FINITE QUOTIENTS OF BRAID GROUPS

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ABSTRACT. We derive a lower bound on the size of finite non-cyclic quotients of the braid group that is superexponential in the number of strands. We also derive a similar lower bound for nontrivial finite quotients of the commutator subgroup of the braid group.

1. INTRODUCTION

Let B_n denote the braid group on n strands and let B_n' denote its commutator subgroup. It is a basic problem to describe all homomorphisms from B_n or B_n' to a given group G. The first results are due to Artin [1], who described all transitive homomorphisms from B_n to the symmetric group S_n. Lin [8] extended Artin's results in various ways and proved analogous results for B_n'.

Since the abelianization of B_n is infinite cyclic, it is always possible to construct homomorphisms $B_n \to G$ that factor through $\mathbb{Z}$. Such a homomorphism is said to be cyclic. On the other hand, the abelianization of B_n' trivial for $n \geq 5$ (see [8]) and so no such construction is possible for B_n'. In general, for fixed n and G it is often not clear whether there exist non-cyclic homomorphisms $B_n \to G$ or non-trivial homomorphisms $B_n' \to G$. Our main results are a necessary condition for the existence of non-cyclic homomorphisms $B_n \to G$ and a necessary condition for the existence of non-trivial homomorphisms $B_n' \to G$.

Theorem 1.1. *Let G be a finite group and let $n \geq 5$. If $B_n \to G$ is not a cyclic homomorphism then*

$$|G| \geq 2^{\lfloor n/2 \rfloor - 1} \left(\lfloor n/2 \rfloor \right)!$$

Theorem 1.2. *Let G be a finite group and let $n \geq 5$. If $B_n' \to G$ is not the trivial homomorphism then*

$$|G| \geq 2^{\lfloor n/2 \rfloor - 2} \left(\lfloor n/2 \rfloor - 1 \right)!$$

Perhaps the best known finite, non-cyclic (respectively nontrivial) quotient of B_n (respectively B_n') is the symmetric group S_n (respectively the alternating group A_n). It does not appear to be known whether there are any such quotients of B_n (respectively B_n') of smaller cardinality for $n \geq 5$ (respectively $n \geq 6$), although larger finite non-cyclic quotients do exist. We remark here that B_n, and hence B_n', is residually finite and so they both possess plenty of finite quotients.

After learning about our Theorem 1.1, Margalit asked the following.

Question 1. *For $n \geq 5$, is S_n the smallest finite, non-cyclic quotient of B_n?*

We also have the following related question.

Question 2. *For $n \geq 6$, is A_n the smallest finite, non-trivial quotient of B_n'?*

The first question above has a positive answer for $n = 2$ and $n = 3$ as can be checked by examining the list of groups of order at most 6. The second question has a positive answer for $n = 5$ because any quotient of B_5' is perfect (see [8, p.7]) and A_5 is the smallest non-trivial perfect group. The answer to the first question is negative for $n = 4$ because there is an exceptional surjective homomorphism $S_4 \to S_3$. The second question has a negative answer for $n \in \{3, 4\}$ also for exceptional reasons: the group B_3' is a free group and so surjects onto $\mathbb{Z}/2$, while B_4' surjects onto the commutator subgroup of S_4 which in turn surjects onto the (cyclic) commutator subgroup of S_3.

The analogue of the above questions for mapping class groups asks for the minimal non-trivial quotient of the genus g mapping class group $\mathrm{Mod}(S_g)$. Zimmerman [9] proved that the smallest non-trivial quotient of $\mathrm{Mod}(S_g)$ is the symplectic group $\mathrm{Sp}_g(\mathbb{Z}/2)$ provided $g \in \{3, 4\}$, and conjectured that the result held for all $g \geq 3$. Zimmermann's conjecture was later proven by Kielak–Pierro [5]. In a slightly different direction, Berrick–Gebhardt–Paris [2] proved that the minimal index of a proper subgroup of $\mathrm{Mod}(S_g)$ is equal to $2^{g-1}(2^g - 1)$ and that, up to conjugation, there is exactly one subgroup with this index. We point out, though, that the subgroups of this index are not normal.

