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Design and optimization of minimum-order compensators of distributed parameter systems via functional observers and unknown input functional observers

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Abstract— **This paper revisits the design of compensator-based controller for a class of infinite dimensional systems. In order to save computational time, a functional observer is employed to reconstruct a functional of the state which coincides with the full state feedback control signal. Such a full-state feedback corresponds to an idealized case wherein the state is available. Instead of reconstructing the entire state via a state-observer and then use this state estimate in a controller expression, a functional observer is used to estimate the product of the state and the feedback operator, thus resulting in a significant reduction in computational load. This observer design is subsequently integrated with a sensor selection in order to improve controller performance. An appropriate metric is used to optimize the sensor location resulting in improved performance of the functional observer-based compensator. The integrated design is further extended to include a controller with an unknown input functional observer. The results are applied to 2D partial differential equations and detailed numerical studies are included to provide an appreciation in the significant savings in both operational and computational costs.**

I. Introduction

Most of the work on *functional observers* (FO) and *unknown input functional observers* (UIFO) has been applied to finite dimensional systems e.g. [1], [2], [3] and more recently in the research monograph [4] and references therein. Functional observers are observers that provide an estimate *not* of the entire state of a dynamical system, *but* a functional of the state. Such an estimate of a function of the state (linear or nonlinear) can subsequently be used in lieu of a full-state feedback controller and thus the functional observer in this case will essentially provide an estimate of the full-state feedback-based controller signal.

The benefit of using a functional observer to produce the control signal is mainly computational. Normally, one implements an observer-based feedback whereby the full state feedback control gain matrix is designed as if the full state were available. Then a state observer, based either on Luenberger observer design of Kalman filter design, is implemented to provide in real-time the estimate of the state process. The last step involves the multiplication of the feedback gain by the state estimate to realize the control signal. Even in the time-invariant case where the controller and filter gains are constants, one still has to simulate a state estimator in real time. When a functional observer is utilized in place of an observer-based controller, then the only system simulated is the state of the functional observer which has dimension significantly lower than the dimension of the dynamical system.

An unknown input observer aims at providing a state estimate of a dynamical process with disturbances by utilizing the knowledge of the distribution matrix of the unknown disturbance input (unknown input) [5]. If the distribution matrix of the unknown disturbance input is known, this information can be utilized in the observer design, via the solution to an associated Sylvester equation, to ensure that regardless of the presence of an unknown input, the estimation error converges to zero asymptotically. This framework has been used in fault detection by designing fault detection observers that are sensitive to faults but are not affected by the unknown inputs [6], [7], [8], [9]. An unknown input functional observer combines the above two designs and produces an estimate of a functional of the process state despite the presence of unknown inputs (disturbances), [4].

The migration and extension of these observers to infinite dimensional systems is rather scant. Early works [10], [11], [12] extended FO and UIFO concepts to a class of partial differential equations and provided an optimization scheme for the selection of the best sensor location. A more systematic approach to the use of Sylvester equations appearing in the design of functional observer was considered in a sequence of papers by Emirsajlow [13], [14], [15].

Combining the earlier results and allowing for a further optimization of the sensor location, if one has the freedom to select a sensor location, is considered here. The conditions that require the realization of a functional observer (FO) and an unknown input functional observer (UIFO) for a class of infinite dimensional systems are presented and the well-posedness of the associate observer (FO and UIFO) are summarized. The use of the state of the functional observer as a substitute of a full-state feedback controller is also examined here along with the resulting closed-loop stability. Such a compensator reveals the important benefit of a functional observer in controller design which is more prevalent in infinite dimensional systems. The use of an observer-based feedback requires the implementation of the finite dimensional representation of the state observer. With the proposed use of functional observer as the control signal, one has to only implement a significantly lower-dimensional

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system whose dimension is not dictated by numerical convergence but by the dimension of the control signal; equivalently this is the same as the rank of the input operator of the infinite dimensional system.

A. Contributions

The contributions of this paper are as follows

1) Extend the FO and UIFO designs to a class of infinite dimensional systems.
2) Provide the well-posedness of the FO and UIFO when the state of the observer is used as a controller.
3) Develop a sensor optimization scheme to find the best sensor locations for FO and UIFO.
4) Demonstrate on a 2D diffusion PDE.

Section II presents a review of the finite dimensional FO and UIFO design. Section III describes the proposed FO for PDEs and its implementation as a compensator along with a sensor optimization scheme. Similarly, Section IV describes the proposed UIFO for DPS and its implementation as a compensator along with a sensor optimization scheme. Section V includes the numerical results for various cases and Section VI concludes the paper with a brief summary of the contributions and future work.

