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Controlling PDEs with mobile actuators constrained over time-varying reachability sets

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***Abstract*— The use of mobile actuators for the control of spatially distributed systems governed by PDEs results in both implementational and computational challenges. First it requires the backward-in-time solution to the actuator guidance and the backward-in-time solution to the control operator Riccati equation. A way to address this computational challenge is to consider a continuous-discrete alternative whereby the mobile actuator is repositioned at discrete instances and resides in a specific spatial location for a certain time interval. In order to find optimal paths for a given time interval, a set of feasible locations is derived using the reachability set. These reachability sets are further constrained to take into account the time it takes to travel to any spatial position with a prescribed maximum velocity. The proposed hybrid continuous-discrete control and actuator guidance is demonstrated for a 2D diffusion PDE that uses no constraints and angular constraints on the actuator motion.**

I. Introduction

One of the earliest treatments of partial differential equations (PDEs) with moving actuators and sensors was the work by Butkovskiĭ and Pustyl'nikov in [1]. The idea was to improve performance of controllers and estimators by appropriately moving actuators and/or sensors. This gave rise to the need for system theoretic aspects of controllability and observability for PDEs with moving actuators/sensors, [2].

More recently, Lyapunov-based guidances for moving actuators used for improving controller performance and moving sensors for improving estimator performance were presented in [3], [4], [5], [6]. The guidance was based on stability arguments and repositioned the mobile actuator or sensor so that the resulting error would converge to zero faster. Optimality of the motion of a sensor in order to improve the performance of a state estimator for a class of PDEs was presented in [7], [8]. The associated mobile actuator problem for the optimal control of PDEs can be derived by duality using the results in [7], [8]. However, one of the challenges is the computational costs associated with the backwards-in-time integration of the actuator guidance and the associated differential Riccati equation.

A computationally efficient way to circumvent the excessive computational costs associated with backwards-in-time integrations, is to consider suboptimal schemes whereby the mobile actuator dynamics are incorporated in the decision policy. A decision to reposition the mobile actuator is made at an a priori defined time instant and the optimal location is found via the minimization of an infinite horizon linear quadratic cost. The mobile actuator must then traverse to a new position and reside for a given time interval. This process is repeated for the next (sub)optimal position. The benefit for this suboptimal policy it two-fold: (a) it allows the computation of algebraic Riccati equations, as opposed to differential Riccati equations, and (ii) provides a real-time implementable policy by taking into account the time-varying reachability sets defined by the mobile platform dynamics. Optimization of the cost-to-go is now over a time varying reachability set, as opposed to the entire spatial domain.

The above modifications are presented for a class of 2D PDEs with the 2D kinematic equations serving as the motion dynamics of a mobile platform carrying an actuator.

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II. Problem formulation

Advection-diffusion PDEs, or more generally parabolic PDEs, in 2D spatial domain can be used to model a wide range of engineering applications; single species mass distribution (salinity, biological mass dispersion), temperature distribution, multi-agent model interaction, etc. When considered over a 2D domain $\Omega \subset \mathbb{R}^2$ with boundary $\partial\Omega$,

$$\frac{\partial x(t,\xi,\psi)}{\partial t} = \mathcal{A}x(t,\xi,\psi) + b(\xi,\psi;\xi_a(t),\psi_a(t))u(t). \qquad (1)$$

In the above, $x(t,\xi,\psi)$ denotes the state at time $t \in \mathbb{R}^+$ and spatial coordinates $\chi = (\xi,\psi) \in \Omega$. The spatial operator is

$$\mathcal{A}\phi = \sum_{i,j}^{n} \alpha_{ij}(t,\chi)\frac{\partial^2\phi}{\partial\chi_i\partial\chi_j} + \sum_{i=1}^{n} \beta(t,\chi)\frac{\partial\phi}{\partial\chi_i} + \gamma(t,\chi)\phi,$$

and is assumed to be uniformly elliptic in $]0,T[\times\Omega$ [9]. In the above, the spatial dimension is taken as $n = 2$ and the coefficients α,β,γ are uniformly Hölder continuous over $]0,T[\times\Omega$; a condition required for well-posedness [10], [11]. In fact, a further simplifying assumption is being made in which the coefficients α,β,γ are time invariant. The spatial domain Ω is assumed to be simply connected, open, bounded with sufficiently smooth boundary $\partial\Omega$. The function $b(\xi,\psi;\xi_a(t),\psi_a(t))$ denotes the spatial distribution of the actuator with $\chi_a(t) = (\xi_a(t),\psi_a(t)) \in \Omega$ denoting its time-varying centroid, and u its corresponding control signal. To completely describe the PDE system, boundary and initial conditions are required; mixed boundary conditions may be assumed to represent a more general PDE system and thus