Obstructions. Various conditions are known to obstruct the existence of non-cyclic homomorphisms $B_n \to G$ or non-trivial homomorphisms $B_n' \to G$. For example, it follows from the fact that B_n' is perfect for $n \geq 5$ (see [8, p.7]) that any homomorphism from B_n or B_n' to a solvable group is cyclic or trivial, respectively, for $n \geq 5$. More generally, B_n' does not admit non-trivial homomorphisms to residually solvable groups for $n \geq 5$.

Another obstruction arises from sizes of generating sets. The braid group B_n admits a generating set of size 2 (consisting of a rotation and a half-twist), and hence any non-trivial quotient of B_n (in particular any non-cyclic quotient of B_n) can also be generated by two elements. Likewise, Gorin–Lin [8, p.6] proved that B_3' has a generating set of size 2, and the second author proved [6] that B_n' is generated by two elements for $n \in \{5\} \cup [7, \infty)$ and by three elements for $n \in \{4, 6\}$. It follows that any quotient of B_n' is generated by two elements for $n \in \{5\} \cup [7, \infty)$ or by three element for $n \in \{4, 6\}$.

Prior results. Since any finite group G embeds into a sufficiently large symmetric group S_k, it is sometimes possible to understand all homomorphisms $B_n, B_n' \to G$ by classifying all homomorphisms $B_n, B_n' \to S_k$. The first result of the latter type is due to Artin [1], who proved that all homomorphisms $B_n \to S_n$ with transitive image and with $n \neq 4, 6$ are either cyclic or conjugate to the standard projection. Artin also completely described all of the exceptional homomorphisms that arise for $n \in \{4, 6\}$.

These results were later greatly extended by Lin [8]. Among other results, he proved that any homomorphism $B_n \to S_k$ with $k < n$ and $n \geq 5$ is cyclic, that any transitive homomorphism $B_n \to S_m$ with $6 < n < m < 2n$ is cyclic, that all transitive homomorphisms $B_n \to S_{n+1}$ with $n \geq 6$ are cyclic, and that all transitive homomorphisms $B_n \to S_{n+2}$ with $n \geq 5$ are cyclic. Lin also completely characterized all homomorphisms $B_n \to S_{2n}$ with $n \geq 7$, and gave explicit formulas for the non-cyclic homomorphisms that arise.

Lin also proved several results about homomorphisms from B_n' to symmetric groups. For example, he proved that any homomorphism $B_n' \to S_n$ with $n \geq 5$ is the restriction of a homomorphism $B_n \to S_n$, that any homomorphism $B_n' \to S_k$ with $n \geq 5$ and $k < n$ is trivial, and that any transitive homomorphism $B_n' \to S_k$ with $k < 2n$ is primitive.

Overview. In Section 2, we first review the basic properties of totally symmetric subsets and give examples of totally symmetric subsets of braid groups. We then proceed to Proposition 2.2 and its proof, which form the technical core of the paper. In Section 3 we prove Theorem 1.1 and Theorem 1.2.

2. Totally symmetric sets

To prove Theorems 1.1 and 1.2 we will use the theory of totally symmetric sets, which were introduced by Margalit and the second author [7]. A totally symmetric subset of a group G is a finite subset $\{g_1, \ldots, g_n\}$ of G such that

(1) The elements g_i pairwise commute, and
(2) For any permutation $\sigma \in S_n$, there exists $h \in G$ such that

$$hg_ih^{-1} = g_{\sigma(i)} \quad \text{for all} \quad 1 \le i \le n.$$

The theory of totally symmetric sets is particularly powerful as a tool for analyzing group homomorphisms. This stems from the following fact: If $f : G \to H$ is a homomorphism and S is a totally symmetric subset of G, then $f(S)$ is a totally symmetric subset of H.