II. REVIEW OF FINITE DIMENSIONAL RESULTS

Consider the finite dimensional system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1a}$$

$$y(t) = Cx(t), \tag{1b}$$

$$z(t) = Kx(t), \tag{1c}$$

where $x \in \mathbb{R}^n$ is the process state vector, $u \in \mathbb{R}^m$ is the control vector, $y \in \mathbb{R}^p$ is the output measurement vector and $z \in \mathbb{R}^r$ is the state functional that is desired to be estimated; the dimension of z is $r \leq n$. When $r = n$ with $K = \mathbf{I}_n$, or more generally when $\text{rank}(K) = n$, then the problem reverts to the standard state estimator.

A. Functional Observer

The proposed functional observer, as taken from Darouach [2] is given by

$$\dot{w}(t) = Nw(t) + Jy(t) + Hu(t), \tag{2a}$$

$$\widehat{z}(t) = w(t) + Ey(t). \tag{2b}$$

The enabling conditions are given by

$$PA - NP - JC = \mathbf{0}_{r\times n}, \tag{3}$$

$$H = PB \tag{4}$$

with $P = K - EC$. Central to the stability of the functional observer is the estimation error

$$\begin{aligned} e(t) &= z(t) - \widehat{z}(t) \\ &= Kx(t) - \widehat{z}(t) \\ &= Px(t) - w(t) \end{aligned} \tag{5}$$

Using the above, the estimation error dynamics are

$$\begin{aligned} \dot{e}(t) &= Ne(t) + (PA - NP - JC)x(t) + (PB - H)u(t), \\ &= Ne(t), \end{aligned} \tag{6}$$

where the $r \times r$ matrix N is designed to be Hurwitz.

When $r = m$ and K is designed to be a state feedback gain with $A + BK$ Hurwitz, then the compensator

$$u(t) = \widehat{z}(t), \tag{7}$$

results in the closed-loop system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ &= Ax(t) + B\widehat{z}(t) \\ &= (A + BK)x(t) - Be(t). \end{aligned}$$

Closed loop stability is examined for the augmented system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} (A+BK) & -B \\ \mathbf{0}_{r\times n} & N \end{bmatrix}\begin{bmatrix} x(t) \\ e(t) \end{bmatrix}. \tag{8}$$

The spectrum of the augmented state matrix consist of the spectra of $\sigma(A + BK) \cup \sigma(N)$. Since both matrices are Hurwitz, then (8) is exponentially stable, [4].

B. Unknown Input Functional Observer

Now consider

$$\dot{x}(t) = Ax(t) + Bu(t) + Fd(t), \tag{9a}$$

$$y(t) = Cx(t), \tag{9b}$$

$$z(t) = Kx(t), \tag{9c}$$

where $d \in \mathbb{R}^q$ denotes the unknown disturbance signal and F is the known $n \times q$ disturbance distribution matrix. The goal is to design an UIFO that will estimate z despite the presence of the disturbance signal d. The proposed UIFO is also given by (2) where now one imposes

$$PA - NP - JC = \mathbf{0}_{r\times n}, \quad H = PB, \quad PF = \mathbf{0}_{r\times q}. \tag{10}$$

Similar to (5), the estimation error is governed by

$$\begin{aligned} \dot{e}(t) = Ne(t) &+ (PA - NP - JC)x(t) + (PB - H)u(t) \\ &+ PFd(t). \end{aligned} \tag{11}$$

When the conditions in (10) are satisfied, then (11) becomes $\dot{e} = Ne$ which establishes the exponential convergence of e to zero. When the estimated functional $\widehat{z}$ is used as a compensator, then the closed-loop system becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} (A+BK) & -B \\ \mathbf{0}_{r\times n} & N \end{bmatrix}\begin{bmatrix} x(t) \\ e(t) \end{bmatrix} + \begin{bmatrix} F \\ \mathbf{0}_{r\times q} \end{bmatrix} d(t) \tag{12}$$

Closed-loop stability can be established when additional conditions on d are imposed, e.g. $d \in L^2$, [16], [4].

Remark 1: When the unknown input d is present but its distribution F is unknown, then the FO-based controller will result in a closed-loop system given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} (A+BK) & -B \\ \mathbf{0}_{r\times n} & N \end{bmatrix}\begin{bmatrix} x(t) \\ e(t) \end{bmatrix} + \begin{bmatrix} F \\ PF \end{bmatrix} d(t).$$

The estimation error in this case is governed by

$$\dot{e}(t) = Ne(t) + PFd(t)$$

and may not converge to zero. If the additional assumption $d \in L^2$ is made, then the convergence may not be exponential. In both the FO and UIFO cases with $\widehat{z}$ used as the estimate of $z = u$, one arrives at a reduced order compensator since the traditional observer-based feedback has order n while the FO or UIFO-based compensator has order equal to the rank of the input matrix, m.

