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Controlling 2D PDEs using mobile collocated actuators-sensors and their simultaneous guidance constrained over path-dependent reachability regions

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Abstract— **Employing mobile actuators and sensors for control and estimation of spatially distributed processes offers a significant advantage over immobile actuators and sensors. In addition to the control performance improvement, one also comes across the economic advantages since fewer devices, if allowed to be repositioned within a spatial domain, must be employed. While simulation studies of mobile actuators report superb controller performance, they are far from reality as the mechanical constraints of the mobile platforms carrying actuators and sensors have to satisfy motional constraints. Terrain platforms cannot behave as point masses without inertia; instead they must satisfy constraints which are adequately represented as path-dependent reachability sets. When the control algorithm commands a mobile platform to reposition itself in a different spatial location within the spatial domain, this does not occur instantaneously and for the most part the motion is not omnidirectional. This constraint is combined with a computationally feasible and suboptimal control policy with mobile actuators to arrive at a numerically viable control and guidance scheme. The feasible control decision comes from a continuous-discrete control policy whereby the mobile platform carrying the actuator is repositioned at discrete times and dwells in a specific position for a certain time interval. Moving to a subsequent spatial location and computing its associated path over a physics-imposed time interval, a set of candidate positions and paths is derived using a path-dependent reachability set. Embedded into the path-dependent reachability sets that dictate the mobile actuator repositioning, a scheme is proposed to integrate collocated sensing measurements in order to minimize costly state estimation schemes. The proposed scheme is demonstrated with a 2D PDE having two sets of collocated actuator-sensor pairs onboard mobile platforms.**

I. Introduction

The idea of using mobile actuators and sensors for the improvement of controllers and estimators for spatially distributed processes, is not new and can be traced back to the 70's [1], [2]. The earlier work [3] provides an account in the various aspects of mobile controls for distributed parameter systems. However, the practical aspects arising from the incorporation of vehicle dynamics and the expensive computational costs associated with implementing integrated actuator guidance and control are not considered.

This paper takes on the earlier work [3] which uses realistic motional constraints in the guidance of mobile actuators used for the control of spatially distributed systems. Using the motion constraints of the mobile platforms carrying the actuating devices, it calculated the time-varying reachability sets, which consisted of the spatial points within the spatial domain that can be reached by the platforms over a time interval. The assumption was that a mobile platform can travel to its new spatial position within a prescribed travel time and reside in this position for a longer time while dispensing the control signal to the spatially distributed process. The decision to reposition a given actuator was dictated by controller performance requirements. These were based on the relaxation of the finite horizon LQR problem, which resulted in a significant reduction in the computational load. Instead of opting for an integrated actuator guidance and control design, the suboptimal approach considered the infinite horizon problem with an adjusted cost-to-go whereby the lower time limit of the performance metric was replaced by the new time instance. This avoided the backwards in time solution to the Differential Riccati equation and the associated guidance, and resulted in the solution to Algebraic Riccati equations over the reachability sets. At the beginning of a new time interval, the lower limit of the cost-to-go was updated and which resulted in a different reachability set. Then the search of the next spatial position that the platform could relocated was dictated by the points that could be reached within the prescribed travel time, namely the reachability set. By parameterizing the ARE solutions by points of the reachability set, a location-dependent optimal cost was generated as a function of the solution to the ARE's and the value of the state at the beginning of the time interval. Minimization of these location-parameterized optimal costs resulted in the guidance of the mobile actuator platforms.

When the full state is not available, then the work [3] can implement neither the controller signal nor the guidance. Working within the confines of a real-time implementation with reduced computational costs while respecting motion constraints, this paper considers the use of sensors to provide state information. To simplify the controller architecture, a collocated actuator-sensor pairs are assumed and further, static output feedback controllers are used to reduced further the computational and implementational costs.

This paper considers collocated actuator-sensor pairs onboard mobile platforms and uses the time-varying reachability sets to reposition the platforms to their new spatial locations. Avoiding full state information which was assumed in [3] for both the controller and the platform guidance, it harnesses the total energy of the closed-loop system to compute location-dependent solutions to operator Lyapunov equations. The minimization of these performance metrics

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provide, in an analog manner to [3], the repositioning of these platforms using only sensor measurements.

The problem is formulated in Section II and the proposed guidance policy based on the time-varying reachability sets is presented in Section III. Numerical results are shown in Section IV and conclusions follow in Section V.

