



Three-loop corrections to lamb shift in positronium: Electron factor and polarization

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ABSTRACT

Hard spin-independent three-loop radiative contributions to energy levels in positronium generated by the two-photon-exchange diagrams with one-loop radiative insertions in the fermion lines and exchanged photons are calculated. This is a next step in calculations of all corrections of order $m\alpha^7$ inspired by the new generation of precise $1S - 2S$ and $2S - 2P$ measurements in positronium.

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A new generation of precise $1S - 2S$ and $2S - 2P$ measurements in muonium and positronium are now either in progress or planned [1–7]. Inspired by the current and forthcoming experimental achievements we recently calculated a series of hard spin-independent corrections of order $m\alpha^7$ in muonium and positronium [8]. We report below results of the calculation of a new hard spin-independent three-loop contribution to the energy levels in positronium.

Let us consider a system of two electromagnetically bound leptons, which in the general case have unequal masses, $m \leq M$. At $m = M$ one can think about this system as positronium, while at $m \ll M$ it can be interpreted as muonium (or hydrogen). In the calculations below we will build on our recent results for muonium [8] and refer to the two cases above as positronium and muonium, respectively. The two-loop contribution to the spin-independent energy shift in this system generated by the diagrams in Fig. 1 is given by the integral [9]

$$\Delta E = -\frac{(Z\alpha)^5}{\pi n^3} m_r^3 \int \frac{d^4 k}{i\pi^2 k^4} \frac{1}{4} \text{Tr}[(1 + \gamma_0)L_{\mu\nu}] \frac{1}{4} \text{Tr}[(1 + \gamma_0)H_{\mu\nu}] \delta_{l0}, \quad (1)$$

where $L_{\mu\nu}$ and $H_{\mu\nu}$ are the light and heavy fermion factors, respectively, $m_r = mM/(m + M)$ is the reduced mass, $Z = 1$ is the charge of the heavy fermion in terms of the positron charge, n and l are the principal quantum number and the orbital momentum, respectively.

The light (electron) factor is equal to the sum of the self-energy, vertex, and spanning photon insertions in the fermion line [9],

$$L_{\mu\nu} = L_{\mu\nu}^{\Sigma} + 2L_{\mu\nu}^{\Lambda} + L_{\mu\nu}^{\Xi}, \quad (2)$$

$$\frac{1}{4} \text{Tr}[(1 + \gamma_0)L_{\mu\nu}] \equiv \frac{\alpha}{\pi m} \mathcal{L}_{\mu\nu} \left(\frac{k}{m} \right) = \frac{\alpha}{\pi m} \left[\mathcal{L}_{\mu\nu}^{\Sigma} \left(\frac{k}{m} \right) + 2\mathcal{L}_{\mu\nu}^{\Lambda} \left(\frac{k}{m} \right) + \mathcal{L}_{\mu\nu}^{\Xi} \left(\frac{k}{m} \right) \right], \quad (3)$$

and the heavy-line (muon) factor is given by the expression



Fig. 1. Electron-line radiative-recoil corrections.

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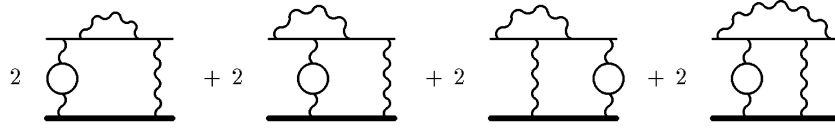


Fig. 2. Graphs with radiative insertions in the electron line and one-loop polarization in the exchanged photons.

$$H_{\mu\nu} = \gamma_\mu \frac{\not{p} + \not{k} + M}{k^2 + 2Mk_0 + i0} \gamma_\nu + \gamma_\nu \frac{\not{p} - \not{k} + M}{k^2 - 2Mk_0 + i0} \gamma_\mu, \quad (4)$$

where $P = (M, \mathbf{0})$ is the momentum of the heavy fermion.

The energy shift in Eq. (1) contains both recoil and nonrecoil contributions to the Lamb shift of order $\alpha(Z\alpha)^5 m$ generated by the diagrams in Fig. 1. The nonrecoil correction was calculated analytically long time ago [10–12]. The respective recoil correction in hydrogen (and muonium) was obtained analytically in [9].

