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ORIGINAL PAPER

Topology of the tropical moduli spaces $\Delta_{2,n}$

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Abstract
We study the topology of the link $\Delta_{2,n}$ of the moduli spaces of n-marked, genus 2 tropical curves. As an application, we calculate the top-weight rational homology of the moduli spaces $\mathcal{M}_{2,n}$ of n-marked algebraic curves of genus 2 for $n \leq 8$.

Keywords Tropical geometry · Tropical curves · Moduli spaces

Mathematics Subject Classification 14T20 · 14H10

1 Introduction

The tropical moduli spaces of curves $M_{g,n}^{\mathrm{trop}}$ are topological spaces that parametrize isomorphism classes of n-marked stable abstract tropical curves of genus g. These spaces were introduced in tropical geometry by Brannetti–Melo–Viviani (2011) and Caporaso (2013), building on earlier work of Mikhalkin (2006, §5.4). They have antecedents in related constructions of Gathmann–Markwig (Gathmann and Markwig 2007; Markwig 2006) and, even further back, in the work of Culler–Vogtmann on Outer Space (Culler and Vogtmann 1986). They have played an important role in many advances in tropical geometry; see for example (Abramovich et al. 2015; Brannetti et al. 2011; Caporaso 2012, 2013, 2014; Chan 2012, 2013; Len 2014; Viviani 2013). Moreover, tropical moduli spaces have already arisen very naturally in other kinds of geometry. For example, $M_{0,n}^{\mathrm{trop}}$ is the Billera–Holmes–Vogtmann space of phylogenetic trees (Billera et al. 2001), while $M_{g,0}^{\mathrm{trop}}$ is an infinite cone over a compactified quotient of Culler–Vogtmann Outer Space X_g of rank g, with the property that the Torelli map on $M_{g,0}^{\mathrm{trop}}$ is compatible with a period map on X_g (Baker 2011; Chan et al. 2013). Recently the space $M_{g,n}^{\mathrm{trop}}$ has been realized as a stack over the category of rational

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polyhedral cones (Cavalieri et al. 2020), but for the current applications it suffices to consider $M_{g,n}^{\mathrm{trop}}$ as a topological space, albeit with some combinatorial structure.

While $M_{g,n}^{\mathrm{trop}}$ itself is contractible, since it is a generalized cone complex, the *link* of $M_{g,n}^{\mathrm{trop}}$, which we will denote $\Delta_{g,n}$, has interesting topology. The space $\Delta_{g,n}$ has real dimension $3g-4+n$, and can be regarded as the cross-section of $M_{g,n}^{\mathrm{trop}}$ that parametrizes tropical curves of total edge length 1. The work of Abramovich–Caporaso–Payne (Abramovich et al. 2015), extending the work of (Thuillier 2007), identifies

$$\Delta_{g,n} = \Delta(\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}),$$

where $\Delta(\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n})$ denotes the *boundary complex* of the Deligne-Mumford-Knudsen compactification by stable curves.

Boundary complexes, first studied by Danilov (1975), encode the intersections of the components of a normal crossings compactification. Furthermore, over $\mathbb{C}$, recall that they encode the top-weight rational cohomology of the space being compactified. The work of Deligne (1971, 1974) implies that for any compactification $\mathcal{X} \subset \overline{\mathcal{X}}$ of smooth, separated Deligne–Mumford stacks of dimension d with normal crossings boundary,

$$\mathrm{Gr}_{2d}^{W} H^{2d-i}(\mathcal{X};\mathbb{Q}) \;\cong\; \widetilde{H}_{i-1}(\Delta(\mathcal{X} \subset \overline{\mathcal{X}});\mathbb{Q}), \tag{1}$$

where $\Delta(\mathcal{X} \subset \overline{\mathcal{X}})$ denotes the boundary complex (see Chan et al. 2021). We refer to the cohomology $\mathrm{Gr}_{2d}^{W} H^{j}$ as *top-weight*, since cohomology does not appear in weights above $2d$.

In fact, the homotopy type of $\Delta(\mathcal{X} \subset \overline{\mathcal{X}})$, even its *simple* homotopy type, is known to be an invariant of $\mathcal{X}$ itself (Payne 2013; Harper 2017). This means that homotopy invariants, e.g., torsion in the integral homology of $\Delta(\mathcal{X} \subset \overline{\mathcal{X}})$, gain interest as invariants of $\mathcal{X}$ itself—even while we currently may not have a direct interpretation of what they mean about $\mathcal{X}$. These are our main motivations for studying the topology of $\Delta_{g,n}$.

The cases $g=0$ and $g=1$ are known. When $g=0$ and $n \geq 4$, the space $\Delta_{0,n}$ has the homotopy type of a wedge of $(n-2)!$ spheres of dimension $n-4$ (Billera et al. 2001). The case $g=1$ and $n \geq 1$ is settled in [CGP], where we prove that $\Delta_{1,n}$ is a wedge of $(n-1)!/2$ spheres of dimension $n-1$. The goal of this paper is to study the topology of $\Delta_{2,n}$. These spaces are not fully understood and already exhibit more complicated behavior than in the lower genus cases.

The first main theorem is on the connectivity of $\Delta_{2,n}$.

Theorem 1.1 *We have*

$$\widetilde{H}_i(\Delta_{2,n};\mathbb{Z}) = 0$$

for $i = 0, \ldots, n$.

In fact, $\Delta_{2,n}$ is an n-connected space.

Table 1 The rational homology of $\Delta_{2,n}$ for $n \leq 8$

n	0	1	2	3	4	5	6	7	8
$\dim H_{n+2}(\Delta_{2,n}, \mathbb{Q})$	0	0	1	0	3	15	86	575	4426
$\dim H_{n+1}(\Delta_{2,n}, \mathbb{Q})$	0	0	0	0	1	5	26	155	1066

Thus, the reduced $\mathbb{Z}$-homology of $\Delta_{2,n}$ is concentrated in the top two degrees. The fact that the *rational* reduced homology of $\Delta_{2,n}$ is concentrated in top two degrees can be deduced, using the comparison theorem (1), from known vanishing of rational cohomology on $\mathcal{M}_{g,n}$ in appropriate degrees: we have $\widetilde{H}_k(\Delta_{g,n}; \mathbb{Q}) = 0$ for $k < \max\{2g-1, 2g-3+n\}$, see (Chan et al. 2019b, §6.2). Note also that the statement on the connectivity of $\Delta_{2,n}$ strengthens, for $g = 2$, the connectivity bound obtained for all (g, n) in (Chan et al. 2016, Theorem 1.3) that $\Delta_{g,n}$ is $(n - 5g + 4)$-connected.

Interestingly, $\Delta_{2,n}$ has 2-torsion in its integral homology, at least when n is odd and $n \geq 5$; this is a new phenomenon. See Proposition 4.9 for the proof. As remarked earlier, I do not have a direct interpretation for this torsion in terms of $\mathcal{M}_{2,n}$.

Second, we present computational results for n up to 8.

Theorem 1.2 *For $n = 0, \ldots, 8$, the $\mathbb{Q}$-homology of $\Delta_{2,n}$ in the top two degrees is given in* Table 1. *Therefore, the top-weight $\mathbb{Q}$-cohomology of $\mathcal{M}_{2,n}$, for $n \leq 8$ is concentrated in degrees $n+3$ and $n+4$ with ranks given in the table.*

The computations obtained in Theorem 1.2 were done in sage (The Sage Developers 2020). They are explained at the end of Sect. 5. Recently, Yun reproved Theorem 1.2 and in fact promoted the calculations to be S_n-equivariant; see Yun (2020). Note agreement of Table 1 with the fact that the top-weight Euler characteristic of $\mathcal{M}_{2,n}$ is $(-1)^{n+1}\, n!/12$. More generally, in Chan et al. (2019a) a formula for the top-weight Euler characteristic of $\mathcal{M}_{g,n}$, as a virtual S_n-representation, was recently proved; the calculations for all $g \leq 9$ were known to Faber prior to that work. The factorial growth in n precludes representation stability on the nose, but it would be interesting to know if some other stability phenomenon can be established. See the discussion in Chan et al. (2019b), §7) for further comments.

This paper, on the case $g = 2$, is a side dish for the series of papers (Chan et al. 2019a, b, 2021), as well as the earlier preprint (Chan et al. 2016), on tropical moduli spaces, graph complexes, and the cohomology of $\mathcal{M}_{g,n}$. It was drafted before that series of papers appeared, and initially included some results for $g = 2$ that have been superceded since. For example, contractibility of the bridge locus for all $g \geq 1$ is now proved in Chan et al. (2019b) and the top-weight Euler characteristic of $\mathcal{M}_{g,n}$ is established in Chan et al. (2019a). The spaces $\Delta_{2,n}$ are still not fully understood and exhibit new phenomena: the first instances of reduced homology below top degree, torsion in integral homology, and new computations up to 8 marked points. So I have revised this paper to catch up with the state of the art presented in those papers.

Before diving in, we refer to the preprint (Allcock et al. 2019) which studies $\Delta_{g,n}$ using computations with *symmetric Δ-complexes* (in the sense of Chan et al. (2021)). There it is proven that $\Delta_{g,n}$ is simply connected for $g \geq 1$, that $\Delta_{3,0}$ is homotopy equivalent to a 5-sphere, and that there are instances of 2- and 3-torsion in the integral homology of $\Delta_{4,0}$. We also refer the reader to previous work by Kozlov, who studied a different but highly related space: the moduli space of tropical curves of genus g with n marked points and with no vertex weights (Kozlov 2009, 2011). These have been called *pure* tropical moduli spaces elsewhere. These results seem to be essentially independent. Basically, allowing nontrivial vertex-weights changes the spaces significantly: see (Chan et al. 2019b, §7.1) for more discussion and references.

Finally, I refer to work of Tommasi and Petersen, in progress, studying the top-weight cohomology of $\mathcal{M}_{2,n}$ via a different approach using local systems, as reported by Tommasi at the MSRI "Recent progress on moduli theory" workshop in 2019.

2 The tropical moduli spaces $M_{g,n}^{\text{trop}}$

In this section we review the general theory of tropical curves, of any genus, and their moduli spaces. In subsequent sections, we specialize to the case of genus 2.

