

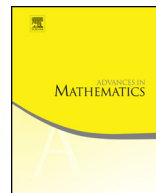


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A Mikhlin multiplier theory for free groups and amalgamated free products of von Neumann algebras

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ARTICLE INFO

Article history:

Received 11 April 2021

Received in revised form 22 October 2021

Accepted 31 March 2022

Available online 14 April 2022

Communicated by Dan Voiculescu

MSC:

primary 46L07, 46L54

secondary 46L52, 42B15

Keywords:

Free groups

Free products of groups

Amalgamated free products of von Neumann algebras

Non-commutative L_p -spaces

Hörmander-Mikhlin multipliers

Completely bounded maps

ABSTRACT

We obtain a Mikhlin multiplier theory for the nonabelian free groups. Let $\mathbb{F}_\infty$ be a free group on infinite many generators $\{g_1, g_2, \dots\}$. Given $d \geq 1$ and a bounded symbol m on $\mathbb{Z}^d$ satisfying the classical Mikhlin condition, the linear map $M_m : \mathbb{C}[\mathbb{F}_\infty] \rightarrow \mathbb{C}[\mathbb{F}_\infty]$ defined by $\lambda(g) \mapsto m(k_1, \dots, k_d)\lambda(g)$ for $g = g_1^{k_1} \dots g_n^{k_n} \in \mathbb{F}_\infty$ in reduced form (with $k_l = 0$ in $m(k_1, \dots, k_d)$ for $l > n$), extends to a completely bounded map on $L_p(\mathbb{F}_\infty)$ for all $1 < p < \infty$, where $\mathbb{F}_\infty$ is the group von Neumann algebra of $\mathbb{F}_\infty$. In the process, we establish a platform to transfer L_p -completely bounded maps on tensor products of von Neumann algebras to L_p -completely bounded maps on the corresponding amalgamated free products. A similar result holds for any free product of discrete groups.

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1. Introduction

Fourier multipliers are by far the most important operators in analysis. One can say that classical harmonic analysis has been developed around Fourier multipliers. The main task there is to find criteria for the boundedness of Fourier multipliers on various function spaces, notably on L_p -spaces. However, it is in general impossible to characterize their L_p -boundedness for finite $p \neq 2$. In this regard, the most fundamental result is the celebrated Hörmander-Mikhlin multiplier theorem which asserts that if m is a $C^{[\frac{d}{2}]+1}$ function on $\mathbb{R}^d \setminus \{0\}$ such that

$$\sup_{0 \leq |\alpha| \leq [\frac{d}{2}]+1} \sup_{\xi \in \mathbb{R}^d \setminus \{0\}} |\xi|^{|\alpha|} |\partial^\alpha m(\xi)| < \infty,$$

where $\alpha = (\alpha_1, \dots, \alpha_d)$ denotes a multi-index of nonnegative integers and $|\alpha| = \alpha_1 + \dots + \alpha_d$, then the associated Fourier multiplier T_m , defined via Fourier transform by $\widehat{T_m(f)} = m\widehat{f}$, is a bounded map on $L_p(\mathbb{R}^d)$ for all $1 < p < \infty$. A similar statement holds for the periodic case, i.e., for $\mathbb{T}^d$ instead of $\mathbb{R}^d$. More precisely, if $m : \mathbb{Z}^d \rightarrow \mathbb{C}$ satisfies

$$\|m\|_{\text{HM}} = \sup_{0 \leq |\alpha| \leq [\frac{d}{2}]+1} \sup_{k \in \mathbb{Z}^d} |k|^{|\alpha|} |\partial^\alpha m(k)| < \infty, \quad (1)$$

where $\partial^\alpha = \partial_{k_1}^{\alpha_1} \dots \partial_{k_d}^{\alpha_d}$ denotes the partial discrete difference operator of order $\alpha = (\alpha_1, \dots, \alpha_d)$, then the associated Fourier multiplier T_m is a bounded map on $L_p(\mathbb{T}^d)$ for all $1 < p < \infty$. A simple argument by Fubini's theorem gives that T_m extends to a bounded map on $L_p(\mathbb{Z}^\infty)$ for all $1 < p < \infty$ if $m : \mathbb{Z}^\infty \rightarrow \mathbb{C}$ depends only on the first d variables, and satisfies (1). The main theorem of this article says, in particular, that the aforementioned Hörmander-Mikhlin theorem holds for the nonabelian free groups $\mathbb{F}_\infty$ instead of $\mathbb{Z}^\infty$, that is replacing the independence by the freeness of the coordinates.

In noncommutative harmonic analysis and operator algebras, the study of Fourier multipliers on general groups is of paramount importance. These multipliers are also called Herz-Schur multipliers in the literature. The underlying L_p -spaces are then the noncommutative L_p -spaces on the group von Neumann algebra. This line of investigation finds one of its origins in the study of approximation properties of operator algebras inaugurated by Haagerup's pioneering work [6] on Fourier multipliers on free groups in which he solved the longstanding problem on Grothendieck's approximation property for the reduced C^* -algebra of a free group: this algebra has the completely bounded approximation property. Later, together with De Cannière [2] and Cowling [5], Haagerup studied completely bounded multipliers on the Fourier algebras of classical Lie groups.

One of the remarkable recent achievements in the interplay between multipliers and noncommutative L_p -spaces is the result of Lafforgue and de la Salle [20] asserting that the noncommutative L_p -space with $p > 4$ associated to the von Neumann algebra of $SL_3(\mathbb{Z})$ fails the completely bounded approximation property. Haagerup and coauthors [9,10,8] extended their result to connected simple Lie groups; later, de Laat and de la

Salle [19] discovered that L_p -multipliers are very tightly connected with Banach space geometry and group representations. Other striking illustrations of this interplay are Junge and Parcet's work [17] on the Maurey-Rosenthal factorization for noncommutative L_p -spaces which relies on Schur multiplier estimates. Multipliers are also intimately related to functional calculus in von Neumann algebras as shown by the resolution of Krein's celebrated Lipschitz continuity problem in noncommutative L_p -spaces by Potapov and Sukochev [34], see also [3] where the crucial use of Fourier multipliers is more apparent.

In a recent series of articles [7,11], Haagerup and co-authors obtained a beautiful handy characterization of the completely bounded radial multipliers on the von Neumann algebras (i.e. L_∞) of free groups and p -adic groups, and, more generally, on homogeneous trees and free products; see also Wysocański's previous work [37]. All this has motivated many other publications of the same nature, in particular, building on Ozawa's work [28], Mei and de la Salle [24] obtained a similar characterization for hyperbolic groups. Nevertheless, little has been understood for non-radial multipliers,¹ nor for L_p -spaces with $2 < p < \infty$. Here, the most challenging problem is to find a Hörmander-Mikhlin type criterion for Fourier multipliers on L_p -spaces to be completely bounded.

The recent papers [15,16,29] provide a first advance towards this direction. The work [25] by the first and second named authors is more related to the present article, it introduces the free Hilbert transforms on free groups and free products of von Neumann algebras and showed their L_p -boundedness for all $1 < p < \infty$. We pursue this line of investigation by going considerably beyond, and will exhibit a large family of L_p -multipliers in the free setting.

To proceed further, we need some notation and refer to the next section for all unexplained notions. Let Γ be a discrete group and λ its left regular representation on $\ell_2(\Gamma)$. The group von Neumann algebra $vN(\Gamma)$ is the von Neumann algebra generated by $\lambda(\Gamma)$, it is also equal to the weak* closure of the group algebra $\mathbb{C}[\Gamma]$ that consists of all polynomials in λ :

$$\mathbb{C}[\Gamma] = \left\{ \sum_{g \in \Gamma} \alpha(g) \lambda(g) : \alpha(g) \in \mathbb{C} \right\}.$$

Following notation in quantum groups, we will denote $vN(\Gamma)$ by $\widehat{\Gamma}$ in this article. $\widehat{\Gamma}$ is equipped with its canonical tracial state τ defined by $\tau(x) = \langle \delta_e, x \delta_e \rangle$ for any $x \in \widehat{\Gamma}$, where δ_e denotes the Dirac mass at the identity e of Γ . $L_p(\widehat{\Gamma})$ is the noncommutative L_p -space based on $(\widehat{\Gamma}, \tau)$, it is equipped with the natural operator space structure introduced by Pisier.

For a complex function m on Γ , the associated Fourier multiplier T_m is a linear map on $\mathbb{C}[\Gamma]$ determined by $\lambda(g) \mapsto m(g) \lambda(g)$. We call m a (completely) bounded L_p -multiplier on Γ if T_m extends to a (completely) bounded map on $L_p(\widehat{\Gamma})$. Our main

¹ A better understanding of non-radial multipliers seems to be in order to answer some fundamental open questions, such as the existence of a Schauder basis for the free group reduced C^* -algebras and the construction of a concrete Schauder basis for the corresponding noncommutative L_p -spaces.

aim is to find sufficient conditions for m to be such a multiplier. Note that the periodic Hörmander-Mikhlin theorem recalled at the beginning concerns the case $\Gamma = \mathbb{Z}^d$ in which $\widehat{\Gamma} = L_\infty(\mathbb{T}^d)$, $\mathbb{T}$ being the unit circle equipped with the normalized Haar measure. In this case, every function m on $\mathbb{Z}^d$ satisfying (1) is a completely bounded L_p -multiplier on $\mathbb{Z}^d$.

Throughout the article, $\mathbb{F}_\infty$ will denote a free group on infinite generators $\{g_1, g_2, \dots\}$. An element $g \in \mathbb{F}_\infty$ different from the identity e is written as a reduced word on $\{g_1, g_2, \dots\}$:

$$g = g_{i_1}^{k_1} g_{i_2}^{k_2} \cdots g_{i_n}^{k_n} \text{ with } i_1 \neq i_2 \neq \cdots \neq i_n, k_j \in \mathbb{Z} \setminus \{0\}, 1 \leq j \leq n. \quad (2)$$

Given a complex function m on $\mathbb{Z}^d$ and a fixed $d \in \mathbb{N}$, we define $M_m : \mathbb{C}[\mathbb{F}_\infty] \rightarrow \mathbb{C}[\mathbb{F}_\infty]$ by

$$M_m(\lambda(g)) = \begin{cases} m(k_1, \dots, k_d) \lambda(g) & \text{if } d \leq n, \\ m(k_1, \dots, k_n, 0, \dots, 0) \lambda(g) & \text{if } d > n \end{cases}$$

for every g as in (2). Below is our first main theorem. It is contained in [26] for $d = 1$.

Theorem 1.1. *Let $d \in \mathbb{N}$ and $1 < p < \infty$. If m is a completely bounded L_p -multiplier on $\mathbb{Z}^d$, then M_m extends to a completely bounded map on $L_p(\widehat{\mathbb{F}_\infty})$. In particular, the conclusion holds if m satisfies (1).*

To achieve Theorem 1.1, we establish a new platform to transfer the L_p -complete boundedness of Fourier multipliers on tori to Fourier multipliers on free groups. The key tool is a group of unitary actions on $L_2(\widehat{\mathbb{F}_\infty})$ and its complete boundedness on $L_p(\widehat{\mathbb{F}_\infty})$ for all $1 < p < \infty$. Given $z = (z_{j,i})_{1 \leq j \leq d, 1 \leq i < \infty} \in \mathbb{T}^d \times \mathbb{T}^\infty$, let

$$\alpha_z^{Ld}(\lambda(g)) = z_{1,i_1}^{k_1} z_{2,i_2}^{k_2} \cdots z_{d,i_d}^{k_d} \lambda(g)$$

for g as in (2) (with $k_\ell = 0$ if $\ell > n$). It is obvious that α^{Ld} extends to a unitary action of the group $\mathbb{T}^d \times \mathbb{T}^\infty$ on $L_2(\widehat{\mathbb{F}_\infty})$.

Theorem 1.2. *The above action α^{Ld} of $\mathbb{T}^d \times \mathbb{T}^\infty$ on $L_2(\widehat{\mathbb{F}_\infty})$ extends to a uniformly completely bounded action on $L_p(\widehat{\mathbb{F}_\infty})$ for every $1 < p < \infty$ with constant depending only on d and p .*

Theorem 1.1 will follow from Theorem 1.2 and a standard transference argument. Our initial proof of Theorem 1.2 was very long and technical. We have then luckily found a new and shorter argument; the new one depends, in a crucial way, on an analogous result on amalgamated free products of von Neumann algebras. The new approach has the extra advantage that it allows us to easily extend Theorems 1.1 and 1.2 to free

products of general discrete groups. The basic idea is to show that we can exchange the first letters by a free copies of them.

In this context, let $(\mathcal{A}_i, \tau_i)_{i \in I}$ be a family of finite von Neumann algebras equipped with normal faithful tracial states. Let $\mathcal{B}$ be a common von Neumann subalgebra of the $\mathcal{A}_i$'s and $\mathbb{E}_i : \mathcal{A}_i \rightarrow \mathcal{B}$ be the corresponding trace preserving conditional expectation. We denote by $(\mathcal{A}, \tau) = \ast_{i \in I, \mathcal{B}} (\mathcal{A}_i, \tau_i)$ the amalgamated free product of the $(\mathcal{A}_i, \tau_i)$'s over $\mathcal{B}$. Let $\mathbb{E}$ be the conditional expectation from $\mathcal{A}$ to $\mathcal{B}$. As usual, for $X \subset L_p(\mathcal{A})$ we write $\mathring{X}$ for the set $\{x - \mathbb{E}(x) : x \in X\}$.

Assume that we are given maps $T_{j,i} : L_p(\mathcal{A}_i) \rightarrow L_p(\mathcal{A}_i)$ with $1 \leq j \leq d$ and $i \in I$ satisfying the following conditions

- $T_{j,i}$ is $\mathcal{B}$ -bimodular, that is, $T_{j,i}(axb) = aT_{j,i}(x)b$ for $a, b \in \mathcal{B}$ and $x \in L_p(\mathcal{A}_i)$;
- $T_{j,i}(\widehat{L_p(\mathcal{A}_i)}) \subset \widehat{L_p(\mathcal{A}_i)}$.