Three-loop spin-independent radiative-recoil corrections in hydrogen and muonium due to the diagrams in Fig. 2 were recently calculated numerically [8]. Our current goal is to calculate the total spin-independent contribution of the diagrams in Fig. 2 to the energy levels in positronium. We build on calculations of the recoil corrections in [8,9], and first review the basic steps in those calculations. We start with the fermion factor in Eq. (4)

$$\frac{1}{4} \text{Tr}[(1 + \gamma_0)H_{\mu\nu}] = -\frac{1}{M} \left[k^2 g_{\mu 0} g_{\nu 0} - k_0 (g_{\mu 0} k_\nu + g_{\nu 0} k_\mu) + k_0^2 g_{\mu\nu} \right] \frac{1}{k_0^2 - \frac{k^4}{4M^2}}. \quad (5)$$

The characteristic integration momenta in Eq. (1) are in the interval $m \leq k \ll M$, and to extract the linear in the heavy mass recoil contribution it is sufficient to make the substitution

$$\begin{aligned} & \frac{1}{4} \text{Tr}[(1 + \gamma_0)H_{\mu\nu}] \\ & \rightarrow -M \frac{\partial}{\partial M} \left\{ \frac{1}{4} \text{Tr} \left[(1 + \gamma_0) \left[\gamma_\mu \frac{\not{p} - \not{k} + M}{k^2 - 2Mk_0 + i0} \gamma_\nu + \gamma_\nu \frac{\not{p} + \not{k} + M}{k^2 + 2Mk_0 + i0} \gamma_\mu \right] \right\} \right\} \\ & = -\frac{1}{M} \left[k^2 g_{\mu 0} g_{\nu 0} - k_0 (g_{\mu 0} k_\nu + g_{\nu 0} k_\mu) + k_0^2 g_{\mu\nu} \right] \frac{k_0^2 + \frac{k^4}{4M^2}}{\left(k_0^2 - \frac{k^4}{4M^2} \right)^2}. \end{aligned} \quad (6)$$

Due to the explicit factor $1/M$ on the RHS we can let $k/M \rightarrow 0$ in all other terms. To avoid possible singularities in the momentum integration in Eq. (1) in this limit we introduced [9,13] a principal value integral defined as

$$k^2 \wp \left(\frac{1}{k_0^2} \right) = k^2 \lim_{\frac{k}{M} \rightarrow 0} \frac{k_0^2 + \frac{k^4}{4M^2}}{\left(k_0^2 - \frac{k^4}{4M^2} \right)^2}. \quad (7)$$

This definition works both in Minkowski and Euclidean space and preserves all momentum integrations finite and unambiguous. In Euclidean space it reduces to

$$\wp \left(\frac{1}{\cos^2 \theta} \right) = \lim_{\varepsilon \rightarrow 0} \frac{\cos^2 \theta - \varepsilon^2}{(\cos^2 \theta + \varepsilon^2)^2}, \quad (8)$$

where the Euclidean momentum is parameterized as $(k \cos \theta, k \sin \theta)$.

In terms of the principal value the recoil trace in Eq. (6) simplifies

$$\begin{aligned} \frac{1}{4} \text{Tr}[(1 + \gamma_0)H_{\mu\nu}] & \rightarrow -\frac{1}{M} \left[k^2 g_{\mu 0} g_{\nu 0} \wp \left(\frac{1}{k_0^2} \right) - (g_{\mu 0} k_\nu + g_{\nu 0} k_\mu) \frac{1}{k_0} + g_{\mu\nu} \right] \\ & \equiv -\frac{1}{M} \mathcal{H}_{\mu\nu}(k), \end{aligned} \quad (9)$$

where $\mathcal{H}_{\mu\nu}(k)$ is a dimensionless function.