2.1 Graph theory

In this paper, all graphs will be finite and connected multigraphs, that is, loops and multiple edges are allowed. We write $V(G)$ and $E(G)$ for the vertex set and the edge multiset of G, respectively.

Edges $e, f \in E(G)$ are called *parallel* if they are distinct nonloop edges incident to the same pair of vertices; we write $e \parallel f$ if so. If $e \in E(G)$ is not parallel to any edge, we will say that e is a *singleton.* A *bridge* in G is an edge whose deletion disconnects G.

The *valence* of a vertex v, denoted $\mathrm{val}(v)$, is the number of half-edges at v; thus loops based at v contribute twice to this sum. A *cut vertex* in G is a vertex $v \in V(G)$ whose removal disconnects the graph G, considered now as a 1-dimensional CW complex. In other words, deleting a vertex does not delete the points on incident edges. Our definition of a cut vertex differs slightly from the standard one in graph theory. As an example, let R_g denote the graph consisting of g loops based at a single vertex v. Then v is a cut vertex if and only if $g \geq 2$.

A vertex-weighted graph is a pair (G, w) where G is a graph and $w\colon V(G) \to \mathbb{Z}_{\geq 0}$ is an arbitrary weight function. Following (Amini and Caporaso 2013), we define the *virtual graph* G^w to be the (unweighted) graph obtained from G by adding $w(v)$ distinct loops to each vertex v. We say that a vertex of v is a *virtual cut vertex* of (G, w) if v is a cut vertex of G^w. Thus v is a virtual cut vertex of (G, w) if and only if v is a cut vertex of G, or $w(v) > 0$ and G^w has at least two edges. For example, of the seven vertex-weighted graphs in Fig. 1, all but the first have a virtual cut vertex.

2.2 Tropical curves with marked points

Definition 2.1 (Brannetti et al. 2011; Caporaso 2013; Mikhalkin 2006) A *stable tropical curve* with n marked points is a quadruple $\Gamma = (G, l, m, w)$ where:

- G is a graph (as in §2.1),

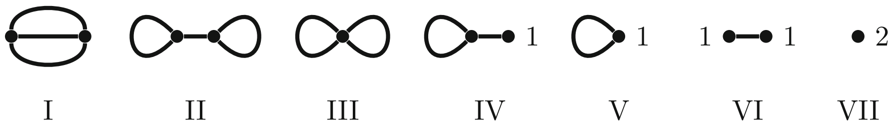

Fig. 1 The seven combinatorial types in $T_{2,0}$. The vertices have weight zero unless otherwise indicated

- $l\colon E(G) \to \mathbb{R}_{>0}$ is any function, called a *length function*, on the edges,
- $m\colon \{1, \dots, n\} \to V(G)$ is any function, called a *marking function*, and
- $w\colon V(G) \to \mathbb{Z}_{\geq 0}$ is any function;

such that the following condition, called the *stability condition*, holds: for every vertex $v \in V(G)$, we have

$$2w(v) - 2 + \mathrm{val}(v) + |m^{-1}(v)| > 0. \tag{2}$$

We remark that our marking function m is combinatorially equivalent to the more common setup of attaching infinite rays to a graph, labeled $\{1, \dots, n\}$, and we use m just for convenience. We will sometimes refer to a stable tropical curve as simply a *tropical curve.*

The *genus* of a tropical curve Γ is

$$g(\Gamma) := |E(G)| - |V(G)| + 1 + \sum_{v \in V(G)} w(v). \tag{3}$$

The *total length* of Γ is the sum of its edge lengths. The total length is called the *weight* of the tropical curve elsewhere, but we won't use this terminology since we already have weights on vertices.

Definition 2.2 Let $\Gamma = (G, l, m, w)$ be a stable tropical curve of genus g.

1. The *combinatorial type* of Γ is the triple (G, m, w). We will write $\mathbf{G} = (G, m, w)$ for short, and we define the genus of a combinatorial type as in Eq. (3). Let $T_{g,n}$ denote the set of combinatorial types of tropical curves of genus g with n marked points. In principle, there are infinitely many distinct combinatorial types, e.g., by renaming vertices of graphs; but there are only finitely many up to isomorphism. We shall choose one for each isomorphism class, making $T_{g,n}$ is a finite set.
2. If $g \geq 2$, we define the *unmarked type* or *underlying type* of Γ to be the combinatorial type $(G', w') \in T_{g,0}$ obtained as follows: (G', w') is the smallest connected, vertex-weighted subgraph of (G, w) that contains all cycles of G and all vertices of positive weight. Here, we assume that we have suppressed all vertices of valence 2, so that (G', w') is stable of genus g.

Example 2.3 There are exactly seven possible unmarked types of tropical curves of genus 2; they are shown in Fig. 1.

Remark 2.4 There is an obvious notion of isomorphism of combinatorial types $(G_1, m_1, w_1) \cong (G_2, m_2, w_2)$, i.e. a multigraph isomorphism between G_1 and G_2 carrying m_1 to m_2 and w_1 to w_2. Then there are only finitely many types in $T_{g,n}$ up to isomorphism. Furthermore, $T_{g,n}$ is nonempty if and only if $2g - 2 + n > 0$, due to the stability condition in Eq. (2).

Let $\mathbf{G} = (G, m, w) \in T_{g,n}$, and let e be any edge of G. The *contraction* $\mathbf{G}/e$ is defined to be the combinatorial type $\mathbf{G}' = (G', m', w') \in T_{g,n}$ obtained as follows: if e is a loop based at $v \in V(G)$, then $\mathbf{G}'$ is obtained by deleting e and increasing $w(v)$ by 1. Otherwise, if e has endpoints v_1 and v_2, then $\mathbf{G}'$ is obtained by deleting e and identifying v_1 and v_2 into a new vertex v. We set $w'(v) = w(v_1) + w(v_2)$.

Now for any subset $S \subseteq E(G)$, we define $\mathbf{G}/S$ to be the contraction of all edges in S. It is well-known that they can be contracted in any order. Note that the new type $(G, m, w)/S$ also lies in $T_{g,n}$.

2.3 Construction of $M_{g,n}^{\mathrm{trop}}$

Suppose $2g - 2 + n > 0$. Given $\mathbf{G} = (G, m, w) \in T_{g,n}$, write

$$\overline{C}(\mathbf{G}) := \mathbb{R}_{\geq 0}^{E(G)},$$
$$\Delta(\mathbf{G}) := \left\{ l \colon E(G) \to \mathbb{R}_{\geq 0} \} \mid \sum l(e) = 1 \right\} \subset \overline{C}(\mathbf{G}).$$

So $\Delta(\mathbf{G})$ is a simplex of dimension $|E(G)|-1$ and $\overline{C}(\mathbf{G})$ is an infinite cone over it.

Now for types $\mathbf{G}, \mathbf{G}' \in T_{g,n}$ and an isomorphism

$$\alpha \colon \mathbf{G}' \xrightarrow{\cong} \mathbf{G}/S$$

for some $S \subseteq E(G)$, we associate the linear map

$$L_\alpha \colon \overline{C}(\mathbf{G}') \to \overline{C}(\mathbf{G})$$

that identifies $\overline{C}(\mathbf{G}')$ with the face of $\overline{C}(\mathbf{G})$ consisting of those $l \in \mathbb{R}_{\geq 0}^{E(G)}$ that are zero on S. That is, it sends $l' \colon E(G') \to \mathbb{R}_{\geq 0}$ to $l \colon E(G) \to \mathbb{R}_{\geq 0}$ with

$$l(e) = \begin{cases} 0 & \text{if } e \in S, \\ l'(\alpha^{-1}(e)) & \text{otherwise.} \end{cases}$$

Call such a linear map L_α a *face identification.* Note that we specifically allow S to be empty, in which case an automorphism of $\mathbf{G}$ produces a map $\overline{C}(\mathbf{G}) \to \overline{C}(\mathbf{G})$.

Definition 2.5 (Brannetti et al. 2011; Caporaso 2013) The *tropical moduli space of curves*, denoted $M_{g,n}^{\mathrm{trop}}$, is the colimit in the category of topological spaces

$$\varinjlim(\{\overline{C}(\mathbf{G})\}, \{L_\alpha\})$$

as $\mathbf{G}$ ranges over $T_{g,n}$ and $\{L_\alpha\}$ is the set of all face identifications.

Note that the points of $M_{g,n}^{\text{trop}}$ are in bijection with isomorphism classes of tropical curves of genus g and n marked points.

Definition 2.6 Let $\Delta_{g,n}$ denote the subset of $M_{g,n}^{\text{trop}}$ parametrizing tropical curves of total length 1. Thus

$$\Delta_{g,n} = \varinjlim(\{\Delta(\mathbf{G})\}, \{L_\alpha\}),$$

where by abuse of notation we write L_α for the restrictions of the maps L_α to the simplices $\Delta(\mathbf{G})$.

Thus $M_{g,n}^{\text{trop}}$ is an infinite cone over $\Delta_{g,n}$; the cone point is the unique tropical curve in $M_{g,n}^{\text{trop}}$ that is a single vertex of weight g and n markings.

Let $M_{g,n}^{\text{br}}$ to be the closure in $M_{g,n}^{\text{trop}}$ of the set of tropical curves containing a bridge. We let $\Delta_{g,n}^{\text{br}}$ be the closed subset of $M_{g,n}^{\text{br}}$ of curves of total length 1, called the *bridge locus*. Then recall:

Theorem 2.7 (Chan et al. 2016, Theorem 1.1(3)) *For all $g > 0$, $\Delta_{g,n}^{\text{br}}$ is contractible.*

Note (Allcock et al. 2019, Theorem 6.1) proves that a larger locus, consisting of $\Delta_{g,n}^{\text{br}}$ together with the locus of tropical curves whose underlying graph has multiedges (i.e., parallel edges), is also contractible.

It will be useful to characterize the set of combinatorial types of curves occurring in the bridge locus. That is, let $T_{g,n}^{\text{br}}$ be the closure under the contraction operation of the combinatorial types that have bridges. Call a type *nonrepeating* if the marking function m is injective, and *repeating* otherwise.