We can define a map T^{Ld} as follows. $T^{Ld}(b) = b$ for $b \in \mathcal{B}$ and

$$T^{Ld}(a) = \begin{cases} T_{1,i_1}(a_1) \otimes \cdots \otimes T_{d,i_d}(a_d) \otimes a_{d+1} \otimes \cdots \otimes a_n & \text{if } d < n, \\ T_{1,i_1}(a_1) \otimes \cdots \otimes T_{n,i_n}(a_n) & \text{if } d \geq n \end{cases}$$

for $a = a_1 \otimes \cdots \otimes a_n \in \mathring{\mathcal{A}}_{i_1} \otimes_{\mathcal{B}} \cdots \otimes_{\mathcal{B}} \mathring{\mathcal{A}}_{i_n}$ with $i_1 \neq i_2 \neq \cdots \neq i_n$.

The following theorem and its proof contain the main novelty of the article, it gives a surprisingly simple condition to ensure the complete boundedness of the map T^{Ld} on $L_p(\mathcal{A})$.

Theorem 1.3. *Let $1 < p < \infty$. Assume that the $T_{j,i}$'s extend to completely bounded maps on $L_p(\mathcal{A}_i)$ and on $L_2(\mathcal{A}_i)$ uniformly in i, j . Then T^{Ld} extends to a completely bounded map on $L_p(\mathcal{A})$ with*

$$\|T^{Ld}\|_{\text{cb}(L_p(\mathcal{A}))} \lesssim_{p,d} \prod_{j=1}^d \sup_{i \in I} \left(\|T_{j,i}\|_{\text{cb}(L_p(\mathcal{A}_i))} + \|T_{j,i}\|_{\text{cb}(L_2(\mathcal{A}_i))} \right).$$

It is clear that the assumption above is necessary for the validity of the conclusion. Thus the theorem gives a characterization of the complete boundedness of T^{Ld} on $L_p(\mathcal{A})$. Another aspect to be emphasized is the fact that p is a single fixed index. One of the main ingredients in our argument is a length reduction formula for the L_p -norms associated with amalgamated free products that we state as Theorem 4.6. This formula is interesting for its own right, it was previously proved in [18, Theorem B] for homogeneous free polynomials.

The article is organized as follows. After a brief introduction to necessary preliminaries, we give in section 3 some norms on Hilbert $\mathcal{B}$ -modules that are the modular extensions of the usual column and row noncommutative L_p -norms; we show that bounded modular

maps extend to these modular noncommutative L_p -spaces. The results in this section provide crucial tools for section 4 that is the core of the article and contains our major ideas. Section 4 gives the proof of Theorem 1.3 that is done through a reduction formula for polynomials in a free product of von Neumann algebras, the latter is proved in its turn with the help of an intermediate result (Theorem 4.5) that is a special case of Theorem 1.3 where $d = 1$, $p > 2$ and every T_i is a $*$ -representation on $\mathcal{A}_i$ leaving $\mathcal{B}$ invariant and commuting with $\mathbb{E}_i$ for all $i \in I$. Theorems 1.1 and 1.2 easily follow from the results in section 4; moreover, the arguments can be extended to the free products of general (non abelian) discrete groups. We do all this in section 5. This section also contains some more paraproducts which might be interesting in free analysis. We end the article with an appendix on the endpoint $p = \infty$.

We will use the following convention through the article: $A \lesssim B$ (resp. $A \lesssim_p B$) means that $A \leq cB$ (resp. $A \leq c_p B$) for some absolute positive constant c (resp. a positive constant c_p depending only on p). $A \simeq B$ or $A \simeq_p B$ means that these inequalities as well as their inverses hold.

2. Preliminaries

This section presents some preliminaries on noncommutative L_p -spaces and amalgamated free products.

Murray and von Neumann's work [27] demonstrates von Neumann algebras as a natural framework to do noncommutative analysis. The elements in a von Neumann algebra $\mathcal{M}$ can be integrated over the equipped trace τ and measured by the associated L_p -norms. In the sequel, $\mathcal{M}$ will denote a von Neumann algebra with a normal faithful semifinite trace τ . For each $1 \leq p \leq \infty$, let $L_p(\mathcal{M})$ be the noncommutative L_p -space associated with the pair $(\mathcal{M}, \tau)$. Note that $L_\infty(\mathcal{M}) = \mathcal{M}$ with the operator norm. We refer to [33] for details and more historical references.

Recall a C^* -algebra A has the weak expectation property of Lance (in short WEP) if for the universal representation $A \subset A^{**} \subset B(H)$, there is a contraction P from $B(H)$ to A such that the restriction of P on A is the identity map. A C^* -algebra $\mathcal{N}$ is QWEP if it is a quotient of a WEP C^* -algebra by a two sided ideal. See [32, Chp 15] for more information. It is an unpublished result of Junge that when M is a QWEP von Neumann algebra, $L_p(\mathcal{M})$ is completely finitely representable in the Schatten p -class S_p for $1 \leq p < \infty$.

All L_p -spaces in this article will be equipped with their natural operator space structure introduced by Pisier in [31, 32]. We also refer to [32] for operator space theory. In fact, what we will need from the latter theory is only [31, Lemma 1.7] that asserts that a linear map $T : L_p(\mathcal{M}) \rightarrow L_p(\mathcal{M})$ is completely bounded iff $\text{Id}_{S_p} \otimes T : L_p(B(\ell_2) \overline{\otimes} \mathcal{M}) \rightarrow L_p(B(\ell_2) \overline{\otimes} \mathcal{M})$ is bounded; and the completely bounded norm of T is equal to the usual norm of $\text{Id}_{S_p} \otimes T$ and denoted by $\|T\|_{\text{cb}(L_p(\mathcal{M}))}$ or simply $\|T\|_{\text{cb}}$ when no ambiguity is possible. As explained just above, here the Schatten p -class S_p can be replaced by $L_p(\mathcal{N})$ for any QWEP von Neumann algebra $\mathcal{N}$ without changing $\|T\|_{\text{cb}}$. Note that $L_2(\mathcal{M})$ is

an operator Hilbert space and every bounded map on it is automatically completely bounded with the same norm.

We will often be concerned with group von Neumann algebras. Let Γ be a discrete group. Recall the definition of its left regular representation λ : for any $g \in \Gamma$, $\lambda(g)$ is just the left translation by g on $\ell_2(\Gamma)$, i.e., $\lambda(g)(f)(h) = f(g^{-1}h)$ for any $h \in \Gamma$ and $f \in \ell_2(\Gamma)$. Thus $\lambda(g)(\delta_h) = \delta_{gh}$, where δ_h is the Dirac mass at h . $\mathbb{C}[\Gamma]$ denotes the linear span of $\lambda(\Gamma)$. Then the group von Neumann algebra $\widehat{\Gamma}$ is the weak* closure of $\mathbb{C}[\Gamma]$ in $B(\ell_2(\Gamma))$. The canonical trace of $\widehat{\Gamma}$ is the vector state induced by δ_e with e the identity of Γ . When Γ is abelian (e.g. $\Gamma = \mathbb{Z}^d$) $L_p(\widehat{\Gamma})$ coincides with the usual L_p -space on the dual group of Γ . We will need the following result in Section 5.3, which is usually called Fell's "absorption principle".

Lemma 2.1. *Let Id be the trivial representation of Γ on $B(H)$ for some Hilbert space H , e.g. $\text{Id}(g) = \text{Id}_{B(H)}$ for any $g \in \Gamma$. Then, for any unitary representation $U : \Gamma \rightarrow B(H)$, $\lambda \otimes U$ is unitary equivalent to $\lambda \otimes \text{Id}$. In particular, for any finite supported function c on G , we have that*

$$\left\| \sum_g c(g) \lambda(g) \right\|_{\widehat{\Gamma}} = \left\| \sum_g c(g) \lambda(g) \otimes U(g) \right\|_{\widehat{\Gamma} \overline{\otimes} B(H)}.$$

We turn to the second part on amalgamated free products. Let $(\mathcal{A}_i, \tau_i)_{i \in I}$ be a family of finite von Neumann algebras equipped with normal faithful normalized traces. Let $\mathcal{B}$ be a common von Neumann subalgebra of $\mathcal{A}_i$ for all $i \in I$, and $\mathbb{E}_i : \mathcal{A}_i \rightarrow \mathcal{B}$ the trace preserving conditional expectation. Denote $(\mathcal{A}, \tau) = *_{i \in I} (\mathcal{A}_i, \tau_i)$, the amalgamated free product of $(\mathcal{A}_i, \tau_i)_{i \in I}$ over $\mathcal{B}$. We will briefly recall the construction to fix notation.

For any $x \in \mathcal{A}_i$, we denote by $\dot{x} = x - \mathbb{E}_i x$ and $\dot{\mathcal{A}}_i = \{\dot{x} : x \in \mathcal{A}_i\}$, which yields a natural decomposition $\mathcal{A}_i = \mathcal{B} \oplus \dot{\mathcal{A}}_i$. We use the multi-index notation: $\dot{i} = (i_1, \dots, i_n) \in I^n$. The space

$$\mathcal{W} = \bigoplus_{\substack{n \geq 0 \\ (i_1, \dots, i_n) \in I^n \\ i_1 \neq i_2 \neq \dots \neq i_n}} \bigoplus_{\substack{n \geq 1 \\ (i_1, \dots, i_n) \in I^n \\ i_1 \neq i_2 \neq \dots \neq i_n}} \mathcal{W}_{\dot{i}} = \mathcal{B} \oplus \bigoplus_{\substack{n \geq 1 \\ (i_1, \dots, i_n) \in I^n \\ i_1 \neq i_2 \neq \dots \neq i_n}} \dot{\mathcal{A}}_{i_1} \otimes_{\mathcal{B}} \dots \otimes_{\mathcal{B}} \dot{\mathcal{A}}_{i_n}$$

is a $*$ -algebra by using concatenation and centering with respect to $\mathcal{B}$. The natural projection $\mathbb{E}$ from $\mathcal{W}$ onto $\mathcal{B}$ is a conditional expectation. $\tau_i \circ \mathbb{E}$ is a trace on $\mathcal{W}$ that is independent of i and will be denoted by τ . Then $(\mathcal{A}, \tau)$ is the finite von Neumann algebra obtained by the GNS construction from $(\mathcal{W}, \tau)$. Thus $\mathcal{W}$ is weak* dense in $\mathcal{A}$ and dense in $L_p(\mathcal{A})$ for $p < \infty$. Moreover, $\mathbb{E}$ extends to a trace preserving conditional expectation from $\mathcal{A}$ onto $\mathcal{B}$. As $\mathbb{E}$ restricts to $\mathbb{E}_i$ on $\mathcal{A}_i$, from now on we skip the index i for the conditional expectations.

For $n \geq 0$, we denote by P_n the natural projection

$$P_n : \mathcal{W} \rightarrow \mathcal{W}_n = \bigoplus_{\substack{(i_1, \dots, i_n) \in I^n \\ i_1 \neq i_2 \neq \dots \neq i_n}} \dot{\mathcal{A}}_{i_1} \otimes_{\mathcal{B}} \dots \otimes_{\mathcal{B}} \dot{\mathcal{A}}_{i_n}$$

and $P_{\geq n} = \text{Id} - (P_1 + \cdots + P_{n-1})$. Note that P_n extends to a completely bounded projection on $L_p(\mathcal{A})$ with cb-norm $\leq 2n + 1$ for all $1 \leq p \leq \infty$ (see [36]).

For $k \in I$ let

$$\mathcal{L}_k = \bigoplus_{\dot{i}: i_1=k} \mathcal{W}_{\dot{i}} \text{ and } \mathcal{R}_k = \mathcal{L}_k^*.$$

We denote the associated orthogonal projections on $\mathcal{W}$ by L_k and R_k . Given a family $\varepsilon = (\varepsilon_i)_{i \in \{0\} \cup I}$ of elements in $\mathcal{B}$, and $x \in \mathcal{W}$, we define

$$\mathcal{H}_\varepsilon(x) = \varepsilon_0 \mathbb{E}(x) + \sum_{k \in I} \varepsilon_k L_k(x) \text{ and } \mathcal{H}_\varepsilon^{\text{op}}(x) = \mathbb{E}(x) \varepsilon_0^* + \sum_{k \in I} R_k(x) \varepsilon_k^*. \quad (3)$$

The following is the heart of [25], we state it as a lemma for later reference.

Lemma 2.2. *Let $1 < p < \infty$ and $\varepsilon = (\varepsilon_i)_{i \in \{0\} \cup I}$ be a family of unitaries in the center $\mathcal{Z}(\mathcal{B})$ of $\mathcal{B}$. Then $\mathcal{H}_\varepsilon$ and $\mathcal{H}_\varepsilon^{\text{op}}$ extend to completely bounded maps on $L_p(\mathcal{A})$. Moreover,*

$$\|\mathcal{H}_\varepsilon(x)\|_p \simeq_p \|x\|_p \simeq_p \|\mathcal{H}_\varepsilon^{\text{op}}(x)\|_p, \quad x \in L_p(\mathcal{A}).$$

3. Norms on modules

This section is a preparatory part for the next one. Here we will introduce the modular versions of the usual row and column p -operator spaces and show the corresponding boundedness results. This is quite tedious but we have to cope with it since they will be the key tools for the proofs of our main results on amalgamated free products in the next section. Our references for Hilbert C^* -modules are [21,30].

Let $\mathcal{B}$ be a von Neumann subalgebra of a semifinite von Neumann algebra $(\mathcal{M}, \tau)$ such that the restriction of τ to $\mathcal{B}$ is semifinite too. We need to introduce several norms on tensor products related to L_p -modules as in [18].