III. Problem formulation: FO for infinite dimensional systems

The above functional observers (FO and UIFO) are extended for a class of infinite dimensional systems and the estimated z is subsequently used in lieu of the full state feedback controller, i.e. use $u(t)=\widehat{z}(t)$ instead of $u(t)=Kx(t)$. When the counterpart of (3), (4) or (10) that enable the realization of the functional observer can be satisfied with different output operators C, then one may proceed with sensor optimization.

Let $\mathcal{X}$ and $\mathcal{U}$ be Hilbert spaces. Assume that $A\colon D(A)\to\mathcal{X}$ generates a C_0-semigroup on $\mathcal{X}$ and $B\colon \mathcal{U}=\mathbb{R}^m\to\mathcal{X}$ is a bounded operator of rank m. If (A,B) is feedback stabilizable, then K can be solved from a feedback Algebraic Riccati equation such that $A+BK$ generates an exponentially stable C_0-semigroup on $\mathcal{X}$ (*cf.* p. 485 Theorem 3.1 [17]). In this case $K\colon\mathcal{X}\to\mathbb{R}^m$ is bounded, the state $z\in\mathbb{R}^m$, and N is a $m\times m$ dimensional matrix. Further assume that $F\in\mathcal{L}(\mathcal{X})$, a bounded operator on $\mathcal{X}$, and $d\in L^2(0,\infty;\mathcal{X})$.

The infinite dimensional counterpart of (1) is given by

$$\begin{aligned}\dot{x}(t)&=Ax(t)+Bu(t)+Fd(t),\qquad x(0)\in D(A),\\ y(t)&=Cx(t),\\ z(t)&=Kx(t).\end{aligned}\tag{13}$$

If the operator F is not known, one may still design a functional observer that will ensure the error $e(t)=z(t)-\widehat{z}(t)$ will converge to zero in the appropriate norm. The functional observer that estimates $z(t)=Kx(t)$ in (13) is given by

$$\begin{aligned}\dot{w}(t)&=Nw(t)+Jy(t)+Hu(t),\\ \widehat{z}(t)&=w(t)+Ey(t).\end{aligned}\tag{14}$$

To obtain a system similar to (8), one must assume that the derivative of the output satisfies

$$\dot{y}(t)=C\dot{x}(t).\tag{15}$$

This condition is only found in infinite dimensional systems since one does not always have that $\frac{d}{dt}\left(Cx(t)\right)=C\left(\frac{d}{dt}x\right)$. Combining (13),(14) and using (15) one arrives at

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}x(t)\\ e(t)\end{bmatrix}&=\begin{bmatrix}A&0\\0&N\end{bmatrix}\begin{bmatrix}x(t)\\ e(t)\end{bmatrix}\\ &\quad+\begin{bmatrix}B\\ \mathbf{0}_{r\times m}\end{bmatrix}u(t)+\begin{bmatrix}F\\ PF\end{bmatrix}d(t).\end{aligned}\tag{16}$$

When the control signal is taken at $u(t)=\widehat{z}(t)$ with $r=m$, then (16) becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}x(t)\\ e(t)\end{bmatrix}&=\begin{bmatrix}A+BK&-B\\0&N\end{bmatrix}\begin{bmatrix}x(t)\\ e(t)\end{bmatrix}\\ &\quad+\begin{bmatrix}F\\ PF\end{bmatrix}d(t).\end{aligned}\tag{17}$$

Both the open loop (16) and closed-loop (17) systems require the solution to the operator equalities

$$PA-NP=JC,\qquad PB=H,\tag{18}$$

where the solution to the Sylvester operator equation $P:\mathcal{X}\to\mathbb{R}^r$, J is an $r\times q$ matrix, H is an $r\times m$ matrix and E is an $r\times q$ matrix. If the control is selected as the estimated functional $u(t)=\widehat{z}(t)$, then $r=m$.