Lemma 2.8 *Fix $g, n \geq 0$ with $2g-2+n > 0$. A type $\mathbf{G} \in T_{g,n}$ lies in $T_{g,n}^{\text{br}}$ if and only if*

1. $\mathbf{G}$ *is repeating, or*
2. $\mathbf{G}$ *has a virtual cut vertex* (see §2.1).

Proof Suppose $\mathbf{H} \in T_{g,n}^{\text{br}}$. Then $\mathbf{H}$ is a contraction of some type $\mathbf{G} = (G, m, w) \in T_{g,n}^{\text{br}}$ that has a bridge, say $e = v_1v_2 \in E(G)$. We will show that (1) or (2) holds for $\mathbf{H}$ by analyzing $\mathbf{G}$. Basically, if e separates two subcurves of positive genus, then $\mathbf{G}$ and hence $\mathbf{H}$ must have a virtual cut vertex. If instead one subcurve has genus 0 then $\mathbf{G}$ and hence $\mathbf{H}$ must be repeating.

More formally: consider the following operation on $\mathbf{G}$. We delete e and add a new marked point to each of v_1 and v_2, obtaining two types $\mathbf{G}_1 = (G_1, m_1, w_1)$ and $\mathbf{G}_2 = (G_2, m_2, w_2)$, which are each stable by the stability condition (2).

Now, if $g(\mathbf{G}_i) > 0$ for both i, then e must have been a bridge in the unmarked type for $\mathbf{G}$. Then any contraction of $\mathbf{G}$ has a virtual cut vertex. Otherwise, one of the graphs, say G_1, is a tree. If $G_1 = \{v_1\}$ then it supports at least three markings, of which at least two are original to $\mathbf{G}$. Consider any leaf of G_1 other than v_1; then it supports at least two markings in G. So $\mathbf{G}$ was repeating, and any contraction of $\mathbf{G}$ must be repeating. This proves that if $\mathbf{H} \in T^{\mathrm{br}}_{g,n}$ then (1) or (2) holds.

For the converse, suppose that $\mathbf{H}$ has a virtual cut vertex. Then this vertex can be expanded into a bridge that separates two types whose genera sum to g. (For example, the cut vertex in the third graph of Fig. 1 can be expanded to obtain the second graph in Fig. 1.) Similarly, if $\mathbf{H}$ is repeating, then $\mathbf{H}$ has a vertex v supporting markings $i \neq j$. So the vertex v can be expanded into a bridge that separates markings i and j from the rest of the curve. □

3 A CW structure on the quotient by the bridge locus

Throughout the rest of this paper, set $g = 2$. We now study the quotient

$$\Delta'_{2,n} := \Delta_{2,n}/\Delta^{\mathrm{br}}_{2,n}.$$

The space $\Delta_{2,n}$ can certainly be given a CW complex structure in which $\Delta^{\mathrm{br}}_{2,n}$ is a subcomplex. For example, one can first take a barycentric subdivision of the simplices $\Delta(\mathbf{G})$. Then $\Delta_{2,n}$ is in fact a Δ-complex on the barycentric cells. (It is not a *simplicial* complex on the barycentric cells unless $n = 0$. For an example, consider the triangle corresponding to the graph with vertices $\{x, y\}$, edges $\{xy, xy, yy\}$, no vertex weights, and a single marking at x.) In any case, $\Delta'_{2,n}$ is a CW complex which is homotopy equivalent to $\Delta_{2,n}$ since $\Delta^{\mathrm{br}}_{2,n}$ is contractible.

We will immediately dispense with the barycentric cell structure on $\Delta_{2,n}$, because it is impractical. (Note, however, that baryenctric subdivisions on tropical moduli spaces are used successfully in computations in Allcock et al. (2019).) Instead, in this section we will equip $\Delta'_{2,n}$ with a simpler CW structure. It will take advantage of the fact that the tropical curves in the part of $\Delta_{2,n}$ that has not been collapsed have very limited automorphisms, namely, as long as $n \geq 4$, at most a single transposition of two parallel edges. So, in what follows, we take $n \geq 4$. The cases $n \leq 3$ are small enough to be handled directly, as in Sect. 5.

Write $T^{\Theta}_{2,n}$ for those nonrepeating types $\mathbf{G} \in T_{2,n}$ whose underlying type is a theta graph, i.e. of type I in Fig. 1. We call these *theta types*. We emphasize that by definition they must be nonrepeating. See Fig. 2 for some examples.

Lemma 3.1 *We have*

$$T_{2,n} = T^{\mathrm{br}}_{2,n} \amalg T^{\Theta}_{2,n}.$$

Proof This is immediate from Lemma 2.8, noting that of the seven types in $T_{2,0}$, shown in Fig. 1, all but type I have a virtual cut vertex. □

Now we characterize the automorphisms of theta types. Note that every vertex of a theta type must have weight zero, so we write $\mathbf{G} = (G, m, 0) = (G, m)$ for short.

Lemma 3.2 *Let $n \geq 4$ and let $\mathbf{G} = (G, m) \in T^{\Theta}_{2,n}$. Then* $\mathrm{Aut}(\mathbf{G})$ *is either trivial or is generated by a single transposition of parallel edges.*

Proof Consider first an unmarked graph of type I, which we will henceforth call Θ as a memory aid. Let e_1, e_2, e_3 denote the three edges of Θ. Now think of constructing $\mathbf{G}$ by adding n markings at distinct points anywhere on Θ, i.e., either on the two vertices of Θ or in the interiors of the three edges. Formally, of course, adding a marked point to the interior of an edge should be viewed as subdividing the edge and then marking the new vertex.

First suppose some e_i had its interior marked twice. Then any automorphism of $\mathbf{G}$ must fix pointwise the path corresponding to e_i, including its endpoints. Then $\mathbf{G}$ has at most an automorphism swapping parallel edges: this automorphism would occur precisely if all n markings were to lie on a single e_i.

Now suppose instead that each e_i has its interior marked at most once. Then $n \geq 4$ implies that one or both of the vertices of Θ were marked. Thus any automorphism of $\mathbf{G}$ must fix its two 3-valent vertices. Furthermore, at least two of the edges of Θ, say e_1 and e_2, have their interiors marked; so any automorphism of $\mathbf{G}$ must fix the two paths corresponding to e_1 and e_2. Hence all of $\mathbf{G}$ must be fixed. So in this case, $\mathbf{G}$ has trivial automorphism group. $\square$

Definition 3.3 Let $\mathbf{G} = (G, m) \in T^{\Theta}_{2,n}$ with $n \geq 4$.

1. A *decoration* δ on $\mathbf{G}$ is a choice, for any pair of parallel edges e and e' in G, of a formal relation
$$e \leq_\delta e' \qquad \text{or} \qquad e =_\delta e' \qquad \text{or} \qquad e \geq_\delta e'.$$
2. A *decorated type* is a pair $(\mathbf{G}, \delta)$. If $\mathrm{Aut}(\mathbf{G})$ is trivial, then G has no pairs of parallel edges, and we may simply write $\mathbf{G}$ instead of $(\mathbf{G}, \delta)$. An isomorphism of decorated types $(\mathbf{G}, \delta)$ and $(\mathbf{G}', \delta')$ is an isomorphism $\mathbf{G} \cong \mathbf{G}'$ carrying δ to δ'. Write $DT^{\Theta}_{2,n}$ for the set of (isomorphism classes of) decorated types.
3. For each $(\mathbf{G}, \delta) \in DT^{\Theta}_{2,n}$, we let
$$\Delta(\mathbf{G}, \delta) \subset \mathbb{R}^{E(G)}$$
be the polytope parametrizing those $l \colon E(G) \to \mathbb{R}_{\geq 0} \in \Delta(\mathbf{G})$ such that if e and e' are parallel then $l(e)$ and $l(e')$ satisfy the relation given by δ.

For any $e \in E(G)$, write l_e for the indicator length function for e, i.e. $l_e(e) = 1$ and $l_e(e') = 0$ for $e' \neq e$.

Lemma 3.4 *Let* $(\mathbf{G}, \delta) \in DT^{\Theta}_{2,n}$ *be any decorated type.*

1. $\Delta(\mathbf{G}, \delta)$ *is a simplex of dimension* $|E(G)| - 1-$ *#equal signs in* δ.
2. *The vertices of* $\Delta(\mathbf{G}, \delta)$ *are*
$$\left\{l_e \mid e \text{ is a singleton or } e \geq_\delta e'\right\} \cup \left\{\frac{1}{2}(l_e + l_{e'}) \mid e =_\delta e' \text{ or } e \leq_\delta e'\right\}.$$
3. *The codimension-1 faces of* $\Delta(\mathbf{G}, \delta)$ *are cut out by equations*

- $l(e) = 0$, *where* e *is a singleton or* $e \leq_\delta e'$, *or*
- $l(e) = l(e') = 0$, *where* $e =_\delta e'$, *or*
- $l(e) = l(e')$, *where* $e \geq_\delta e'$.

4. *A point* $l\colon E(G) \to \mathbb{R}_{\geq 0}$ *is in the relative interior of* $\Delta(\mathbf{G}, \delta)$ *if and only if* $l(e) \neq 0$ *for each* $e \in E(G)$, *and further*

$$l(e) < l(e') \quad \textit{if} \quad e \leq_\delta e' \qquad \textit{and} \qquad l(e) = l(e') \quad \textit{if} \quad e =_\delta e'.$$

Proof From Definition 3.3, it follows that the points in $\Delta(\mathbf{G}, \delta)$ are exactly those that can be expressed as a convex combination of the finitely many points listed in part (2), and that this expression is unique if it exists. Thus $\Delta(\mathbf{G}, \delta)$ is a simplex. It has a vertex for every singleton e, two vertices for every $e \parallel e'$ with $e \leq_\delta e'$ or $e \geq_\delta e'$, and a single vertex for every $e \parallel e'$ with $e =_\delta e'$. Thus the number of vertices in $\Delta(\mathbf{G}, \delta)$ is $|E(G)|$ − #equal signs in δ. Next, one may check that each of the linear slices of $\Delta(\mathbf{G}, \delta)$ listed in (3) contains all vertices but one, and that they are all distinct. Finally, the points described in (4) are precisely those points of $\Delta(\mathbf{G}, \delta)$ not contained in any facet. □

Let $n \geq 4$. We now describe a CW structure on $\Delta'_{2,n} = \Delta_{2,n}/\Delta^{\mathrm{br}}_{2,n}$. We will give a finite number of maps of simplices into $\Delta'_{2,n}$, and then prove that they are characteristic maps.