Let E be a right Hilbert $\mathcal{B}$ -module with $\mathcal{B}$ -inner product $\langle \cdot, \cdot \rangle$ (or $\langle \cdot, \cdot \rangle_E$ if we deal with several modules). E is equipped with the norm induced by its inner product. However, we do not assume that E is complete. A typical example is, given an index set $\mathcal{I}$, the module

$$\mathcal{C}_{\mathcal{I}}(\mathcal{B}) = \left\{ (x_\alpha)_{\alpha \in \mathcal{I}} \subset \mathcal{B} : \sup_{S \subset \mathcal{I} \text{ finite}} \left\| \sum_{\alpha \in S} x_\alpha^* x_\alpha \right\|_{\mathcal{B}} < \infty \right\}$$

with inner product $\langle x, y \rangle = \lim_S \sum_{\alpha \in S} x_\alpha^* y_\alpha$ which is well defined as a weak* limit over finite subsets of $\mathcal{I}$ for inclusion. By [21,30], a general module E can be embedded into a self-dual Hilbert module and there exist an index set $\mathcal{I}$ and a right $\mathcal{B}$ -module map $u = (u_\alpha)_{\alpha \in \mathcal{I}} : E \rightarrow \mathcal{C}_{\mathcal{I}}(\mathcal{B})$ so that $\langle x, y \rangle = \langle u(x), u(y) \rangle$.

Given $2 \leq p < \infty$, we introduce a norm on the amalgamated tensor product $E \otimes_{\mathcal{B}} L_p(\mathcal{M})$ as follows:

$$\|x\|_{E^c \otimes_p L_p(\mathcal{M})} = \left\| \left(\sum_{i,j=1}^n a_i^* \langle x_i, x_j \rangle a_j \right)^{1/2} \right\|_{L_p(\mathcal{M})}$$

for $x = \sum_{i=1}^n x_i \otimes a_i \in E \otimes_{\mathcal{B}} L_p(\mathcal{M})$. Equipped with this norm, $E \otimes_{\mathcal{B}} L_p(\mathcal{M})$ is denoted by $E^c \otimes_p L_p(\mathcal{M})$ (the superscript c refers to column). In [18], this norm is denoted by $\|\sum_i |x_i\rangle a_i\|_p$. Via the above concrete embedding of E into $\mathcal{C}_{\mathcal{I}}(\mathcal{B})$, the map

$$u \otimes \text{Id}_{L_p(\mathcal{M})} : E^c \otimes_{\mathcal{B}} L_p(\mathcal{M}) \rightarrow \mathcal{C}_{\mathcal{I}}(\mathcal{B}) \otimes_{\mathcal{B}} L_p(\mathcal{M}) \subset L_p(B(\ell_2(\mathcal{I})) \overline{\otimes} \mathcal{M})$$

is then an isometry that allows us to view $E^c \otimes_p L_p(\mathcal{M})$ as a subspace of the column subspace of the last space. This also fully justifies that $\|\cdot\|_{E^c \otimes_p L_p(\mathcal{M})}$ is indeed a norm.

Similarly, given F a left Hilbert $\mathcal{B}$ -module, we define a norm on $L_p(\mathcal{M}) \otimes_{\mathcal{B}} F$ by

$$\|y\|_{L_p(\mathcal{M}) \otimes_p F^r} = \left\| \left(\sum_{i,j=1}^n b_i \langle y_i, y_j \rangle b_j^* \right)^{1/2} \right\|_{L_p(\mathcal{M})}$$

for $y = \sum_{i=1}^n b_i \otimes y_i$. This norm is denoted by $\|\sum_i b_i \langle y_i | \cdot \|_p$ in [18]. As above, $L_p(\mathcal{M}) \otimes_p F^r$ can be identified with a subspace of the row subspace of $L_p(B(\ell_2(\mathcal{J})) \overline{\otimes} \mathcal{M})$ for some index set $\mathcal{J}$.

We can gather the two definitions together and introduce a norm on $E \otimes_{\mathcal{B}} L_p(\mathcal{M}) \otimes_{\mathcal{B}} F$; the resulting space is denoted by $E^c \otimes_p L_p(\mathcal{M}) \otimes_p F^r$, it isometrically embeds into $L_p(B(\ell_2(\mathcal{J}, \mathcal{I})) \overline{\otimes} \mathcal{M})$. This is independent of the choice of the sets $\mathcal{I}, \mathcal{J}$.

We will need extra notions when we assume that E and F are also bimodules (i.e., $\mathcal{B} \subset \mathcal{L}_{\mathcal{B}}(E)$, the algebra of adjointable right $\mathcal{B}$ -modular maps on E , and $\mathcal{B} \subset {}_{\mathcal{B}}\mathcal{L}(F)$, the algebra of adjointable left $\mathcal{B}$ -modular maps on F). Given E' another right Hilbert $\mathcal{B}$ -module, we can consider $E' \otimes_{\mathcal{B}} E$ as a right $\mathcal{B}$ -module with the internal inner product; consequently, we have the space $(E' \otimes_{\mathcal{B}} E)^c \otimes_p L_p(\mathcal{M})$. Note that the norm of $(E' \otimes_{\mathcal{B}} E)^c \otimes_p L_p(\mathcal{M})$ coincides with that of $E'^c \otimes_p (E^c \otimes_p L_p(\mathcal{M}))$ when viewing $E^c \otimes_p L_p(\mathcal{M})$ in the column subspace of $L_p(B(\ell_2(\mathcal{I})) \overline{\otimes} \mathcal{M})$ as above. Similar constructions apply to the row case too. It is clear that all these operations are naturally associative.

Remark 3.1. We will often use without any reference that the above norms are injective. We mean that if $E' \subset E$, $F' \subset F$ are submodules and $(\mathcal{M}', \tau) \subset (\mathcal{M}, \tau)$ is a semifinite von Neumann subalgebra, then $E'^c \otimes_p L_p(\mathcal{M}') \otimes_p F'^r$ isometrically sits in $E^c \otimes_p L_p(\mathcal{M}) \otimes_p F^r$.

Proposition 3.2. Assume that $T : E \rightarrow E$ is a bounded right $\mathcal{B}$ -modular map. Then

$$\|T \otimes \text{Id}_{L_p(\mathcal{M})} : E^c \otimes_p L_p(\mathcal{M}) \rightarrow E^c \otimes_p L_p(\mathcal{M})\| \leq \|T\|.$$

If additionally E is equipped with a left $\mathcal{B}$ -action and T is $\mathcal{B}$ -bimodular, then for any right Hilbert $\mathcal{B}$ -module E'

$$\|\mathrm{Id}_{E'} \otimes T \otimes \mathrm{Id}_{L_p(\mathcal{M})}\|_{B((E' \otimes_{\mathcal{B}} E)^c \otimes_p L_p(\mathcal{M}))} \leq \|T\|.$$

Similar statements hold for left modules too.

Proof. By [30, Theorem 2.8], we have $\langle T(x), T(x) \rangle \leq \|T\|^2 \langle x, x \rangle$ for $x \in E$. Let $x_i \in E$ and $b_i \in \mathcal{B}$ for $1 \leq i \leq n$. Then

$$\left\langle \sum_i T(x_i) b_i, \sum_i T(x_i) b_i \right\rangle \leq \|T\|^2 \left\langle \sum_i x_i b_i, \sum_i x_i b_i \right\rangle,$$

so we can reinterpret this inequality in the matrix algebra $\mathbb{M}_n(\mathcal{B})$ as

$$0 \leq (\langle T(x_i), T(x_j) \rangle)_{i,j} \leq \|T\|^2 (\langle x_i, x_j \rangle)_{i,j}.$$

The latter immediately implies

$$\left(\sum_{i,j} a_i^* \langle T(x_i), T(x_j) \rangle a_j \right)^{1/2} \leq \|T\|^2 \left(\sum_{i,j} a_i^* \langle x_i, x_j \rangle a_j \right)^{1/2}$$

for any $a_i \in L_p(\mathcal{M})$. This yields the first assertion.

The second follows from the same argument by noticing that $\mathrm{Id}_{E'} \otimes T$ is bounded on the right Hilbert $\mathcal{B}$ -module $E' \otimes_{\mathcal{B}} E$ with norm $\|T\|$. Indeed, for finite families $x_i \in E$ and $y_i \in E'$ with $1 \leq i \leq n$

$$\left\langle \sum_i y_i \otimes T(x_i), \sum_i y_i \otimes T(x_i) \right\rangle_{E' \otimes_{\mathcal{B}} E} = \sum_{i,j} \langle T(x_i), \langle y_i, y_j \rangle_{E'} T(x_j) \rangle_E.$$

As a positive element in $\mathbb{M}_n(\mathcal{B})$, the matrix $(\langle y_i, y_j \rangle_{E'})_{i,j}$ can be written as

$$(\langle y_i, y_j \rangle_{E'})_{i,j} = \sum_k (b_{k,i}^* b_{k,j})_{i,j} \quad \text{with } b_{k,j} \in \mathcal{B}.$$

Thus we deduce

$$\begin{aligned} \left\langle \sum_i y_i \otimes T(x_i), \sum_i y_i \otimes T(x_i) \right\rangle_{E' \otimes_{\mathcal{B}} E} &\leq \|T\|^2 \sum_k \left\langle \sum_i b_{k,i} x_i, \sum_i b_{k,i} x_i \right\rangle_E \\ &= \|T\|^2 \left\langle \sum_i y_i \otimes x_i, \sum_i y_i \otimes x_i \right\rangle_{E' \otimes_{\mathcal{B}} E}. \end{aligned}$$

This is the desired boundedness of $\mathrm{Id}_{E'} \otimes T$ on $E' \otimes_{\mathcal{B}} E$. Thus the proposition is proved. $\square$

Thanks to the identifications recalled at the beginning of this section, the previous proposition immediately implies the following

Remark 3.3. If $T : E \rightarrow E$ is a bounded right $\mathcal{B}$ -modular map, then for any left $\mathcal{B}$ -module F

$$\|T \otimes \text{Id}_{L_p(\mathcal{M})} \otimes \text{Id}_F : E^c \otimes_p L_p(\mathcal{M}) \otimes_p F^r \rightarrow E^c \otimes_p L_p(\mathcal{M}) \otimes_p F^r\| \leq \|T\|.$$

A similar statement holds for bimodules, namely, $\text{Id}_{E'} \otimes T \otimes \text{Id}_{L_p(\mathcal{M})} \otimes \text{Id}_F$ is bounded on $(E' \otimes_{\mathcal{B}} E)^c \otimes_p L_p(\mathcal{M}) \otimes_p F^r$ with norm less than or equal to $\|T\|$ if additionally E is equipped with a left $\mathcal{B}$ -action and T is $\mathcal{B}$ -bimodular.

Remark 3.4. Proposition 3.2 is the modular version of the fact that the row and column p -operator spaces are homogeneous (i.e., when $\mathcal{B} = \mathbb{C}$ and E and F are just Hilbert spaces).

A typical situation in which we will apply the previous results is the case when $E = F = \mathcal{A}$ with $\mathcal{A}$ a finite von Neumann algebra containing $\mathcal{B}$. The right inner product on $\mathcal{A}$ is $\langle x, y \rangle = \mathbb{E}(x^*y)$ and the left $\langle\langle x, y \rangle\rangle = \mathbb{E}(xy^*)$, $\mathbb{E}$ being the trace preserving conditional expectation from $\mathcal{A}$ onto $\mathcal{B}$. In this case, we have $\mathcal{A}^c \otimes_2 L_2(\mathcal{B}) = L_2(\mathcal{A})$ isometrically. Thus if $T : \mathcal{A} \rightarrow \mathcal{A}$ is right $\mathcal{B}$ -modular and bounded for $\langle \cdot, \cdot \rangle$, it automatically extends to a map $\tilde{T} : L_2(\mathcal{A}) \rightarrow L_2(\mathcal{A})$ with $\|\tilde{T}\|_{B(L_2(\mathcal{A}))} \leq \|T\|$. Since $\|\langle x, x \rangle\|_{\mathcal{B}} = \sup_{b \in L_2(\mathcal{B}), \|b\|_2 \leq 1} \|xb\|_2$ for $x \in \mathcal{A}$, we actually have $\|\tilde{T}\|_{B(L_2(\mathcal{A}))} = \|T\|$. We may still write T instead of $\tilde{T}$.

We state this fact as a lemma for later use.

Lemma 3.5. *Let $(\mathcal{A}, \tau)$ be a finite von Neumann and $\mathcal{B} \subset \mathcal{A}$ a von Neumann subalgebra with the associated conditional expectation $\mathbb{E}$. Assume that $\mathcal{A}$ is equipped with the right $\mathcal{B}$ -module inner product $\langle x, y \rangle = \mathbb{E}(x^*y)$. Then any bounded right $\mathcal{B}$ -modular map $T : \mathcal{A} \rightarrow \mathcal{A}$ extends to a bounded map on $L_2(\mathcal{A})$ with $\|T\|_{B(L_2(\mathcal{A}))} = \|T\|$.*

If $T : \mathcal{A} \rightarrow \mathcal{A}$ is completely positive and leaves $\mathcal{B}$ invariant, then it satisfies the above conditions. If additionally $\mathbb{E} \circ T \leq \mathbb{E}$, then T also extends to a completely bounded map on $L_p(\mathcal{A})$.

We will need the following

Proposition 3.6. *Let $T : L_p(\mathcal{M}) \rightarrow L_p(\mathcal{M})$ be a completely bounded map. Then for any right Hilbert $\mathcal{B}$ -module E and left Hilbert $\mathcal{B}$ -module F we have*

$$\|\text{Id}_E \otimes T \otimes \text{Id}_F\|_{\text{cb}(E^c \otimes_p L_p(\mathcal{M}) \otimes_p F^r)} \leq \|T\|_{\text{cb}}.$$

Proof. This is a direct consequence of the fact that identifying $E^c \otimes_p L_p(\mathcal{M}) \otimes_p F^r$ with a subspace of $L_p(B(\ell_2(\mathcal{J}, \mathcal{I})) \overline{\otimes} \mathcal{M})$, then $\text{Id}_E \otimes T \otimes \text{Id}_F$ acts like $\text{Id} \otimes T$. $\square$

4. Multipliers on amalgamated free products

This section is the core of the article and contains our major novelty. The principal result is Theorem 4.9 that will imply Theorem 1.3 by iteration. Throughout this section, $(\mathcal{A}_i, \tau_i)_{i \in I}$ will denote a family of finite von Neumann algebras containing $\mathcal{B}$ as a common subalgebra. We will use notation introduced in section 2 on amalgamated free products, in particular, $(\mathcal{A}, \tau) = *_{i \in I, \mathcal{B}} (\mathcal{A}_i, \tau_i)$.

