb{I}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!].$$

Remark 5.3. In the proof of Theorem 5.1 we used [64, Thm. 1.1], which assumes that the underlying CM form $\boldsymbol{g}$ is residually irreducible and p-distinguished. Without these hypotheses on $\boldsymbol{g}$, the argument in the proof of [64, Thm. 1.1] establishes the divisibility

$$\mathrm{Char}_{\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]}(X_{\emptyset,0}(\mathcal{K}_\infty, A_{\boldsymbol{f}})) \cdot \mathbb{I}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!] \subset (\mathscr{L}_{\mathfrak{p}}(\boldsymbol{f}/\mathcal{K})) \tag{5.4}$$

in $\mathbb{I}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!]$, up to certain height one primes of $\mathbb{I}^{\mathrm{ur}}$. However, that such ambiguity can be removed follows from the integrality of $\mathscr{L}_{\mathfrak{p}}(\boldsymbol{f}/\mathcal{K})$ established in Proposition 2.7 together with the vanishing of the μ-invariant of its anticyclotomic restriction [9, Thm. 5.7] (see Corollary 2.10), and hence the divisibility (5.4) also holds for our underlying CM form $\boldsymbol{g}$ in (2.1).

Remark 5.4. A key ingredient in the proof of (5.3) is the Heegner point "explicit reciprocity law" of [14] (see also [7, §4.2] for the additional arguments in the case $N^- \neq 1$). Indeed, as explained in [12, Appendix], this allows one to relate the main conjecture for $\mathscr{L}_{\mathfrak{p}}^{\mathsf{BDP}}(f/\mathcal{K})^2$ to Perrin-Riou's Heegner point main conjecture [48], whose "upper bound divisibility" was established by Howard [33,34] using the Euler system of Heegner points.

5.2. Converse to Howard's theorem

As shown in [35, §§2.3-4], for varying c prime to N the big Heegner points $\mathfrak{X}_c \in \mathrm{H}^1(H_c, T_{\boldsymbol{f}}^\dagger)$ form an anticyclotomic Euler system for $T_{\boldsymbol{f}}^\dagger$. Setting

$$\mathfrak{Z}_0 := \mathrm{Cor}_{H_1/\mathcal{K}}(\mathfrak{X}_1) \in \mathrm{H}^1(\mathcal{K}, T_{\boldsymbol{f}}^\dagger),$$

Kolyvagin's methods thus yield a proof of the implication

$$\mathfrak{Z}_0 \notin \mathrm{Sel}_{\mathtt{Gr}}(\mathcal{K}, T_{\boldsymbol{f}}^\dagger)_{\mathrm{tors}} \quad \Longrightarrow \quad \mathrm{rank}_{\mathbb{I}}\ \mathrm{Sel}_{\mathtt{Gr}}(\mathcal{K}, T_{\boldsymbol{f}}^\dagger) = 1, \tag{5.5}$$

where the subscript tors denotes the $\mathbb{I}$-torsion submodule (see [35, Cor. 3.4.3]). In the spirit of Skinner's converse to the theorem of Gross–Zagier and Kolyvagin, [56], in this section we prove a result in the converse direction. Similarly as in [65], our converse to (5.5) will be deduced from progress on the "big Heegner point main conjecture" (see [35, Conj. 3.3.1] and [42, Conj. 10.8]), as recorded in the next result.

Theorem 5.5. *Assume hypotheses* (h0)–(h6)*. Then both* $X_{\mathtt{Gr}}(\mathcal{K}_\infty^{\mathrm{ac}}, A_{\boldsymbol{f}}^\dagger)$ *and* $\mathrm{Sel}_{\mathtt{Gr}}(\mathcal{K}, \mathbf{T}^{\dagger,\mathrm{ac}})$ *have* $\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]$*-rank one, and*

$$\mathrm{Char}_{\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]}(X_{\mathtt{Gr}}(\mathcal{K}_\infty^{\mathrm{ac}}, A_{\boldsymbol{f}}^\dagger)_{\mathrm{tors}}) = \mathrm{Char}_{\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]}\left(\frac{\mathrm{Sel}_{\mathtt{Gr}}(\mathcal{K}, \mathbf{T}^{\dagger,\mathrm{ac}})}{\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!] \cdot \mathfrak{Z}_\infty}\right)^2,$$

where the subscript tors *denotes the* $\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]$*-torsion submodule.*