II. Problem formulation

The parabolic PDE over a 2D domain $\Omega \subset \mathbb{R}^2$ is given by

$$\frac{\partial x(t,\xi,\psi)}{\partial t} = \mathcal{A}x(t,\xi,\psi) + \sum_{i=1}^{N} b_i(\xi,\psi;\xi_{ai}(t),\psi_{ai}(t))u_i(t). \quad (1)$$

The state is denoted by $x(t,\xi,\psi)$ at time $t \in \mathbb{R}^+$ and spatial coordinates $\chi = (\xi,\psi) \in \Omega$. A finite number N of mobile actuators with time-varying centroids $(\xi_{ai}(t),\psi_{ai}(t)) \in \Omega$ is assumed to dispense the control signals to the spatially distributed process. The state operator is given by

$$\mathcal{A}\phi = \sum_{i,j=1}^{2} \alpha_{ij}(t,\chi)\frac{\partial^2\phi}{\partial\chi_i\partial\chi_j} + \sum_{i=1}^{2} \beta(t,\chi)\frac{\partial\phi}{\partial\chi_i} + \gamma(t,\chi)\phi,$$

and is assumed to satisfy the minimum conditions for the well-posedness of (1). These are the uniform ellipticity in the cylinder $(0,T)\times\Omega$ [4], [5], the uniform Hölder continuity of the spatially and temporally varying coefficients α,β,γ over the cylinder $(0,T)\times\Omega$ and the square integrability of the non-homogenous term represented by the input term due to the mobile actuators. The spatial distribution of the actuating devices is denoted by $b_i(\xi,\psi;\xi_{ai}(t),\psi_{ai}(t))$ and describes the manner in which the actuating devices are dispensing the control signal $u_i(t)$ to the process. For simplicity, a further assumption is made regarding the spatial domain, namely that Ω is simply connected, open, bounded and with sufficiently smooth boundary $\partial\Omega$. Finally, initial and boundary conditions must be given. While all types of boundary conditions are allowed, which for the case of Neumann conditions it will enable the actuators to move along the boundary, we have selected Dirichlet conditions for simplicity. In the case of boundary actuators, they can stay at the boundary or move in the interior of the spatial domain. The proposed scheme allows for actuators to be at the boundary as well. Thus we have the conditions

$$x(t_0,\xi,\psi) = x_0(\xi,\psi), \;\; x(t,\xi,\psi)\Big|_{(\xi,\psi)\in\partial\Omega} = 0. \quad (2)$$

Following the earlier contribution on the full-state availability [3], we assume that the spatial distribution of the actuating devices is given by the spatial Dirac delta functions.

Assumption 1 (actuator spatial distribution): The spatial distribution of the actuating devices is that of a Dirac delta function centered at the centroids

$$b_i(\xi,\psi;\xi_{ai}(t),\psi_{ai}(t)) = \delta(\xi-\xi_{ai})\delta(\psi-\psi_{ai}(t)), i=1,\dots,N.$$

If one has fixed actuators, namely all the centroids are fixed with $\chi_{ai}(t) = (\xi_{ai}(t),\psi_{ai}(t)) = (\xi_{ai},\psi_{ai})$, $i=1,\dots,N$, for all $t>0$, then a full state feedback controller based on the LQR design can provide a desired controller performance. In this case, the PDE in (1) and (2) must be placed in an abstract form in terms of an evolution equation in a functional space. The evolution equation in a Hilbert space $\mathcal{X}$ is given by

$$\begin{aligned} \dot{x}(t) &= \mathcal{A}x(t) + \mathcal{B}(\chi_a)u(t) \\ u(t) &= [u_1(t),\dots,u_N(t)]^T, \quad x(t_0) = x_0, \end{aligned} \quad (3)$$

where with a slight abuse of notation, $x(t) = x(t,\cdot,\cdot)$ is used to denote the state as an element of the abstract space $\mathcal{X}$. The state space $\mathcal{X}$, a Hilbert space, serves as the interpolating space for the reflexive Banach space $\mathcal{V}$ that is continuously and densely embedded in $\mathcal{X}$. The conjugate dual of $\mathcal{V}$ is denoted $\mathcal{V}^*$. In this case, one has the Gelf'and triple space $\mathcal{V} \hookrightarrow \mathcal{X} \hookrightarrow \mathcal{V}^*$ with both embeddings dense and continuous. We have that the state operator $\mathcal{A} \in L(\mathcal{V},\mathcal{V}^*)$, and the input operator $\mathcal{B} \in L(\mathcal{U},\mathcal{V}^*)$, with $\mathcal{U}$ the input space, for each fixed centroid χ_a in an admissible set of actuator locations that ensure the operator pair $(\mathcal{A},\mathcal{B})$ is exponentially stabilizable, [6].

For the system in (1), (2), the spaces are identified by $\mathcal{X} = L_2(\Omega)$, $\mathcal{V} = H_0^1(\Omega)$ and $\mathcal{V}^* = H^{-1}(\Omega)$ with the Sobolev spaces $H_0^1(\Omega), H^{-1}(\Omega)$ defined in [5]. The operators in weak form are $\mathcal{A} : \mathcal{V} \to \mathcal{V}^*$

$$\begin{aligned} \langle \mathcal{A}\phi,\psi\rangle = &\int_\Omega \nabla\cdot(\alpha(\xi,\psi)\nabla\phi_1(\xi,\psi))\phi_2(\xi,\psi)\,\mathrm{d}\omega \\ &+ \int_\Omega \beta(\xi,\psi)\nabla\phi_1(\xi,\psi)\phi_2(\xi,\psi)\,\mathrm{d}\omega \\ &+ \int_\Omega \gamma(\xi,\psi)\phi_1(\xi,\psi)\phi_2(\xi,\psi)\,\mathrm{d}\omega, \end{aligned}$$

for $\phi_1,\phi_2 \in \mathcal{V}^*$. The location-dependent input operator $\mathcal{B}(\cdot) : \mathbb{R}^N \to \mathcal{V}^*$ is given by

$$\langle \mathcal{B}(\chi_a)u,\phi_1\rangle = \int_\Omega b(\xi,\psi;\xi_a,\psi_a)u(t)\phi_1(\xi,\psi)\,\mathrm{d}\omega.$$