Lemma 1: If the pair (A,B) is feedback stabilizable, the output $y(t)$ satisfies (15), $d\in L^2(0,\infty;\mathcal{X})$ and the Sylvester equation (18) is satisfied, then the FO in (16) is well-posed and for $u\in L^2(0,\infty;\mathbb{R}^m)$ the state x is bounded with

$$\lim_{t\to\infty}|e(t)|_{R^r}=0.$$

Further, if the controller is selected as $u(t)=\widehat{z}(t)$, then the closed-loop system (17) is well-posed and

$$\lim_{t\to\infty}\|x(t)\|=0,\quad \lim_{t\to\infty}|e(t)|_{R^m}=0.\tag{19}$$

Proof The state operator

$$\mathcal{A}=\begin{bmatrix}(A+BK)&-B\\0&N\end{bmatrix}\tag{20}$$

generates an exponentially stable C_0 semigroup on $\mathcal{X}\times\mathbb{R}^m$ since N is Hurwitz and $A+BK$ generates an exponentially stable C_0 semigroup, [18]. Using the fact that $d\in L^2(0,\infty;\mathcal{X})$, then the perturbed system (17) is stable leading to (19). □

A. Sensor optimization and design of Functional Observer

Now parameterizing the output measurement operator C in terms of sensor location $\xi\in\mathbb{R}^l$, then N solved from the Sylvester equation (18) can be also parameterized by ξ.

Let the operator (20) be parameterized by ξ

$$\mathcal{A}(\xi)=\begin{bmatrix}(A+BK)&-B\\0&N(\xi)\end{bmatrix}.\tag{21}$$

with domain $D(\mathcal{A}(\xi))=D(A)\times\mathbb{R}^m\to\mathcal{X}\times\mathbb{R}^m$. Note that if ξ is chosen such that $\sigma_p(N(\xi))\subset\mathbb{C}^-$, then $\mathcal{A}(\xi)$ generates an exponentially stable C_0-semigroup on $\mathcal{X}\times\mathbb{R}^m$. Further let

$$\vec{x}=\begin{bmatrix}x\\e\end{bmatrix},\qquad \mathcal{F}(\xi)=\begin{bmatrix}F\\P(\xi)F\end{bmatrix}.$$

The closed-loop system becomes

$$\dot{\vec{x}}(t)=\mathcal{A}(\xi)\vec{x}(t)+\mathcal{F}(\xi)d(t).\tag{22}$$

The objective is to choose the sensor location to determine C and hence N, such that

$$\sup_{\vec{x}_0\in\mathcal{X},\|\vec{x}_0\|_2=1}J(\vec{x}_0)=\int_0^\infty\|D\vec{x}(\xi)\|_2^2\,\mathrm{d}t\tag{23}$$

is minimized, where $D\in\mathcal{L}(\mathcal{X})$ is a weight operator. Note that in this case,

$$\sup_{\vec{x}_0\in\mathcal{X},\|\vec{x}_0\|_2=1}J(\vec{x}_0)=\sup_{\vec{x}_0\in\mathcal{X},\|\vec{x}_0\|_2=1}(\mathcal{P}(\xi)\vec{x}_0,\vec{x}_0).$$

Here $\mathcal{P}(\xi)$ satisfies the ξ-parameterized Lyapunov equation

$$\mathcal{P}(\xi)\mathcal{A}(\xi)+\mathcal{A}^*(\xi)\mathcal{P}(\xi)+D^*D=\mathbf{0}.\tag{24}$$

As a result, the optimal sensor location is given by

$$\boxed{\xi = \inf_{\xi\in\mathbb{R}^l} \|\mathcal{P}(\xi)\|_1,} \tag{25}$$

where $\|\cdot\|_1$ stands for the nuclear norm, i.e., $\|\mathcal{P}\|_1 = \operatorname{trace}(\sqrt{\mathcal{P}^*\mathcal{P}})$. However, the robustness of the sensor location is not guaranteed with unknown disturbance distribution F. At this case, one has to assume a "worst" distribution of disturbance as was considered in [19], [20], [21] or if F is known, to incorporate this knowledge in the functional observer design and the subsequent sensor optimization.

IV. Problem formulation: UIFO for infinite dimensional systems

When one has knowledge of the distribution of disturbances, via the operator F, then the conditions for the solvability of the unknown input functional observer become

$$PA - NP = JC, \qquad PB = H, \qquad PF = 0. \tag{26}$$

The general UIFO, which is the counterpart to (16), is given by

$$\dot{\vec{x}}(t) = \begin{bmatrix} A & 0 \\ 0 & N \end{bmatrix}\vec{x}(t) + \begin{bmatrix} B \\ \mathbf{0}_{r\times m} \end{bmatrix}u(t) + \begin{bmatrix} F \\ 0 \end{bmatrix}d(t). \tag{27}$$

When the control signal is taken as $u(t) = \widehat{z}(t)$ with $r = m$, then (27) becomes

$$\dot{\vec{x}}(t) = \begin{bmatrix} A+BK & -B \\ 0 & N \end{bmatrix}\vec{x}(t) + \begin{bmatrix} F \\ 0 \end{bmatrix}d(t). \tag{28}$$

Comparing (17) (unknown F) and (28) (known F), one notices that $PF = 0$ is enforced via (26) and thus the input distribution operator can be used in the sensor optimization.