Proof. Since $\mathfrak{Z}_\infty$ is not $\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]$-torsion by Cornut–Vatsal, part (iii) of [19, Thm. B] implies that $X_{\mathtt{Gr}}(\mathcal{K}_\infty^{\mathrm{ac}}, A_{\boldsymbol{f}}^\dagger)$ and $\mathrm{Sel}_{\mathtt{Gr}}(\mathcal{K}, \mathbf{T}^{\dagger,\mathrm{ac}})$ have both $\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]$-rank one, and that the divisibility

$$\mathrm{Char}_{\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]}(X_{\mathtt{Gr}}(\mathcal{K}_\infty^{\mathrm{ac}}, A_{\boldsymbol{f}}^\dagger)_{\mathrm{tors}}) \supset \mathrm{Char}_{\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]}\left(\frac{\mathrm{Sel}_{\mathtt{Gr}}(\mathcal{K}, \mathbf{T}^{\dagger,\mathrm{ac}})}{\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!] \cdot \mathfrak{Z}_\infty}\right)^2 \tag{5.6}$$

holds in $\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]$. Concerning the additional hypotheses in Fouquet's result, we note that:

- Assumption 3.4, that $\bar{\rho}_{\boldsymbol{f}}$ is irreducible, is our (h2),
- Assumption 3.5, that $\bar{\rho}_{\boldsymbol{f}}$ is p-distinguished, follows from (h1) (see [40, Rem. 7.2.7]),
- Assumption 3.10, that the tame character of $\boldsymbol{f}$ admits a square-root, is satisfied by (h1),
- Assumption 5.10, that all primes $\ell | N$ for which $\bar{\rho}_{\boldsymbol{f}}$ is not ramified have infinite decomposition group in $\mathcal{K}_\infty^{\mathrm{ac}}/\mathcal{K}$, is a reformulation of (h5),
- Assumption 5.13, that $\bar{\rho}_{\boldsymbol{f}}|_{G_{\mathcal{K}}}$ is irreducible, follows from (h2), (h4) and (h5) (see [56, Lem. 2.8.1]).

Let $\phi \in \mathcal{X}_a(\mathbb{I})$ be such that $\boldsymbol{f}_\phi$ is the ordinary p-stabilization of a newform $f \in S_2(\Gamma_0(N))$ as in hypothesis (h1). Letting $X \supset Y$ stand for the divisibility (5.6), by [12, Thm. 3.4] (or [65, Thm. 1.2]) we have the equality

$$X = Y \pmod{\ker(\phi)\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]}$$

(note that this is the source of the additional hypotheses (h3) and (h6)), from where the result follows by an application of Lemma 3.5. □

Theorem 5.6. *Assume hypotheses* (h0)–(h6). *Then the following implication holds:*

$$\mathrm{rank}_{\mathbb{I}}\,\mathrm{Sel}_{\mathtt{Gr}}(\mathcal{K}, T_{\boldsymbol{f}}^\dagger) = 1 \quad \Longrightarrow \quad \mathfrak{Z}_0 \notin \mathrm{Sel}_{\mathtt{Gr}}(\mathcal{K}, T_{\boldsymbol{f}}^\dagger)_{\mathrm{tors}}$$

where the subscript tors *denotes the* $\mathbb{I}$*-torsion submodule.*

Proof. Let $\gamma_{\mathrm{ac}} \in \Gamma_{\mathcal{K}}^{\mathrm{ac}}$ be a topological generator. The dual of the restriction map for the extension $\mathcal{K}_\infty^{\mathrm{ac}}/\mathcal{K}$ induces a surjective homomorphism

$$X_{\mathtt{Gr}}(\mathcal{K}_\infty^{\mathrm{ac}}, A_{\boldsymbol{f}}^\dagger)/(\gamma_{\mathrm{ac}} - 1)X_{\mathtt{Gr}}(\mathcal{K}_\infty^{\mathrm{ac}}, A_{\boldsymbol{f}}^\dagger) \twoheadrightarrow X_{\mathtt{Gr}}(\mathcal{K}, A_{\boldsymbol{f}}^\dagger)$$

with $\mathbb{I}$-torsion kernel. Since $X_{\mathtt{Gr}}(\mathcal{K}, A_{\boldsymbol{f}}^\dagger)$ and $\mathrm{Sel}_{\mathtt{Gr}}(\mathcal{K}, T_{\boldsymbol{f}}^\dagger)$ have the same $\mathbb{I}$-rank by Lemma 3.2, this shows that if $\mathrm{Sel}_{\mathtt{Gr}}(\mathcal{K}, T_{\boldsymbol{f}}^\dagger)$ has $\mathbb{I}$-rank one, then so do the $\Gamma_{\mathcal{K}}^{\mathrm{ac}}$-coinvariants of $X_{\mathtt{Gr}}(\mathcal{K}_\infty^{\mathrm{ac}}, A_{\boldsymbol{f}}^\dagger)$, and hence by Theorem 5.5 we deduce that

$$(\gamma_{\mathrm{ac}} - 1) \nmid \mathrm{Char}_{\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]}\left(\frac{\mathrm{Sel}_{\mathtt{Gr}}(\mathcal{K}, \mathbf{T}^{\dagger,\mathrm{ac}})}{\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!] \cdot \mathfrak{Z}_\infty}\right).$$