$$x(t_0,\xi,\psi) = x_0(\xi,\psi), \qquad x\Big|_{\partial\Omega_d} = 0, \quad \frac{\partial x}{\partial n}\Big|_{\partial\Omega_n} = 0,$$

where $\partial\Omega_d \cup \partial\Omega_n = \partial\Omega$ is the smooth boundary and $\frac{\partial}{\partial n}$ denotes the outward normal. However, in the remainder, it will be taken that Dirichlet boundary conditions are assumed for ease of presentation and thus

$$x(t_0,\xi,\psi)=x_0(\xi,\psi), \qquad x\Big|_{\partial\Omega}=0. \tag{2}$$

A pointwise actuator is assumed here which reflects a realistic and widely used actuator spatial distribution. Thus, the following assumption is made.

Assumption 1: The actuator distribution is that of a spatial Dirac delta function centered at $\chi_a(t)=(\xi_a(t),\psi_a(t))$ with

$$b(\xi,\psi;\xi_a(t),\psi_a(t))=\delta(\xi-\xi_a(t))\delta(\psi-\psi_a(t)).$$

When the actuator is fixed with $(\xi_a(t),\psi_a(t))=(\xi_a,\psi_a)$ for all $t>0$, one can resort to established results for controller design. The PDE in (1), (2) in state space form is

$$\begin{aligned}\dot{x}(t)&=Ax(t)+B(\chi_a)u(t)\\ x(t_0)&=x_0\end{aligned} \tag{3}$$

where the operator $A: H_0^1(\Omega)\to H^{-1}(\Omega)$ in weak form is

$$\begin{aligned}\langle A\phi,\psi\rangle=&\int_\Omega \nabla\cdot(\alpha(\xi,\psi)\nabla\phi_1(\xi,\psi))\phi_2(\xi,\psi)\,\mathrm{d}\omega\\ &+\int_\Omega \beta(\xi,\psi)\nabla\phi_1(\xi,\psi)\phi_2(\xi,\psi)\,\mathrm{d}\omega\\ &+\int_\Omega \gamma(\xi,\psi)\phi_1(\xi,\psi)\phi_2(\xi,\psi)\,\mathrm{d}\omega,\end{aligned}$$

for $\phi_1,\phi_2\in H_0^1(\Omega)$, and the location-dependent input operator $B(\cdot):\mathbb{R}^1\to H^{-1}(\Omega)$ is given by

$$\langle B(\chi_a)u,\phi_1\rangle=\int_\Omega b(\xi,\psi;\xi_a,\psi_a)u(t)\phi_1(\xi,\psi)\,\mathrm{d}\omega.$$

When the actuator centroid χ_a is such that the pair $(A,B(\chi_a))$ is approximately controllable, [12], then one can obtain the control signal that minimizes the infinite horizon cost

$$J(x_0;t_0)=\int_{t_0}^{\infty}\langle x(\tau),Q_1x(\tau)\rangle+R_1u^2(\tau)\,\mathrm{d}\tau. \tag{4}$$

The solution to this optimization problem is given by

$$u(t)=-R_1^{-1}B^*(\chi_a)P(\chi_a)x(t) \tag{5}$$

where $P(\chi_a)$ solves the location-parameterized Riccati operator equation

$$\begin{aligned}0=&\langle A\phi_1,P(\chi_a)\phi_2\rangle+\langle P(\chi_a)\phi_1,A\phi_2\rangle\\ &-\langle P(\chi_a)B(\chi_a)(\tfrac{1}{R_1})B^*(\chi_a)P(\chi_a)\phi_1,\phi_2\rangle+\langle Q_1\phi_1,\phi_2\rangle,\end{aligned} \tag{6}$$

for $\phi_1,\phi_2\in D(A)$. The above is conducive to optimization of the actuator location, [13]. If there is a set Θ_{ad} of admissible locations that ensures the solvability of the location-parameterized Riccati equation (6), then the optimal is

$$\chi_a^{opt}=\arg\min_{\chi_a\in\Theta_{ad}}\langle x(t_0),P(\chi_a)x(t_0)\rangle. \tag{7}$$

In the above, it is assumed that the actuator centroid $\chi_a\in\Theta_{ad}$ is fixed in space for all times $t\geq t_0$ and the set Θ_{ad} has constant elements (fixed spatial locations).