The convergence and well-posedness results in Lemma 1 for (27) and (28) extend to this case. Due to the similarities in the arguments, the proof is omitted.

A. Sensor optimization and design of Unknown Input Functional Observer

In this case, the closed-loop ξ-parameterized system (28) is given by (*cf.* (22))

$$\dot{\vec{x}}(t) = \mathcal{A}(\xi)\vec{x}(t) + \mathcal{G}d(t) \tag{29}$$

where

$$\mathcal{G} = \begin{bmatrix} F \\ 0 \end{bmatrix}.$$

With F being known, the sensor location is chosen such that

$$\sup_{\vec{x}_0\in X, \|\vec{x}_0\|_2=1} \sup_{d\in X} J(\vec{x}_0, d) = \int_0^\infty \left(\|D\vec{x}(\xi)\|_2^2 - \gamma^2\|d\|_2^2\right)\mathrm{d}t \tag{30}$$

is minimized, where parameter $\gamma > 0$. If γ is chosen properly, then there exists a unique solution to (29), and

$$\sup_{\vec{x}_0\in X, \|\vec{x}_0\|_2=1} \sup_{d\in X} J(\vec{x}_0, d) = \sup_{\vec{x}_0\in X, \|\vec{x}_0\|_2=1} (\mathcal{S}\vec{x}_0, \vec{x}_0).$$

Here $\mathcal{S}(\xi)$ satisfies the filter Riccati equation (*cf.* (24))

$$\mathcal{S}(\xi)\mathcal{A}(\xi) + \mathcal{A}^*(\xi)\mathcal{S}(\xi) + D^*D = -\gamma^{-2}\mathcal{G}\mathcal{G}^*. \tag{31}$$

The optimal sensor location is again given by

$$\boxed{\xi = \inf_{\xi\in\mathbb{R}^l} \|\mathcal{S}(\xi)\|_1.} \tag{32}$$

Remark 2: Note that the min-max problem (30) is of non-definite quadratic cost, which is related to H^∞-robust, state feedback stabilization problem. For infinite dimensional systems, it is shown in [22, Chp. 6] that there exists a critical $\gamma_c \geq 0$ such that if $\gamma > \gamma_c$, then the existence of a unique positive definite solution to the Riccati equation (31) is guaranteed. However, the maximization problem over the disturbances d does not have a finite solution if $0 < \gamma < \gamma_c$.

Remark 3: The knowledge of the distribution of disturbances operator F affects the integrated design of functional observer and sensor optimization twofold: When F is known, then the functional observer can utilize the information so that the equation for the error $z(t) - \widehat{z}(t)$ in (17) with F unknown

$$\dot{e}(t) = Ne(t) + PFd(t),$$

can have an exponential convergence as seen in its counterpart from (28)

$$\dot{e}(t) = Ne(t).$$

Additionally, when F is known, then the UIFO enforces $PF = 0$ in (26) and thus the ξ-parameterized closed-loop system (29) can use the knowledge of F to find the optimal sensor location that yields a closed-loop system that is robust with respect to the disturbances.

V. Numerical Results

We consider the diffusion PDE over the 2D domain $[0, L_\chi] \times [0, L_\psi] = [0, 100] \times [0, 60]\, m$

$$\frac{\partial x(t,\chi,\psi)}{\partial t} = a\left(\frac{\partial^2 x(t,\chi,\psi)}{\partial\chi^2} + \frac{\partial^2 x(t,\chi,\psi)}{\partial\psi^2}\right) + b(\chi,\psi)u(t) + f(\chi,\psi)d(t)$$

with Dirichlet boundary conditions and measurements

$$y(t) = \int_0^{L_\chi}\int_0^{L_\psi} \delta(\chi-\chi_s)\delta(\psi-\psi_s)x(t,\chi,\psi)\,\mathrm{d}\psi\,\mathrm{d}\chi.$$