The linear in mass ratio radiative-recoil contribution [9] is obtained from Eq. (1) by the substitution in Eq. (9)

$$\Delta E_{\text{rec}} = \frac{\alpha(Z\alpha)^5}{\pi^2 n^3} \frac{m_r^3}{Mm} \int \frac{d^4 k}{i\pi^2 k^4} \mathcal{L}_{\mu\nu} \left(\frac{k}{m} \right) \mathcal{H}_{\mu\nu}(k). \quad (10)$$

We notice that in the case of equal masses, $M = m$, the integrand for the total (recoil and nonrecoil) corrections in Eq. (5) and the integrand for the recoil corrections in Eq. (9) are connected by the relationship

$$\mathcal{H}_{\mu\nu}(k) \frac{k_0^2}{k_0^2 - \frac{k^4}{4m^2}} = \frac{k^2 g_{\mu 0} g_{\nu 0} - (g_{\mu 0} k_\nu + g_{\nu 0} k_\mu) k_0 + g_{\mu\nu} k_0^2}{k_0^2 - \frac{k^4}{4m^2}}, \quad (11)$$

where we used $k_0^2 \wp(1/k_0^2) = 1$, see definition in Eq. (7).

In Euclidean space

$$\frac{k_0^2}{k_0^2 - \frac{k^4}{4m^2}} \longrightarrow \frac{4m^2 \cos^2 \theta}{k^2 + 4m^2 \cos^2 \theta}. \quad (12)$$

We can go a step further and modify the integrand in Eq. (10) in such way that in the case of unequal masses it would produce radiative-recoil contribution and in the case of equal masses it would generate the total (sum of nonrecoil and recoil) contribution of the diagrams in Fig. 1

$$\mathcal{H}_{\mu\nu}(k) \rightarrow \mathcal{H}_{\mu\nu}(k) \tilde{G}(k, M), \quad (13)$$

where

$$\tilde{G}(k, M) = \frac{k_0^2 \left(\frac{k^4}{4M^2} + k_0^2 \right)}{\left(k_0^2 - \frac{k^4}{4M^2} \right)^2} - \frac{k_0^2 k^4 m^2}{2M^4 \left(k_0^2 - \frac{k^4}{4M^2} \right)^2}. \quad (14)$$

The interpolating factor $\tilde{G}(k, M)$ reduces to $\tilde{G}(k, m) = k_0^2 / (k_0^2 - k^4/4m^2)$ at $M = m$ and then the factor on the RHS in Eq. (13) coincides with the one in Eq. (11). At $k/M \rightarrow 0$ the interpolating factor $\tilde{G}(k, M) \rightarrow 1$ and we return to $\mathcal{H}_{\mu\nu}(k)$.

In Euclidean space

$$\begin{aligned} \tilde{G}(k, m) &\rightarrow k_0^2 \frac{-\frac{k^4}{4m^2} + k_0^2}{\left(k_0^2 + \frac{k^4}{4m^2} \right)^2} + \frac{k_0^2 k^4 m^2}{2m^4 \left(k_0^2 + \frac{k^4}{4m^2} \right)^2} = \frac{4m^2 \cos^2 \theta}{4m^2 \cos^2 \theta + k^2}, \\ \tilde{G}(k, M) &\rightarrow k_0^2 \frac{-\frac{k^4}{4M^2} + k_0^2}{\left(k_0^2 + \frac{k^4}{4M^2} \right)^2} + \frac{k_0^2 k^4 m^2}{2M^4 \left(k_0^2 + \frac{k^4}{4M^2} \right)^2} \Big|_{M \rightarrow \infty} = 1. \end{aligned} \quad (15)$$

Next we rescale the integration momentum $k \rightarrow km$ in Eq. (1) and insert the dimensionless factor $\tilde{G}(k, M)$ in the integrand in Eq. (10)

$$\begin{aligned} \Delta E &= \frac{\alpha(Z\alpha)^5}{\pi^2 n^3} \frac{m_r^3}{mM} \int \frac{d^4 k}{i\pi^2 k^4} \mathcal{L}_{\mu\nu}(k) \mathcal{H}_{\mu\nu}(k) \tilde{G}(km, M) \delta_{l0} \\ &\equiv \frac{\alpha(Z\alpha)^5}{\pi^2 n^3} \frac{m_r^3}{mM} \Delta \mathcal{E}(M). \end{aligned} \quad (16)$$

The linear in the mass ratio spin-independent radiative-recoil contribution of the diagrams in Fig. 1 (see Eq. (10)) in muonium and hydrogen can be written as

$$\Delta E_{rec}^{Mu} = \frac{\alpha(Z\alpha)^5}{\pi^2 n^3} \frac{m_r^3}{mM} \Delta \mathcal{E}(M \rightarrow \infty), \quad (17)$$

while the respective total (recoil and nonrecoil) spin-independent contribution in positronium has the form

$$\Delta E^{Ps} = \frac{\alpha^6}{\pi^2 n^3} \frac{m}{4} \Delta \mathcal{E}(M = m), \quad (18)$$

where we inserted an extra factor two to account for radiative insertions in both fermion lines.