4.1. An intermediate result

Given a family $\pi = (\pi_i)_{i \in I}$ of $*$ -representations $\pi_i : \mathcal{A}_i \rightarrow \mathcal{A}_i$ such that $\pi_i(b) = b$ for all $b \in \mathcal{B}$ and $\mathbb{E} \circ \pi_i = \mathbb{E}$, we introduce a linear map T_π on $\mathcal{W}$ by $T_\pi(b) = b$ for $b \in \mathcal{B}$ and

$$T_\pi(a_1 \otimes \cdots \otimes a_n) = \pi_{i_1}(a_1) \otimes a_2 \otimes \cdots \otimes a_n$$

for $n \geq 1$ and $a_1 \otimes \cdots \otimes a_n \in \mathcal{W}_{\underline{i}}$ where $\underline{i} = (i_1, \dots, i_n)$ and $i_1 \neq \cdots \neq i_n$. Define T_π^{op} on $\mathcal{W}$ as $T_\pi^{\text{op}}(x) = T_\pi(x^*)^*$. Note that both T_π and T_π^{op} commute with the projections P_n .

We aim to show that T_π extends to a bounded map on $L_p(\mathcal{A})$ for $1 < p < \infty$. We will first need some elementary free algebraic facts.

Lemma 4.1. *Let $g \in \mathcal{W}_l$, $h \in \mathcal{W}_n$ with $l, n \geq 0$.*

- i) *If $l > n$, then $T_\pi(gh) = T_\pi(g)h$.*
- ii) *If $n > l$, then $T_\pi^{\text{op}}(gh) = gT_\pi^{\text{op}}(h)$.*
- iii) *If $l = n$, then $P_{\geq 2}[T_\pi(gh)] = P_{\geq 2}[T_\pi(g)h]$ and $P_{\geq 2}[T_\pi^{\text{op}}(gh)] = P_{\geq 2}[gT_\pi^{\text{op}}(h)]$.*

Proof. One can assume that g and h are elementary tensors. The first item is then clear as the first letter of g cannot be canceled if $l > n$. The second is obtained by passing to adjoints. Similarly, if $n = l$, $P_{\geq 2}(gh)$ is a sum of elementary tensors that all start with the first letter of g and end with the last letter of h , up to a multiplication by an element of $\mathcal{B}$. $\square$

The following Cotlar type formula immediately follows from the previous lemma.

Lemma 4.2. *For $g, h \in \mathcal{W}$ we have*

$$P_{\geq 2}[T_\pi(g)T_\pi^{\text{op}}(h)] = P_{\geq 2}[T_\pi(gT_\pi^{\text{op}}(h)) + T_\pi^{\text{op}}(T_\pi(g)h) - T_\pi T_\pi^{\text{op}}(gh)]. \quad (4)$$

Proof. By linearity, it suffices to show the formula for $g \in \mathcal{W}_l$, $h \in \mathcal{W}_n$. Then it remains to check it by using Lemma 4.1 according to the different cases. We omit the details. $\square$

Lemma 4.3. *Let $g = g_1 \otimes \cdots \otimes g_l \in \mathcal{W}_l$, $h = h_n \otimes \cdots \otimes h_1 \in \mathcal{W}_n$ with $l, n \geq 0$, and let $g' = g_2 \otimes \cdots \otimes g_l$ and $h' = h_n \otimes \cdots \otimes h_2$.*

- i) If $n > l$, then $P_1(gh) = \delta_{n,l+1} \mathbb{E}(gh')h_1$.
- ii) If $l > n$, then $P_1(gh) = \delta_{n+1,l} g_1 \mathbb{E}(g'h)$.
- iii) If $l = n$, then $P_1(gh) = P_1(g_1 \mathbb{E}(g'h')h_1)$.

Proof. This proof is easy. Let us verify only i). If $n > l + 1$, then both $P_1(gh)$ and $\mathbb{E}(gh')h_1$ vanish. The case $n = l + 1$ is checked by induction on l and n thanks to the following simplification formula:

$$gh = g^- \otimes \widehat{(g_l h_n)} \otimes h^- + g^- \mathbb{E}(g_l h_n) h^-$$

where $g^- = g_1 \otimes \cdots \otimes g_{l-1}$ and $h^- = h_{n-1} \otimes \cdots \otimes h_1$. It then follows that $P_1(gh) = P_1(g^- \mathbb{E}(g_l h_n) h^-)$. $\square$

We will also need the notation of paraproducts on $\mathcal{W} \times \mathcal{W}$ in the manner of [25] to prove the boundedness of T_π . Given $x, y \in \mathcal{W}$, let

$$\begin{aligned} x \overset{1,0}{\dagger} y &= \mathbb{E}_\varepsilon [H_\varepsilon^{\text{op}}(x H_\varepsilon^{\text{op}}(y))], \\ x \overset{0,1}{\dagger} y &= \mathbb{E}_\varepsilon [H_\varepsilon(H_\varepsilon(x)y)], \\ x \overset{1,1}{\dagger} y &= \mathbb{E}_\varepsilon \mathbb{E}_{\varepsilon'} [H_\varepsilon H_{\varepsilon'}^{\text{op}}(H_\varepsilon(x) H_{\varepsilon'}^{\text{op}}(y))], \\ x \overset{1,0}{\dagger} y &= xy - \mathbb{E}_\varepsilon [H_\varepsilon^{\text{op}}(x H_\varepsilon^{\text{op}}(y))], \\ x \overset{0,1}{\dagger} y &= xy - \mathbb{E}_\varepsilon [H_\varepsilon(H_\varepsilon(x)y)], \end{aligned}$$

where H_ε and $H_\varepsilon^{\text{op}}$ are the free Hilbert transforms defined in (3) and $\mathbb{E}_\varepsilon, \mathbb{E}_{\varepsilon'}$ are the expectation over all possible choices of symmetric independent signs $\varepsilon = (\varepsilon_i), \varepsilon' = (\varepsilon'_i)$.

Note that when x and y are elementary tensors, then $x \overset{1,0}{\dagger} y$ collects in xy the parts that end in the same algebra as y and $x \overset{1,0}{\dagger} y$ collects in xy the parts that do not end in the same algebra as y , that is, all letters in y must have been simplified. Similarly, $x \overset{0,1}{\dagger} y$ collects in xy the parts that start in the same algebra as x , $x \overset{0,1}{\dagger} y$ collects in xy the parts that do not start in the same algebra as x , and $x \overset{1,1}{\dagger} y$ collects in xy the parts that start in the same algebra as x and end in the same algebra as y . We have the identity

$$xy = x \overset{1,0}{\dagger} y + x \overset{0,1}{\dagger} y + x \overset{1,1}{\dagger} y.$$

By Lemma 2.2, all these bilinear operators are bounded from $L_p(\mathcal{A}) \times L_q(\mathcal{A})$ to $L_r(\mathcal{A})$ for all $1 < r, p, q < \infty, \frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.

The following is a complement to the Cotlar formula (4).

Lemma 4.4. For $g, h \in \mathcal{W}$ we have

$$P_1(T_\pi(g)T_\pi^{\text{op}}(h)) = T_\pi[P_1(g \overset{1,0}{\uparrow} T_\pi^{\text{op}}(h))] + T_\pi^{\text{op}}[P_1(T_\pi(g) \overset{0,1}{\uparrow} h)] + T_\pi[P_1(g \overset{1,1}{\uparrow} h)]. \quad (5)$$

Proof. Note that the identity is bilinear, we will only check the formula for elementary tensors $g \in \mathcal{W}_l, h \in \mathcal{W}_n$ with $l, n \geq 0$ according to the three cases in Lemma 4.3 and use the notations g', h' there.

For the case i), we can assume $n = l + 1$, otherwise all terms are 0. Then

$$P_1(T_\pi(g)T_\pi^{\text{op}}(h)) = \mathbb{E}(T_\pi(g)h') \pi_{j_1}(h_1).$$

On the right side, we have $P_1(g \overset{1,1}{\uparrow} h) = 0$ and $g \overset{1,0}{\uparrow} h = 0$ as the last letter of h in gh is not canceled. To deal with $g \overset{0,1}{\uparrow} h$, note that

$$H_\varepsilon[P_1(H_\varepsilon(g)h)] = \varepsilon_{i_1}\varepsilon_{j_1}P_1(gh),$$

with $i_1 = 0$ if $l = 0$. Note that $\mathbb{E}(gh') = 0$ unless $i_1 = j_2$, but then $j_1 \neq j_2$ and $\mathbb{E}_\varepsilon(\varepsilon_{i_1}\varepsilon_{j_1}) = 0$. It follows that $P_1(g \overset{0,1}{\uparrow} h) = P_1(gh)$. Hence the right hand side of (5) is exactly $T_\pi^{\text{op}}[P_1(T_\pi(g)h)] = \mathbb{E}(T_\pi(g)h')\pi_{j_1}(h_1)$.

The case ii) is obtained from i) by passing to adjoints.

For the case iii), we have $P_1(gh) = 0$ unless $i_k = j_k$ for all $1 \leq k \leq n$. Hence we can assume that $i_1 = j_1$. Then noting that

$$H_\varepsilon[P_1(H_\varepsilon(g)h)] = \varepsilon_{i_1}^2 P_1(gh) = P_1(gh),$$

we deduce $P_1(g \overset{0,1}{\uparrow} h) = 0$; by symmetry, $P_1(g \overset{1,0}{\uparrow} h) = 0$ too. Thus the right hand side of (5) becomes

$$T_\pi[P_1(g \overset{1,1}{\uparrow} h)] = T_\pi[P_1(g_1\mathbb{E}(g'h')h_1)] = \pi_{i_1}[P_1(g_1\mathbb{E}(g'h')h_1)].$$

However, the left hand side is

$$P_1[\pi_{i_1}(g_1)\mathbb{E}(g'h')\pi_{i_1}(h_1)] = P_1[\pi_{i_1}(g_1\mathbb{E}(g'h')h_1)] = \pi_{i_1}[P_1(g_1\mathbb{E}(g'h')h_1)]$$

because π_{i_1} is a $*$ -representation that leaves the elements of $\mathcal{B}$ invariant and $P_1\pi_{i_1} = \pi_{i_1}P_1$. (5) is thus proved in the case $l = n$ too. $\square$

The following is an intermediate result to Theorem 4.9, it will be the key for the reduction formula in Theorem 4.6 below.

Theorem 4.5. The map T_π extends to a completely bounded map on $L_p(\mathcal{A})$ for all $2 \leq p < \infty$ with cb-norm majorized by a constant depending only on p .

Proof. As the π_i 's are trace preserving and leave $\mathcal{B}$ invariant, T_π is an isometry on $\mathcal{W}$ for the $L_2(\mathcal{A})$ -norm, thus it extends to an isometry on $L_2(\mathcal{A})$. We need only to prove the boundedness of T_π since the complete boundedness will then be automatic thanks to the usual trick of replacing $\mathcal{B}$ by the matrix algebra $\mathbb{M}_n(\mathcal{B})$.

As in [25], we show that the L_p -boundedness of T_π implies its L_{2p} -boundedness for $2 \leq p < \infty$. Starting with $p = 2$, using iteration and interpolation, we will deduce the assertion for the full range $2 < p < \infty$. In the following, we will denote by γ_p the norm of T_π and T_π^{op} on $\mathcal{W}$ equipped with the L_p -norm.

For $x \in \mathcal{W}$ we write

$$\begin{aligned} T_\pi(x)T_\pi(x)^* &= T_\pi(x)T_\pi^{\text{op}}(x^*) \\ &= P_{\geq 2}[T_\pi(x)T_\pi^{\text{op}}(x^*)] + P_1[T_\pi(x)T_\pi^{\text{op}}(x^*)] + \mathbb{E}[T_\pi(x)T_\pi^{\text{op}}(x^*)]. \end{aligned}$$

By (4) and the fact that $P_{\geq 2}$ has norm less than 5 on $L_p(\mathcal{A})$, we have

$$\|P_{\geq 2}[T_\pi(x)T_\pi^{\text{op}}(x^*)]\|_p \leq 5 [2\gamma_p\|x\|_{2p}\|T_\pi(x)\|_{2p} + \gamma_p^2\|x\|_{2p}^2].$$

On the other hand, by [25, Proposition 3.14] or Lemma 2.2, the paraproducts $\overset{i,j}{\dagger}$ are bounded from $L_{2p}(\mathcal{A}) \times L_{2p}(\mathcal{A})$ to $L_p(\mathcal{A})$ with norm less than η_p . Thus using (5) and the fact that P_1 has norm less than 3 on $L_p(\mathcal{A})$, we get

$$\|P_1[T_\pi(x)T_\pi^{\text{op}}(x^*)]\|_p \leq 3\eta_p\gamma_p[2\|x\|_{2p}\|T_\pi(x)\|_{2p} + \|x\|_{2p}^2].$$

Clearly,

$$\|\mathbb{E}[T_\pi(x)T_\pi^{\text{op}}(x^*)]\|_p = \|\mathbb{E}(xx^*)\|_p \leq \|x\|_p^2.$$

So combining all the estimates yields

$$\|T_\pi(x)\|_{2p}^2 \leq \gamma_p(10 + 6\eta_p)\|x\|_{2p}\|T_\pi(x)\|_{2p} + (1 + 3\eta_p\gamma_p + 5\gamma_p^2)\|x\|_p^2.$$

It follows that

$$\|T_\pi(x)\|_{2p} \leq C\gamma_p\eta_p\|x\|_{2p}$$

for some absolute constant C , whence $\gamma_{2p} \leq C\gamma_p\eta_p$. This finishes the proof. $\square$

4.2. A length reduction formula

We show here how to recursively estimate the L_p -norm of an element in $\mathcal{W}$ in the spirit of [18]. The space $\mathcal{W}$ is naturally a right Hilbert $\mathcal{B}$ -module with inner product $\langle x, y \rangle = \mathbb{E}(x^*y)$ and also a left Hilbert $\mathcal{B}$ -module with $\langle\langle x, y \rangle\rangle = \mathbb{E}(xy^*)$. The same

holds for $\mathcal{W}_1$ and $\dot{\mathcal{W}}$ as submodules. A typical element in $\mathcal{W}$ can be written as a finite sum

$$x = x_0 + x_1 + \sum_{i,\alpha} a_i(\alpha) \otimes b_i(\alpha) \stackrel{\text{def}}{=} x_0 + x_1 + z, \quad (6)$$

where $x_0 \in \mathcal{B}$, $x_1 \in \mathcal{W}_1$ and $a_i(\alpha) \in \dot{\mathcal{A}}_i$ and $b_i(\alpha) \in \dot{\mathcal{W}}$ with $L_i(b_i(\alpha)) = 0$.