The spatial distribution of the controller was selected as $b(\chi,\psi) = \delta(\chi - 0.5L_\chi)\delta(\psi - 0.5L_\psi)$, the diffusivity was set to $a = 10$, and the spatial distribution of the unknown input was selected as

$$f(\chi,\psi) = 100\chi\psi(L_\chi-\chi)^3(L_\psi-\psi)^3.$$

Both the actuator and sensor distribution functions, selected as the spatial Dirac delta functions, do not result in bounded operators. However to numerically realize these functions, the boxcar function

$$\delta(\chi-\chi_s) \approx \begin{cases} \frac{1}{2\varepsilon} & \text{if } \chi_s - \varepsilon \leq \chi \leq \chi_s + \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

was used, which ensures that the associated operators are bounded. The initial conditions was selected as

$$x(0,\chi,\psi) = 10^4\left(\frac{\chi}{L_\chi}\right)^3\left(\frac{\psi}{L_\psi}\right)^3\left(1-\frac{\chi}{L_\chi}\right)^3\left(1-\frac{\psi}{L_\psi}\right)^3.$$

case	$\int_0^T \|x(t)\|^2\,\mathrm{d}t$
full state $u(t) = Kx(t)$	140.23
UIFO optimal sensor $u(t) = \widehat{z}(t)$	165.88
UIFO non-optimal sensor $u(t) = \widehat{z}(t)$	682.26

TABLE I

CLOSED-LOOP SYSTEM STATE NORM.

The optimal gain K was designed using an LQR controller design that minimized

$$\int_0^\infty \langle x(\tau), Qx(\tau)\rangle + u^T(\tau)Ru(\tau)\,\mathrm{d}\tau,$$

with $Q = 10\mathbf{I}, R = 0.01$.

To simulate the above PDE, a Galerkin-based finite element scheme was used with $n_\chi = 26$ linear elements in the χ direction and $n_\psi = 16$ linear elements in the ψ direction. The spatial integrals were numerically evaluated using a composite two-point Gauss-Legendre quadrature rule, [23]. The resulting semidiscrete system was numerically integrated in the time interval $[0, 100]s$ using the Matlab ode solver `ode23s`. The integrated UIFO and sensor optimization scheme was used with the knowledge of $f(\chi, \psi)$. However, to simplify the numerical simulations, the closed-loop system (28) was simulated with $d = 0$.

The optimization scheme produced an optimal sensor location at $(65.67, 20.61)$. As a non-optimal sensor, the locations was selected as the one that enabled the UIFO but yielded the largest value of $\|\mathcal{S}(\xi)\|_1$ and which placed the sensor at $(18.20, 4.85)$.

The evolution of the L^2 state norm for the case of a full-state feedback controller $u(t) = Kx(t)$, the case of an UIFO based controller $u(t) = \widehat{z}(t)$ with an optimal sensor and an UIFO based controller $u(t) = \widehat{z}(t)$ with a non-optimal sensor are depicted in Figure 1. It is observed that when the sensor is optimized, then the performance of the UIFO-based controller is comparable to the full-state controller. The difference in terms of information is that the ideal case of a full-state controller requires the entire infinite dimensional state, whereas the UIFO-based controller requires a single scalar output measurement and the integration of the scalar state of the UIFO.

Comparing the performance of the optimal versus the non-optimal sensor location of the UIFO-based controller is depicted in Figure 2. At the two different time instances of $t = 25s$ and $t = 75s$, the spatial distribution of the state is shown. The positive effects of the optimal sensor location are highlighted, where one can observe the significant difference of the amplitude for the two cases. Finally, the performance of the UIFO in reconstructing the functional $Kx(t)$ is shown in Figure 3. The functional estimate $\widehat{z}(t)$ converges, as shown theoretically, to the true functional $Kx(t)$ exponentially.

The performance of the UIFO-based controller is also summarized in Table I where the energy norm in the interval $[0, 100]s$ is presented for the three cases.

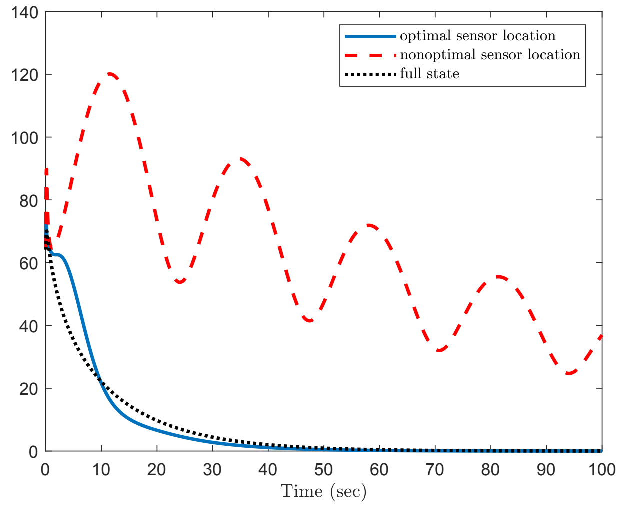

Fig. 1. Evolution of the L_2 state norm using the functional observer output for control $u(t) = \widehat{z}(t)$ with an optimal sensor (blue), the functional observer output for control $u(t) = \widehat{z}(t)$ with a non-optimal sensor (red) and the system with the full state feedback controller $u(t) = Kx(t)$ (black).