Some examples of totally symmetric sets. Totally symmetric sets occur naturally in the study of braid groups and, more generally, mapping class groups of surfaces. We now describe two totally symmetric subsets $X \subset B_n$ and $X' \subset B_n'$ that will play critical roles in the proofs of Theorems 1.1 and 1.2.

Recall that B_n is generated by a standard set of half-twists $\sigma_1, \ldots, \sigma_{n-1}$. The usual commutation relations along with the change-of-coordinates principle from mapping class group theory imply that the following subset of B_n is totally symmetric of size $\lfloor n/2 \rfloor$:

$$X = \{\sigma_{2i-1}\}_{i=1}^{\lfloor n/2 \rfloor}.$$

The elements of B_n' are exactly those elements of B_n with vanishing signed word length. We claim that the following subset of B_n' is totally symmetric of size $\lfloor n/2 \rfloor - 1$:

$$X' = \{\sigma_1\sigma_{2i-1}^{-1}\}_{i=2}^{\lfloor n/2 \rfloor}.$$

It is clear that the elements of this set commute pairwise and that they can be permuted in an arbitrary fashion via conjugation by elements of B_n. However, in order for X' to be a totally symmetric subset of B_n' we require the conjugating elements to lie in B_n'. This can be arranged as follows. Suppose that g permutes the elements of X' according to a permutation $\sigma \in S_{\lfloor n/2 \rfloor - 1}$, and let ℓ denote the signed word length of g. Then $g\sigma_1^{-\ell}$ has vanishing signed word length and permutes the elements of X' according to σ, since for each $1 \le i \le \lfloor n/2 \rfloor$ we have

$$(g\sigma_1^{-\ell})(\sigma_1\sigma_{2i-1}^{-1})(g\sigma_1^{-\ell})^{-1} = g(\sigma_1\sigma_{2i-1}^{-1})g^{-1}.$$

The image of a totally symmetric set. The following lemma establishes one of the most useful properties possessed by totally symmetric sets. It is a main ingredient in our proof of Theorems 1.1 and 1.2. The proof is due to Margalit and the second author [7, Lemma 2.1], but we also give it here for the reader's convenience.

Lemma 2.1. *Let G, H be groups and let $f : G \to H$ be a homomorphism. If $S \subset G$ is a totally symmetric set, then $|f(S)|$ is equal to either 1 or $|S|$.*

Proof. Let $S = \{g_1, \ldots, g_n\}$ and assume that $|f(S)| < |S|$. After relabeling the elements of S we may assume that $f(g_1) = f(g_2)$, and hence that $f(g_1 g_2^{-1}) = 1$. Since S is totally symmetric, for each $i > 2$ there exists $h \in G$ with $h g_1 h^{-1} = g_1$ and $h g_2 h^{-1} = g_i$. We have

$$f(g_1 g_i^{-1}) = f(h(g_1 g_2^{-1})h^{-1}) = f(h)\left(f(g_1 g_2^{-1})\right) f(h)^{-1}$$
$$= f(h)\,(1)\, f(h)^{-1}$$
$$= 1.$$

That is, $f(g_i) = f(g_1)$. This shows that $|f(S)| = 1$, as desired.   □

The lower bound. The remainder of this section is dedicated to the proof of Proposition 2.2, which gives a lower bound on the size of a group in terms of the size of a totally symmetric subset whose members have finite order.

Proposition 2.2. *Let $n \geq 1$, and suppose that S is a totally symmetric subset of a group G with $|S| = n$. If the elements of S have finite order, then $|G| \geq 2^{n-1} n!$.*

We remark that Proposition 2.2 is sharp in the sense that, for each $n \geq 1$, there exists a group G of cardinality $2^{n-1} n!$ that contains a totally symmetric set of size n all of whose elements have finite order. For $n = 1$ this is trivial. For $n \geq 2$, we may take $G = S_n \ltimes V$, where

$$V = (\mathbb{Z}/2)^n / \langle e_1 + e_2 + \cdots + e_n \rangle$$

is the standard representation of the symmetric group over $\mathbb{Z}/2$. It follows that $\{e_1, e_2, \ldots, e_n\} \subset G$ is a totally symmetric set of size n, and since V is a $\mathbb{Z}/2$-vector space of dimension $n - 1$ we have that $|G| = 2^{n-1} n!$.