Thus the image of $\mathfrak{Z}_\infty$ in $\mathrm{Sel}_{\mathtt{Gr}}(\mathcal{K}, \mathbf{T}^{\dagger,\mathrm{ac}})/(\gamma_{\mathrm{ac}} - 1)\mathrm{Sel}_{\mathtt{Gr}}(\mathcal{K}, \mathbf{T}^{\dagger,\mathrm{ac}})$ is not $\mathbb{I}$-torsion, and since this image is sent to $\mathfrak{Z}_0$ under the natural injection

$$\mathrm{Sel}_{\mathtt{Gr}}(\mathcal{K}, \mathbf{T}^{\dagger,\mathrm{ac}})/(\gamma_{\mathrm{ac}} - 1)\mathrm{Sel}_{\mathtt{Gr}}(\mathcal{K}, \mathbf{T}^{\dagger,\mathrm{ac}}) \hookrightarrow \mathrm{Sel}_{\mathtt{Gr}}(\mathcal{K}, T_{\boldsymbol{f}}^\dagger),$$

the result follows. □

Remark 5.7. Replacing the appeal to [12, Thm. 3.4] (or [65, Thm. 1.2]) in the proof of Theorem 5.5 by an appeal to [7, Thm. 5.1] the same argument as above gives a proof of Theorems 5.5 and 5.6 with hypotheses (h3)–(h6) replaced by "Hypothesis ♡" from [68], i.e., letting $\mathrm{Ram}(\bar{\rho}_{\boldsymbol{f}})$ be the set of primes $\ell \| N$ such that $\bar{\rho}_{\boldsymbol{f}}$ is ramified at ℓ:

- $\mathrm{Ram}(\bar{\rho}_{\boldsymbol{f}})$ contains all primes $\ell \| N^+$, and all primes $\ell | N^-$ such that $\ell \equiv \pm 1 \pmod{p}$,
- If N is not squarefree, then $\mathrm{Ram}(\bar{\rho}_{\boldsymbol{f}})$ contains either a prime $\ell | N^-$ or at least two primes $\ell \| N^+$,
- If $\ell^2 | N^+$, then $\mathrm{H}^0(\mathbf{Q}_\ell, \bar{\rho}_{\boldsymbol{f}}) = \{0\}$,

and the assumption that $\bar{\rho}_{\boldsymbol{f}}$ is surjective and $\boldsymbol{a}_p \not\equiv \pm 1 \pmod{p}$.

5.3. $\mathbb{I}$-adic Gross–Zagier formula

In this section we prove a $\mathbb{I}$-adic Gross–Zagier formula for the big Heegner point $\mathfrak{Z}_0$ which will be a key ingredient in our application to Greenberg's nonvanishing conjecture. More generally, we shall prove a Gross–Zagier type formula for the $\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]$-adic family $\mathfrak{Z}_\infty$; the result for $\mathfrak{Z}_0$ then follows by specialization at the trivial character.

Define the cyclotomic $\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]$-adic height pairing

$$\langle -, - \rangle_{\mathcal{K}_\infty^{\mathrm{ac}},\mathbb{I}}^{\mathrm{cyc}} : \mathrm{Sel}_{\mathtt{Gr}}(\mathcal{K}, \mathbf{T}^{\dagger,\mathrm{ac}}) \otimes_{\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]} \mathrm{Sel}_{\mathtt{Gr}}(\mathcal{K}, \mathbf{T}^{\dagger,\mathrm{ac}})^{\iota} \longrightarrow \mathcal{J}^{\mathrm{cyc}} \otimes_{\mathbb{I}} \mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!] \otimes_{\mathbb{I}} F_{\mathbb{I}} \tag{5.7}$$

by

$$\langle a_\infty, b_\infty \rangle_{\mathcal{K}_\infty^{\mathrm{ac}},\mathbb{I}}^{\mathrm{cyc}} = \varprojlim_n \sum_{\sigma \in \mathrm{Gal}(\mathcal{K}_n^{\mathrm{ac}}/\mathcal{K})} \langle a_n, b_n^\sigma \rangle_{\mathcal{K}_n^{\mathrm{ac}},\mathbb{I}}^{\mathrm{cyc}} \cdot \sigma,$$

and define the cyclotomic regulator $\mathcal{R}_{\mathrm{cyc}} \subset \mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!] \otimes_{\mathbb{I}} F_{\mathbb{I}}$ to be the characteristic ideal of the cokernel of (5.7) (after dividing by the image of $(\gamma_{\mathrm{cyc}} - 1)$ in $\mathcal{J}^{\mathrm{cyc}}$).