The following result extends the main result of [18] on homogeneous polynomials to any polynomials, it is the key tool for the argument in the next subsection.

Theorem 4.6. *With the notation above, for $2 \leq p < \infty$, we have (considering $z \in \mathcal{W}_1 \otimes \dot{\mathcal{W}}$)*

$$\|x\|_p \approx_p \|x_0\|_p + \|x_1\|_p + \|z\|_{\mathcal{W}_1^c \otimes_p L_p(\mathcal{A})} + \|z\|_{L_p(\mathcal{A}) \otimes_p \dot{\mathcal{W}}}. \quad (7)$$

Proof. By the boundedness of the projections P_0 and P_1 , it suffices to prove the estimate for z . To this end, we consider two copies of $\mathcal{A}$, and put a superscript to distinguish them. We use associativity of the free product to write

$$\mathcal{A}^{(1)} *_{\mathcal{B}} \mathcal{A}^{(2)} = (*_{i \in I, \mathcal{B}} \mathcal{A}_i^{(1)}) *_{\mathcal{B}} (*_{i \in I, \mathcal{B}} \mathcal{A}_i^{(2)}) = *_{i \in I, \mathcal{B}} (\mathcal{A}_i^{(1)} *_{\mathcal{B}} \mathcal{A}_i^{(2)}) \stackrel{\text{def}}{=} *_{i \in I, \mathcal{B}} \widetilde{\mathcal{A}}_i.$$

Since the traces are compatible, $\|x\|_{L_p(\mathcal{A})} = \|x^{(1)}\|_{L_p(\mathcal{A}^{(1)} *_{\mathcal{B}} \mathcal{A}^{(2)})}$ for every $x \in L_p(\mathcal{A})$.

For each i , we define the swap map π_i on $\widetilde{\mathcal{A}}_i$ by

$$\begin{aligned} a_1^{(1)} \otimes a_2^{(2)} \otimes a_3^{(1)} \otimes \cdots &\mapsto a_1^{(2)} \otimes a_2^{(1)} \otimes a_3^{(2)} \otimes \cdots \\ a_1^{(2)} \otimes a_2^{(1)} \otimes a_3^{(2)} \otimes \cdots &\mapsto a_1^{(1)} \otimes a_2^{(2)} \otimes a_3^{(1)} \otimes \cdots \end{aligned}$$

for $\mathcal{B}$ -centered elements $a_k^{(j)}$ and by $\pi_i(b) = b$ for $b \in \mathcal{B}$. It is clear that π_i is a $*$ -representation, $\mathbb{E}$ -preserving and leaves the elements of $\mathcal{B}$ invariant. Thus, noticing that $\pi_i^2 = \text{Id}_{\widetilde{\mathcal{A}}_i}$, we use Theorem 4.5 to get

$$\left\| \sum_{i,\alpha} a_i(\alpha) \otimes b_i(\alpha) \right\|_{L_p(\mathcal{A})} \simeq_p \left\| \sum_{i,\alpha} a_i(\alpha)^{(2)} \otimes b_i(\alpha)^{(1)} \right\|_{L_p(\mathcal{A}^{(1)} *_{\mathcal{B}} \mathcal{A}^{(2)})}.$$

The element $\sum_{i,\alpha} a_i(\alpha)^{(2)} \otimes b_i(\alpha)^{(1)}$ is homogeneous of degree 2 with respect to the length of $\mathcal{A}^{(1)} *_{\mathcal{B}} \mathcal{A}^{(2)}$. Thus by [18, Theorem B]

$$\begin{aligned} \left\| \sum_{i,\alpha} a_i(\alpha)^{(2)} \otimes b_i(\alpha)^{(1)} \right\|_{L_p(\mathcal{A}^{(1)} *_{\mathcal{B}} \mathcal{A}^{(2)})} &\simeq \left\| \sum_{i,\alpha} |a_i(\alpha)^{(2)} \rangle \langle b_i^{(1)}(\alpha) | \right\|_p \\ &\quad + \left\| \sum_{i,\alpha} a_i(\alpha)^{(2)} \langle b_i^{(1)}(\alpha) | \right\|_p. \end{aligned}$$

Since taking copies clearly does not change norms, we have

$$\begin{aligned} \left\| \sum_{i,\alpha} |a_i(\alpha)^{(2)} \rangle b_i^{(1)}(\alpha) \right\|_p &= \left\| \sum_{i,\alpha} a_i(\alpha)^{(2)} \otimes b_i^{(1)}(\alpha) \right\|_{(\mathcal{W}_1^{(2)})^c \otimes_p L_p(\mathcal{A}^{(1)} *_{\mathcal{B}} \mathcal{A}^{(2)})} \\ &= \left\| \sum_{i,\alpha} a_i(\alpha) \otimes b_i(\alpha) \right\|_{\mathcal{W}_1^c \otimes_p L_p(\mathcal{A})}. \end{aligned}$$

Similarly,

$$\left\| \sum_{i,\alpha} a_i(\alpha)^{(2)} \langle b_i^{(1)}(\alpha) \right\|_p = \left\| \sum_{i,\alpha} a_i(\alpha) \otimes b_i(\alpha) \right\|_{L_p(\mathcal{A}) \otimes_p \dot{\mathcal{W}}^r}.$$

This concludes the proof of the theorem. $\square$

Remark 4.7. We could also have used [18, Theorem C] to make some terms more explicit. Namely,

$$\|x_1\|_p \simeq \|(\mathbb{E}(x_1^* x_1))^{1/2}\|_p + \left(\sum_i \|L_i(x_1)\|_p^p \right)^{1/p} + \|(\mathbb{E}(x_1 x_1^*))^{1/2}\|_p,$$

and

$$\begin{aligned} \|z\|_{L_p(\mathcal{A}) \otimes_p \dot{\mathcal{W}}^r} &\simeq_p \|z\|_{\mathcal{W}_1^c \otimes_p \dot{\mathcal{W}}^r} + \|(\mathbb{E}(z z^*))^{1/2}\|_p \\ &\quad + \left(\sum_i \left\| \left(\sum_{\alpha, \beta} a_i(\alpha) \mathbb{E}(b_i(\alpha) b_i(\beta)^*) a_i(\beta) \right)^{1/2} \right\|_p^p \right)^{1/p}. \end{aligned}$$

Here $\mathcal{W}_1^c \otimes_p \dot{\mathcal{W}}^r$ has to be understood as $\mathcal{W}_1^c \otimes_p L_p(\mathcal{B}) \otimes_p \dot{\mathcal{W}}^r$.

Remark 4.8. It is now rather easy to get an analogue of [18, Theorem C]. The norm of the last term $\|z\|_{\mathcal{W}_1^c \otimes_p L_p(\mathcal{A})}$ corresponds to that of an element in $S_p \otimes_p L_p(\mathcal{A})$ and we can formally iterate the argument. Thus, we can write the norm of x in $P_{\geq k}(L_p(\mathcal{A}))$ as a sum of $2k+1$ norms. For simplicity assume $x \in P_{\geq k}(\mathcal{W})$, they are given by

$$\begin{aligned} \|x\|_{\mathcal{W}_l^c \otimes_p \mathcal{W}^r}, \quad 0 \leq l \leq k, \\ \|x\|_{\mathcal{W}_l^c \otimes_p (\oplus_p L_p(\mathcal{A}_i)) \otimes_p \mathcal{W}^r}, \quad 0 \leq l \leq k-2 \\ \|x\|_{\mathcal{W}_{k-1}^c \otimes_p L_p(\mathcal{A})}. \end{aligned}$$

The last one being recursive. We leave the details to the interested reader.

4.3. Maps of the d -th letters and the proof of Theorem 1.3

This subsection contains our principal result that is the key step of the proof of Theorem 1.3. Fix $1 < p < \infty$. Given a family of maps $T_i : L_p(\mathcal{A}_i) \rightarrow L_p(\mathcal{A}_i)$ we will define an associated map of the d -th letters of reduced words in $\mathcal{W}$ for $d \geq 1$. The minimal assumption required for the T_i 's is the following

- (H₁) T_i is $\mathcal{B}$ -bimodular and $T_i(\widehat{L_q(\mathcal{A}_i)}) \subset \widehat{L_q(\mathcal{A}_i)}$ for $q = 2$ and $q = p$.
 (H₂) $T_i : L_q(\mathcal{A}_i) \rightarrow L_q(\mathcal{A}_i)$ is completely bounded and

$$\text{cb}_q = \sup_{i \in I} \|T_i\|_{\text{cb}(L_q(\mathcal{A}_i))} < \infty \quad \text{for } q = 2 \text{ and } q = p.$$

Note that the cb-norm of T_i on $L_2(\mathcal{A}_i)$ coincides with its usual norm since $L_2(\mathcal{A}_i)$ is a homogeneous operator space. On the other hand, it is obvious that

$$\text{cb}_q = \left\| \oplus_i T_i : L_q(\oplus_i \mathcal{A}_i) \rightarrow L_q(\oplus_i \mathcal{A}_i) \right\|_{\text{cb}}.$$

Now we define a linear map $T^{(d)}$ on $\mathcal{W}$ by $T^{(d)}(b) = b$ for $b \in \mathcal{B}$ and

$$T^{(d)}(a_1 \otimes \cdots \otimes a_n) = \begin{cases} a_1 \otimes \cdots \otimes a_{d-1} \otimes T_{i_d}(a_d) \otimes a_{d+1} \otimes \cdots \otimes a_n & \text{if } d \leq n, \\ a_1 \otimes \cdots \otimes a_n & \text{if } d > n \end{cases}$$

for $n \geq 1$ and $a_1 \otimes \cdots \otimes a_n \in \mathcal{W}_{\underline{i}}$ with $\underline{i} = (i_1, \dots, i_n)$ and $i_1 \neq \cdots \neq i_n$. Note that the range of $T^{(d)}$ is not inside $\mathcal{W}$ but clearly in $L_p(\mathcal{A})$.

Theorem 4.9. *Under the hypotheses (H₁) and (H₂), $T^{(d)}$ extends to a completely bounded map on $L_p(\mathcal{A})$ with*

$$\|T^{(d)}\|_{\text{cb}(L_p(\mathcal{A}))} \lesssim_{p,d} \text{cb}_2 + \text{cb}_p.$$

Proof. We start the proof by a crucial observation related to Lemma 3.5. We view $\mathcal{A}_i$ as a right Hilbert $\mathcal{B}$ -module with the inner product $\langle x, y \rangle = \mathbb{E}(x^*y)$. We claim that $\mathbb{E}(T_i(x)^*T_i(y))$ belongs to $\mathcal{B}$ for any $x, y \in \mathcal{A}$. By polarization, we can assume $x = y$. Then for $b \in L_2(\mathcal{A}_i)$, by the modularity of T_i ,

$$\tau[b^*\mathbb{E}(T_i(x)^*T_i(x))b] = \tau[\mathbb{E}(T_i(xb)^*T_i(xb))] = \|T_i(xb)\|_2 \leq \|T_i\| \|xb\|_2.$$

This implies the claim, as well as $\mathbb{E}(T_i(x)^*T_i(x)) \leq \|T_i\| \langle x, x \rangle$. Thus $T_i(\mathcal{A}_i)$ is a $\mathcal{B}$ -bimodule. A similar statement holds when $\mathcal{A}_i$ is viewed as a left $\mathcal{B}$ -module with the inner product $\langle \langle x, y \rangle \rangle = \mathbb{E}(xy^*)$.

We have stated the results in section 3 for maps $T : E \rightarrow E$ but they clearly remain true if $T : E \rightarrow E'$. We can thus apply them to the restriction of T_i to $\mathcal{A}_i$ with range $T_i(\mathcal{A}_i)$. The same holds for direct sums.

An immediate consequence is the boundedness of $T^{(d)}$ on $\mathcal{W}$ equipped with the L_2 -norm, this follows from the orthogonality of the $\mathcal{W}_n$'s and Proposition 3.2 combined with the above observation. So the theorem holds for $p = 2$. On the other hand, by duality, we need only to consider the case $2 < p < \infty$ that will be assumed in the remainder of the proof.

Before going into the core of the proof, we point out that the result of Theorem 4.6 can be applied to $T^{(d)}(x)$ when $x \in \mathcal{W}$ by using an obvious approximation argument. One just need to adapt correctly the modules, replacing $\mathcal{W}_1^c$ by $T^{(1)}(\mathcal{W}_1)^c$ if $d = 1$ and $\dot{\mathcal{W}}^r$ by $T^{(d-1)}(\dot{\mathcal{W}})^r$ if $d \geq 1$.

We continue the proof by induction on d . The main part is the initial step: $d = 1$. By the usual argument of tensoring with the matrix algebras $\mathbb{M}_n$, it suffices to prove the boundedness of $T^{(1)}$. We will apply Theorem 4.6. Let $x = x_0 + x_1 + z \in \mathcal{W}$ as (6).