To prove Proposition 2.2, we require the following lemma. It is due to Chen, Margalit, and the second author [4]. Since their paper has not yet appeared, we give the proof here.

Lemma 2.3. *Let $n \geq 1$, let G be a group, and let $S \subseteq G$ be a totally symmetric subset with $|S| = n$. Suppose that each element of S has finite order. Then $\langle S \rangle$ is a finite group whose order is greater than or equal to 2^{n-1}.*

Proof. If $n = 1$, then the lemma is trivially true, so we may assume that $n \geq 2$. Let $S = \{g_1, g_2, \ldots, g_n\}$. Since S is totally symmetric, the elements g_i all have the same order, which we denote by m. It follows that the subgroup $\langle S \rangle$ generated by S is a quotient of $(\mathbb{Z}/m)^n$, which is finite. Thus $\langle S \rangle$ is finite. Our aim now is to show that $|\langle S \rangle| \geq 2^{n-1}$.

Let p be the smallest integer in the interval $[1, m]$ such that $g_i^p = g_j^p$ for each i, j (note that such a p exists because $g_i^m = g_j^m = 1$ for all i, j). Observe that in fact $p > 1$, since $p = 1$ woud imply that the elements g_i were not all distinct.

Now let A denote the free abelian group of rank n with generators $e_1, \ldots, e_n$. Let $\bar{A}$ denote the quotient of A by the subgroup of relations generated by the elements

$$e_1 + \cdots + e_n \quad \text{and} \quad pe_i \text{ with } 1 \leq i \leq n.$$

It follows that $|\bar{A}| = p^{n-1} \geq 2^{n-1}$. To prove the lemma, it suffices to demonstrate the existence of a surjective homomorphism $\langle S \rangle \to \bar{A}$. We will do this by showing that the function $\phi : S \to \bar{A}$ defined by $\phi(g_i) = \bar{e}_i$ extends to a well-defined surjective homomorphism $\langle S \rangle \to \bar{A}$ (surjectivity follows from the fact that ϕ maps elements of S to generators of $\bar{A}$).

To this end, we consider the map $\phi' : \mathbb{Z}S \to A$ induced by ϕ, where $\mathbb{Z}S$ denotes the free abelian group generated by the elements of S. We will show that any relation satisfied by the elements of S (that is, any element of the kernel of the quotient $\mathbb{Z}S \to \langle S \rangle$) maps to a defining relation for $\bar{A}$. We will then conclude that ϕ' descends to a well-defined surjective homomorphism $\langle S \rangle \to \bar{A}$, as desired. We first carry out these steps for $n \geq 3$ and then proceed to the case of $n = 2$.

Let $R = g_1^{q_1} g_2^{q_2} \cdots g_n^{q_n} \in \mathbb{Z}S$ be a relation in $\langle S \rangle$ (here we write the elements of $\mathbb{Z}S$ in multiplicative notation). Since $g_1^p = g_i^p$ in $\langle S \rangle$ for each i, we may assume that $0 \leq q_i \leq p - 1$ for $2 \leq i \leq n$.

Assume that $n \geq 2$. First, we claim that the q_i are independent of i for $2 \leq i \leq n$. The claim is trivially true if $n = 2$, so from now on we assume that $n \geq 3$. By the total symmetry of S, there exists $h \in G$ that conjugates g_2 to g_3, that conjugates g_3 to g_2, and that commutes with all other g_i. This implies that the element $R' = g_1^{q_1} g_2^{q_3} g_3^{q_2} g_4^{q_4} \cdots g_n^{q_n} \in \mathbb{Z}S$ is also relation in $\langle S \rangle$. It follows now that