The principal value prescription in Eq. (7), introduced originally [9,13] in order to simplify calculations of radiative-recoil corrections in the principal, allowed us to derive a universal formula which describes both the radiative-recoil correction in the case of unequal masses and the total correction when the masses of the leptons are equal. The dimensionless function $\Delta \mathcal{E}(M)$ smoothly interpolates between the radiative-recoil contribution of the diagrams in Fig. 1 in muonium and the total contribution of these diagrams in positronium. This property survives insertion of polarization graphs in Fig. 2 and provides an additional control of the calculations below.

Let us turn to the diagrams in Fig. 2. To account for the polarization insertions it is sufficient to insert the factor $(\alpha/\pi)2k^2 I_1(k)$ in the integrand in Eq. (16)

$$\begin{aligned} \Delta E_{pol} &= \frac{\alpha^2(Z\alpha)^5}{\pi^3 n^3} \frac{m_r^3}{mM} \int \frac{d^4 k}{i\pi^2 k^4} \mathcal{L}_{\mu\nu}(k) \mathcal{H}_{\mu\nu}(k) 2k^2 I_1(k) \tilde{G}(km, M) \delta_{l0} \\ &\equiv \frac{\alpha^2(Z\alpha)^5}{\pi^3 n^3} \frac{m_r^3}{mM} \Delta \mathcal{E}_{pol}(M), \end{aligned} \quad (19)$$

where the polarization operator $I_1(k)$ after the Wick rotation has the form

$$I_1(k) = \int_0^1 dv \frac{v^2(1-v^2/3)}{4 + (1-v^2)k^2}. \quad (20)$$

The linear in the mass ratio spin-independent recoil contribution of the diagrams in Fig. 1 in muonium (compare Eq. (17)) can be written as

$$\Delta E_{rec,pol}^{Mu} = \frac{\alpha^2 (Z\alpha)^5}{\pi^3 n^3} \frac{m_r^3}{mM} \Delta \mathcal{E}_{pol}(M \rightarrow \infty). \quad (21)$$

The respective total (recoil and nonrecoil) spin-independent contribution in positronium has the form

$$\Delta E_{pol}^{Ps} = \frac{\alpha^2 (Z\alpha)^5}{\pi^3 n^3} \frac{m}{4} \Delta \mathcal{E}_{pol}(M = m) \equiv (J_{\Sigma P} + 2J_{\Lambda P} + J_{\Xi P}) \frac{\alpha^7 m}{\pi^3 n^3} \delta_{l0}, \quad (22)$$

where we again inserted an extra factor two to account for radiative insertions in both fermion lines, and in the last line separated contributions of different diagrams.

The recoil correction in muonium in Eq. (21) was calculated recently [8]. Now we calculated the total contribution of order $m\alpha^7$ in Eq. (22) of the diagrams in Fig. 2 to the energy levels in positronium. Calculations are similar to the ones in [9], and the contributions of the separate diagrams in the Yennie gauge are

$$J_{\Sigma P} = -0.114395(1), \quad 2J_{\Lambda P} = 0.977677(1), \quad J_{\Xi P} = -0.293172(1). \quad (23)$$

Finally, the total gauge independent contribution of order α^7 to the Lamb shift in positronium generated by the four diagrams in Fig. 2 is

$$\Delta E^{(Ps)} = 0.5701(2) \frac{\alpha^7 m}{\pi^3 n^3} \delta_{l0}. \quad (24)$$

This contribution should be added to two other corrections of order $m\alpha^7$ in positronium, which have been calculated recently in [8]² and have comparable magnitude. We hope to report results for the remaining hard contributions of this order in muonium and positronium in the near future.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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² Comparison of the contributions of order $m\alpha^7$ with the accuracy of present and prospective experimental results is also discussed in [8]. Phenomenologically important role will also play spin-independent corrections of order $m\alpha^7$ in positronium originating from the annihilation diagrams, which were calculated as a side result in the works on spin-dependent corrections, see, e.g., review in [14] and references therein.