First, let $\bullet_{\mathrm{br}}$ be a 0-simplex, i.e. a point, and consider the map $\phi_{\mathrm{br}}\colon \bullet_{\mathrm{br}} \to \Delta'_{2,n}$ sending $\bullet_{\mathrm{br}}$ to the collapse of $\Delta^{\mathrm{br}}_{2,n}$. Next, for each $(\mathbf{G}, \delta) \in DT^{\Theta}_{g,n}$, consider the map

$$\phi_{\mathbf{G},\delta}\colon \Delta(\mathbf{G}, \delta) \longrightarrow \Delta(\mathbf{G}) \longrightarrow \Delta_{2,n} \longrightarrow \Delta'_{2,n}$$

that factors through the inclusion of polytopes $\Delta(\mathbf{G}, \delta) \subseteq \Delta(\mathbf{G})$ and the canonical map $\iota_{\mathbf{G}}\colon \Delta(\mathbf{G}) \to \Delta_{2,n}$.

Proposition 3.5 *Let* $n \geq 4$. *The maps* ϕ_{br} *and* $\{\phi_{\mathbf{G},\delta} \mid (\mathbf{G}, \delta) \in DT^{\Theta}_{2,n}\}$ *are the characteristic maps for a CW complex structure on* $\Delta'_{2,n}$.

Proof We will show two claims:

1. the maps ϕ_{br} and $\{\phi_{\mathbf{G},\delta}\}$, restricted to the interiors of their domains, are homeomorphisms onto their images. We call the images of these interiors *cells* and denote them ϵ_{br} and $\epsilon_{\mathbf{G},\delta}$. Furthermore, we have
$$\Delta'_{2,n} = \left(\coprod \epsilon_{\mathbf{G},\delta}\right) \amalg (\epsilon_{\mathrm{br}}) \tag{4}$$
as an equality of sets.
2. For each $(\mathbf{G}, \delta)$, we have that $\phi_{\mathbf{G},\delta}(\partial\Delta(\mathbf{G}, \delta))$ is contained in a union of cells of smaller dimension.

Then the proposition follows immediately. Actually we will prove more in the process: we will analyze the contribution that each facet of a cell $\Delta(\mathbf{G}, \delta)$ makes to the boundary

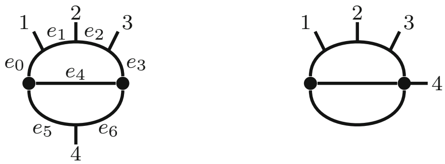

Fig. 2 Two examples of cyclic theta types in $T^{\sigma}_{2,4}$ for $\sigma = (1\ 2\ 3\ 4)$, of form $\binom{3}{1}$ and $\binom{3}{0}$, respectively. Our convention is that the edges are ordered in reading order, as shown on the left. The type on the right is obtained by contracting edge e_6; note that this operation creates a new pair of parallel edges

maps in the cellular chain complex. In order to talk about the chain maps, we shall assume that the vertices of each simplex $\Delta(\mathbf{G}, \delta)$ are equipped with a total ordering, and the faces of $\Delta(\mathbf{G}, \delta)$ are given the corresponding orientation.

Let us start by proving (1). The point $\bullet_{\mathrm{br}}$ is mapped in $\Delta'_{2,n}$ to the collapse of $\Delta^{\mathrm{br}}_{2,n}$. Next, if (G, l, m, w) is a tropical curve in $\Delta_{2,n}$ of theta type, then by Lemma 3.4(4), there is a unique decoration δ of $\mathbf{G} = (G, m, w)$ for which l lies in the relative interior of $\Delta(\mathbf{G}, \delta)$. So equation (4) holds. Now for any $(\mathbf{G}, \delta) \in DT^{\Theta}_{2,n}$, consider the map

$$\Delta(\mathbf{G}, \delta)/\sim \;\longrightarrow \Delta'_{2,n}$$

induced from identifying points in $\Delta(\mathbf{G}, \delta)$ in the same fiber of $\phi_{\mathbf{G},\delta}$. Note that $\Delta'_{2,n}$ is Hausdorff, as it is the quotient of a CW complex by a subcomplex. Thus this map is an injection with compact domain and Hausdorff codomain, so it is a homeomorphism onto its image. We already argued that $\sim$ is trivial on the interior of $\Delta(\mathbf{G}, \delta)$, so $\phi_{\mathbf{G},\delta}$ restricts to a homeomorphism on the interior of $\Delta(\mathbf{G}, \delta)$. This proves (1).

For part (2), let $(\mathbf{G} = (G, m, w), \delta) \in DT^{\Theta}_{2,n}$ and let F be a facet of $\Delta(\mathbf{G}, \delta)$. By Lemma 3.4 we have the following three cases.

Case 1: F is defined by $l(e) = 0$ for some $e \in E(G)$.

First, if $\mathbf{G}/e \in T^{\mathrm{br}}_{2,n}$, then in $\Delta'_{2,n}$, we have that F is collapsed to the 0-cell ϵ_{br}. Note that F does not contribute to the boundary map, because the codimension of ϵ_{br} is too large.

So suppose that $\mathbf{G}/e \notin T^{\mathrm{br}}_{2,n}$. Then $\mathbf{G}/e \in T^{\Theta}_{2,n}$ by Lemma 3.1. We have two possibilities. The first possibility is that $\mathbf{G}$ had no parallel edges, but $\mathbf{G}/e$ does, say edges $f_1, f_2 \in E(G/e)$. See Fig. 2.

By abuse of notation, the edge in G giving rise to f_i will also be called f_i. Now F obviously decomposes as the union of the two closed simplices

$$F \cap \{l(f_1) \le l(f_2)\} \quad \text{and} \quad F \cap \{l(f_1) \ge l(f_2)\}.$$

In $\Delta'_{2,n}$, each of these halves is identified with $\Delta(\mathbf{G}/e, f_1 \le f_2)$, *but with opposite signs,* since the two identifications differ by a flip of parallel edges. Thus F is sent to the image of $\Delta(\mathbf{G}/e, f_1 \le_{\delta} f_2)$ in $\Delta'_{2,n}$, which does indeed have codimension 1; however, its total contribution to the cellular boundary map is zero due to cancellation.

The other possibility is that contracting e did not create new parallel edges. Then F is identified with $\Delta(\mathbf{G}/e, \delta|_{\mathbf{G}/e})$ with a contribution of ± 1 to the boundary map. The sign simply depends on our choice of vertex-ordering for the two simplices.

Case 2: *F is defined by $l(e) = l(e') = 0$*, for $e =_\delta e'$. Then F is sent to ϵ_{br} in $\Delta'_{2,n}$, because contracting e and e' produces a vertex of positive weight.

Case 3: *F is defined by $l(e) = l(e')$*, for $e \leq_\delta e'$. Then F is identified with $\Delta(\mathbf{G}, e =_\delta e')$ and the contribution to the boundary map is again ± 1.

In each case, F was identified with the closure of a cell of smaller dimension, so Proposition 3.5(2) is proved. □

4 Homology of the cyclic theta complex

Let $\mathbf{G} = (G, m, w) \in T^{\Theta}_{2,n}$. We say $\mathbf{G}$ is a *cyclic theta* if there is a cycle in $\mathbf{G}$ passing through all the marked points. If not, we say $\mathbf{G}$ is a *full theta*. We let $T^{\text{cyc}}_{2,n}$ and $T^{\text{full}}_{2,n}$ denote the set of all cyclic theta types and the set of all full theta types, respectively. In this section, we compute the integral homology of the subcomplex of $\Delta_{2,n}$ supported on cyclic theta types. In the next section, we will study the remaining full types, which will allow us to prove Theorems 1.1 and 1.2.

Definition 4.1 Let $C_{2,n}$ denote the subcomplex of $\Delta'_{2,n}$

$$C_{2,n} = \epsilon_{\text{br}} \cup \bigcup_{\mathbf{G} \in T^{\text{cyc}}_{2,n}} \epsilon_{\mathbf{G},\delta},$$

called the cyclic theta complex.

This is a subcomplex of $\Delta'_{2,n}$ because every contraction of a cyclic theta either is in the bridge locus or is again a cyclic theta type. The next theorem computes the integral homology of $C_{2,n}$, and we will prove it in the rest of the section. Our proof also explicitly constructs representatives for the nonzero homology classes in degree $n + 1$.

Theorem 4.2 *For all $i \geq 0$,*

$$\widetilde{H}_i(C_{2,n}, \mathbb{Z}) = \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{\frac{(n-1)!}{2}} & \text{if } i = n+1, \\ 0 & \text{else.} \end{cases}$$

Proof Consider the $(n-1)!/2$ possible unoriented cyclic orderings of the set $\{1, \ldots, n\}$, i.e. ways to place n numbers around a circle in either direction. For example, when $n = 4$ the three cyclic orderings are (1234), (1243), (1324). Suppose $\mathbf{G} = (G, m, w) \in T^{\text{cyc}}_{2,n}$. Notice that $\mathbf{G}$ determines a unique cyclic ordering on $\{1, \ldots, n\}$. Given a cyclic ordering σ, write $T^{\sigma}_{2,n}$ for the types in $T^{\text{cyc}}_{2,n}$ inducing σ.

Now suppose $e \in E(G)$. As noted above, either $\mathbf{G}/e \in T^{\text{br}}_{2,n}$ or $\mathbf{G}/e$ is again a cyclic theta. The crucial observation is that in the latter case, $\mathbf{G}$ and $\mathbf{G}/e$ determine the *same* cyclic ordering on $\{1, \ldots, n\}$. Thus, for any cyclic ordering σ of $\{1, \ldots, n\}$, we may define $C^\sigma_{2,n}$ to be the subcomplex on cells

$$C^\sigma_{2,n} = \epsilon_{\mathrm{br}} \cup \bigcup_{\mathbf{G} \in T^\sigma_{2,n}} \epsilon_{\mathbf{G},\delta}.$$

Then we have

$$C_{2,n} = \bigvee_\sigma C^\sigma_{2,n},$$

where the wedge is over all cyclic orderings of $\{1, \dots, n\}$ and the identification is at the 0-cell ϵ_{br}. Therefore

$$\widetilde{H}_i(C_{2,n}, \mathbb{Z}) \cong \bigoplus_\sigma \widetilde{H}_i(C^\sigma_{2,n}, \mathbb{Z}).$$

Therefore the following claim implies the theorem.