When the actuator is to be repositioned within the domain Ω in an optimal way and/or the control cost (4) is over a finite horizon, then one has to solve a *differential* Riccati equation. The cost (4) is now over a finite time interval

$$J(x_0;t_0)=\int_{t_0}^{T}\langle x(\tau),Q_1x(\tau)\rangle+R_1u^2(\tau)\,\mathrm{d}\tau, \tag{8}$$

and when a fixed actuator location is to be selected for the entire time interval $[0,T]$, one can then obtain it in a similar fashion to (7); the difference now is that the Riccati operator is the solution to the associated differential equation. The optimal location is

$$\chi_a^{opt}=\arg\min_{\chi_a\in\Theta_{ad}}\langle x(t_0),P(t_0;\chi_a)x(t_0)\rangle \tag{9}$$

where the location-parameterized Riccati operator solves

$$\begin{aligned}-\langle\dot{P}(t;\chi_a)\phi_1,\phi_2\rangle=&\langle A\phi_1,P(t;\chi_a)\phi_2\rangle\\ &+\langle P(t;\chi_a)\phi_1,A\phi_2\rangle+\langle Q_1\phi_1,\phi_2\rangle\\ &-\langle P(t;\chi_a)B(\chi_a)(\tfrac{1}{R_1})B^*(\chi_a)P(t;\chi_a)\phi_1,\phi_2\rangle,\end{aligned} \tag{10}$$

with terminal condition $P(T;\chi_a)=0$.

III. Optimal control and mobile actuator guidance

When the actuator is mobile, with the time-varying centroid $\chi_a(t)$ having linear dynamics of the form

$$\dot{\chi}_a(t)=F\chi_a(t)+\upsilon(t), \quad \chi_a(0)\in\Omega, \tag{11}$$

where $\upsilon(t)$ denotes the controls for the mobile platform carrying the actuator and the matrix F captures the (linearized) motion dynamics, then the cost to be optimized is now

$$\begin{aligned}J(x_0;t_0)\;=&\int_{t_0}^{T}\langle x(\tau),Q_1x(\tau)\rangle+R_1u^2(\tau)\,\mathrm{d}\tau\\ &+\int_{t_0}^{T}\chi_a^T(\tau)Q_2\chi_a(\tau)+\upsilon^T(\tau)R_2\upsilon(\tau)\,\mathrm{d}\tau.\end{aligned} \tag{12}$$

The optimal solution to the minimum of (12) is given in terms of the adjoint states $(p(t),q(t))$ by

$$\begin{cases}u(t)=-R_1^{-1}p(t;\chi_a(t))\\ \upsilon(t)=-\frac{1}{2}R_2^{-1}q(t)\end{cases} \tag{13}$$

where the adjoint states $(p(t),q(t))$ satisfy

$$\begin{cases}\dfrac{\partial p}{\partial t}=-A^*p-2x\\ p\Big|_{\partial\Omega}=0\\ p(T,\xi,\psi)=0\\ F^T\dot{q}=-q-2Q_2\chi_a(t)-u\nabla p\Big|_{\chi=\chi_a(t)}\\ \chi_a(T)=0\end{cases} \tag{14}$$

The use of the method of transportation isomorphism in [14] is used to provide the existence of the solutions (14) to the optimal control problem described by (1), (11), (12).

Remark 1: Similar to the fixed actuator case, one can arrive at a location-parameterized operator Riccati differential equation that must be solved backwards in time. As can be observed from (14), the mobile actuator guidance law must also be integrated backwards in time. This of course poses an algorithmic and implementational challenge commanding and committing prohibitively large computational resources.

IV. Suboptimal control and mobile actuator motion

In response to Remark 1, one may consider suboptimal control and actuator guidance laws that are at the same time real-time implementable with realistic computational costs.

1) One can consider the optimal control problem (12) over each of the smaller time intervals $[t_0,t_1]\cup[t_1,t_2]\cup\ldots\cup[t_{n-1},T]$; in particular, one first solves (14) backwards in time from t_1 to t_0 and then propagates the state $x(t,\xi,\psi)$ in (1) and $\chi_a(t)$ in (11) from t_0 to t_1. Then the optimal control problem is solved in the interval $[t_1,t_2]$ by solving (14) backwards in time from t_2 to t_1 and propagate the state $x(t,\xi,\psi)$ in (1) and $\chi_a(t)$ in (11) from t_1 to t_2. This is repeated till the last subinterval $[t_{n-1},T]$. This is suboptimal and still an open loop policy. However, because of the updated actuator locations every $(t_{i+1}-t_i)$ time units that change in response to (9), a certain closed-loop performance can be argued.
2) An alternative to the suboptimal policy that considers (13), (14) over smaller time intervals, is to consider a policy that requires the solution to algebraic Riccati equations (ARE) and additionally does not require backwards-in-time integration of the mobile actuator guidance. The computational costs is significantly reduced and a motion controller can be selected to reposition the actuator to a desired spatial location.

The latter is detailed below as a first approach in addressing computationally efficient schemes for switching and/or moving actuators.