Different to the earlier work [3], to obtain the control signal we can minimize the finite horizon performance index cost

$$\begin{aligned} J(x_0;t_0) = &\langle x(T),\mathcal{M}_1 x(T)\rangle \\ &+ \int_{t_0}^{T} \langle x(\tau),Q_1 x(\tau)\rangle + u^T(\tau)R_1 u(\tau)\,\mathrm{d}\tau. \end{aligned} \quad (4)$$

The solution to this optimization problem is given by

$$u(t) = -R_1^{-1}\mathcal{B}^*(\chi_a)\mathcal{P}(t;\chi_a)x(t) \quad (5)$$

where $\mathcal{P}(t;\chi_a)$ solves the location-parameterized DRE

$$\begin{aligned} -\langle\dot{\mathcal{P}}(t;\chi_a)\phi_1,\phi_2\rangle &= \langle \mathcal{A}\phi_1,\mathcal{P}(\chi_a)\phi_2\rangle + \langle \mathcal{P}(\chi_a)\phi_1,\mathcal{A}\phi_2\rangle \\ &-\langle \mathcal{P}(\chi_a)\mathcal{B}(\chi_a)R_1^{-1}\mathcal{B}^*(\chi_a)\mathcal{P}(\chi_a)\phi_1,\phi_2\rangle + \langle Q_1\phi_1,\phi_2\rangle, \end{aligned} \quad (6)$$

for $\phi_1,\phi_2 \in D(\mathcal{A})$ with $D(\mathcal{A})$ denoting the domain of $\mathcal{A}$, and having terminal condition $\mathcal{P}(T;\chi_a) = \mathcal{M}$. The optimal value of the performance index is

$$J^{opt}(x_0;t_0) = \langle x_0,\mathcal{P}(t_0;\chi_a)x_0\rangle \quad (7)$$

and this can be used to obtain the optimal values of the fixed-in-space actuators via

$$\chi_a^{opt} = \arg\min\langle x_0,\mathcal{P}(t_0;\chi_a)x_0\rangle. \quad (8)$$

When the actuators are desired to be repositioned within the spatial domain, another layer of optimization is included in the formulation of the control problem. The solution to the optimal guidance of moving actuators of the system (3) with a finite horizon cost (4) was summarized in [3] and which included the actuator motion dynamics

$$\dot{\chi}_{ai}(t) = F\chi_{ai}(t) + \upsilon_i(t), \quad \chi_{ai}(0) \in \Omega, \quad (9)$$

where $\upsilon_i(t)$ denotes the control signal for the i^{th} mobile platform carrying the i^{th} actuator and the matrix F is the state matrix for the motion dynamics.

This optimal guidance-plus-control solution requires the backwards in time solution to the Operator Differential Riccati Equation (6) and the actuator guidance. The backwards in time integration and knowledge of the full state are the two requirements that impede the real-time implementation of the optimal control-and-actuator-guidance of advection PDEs like (1) with mobile actuators. To alleviate some of the computational burdens that hamper this real-time implementation, two suboptimal policies are proposed:

1) Modify the performance index in (4) to
$$J(x_0;t_k) = \langle x(T), \mathcal{M}_1 x(T)\rangle + \int_{t_k}^{T} \langle x, Q_1 x\rangle + u^T R_1 u \,\mathrm{d}\tau,$$
which essentially changes the cost-to-go at discrete time instances t_k with $t_{k+1} = t_k + \Delta t$, $k = 0, 1, \ldots, n-1$. In each of the time subintervals $[t_0, T], [t_1, T], [t_2, T], \ldots, [t_{n-1}, T]$, one solves the operator DRE's and then selects the optimal actuator location for the interval $[t_k, t_{k+1}]$ by optimizing the performance index
$$J^{opt}(x(t_k);t_k) = \langle x(t_k), \mathcal{P}(t_k;\chi_a) x(t_k)\rangle$$
via $\chi_a^{opt} = \arg\min \langle x(t_k), \mathcal{P}(t_k;\chi_a) x(t_k)\rangle$. Thus, at the beginning of a new subinterval t_k, the actuator positions change according to the above optimization and reside in that position throughout each $[t_k, t_{k+1})$. This optimization requires the solution to operator DREs in each diminishing time interval $[t_k, T]$. While it takes into consideration the delays due to the platform motion, it requires the state $x(t_k)$ at each switch time t_k.
2) An improvement to the above, considers the infinite horizon problem and changes the lower time limit at each time instance t_k. Thus one considers at each new time instance t_k the infinite horizon performance index
$$J(x_{t_k};t_k) = \int_{t_k}^{\infty} \langle x(\tau), Q_1 x(\tau)\rangle + u^T(\tau) R_1 u(\tau)\,\mathrm{d}\tau.$$
The optimal value is given by
$$J^{opt}(x(t_k);t_k) = \langle x(t_k), \mathcal{S}(\chi_a) x(t_k)\rangle$$
where $\mathcal{S}(\chi_a)$ solves the operator ARE
$$\begin{aligned} 0 &= \langle \mathcal{A}\phi_1, \mathcal{S}(\chi_a)\phi_2\rangle + \langle \mathcal{S}(\chi_a)\phi_1, \mathcal{A}\phi_2\rangle \\ &- \langle \mathcal{S}(\chi_a)\mathcal{B}(\chi_a) R_1^{-1}\mathcal{B}^*(\chi_a)\mathcal{S}(\chi_a)\phi_1, \phi_2\rangle + \langle Q_1\phi_1, \phi_2\rangle, \end{aligned}$$
Thus, at the beginning of each time interval $[t_k, t_{k+1})$ one repositions the actuators via
$$\chi_a^{opt,k} = \arg\min \langle x(t_k), \mathcal{S}(\chi_a) x(t_k)\rangle.$$
While this is a significant improvement to the previous case, in terms of the computational load that only require solution to operator ARE's instead of operator DRE's, it still requires the state $x(t_k)$ at each time instance to realize the optimization. Additionally, it requires the full state at each time in each time subinterval $[t_k, t_{k+1})$ in order to realize the control signal
$$u^{opt,k}(t) = -R_1^{-1}\mathcal{B}^*(\chi_a^{opt,k})\mathcal{S}(\chi_a^{opt,k})x(t), \quad t \in [t_k, t_{k+1}).$$

The second option was considered in [3] for a single actuator. The above accounts for multiple actuators, but is still infeasible since it requires access to the state $x(t)$. However, the guidance scheme presented in [3] allowed for actuator guidance (continuous motion) in each time subinterval $[t_k, t_{k+1})$ combined with discrete time updates on the cost-to-go.

The case of static feedback with actuator guidance that takes into account the motional constraints is presented in the next section and constitutes the main result in this paper.

III. SUBOPTIMAL STATIC OUTPUT FEEDBACK CONTROL AND ACTUATOR GUIDANCE WITH ACTUATOR MOTION CONSTRAINED OVER TIME VARYING REACHABILITY SETS

We first consider the platform kinematics given by
$$\begin{aligned} \dot{\xi}_{ai}(t) &= v_i(t)\cos(\theta_{ai}(t)), \\ \dot{\psi}_{ai}(t) &= v_i(t)\sin(\theta_{ai}(t)), \\ \dot{\theta}_{ai}(t) &= \omega_{ai}(t), \end{aligned} \tag{10}$$
for $i = 1, \ldots, N$, where $v_i(t)$ and $\omega_{ai}(t)$ are the speed and turning rate for each mobile platform. While both the speed and turning rate are the control signals, for simplicity it is assumed that the speeds of the actuator platforms are constant and only the turning rates are used as the control signals.

As argued in [3], the repositioning of each platform at the beginning of each time subinterval $[t_k, t_{k+1})$ cannot occur instantaneously and may not be able to solve the operator Riccati (algebraic or differential) for every single candidate position in Ω. Instead, one considers the motional constraints and searches for the new actuator position in each time subinterval, over the current reachability set for each actuator platform. In a given time subinterval, an actuator platform can only traverse to a region surrounding its current position, which is defined as its time varying reachability set.

The residence time t_{res}, i.e the duration $t_{k+1} - t_k$ that an actuator platform resides at a given spatial location $\chi_a(t)$ for $t \in [t_i, t_i + \Delta t[$, satisfies
$$t_{res} = t_{k+1} - t_k = \Delta t.$$
This assumes that all time subintervals are uniform with duration Δt. One can easily express the beginning of a new subinterval via $t_k = t_0 + k t_{res}$, $k = 1, \ldots, \frac{T - t_0}{t_{res}} - 1$.