Claim 4.3 *For any cyclic ordering σ of $\{1, \dots, n\}$, we have*

$$\widetilde{H}_i(C^\sigma_{2,n}, \mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } i = n+1, \\ 0 & \text{else.} \end{cases}$$

Now we prove the claim. Clearly the choice of σ does not matter; choose $\sigma = (1 \cdots n)$ once and for all. Write $C^{\sigma,d}_{2,n}$ for the d-skeleton of $C^\sigma_{2,n}$. For simplicity write

$$V_i = H_i(C^{\sigma,i}_{2,n}, C^{\sigma,i-1}_{2,n}; \mathbb{Z}) \tag{5}$$

Consider the cells of $C^\sigma_{2,n}$. A type $\mathbf{G} = (G, m) \in T^\sigma_{2,n}$ is obtained from an unmarked graph of type I in Fig. 1, which we will once again call Θ, by marking n distinct points on it. (If the interior of an edge is marked then we consider it to be subdivided there.) Thus, G has $n+1, n+2$, or $n+3$ edges, depending on whether 0, 1, or 2 of the vertices of Θ are marked.

Then by Lemma 3.4 (1), the cell $\epsilon_{\mathbf{G},\delta}$ has dimension $n-1, n, n+1$, or $n+2$. This already shows that $\widetilde{H}_i(C^\sigma_{2,n}, \mathbb{Z}) = 0$ for $i < n-1$. We are left to compute the homology of the complex of $\mathbb{Z}$-modules

$$0 \longrightarrow V_{n+2} \xrightarrow{\partial_{n+2}} V_{n+1} \xrightarrow{\partial_{n+1}} V_n \xrightarrow{\partial_n} V_{n-1} \longrightarrow 0 \tag{6}$$

with the modules V_i as in (5).

We need to set some notation for the types $\mathbf{G}$ involved, and set a convention for ordering the vertices of the relevant simplices $\Delta(\mathbf{G})$ and $\Delta(\mathbf{G}, \delta)$. Choose, once and for all, one of the two possible orientations of σ, say the oriented cyclic order $1, \dots, n$. We will say that $\mathbf{G}$ is of the form $(\epsilon_1, k_1, \epsilon_2, k_2)$, with $\epsilon_i \in \{0, 1\}$ and $\epsilon_1 + k_1 + \epsilon_2 + k_2 = n$, if $\mathbf{G}$ is obtained by adding markings to a Θ graph such that, in the (oriented) cyclic order, there are ϵ_1 marked points on a vertex of Θ, then k_1 marked points on an edge, then ϵ_2 marked points on the other vertex of Θ, then k_2 marked points on another edge. Obviously we have an equivalence of such forms via cyclic reordering: $(\epsilon_1, k_1, \epsilon_2, k_2) \sim (\epsilon_2, k_2, \epsilon_1, k_1)$. For example, the types in Fig. 2 have forms (0, 3, 0, 1) and (0, 3, 1, 0) respectively.

We adopt the convention that the edges of a type $\mathbf{G}$ of form $(\epsilon_1, k_1, \epsilon_2, k_2)$ are ordered as follows: we place the k_1 marked points in order, left to right, on the top edge of Θ, and the k_2 marked points right to left on the bottom edge of Θ; then we put the edges in reading order. See Fig. 2 for an example. Note that our choice of ordering of the edges of $\mathbf{G}$ induces an order on the vertices of $\Delta(\mathbf{G})$, when $\mathbf{G}$ is an automorphism-free type. We will take this ordering throughout.

In keeping with the above convention, we will sometimes notate the forms

$$(0, k_1, 0, k_2) \qquad (1, k_1, 0, k_2) \qquad (0, k_1, 1, k_2) \qquad (1, k_1, 1, k_2)$$

as

$$\binom{k_1}{k_2} \qquad \text{-}\!\binom{k_1}{k_2} \qquad \binom{k_1}{k_2}\!\text{-} \qquad \text{-}\!\binom{k_1}{k_2}\!\text{-}$$

respectively, simply as a visual aid. Furthermore we notate a specific type in the same way. For example the type on the left in Fig. 2 will be written $\binom{123}{4}$ and it has form $\binom{3}{1}$.

Suppose $\mathbf{G}$ has a nontrivial automorphism. That is, $\mathbf{G}$ has form $(\epsilon_1, k_1, \epsilon_2, k_2)$ and $k_1 = 0$ or $k_2 = 0$. Without loss of generality, suppose $k_2 = 0$. Then $\mathbf{G}$ has a pair of parallel edges e and e', and $\mathbf{G}$ thus admits two possible decorations δ, namely $e = e'$ or $e \le e'$. Then by slight abuse of notation we will write $(\mathbf{G}, \delta)$ as having form $(\epsilon_1, k_1, \epsilon_2, =)$ or $(\epsilon_1, k_1, \epsilon_2, \le)$. We extend this notation to the shorthand above. For example $(1, k_1, 0, \le)$ will be written $\text{-}\!\binom{k_1}{\le}$.

Furthermore, we adopt the following convention on the order of the vertices of $\Delta(\mathbf{G}, \delta)$ in this case. Write $e_0, \ldots, e_{n'}$ for the edges $E(G) - \{e, e'\}$ taken as they appear in the directed path compatible with the chosen orientation of σ. See Fig. 3. So $n' = n - \epsilon_1 - \epsilon_2$. If the decoration δ is $e = e'$ then the vertices of $\Delta(\mathbf{G}, \delta)$ are

$$\left\{ \frac{1}{2}(l_e + l_{e'}), l_{e_0}, \ldots, l_{e_{n'}} \right\}$$

by Lemma 3.4(2), and we take them in that order. If the decoration is $e \ge e'$ then the vertices of $\Delta(\mathbf{G}, \delta)$ are

$$\{l_e, \frac{1}{2}(l_e + l_{e'}), l_{e_0}, \ldots, l_{e_{n'}}\}$$

and we take them in that order. We use analogous conventions if $\mathbf{G}$ has the form $(\epsilon_1, 0, \epsilon_2, k_2)$.

In this way, we have chosen orientations for each of the cells of $C^{\sigma}_{2,n}$. We will write down the boundary maps ∂_i in (6) explicitly in terms of them. Of course, they are just conventions chosen for the sake of explicit computation; they do not affect the Smith normal forms of ∂_i. $\square$

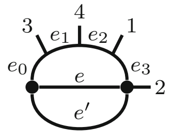

Fig. 3 The edges $e_0, \dots, e_{n'}$ in the ordering compatible with the oriented cyclic order $\sigma = (1234)$. Here $n' = n - 1 = 3$

Lemma 4.4 *Suppose* $(\mathbf{G}, \delta)$ *is a decorated type that is a cyclic theta, with cyclic order* σ*. Then we have the following case analysis for the possible forms of* $(\mathbf{G}, \delta)$*:*

1. $(\mathbf{G}, \delta)$ *has one of the forms*
$$\binom{n}{\leq} \quad \binom{n-1}{1} \quad \cdots \quad \binom{\lceil n/2 \rceil}{\lfloor n/2 \rfloor}$$
and in these cases $\epsilon_{\mathbf{G},\delta}$ *has dimension* $n + 2$*;*
2. $(\mathbf{G}, \delta)$ *has one of the forms*
$$\binom{n}{=} \quad {-}\!\binom{n-1}{\leq} \quad {-}\!\binom{n-2}{1} \quad \cdots \quad {-}\!\binom{1}{n-2} \quad {-}\!\binom{\leq}{n-1},$$
and in these cases $\epsilon_{\mathbf{G},\delta}$ *has dimension* $n + 1$*;*
3. $(\mathbf{G}, \delta)$ *has one of the forms*
$${-}\!\binom{n-1}{=} \quad {-}\!\binom{=}{n-1} \quad {-}\!\binom{n-2}{\leq}\!{-} \quad {-}\!\binom{n-3}{1}\!{-} \quad \cdots \quad {-}\!\binom{\lceil n/2 \rceil - 1}{\lfloor n/2 \rfloor - 1}\!{-},$$
and in these cases $\epsilon_{\mathbf{G},\delta}$ *has dimension* n*;*
4. $(\mathbf{G}, \delta)$ *has the form*
$${-}\!\binom{n-2}{=}\!{-}$$
and in this case $\epsilon_{\mathbf{G},\delta}$ *has dimension* $n - 1$.

Proof This is a straightforward case analysis; the dimensions of the cells were computed in Lemma 3.4(1). □

By counting we deduce the following.

Lemma 4.5 *The modules* V_i *in* (5) *have ranks:*

1. rank $V_{n+2} = \binom{n+1}{2}$,
2. rank $V_{n+1} = 2\binom{n+1}{2}$,
3. rank $V_n = \binom{n+1}{2} + n$,
4. rank $V_{n-1} = n$.

In this lemma, $\binom{n+1}{2}$ has its usual meaning "$n+1$ choose 2," not to be confused with the previous notation.

Proof of Lemma 4.5 This is a routine count in each case. For example, part (1) is true because there are $\binom{n+1}{2} = n(n+1)/2$ ways to cut the numbers $1, \dots, n$, arranged around a circle, into two segments (with length zero segments permitted). Alternatively, the counts can be derived directly from Lemma 4.4. □

We can now compute the homology of (6). The proof is by direct computation. We compute the Smith normal forms over $\mathbb{Z}$ of each boundary map ∂_{n+2}, ∂_{n+1}, and ∂_n. The boundary maps themselves are computed using the case analysis in the proof of Proposition 3.5. We compute them with respect to the choices of orientations established earlier in the section. The homology computations then immediately follow from the Smith normal forms. From here on, SNF denotes Smith normal form, and **1** and **0** denote identity and zero matrices respectively.