Since we assume that $\mathcal{K}$ satisfies (gen-H), the constant term $L_{p,0}^{\mathtt{Hi}}(\boldsymbol{f}^\dagger/\mathcal{K})_{\mathrm{ac}}$ in the expansion (3.5) vanishes (see Proposition 2.3). We next consider the linear term $L_{p,1}^{\mathtt{Hi}}(\boldsymbol{f}^\dagger/\mathcal{K})_{\mathrm{ac}}$.

Theorem 5.8. *Assume hypotheses* (h0)–(h6), *and denote by* $\mathcal{X}_{\mathrm{tors}}$ *the characteristic ideal of* $X_{\mathtt{Gr}}(\mathcal{K}_\infty^{\mathrm{ac}}, A_{\boldsymbol{f}}^\dagger)_{\mathrm{tors}}$. *Then*

$$\mathcal{R}_{\mathrm{cyc}} \cdot \mathcal{X}_{\mathrm{tors}} = (L_{p,1}^{\mathtt{Hi}}(\boldsymbol{f}^\dagger/\mathcal{K})_{\mathrm{ac}})$$

as ideals in $\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!] \otimes_{\mathbb{I}} F_{\mathbb{I}}$.

Proof. Since $\mathrm{Sel}_{\mathtt{Gr}}(\mathcal{K}, \mathbf{T}^{\dagger,\mathrm{ac}})$ has $\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]$-rank one by Theorem 5.5, the height formula of Theorem 3.10 and Lemma 3.9 immediately yield the equality

$$\mathcal{R}_{\mathrm{cyc}} \cdot \mathrm{Char}_{\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]}\left(\frac{\mathrm{Sel}_{\mathtt{Gr}}(\mathcal{K}, \mathbf{T}^{\dagger,\mathrm{ac}})}{\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!] \cdot \mathcal{BF}^{\dagger,\mathrm{ac}}}\right) = (L_{p,1}^{\mathtt{Hi}}(\boldsymbol{f}^\dagger/\mathcal{K})_{\mathrm{ac}}) \cdot \eta^{\iota}, \tag{5.8}$$

where $\eta \subset \mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]$ is the characteristic ideal of $\mathrm{H}^1_{\mathtt{Gr}}(\mathcal{K}_{\overline{\mathfrak{p}}}, \mathbf{T}^{\dagger,\mathrm{ac}})/\mathrm{loc}_{\overline{\mathfrak{p}}}(\mathrm{Sel}_{\mathtt{Gr}}(\mathcal{K}, \mathbf{T}^{\dagger,\mathrm{ac}}))$. We shall argue below that $\eta \neq 0$. Global duality yields the exact sequence

$$0 \longrightarrow \frac{\mathrm{H}^1_{\mathtt{Gr}}(\mathcal{K}_{\mathfrak{p}}, \mathbf{T}^{\dagger,\mathrm{ac}})}{\mathrm{loc}_{\mathfrak{p}}(\mathrm{Sel}_{\mathtt{Gr}}(\mathcal{K}, \mathbf{T}^{\dagger,\mathrm{ac}}))} \longrightarrow X_{\emptyset,\mathtt{Gr}}(\mathcal{K}_\infty^{\mathrm{ac}}, A_{\boldsymbol{f}}^\dagger) \longrightarrow X_{\mathtt{Gr}}(\mathcal{K}_\infty^{\mathrm{ac}}, A_{\boldsymbol{f}}^\dagger) \longrightarrow 0. \tag{5.9}$$

The left term in (5.9) is $\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]$-torsion, since by Corollary 3.7 the image of the map $\mathrm{loc}_{\mathfrak{p}} : \mathrm{Sel}_{\mathtt{Gr}}(\mathcal{K}, \mathbf{T}^{\dagger,\mathrm{ac}}) \to \mathrm{H}^1_{\mathtt{Gr}}(\mathcal{K}_{\mathfrak{p}}, \mathbf{T}^{\dagger,\mathrm{ac}})$ is nonzero and the target has $\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]$-rank one. On the other hand, by Theorem 5.5 the module $X_{\mathtt{Gr}}(\mathcal{K}_\infty^{\mathrm{ac}}, A_{\boldsymbol{f}}^\dagger)$ has $\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]$-rank one. Hence it follows that the middle term in (5.9) has $\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]$-rank one, and by the action of complex conjugation the same is true for $X_{\mathtt{Gr},\emptyset}(\mathcal{K}_\infty^{\mathrm{ac}}, A_{\boldsymbol{f}}^\dagger)$. Thus the nonvanishing of η follows from the analogue of (5.9) for the prime $\overline{\mathfrak{p}}$ (see (5.12) below).