The requirement is: *at the beginning t_k of each subinterval $[t_k, t_k + t_{res})$, the actuator platforms must move to their new commanded positions while obeying the kinematics* (10) *in a travel time t_{travel} that is significantly less than the residence time t_{res}. The additional requirement is that $t_{travel} \ll t_{res}$.*

For each actuator platform, the set of locations that can be traversed over the travel time t_{travel} is described by the spatial points that are within a distance $v_i t_{travel}$ from the current position $\chi_{ai}(t_k)$. Taking into account the dynamics (10) for each mobile platform, we have that the possible locations of each platform over the interval $[t_k, t_{k+1})$ are given by
$$\begin{aligned} \xi_i(t) &= \xi_{ai}(t_i) + (v t_{travel})\cos(\theta_i(t)) \\ \psi_i(t) &= \psi_{ai}(t_i) + (v t_{travel})\sin(\theta_i(t)) \end{aligned} \quad i = 1, \ldots, N, \tag{11}$$
where the angle $\theta_i(t)$ for each platform satisfies $\theta_{ai}(t_k) - \pi \le \theta_i(t) \le \theta_{ai}(t_k) + \pi$, $i = 1, \ldots, N$. This of course assumes that there are no angle constraints and that each mobile

platform can traverse during the time subinterval $[t_k, t_{k+1})$ at any point in the circle centered at the current position $\chi_{ai}(t_k) = (\xi_{ai}(t_k), \psi_{ai}(t))$ with radius vt_{travel}.

Prior to the definition of the reachability sets for each of the actuator platforms, we provide the definition of the admissible set of locations for each platform.

Definition 1 (admissible sets): The set of admissible candidate locations for each platform Θ^i_{ad}, $i = 1, \ldots, N$ is the set of points within the spatial domain Ω such that a parameterized operator Riccati or Lyapunov equation is solvable.

Equipped with the admissible set of points for each platform, we can now proceed with the time varying reachability sets for each platform.

Definition 2 (Reachability sets): The reachability set for each platform is given by

$$\mathcal{R}_{ai}(t_k) = \Theta^i_{ad} \cap \Big\{ (\xi_i, \psi_i, \theta_i) : \\ (\xi_i, \psi_i) \text{ satisfy (11)}, \; -\pi \le \theta_i(t) \le \pi \Big\}$$

It should be noted that in the case of no angular constraints, the reachability sets consist of all spatial points in the admissible sets Θ^i_{ad} that are within the circle of radius vt_{travel} from the current location $\chi_{ai}(t_k)$. If there are angular constraints of the form $\pm\Delta\theta$ with $\Delta\theta \ll \pi$, then each actuator platform can travel a distance $v \cdot t_{travel}$ from its position $\chi_{ai}(t_k) = (\xi_{ai}(t_k), \psi_{ai}(t_k)$ with the circular regions now becoming sectors given by $\theta_i(t) \in [\theta_{ai}(t_k) - \Delta\theta, \theta_{ai}(t_k) + \Delta\theta]$. In this case, the reachability sets are reduced and given by

$$\mathcal{R}_{bi}(t_k) = \Theta^i_{ad} \cap \Big\{ (\xi_i, \psi_i, \theta_i) : (\xi_i, \psi_i) \text{ satisfy (11)}, \\ \theta_{ai}(t_k) - \Delta\theta \le \theta(t) \le \theta_{ai}(t_k) + \Delta\theta \Big\}.$$

There is another reachability set that was considered in [3] and which included both angular and angular rate constraints. The time required by an actuator platform to turn by an angle $\Delta\theta$ is given by $t_{turn} = \frac{|\Delta\theta|}{\omega}$. This constraint leaves $(t_{travel} - t_{turn})$ time units to travel. In this case, the spatial points in the time-varying sector that can be travelled over the time interval $[t_k, t_{k+1})$ are

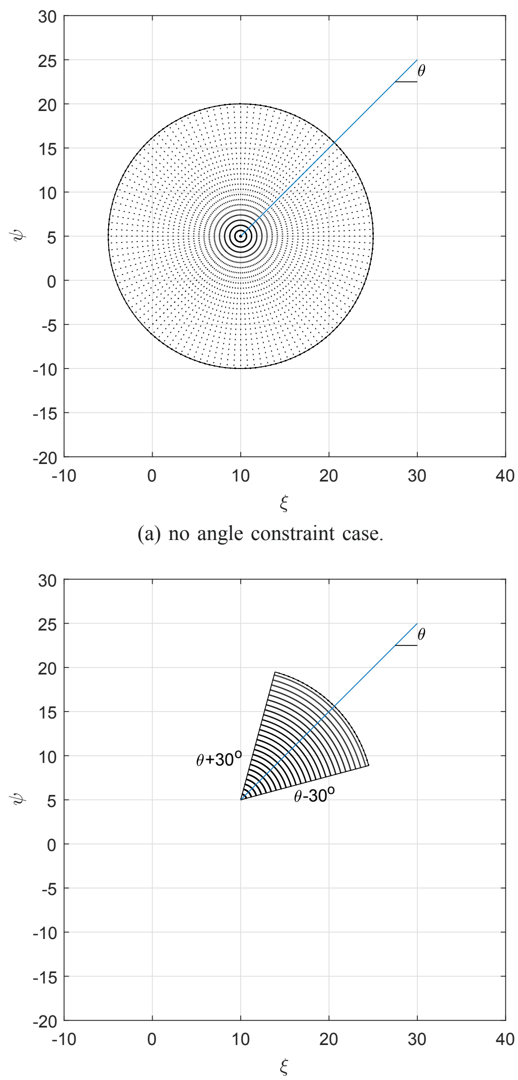
$$\begin{aligned} \xi_i(t) &= \xi_{ai}(t_k) + v(t_{travel} - t_{turn}) \cos(\theta_i(t)) \\ \psi_i(t) &= \psi_{ai}(t_k) + v(t_{travel} - t_{turn}) \sin(\theta_i(t)). \end{aligned} \tag{12}$$