To deal with $\|T^{(1)}(x_1)\|_p$, we use the Khintchine inequality from Remark 4.7. We have that $T^{(1)}$ is bounded on $L_p(\oplus_i \mathcal{A}_i)$ with norm majorized by cb_p , that is,

$$\left(\sum_i \|L_i(T^{(1)}(x_1))\|_p^p \right)^{1/p} = \left(\sum_i \|T_i(L_i(x_1))\|_p^p \right)^{1/p} \leq \text{cb}_p \left(\sum_i \|L_i(x_1)\|_p^p \right)^{1/p}.$$

On the other hand, thanks to the previous observation, $\oplus_i T_i$ can be viewed as a bounded modular map on $\mathcal{W}_1$ with respect to both inner products $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle$ with norm bounded by cb_2 . Thus by Proposition 3.2,

$$\begin{aligned} \left\| \left(\mathbb{E}[T^{(1)}(x_1)^* T^{(1)}(x_1)] \right)^{1/2} \right\|_p &= \left\| \left(\mathbb{E}[(\oplus_i T_i)(x_1)]^* [\oplus_i T_i](x_1) \right)^{1/2} \right\|_p \\ &\leq \text{cb}_2 \left\| \left(\mathbb{E}(x_1^* x_1) \right)^{1/2} \right\|_p \end{aligned}$$

and similarly for the second inner product. Hence,

$$\|T^{(1)}(x_1)\|_p \lesssim (\text{cb}_2 + \text{cb}_p) \|x_1\|_p.$$

For the remaining part z , note that

$$T^{(1)}(z) = \sum_{i,\alpha} [\oplus_j T_j](a_i(\alpha)) \otimes b_i(\alpha).$$

Thus by the observation that $\oplus_i T_i : \mathcal{W}_1 \rightarrow T^{(1)}(\mathcal{W}_1)$ is $\mathcal{B}$ -bimodular and Proposition 3.2, we again have

$$\left\| \sum_{i,\alpha} T_i(a_i(\alpha)) \otimes b_i(\alpha) \right\|_{T^{(1)}(\mathcal{W}_1)^c \otimes_p L_p(\mathcal{A})} \leq \text{cb}_2 \left\| \sum_{i,\alpha} a_i(\alpha) \otimes b_i(\alpha) \right\|_{\mathcal{W}_1^c \otimes_p L_p(\mathcal{A})}.$$

For the other norm of z , we use Proposition 3.6 to obtain

$$\left\| \sum_{i,\alpha} T_i(a_i(\alpha)) \otimes b_i(\alpha) \right\|_{L_p(\mathcal{A}) \otimes_p \dot{\mathcal{W}}^r} \leq \text{cb}_p \left\| \sum_{i,\alpha} a_i(\alpha) \otimes b_i(\alpha) \right\|_{L_p(\mathcal{A}) \otimes_p \dot{\mathcal{W}}^r}.$$

Thus $T^{(1)}$ extends to a (completely) bounded map on $L_p(\mathcal{A})$.

Now assume that $d \geq 2$ and $T^{(d-1)}$ is completely bounded on $\mathcal{W}$ for the L_p -norm. For $x = x_0 + x_1 + z$ as above, we have

$$T^{(d)}(x_0 + x_1) = x_0 + x_1 \quad \text{and} \quad T^{(d)}(z) = \sum_{i,\alpha} a_i(\alpha) \otimes T^{(d-1)}(b_i(\alpha)).$$

Using the boundedness of $T^{(d-1)}$ on $L_2(\mathcal{A})$ and Proposition 3.2, we have

$$\left\| T^{(d)} \left(\sum_{i,\alpha} a_i(\alpha) \otimes b_i(\alpha) \right) \right\|_{L_p(\mathcal{A}) \otimes_p T^{(d-1)}(\mathcal{W})^r} \leq \text{cb}_2 \left\| \sum_{i,\alpha} a_i(\alpha) \otimes b_i(\alpha) \right\|_{L_p(\mathcal{A}) \otimes_p \mathcal{W}^r}.$$

Similarly, the complete boundedness of $T^{(d-1)}$ on $L_p(\mathcal{A})$ and Proposition 3.6 imply

$$\left\| T^{(d)} \left(\sum_{i,\alpha} a_i(\alpha) \otimes b_i(\alpha) \right) \right\|_{\mathcal{W}_1^c \otimes_p L_p(\mathcal{A})} \lesssim_{p,d} \text{cb}_p \left\| \sum_{i,\alpha} a_i(\alpha) \otimes b_i(\alpha) \right\|_{\mathcal{W}_1^c \otimes_p L_p(\mathcal{A})}.$$

This concludes the induction, and the proof of the theorem too. $\square$

Theorem 1.3 immediately follows from Theorem 4.9.

Proof of Theorem 1.3. For $1 \leq k \leq d$, let $T_k^{(k)}$ be the map $T^{(k)}$ in Theorem 4.9 associated to the family $(T_{k,i})_{i \in I}$. Then

$$T^{Ld} = T_1^{(1)} T_2^{(2)} \dots T_d^{(d)}.$$

This yields the assertion. $\square$

We extend Lemma 2.2 to the Hilbert transform of the d -th letters in the spirit of Theorem 4.9 and Theorem 1.3. Let $\varepsilon = (\varepsilon_i)_{i \in I}$ be a family of elements in the unit ball of $\mathcal{Z}(\mathcal{B})$. Let T_i be the left multiplication map on $L_q(\mathcal{A}_i)$ by ε_i . Clearly, the T_i 's satisfy the hypotheses $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$. Denote the corresponding $T^{(d)}$ by $\mathcal{H}_\varepsilon^{(d)}$. If $d = 1$, this coincides with the free Hilbert transform in (3). More generally, given d let $\varepsilon = (\varepsilon_{j,i})_{1 \leq j \leq d, i \in I}$ be a family in the unit ball of $\mathcal{Z}(\mathcal{B})$. The corresponding map T^{Ld} as in Theorem 1.3 is denoted by $\mathcal{H}_\varepsilon^{Ld}$.

The following is a particular case of Theorem 1.3, it extends [25, Theorem 4.7] to the amalgamated free product case.

Corollary 4.10. *Both $\mathcal{H}_\varepsilon^{(d)}$ and $\mathcal{H}_\varepsilon^{Ld}$ extend to completely bounded maps on $L_p(\mathcal{A})$ for all $1 < p < \infty$ with cb-norms controlled by constants depending only on p and d .*

We conclude this section with the boundedness of some paraproducts that generalize those introduced at the beginning of subsection 4.1. These paraproducts are of independent interest in free analysis.

Let $\varepsilon = (\varepsilon_{j,i})_{j \geq 1, i \in I}$ be an independent family of symmetric random variables with values ± 1 . Let $\mathcal{H}_\varepsilon^{Ld}$ be the map associated to $(\varepsilon_{j,i})_{1 \leq j \leq d, i \in I}$ as in the previous corollary, and let $\mathcal{H}_\varepsilon^{Ld, \text{op}}(x) = [\mathcal{H}_\varepsilon^{Ld}(x^*)]^*$. We use the convention that $\mathcal{H}_\varepsilon^{L0} = \text{Id}$. For $j, k \geq 0$ and $x, y \in \mathcal{W}$, define

$$x \overset{j,k}{\sharp} y = \mathbb{E}_\varepsilon \mathbb{E}_{\varepsilon'} \mathcal{H}_\varepsilon^{Lj} \mathcal{H}_{\varepsilon'}^{Lk, \text{op}} [\mathcal{H}_\varepsilon^{Lj}(x) \mathcal{H}_{\varepsilon'}^{Lk, \text{op}}(y)],$$

where $\mathbb{E}_\varepsilon$ denotes the underlying expectation and ε' is an independent copy of ε . This paraproduct is easily understood for elementary tensors x and y : $x \overset{j,k}{\sharp} y$ then collects all those terms in the development of xy into elementary tensors whose first j letters come from the same algebras of the first j letters of x , and whose last k letters from the same algebras of the last k letters of y .

The previous corollary implies the following

Corollary 4.11. *The paraproduct $\overset{j,k}{\sharp}$ extends to a bounded bilinear map from $L_{2p}(\mathcal{A}) \times L_{2p}(\mathcal{A})$ to $L_p(\mathcal{A})$ for all $1 < p < \infty$ with norm majorized by a constant depending only on p, j and k .*

Remark 4.12. We leave to the reader familiar with Haagerup noncommutative L_p -spaces to extend, with necessary modifications, the results of this section to the type III case, that is, to amalgamated free products of von Neumann algebras equipped with faithful normal states instead of traces.

5. Multipliers on free products of groups

In this section we will first prove Theorems 1.1 and 1.2, then consider Fourier multipliers on free products of general discrete groups. Recall that $\widehat{\Gamma}$ denotes the group von Neumann algebra of a discrete group Γ generated by the left regular representation λ .

5.1. Proofs of Theorems 1.1 and 1.2

We start by the results on the free group $\mathbb{F}_\infty$.

Proof of Theorem 1.2. We will apply Theorem 1.3 to the special case where $\mathcal{A}_i = \widehat{\mathbb{Z}} = L_\infty(\mathbb{T})$ for all $i \in \mathbb{N}$. Fix a family $z = (z_{j,i})_{1 \leq j \leq d, 1 \leq i < \infty}$ of complex numbers with modulus 1. Define $T_{j,i}$ to be the measure preserving $*$ -representation on $\mathcal{A}_i$ given by $T_{j,i}(\lambda(n)) = z_{j,i}^n \lambda(n)$ for any $n \in \mathbb{Z}$, or equivalently in terms of the generator $\zeta \in L_\infty(\mathbb{T})$, $T_{j,i}(\zeta^n) = z_{j,i}^n \zeta^n$. $T_{j,i}$ extends to a complete isometry on $L_p(\mathbb{T})$ for $1 \leq p \leq \infty$. The corresponding map T^{Ld} is exactly the map α_z^{Ld} in Theorem 1.2. Thus Theorem 1.3 implies that α_z^{Ld} is completely bounded on $L_p(\widehat{\mathbb{F}}_\infty)$ for $1 < p < \infty$, whence Theorem 1.2. $\square$

Proof of Theorem 1.1. We will use α_z^{Ld} in the previous proof for the special case where $z_{j,i} = z_j$ for all i , and write $\alpha_z = \alpha_z^{Ld}$ for $z \in \mathbb{T}^d$. Thus α is a uniformly completely bounded action of $\mathbb{T}^d$ on $L_p(\widehat{\mathbb{F}}_\infty)$. We then easily deduce Theorem 1.1 by the standard transference argument as presented in [4]. Let us give the details.

Let m be a Hörmander-Mikhlin multiplier on $\mathbb{Z}^d$, that is, m satisfies (1). Then the associated Fourier multiplier T_m on $L_p(\mathbb{T}^d)$ is completely bounded with cb-norm majorized

by $C_{p,d}\|m\|_{\text{HM}}$. This follows from [1] for $d = 1$ and [22,39] for $d \geq 2$ since the Schatten p -class S_p is a UMD space. Note that valid for general UMD spaces, the results in these papers require more regularity on m than the condition (1), that is, the partial discrete derivations should run to all orders up to d instead of $\lfloor \frac{d}{2} \rfloor + 1$ in (1). However, using the arguments of [23, Theorem 6.4] and the interpolation or the argument of [38, section 4.1], we can show that when the UMD space in consideration is a noncommutative $L_p(\mathcal{M})$, we can go down again to the classical order $\lfloor \frac{d}{2} \rfloor + 1$. Hence

$$\|T_m \otimes \text{Id}_{S_p} : L_p(\mathbb{T}^d; S_p) \rightarrow L_p(\mathbb{T}^d; S_p)\|_{\text{cb}} \lesssim_{p,d} \|m\|_{\text{HM}}.$$

The Schatten p -class S_p here can be replaced by $L_p(\mathcal{M})$ for any QWEP $\mathcal{M}$, in particular, by $L_p(\widehat{\mathbb{F}}_\infty)$. Thus

$$\|T_m \otimes \text{Id}_{L_p(\widehat{\mathbb{F}}_\infty)} : L_p(\mathbb{T}^d; L_p(\widehat{\mathbb{F}}_\infty)) \rightarrow L_p(\mathbb{T}^d; L_p(\widehat{\mathbb{F}}_\infty))\|_{\text{cb}} \lesssim_{p,d} \|m\|_{\text{HM}}.$$

Now given $x \in L_p(\widehat{\mathbb{F}}_\infty)$ define $f \in L_p(\mathbb{T}^d; L_p(\widehat{\mathbb{F}}_\infty))$ by $f(z) = \alpha_z(x)$ for $z \in \mathbb{T}^d$. Then by Theorem 1.2

$$\|f(z)\|_{L_p(\widehat{\mathbb{F}}_\infty)} \simeq_{p,d} \|x\|_{L_p(\widehat{\mathbb{F}}_\infty)}, \quad z \in \mathbb{T}^d.$$

Clearly, we have the intertwining identity:

$$[T_m \otimes \text{Id}_{L_p(\widehat{\mathbb{F}}_\infty)}](f)(z) = \alpha_z(M_m(x)), \quad z \in \mathbb{T}^d.$$

Thus we deduce

$$\begin{aligned} \|M_m(x)\|_{L_p(\widehat{\mathbb{F}}_\infty)}^p &\lesssim_{p,d} \int_{\mathbb{T}^d} \|\alpha_z(M_m(x))\|_{L_p(\widehat{\mathbb{F}}_\infty)}^p dz \\ &= \|[T_m \otimes \text{Id}_{L_p(\widehat{\mathbb{F}}_\infty)}](f)\|_{L_p(\mathbb{T}^d; L_p(\widehat{\mathbb{F}}_\infty))}^p \\ &\lesssim_{p,d} \|m\|_{\text{HM}}^p \|f\|_{L_p(\mathbb{T}^d; L_p(\widehat{\mathbb{F}}_\infty))}^p \\ &\lesssim_{p,d} \|m\|_{\text{HM}}^p \|x\|_{L_p(\widehat{\mathbb{F}}_\infty)}^p. \end{aligned}$$

Therefore, M_m is bounded on $L_p(\widehat{\mathbb{F}}_\infty)$ with norm controlled by $C_{p,d}\|m\|_{\text{HM}}$. The complete boundedness follows from the usual argument of tensoring with S_p . $\square$

We end this subsection with some examples of Fourier multipliers on the free group. The free Hilbert transform of [25] is a typical example of Fourier multipliers studied in this article. Theorem 1.1 allows us to exhibit plenty of examples of Fourier multipliers on the free group. We give here just some typical ones.