By Lemma 3.4 the above also shows that $X_{\mathtt{Gr},0}(\mathcal{K}_\infty^{\mathrm{ac}}, A_{\boldsymbol{f}}^\dagger)$ is $\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]$-torsion, and counting ranks in the exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Sel}_{\mathtt{Gr}}(\mathcal{K}, \mathbf{T}^{\dagger,\mathrm{ac}}) \longrightarrow \mathrm{Sel}_{\mathtt{Gr},\emptyset}(\mathcal{K}, \mathbf{T}^{\dagger,\mathrm{ac}}) \longrightarrow \frac{\mathrm{H}^1(\mathcal{K}_{\overline{\mathfrak{p}}}, \mathbf{T}^{\dagger,\mathrm{ac}})}{\mathrm{H}^1_{\mathtt{Gr}}(\mathcal{K}_{\overline{\mathfrak{p}}}, \mathbf{T}^{\dagger,\mathrm{ac}})} \\ \longrightarrow X_{\mathtt{Gr}}(\mathcal{K}_\infty^{\mathrm{ac}}, A_{\boldsymbol{f}}^\dagger) \longrightarrow X_{\mathtt{Gr},0}(\mathcal{K}_\infty^{\mathrm{ac}}, A_{\boldsymbol{f}}^\dagger) \longrightarrow 0, \end{aligned} \tag{5.10}$$

we see that the first two terms in (5.10) have $\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]$-rank one. Since the quotient $\mathrm{H}^1(\mathcal{K}_{\overline{\mathfrak{p}}}, \mathbf{T}^{\dagger,\mathrm{ac}})/\mathrm{H}^1_{\mathtt{Gr}}(\mathcal{K}_{\overline{\mathfrak{p}}}, \mathbf{T}^{\dagger,\mathrm{ac}})$ has no $\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]$-torsion, it follows that

$$\mathrm{Sel}_{\mathtt{Gr}}(\mathcal{K}, \mathbf{T}^{\dagger,\mathrm{ac}}) = \mathrm{Sel}_{\mathtt{Gr},\emptyset}(\mathcal{K}, \mathbf{T}^{\dagger,\mathrm{ac}}). \tag{5.11}$$

Taking $\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]$-torsion in the analogue of (5.9) for $\overline{\mathfrak{p}}$, that is

$$0 \longrightarrow \frac{\mathrm{H}^1_{\mathtt{Gr}}(\mathcal{K}_{\overline{\mathfrak{p}}}, \mathbf{T}^{\dagger,\mathrm{ac}})}{\mathrm{loc}_{\overline{\mathfrak{p}}}(\mathrm{Sel}_{\mathtt{Gr}}(\mathcal{K}, \mathbf{T}^{\dagger,\mathrm{ac}}))} \longrightarrow X_{\mathtt{Gr},\emptyset}(\mathcal{K}_\infty^{\mathrm{ac}}, A_{\boldsymbol{f}}^\dagger) \longrightarrow X_{\mathtt{Gr}}(\mathcal{K}_\infty^{\mathrm{ac}}, A_{\boldsymbol{f}}^\dagger) \longrightarrow 0, \tag{5.12}$$

and applying Lemma 3.4 and the "functional equation" $\mathcal{X}_{\mathrm{tors}}^\iota = \mathcal{X}_{\mathrm{tors}}$ of [33, p. 1464] we obtain

$$\eta^\iota \cdot \mathcal{X}_{\mathrm{tors}} = \mathrm{Char}_{\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]}(X_{\mathtt{Gr},0}(\mathcal{K}_\infty^{\mathrm{ac}}, A_{\boldsymbol{f}}^\dagger)). \tag{5.13}$$

On the other hand, by the equivalence (i)'$\Longleftrightarrow$(ii)' in Theorem 3.8, the second part of Theorem 5.2 implies that

$$\mathrm{Char}_{\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]}(X_{\mathtt{Gr},0}(\mathcal{K}_\infty^{\mathrm{ac}}, A_{\boldsymbol{f}}^\dagger)) = \mathrm{Char}_{\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]}\left(\frac{\mathrm{Sel}_{\mathtt{Gr},\emptyset}(\mathcal{K}, \mathbf{T}^{\dagger,\mathrm{ac}})}{\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!] \cdot \mathcal{BF}^{\dagger,\mathrm{ac}}}\right)$$

as ideals in $\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!] \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$, and so the result follows from the combination of (5.8), (5.11), and (5.13). □

The aforementioned $\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]$-adic Gross–Zagier formula for $\mathfrak{Z}_\infty$ is the following.