The reachability set is further reduced and is given by

$$\mathcal{R}_{ci}(t_k) = \Theta^i_{ad} \cap \Big\{ (\xi_i, \psi_i, \theta_i) : (\xi_i, \psi_i) \text{ satisfy (12)}, \\ \theta_{ai}(t_k) - \Delta\theta \le \theta_i \le \theta_{ai}(t_k) + \Delta\theta, \; |\Delta\theta| \le \omega t_{turn} \Big\}.$$

One obviously has $\mathcal{R}_{ci}(t_k) \subseteq \mathcal{R}_{bi}(t_k) \subseteq \mathcal{R}_{ai}(t_k)$, $\forall t_k$, $\forall i = 1, \ldots, N$. Figure 1a depicts the reachability set $\mathcal{R}_{ai}(t_k)$ which is a circle centered at the current actuator location $\chi_{ai}(t_k) = (\xi_{ai}(t_k), \psi_{ai}(t_k))$. Figure 1b depicts the reachability set $\mathcal{R}_{bi}(t_k)$ with $\Delta\theta = 30^o$. The spatial region in this case is restricted to the interior of the arc.

Once the equations of motion and the associated reachability sets are provided, one must design the controller and the actuator guidance for each actuator platform. Since a static output feedback gain is to be considered, then an expression for the output signals must be provided. To simplify further



(a) no angle constraint case.

(b) angle constraint case.

Fig. 1: Reachability regions $\mathcal{R}(t_i)$: (a) the interior of circle; (b) the interior of sector defined by $[\theta - 30^o, \theta + 30^o]$.

the controller design, it is assumed that sensors are collocated to the actuators onboard the mobile platforms. Thus the output of each sensor is given by

$$y_i(t) = \mathcal{B}^*(\chi_{ai}(t))x(t), \quad i = 1, \ldots, N. \tag{13}$$

With regards to the PDE in (1), the measurements are

$$y_i(t) = \int_{\Omega} b_i(\xi, \psi; \xi_{ai}(t), \psi_{ai}(t)) x(t, \xi, \psi) \, d\omega. \tag{14}$$

In the particular case of a spatial Dirac delta function $b_i(\xi, \psi; \xi_{ai}(t), \psi_{ai}(t)) = \delta(\xi - \xi_{ai}(t))\delta(\psi - \psi_{ai}(t))$, then the sensor measurements are given by

$$y_i(t) = x(t, \xi_{ai}(t), \psi_{ai}(t)), \quad i = 1, \ldots, N. \tag{15}$$

The proposed static feedback controllers take the form

$$u_i(t) = -\sum_{j=1}^{N} \Gamma_{ij} y_j(t), \quad i = 1, \ldots, N. \tag{16}$$

The proposed control (16) requires only the output measurements $y_i(t)$. A careful examination of (16) reveals that it imposes a heavy communication burden amongst the N mobile platforms that have to share (transmit) their own measurements with the remaining platforms. Certainly, an optimal static feedback gain Γ_{ij} can be calculated by the infinite dimensional analogue of the optimal static feedback design first presented in [7], and which entails the iterative solution of Riccati and Lyapunov equations. This however does not take into account the communication topology between the mobile platforms. An optimization scheme for computing both the gains and minimizing the communication links was proposed in [8], but that would not apply here, since the actuator are moving at the beginning of each time interval $[t_k, t_{k+1})$. Presumably, one can extend to the time varying case by solving coupled operator differential Lyapunov and differential Riccati equations. That, also, would not work as it would require a large computational load. While not considered here, one can select for simplicity an adaptive scheme to update the gains using available signals as was first presented in [9].

For simplicity, the static controllers considered in this paper are given by

$$u_i(t) = -\Gamma_{ii} y_i(t), \quad i = 1, \dots, N. \tag{17}$$

In other words, a completely decentralized controller with a complete loss of inter-agent communication is considered. Further, a uniform gain is assumed for all platforms and all time intervals with $\Gamma_{ii} = \gamma$. Since now the controllers are

$$u(t) = -\Gamma y(t), \tag{18}$$

with $\Gamma = \gamma I_N$, then one must now provide the last requirement of spatial repositioning (guidance) of the actuator platforms. The closed-loop system is given by

$$\dot{x}(t) = (\mathcal{A} - \mathcal{B}(\chi_a(t)) \Gamma \mathcal{B}^*(\chi_a(t))) x(t). \tag{19}$$