Claim 4.6 *The normal form of* ∂_{n+2} *is*

$$\mathrm{SNF}(\partial_{n+2}) = \begin{array}{cl} n(n+1)/2 & \\ \left(\begin{array}{c} \begin{matrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 2 \end{matrix} \\ \hline \\ \mathbf{0} \\ \\ \end{array}\right) & \begin{array}{l} n(n+1)/2 \\ \\ \\ \\ n(n+1)/2 \end{array} \end{array} \tag{7}$$

Claim 4.7 *The normal form of* ∂_{n+1} *is*

$$\mathrm{SNF}(\partial_{n+1}) = \begin{array}{cl} \begin{array}{cc} n(n+1)/2 & n(n+1)/2 \end{array} & \\ \left(\begin{array}{c|c} \mathbf{1} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right) & \begin{array}{l} n(n+1)/2 \\ n \end{array} \end{array} \tag{8}$$

Claim 4.8 *The normal form of* ∂_n *is*

$$\mathrm{SNF}(\partial_n) = \begin{array}{cl} \begin{array}{cc} n & n(n+1)/2 \end{array} & \\ \left(\begin{array}{c|c} \mathbf{1} & \mathbf{0} \end{array}\right) & n \end{array} \tag{9}$$

Now we prove Claims 4.6, 4.7, 4.8. The boundary maps look different according to the parity of n, so we split our analysis into cases.

Proof (Proof of Claim 4.6 for n odd) Let $n = 2k + 1$. First, consider the $n \times n$ square submatrix of ∂_{n+2} whose columns are indexed by all types of the form $\binom{k+1}{k}$ and whose rows are indexed by all types of the form $-\binom{k}{k}$. Call this submatrix A. We now show that $\det(A) \neq 0$, indeed, that A is an $n \times n$ double diagonal matrix, with two entries 1 in each column, corresponding to two permutations in S_n of the same sign.

For example, when $n = 5$, we have

$$A = \begin{array}{cc} & \begin{array}{ccccc} \binom{123}{54} & \binom{234}{15} & \binom{345}{21} & \binom{451}{32} & \binom{512}{43} \end{array} \\ \begin{array}{c} 1\binom{23}{54} \\ 2\binom{34}{15} \\ 3\binom{45}{21} \\ 4\binom{51}{32} \\ 5\binom{12}{43} \end{array} & \left(\begin{array}{ccccc} 1 & & & 1 & \\ & 1 & & & 1 \\ 1 & & 1 & & \\ & 1 & & 1 & \\ & & 1 & & 1 \end{array}\right) \end{array}.$$

More generally, for each $i \in \{1, \dots, n\}$, write

$$\mathbf{G}_i = \binom{i \ \cdots \ k+i}{i-1 \ \cdots \ k+i+1}, \qquad \mathbf{H}_i = {}_i\binom{i+1 \ \cdots \ k+i}{i-1 \ \cdots \ k+i+1}.$$

Then the column of A indexed by $\mathbf{G}_i$ has an entry 1 in row $\mathbf{H}_i$ and in row $\mathbf{H}_{k+i}$ (with indices modulo n), and no other nonzero entries. This is because $\mathbf{H}_i$ is obtained by contracting edge e_0 of $\mathbf{G}_i$, i.e. $\Delta(\mathbf{H}_i)$ is identified with facet 0 of $\Delta(\mathbf{G}_i)$. Similarly, $\mathbf{H}_{k+i}$ is obtained by contracting edge e_{k+1} of $\mathbf{G}_i$, which gives sign $(-1)^{k+1}$; and then reversing the order of the $n + 2 = 2k + 3$ edges. This reversal is a product of $k + 1$ transpositions. Hence the total contribution is $(-1)^{2(k+1)} = 1$. Then it follows that

$$\det(A) = 1 + \operatorname{sgn} \tau = 2 \neq 0, \tag{10}$$

where $\tau \in S_n$ is the n-cycle $\tau(i) = i + k \mod n$.

Then ∂_{n+2} has the following form, for suitable orderings of rows and columns:

$$\begin{array}{cc} & \begin{array}{ccccc} \binom{k+1}{k} & \binom{k+2}{k-1} & \cdots & \binom{2k}{1} & \binom{2k+1}{\le} \end{array} \\ \begin{array}{c} \text{-}\binom{k}{k} \\ \text{-}\binom{k+1}{k-1} \\ \vdots \\ \text{-}\binom{2k-1}{1} \\ \text{-}\binom{2k}{\le} \\ \binom{k+1}{k-1}\text{-} \\ \binom{k+2}{k-2}\text{-} \\ \vdots \\ \binom{2k}{\le}\text{-} \\ \binom{2k+1}{=} \end{array} & \left(\begin{array}{ccccc} A & & & \cdots & \mathbf{0} \\ * & \mathbf{1} & & & \vdots \\ & * & \mathbf{1} & & \\ \vdots & & * & \mathbf{1} & \\ \mathbf{0} & \cdots & & \mathbf{0} & \mathbf{1} \\ \hline -\mathbf{1} & * & & \cdots & \mathbf{0} \\ & -\mathbf{1} & * & & \vdots \\ & & -\mathbf{1} & * & \\ \vdots & & & & -\mathbf{1} \\ \mathbf{0} & \cdots & & & \mathbf{1} \end{array}\right) \end{array}. \tag{11}$$

Here, each block represents an $n \times n$ submatrix whose rows and columns are indexed by types of the indicated forms. The blocks notated $*$ are in fact $n \times n$ identity matrices up to row/column permutation and up to multiplication by $-\mathbf{1}$.

As a brief example of how to compute the entries in ∂_{n+2}, we explain why the $n \times n$ submatrix with columns of form $\binom{2k}{1}$ and rows of form $\binom{2k}{\leq}\!\!-$ is $\mathbf{0}$. Indeed, it is possible to contract an edge of a type of form $\binom{2k}{1}$ to obtain a type of form $\binom{2k}{\leq}\!\!-$ with two parallel edges e and e'. An example is shown in Fig. 2. (In the figure, n is even, but the essential point is the same.) However, there are two possible choices of decoration for the new type: $e \geq e'$ and $e' \geq e$. So the corresponding codimension 1 cell occurs once with each orientations, and the total contribution is zero. This was already argued in the proof of Proposition 3.5, and all of the other claimed entries of ∂_{n+2} similarly follow from the case analysis in that proof.

Now we compute $\mathrm{SNF}(\partial_{n+2})$. Consider the upper $(n+1)n/2 \times (n+1)n/2$ square submatrix of ∂_{n+2}. It is the submatrix above the line in (7). It is invertible since A is, and by (10) its Smith normal form is $\mathrm{diag}(1, \dots, 1, 2)$. Therefore either $\mathrm{SNF}(\partial_{n+2})$ is as claimed in (7), or it is instead a $(n+1)n/2$ square *identity* matrix, padded with zeroes. We show that the latter is not possible by exhibiting a torsion element in $V_{n+1}/\operatorname{im}(\partial_{n+2})$. Let v be the sum of the first $(n-1)n/2$ columns of ∂_{n+2} in (11). That is, v is sum the column vectors corresponding to all types except those of the form $\binom{2k+1}{\leq}$. By inspecting (11), we see $v \equiv 0 \mod 2$. The columns of ∂_{n+2} are linearly independent over $\mathbb{Q}$, from which it follows that $v/2 \in V_{n+1}$ is not in the image of ∂_{n+2} considered as a $\mathbb{Z}$-module map. This shows Claim 4.6 for n odd, and gives an explicit representative for the nonzero homology class of $C^{\sigma}_{2,n}$ in degree $n+1$. $\square$

Proof (Proof of Claim 4.6 for n even) Let $n = 2k$. First, some temporary notation: write $-\!\!\binom{k}{k-1}'$, respectively $-\!\!\binom{k}{k-1}''$, for the types of the form $-\!\!\binom{k}{k-1}$ in which the marking on the left vertex of Θ is in $\{1, \dots, k\}$, respectively in $\{k+1, \dots, n\}$. For example, when $n = 4$ the types of form $-\!\!\binom{2}{1}'$ are ${}_1\binom{23}{4}$ and ${}_2\binom{34}{1}$, and the types of form $-\!\!\binom{2}{1}''$ are ${}_3\binom{41}{2}$ and ${}_4\binom{12}{3}$.

Now let A be the $(3n/2) \times (3n/2)$ square submatrix of ∂_{n+2} with columns indexed by all types of the form $\binom{k+1}{k-1}$ or $\binom{k}{k}$ and rows indexed by all types of the form $\binom{k}{k-1}\!\!-$ or $-\!\!\binom{k}{k-1}'$. For example, when $n = 4$, we have

$$A = \begin{array}{c} \\ \binom{12}{4}3 \\ \binom{23}{1}4 \\ \binom{34}{2}1 \\ \binom{41}{3}2 \\ 1\binom{23}{4} \\ 2\binom{34}{1} \end{array} \!\!\begin{array}{c} \begin{array}{cccccc} \binom{123}{4} & \binom{234}{1} & \binom{341}{2} & \binom{412}{3} & \binom{12}{43} & \binom{23}{14} \end{array} \\ \begin{pmatrix} -1 & & & & 1 & \\ & -1 & & & & 1 \\ & & -1 & & -1 & \\ & & & -1 & & -1 \\ 1 & & & & & 1 \\ & 1 & & & -1 & \end{pmatrix} \end{array}. \tag{12}$$

We claim that $\det A = \pm 2$. First, consider the k types

$$\binom{k+1 \,\cdots\, 1}{k \,\cdots\, 2} \quad \cdots \quad \binom{n \;1 \cdots k}{n-1 \,\cdots\, k+1}.$$

Each of the columns they index has a single nonzero entry which is ± 1, i.e. in the k rows defined by

$$\binom{k{+}1\ \cdots\ n}{k\ \cdots\ 2}1 \quad \cdots \quad \binom{n\ 1\ \cdots\ k{-}1}{n{-}1\ \cdots\ k{+}1}k.$$

So we may delete these k rows and k columns to obtain a matrix A' with $\det A' = \pm \det A$.