Corollary 5.9. *Assume hypotheses* (h0)–(h6). *Then we have the equality*

$$(L_{p,1}^{\mathtt{Hi}}(\boldsymbol{f}^\dagger/\mathcal{K})_{\mathrm{ac}}) = (\langle \mathfrak{Z}_\infty, \mathfrak{Z}_\infty \rangle_{\mathcal{K}_\infty^{\mathrm{ac}},\mathbb{I}}^{\mathrm{cyc}})$$

as ideals of $\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!] \otimes_{\mathbb{I}} F_{\mathbb{I}}$.

Proof. Since $\mathrm{Sel}_{\mathtt{Gr}}(\mathcal{K}, \mathbf{T}^{\dagger,\mathrm{ac}})$ has $\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]$-rank one by Theorem 5.5 and $\mathfrak{Z}_\infty$ is not $\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]$-torsion, the regulator $\mathcal{R}_{\mathrm{cyc}}$ of (5.7) satisfies

$$(\langle \mathfrak{Z}_\infty, \mathfrak{Z}_\infty\rangle^{\mathrm{cyc}}_{\mathcal{K}_\infty^{\mathrm{ac}},\mathbb{I}}) = \mathcal{R}_{\mathrm{cyc}} \cdot \mathrm{Char}_{\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]}\left(\frac{\mathrm{Sel}_{\mathtt{Gr}}(\mathcal{K}, \mathbf{T}^{\dagger,\mathrm{ac}})}{\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]\cdot\mathfrak{Z}_\infty}\right)\cdot \mathrm{Char}_{\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]}\left(\frac{\mathrm{Sel}_{\mathtt{Gr}}(\mathcal{K}, \mathbf{T}^{\dagger,\mathrm{ac}})}{\mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{ac}}]\!]\cdot\mathfrak{Z}_\infty}\right)^{\iota}.$$

By the “functional equation” of [33, p. 1464], the result thus follows from the combination of Theorem 5.8 and the equality of characteristic ideals in Theorem 5.5. □

5.4. Proof of Theorem B

As in the Introduction, let $-w \in \{\pm 1\}$ be the generic sign in the functional equation of the p-adic L-functions $L_p^{\mathtt{MTT}}(\boldsymbol{f}_\phi, s)$ for varying $\phi \in \mathcal{X}_a^o(\mathbb{I})$. For comparison, before giving the proof of our application to Greenberg's nonvanishing conjecture in the case of rank one, we record a result in the rank zero case that follows immediately from [57].

Theorem 5.10 *(Skinner–Urban). Assume that:*

- $\bar\rho_{\boldsymbol{f}}$ *is irreducible and p-distinguished,*
- $\boldsymbol{f}$ *has trivial tame character,*
- *there is a prime* $\ell \| N$ *such that* $\bar\rho_{\boldsymbol{f}}$ *is ramified at* ℓ.

If $\mathrm{Sel}_{\mathtt{Gr}}(\mathbf{Q}, T_{\boldsymbol{f}}^\dagger)$ *is* $\mathbb{I}$*-torsion, then* $L(\boldsymbol{f}_\phi, k_\phi/2) \neq 0$ *for all but finitely many* $\phi \in \mathcal{X}_a^o(\mathbb{I})$.

Proof. Since the $\mathbb{I}$-modules $\mathrm{Sel}_{\mathtt{Gr}}(\mathbf{Q}, T_{\boldsymbol{f}}^\dagger)$ and $X_{\mathtt{Gr}}(\mathbf{Q}, A_{\boldsymbol{f}}^\dagger)$ have the same rank by Lemma 3.2, our hypothesis implies that $X_{\mathtt{Gr}}(\mathbf{Q}, A_{\boldsymbol{f}}^\dagger)$ is $\mathbb{I}$-torsion. Thus in particular $\mathrm{Sel}_{\mathtt{Gr}}(\mathbf{Q}, A_{\boldsymbol{f}_\phi}(1-k_\phi/2))$ is finite for all but finitely many $\phi \in \mathcal{X}_a^o(\mathbb{I})$, and so the result follows from [57, Thm. 3.6.13]. □

The following is Theorem B in the Introduction.

Theorem 5.11. *Assume that:*

(i) $\mathbb{I}$ *is regular,*
(ii) $\bar\rho_{\boldsymbol{f}}$ *is irreducible,*
(iii) *some specialization* $\boldsymbol{f}_\phi$ *is the p-stabilization of a newform* $f \in S_2(\Gamma_0(N))$,
(iv) N *is squarefree,*
(v) *there are at least two primes* $\ell | N$ *at which* $\bar\rho_{\boldsymbol{f}}$ *is ramified.*

If $\mathrm{Sel}_{\mathtt{Gr}}(\mathbf{Q}, T_{\boldsymbol{f}}^\dagger)$ *has* $\mathbb{I}$*-rank one and the* $\mathbb{I}$*-adic height pairing* $\langle -, -\rangle^{\mathrm{cyc}}_{\mathbf{Q},\mathbb{I}}$ *is non-degenerate, then*

$$\frac{d}{ds} L_p^{\mathtt{MTT}}(\boldsymbol{f}_\phi, s)\Big|_{s=k_\phi/2} \neq 0,$$

for all but finitely many $\phi \in \mathcal{X}_a^o(\mathbb{I})$.