A suitable metric is the energy-to-go given by

$$J(t_k; (\chi_a)) = \int_{t_k}^{\infty} \langle x(\tau), \mathcal{M} x(\tau) \rangle \, \mathrm{d}\tau, \tag{20}$$

and whose value is given by

$$J^{opt}(t_k; \chi_a) = \langle x(t_k), \Sigma(\chi_a) x(t_k) \rangle. \tag{21}$$

The optimal value (21) can be used to find the actuator locations for each subinterval $[t_k, t_{k+1})$, but it requires access to the state $x(t_k)$. A final assumption is made on the nuclearity of the Lyapunov operator $\Sigma(\chi_a)$ and thus the optimal value, parameterized by the actuator locations χ_a, is given by

$$J^{opt}(t_k; (\chi_a)) = \mathrm{trace}\,(\Sigma(\chi_a)). \tag{22}$$

The Lyapunov operator satisfies the Lyapunov equation

$$\begin{aligned} &(\mathcal{A} - \mathcal{B}(\chi_a(t)) \Gamma \mathcal{B}^*(\chi_a(t)))^* \Sigma(\chi_a) \\ &+ \Sigma(\chi_a) (\mathcal{A} - \mathcal{B}(\chi_a(t)) \Gamma \mathcal{B}^*(\chi_a(t))) = -\mathcal{M} \end{aligned} \tag{23}$$

Having presented all three parts of the output feedback control and guidance of actuators-sensors onboard mobile platforms, we summarize all steps in Algorithm 1. Based on the motional constraints of the mobile platforms, all three reachability sets $\mathcal{R}_a(t_k)$, $\mathcal{R}_b(t_k)$ or $\mathcal{R}_c(t_k)$ can be used. Algorithm 1 reflects this choice via the generic notation $\mathcal{R}_{\Box}(t_k)$ for any of the three reachability sets.

Algorithm 1 Actuator-sensor guidance in $[t_k, t_{k+1}[$

1: **initialize:** Determine the set of admissible actuator locations Θ_{ad} that render the location-parameterized operators $(\mathcal{A}, \mathcal{B}(\chi_a))$ approximately controllable.
2: **initialize:** Select Δt based on hardware and processor requirements. Divide $[t_0, T]$ into n uniform subintervals $[t_k, t_{k+1}]$ with $t_k = t_0 + k\Delta t$ and $\Delta t = (T - t_0)/n$.
3: **initialize:** Select γ in the control laws $u(t) = -\gamma I_N y(t)$.
4: **iterate:** $k = 0$
5: **loop**
6: minimize the location-parameterized cost-to-go

$$J(x(t_k); t_k) = \int_{t_k}^{\infty} \langle x(\tau), \mathcal{M} x(\tau) \rangle \, \mathrm{d}\tau,$$

associated with the closed-loop system

$$\dot{x}(t) = \left(\mathcal{A} - \mathcal{B}(\chi_a^{opt,t_k}) \Gamma \mathcal{B}^*(\chi_a^{opt,t_k}) \right) x(t)$$

7: select the actuator locations for $[t_k, t_{k+1}[$ using

$$\chi_a^{opt,t_k} = \arg \min_{\chi_a \in \mathcal{R}(t_i)} \mathrm{trace}\,(\Sigma(\chi_a))$$

where $\Sigma(\chi_a)$ is the solution to the Lyapunov equation

$$\begin{aligned} &(\mathcal{A} - \mathcal{B}(\chi_a(t)) \Gamma \mathcal{B}^*(\chi_a(t)))^* \Sigma(\chi_a) \\ &+ \Sigma(\chi_a) (\mathcal{A} - \mathcal{B}(\chi_a(t)) \Gamma \mathcal{B}^*(\chi_a(t))) = -\mathcal{M} \end{aligned}$$

8: for $t \in [t_k, t_{k+1}[$, using (11) move to actuator location χ_a^{opt,t_k} within the appropriate reachability set $\mathcal{R}_{\Box}(t_k)$ and implement controller

$$u(t) = -\Gamma \mathcal{B}^*(\chi_a^{opt,t_k}) x(t) = -\Gamma y(t)$$

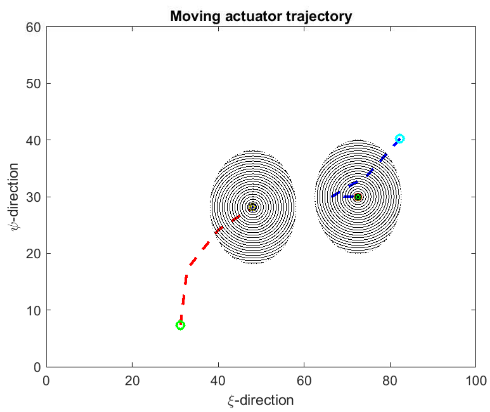
9: propagate (1) in the subinterval $[t_k, t_{k+1}[$
10: **if** $k \leq n - 2$ **then**
11: $k \leftarrow k + 1$
12: **goto** 4
13: **else**
14: terminate
15: **end if**
16: **end loop**