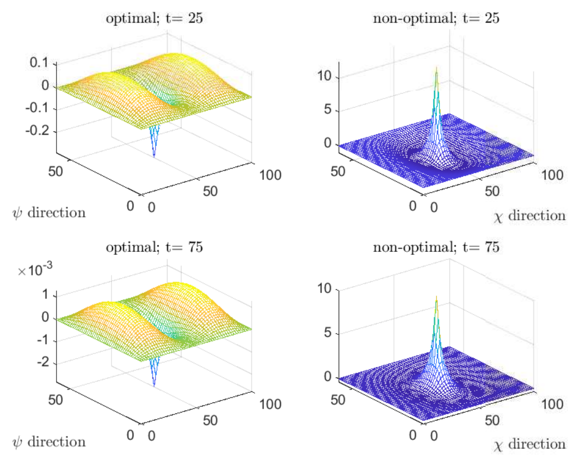

Fig. 2. Spatial distribution of the state $x(t, \chi, \psi)$ at different time instances using the optimal sensor (left) and the non-optimal sensor (right).

VI. CONCLUSION

This paper examined the use of functional observer output as a substitute to a full-state feedback for a class of infinite dimensional systems. This has a significant impact on the computational costs for the implementation of the controller. While in the traditional approach of an observer-based controller one has to realize and implement a finite dimensional approximation of the full-order observer with a dimension that is dictated by numerical stability and convergence, the proposed functional observer-based controller has dimension equal to the rank of the input operator.

When a disturbance input is not present, the functional observer output can be used in lieu of a full state controller and when certain conditions are satisfied, then the resulting closed-loop system is shown to be exponentially stable. Adding another level of performance improvement,

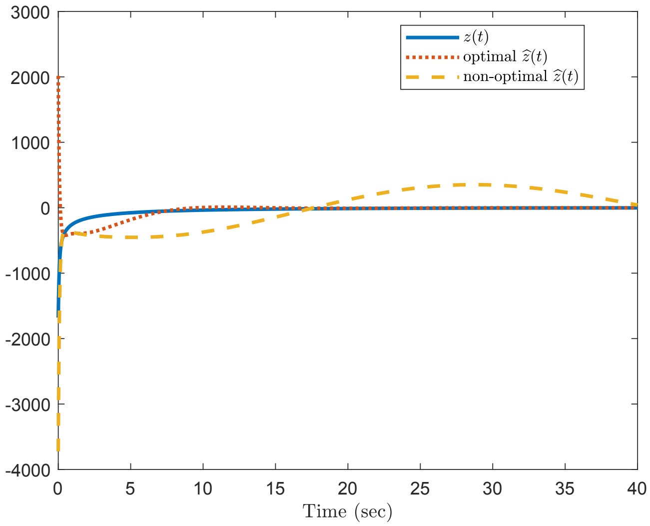

Fig. 3. Evolution of the functional $z = Kx$ and its estimate $\hat{z}(t)$.

the sensor location optimization was presented. By using an appropriate metric of the resulting closed-loop system, the resulting sensor optimization produced the optimal sensor location that resulted in the smallest state energy. With the assumption of nuclearity of a location-parameterized Lyapunov operator, the optimal sensor location was computed via the minimization of the trace of the location-parameterized solution to a operator Lyapunov equation.

When an unknown disturbance input is present in the system, but distribution operator is known, then this knowledge was used to extend the Unknown Input Functional Observer to the infinite dimensional case. When similar algebraic conditions were satisfied, including an operator Sylvester equation, were satisfied, the resulting UIFO output was used as a substitute to the control signal and the closed-loop system was also shown to be well-posed with controller performance similar to the full-state feedback. A sensor optimization scheme was also presented for the case of an unknown input with known distribution operator. The optimization metric in this case was expressed in terms of a location-parameterized filter operator Riccati equation ($\mathbb{H}^\infty$). The resulting optimal sensor resulted in a closed-loop system with robustness with respect to unknown disturbance inputs.

Extensive numerical studies involving a diffusion PDE in two spatial dimensions was considered and which provided insights on the performance of an UIFO-based controller with optimal sensor. Such a controller performance was comparable to a full-state feedback.

Possible extensions to the FO and UIFO based controller with sensor optimization include the case of unbounded input and output operators. This along with the case of joint actuator-and-sensor selection for both FO and UIFO for a class of infinite dimensional systems are currently considered by the authors and will appear in a forthcoming publication.

References

[1] M. Aldeen and H. Trinh, "Reduced-order linear functional observer for linear systems," *IEE Proceedings*, pp. 399–405, 1999.