$$R(R')^{-1} = g_2^{q_2 - q_3} g_3^{q_3 - q_2}$$

is a relation in $\langle S \rangle$ and hence that

$$g_2^{q_2 - q_3} = g_3^{q_2 - q_3}$$

in $\langle S \rangle$. After possibly relabeling g_2 and g_3 we may assume that $q_2 \geq q_3$, and since $0 \leq q_2, q_3 < p$, we also have $0 \leq q_2 - q_3 < p$. By total symmetry, we have that $g_i^{q_2 - q_3} = g_j^{q_2 - q_3}$ in $\langle S \rangle$ for all i, j. By the minimality of p, we have that $q_2 - q_3 = 0$. Total symmetry further implies that $q_3 = q_4 = \cdots = q_n$. This proves the claim.

Next, we claim that there exists $\ell \in \mathbb{Z}$ such that $q_1 = q_2 + \ell p$. By the preceding claim, we can now write

$$R = g_1^{q_1} (g_2 g_3 \cdots g_n)^{q_2}.$$

As in the proof of the preceding claim, we may transpose g_1 and g_2 to obtain a further relation

$$g_2^{q_1} (g_1 g_3 \cdots g_n)^{q_2}.$$

Combining these two relations shows that $g_1^{q_1 - q_2} g_2^{-(q_1 - q_2)}$ is also a relation, and hence that

$$g_1^{q_1 - q_2} = g_2^{q_1 - q_2}$$

in $\langle S \rangle$. Since $g_i^p = g_j^p$ in $\langle S \rangle$ for all i, j, we further have that

$$g_1^{q_1 - q_2 - \ell p} = g_2^{q_1 - q_2 - \ell p} \qquad \text{for all } \ell \geq 0.$$

The total symmetry of S then implies that $g_i^{q_1 - q_2 - \ell p} = g_j^{q_1 - q_2 - \ell p}$ in $\langle S \rangle$ for all i, j. By the division algorithm, we may choose ℓ so that $0 \leq q_1 - q_2 - \ell p \leq p - 1$. Again by the minimality of p, we must have that $q_1 - q_2 - \ell p$. That is, $q_1 = q_2 + \ell p$. The claim is proven.

Combining the preceding two claims, we may now write

$$R = g_1^{q_1} \cdots g_n^{q_n} = g_1^{q_2 + \ell p} (g_2 \cdots g_n)^{q_2} = g_1^{\ell p} (g_1 \cdots g_n)^{q_2}.$$

It follows now that R maps under ϕ' to the relation

$$\ell(p e_1) + q_2 (e_1 + \cdots + e_n)$$

in $\bar{A}$. Since R was chosen arbitrarily, this shows that ϕ' descends to a homomorphism $\langle S \rangle \to \bar{A}$. This completes the proof of the lemma. $\qquad\square$

We can now give the proof of Proposition 2.2.

Proof of Proposition 2.2. We begin with the claim that if G is a group and $S \subset G$ is totally symmetric of size n, then there is subgroup $\Gamma < G$ and a surjective homomorphism $\Gamma \to S_n$. To prove the claim, we first observe that G acts by conjugation on the set of totally symmetric subsets of G of cardinality n. Explicitly, if $S = \{x_1, \ldots, x_n\} \subset G$ is totally symmetric then

$$g \cdot S = \{gx_1 g^{-1}, \ldots, gx_n g^{-1}\}.$$

Let Γ denote the stabilizer of S in G. There is a homomorphism $\phi : \Gamma \to \mathrm{Sym}(S) \cong S_n$ defined by sending $\gamma \in \Gamma$ to the automorphism defined by $s \to \gamma s \gamma^{-1}$. Since S is totally symmetric, every permutation of S can be realized as the image of some element $\gamma \in \Gamma$. Thus ϕ is surjective and the claim follows.