A. Actuator switching/activation in $[t_i,t_{i+1}]$ with discrete time updates on the cost-to-go

The idea has been explored earlier and entails the use of optimal static actuator in (7) with updated cost-to-go. Initially, one considers the infinite horizon cost

$$J(x_0;t_0)=\int_{t_0}^{\infty}\langle x(\tau),Q_1x(\tau)\rangle+R_1u^2(\tau)\,\mathrm{d}\tau$$

and the optimal actuator location for the interval $[t_0,\infty[$ is

$$\chi_a^{opt}=\arg\min_{\chi_a\in\Theta_{ad}}\langle x(t_0),P(\chi_a)x(t_0)\rangle,\quad t\in[t_0,\infty[.$$

Now, at a later time $t_1=t_0+\delta t$, one <u>re-examines</u> the optimal location and considers the *updated cost-to-go*

$$J(x(t_1);t_1)=\int_{t_1}^{\infty}\langle x(\tau),Q_1x(\tau)\rangle+R_1u^2(\tau)\,\mathrm{d}\tau.$$

The optimal actuator location for the interval $[t_1,\infty[$ is now

$$\chi_a^{opt}=\arg\min_{\chi_a\in\Theta_{ad}}\langle x(t_1),P(\chi_a)x(t_1)\rangle,\quad t\in[t_1,\infty[.$$

The above procedure is continued for subsequent time intervals and the cost-to-go is examined with a lower time limit t_i. At each time instant t_i, the actuator is switched to a new location according to

$$\chi_a^{opt}=\arg\min_{\chi_a\in\Theta_{ad}}\langle x(t_i),P(\chi_a)x(t_i)\rangle,\quad t\in[t_i,\infty[.\qquad(15)$$

At every instant t_i, the infinite horizon problem is *re-evaluated* with an updated cost-to-go which itself produces a new actuator location according to (15). This switching policy is suboptimal, but it is closer to being closed-loop. Algorithm 1 summarizes this actuator switching policy.
<u>*Advantage:*</u> The actuator location optimization (15) is performed over the admissible set Θ_{ad} and requires the solution to the *algebraic* operator Riccati equations (6). At the onset of a new time interval, one can use the set of algebraic Riccati equations computed for the first interval to form the costs (15) for subsequent times.
<u>*Disadvantage:*</u> This policy assumes that the actuating device can hop at the discrete times $t_0,t_1,t_2,\ldots,$ instantaneously. Another way to view this is to assume an infinite number of pointwise actuators within the set $\Theta_{ad}\in\Omega$ with only one of them kept active at all times while the remaining ones are kept inactive (dormant). No consideration for the motion of the mobile platform carrying the actuating devices is made.

Algorithm 1 Actuator switching in $[t_i,t_{i+1}[$

1: **initialize:** Determine the set Θ_{ad} consisting of all locations that render the pairs $(A,B(\chi_a))$ approximately controllable. Divide the interval $[t_0,T]$ into n uniform subintervals $[t_i,t_{i+1}]$ with $t_i=t_0+i\delta t$ and $\delta t=(T-t_0)/n$.
2: **iterate:** $i=0$
3: **loop**
4: minimize the location-parameterized cost-to-go
$$J(x(t_i);t_i)=\int_{t_i}^{\infty}\langle x(\tau),Q_1x(\tau)\rangle+R_1u^2(\tau)\,\mathrm{d}\tau,$$
5: select the actuator location for $[t_i,t_{i+1}[$ using
$$\chi_a^{opt,t_i}=\arg\min_{\chi_a\in\Theta_{ad}}\langle x(t_i),P(\chi_a)x(t_i)\rangle$$
with $P(\chi_a)$ the solution to the ARE
$$0=\langle A\phi_1,P(\chi_a)\phi_2\rangle+\langle P(\chi_a)\phi_1,A\phi_2\rangle$$
$$-\langle P(\chi_a)B(\chi_a)(\tfrac{1}{R_1})B^*(\chi_a)P(\chi_a)\phi_1,\phi_2\rangle+\langle Q_1\phi_1,\phi_2\rangle,$$
6: for $t\in[t_i,t_{i+1}[$, switch to actuator with location χ_a^{opt,t_i} and implement controller
$$u(t)=-R_1^{-1}B^*(\chi_a^{opt,t_i})P(\chi_a^{opt,t_i})x(t)$$
7: propagate (1) or (3) in the interval $[t_i,t_{i+1}[$
8: **if** $i\le n-2$ **then**
9: $i\leftarrow i+1$
10: **goto** 3
11: **else**
12: terminate
13: **end if**
14: **end loop**