Now we may check that A' has the form

$$\begin{array}{c|c}
 & \begin{array}{ccc|ccc} \binom{1\ \cdots\ k+1}{n\ \cdots\ k+2} & \cdots & \binom{k\ \cdots\ n}{k-1\ \cdots\ 1} & \binom{1\ \cdots\ k}{n\ \cdots\ k+1} & \cdots & \binom{k\ \cdots\ n-1}{k-1\ \cdots\ n} \end{array} \\
\begin{array}{c} \binom{1\ \cdots\ k}{n\ \cdots\ k+2}k{+}1 \\ \vdots \\ \binom{k\ \cdots\ n-1}{k-1\ \cdots\ 1}n \\ 1\binom{2\ \cdots\ k+1}{n\ \cdots\ k+2} \\ \vdots \\ k\binom{k+1\ \cdots\ n}{k-1\ \cdots\ 1} \end{array} & \left(\begin{array}{c|c} (-\mathbf{1})^{k+1} & \mathbf{1} \\ \hline \mathbf{1} & \begin{matrix} (-1)^k & & \\ & \ddots & \\ -1 & & (-1)^k \end{matrix} \end{array}\right)
\end{array}$$

For example, when $n = 4$, A' is the 4×4 matrix obtained from (12) by deleting the middle two rows and columns. The computations of each entry follow directly from our choice of ordering of the edges for each type. Then we have

$$\begin{aligned} \det A' &= (-1)^{k(k+1)+(k-1)k+1}\,\mathrm{sgn}(\tau_1) + \mathrm{sgn}(\tau_2) \\ &= (-1)^k + (-1)^k = \pm 2, \end{aligned} \tag{13}$$

where $\tau_1 = (n\ n{-}1\ \cdots\ k{+}1)$ is a k-cycle, and $\tau_2 = (1\ k{+}1)(2\ k{+}2)\cdots(k\ n)$ is a product of k disjoint transpositions.

Next, we can see that ∂_{n+2} has a square submatrix of the following form, again for suitable orderings of rows and columns:

$$\begin{array}{c|c}
 & \begin{array}{cccc|cc} \binom{2k}{\le} & \binom{2k-1}{1} & \cdots & \binom{k+2}{k-2} & \binom{k+1}{k-1} & \binom{k}{k} \end{array} \\
\begin{array}{c} {-}\binom{2k-1}{\le} \\ {-}\binom{2k-2}{1} \\ \vdots \\ {-}\binom{k+1}{k-2} \\ \binom{k}{k-1}{-} \\ {-}\binom{k}{k-1}' \end{array} & \left(\begin{array}{cccc|cc} \mathbf{1} & & & & \cdots & \mathbf{0} \\ & \mathbf{1} & * & & & \vdots \\ & & \mathbf{1} & * & & \\ & & & \mathbf{1} & * & \\ \hline \vdots & & & & \multicolumn{2}{c}{A} \\ \mathbf{0} & \cdots & & & & \end{array}\right)
\end{array} \tag{14}$$

Note that the submatrix (14) uses all columns of ∂_{n+2}, and moreover by (13) its normal form is a diagonal matrix with entries $(1, \dots, 1, 2)$. It follows that $\mathrm{SNF}(\partial_{n+2})$ is either of the form claimed in (7), or it is an $\binom{n+1}{2} \times \binom{n+1}{2}$ identity matrix, padded with zeroes. Again we argue that the latter is not possible by exhibiting a nontrivial torsion element of $V_{n+1}/\operatorname{im}(\partial_{n+2})$. Consider the vector v that is the sum of all of the columns of ∂_{n+2} except those indexed by types of the form $\binom{2k}{\leq}$. Note $v \equiv 0 \mod 2$. This is because v does not involve any rows indexed by $\text{-}\binom{2k-1}{\leq}$, $\binom{2k-1}{\leq}\text{-}$, or $\binom{2k}{=}$, and every row other than these $3n$ rows has exactly two ± 1 nonzero entries. So we have produced a nontrivial torsion element of $V_{n+1}/\operatorname{im}(\partial_{n+2})$, namely $v/2$. This constructs a representative for a nonzero homology class in degree $n+1$, and proves the claim. $\square$

Proof (Proof of Claim 4.7 for n odd) Let $n = 2k + 1$. The matrix for ∂_{n+1} has the following form. Again, each symbol represents an $n \times n$ submatrix, with rows and columns indexed by the types of the indicated forms.

$$
\begin{array}{cc}
 & \begin{array}{cccc|ccc|ccc}
\text{-}\binom{k}{k} & \text{-}\binom{k+1}{k-1} & \cdots & \text{-}\binom{2k-1}{1} & \binom{k+1}{k-1}\text{-} & \cdots & \binom{2k-1}{1}\text{-} & \text{-}\binom{2k}{\leq} & \binom{2k}{\leq}\text{-} & \binom{2k+1}{=}
\end{array} \\
\begin{array}{c}
\text{-}\binom{k}{k-1}\text{-} \\ \vdots \\ \text{-}\binom{2k-2}{1}\text{-} \\ \text{-}\binom{2k-1}{\leq}\text{-} \\ \text{-}\binom{2k}{=} \\ \binom{2k}{=}\text{-}
\end{array} &
\left(\begin{array}{cccc|ccc|ccc}
B & * & & & \mathbf{1} & & & & & \\
 & * & * & & * & \mathbf{1} & & & & \\
 & & * & * & & * & \mathbf{1} & & & \\
\hline
 & & & & & & & \mathbf{1} & \mathbf{1} & \\
 & & & & & & & \mathbf{1} & & \mathbf{-1} \\
 & & & & & & & & \mathbf{1} & \mathbf{1}
\end{array}\right)
\end{array}
$$

Here B is a double diagonal matrix and the matrices $*$ are signed identity matrices up to row/column permutation. After column operations preserving Smith normal form, we may assume instead that B and all the matrices $*$ are zero. Moreover the lower right $3n \times 3n$ submatrix has Smith normal form

$$
\begin{pmatrix} \mathbf{1} & & \\ & \mathbf{1} & \\ & & \mathbf{0} \end{pmatrix}.
$$

This shows Claim 4.7 for n odd. $\square$

Proof (Proof of Claim 4.7 for n even) Let $n = 2k$. The matrix for ∂_{n+1} has the following form:

$$
\begin{array}{cc}
 & \begin{array}{ccccccccc} \text{-}\binom{k}{k-1} & \cdots & \text{-}\binom{2k-2}{1} & \binom{k}{k-1}\text{-} & \cdots & \binom{2k-2}{1}\text{-} & \text{-}\binom{2k-1}{\leq} & \binom{2k-1}{\leq}\text{-} & \binom{2k}{=} \end{array} \\
\begin{array}{c} \text{-}\binom{k-1}{k-1}\text{-} \\ \vdots \\ \text{-}\binom{2k-3}{1}\text{-} \\ \text{-}\binom{2k-2}{\leq}\text{-} \\ \text{-}\binom{2k-1}{=} \\ \binom{2k-1}{=}\text{-} \end{array} &
\left(\begin{array}{ccc|ccc|ccc}
B' & & & B'' & & & & & \\
* & * & & * & \mathbf{1} & & & & \\
 & * & * & & * & \mathbf{1} & & & \\
\hline
 & & & & & & -\mathbf{1} & \mathbf{1} & \\
 & & & & & & \mathbf{1} & & -\mathbf{1} \\
 & & & & & & & \mathbf{1} & -\mathbf{1}
\end{array}\right)
\end{array}
$$

where B' and B'' are the $k \times n$ matrices

$$
\left(\begin{array}{c|c} (-\mathbf{1})^k & \mathbf{1} \end{array}\right) \quad \text{and} \quad \left(\begin{array}{c|c} \mathbf{1} & (-\mathbf{1})^k \end{array}\right)
$$

and the matrices $*$ are signed $n \times n$ identity matrices up to row/column permutation. After column operations preserving Smith normal form, we may assume instead that B' and all the matrices $*$ are zero, and that

$$
B'' = \left(\begin{array}{c|c} \mathbf{0} & (-\mathbf{1})^k \end{array}\right).
$$

Moreover the lower right $3n \times 3n$ submatrix has Smith normal form

$$
\begin{pmatrix} \mathbf{1} & & \\ & \mathbf{1} & \\ & & \mathbf{0} \end{pmatrix}.
$$

Putting these statements together shows Claim 4.7 for n even. □

Proof (Proof of Claim 4.8 for all n) This claim is clear from the fact that ∂_n restricted to types of the form $\text{-}\binom{n-2}{\leq}\text{-}$ is $\mathrm{Id}_{n\times n}$. □

Claims 4.6, 4.7, and 4.8 , which we just proved, imply Claim 4.3 immediately, and Claim 4.3 proves Theorem 4.2. □

We remark that using the proof of Theorem 4.2, we can also exhibit torsion in the integral homology of $\Delta_{2,n}$.

Proposition 4.9 *For $n \geq 5$ odd, there exists $0 \neq \alpha \in H_{n+1}(\Delta_{2,n}, \mathbb{Z})$ with $2\alpha = 0$.*

Proof We continue to use the CW structure on $\Delta'_{2,n} \simeq \Delta_{2,n}$ in Proposition 3.5. Let $\alpha := v/2$ be the 2-torsion element of $H_{n+1}(C_{2,n}; \mathbb{Z})$ constructed in the proof of Claim 4.6. We claim that it is still nonzero in $H_{n+1}(\Delta'_{2,n}; \mathbb{Z})$. Indeed, the sum of coefficients of α attached to types of the form

$$
\text{-}\binom{k}{k},
$$

over all $(n-1)!/2$ cyclic orders, is $\pm n$. This follows by inspecting (11). In particular the sum is odd. On the other hand, for any decorated type $(\mathbf{G}, \delta)$, consider the boundary of the cell corresponding to $(\mathbf{G}, \delta)$. The sum of all coefficients of this boundary attached to types of form $\text{-}\!\binom{k}{k}$ is *even*: in particular, if G has k, k, and 1 marked points decorating the interiors of the three edges of Θ, then G has exactly two edge-contractions of the form $\text{-}\!\binom{k}{k}$, with respect to two distinct cyclic orders. □

5 The $\mathbb{Z}$-homology of $\Delta_{2,n}$ vanishes in codimension > 1

In this section, we will study the final piece of the complex $\Delta'_{2,n}$, the locus of *full theta types*. Recall that a type $\mathbf{G} \in T^{\Theta}_{2,n}$ is *full* if its marked vertices do not lie on a single cycle. Equivalently, $\mathbf{G}$ is obtained from the unmarked graph Θ by adding markings such that at least one marked point lands on the interior of each of the three edges of Θ.