IV. Numerical Studies

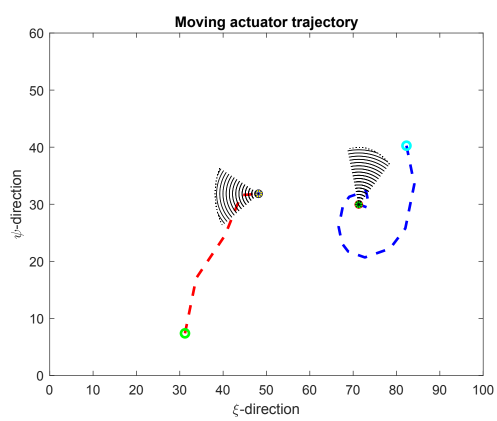
We consider the PDE in (1) with the spatial domain given by the rectangle $\Omega = [0, L_X] \cup [0, L_Y] = [0, 100] \cup [0, 60]$. For ease of implementation, the parameters in the elliptic operator were assumed constant $\alpha = 0.1, \beta = \gamma = 0$. For initial conditions the function $x_0(\xi, \psi) = 10^4 (\xi/L_X)^3 (\psi/L_Y)^3 (1 - (\xi/L_X))^3 (1 - (\psi/L_Y))^3$ was selected. A total of $N = 2$ collocated actuators-sensors were considered.

To simulate (1) and the controller (18), a finite dimensional approximation scheme based on Galerkin methods was used with $n_x = 26, n_y = 16$ linear elements. The approximation scheme ensured that exponential stabilizability is preserved [10], [11]. The spatial integrals required for the numerical computation of the matrix representation of the PDE in (1) were computed using a composite two-point Gauss-Legendre quadrature rule [12]. The finite dimensional state space model resulting from the Galerkin approximation of (1) was integrated using the stiff ODE solver from the Matlab® ODE library, routine `ode23s`, a 4$^{\text{th}}$ order Runge-Kutta scheme.

In the numerical studies, the circle-based $\mathcal{R}_{ai}(t_k)$ and sector-based $\mathcal{R}_{bi}(t_k)$ reachability sets were considered. Figure 2(a) depicts the actuator-sensor trajectories along with the circle reachability set at the penultimate time. Figure 2(b) depicts the actuator-sensor trajectories corresponding to the sector reachability set. From these two figures, it can be observed that the actuator-sensor trajectories are different for different reachability sets. The controller performance when the mobile actuator-sensor pairs move over a circle versus a sector reachability sets is presented in Figure 3, which depicts the evolution of the L_2 state norm. The use of a circle reachability set has a slight advantage over the sector reachability set. Both of course perform better when the actuator-sensor pairs are fixed in space, positioned at the same location as the initial position of the mobile actuators.



(a) circular reachability set.



(b) sector reachability set

Fig. 2: Mobile actuator trajectories including the reachability set shown at the penultimate switch time; yellow circles (○) designate the penultimate actuator locations $\chi_{a1}(t_{n-2})$ and $\chi_{a2}(t_{n-2})$, blue asterix ($*$) denotes the final locations $\chi_{a1}(t_{n-1})$, $\chi_{a2}(t_{n-1})$, and the green circle (○) and cyan circle (○) denote the initial location $\chi_{a1}(t_0)$, $\chi_{a2}(t_0)$.

V. Conclusions

A guidance policy for collocated mobile actuator-sensor using realistic motional constraints was presented. The constraints incorporated platform kinematics with restrictions on

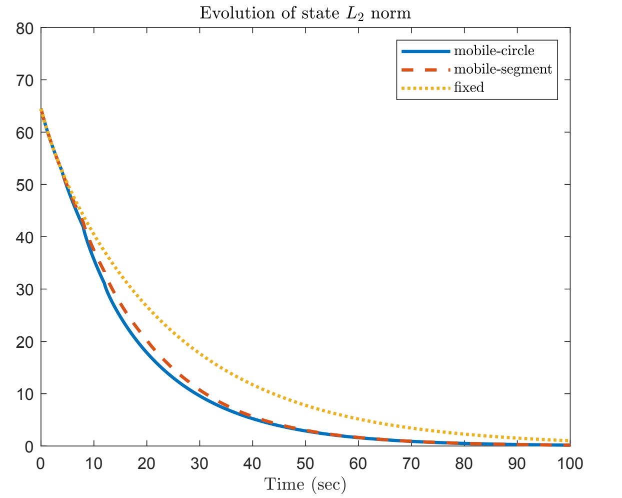


Fig. 3: Evolution of the L_2 state norm using the actuator guidance with circular reachability set (blue), with sector reachability set (red) and using fixed actuators (orange).

the set of spatial points that each platform can visit over a time interval and which were dictated by time-varying reachability sets. To simplify the computational requirements a static controller was selected and which provided realistic control architectures. The proposed guidance and control scheme was demonstrated on a 2D PDE. An immediate extension involves the information sharing between the actuator-sensor pairs and their modified guidance for collision avoidance and performance improvement.

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