[2] M. Darouach, "Existence and Design of Functional Observers for Linear Systems," *IEEE Trans. Automatic Control*, vol. 45, no. 5, pp. 940–943, 2000.

[3] R. D. Gupta, F. Fairman, and T. Hinamoto, "A direct procedure for the design of single functional observers," *IEEE Trans. Circuits and Systems*, vol. 28, no. 4, pp. 294–300, 1981.

[4] H. Trinh and T. Fernando, *Functional observers for dynamical systems*, ser. Lecture Notes in Control and Information Sciences. Springer, Berlin, 2012, vol. 420.

[5] M. Darouach, M. Zasadzinski, and S. J. Xu, "Full-order observers for linear systems with unknown inputs," *IEEE Transactions on Automatic Control*, vol. 39, no. 3, pp. 606–609, 1994.

[6] J. Gertler, *Fault Detection and Diagnosis in Engineering Systems*. CRC Press, 2019.

[7] R. Isermann, *Fault-diagnosis applications*. Springer, Heidelberg, 2011, model-based condition monitoring: actuators, drives, machinery, plants, sensors, and fault-tolerant systems.

[8] J. Chen and R. J. Patton, *Robust Model-Based Fault Diagnosis for Dynamic Systems*. Springer-Verlag, 1998.

[9] S. Simani, C. Fantuzzi, and R. J. Patton, *Model-based Fault Diagnosis in Dynamic Systems Using Identification Techniques*. Springer-Verlag, 2010.

[10] M. A. Demetriou, "Optimization of sensor locations in transport systems using functional observers," in *Proc. of the 9th IEEE International Conference on Methods and Models in Automation and Robotics*, Miedzyzdroje, Poland, August 25-28 2003.

[11] M. A. Demetriou, "Robust sensor location optimization in distributed parameter systems using functional observers," in *Proc. of the 44th IEEE Conference on Decision and Control*, Dec 2005, pp. 7187–7192.

[12] M. A. Demetriou and I. G. Rosen, "Unknown input observers for a class of distributed parameter systems," in *Proc. of the 44th IEEE Conference on Decision and Control*, Dec 2005, pp. 3874–3879.

[13] Z. Emirsajlow and T. Barcinski, "Infinite-dimensional sylvester equation in observers design," in *Proc. of the European Control Conference*, 2007, pp. 5162–5168.

[14] Z. Emirsajł ow, "Infinite-dimensional Sylvester equations: basic theory and application to observer design," *Int. J. Appl. Math. Comput. Sci.*, vol. 22, no. 2, pp. 245–257, 2012.

[15] Z. Emirsajłow, "Remarks on functional observers for distributed parameter systems," in *Proc. of the International Conference on Methods Models in Automation Robotics*, Aug 2018, pp. 144–147.

[16] M. Zasadzinski, M. Darouach, and M. Hayar, "Loop transfer recovery designs with an unknown input reduced-order observer-based controller," *Internat. J. Robust Nonlinear Control*, vol. 5(7), pp. 627–648, 1995.

[17] A. Bensoussan, G. Da Prato, M. C. Delfour, and S. K. Mitter, *Representation and control of infinite dimensional systems*, 2nd ed., ser. Systems & Control: Foundations & Applications. Birkhäuser Boston, Inc., Boston, MA, 2007.

[18] R. F. Curtain and H. J. Zwart, *An Introduction to Infinite Dimensional Linear Systems Theory*, ser. Texts in Applied Mathematics, Vol. 21. Berlin: Springer-Verlag, 1995.

[19] M. A. Demetriou and J. Borggaard, "Optimization of a joint sensor placement and robust estimation scheme for distributed parameter processes subject to worst case spatial disturbance distributions," in *Proc. of the American Control Conference*, vol. 3, 30 June-2 July 2004, pp. 2239–2244 vol.3.

[20] ——, "Design of worst spatial distribution of disturbances for a class of parabolic partial differential equations," in *Proc. of the American Control Conference*, 8-10 June 2005, pp. 3894–3899 vol. 6.

[21] K. Morris, M. A. Demetriou, and S. D. Yang, "Using $\mathbb{H}_2$-control performance metrics for the optimal actuator location of distributed parameter systems," *IEEE Transactions on Automatic Control*, vol. 60, no. 2, pp. 450–462, Feb 2015.

[22] I. Lasiecka and R. Triggiani, *Control theory for partial differential equations: Volume 1, Abstract parabolic systems: Continuous and approximation theories*. Cambridge University Press, 2000, vol. 1.

[23] M. A. Celia and W. G. Gray, *Numerical methods for differential equations*. Englewood Cliffs, NJ: Prentice Hall Inc., 1992, fundamental concepts for scientific and engineering applications.