Proof. Let $\phi \in \mathcal{X}_a^o(\mathbb{I})$ be such that $\boldsymbol{f}_\phi$ is the ordinary p-stabilization of a newform $f \in S_2(\Gamma_0(N))$. Let ℓ_1 and ℓ_2 be two distinct primes as in hypothesis (v), and choose an imaginary quadratic field $\mathcal{K}$ such that the following hold:

- ℓ_1 and ℓ_2 are inert in $\mathcal{K}$,
- every prime dividing $N^+ := N/\ell_1\ell_2$ splits in $\mathcal{K}$,
- p splits in $\mathcal{K}$,
- $L(f \otimes \epsilon_{\mathcal{K}}, 1) \neq 0$, where $\epsilon_{\mathcal{K}}$ is the quadratic character corresponding to $\mathcal{K}$.

Note that the existence of $\mathcal{K}$ is ensured by a result of [63] (see also [20, Thm. B.1]), and that, so chosen, $\mathcal{K}$ satisfies (gen-H) with $N^- = \ell_1\ell_2$. Now, the action of a complex conjugation c combined with the restriction map induces an isomorphism

$$\mathrm{Sel}_{\mathtt{Gr}}(\mathcal{K}, T_{\boldsymbol{f}}^\dagger) \simeq \mathrm{Sel}_{\mathtt{Gr}}(\mathbf{Q}, T_{\boldsymbol{f}}^\dagger) \oplus \mathrm{Sel}_{\mathtt{Gr}}(\mathbf{Q}, T_{\boldsymbol{f}}^\dagger \otimes \epsilon_{\mathcal{K}}), \tag{5.14}$$

where the first and second summands are identified with the $+$ and $-$ eigenspaces for the action of c, respectively (see [57, Lem. 3.1.5]). By Kato's work [38], the nonvanishing of $L(f \otimes \epsilon_{\mathcal{K}}, 1)$ implies that $\mathrm{Sel}(\mathbf{Q}, T_f \otimes \epsilon_{\mathcal{K}})$ is finite, and so by the control theorem for $\mathrm{Sel}_{\mathtt{Gr}}(\mathbf{Q}, T_{\boldsymbol{f}}^\dagger \otimes \epsilon_{\mathcal{K}})$ (see the exact sequence in [35, Cor. 3.4.3]) we conclude that $\mathrm{Sel}_{\mathtt{Gr}}(\mathbf{Q}, T_{\boldsymbol{f}}^\dagger \otimes \epsilon_{\mathcal{K}})$ is $\mathbb{I}$-torsion, and so

$$\mathrm{rank}_{\mathbb{I}}\ \mathrm{Sel}_{\mathtt{Gr}}(\mathcal{K}, T_{\boldsymbol{f}}^\dagger) = \mathrm{rank}_{\mathbb{I}}\ \mathrm{Sel}_{\mathtt{Gr}}(\mathbf{Q}, T_{\boldsymbol{f}}^\dagger) = 1$$

by (5.14) and our assumption. In particular, since hypotheses (i)–(iv) imply hypotheses (h0)–(h3) at the start of this section, and hypotheses (h4)–(h6) hold by our choice of $\mathcal{K}$, Theorem 5.6 yields the non-triviality of the class $\mathfrak{Z}_0$, and so the element $\langle \mathfrak{Z}_0, \mathfrak{Z}_0 \rangle_{\mathcal{K},\mathbb{I}}^{\mathrm{cyc}} \in \mathbb{I}$ is non-zero by our hypothesis of non-degeneracy.

Let $L_p^{\mathtt{Hi}}(\boldsymbol{f}^\dagger/\mathcal{K})_{\mathrm{cyc}}$ be the image of $\mathrm{tw}_{\Theta^{-1}}(L_p^{\mathtt{Hi}}(\boldsymbol{f}/\mathcal{K}))$ under the natural projection $\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!] \twoheadrightarrow \mathbb{I}[\![\Gamma_{\mathcal{K}}^{\mathrm{cyc}}]\!]$. By Theorem 2.4, for every $\phi \in \mathcal{X}_a^o(\mathbb{I})$ we have the factorization