Suppose $\mathbf{G}$ is a theta type. Let us say it has form $(\epsilon_1, \epsilon_2, k_1, k_2, k_3)$, with $1 \geq \epsilon_1 \geq \epsilon_2 \geq 0$ and $k_1 \geq k_2 \geq k_3 \geq 0$, if it is obtained from the unmarked graph Θ by placing ϵ_1 and ϵ_2 markings on each of the two vertices and k_1, k_2, and k_3 markings on the interiors of the three edges. Thus $\epsilon_1 + \epsilon_2 + k_1 + k_2 + k_3 = n$.

Let $F_{2,n}$ be the closure in $\Delta'_{2,n}$ of the locus of full thetas, i.e.

$$F_{2,n} = \overline{\bigcup_{\mathbf{G} \in T^{\mathrm{full}}_{2,n}} \epsilon_{\mathbf{G}}}.$$

Thus $F_{2,n}$ is the subcomplex of $\Delta'_{2,n}$ whose cells are all possible contractions of full theta types, and it inherits a CW structure from $\Delta'_{2,n}$. Then $C_{2,n} \cup F_{2,n} = \Delta'_{2,n}$. Now Theorem 4.2 implies that $\widetilde{H}_i(C_{2,n}; \mathbb{Q}) = 0$ for all i and that $\widetilde{H}_i(C_{2,n}; \mathbb{Z}) = 0$ for all $i \leq n$. Then we have

$$\tilde{H}_*(\Delta'_{2,n}; \mathbb{Q}) \cong \tilde{H}_*(\Delta'_{2,n}/C_{2,n}; \mathbb{Q}) \cong \tilde{H}_*(F_{2,n}/(F_{2,n} \cap C_{2,n}); \mathbb{Q}). \tag{15}$$

and, for $0 \leq i \leq n$,

$$\tilde{H}_i(\Delta'_{2,n}; \mathbb{Z}) \cong \tilde{H}_i(\Delta'_{2,n}/C_{2,n}; \mathbb{Z}) \cong \tilde{H}_i(F_{2,n}/(F_{2,n} \cap C_{2,n}); \mathbb{Z}). \tag{16}$$

So we study $F_{2,n}$ and $F_{2,n}/(F_{2,n} \cap C_{2,n})$ now. The next lemma describes the cells in $F_{2,n}$.

Lemma 5.1 *Suppose $\epsilon_{\mathbf{G},\delta}$ is a cell in $F_{2,n}$ other than ϵ_{br}. Then we have the following case analysis for the possible forms of $(\mathbf{G}, \delta)$:*

1. *$(\mathbf{G}, \delta)$ is a full theta of the form $(0, 0, k_1, k_2, k_3)$, and in this case $\epsilon_{\mathbf{G}}$ has dimension $n + 2$;*
2. *$(\mathbf{G}, \delta)$ is a full theta of the form $(1, 0, k_1, k_2, k_3)$ or a cyclic theta of the form $\text{-}\!\binom{k_1}{k_2}$ for $k_1, k_2 > 0$, and in these cases $\epsilon_{\mathbf{G}}$ has dimension $n + 1$;*

3. $(\mathbf{G}, \delta)$ *is a full theta of the form* $(1, 1, k_1, k_2, k_3)$ *or is a cyclic theta of the form* $\text{-}\!\binom{k_1}{k_2}\!\text{-}$ *with* $k_1, k_2 > 0$*, or is a cyclic theta of the form* $\text{-}\!\binom{n-2}{\leq}\!\text{-}$*, and in these cases* $\epsilon_{\mathbf{G}}$ *has dimension* n*;*
4. $(\mathbf{G}, \delta)$ *is a cyclic theta of the form* $\text{-}\!\binom{n-2}{=}\!\text{-}$*, and in this case* $\epsilon_{\mathbf{G}}$ *has dimension* $n - 1$.

Proof The cells in the aforementioned CW complex structure on $F_{2,n}$, apart from the unique 0-cell ϵ_{br}, are indexed by decorated types $\epsilon_{\mathbf{G},\delta}$ where $\mathbf{G} \in T^{\Theta}_{2,n}$ is obtained from a full theta by contraction (see Lemma 3.1). Recall that any $\mathbf{G} \in T^{\Theta}_{2,n}$ may be obtained from an unmarked graph Θ by adding markings. Let $i \in \{0, 1, 2, 3\}$ be the number of edges of Θ whose interiors remain unmarked in this process; let $j \in \{0, 1, 2\}$ be the number of vertices of Θ that are marked. Then $\mathbf{G} \in T^{\Theta}_{2,n}$ is a contraction of a full theta type if and only if $j \geq i$. For example, a cyclic type of the form $\text{-}\!\binom{n-1}{\leq}$ has $i = 2$ and $j = 1$ and hence fails the criterion above; the point is that even if the single marking on the 3-valent vertex is moved onto an edge, the resulting type is still not full.

The lemma follows from the observation above by a straightforward case analysis. The dimensions were computed in Lemma 3.4. □

Proposition 5.2 *For all* $i \leq n$*, we have*

$$\widetilde{H}_i(F_{2,n}/(F_{2,n} \cap C_{2,n}); \mathbb{Z}) = 0.$$

Proof Write ∂_i for the boundary maps of the cellular chain complex for $F_{2,n}/(F_{2,n} \cap C_{2,n})$. From Lemma 5.1(4), we see that $F_{2,n}/(F_{2,n} \cap C_{2,n})$ has no cells of dimension less than n. So we are reduced to proving the statement for $i = n$. Furthermore, by Lemma 5.1(3), the n-cells of $F_{2,n}/(F_{2,n} \cap C_{2,n})$ correspond to all types obtained from Θ by marking both vertices once and marking the three edges with $k_1 \geq k_2 \geq k_3 \geq 1$ markings. Then $k_1 + k_2 + k_3 = n - 2$ and $k_1 \leq n - 4$. Now suppose $\mathbf{G}$ is of the form we just described. To show the Proposition for $i = n$, we want to show that $\epsilon_{\mathbf{G}} \in \operatorname{im} \partial_{n+1}$. We induct downward on k_1. Suppose $k_1 = n - 4$. Then $k_2 = k_3 = 1$. Consider the type $\mathbf{G}'$ obtained from $\mathbf{G}$ by moving either of the two marked points on the vertices of Θ to the interior of the k_1-marked edge of Θ. We claim that $\partial_{n+1}(\epsilon_{\mathbf{G}'}) = \pm\epsilon_{\mathbf{G}}$. This is because starting with $\mathbf{G}'$ and moving the single marking on the interior of either once-marked edge of Θ to the unmarked vertex of Θ produces a cyclic type and hence no contribution to the boundary map. Of course, any other edge contraction produces a repeating type and again no contribution to the boundary map.

Next, suppose inductively we have already shown that $\epsilon_{\mathbf{H}} \in \operatorname{im} \partial_{n+1}$ for any type $\mathbf{H}$ of the form $(1, 1, m_1, m_2, m_3)$ such that $m_1 \geq m_2 \geq m_3 \geq 1$ and $m_1 > k_1$. Again, we want to show $\epsilon_{\mathbf{G}} \in \operatorname{im} \partial_{n+1}$. Consider the type $\mathbf{G}'$ gotten by moving either one of the marked points on the 3-valent vertices of $\mathbf{G}$ to the interior of the first edge of Θ, i.e. the one supporting k_1 markings. Then $\mathbf{G}'$ has the form $(1, 0, k_1 + 1, k_2, k_3)$. Then

$$\partial_{n+1}(\epsilon_{\mathbf{G}'}) = \pm\epsilon_{\mathbf{G}} \pm \epsilon_{\mathbf{G}_2} \pm \epsilon_{\mathbf{G}_3}$$

for types $\mathbf{G}_2$ and $\mathbf{G}_3$ of the forms $(1, 1, k_1+1, k_2-1, k_3)$ and $(1, 1, k_1+1, k_2, k_3-1)$. But by the inductive hypothesis, both $\epsilon_{\mathbf{G}_2}, \epsilon_{\mathbf{G}_3} \in \operatorname{im} \partial_{n+1}$, so $\epsilon_{\mathbf{G}} \in \operatorname{im} \partial_{n+1}$ as well. □

Now we prove Theorems 1.1 and 1.2. Recall that a path-connected space is called k-connected if its homotopy groups π_i vanish for all $1 \leq i \leq k$.

Proof of Theorem 1.1 From (16), Proposition 5.2, and the fact that $\Delta_{2,n}$ and $\Delta'_{2,n}$ have the same homotopy type, it follows that the reduced $\mathbb{Z}$-homology of $\Delta_{2,n}$ also vanishes in degrees up to n. Furthermore $\Delta_{2,n}$ is clearly 1-connected, since it is homotopy equivalent to a CW complex $\Delta'_{2,n}$ with one 0-cell and no 1-cells. Hurewicz's theorem implies that $\Delta_{2,n}$ is n-connected. □

Proof of Theorem 1.2 This is a computational result. We carried out the computations in `sage` (The Sage Developers 2020) and the code is embedded in the TeX file on arXiv. If $n \leq 3$, the theorem follows from a direct computation in sage. The spaces $\Delta_{2,n}$ for $n \leq 3$ are small enough that no reductions are necessary. For $n \geq 4$, the computations rely on the identification of $\mathbb{Q}$-homology (15) which is computationally very significant. We start by building the top boundary matrix ∂_{n+2} for the cellular homology of $F/(F \cap C)$. The rows and columns of ∂_{n+2} are indexed by full theta types of dimensions $n+1$ and $n+2$, respectively. Then we use sage to compute the rank of ∂_{n+2}. The computation is carried out over $\mathbb{Q}$ as opposed to $\mathbb{Z}$ because it is much faster. The data of the rank of ∂_{n+2}, along with counts of the full theta types of codimension 0, 1, and 2, give the results in Table 1. □

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