B. Actuator guidance in $[t_i,t_{i+1}]$ with discrete time updates on cost-to-go

In this case, the mobile actuator will have to move to its new position at the beginning of a new interval $[t_i,t_i+\delta t]$ according to motion dynamics of the form (11). However, in most cases, the platform dynamics may involve more states than the actuator centroid. A widely used motion dynamic model for the 2D case involves the kinematic equations

$$\begin{aligned}\dot{\xi}_a(t)&=v(t)\cos(\theta_a(t)),\\ \dot{\psi}_a(t)&=v(t)\sin(\theta_a(t)),\\ \dot{\theta}_a(t)&=\omega_a(t).\end{aligned}\qquad(16)$$

The control signals for (16) are the speed $v(t)$ and turning rate $\omega(t)$; often the speed is assumed constant and thus only the turning rate is the control signal. The mobile actuator centroid is taken to be the output of the dynamical system

$$\chi_a(t) = \begin{bmatrix} \xi_a(t) \\ \psi_a(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \xi_a(t) \\ \psi_a(t) \\ \theta_a(t) \end{bmatrix}. \quad (17)$$

With the inclusion of the vehicle motion dynamics, one can derive the equivalent expression to (13), (14) for the nonlinear motion dynamics (16).

Unlike the previous case of actuator switching using (15), here it is expected for the single mobile actuator to reposition itself within Ω to a new location at the beginning of a new time subinterval $[t_i, t_{i+1}]$. Since now the motion is constrained by the kinematic equations (16), this repositioning cannot occur instantaneously. The problems associated with the inclusion of the motion dynamics of the actuator are:

1) The search for the new actuator position in (15) over the set Θ_{ad} may require large computational time. Searching over the large set Θ_{ad} may be time consuming and may even require more time to complete than the duration of the time interval $t_{i+1} - t_i$.
2) The search for the new actuator position in (15) over the set Θ_{ad}, even if it is performed instantaneously using a high-end processor, may produce a position that is far away from the current actuator position $\chi_a(t_i)$; the vehicle may simply be unable to reach the new commanded position in the time interval $t_{i+1} - t_i$.

To address the above two challenges, one can restrict the search over a subset of the admissible set Θ_{ad}; thus the search (17) will be performed over a significantly smaller set thereby rendering the optimization search real-time implementable. To address the second challenge above, one may assume that the time it takes to go to a new commanded actuator location within Ω while obeying (16), is significantly less than the duration of any of the decision intervals $t_{i+1} - t_i$.

V. Main results: suboptimal control and mobile actuator motion constrained over time-varying reachability sets

The duration $t_{i+1} - t_i$ that an actuator resides at a given spatial location $\chi_a(t)$ for $t \in [t_i, t_i + \delta t[$, is termed the *residence time*; assuming that all time subintervals are uniform with duration δt, then the residence time t_{res} satisfies

$$t_{res} = t_{i+1} - t_i = \delta t.$$

Thus one can express the beginning of a new subinterval as

$$t_i = t_0 + i t_{res}, \qquad i = 1, \dots, \frac{T - t_0}{t_{res}} - 1, \quad \forall i. \quad (18)$$

At the beginning of a new interval t_i, the actuator must move to its new commanded position while obeying (16) in a travel time t_{travel} that is significantly less than the residence time t_{res}. Thus one imposes

$$t_{travel} \ll t_{res}. \quad (19)$$

Because of the time constraint the actuator has to reposition itself, the set of actuator locations that can be traversed within t_{travel} time units is characterized by the points that are within a distance $v t_{travel}$ from the current position $\chi_a(t_i)$ where v is the maximum speed in equation (16)

$$\begin{aligned} \xi(t) &= \xi_a(t_i) + (v t_{travel}) \cos(\theta(t)) \\ \psi(t) &= \psi_a(t_i) + (v t_{travel}) \sin(\theta(t)) \end{aligned} \quad (20)$$

where the angle is $\theta_a(t_i) - \pi \le \theta(t) \le \theta_a(t_i) + \pi$. This immediately defines the time-varying *reachability set*

$$\mathcal{R}_a(t_i) = \Theta_{ad} \cap \Big\{ (\xi, \psi, \theta) : (\xi, \psi) \text{ satisfy (20)}, \ -\pi \le \theta(t) \le \pi \Big\} \quad (21)$$

The reachability set $\mathcal{R}_a(t_i)$ consists of all the spatial points within Θ_{ad} that can be reached by the mobile actuator from the current location $\chi_a(t_i) = (\xi_a(t_i), \psi_a(t_i))$ within the time span t_{travel}. Figure 1a depicts the reachability region (21) which is a circle centered at the current actuator location $\chi_a(t_i) = (\xi_a(t_i), \psi_a(t_i))$. When searching for the actuator location in $[t_i, t_{i+1}[$, one optimizes (15) over $\mathcal{R}_a(t_i)$.