Example 5.1. Let $\mathcal{A}_i = L_\infty(\mathbb{T})$ for all $i \in \mathbb{N}$ and $z = (z_{j,i})_{1 \leq j \leq d, i \in \mathbb{N}}$ be a family of complex numbers of modulus 1. Then the corresponding transforms $\mathcal{H}_z^{(d)}$ and $\mathcal{H}_z^{Ld}$ in Corollary 4.10 are completely bounded on $L_p(\widehat{\mathbb{F}}_\infty)$ for all $1 < p < \infty$.

Example 5.2. We give two more examples of similar nature: For $1 \leq j \leq d$ let $T_{j,i}$ be the map on $\mathcal{A}_i = L_\infty(\mathbb{T})$ defined by $T_{j,i}(\zeta^n) = z_{j,i}^{\text{sgn}(n)} \zeta^n$ for $\zeta \in \mathbb{T}$ and $n \in \mathbb{Z}$. Clearly, $T_{j,i}$ is completely bounded on $L_p(\mathbb{T})$ for all $1 < p < \infty$. Let $\widetilde{\mathcal{H}}_z^{Ld}$ be the corresponding map T^{Ld} in Theorem 1.3 and $\widetilde{\mathcal{H}}_z^{(d)}$ the map $T^{(d)}$ in Theorem 4.9 associated to $(T_{d,i})_{i \in \mathbb{N}}$. Again, $\widetilde{\mathcal{H}}_z^{Ld}$ and $\widetilde{\mathcal{H}}_z^{(d)}$ are completely bounded on $L_p(\widehat{\mathbb{F}}_\infty)$ for all $1 < p < \infty$.

Example 5.3. The Riesz transforms R_j , $1 \leq j \leq d$, on $L_p(\mathbb{T}^d)$ are the Fourier multipliers of symbols

$$m_j(k) = \frac{k_j}{\|k\|} \text{ for } k = (k_1, \dots, k_d) \in \mathbb{Z}^d, \quad \|k\| = \sqrt{k_1^2 + \dots + k_d^2}.$$

It is classical that R_j is completely bounded on $L_p(\mathbb{T}^d)$ for $1 < p < \infty$. The corresponding multipliers M_{m_j} in Theorem 1.1 are denoted by R_j^{Ld} and may be called the free Riesz transforms of the first d letters on $\mathbb{F}_\infty$. R_j^{L1} is just the free Hilbert transform of [25]. R_j^{Ld} , $1 \leq j \leq d$, are completely bounded on $L_p(\widehat{\mathbb{F}}_\infty)$ for $1 < p < \infty$.

Example 5.4. Our final example is given by the classical Littlewood-Paley multiplier. By the vector-valued multiplier theory, we know that the following Littlewood-Paley multiplier

$$m = \sum_{i=0}^{\infty} \varepsilon_i \mathbb{1}_{R_i},$$

where $R_i = \{k = (k_1, \dots, k_d) \in \mathbb{Z}^d : 2^i - 1 \leq \max_{1 \leq j \leq d} |k_j| < 2^{i+1}\}$ is a completely bounded Fourier L_p -multiplier on $\mathbb{Z}^d$ for all $1 < p < \infty$ (cf. [39]). The corresponding multiplier M_m in Theorem 1.1, denoted by LP^{Ld} , is completely bounded on $L_p(\widehat{\mathbb{F}}_\infty)$. By changing of variables, one can equally consider the following Littlewood-Paley multiplier:

$$m = \sum_{i=0}^{\infty} \varepsilon_i \mathbb{1}_{\{k \in \mathbb{Z}^d : 2^i - 1 \leq |k| < 2^{i+1}\}} \text{ with } \varepsilon_i = \pm 1, \quad |k| = |k_1| + \dots + |k_d|$$

It gives rise to a completely bounded multiplier on $L_p(\widehat{\mathbb{F}}_\infty)$ too.

5.2. More paraproducts

We make the paraproducts in Corollary 4.11 more precise in the case of free groups. Let $z = (z_{j,i})_{j \in \mathbb{N}, i \in \mathbb{N}} \in \mathbb{T}^{\mathbb{N}} \times \mathbb{T}^{\mathbb{N}}$. Let α_z^{Ld} be the map associated to $(z_{j,i})_{1 \leq j \leq d, i \in \mathbb{N}}$ in Theorem 1.2. Note that

$$\alpha_z^{Ld} = T_z^{(1)} \cdots T_z^{(d)},$$

where $T_z^{(j)}(\lambda(g)) = z_{j,i_j}^{k_j} \lambda(g)$ (with $k_j = 0$ for $j > n$) for $g = g_{i_1}^{k_1} \cdots g_{i_n}^{k_n} \in \mathbb{F}_\infty$ in reduced form. Put

$$\alpha_z^{Ld,\text{op}}(x) = [\alpha_z^{Ld}(x^*)]^* \quad \text{and} \quad T_z^{(j),\text{op}}(x) = [T_z^{(j)}(x^*)]^*.$$

Let $\varepsilon = (\varepsilon_{j,i})_{j \in \mathbb{N}, i \in \mathbb{N}}$ be an independent family of symmetric signs. Recall that we use $\mathcal{H}_\varepsilon^{(j)}$ and $\mathcal{H}_\varepsilon^{(j),\text{op}}$ to denote the free Hilbert transforms on the (last) j -th letters considered in Corollary 4.10,

$$H_\varepsilon^{(j)}(\lambda_g) = \varepsilon_{j,i_j} \lambda_g \quad \text{and} \quad H_\varepsilon^{(j),\text{op}}(\lambda_g) = \varepsilon_{j,i_{m+1-j}} \lambda_g.$$

We use again the convention that $T_z^0 = H_\varepsilon^0 = \alpha_z^{L0} = \text{Id}$. For $j, k \geq 0$ and $x, y \in \mathbb{C}[\mathbb{F}_\infty]$, define

$$x \overset{j,k}{\top} y = \mathbb{E}_z \mathbb{E}_{z'} \alpha_z^{Lj} \alpha_{z'}^{Lk,\text{op}} [\alpha_z^{Lj}(x) \alpha_{z'}^{Lk,\text{op}}(y)],$$

where $\mathbb{E}_z$ and $\mathbb{E}_{z'}$ denote the expectations on z and z' respectively. Let us interpret these paraproducts for $x = \lambda(g)$ and $y = \lambda(h)$. To this end, we say that the first j blocks of g survive in gh if the first j blocks of gh and g are exactly the same, and that the j -th block of g marks in gh if the j -th blocks of gh and g are powers of a same generator. Replacing “first” by “last” (i.e., counting the letters of a reduced word in the reverse order), we get similar notions.

Thus for $g, h \in \mathbb{F}_\infty$

- $\lambda(g) \overset{j,k}{\top} \lambda(h) = \lambda(gh)$ if the first j blocks of g and the last k blocks of h survive in gh ;
 $\lambda(g) \overset{j,k}{\top} \lambda(h) = 0$ otherwise;

Note $\overset{j,k}{\top}$ is different from $\overset{j,k}{\ddagger}$ defined in Section 4 in general. For example, for $x = \lambda(g_1 g_2), y = \lambda(g_2^{-1} g_1^{-2})$, then $x \overset{1,0}{\ddagger} y = \lambda(g_1^{-1})$ but $x \overset{1,0}{\top} y = 0$. There are two more paraproducts that may be useful for future research. Let

$$\begin{aligned} x \overset{j+,k}{\top} y &= \mathbb{E}_\varepsilon H_\varepsilon^{(j+1)} [H_\varepsilon^{(j+1)}(x) \overset{j,k}{\top} y], \\ x \overset{j,k+}{\top} y &= \mathbb{E}_\varepsilon H_\varepsilon^{(k+1),\text{op}} [x \overset{j,k}{\top} H_\varepsilon^{(k+1),\text{op}}(y)]. \end{aligned}$$

Then for $g, h \in \mathbb{F}_\infty$

- $\lambda(g) \overset{j+,k}{\top} \lambda(h) = \lambda(gh)$ if the first j blocks of g and the last k blocks of h survive in gh ,
 and in addition, the $(j+1)$ -th block of g marks in gh ; $\lambda(g) \overset{j+,k}{\top} \lambda(h) = 0$ otherwise.

- $\lambda(g) \overset{j,k+}{\top} \lambda(h) = \lambda(gh)$ if the first j blocks of g and the last k blocks of h survive in gh , and in addition, the last $(k+1)$ -th block of h marks in gh ; $\lambda(g) \overset{j,k+}{\top} \lambda(h) = 0$ otherwise.

Our first approach to Theorem 1.2 heavily relies on the boundedness of the above paraproducts and several variants of them. Now their boundedness immediately follows from Theorem 1.2.

Proposition 5.5. *All the above paraproducts extend to bounded bilinear maps from $L_p(\widehat{\mathbb{F}}_\infty) \times L_q(\widehat{\mathbb{F}}_\infty)$ to $L_r(\widehat{\mathbb{F}}_\infty)$ for all $1 < r, p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ with norm majorized by constants depending only on r, p, q, j and k .*

5.3. Extension to free products of groups

In the proof of Theorem 1.1 one can easily replace $\mathbb{Z}$ by any abelian discrete group Γ . However, to go beyond the abelian case, one needs extra efforts. In this subsection, Γ will denote a general discrete group. Let $\Gamma_\infty = \Gamma^{*\mathbb{N}}$ be the infinite free power of Γ . Each $g \in \Gamma_\infty \setminus \{e\}$ is written as a reduced word:

$$g = g_1 g_2 \cdots g_n$$

with $g_j \neq e$ belonging to the i_j -th copy of Γ in Γ_∞ and $i_1 \neq i_2 \neq \cdots \neq i_n$.

We begin by extending Theorem 1.2 to this general setting. Define a linear map $\alpha^{L^d} : \mathbb{C}[\Gamma_\infty] \rightarrow \mathbb{C}[\Gamma^d \times \Gamma_\infty]$ as follows: for $g = g_1 \cdots g_n \in \Gamma_\infty$ as above in reduced form,

$$\alpha^{L^d}(\lambda(g)) = \lambda_{\Gamma^d}(g_1, \dots, g_d) \otimes \lambda(g)$$

with $g_\ell = e$ in $\lambda_{\Gamma^d}(g_1, \dots, g_d)$ if $\ell > n$. Here we have denoted by λ_{Γ^d} the left regular representation of Γ^d to avoid ambiguity (λ being that of Γ_∞).

Theorem 5.6. *Let $d \in \mathbb{N}$ and $1 < p < \infty$. Then the map α^{L^d} extends to a completely isomorphic embedding of $L_p(\widehat{\Gamma}_\infty)$ into $L_p(\widehat{\Gamma}^d \overline{\otimes} \widehat{\Gamma}_\infty)$.*

Proof. We use an argument similar to that of Theorem 4.6. Let $G = *_{l=1, \dots, d+1} \Gamma$ and $G_\infty = *_{i \geq 1} G$. To avoid confusion we use $\Gamma_{i,l}$ to denote the l -th copy of Γ in the i -th copy of G in G_∞ .

First, let $\pi^l : \widehat{G} \rightarrow \widehat{G}$ be the $*$ -representation given by the cyclic permutation of the copies that sends the l -th copy of Γ to the $l+1 \pmod{d+1}$ -th copy. Consider the map T^{L^d} on $L_p(\widehat{G}_\infty)$ given by Theorem 1.3 associated to $T_k = \pi^k, 1 \leq k \leq d$, as well as its inverse associated to $T_k^{-1} = \pi^{-k}, 1 \leq k \leq d$. By Theorem 1.3, T^{L^d} is a complete isomorphism on $L_p(\widehat{G}_\infty)$. We will need the restriction of T^{L^d} to a copy of $L_p(\widehat{\Gamma}_\infty)$ in $L_p(\widehat{G}_\infty)$ and will make its presentation more precise.

Let us identify Γ_∞ with $*_{i \geq 1} \Gamma_{i,1}$. For an element in $g \in \Gamma_{i,1}$, we denote its copy in $\Gamma_{i,l}$ by $g^{(l)}$. Thus, for $g = g_1 \cdots g_n \in \Gamma_\infty$ in reduced form, we have

$$T^{Ld}(\lambda(g)) = \lambda(g_1^{(2)} \cdots g_d^{(d+1)} g_{d+1}^{(1)} \cdots g_n^{(1)}). \quad (8)$$

As we have explained, (8) defines a complete isomorphic embedding of $L_p(\widehat{\Gamma}_\infty)$ into $L_p(\widehat{G}_\infty)$.

Next, we consider the group morphism ϕ from G_∞ onto Γ^d such that for all i and $g \in \Gamma_{i,k} \subset G_\infty$

$$\phi(g) = e \text{ if } k = 1 \text{ and } \phi(g) = (\underbrace{e, \dots, e}_{k-2}, g, e, \dots, e) \text{ if } 2 \leq k \leq d+1.$$

Let $U = \lambda_{\Gamma^d} \circ \phi$. Then U is a unitary representation of G_∞ on $\ell_2(\Gamma^d)$. Applying the Fell absorption principle (Lemma 2.1) to U , we get a completely isometric embedding of $L_p(\widehat{G}_\infty)$ into $L_p(\widehat{\Gamma^d} \overline{\otimes} \widehat{G}_\infty)$:

$$\Pi_\phi : \sum_{g \in G_\infty} c(g) \lambda(g) \mapsto \sum_{g \in G_\infty} c(g) \lambda_{\Gamma^d}(\phi(g)) \otimes \lambda(g). \quad (9)$$

By (8) and (9), we see that

$$\alpha^{Ld}(x) = \Pi_\phi \cdot T^{Ld}(x), \quad x \in \mathbb{C}[\Gamma_\infty].$$

Thus α^{Ld} extends to a completely isomorphic embedding on $L_p(\widehat{\Gamma}_\infty)$. $\square$

Remark 5.7. There is an alternate proof to Theorem 5.6. First, one may apply Theorem 4.6 and Remark 4.7 for $\mathcal{A}_i = \widehat{\Gamma^d} \overline{\otimes} \widehat{\Gamma}$ for all $i \in \mathbb{N}$, and $\mathcal{B} = \widehat{\Gamma^d} \otimes \mathbb{C}$, and check that

$$\|\alpha^{L1}(x)\|_{L_p(\widehat{\Gamma^d} \overline{\otimes} \widehat{\Gamma}_\infty)} \simeq_p \|x\|_{L_p(\widehat{\Gamma}_\infty)}$$

for any $x \in \mathbb{C}[\Gamma_\infty]$. Next, one may apply Theorem 4.6 repeatedly to get

$$\|\alpha^{Ld}(x)\|_{L_p(\widehat{\Gamma^d} \overline{\otimes} \widehat{\Gamma}_\infty)} \simeq_p \|\alpha^{L(d-1)}(x)\|_{L_p(\widehat{\Gamma^d} \overline{\otimes} \widehat{\Gamma}_\infty)} \simeq_p \cdots \simeq_p \|x\|_{L_p(\widehat{\Gamma}_\infty)}.$$

This alternate proof gives a better constant.