$$\phi(L_p^{\mathtt{Hi}}(\boldsymbol{f}^\dagger/\mathcal{K})_{\mathrm{cyc}}) = \mathrm{tw}_{\Theta_\phi^{-1}}(L_p^{\mathtt{MTT}}(\boldsymbol{f}_\phi)) \cdot \mathrm{tw}_{\Theta_\phi^{-1}}(L_p^{\mathtt{MTT}}(\boldsymbol{f}_\phi \otimes \epsilon_{\mathcal{K}})) \tag{5.15}$$

up to a unit in $\phi(\mathbb{I})[\![\Gamma^{\mathrm{cyc}}]\!]^\times$. Expand

$$\begin{aligned}
\phi(L_p^{\mathtt{Hi}}(\boldsymbol{f}^\dagger/\mathcal{K})_{\mathrm{cyc}}) &= L_{p,0}^{\mathtt{Hi}}(\boldsymbol{f}_\phi^\dagger/\mathcal{K}) + L_{p,1}^{\mathtt{Hi}}(\boldsymbol{f}_\phi^\dagger/\mathcal{K}) \cdot (\gamma_{\mathrm{cyc}} - 1) + \cdots, \\
\mathrm{tw}_{\Theta_\phi^{-1}}(L_p^{\mathtt{MTT}}(\boldsymbol{f}_\phi)) &= L_{p,0}^{\mathtt{MTT}}(\boldsymbol{f}_\phi^\dagger) + L_{p,1}^{\mathtt{MTT}}(\boldsymbol{f}_\phi^\dagger) \cdot (\gamma_{\mathrm{cyc}} - 1) + \cdots, \\
\mathrm{tw}_{\Theta_\phi^{-1}}(L_p^{\mathtt{MTT}}(\boldsymbol{f}_\phi \otimes \epsilon_{\mathcal{K}})) &= L_{p,0}^{\mathtt{MTT}}(\boldsymbol{f}_\phi^\dagger \otimes \epsilon_{\mathcal{K}}) + L_{p,1}^{\mathtt{MTT}}(\boldsymbol{f}_\phi^\dagger \otimes \epsilon_{\mathcal{K}}) \cdot (\gamma_{\mathrm{cyc}} - 1) + \cdots,
\end{aligned}$$

as power series in $\gamma_{\mathrm{cyc}} - 1$, and note that by the p-adic Mellin transform we have

$$\frac{d}{ds} L_p^{\mathtt{MTT}}(\boldsymbol{f}_\phi, s)\bigg|_{s=k_\phi/2} \neq 0 \quad \Longleftrightarrow \quad L_{p,1}^{\mathtt{MTT}}(\boldsymbol{f}_\phi^\dagger) \neq 0$$

(see [62, (24)]). The constant term $L_{p,0}^{\mathtt{Hi}}(\boldsymbol{f}_\phi^\dagger/\mathcal{K}) \in \mathbb{I}$ vanishes by Proposition 2.3, and so the factorization (5.15) yields the following equality up to unit in $\mathcal{O}_\phi^\times$:

$$L_{p,1}^{\mathtt{Hi}}(\boldsymbol{f}_\phi^\dagger/\mathcal{K}) = L_{p,1}^{\mathtt{MTT}}(\boldsymbol{f}_\phi^\dagger) \cdot L_{p,0}^{\mathtt{MTT}}(\boldsymbol{f}_\phi^\dagger \otimes \epsilon_\mathcal{K}). \tag{5.16}$$

Finally, since by definition $L_{p,1}^{\mathtt{Hi}}(\boldsymbol{f}^\dagger/\mathcal{K}) \in \mathbb{I}$ agrees with the image of the linear term $L_{p,1}^{\mathtt{Hi}}(\boldsymbol{f}^\dagger/\mathcal{K})_{\mathrm{ac}}$ in (3.5) under the augmentation map $\mathbb{I}[\![\Gamma_\mathcal{K}^{\mathrm{ac}}]\!] \to \mathbb{I}$, from Corollary 5.9 specialized at the trivial character of $\Gamma_\mathcal{K}^{\mathrm{ac}}$ and (5.16) we see that

$$\begin{aligned} \langle \mathfrak{z}_0, \mathfrak{z}_0 \rangle_{\mathcal{K},\mathbb{I}}^{\mathrm{cyc}} \neq 0 \quad &\Longrightarrow \quad L_{p,1}^{\mathtt{Hi}}(\boldsymbol{f}_\phi^\dagger/\mathcal{K}) \neq 0, \quad \text{for almost all } \phi \in \mathcal{X}_a^o(\mathbb{I}) \\ &\Longrightarrow \quad L_{p,1}^{\mathtt{MTT}}(\boldsymbol{f}_\phi^\dagger) \neq 0, \qquad \text{for almost all } \phi \in \mathcal{X}_a^o(\mathbb{I}), \end{aligned}$$

concluding the proof of Theorem 5.11. □

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