When there is an angle constraint, namely the mobile actuator can traverse a distance $v \cdot t_{travel}$ from the current $\chi_a(t_i) = (\xi_a(t_i), \psi_a(t_i)$ but has an angular constraint $\pm\Delta\theta$ with $\Delta\theta \ll \pi$, the circular region now becomes a sector defined by $\theta(t) \in [\theta_a(t_i) - \Delta\theta, \theta_a(t_i) + \Delta\theta]$ with

$$\begin{aligned} \xi &= \xi_a(t_i) + (v t_{travel}) \cos(\theta(t)) \\ \psi &= \psi_a(t_i) + (v t_{travel}) \sin(\theta(t)) \end{aligned} \quad (22)$$

and the reachability region is

$$\mathcal{R}_b(t_i) = \Theta_{ad} \cap \Big\{ (\xi, \psi, \theta) : (\xi, \psi) \text{ satisfy (22)}, \ \theta_a(t_i) - \Delta\theta \le \theta(t) \le \theta_a(t_i) + \Delta\theta \Big\} \quad (23)$$

Figure 1b depicts the reachability region (22) with $\Delta\theta = 30^o$. The region is constrained to the interior of the arc.

Finally, one may consider the case where there is a constraint in the angular rate as well. In this case, there is time required for turning to an angle $\Delta\theta$ given by

$$t_{turn} = \frac{|\Delta\theta|}{\omega} \quad (24)$$

which then leaves $(t_{travel} - t_{turn})$ time units to travel. The possible points in the sector that can now be reached are

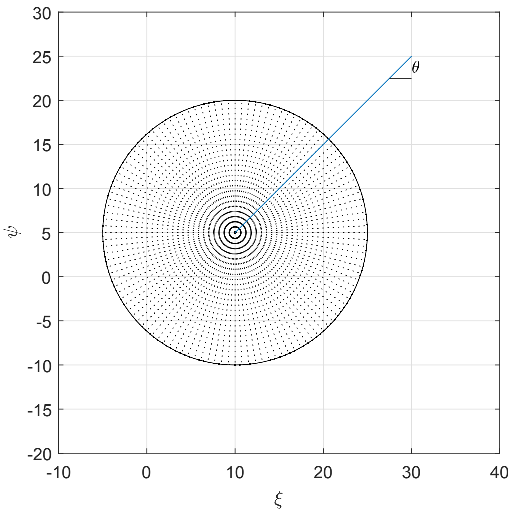
$$\begin{aligned} \xi &= \xi_a(t_i) + v(t_{travel} - t_{turn}) \cos(\theta(t)) \\ \psi &= \psi_a(t_i) + v(t_{travel} - t_{turn}) \sin(\theta(t)) \end{aligned} \quad (25)$$

and the reachability region is now given by

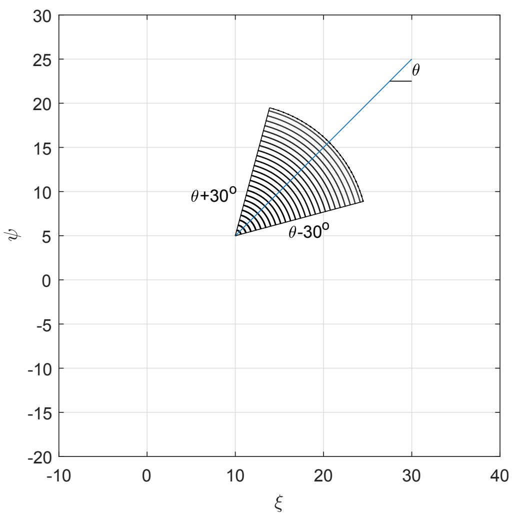
$$\mathcal{R}_c(t_i) = \Theta_{ad} \cap \Big\{ (\xi, \psi, \theta) : (\xi, \psi) \text{ satisfy (25)}, \ \theta_a(t_i) - \Delta\theta \le \theta \le \theta_a(t_i) + \Delta\theta, \ |\Delta\theta| \le \omega t_{turn} \Big\} \quad (26)$$

Figure 1c depicts the reachability region (25) that includes both angle and angular rate constraints. The angular rate constraints result in a significantly smaller region than in Figure 1b since now the mobile platform has to execute the turn within t_{turn} time units, leaving only $t_{travel} - t_{turn}$ time units for traversing.