As in the free group case, the previous theorem, together with transference, easily implies a multiplier result on Γ_∞ . Given a bounded function m on Γ^d , define a linear map M_m on $L_2(\widehat{\Gamma}_\infty)$ by

$$M_m(\lambda(g)) = m(g_1, g_2, \dots, g_d) \lambda(g)$$

with $g_\ell = e$ in $m(g_1, g_2, \dots, g_d)$ if $\ell > n$ for every $g \in \Gamma_\infty$ in reduced form. Note that $\widehat{\Gamma}$ is QWEP iff Γ is hyperlinear (cf. [35]).

Theorem 5.8. *Assume that $\widehat{\Gamma}$ is QWEP. Let $d \in \mathbb{N}$ and $1 < p < \infty$. If the Fourier multiplier T_m is completely bounded on $L_p(\widehat{\Gamma}^d)$, then M_m extends to a completely bounded map on $L_p(\widehat{\Gamma}_\infty)$.*

Proof. This proof is similar to that of Theorem 1.1. As $\widehat{\Gamma}$ is QWEP, so is $\widehat{\Gamma}_\infty$. Thus $T_m \otimes \text{Id}_{L_p(\widehat{\Gamma}_\infty)}$ is completely bounded on $L_p(\widehat{\Gamma}^d \overline{\otimes} \widehat{\Gamma}_\infty)$ by Junge's noncommutative Fubini theorem [12]. Let $x \in \mathbb{C}[\Gamma_\infty]$. Using the action α^{L^d} in Theorem 5.6, we have

$$\|x\|_{L_p(\widehat{\Gamma}_\infty)} \simeq_{d,p} \|\alpha^{L^d}(x)\|_{L_p(\widehat{\Gamma}^d \overline{\otimes} \widehat{\Gamma}_\infty)}.$$

It remains to use the intertwining formula

$$T_m \otimes \text{Id}[\alpha^{L^d}(x)] = a^{L^d}[M_m(x)]$$

to conclude as in the proof of Theorem 1.1. $\square$

In the same line, we conclude this subsection by stating another application. Consider a family of discrete groups Γ_i , $i \in I$, and its free product $\Gamma_\infty = *_{i \in I} \Gamma_i$; consider also a family of finite von Neumann algebras $(\mathcal{M}_i, \tau_i)$, $i \in I$, and its von Neumann tensor product $(\mathcal{M}, \tau) = \overline{\otimes}_{i \in I} (\mathcal{M}_i, \tau_i)$. Let $\{m_{i,g}\}_{g \in \Gamma_i} \subset \mathcal{M}_i$ for all $i \in I$, and let M_i be the operator-valued Fourier multiplier on Γ_i :

$$M_i(\lambda(g)) = m_{i,g} \otimes \lambda(g), \quad g \in \Gamma_i.$$

We construct a map M^{L^d} similar to Theorem 5.8. Given $g = g_1 \cdots g_n \in \Gamma_\infty$ in reduced form, define

$$M^{L^d}(\lambda(g)) = \begin{cases} m_{i_1, g_1} \otimes \cdots \otimes m_{i_d, g_d} \otimes \lambda(g) & \text{if } n \geq d, \\ m_{i_1, g_1} \otimes \cdots \otimes m_{i_n, g_n} \otimes \lambda(g) & \text{if } n < d. \end{cases}$$

Theorem 5.9. *Let $1 < p < \infty$. Then M^{L^d} extends to a completely bounded Fourier multiplier from $L_p(\widehat{\Gamma}_\infty)$ to $L_p(\mathcal{M} \overline{\otimes} \widehat{\Gamma}_\infty)$ iff the family $\{\|M_i\|_{\text{cb}(L_p(\widehat{\Gamma}_i), L_p(\mathcal{M}_i \overline{\otimes} \widehat{\Gamma}_i))}\}_{i \in I}$ is bounded. In this case, we have*

$$\|M^{L^d}\|_{\text{cb}(L_p(\widehat{\Gamma}_\infty), L_p(\mathcal{M} \overline{\otimes} \widehat{\Gamma}_\infty))} \simeq_{p,d} \sup_{i \in I} \|M_i\|_{\text{cb}(L_p(\widehat{\Gamma}_i), L_p(\mathcal{M}_i \overline{\otimes} \widehat{\Gamma}_i))}.$$

Proof. We can easily adapt the proofs of Theorem 5.6 to the present setting, so we omit the details. $\square$

Acknowledgments

The first author is partially supported by NSF award DMS-1700171. The second and third authors are partially supported by the French ANR project No. ANR-19-CE40-0002. The third author is also partially supported by the Natural Science Foundation of China (No. 12031004).

Appendix A. Endpoint boundedness of free Hilbert transforms

Our arguments rely on the L_p -boundedness ($1 < p < \infty$) of the free Hilbert transforms H_ε in Lemma 2.2 and their variants $H_\varepsilon^{(j)}$ in Corollary 4.10. We will discuss their bounds on homogeneous polynomials when $p = \infty$, since they cannot be bounded in full generality at the end point. It is also natural to ask if one can get an L_∞ -BMO boundedness for the BMO spaces studied in [14].

A.1. Bounds on homogeneous polynomials

We work in the setting of amalgamated free products of von Neumann algebras as in section 4. Let $H_\varepsilon^{(j)}$ be the maps introduced in Corollary 4.10 with ε a family of signs.

The case $j = 1$ of the following theorem follows from [18, Proposition 2.8].

Theorem A.1. *Let $d \geq 1$ and $1 \leq j \leq d$. Then for any $x \in \mathcal{W}_d$ we have*

$$\|H_\varepsilon^{(j)}(x)\|_\infty \lesssim \min \{ \log(j+2), \log(d-j+2) \} \|x\|_\infty.$$

Proof. For $z \in \mathbb{T}$, let U_z be the unitary on $L_2(\mathcal{A})$ sending $w \in \mathcal{W}_n$ to $z^n w$ for $n \geq 0$. Given $x \in \mathcal{W}_d, y \in L_2(\mathcal{A})$ and $0 \leq k \leq 2d$, let

$$x \perp^k y = \mathbb{E}_z [z^{d-k} U_{\bar{z}}(x U_z(y))].$$

It is easy to verify that

$$xy = \sum_{k=0}^{2d} x \perp^k y \quad \text{and} \quad \|x \perp^k y\|_2 \leq \|x\|_\infty \|y\|_2.$$

Moreover, we easily show

$$H_\varepsilon^{(j)}(x) \perp^k y = \begin{cases} H_\varepsilon^{(j)}(x \perp^k y) & \text{if } k \leq 2(d-j), \\ x \perp^k H_\varepsilon^{(d+1-j)}(y) & \text{if } k > 2(d-j). \end{cases}$$

Thus

$$H_\varepsilon^{(j)}(x)y = \sum_{0 \leq k \leq 2(d-j)} H_\varepsilon^{(j)}(x \perp^k y) + \sum_{2(d-j) < k \leq 2d} x \perp^k H_\varepsilon^{(d+1-j)}(y).$$

On the other hand, using the elementary estimate

$$\left\| \sum_{k=0}^n z^k \right\|_{L_1(\mathbb{T})} \simeq \log(n+2),$$

we get

$$\left\| \sum_{k=0}^n x \perp^k y \right\|_2 + \left\| \sum_{k=n}^{2d} x \perp^k y \right\|_2 \lesssim \min \{ \log(n+2), \log(2d-n+2) \} \|x\|_\infty \|y\|_2.$$

Therefore, since $H_\varepsilon^{(j)}$ is isometric on $L_2(\mathcal{A})$, we deduce

$$\begin{aligned} \|H_\varepsilon^{(j)}(x)y\|_2 &\leq \left\| \sum_{k=0}^{2(d-j)} x \perp^k y \right\|_2 + \left\| \sum_{k=2(d-j)+1}^{2d} x \perp^k H_\varepsilon^{(d+1-j)}(y) \right\|_2 \\ &\lesssim \min \{ \log(j+2), \log(d-j+2) \} \|x\|_\infty \|y\|_2, \end{aligned}$$

whence the desired estimate on $\|H_\varepsilon^{(j)}(x)\|_\infty$. $\square$

A.2. Failure of the L_∞ -BMO boundedness

Let us restrict ourselves to the free group case. Recall that the Poisson semigroup $(S_t)_{t \geq 0}$ on $\mathbb{F}_\infty$ is the normal unital completely positive semigroup given by

$$S_t(\lambda(g)) = e^{-t|g|} \lambda(g), \quad g \in \mathbb{F}_\infty.$$

We also recall the definitions of various BMO-spaces according to [14]. As usual we denote by $L_p^0(\widehat{\mathbb{F}}_\infty)$ the subspace of centered elements (i.e., elements with vanishing trace) in $L_p(\widehat{\mathbb{F}}_\infty)$. Define

$$\begin{aligned} \text{BMO}^c(S) &= \{x \in L_2^0(\widehat{\mathbb{F}}_\infty) : \|x\|_{\text{BMO}_S} < \infty\}, \\ \text{bmo}^c(S) &= \{x \in L_2^0(\widehat{\mathbb{F}}_\infty) : \|x\|_{\text{bmo}_S} < \infty\}, \end{aligned}$$

where

$$\begin{aligned} \|x\|_{\text{BMO}^c(S)} &= \sup_{t>0} \|S_t[|x - S_t(x)|^2]\|_\infty^{1/2}, \\ \|x\|_{\text{bmo}^c(S)} &= \sup_{t>0} \|S_t(|x|^2) - |S_t(x)|^2\|_\infty^{1/2}. \end{aligned}$$

Similarly, we define the row versions $\text{BMO}^r(S)$ and $\text{bmo}^r(S)$ by passing to adjoints. One of the main results of [14] states that the intersection space $\text{BMO}^c(S) \cap \text{BMO}^r(S)$

behaves well with complex interpolation, i.e., it replaces L_∞ as an endpoint space in the complex interpolation scale $\{L_p(\widehat{\mathbb{F}}_\infty)\}_{p>1}$.

Let H_ε denote the free Hilbert transform of the first letters associated to a sequence of signs, see Lemma 2.2. It is easy to see that $\|H_\varepsilon(x)\|_{\text{bmo}^c(S)} = \|x\|_{\text{bmo}^c(S)}$ for $x \in \mathbb{C}[\mathbb{F}_\infty]$.² We will explain why one cannot hope the boundedness of H_ε from $L_\infty(\widehat{\mathbb{F}}_\infty) = \widehat{\mathbb{F}}_\infty$ to any of $\text{BMO}^r(S)$, $\text{BMO}^c(S)$ or $\text{bmo}^r(S)$.

Lemma A.2. *Let $a, b \in \mathbb{F}_\infty$ be two free elements. Let z be a finite sum $z = \sum_{k \geq 1} c_k \lambda_{a^k}$ and $z_n = z\lambda(b^n)$. Then*

- i) $\lim_{n \rightarrow \infty} \|z_n\|_{\text{bmo}^r(S)} = \|z\|_\infty$,
- ii) $(e^{-1} - e^{-2})\|z\|_\infty \leq \limsup_{n \rightarrow \infty} \|x_n\|_{\text{BMO}^\alpha(S)} \leq 2\|z\|_\infty$, $\alpha \in \{c, r\}$.

Proof. i) It is clear that

$$\|z_n\|_{\text{bmo}^r(S)} \leq \|z_n\|_\infty = \|z\|_\infty.$$

For a fixed $t > 0$, we have $\lim \|S_t(z_n)\|_\infty = 0$. Thus,

$$\limsup_{n \rightarrow \infty} \|z_n\|_{\text{bmo}^r(S)}^2 \geq \sup_t \|S_t(z_n z_n^*)\|_\infty = \|z\|_\infty^2.$$

ii) The upper bound is clear. For the lower, we use the Kadison inequality to ensure

$$\|S_t|z_n^* - S_t(z_n^*)|^2\|_\infty, \|S_t|z_n - S_t(z_n)|^2\|_\infty \geq \|(S_t - S_{2t})z_n\|_\infty^2.$$

But in $\widehat{\mathbb{F}}_\infty$,

$$\lim_{n \rightarrow \infty} ((S_{\frac{1}{n}} - S_{\frac{2}{n}})z_n)\lambda_{b^{-n}} = (e^{-1} - e^{-2})z.$$

This finishes the proof. $\square$

As $H_\varepsilon(z_n) = H_\varepsilon(z)\lambda_{b^n}$, we get

Corollary A.3. *The transform H_ε is unbounded from $L_\infty(\widehat{\mathbb{F}}_\infty)$ to any of $\text{BMO}^r(S)$, $\text{BMO}^c(S)$ and $\text{bmo}^r(S)$.*

² One can verify directly that $S_t|H_\varepsilon x|^2 - |S_t H_\varepsilon x|^2 = S_t|x|^2 - |S_t x|^2$. From this identity, one may obtain directly that $H_\varepsilon x$ belongs to $L_p(\widehat{\mathbb{F}}_\infty)$ if $x \in L_p(\widehat{\mathbb{F}}_\infty)$ and $H_\varepsilon x$ is selfadjoint via the probability model introduced in [13].

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