The kernel of ϕ consists of those elements of Γ that commute with each element of S. Since the elements of S commute pairwise, any element of $\langle S \rangle$ commutes with each element of S. This shows that $\langle S \rangle \subset \ker \phi$. By Proposition 2.2 we have that $|\langle S \rangle| \geq 2^{n-1}$, and so we further have that $|\ker \phi| \geq 2^{n-1}$. The proposition then follows from the fact that $|\ker \phi||S_n| = |\Gamma| \leq |G|$.

$\qquad\square$

3. The proofs of Theorems 1.1 and 1.2

We now have everything we need to prove Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1. Let $f : B_n \to G$ be any homomorphism that is not cyclic. We claim first that $|f(X)| = \lfloor n/2 \rfloor$. The theorem will follow from this claim, since Proposition 2.2 will then imply that $|G| \geq 2^{\lfloor n/2 \rfloor - 1}(\lfloor n/2 \rfloor)!$.

Since $|X| = \lfloor n/2 \rfloor$, we have that $|f(X)| \leq \lfloor n/2 \rfloor$. For the sake of deriving a contradiction, assume that $|f(X)| \leq \lfloor n/2 \rfloor - 1$. Lemma 2.1 then implies that $|f(X)| = 1$, and so we have that

$$f(\sigma_1) = f(\sigma_3).$$

In other words, $\sigma_1 \sigma_3^{-1} \in \ker f$. Since the normal closure of $\sigma_1 \sigma_3^{-1}$ in B_n is equal to B_n' (see [8]) and $\ker f$ is a normal subgroup of B_n, we have that $B_n' \subset \ker f$. In view of the fact that $B_n/B_n' \cong \mathbb{Z}$, we have that f factors through $\mathbb{Z}$. That is, f is cyclic. This contradiction shows that $|f(X)| = \lfloor n/2 \rfloor$, as claimed. This completes the proof. $\qquad\square$

We now give the proof of Theorem 1.2.

Proof of Theorem 1.2. We first dispense with the case $n = 5$. As we pointed out in the introduction, any quotient of B_5' is perfect and the smallest non-trivial perfect group is the alternating group A_5. It follows that if $f : B_5' \to G$ is non-trivial then $|G| \geq |A_5| = 60$. Since $60 > 2^{2-2}(2-1)! = 2^{\lfloor n/2 \rfloor - 2}(\lfloor n/2 \rfloor - 1)!$ the proof is complete in the case $n = 5$.

Assume now that $n \geq 6$. Let $f : B_n' \to G$ be a non-trivial homomorphism. Recall from Section 2 that there is a totally symmetric subset X' of B_n' of size $\lfloor n/2 \rfloor - 1$ defined by

$$X' = \{\sigma_1 \sigma_{2i-1}^{-1}\}_{i=2}^{\lfloor n/2 \rfloor}.$$

Lemma 2.1 implies that either $|f(X')| = |X'| = \lfloor n/2 \rfloor - 1$ or that $|f(X')| = 1$. Assume that $|f(X')| = \lfloor n/2 \rfloor - 1$. It follows directly from Proposition 2.2 that $|G| \geq 2^{\lfloor n/2 \rfloor - 2}(\lfloor n/2 \rfloor - 1)!$. To complete the proof, it therefore suffices to show that if $|f(X')| = 1$ then f is trivial.

We now assume that $|f(X')| = 1$. The assumption implies that $f(\sigma_1\sigma_3^{-1}) = f(\sigma_1\sigma_5^{-1})$ and therefore that $f(\sigma_3\sigma_5^{-1}) = 1$. The element $\sigma_3\sigma_5^{-1}$ is conjugate in B_n to $\sigma_1\sigma_3^{-1}$, and, since the normal closure of $\sigma_1\sigma_3^{-1}$ in B_n is equal to B_n', the normal closure of $\sigma_3\sigma_5^{-1}$ in B_n is also equal to B_n'. It follows that the normal closure of $\sigma_3\sigma_5^{-1}$ in B_n' is also equal to B_n' (see [3, Lemma 8.3]). It further follows that $B_n' \subset \ker f$, and hence that $B_n' = \ker f$. That is, f is trivial. This completes the proof. □

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