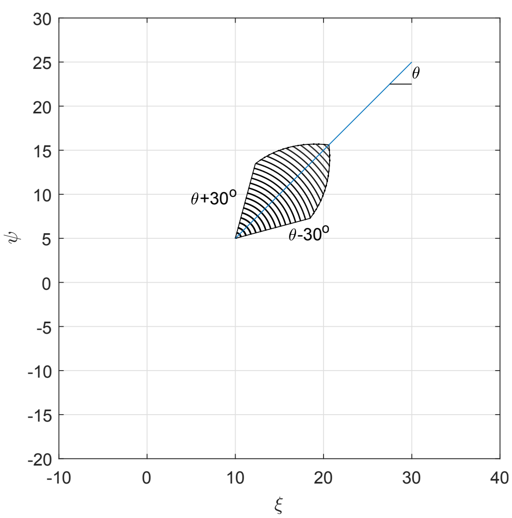
Algorithm 2 summarizes the proposed actuator guidance with the search restricted over the time varying reachability sets $\mathcal{R}_a(t_i)$, $\mathcal{R}_b(t_i)$ or $\mathcal{R}_c(t_i)$.



(a) no angle constraint case.



(b) angle constraint case.



(c) angle and angular rate constraint case.

Fig. 1: Reachability regions $\mathcal{R}(t_i)$: (a) the interior of circle; (b) the interior of sector defined by $[\theta - 30^o, \theta + 30^o]$; (c) the interior of segment defined by $[\theta - 30^o, \theta + 30^o]$.

Algorithm 2 Actuator guidance in $[t_i, t_{i+1}[$

1: **initialize:** Determine the set Θ_{ad} consisting of all locations that render the pairs $(A, B(\chi_a))$ approximately controllable. Divide the interval $[t_0, T]$ into n uniform subintervals $[t_i, t_{i+1}]$ with $t_i = t_0 + i\delta t$ and $\delta t = (T - t_0)/n$.
2: **iterate:** $i = 0$
3: **loop**
4: minimize the location-parameterized cost-to-go

$$J(x(t_i); t_i) = \int_{t_i}^{\infty} \langle x(\tau), Q_1 x(\tau)\rangle + R_1 u^2(\tau)\, d\tau,$$

5: select the actuator location for $[t_i, t_{i+1}[$ using

$$\chi_a^{opt,t_i} = \arg \min_{\chi_a \in \mathcal{R}(t_i)} \langle x(t_i), P(\chi_a) x(t_i)\rangle$$

with $P(\chi_a)$ the solution to

$$0 = \langle A\phi_1, P(\chi_a)\phi_2\rangle + \langle P(\chi_a)\phi_1, A\phi_2\rangle$$
$$-\langle P(\chi_a)B(\chi_a)(\tfrac{1}{R_1})B^*(\chi_a)P(\chi_a)\phi_1, \phi_2\rangle + \langle Q_1\phi_1, \phi_2\rangle,$$

6: for $t \in [t_i, t_{i+1}[$, move to actuator location χ_a^{opt,t_i} within the reachability set $\mathcal{R}(t_i)$ and implement controller

$$u(t) = -R_1^{-1} B^*(\chi_a^{opt,t_i}) P(\chi_a^{opt,t_i}) x(t)$$

7: propagate (1) or (3) in the interval $[t_i, t_{i+1}[$
8: **if** $i \leq n-2$ **then**
9: $i \leftarrow i+1$
10: **goto** 3
11: **else**
12: terminate
13: **end if**
14: **end loop**

VI. Numerical studies

The PDE (2) with $\Omega = [0, L_X] \cup [0, L_Y] = [0, 100] \cup [0, 60]$ and $\alpha = 0.1, \beta = \gamma = 0$ is considered. Initial conditions were set as $x_0(\xi, \psi) = 10^4 \xi^3 \psi^3 (L_X - \xi)^3 (L_Y - \psi)^3$. A Galerkin-based finite dimensional approximation scheme using $n_x = 26, n_y = 16$ linear elements was used to approximate (1).

The duration that a given actuator would reside in a particular location was selected as $t_{res} = 4$ s while a travel time was selected as $t_{travel} = 0.1 t_{res} = 0.4$. The speed in (16) was set to $v(t) = 25$m/s. The initial actuator location was selected as $\chi_a(t_0) = (\xi_a(t_0), \psi_a(t_0)) = (0.312 L_X, 0.123 L_Y)$. The LQR cost values were selected as $Q_1 = 10I$ and $R_1 = 0.01$. The closed-loop system was numerically integrated using the Matlab ode solver `ode45` over the interval $[0, 100]$s.

Algorithm 2 was implemented with the reachability set given by $\mathcal{R}_a(t_i)$ in (21) (circle in Figure 1a) and by $\mathcal{R}_b(t_i)$ in (23) (sector in Figure 1b). As expected the guidance over different reachability sets produced a different actuator trajectory. Figure 2 depicts the trajectory where the instantaneous reachability set is a circle and Figure 3 depicts the trajectory where the instantaneous reachability set is a sector.

As a comparison of the performance of the mobile actuator, a fixed-in-space actuator was selected at the spatial location of the initial position of the moving actuator, i.e. at $(0.312 L_X, 0.123 L_Y)$. The evolution of the state L_2 norm is depicted for the case of a fixed actuator and a moving

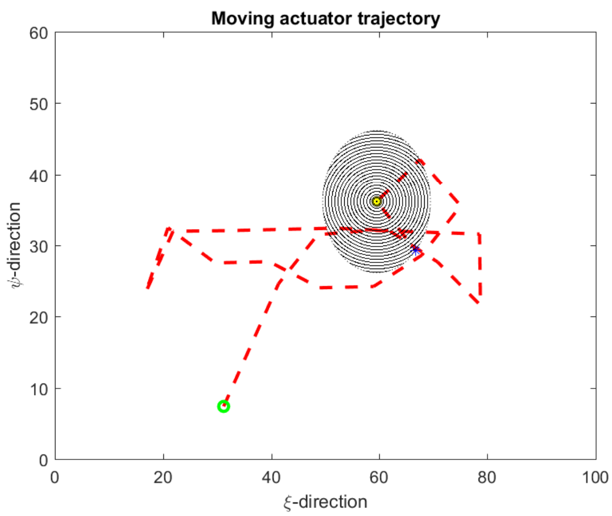


Fig. 2: Moving actuator trajectory including the reachability set (circle) shown at the penultimate switch time; yellow circle ($\circ$) designates the penultimate actuator location $\chi_a(t_{n-2})$, blue asterix ($*$) denotes the final location $\chi_a(t_{n-1})$ and the green circle ($\circ$) denotes the initial location $\chi_a(t_0)$.

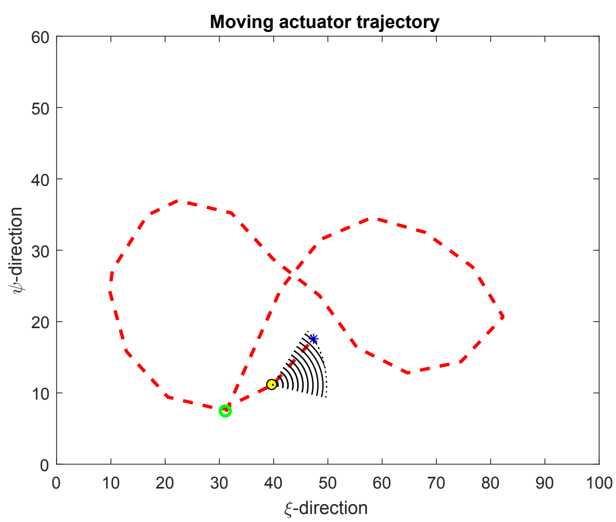


Fig. 3: Moving actuator trajectory including the reachability set (segment) shown at the penultimate switch time; yellow circle ($\circ$) designates the penultimate actuator location $\chi_a(t_{n-2})$, blue asterix ($*$) denotes the final location $\chi_a(t_{n-1})$ and the green circle ($\circ$) denotes the initial location $\chi_a(t_0)$.

actuator using the reachability set $\mathcal{R}_a(t_i)$ in Figure 4. As expected, the case of a moving actuator outperforms the case of a fixed actuator. Similar results are obtained when the reachability set is selected as $\mathcal{R}_c(t_i)$

VII. Conclusions

A suboptimal policy for the repositioning of a mobile actuator was presented as a means to address the computational costs associated with optimal policies that require the backwards-in-time integration of actuator guidances and Riccati equations needed for the control signals.

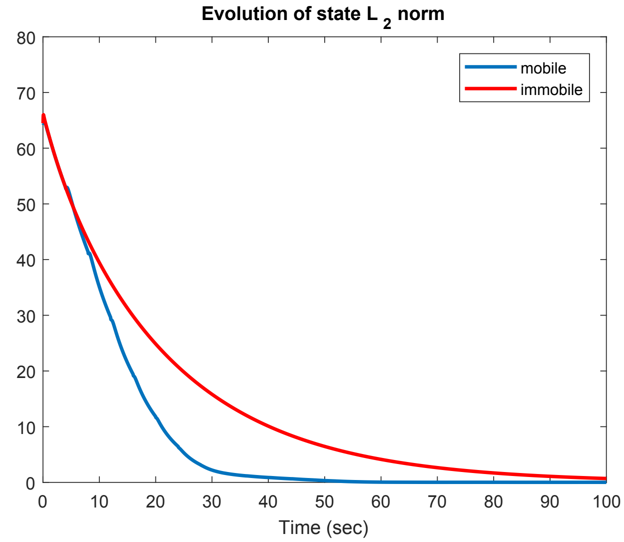


Fig. 4: Evolution of the L_2 state norm using the proposed actuator guidance (blue) and using a fixed actuator (red).

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